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Research article

A regularity criterion of weak solutions to the 3D Boussinesq equations

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Abstract: In this note, a regularity criterion of weak solutions to the 3D-Boussinesq equations with respect to Serrin type condition under the framework of Besov space $\dot{B}^{r}_{\infty,\infty}$. It is shown that the weak solution (u,θ) is regular on $(0,T]$ if u satisfies

$$\int_0^T \|u(\cdot,t)\|^{\frac{2}{1+r}}_{\dot{B}^{r}_{\infty,\infty}}\, dt < \infty,$$

for $0 < r < 1$. This result improves some previous works.

Keywords: Regularity criterion; Boussinesq equations; a priori estimates
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1. Introduction and main result

In this work, we consider the Cauchy problem of 3D viscous incompressible Boussinesq equations [17]:

$$\begin{cases} \partial_t u + u\cdot\nabla u - \Delta u + \nabla\pi = \theta e_3, \\ \partial_t\theta + u\cdot\nabla\theta - \Delta\theta = 0, \\ \nabla\cdot u = 0, \\ (u,\theta)(x,0) = (u_0,\theta_0)(x), \quad x\in\mathbb{R}^3, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ and $\theta = \theta(x,t)$ denote the unknown velocity vector field and the scalar function temperature, while u_0, θ_0 with $\nabla \cdot u_0 = 0$ in the sense of distribution are given initial data. $e_3 = (0,0,1)^T$. $\pi = \pi(x,t)$ the pressure of fluid at the point $(x,t) \in \mathbb{R}^3 \times (0,\infty)$. There are a huge literatures on the incompressible Boussinesq equations such as [1, 3–5, 7, 9–11, 22, 24–27] and the references therein.

When $\theta = 0$, (1.1) reduces to the well-known incompressible Navier-Stokes equations and many results are available. Since Leray [16] and Hopf [12] constructed the so-called well-known Leray-Hopf weak solution $u(x,t)$ of the incompressible Navier-Stokes equation for arbitrary $u_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0(x) = 0$ in last century, the problem on the uniqueness and regularity of the Leray-Hopf weak solutions is one of the most challenging problem of the mathematical community. There are two approaches to tackle this problem : The first is to study the partial regularity of suitable weak solutions to Navier-Stokes equation which was initiated by L. Caffarelli, R. Kohn and L. Nirenberg [2]. The other way is to propose different criteria to guarantee the regularity of the weak solutions which was studied by G. Prodi [19], J. Serrin [20], Struwe [21], etc. However, similar to the Navier-Stokes equations, the question of global regularity of the weak solutions of the 3D Boussinesq equations still remains a big open problem. This paper is concerned with the second approach and is devoted to presenting an improved regularity criterion of weak solutions for the 3D Boussinesq equations in the Besov space.

There has been a lot of work on the regularity theory of Boussinesq equations [6, 22, 24, 25, 27, 28]. In particular, Fan and Ozawa [6] showed that the weak solution becomes regular if the velocity satisfies

$$\int_0^T \|u(\cdot,t)\|^2_{\dot{B}^0_{\infty,\infty}} \, dt < \infty.$$

Before stating our main result, let us first recall the definition of the homogeneous Besov space (see e.g. [23]).

Definition 1.1. *Let $\left\{\varphi_j\right\}_{j\in\mathbb{Z}}$ be the Littlewood-Paley dyadic decomposition of unity that satisfies $\widehat{\varphi} \in C_0^\infty\left(B_2 \backslash B_{\frac{1}{2}}\right)$, $\widehat{\varphi}_j(\xi) = \widehat{\varphi}\left(2^{-j}\xi\right)$ and*

$$\sum_{j\in\mathbb{Z}} \widehat{\varphi}_j(\xi) = 1 \ \textit{for any} \ \ \xi \neq 0,$$

where B_R is the ball in $\mathbb{R}^3$ centered at the origin with radius $R > 0$. The homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^3)$ are defined to be

$$\dot{B}^s_{p,q}(\mathbb{R}^3) = \left\{ f \in \mathcal{S}'(\mathbb{R}^3)/\mathcal{P}(\mathbb{R}^3) : \|f\|_{\dot{B}^s_{p,q}} < \infty \right\}$$

where

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} \left(\sum\limits_{j\in\mathbb{Z}} \left\|2^{js}\varphi_j * f\right\|^q_{L^p} \right)^{\frac{1}{q}} & \textit{if} \ \ 1 < q < \infty, \\ \sup\limits_{j\in\mathbb{Z}} 2^{js} \left\|\varphi_j * f\right\|_{L^p} & \textit{if} \ \ q = \infty, \end{cases}$$

for $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$, where $\mathcal{S}'$ is the space of tempered distributions and $\mathcal{P}$ is the space of polynomials.

