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Research article

Large Deviations for Stochastic Fractional Integrodifferential Equations

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Abstract: In this work we establish a Freidlin-Wentzell type large deviation principle for stochastic fractional integrodifferential equations by using the weak convergence approach. The compactness argument is proved on the solution space of corresponding skeleton equation and the weak convergence is done for Borel measurable functions whose existence is asserted from Yamada-Watanabe theorem. Examples are included which illustrate the theory and also depict the link between large deviations and optimal controllability.

Keywords: Fractional differential equations; Large deviation principle; Stochastic integrodifferential equations
Mathematics subject classification: 34A08, 45J05, 60F10, 60H10

1. Introduction

The subject of fractional calculus deals with the investigations of derivatives and integrals, of any arbitrary real or complex order, which unify and extend the notions of integer-order derivative and n-fold integral. It can be considered as a branch of mathematical analysis which deals with integrodifferential operators and equations where the integrals are of convolution type and exhibit (weakly singular) kernels of power-law type. It is strictly related to the theory of pseudo-differential operators. Fractional order models have the tendency to capture non-local relations in space and time, thus forming an improvised model for analyzing complex phenomena. It is a successful tool for describing complex quantum field dynamical systems, dissipation and long-range phenomena that cannot be well illustrated using ordinary differential and integral operators. For an introductory study on fractional calculus and fractional derivatives, see the literatures [19, 21, 25].

Inducing randomness into the model helps us to analyze better by taking into consideration the effect of uncertainty, thus leading to stochastic fractional differential equations (refer [24] and references therein). The theory of existence, controllability and stability of fractional differential equations has been studied by many authors (for instance, see [1, 2, 15, 16]). However there seems to be possibly limited literature to the study of large deviations for stochastic fractional differential equations.

Large deviation theory is a branch of probability theory that deals with the study of rare events. Though the probability of occurrence of rare events is too small, their impact may be large and so it is significant to study such rare events. Large deviation theory finds its application in many areas such as mathematical finance, statistical mechanics and various fields ranging from physics to biology. The origin of large deviations dates back to the 1930s where there was a necessity to solve the problem of total claim exceeding the reserve fund set aside in an insurance company. The solution was discovered by the Swedish mathematician Cramer via refinement of the central limit theorem. Subsequent developments has been made since then and there was major breakthrough into the subject after Varadhan [31] established a general framework for large deviation principle and formulated the Varadhan's lemma in 1966. In 1970, Wentzell and Freidlin [13] developed a theory to enhance the large deviation principle for differential equations with small stochastic perturbations, which involves time discretization of the original problem and then analyzing the large deviation principle in the limit. Fleming [12] developed a stochastic control approach to establish large deviation principle and then Dupuis and Ellis [11] combined the weak convergence approach with the theory of Fleming. These developments indeed explore the close association of large deviation theory with optimal controllability problems.

Using the weak convergence approach, the large deviations for homeomorphism flows of non-Lipschitz Stochastic Differential Equations (SDEs) was studied by Ren and Zhang [27]; the large deviations for two-dimensional stochastic Navier-Stokes equations by Sritharan and Sundar [28], and for stochastic evolution equations with small multiplicative noise by Liu [18]. For more references on this approach, one may refer [5, 6, 11, 14, 26]. By using the approximating method, Mohammed and Zhang [23] established a Freidlin-Wentzell type large deviation principle for the stochastic delay differential equations. Mo and Luo [22] also studied the large deviations for the stochastic delay differential equations by employing the weak convergence approach. Bo and Jiang [4] analyzed the large deviation for Kuramoto-Sivashinsky stochastic partial differential equation. A large deviation principle for stochastic differential equations with deviating arguments is dealt with in [30].