To aid the introduction of our main result, we recall the definition of weak solutions.

Definition 1.2. *Let* $(u_0, \theta_0) \in L^2(\mathbb{R}^3)$ *with* $\operatorname{div} u_0 = 0$ *in the sense of distributions. A measurable pair* (u, θ) *is said to be a weak solution of (1.1) on* $(0, T)$*, provided that*

a) $(u, \theta) \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$*;*

b) *(1.1)*$_{1,2,3}$ *are satisfied in the sense of distributions;*

c) *the strong energy inequality*

$$\|u(\cdot, t)\|_{L^2}^2 + \|\theta(\cdot, t)\|_{L^2}^2 + 2\int_\epsilon^t (\|\nabla u(\cdot, \tau)\|_{L^2}^2 + \|\nabla\theta(\cdot, \tau)\|_{L^2}^2) d\tau \leq \|u(\cdot, \epsilon)\|_{L^2}^2 + \|\theta(\cdot, \epsilon)\|_{L^2}^2 ,$$

for all $0 \leq \epsilon \leq t \leq T$.

By a strong solution we mean that a weak solution (u, θ) of the Boussinesq equations (1.1) satisfies

$$(u, \theta) \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3)).$$

It is well known that the strong solution is regular and unique.

The main result on the regularity criterion of the weak solutions now reads :

Theorem 1.3. *Suppose* $(u_0, \theta_0) \in L^2(\mathbb{R}^3)$ *with* $\operatorname{div} u_0 = 0$ *in the sense of distributions. Assume that* $(u(x, t), \theta(x, t))$ *is a weak solution of (1.1) on* $\mathbb{R}^3 \times (0, T)$ *and satisfies the strong energy inequality. If* u *satisfies*

$$\int_0^T \|u(\cdot, t)\|_{\dot{B}^r_{\infty,\infty}}^{\frac{2}{1+r}} dt < \infty \quad \textit{with} \;\; 0 < r < 1, \tag{1.2}$$

then the weak solution (u, θ) *becomes a regular solution on* $(0, T]$.

Remark 1.1. *If* $r > 0$*, we have*

$$B^r_{\infty,\infty} = L^\infty \cap \dot{B}^r_{\infty,\infty} \textit{ and } \quad \|f\|_{B^r_{\infty,\infty}} \approx \|f\|_{\dot{B}^r_{\infty,\infty}} + \|f\|_{L^\infty} .$$

Here $B^r_{\infty,\infty}$ *is the inhomogeneous Besov space. Definitions and basic properties of the inhomogeneous Besov spaces can be find in [23]. For concision, we omit them here. So this result is an improvement of the earlier regularity criterion.*

In order to prove our main result, we need the following lemma.

Lemma 1.4. *Let* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *be a function such that* $f(x) = ax^{1-r} + bx^{-r}$*, for all* $0 < r < 1$ *and* $a, b \in \mathbb{R}_+$*. Then there holds*

$$f(x) \leq \left[\left(\frac{r}{1-r}\right)^{1-r} + \left(\frac{1-r}{r}\right)^r\right] a^r b^{1-r}.$$

The proof of this lemma is straight forward and can be obtained by simple calculations.

2. Proof of Theorem 1.3

Proof: Apply ∇ operator to the equation of $(1.1)_1$ and $(1.1)_2$, then taking the inner product with ∇u and $\nabla\theta$, respectively and using integration by parts, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2 \\
= &-\int_{\mathbb{R}^3} \nabla u\cdot\nabla u\cdot\nabla u dx + \int_{\mathbb{R}^3}\nabla(\theta e_3)\cdot\nabla u dx - \int_{\mathbb{R}^3}\nabla u\cdot\nabla\theta\cdot\nabla\theta dx \\
= &\ \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned} \tag{2.1}$$

According to the homogeneous Littlewood-Paley decomposition, ∇u can be written as

$$\nabla u = \sum_{j=-\infty}^{+\infty}\Delta_j(\nabla u) = \sum_{j=-\infty}^{N}\Delta_j(\nabla u) + \sum_{j=N+1}^{+\infty}\Delta_j(\nabla u),$$

where N is a positive integer to be chosen later. We decompose $\mathcal{I}_1$ as follows

$$\begin{aligned}
\mathcal{I}_1 &= -\int_{\mathbb{R}^3}\sum_{j=-\infty}^{N}\Delta_j(\nabla u)\cdot\nabla u\cdot\nabla u dx - \int_{\mathbb{R}^3}\sum_{j=N+1}^{+\infty}\Delta_j(\nabla u)\cdot\nabla u\cdot\nabla u dx \\
&\le \left|\int_{\mathbb{R}^3}\sum_{j=-\infty}^{N}\Delta_j(\nabla u)\cdot\nabla u\cdot\nabla u dx\right| + \left|\int_{\mathbb{R}^3}\sum_{j=N+1}^{+\infty}\Delta_j(\nabla u)\cdot\nabla u\cdot\nabla u dx\right| \\
&\le \sum_{j=-\infty}^{N}\int_{\mathbb{R}^3}\left|\Delta_j(\nabla u)\right| |\nabla u|^2\, dx + 2\sum_{j=-\infty}^{N}\int_{\mathbb{R}^3}\left|\Delta_j u\right| |\nabla u|\, |\Delta u|\, dx \\
&= \mathcal{I}_{11} + \mathcal{I}_{12}.
\end{aligned}$$

For $\mathcal{I}_{11}$, by Hölder inequality and the definition of Besov space, for $0 < r < 1$, we derive that

$$\begin{aligned}
\mathcal{I}_{11} &\le \sum_{j=-\infty}^{N}\left\|\Delta_j(\nabla u)\right\|_{L^\infty}\|\nabla u\|_{L^2}^2 \\
&= \|\nabla u\|_{L^2}^2\sum_{j=-\infty}^{N}2^{(-1+r)j}\left\|\Delta_j(\nabla u)\right\|_{L^\infty}2^{(1-r)j} \\
&\le C\left(\sum_{j=-\infty}^{N}2^{(1-r)j}\right)\sup_{j\in\mathbb{Z}}\left(2^{(-1+r)j}\left\|\Delta_j(\nabla u)\right\|_{L^\infty}\right)\|\nabla u\|_{L^2}^2 \\
&\le C2^{(1-r)N}\|\nabla u\|_{\dot{B}^{-1+r}_{\infty,\infty}}\|\nabla u\|_{L^2}^2 \\
&\le C2^{(1-r)N}\|u\|_{\dot{B}^{r}_{\infty,\infty}}\|\nabla u\|_{L^2}^2\,.
\end{aligned} \tag{2.2}$$

For $\mathcal{I}_{12}$, in view of the definition of Besov space, it follows that

$$\mathcal{I}_{12} \le \sum_{j=N+1}^{+\infty}\left\|\Delta_j u\right\|_{L^\infty}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}$$

$$
\begin{aligned}
&= \|\nabla u\|_{L^2}\|\Delta u\|_{L^2}\sum_{j=N+1}^{+\infty}2^{-rj}\left(2^{rj}\left\|\Delta_j u\right\|_{L^\infty}\right)\\
&\leq C\left(\sum_{j=N+1}^{+\infty}2^{-rj}\right)\left(\sup_{j\in\mathbb{Z}}2^{rj}\left\|\Delta_j u\right\|_{L^\infty}\right)\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}\\
&\leq C2^{-Nr}\|u\|_{\dot{B}^r_{\infty,\infty}}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}. \qquad (2.3)
\end{aligned}
$$

It follows from (2.2)-(2.3) and Lemma 1.4 with $x = 2^N$, $a = \|\nabla u\|_{L^2}$ and $b = \|\Delta u\|_{L^2}$ that

$$
\begin{aligned}
\mathcal{I}_1 &\leq C2^{(1-r)N}\|u\|_{\dot{B}^r_{\infty,\infty}}\|\nabla u\|_{L^2}^2 + C2^{-Nr}\|u\|_{\dot{B}^r_{\infty,\infty}}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}\\
&= C\|u\|_{\dot{B}^r_{\infty,\infty}}\|\nabla u\|_{L^2}\left(2^{(1-r)N}\|\nabla u\|_{L^2} + 2^{-rN}\|\Delta u\|_{L^2}\right)\\
&\leq C\|u\|_{\dot{B}^r_{\infty,\infty}}\|\nabla u\|_{L^2}\left[\left(\frac{r}{1-r}\right)^{1-r} + \left(\frac{1-r}{r}\right)^{r}\right]\|\nabla u\|_{L^2}^r\|\Delta u\|_{L^2}^{1-r}\\
&\leq C\|u\|_{\dot{B}^r_{\infty,\infty}}\|\nabla u\|_{L^2}^{1+r}\|\Delta u\|_{L^2}^{1-r},
\end{aligned}
$$

by choosing

$$
N = \left[\frac{1}{\ln 2}\ln\left(\frac{r}{1-r}\frac{\|\Delta u\|_{L^2}}{\|\nabla u\|_{L^2}}\right)\right].
$$

By Young's inequality, we get

$$
\mathcal{I}_1 \leq \frac{1}{2}\|\Delta u\|_{L^2}^2 + C\|u\|_{\dot{B}^r_{\infty,\infty}}^{\frac{2}{1+r}}\|\nabla u\|_{L^2}^2.
$$

We estimate $\mathcal{I}_3$ in the same way as $\mathcal{I}_1$. We decompose $\mathcal{I}_3$ as follows

$$
\begin{aligned}
\mathcal{I}_3 &= -\int_{\mathbb{R}^3}\sum_{j=-\infty}^{N}\Delta_j(\nabla u)\cdot\nabla\theta\cdot\nabla\theta dx - \int_{\mathbb{R}^3}\sum_{j=N+1}^{+\infty}\Delta_j(\nabla u)\cdot\nabla\theta\cdot\nabla\theta dx\\
&\leq \sum_{j=-\infty}^{N}\int_{\mathbb{R}^3}\left|\Delta_j(\nabla u)\right||\nabla\theta|^2dx + 2\sum_{j=-\infty}^{N}\int_{\mathbb{R}^3}\left|\Delta_j u\right||\nabla\theta||\Delta\theta|dx\\
&= \mathcal{I}_{31} + \mathcal{I}_{32}.
\end{aligned}
$$

Then, by using Lemma 1.4, $\mathcal{I}_3$ can be estimated as

$$
\begin{aligned}
\mathcal{I}_3 &\leq C2^{(1-r)N}\|u\|_{\dot{B}^r_{\infty,\infty}}\|\nabla\theta\|_{L^2}^2 + C2^{-Nr}\|u\|_{\dot{B}^r_{\infty,\infty}}\|\nabla\theta\|_{L^2}\|\Delta\theta\|_{L^2}\\
&= C\|u\|_{\dot{B}^r_{\infty,\infty}}\|\nabla\theta\|_{L^2}\left(2^{(1-r)N}\|\nabla\theta\|_{L^2} + 2^{-rN}\|\Delta\theta\|_{L^2}\right)\\
&\leq C\|u\|_{\dot{B}^r_{\infty,\infty}}\|\nabla\theta\|_{L^2}^{1+r}\|\Delta\theta\|_{L^2}^{1-r}\\
&\leq \frac{1}{2}\|\Delta\theta\|_{L^2}^2 + C\|u\|_{\dot{B}^r_{\infty,\infty}}^{\frac{2}{1+r}}\|\nabla\theta\|_{L^2}^2
\end{aligned}
$$

The term $\mathcal{I}_2$ can be estimated by Cauchy's inequality as

$$
\mathcal{I}_2 \leq \|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2} \leq \frac{1}{2}(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2).
$$

Plugging all the estimates into (2.1) yields that

$$\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2 \le C(1 + \|u\|_{\dot{B}^{r}_{\infty,\infty}}^{\frac{2}{1+r}})(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2).$$

Applying Gronwall's inequality, we get

$$(u,\theta) \in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3)).$$

Therefore, by the standard regularity arguments of weak solutions to drive high-order derivative bounds, which would imply

$$(u,\theta) \in C^\infty(\mathbb{R}^3 \times (0,T))$$

by Sobolev imbedding theorems, as desired. The proof of Theorem 1.3 is completed. □

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Conflict of Interest

All authors declare no conflicts of interest in this paper.

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