A Freidlin-Wentzell type large deviation principle is discussed in Dembo and Zeitouni [8] for the following stochastic differential equation:

$$\left.\begin{array}{rcl} dX(t) &=& b(t, X(t))\mathrm{d}t + \sqrt{\epsilon}\sigma(t, X(t))\mathrm{d}W(t), \quad t \in (0, T], \\ X(0) &=& X_0. \end{array}\right\} \tag{1}$$

In the case that the system is affected by hereditary influences, the drift and diffusion coefficients (b and σ) also depend on an integral component, thus giving rise to stochastic integrodifferential equations. The large deviations for stochastic integrodifferential equations has been carried out in [29]. In this paper, we consider the stochastic fractional integrodifferential equations with Gaussian noise perturbation of multiplicative type and establish the large deviation principle by using the results developed by Budhiraja and Dupuis [7]. The compactness argument is done with the associated control equation and weak convergence result is obtained by observing the nature of the solution of the stochastic control equation as the perturbation of the noise term tends to zero.

2. Preliminaries

Let $\mathbb{X}$ and $\mathbb{H}$ be separable Hilbert spaces. Denote by $L(\mathbb{X})$ the space of all bounded linear operators from $\mathbb{X}$ to $\mathbb{X}$. Denote by J the time interval $[0, T]$. Let $\{\Omega, \mathcal{F}, \mathbb{P}\}$ be a complete filtered probability space equipped with a complete family of right continuous increasing sub σ-algebras $\{\mathcal{F}_t, t \in J\}$ satisfying $\{\mathcal{F}_t \subset \mathcal{F}\}$. Let Q be a symmetric, positive, trace class operator on $\mathbb{H}$ and $W(\cdot)$ be a $\mathbb{H}$-valued Wiener process with covariance operator Q. Denote the space $\mathcal{H}_0 := Q^{1/2}\mathbb{H}$. Then $\mathcal{H}_0$ is a Hilbert space with the inner product $(X, Y)_0 := (Q^{-1/2}X, Q^{-1/2}Y)$ for all $X, Y \in \mathcal{H}_0$ and the corresponding norm is denoted by $\|\cdot\|_0$. Let L_Q denote the space of all Hilbert-Schmidt operators from $\mathcal{H}_0$ to $\mathbb{X}$. Consider the nonlinear stochastic fractional integrodifferential equation in $\mathbb{X}$ of the form

$$\left.\begin{array}{rcl} {}^CD^\alpha X(t) &=& AX(t) + b\left(t, X(t), \int_0^t f(t, s, X(s))\mathrm{d}s\right) + \sigma\left(t, X(t), \int_0^t g(t, s, X(s))\mathrm{d}s\right)\frac{\mathrm{d}W(t)}{\mathrm{d}t},\ t \in J, \\ X(0) &=& X_0, \end{array}\right\} \tag{2}$$

where $1/2 < \alpha \leq 1, X_0 \in \mathbb{X}$ and $A : \mathbb{X} \to \mathbb{X}$ is a bounded linear operator. Also the drift coefficient $b : J \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$, the noise coefficient $\sigma : J \times \mathbb{X} \times \mathbb{X} \to L_Q(\mathcal{H}_0; \mathbb{X})$ and $f, g : J \times J \times \mathbb{X} \to \mathbb{X}$. Assume the following Lipschitz conditions on the drift and noise coefficients: For all $x_1, x_2, y_1, y_2 \in \mathbb{X}$ and $0 \leq s \leq t \leq T$, there exist constants $L_b, L_\sigma, L_f, L_g > 0$ such that

$$\left.\begin{array}{rcl} \|b(t, x_1, y_1) - b(t, x_2, y_2)\|_{\mathbb{X}} &\leq& L_b[\|x_1 - x_2\|_{\mathbb{X}} + \|y_1 - y_2\|_{\mathbb{X}}], \\ \|\sigma(t, x_1, y_1) - \sigma(t, x_2, y_2)\|_{L_Q} &\leq& L_\sigma[\|x_1 - x_2\|_{\mathbb{X}} + \|y_1 - y_2\|_{\mathbb{X}}], \\ \|f(t, s, x_1) - f(t, s, x_2)\|_{\mathbb{X}} &\leq& L_f\|x_1 - x_2\|_{\mathbb{X}}, \\ \|g(t, s, x_1) - g(t, s, x_2)\|_{\mathbb{X}} &\leq& L_g\|x_1 - x_2\|_{\mathbb{X}}. \end{array}\right\} \tag{3}$$

Also assume the following linear growth assumptions on the coefficients: For all $x, y \in \mathbb{X}$ and $0 \leq s \leq t \leq T$, there exist positive constants $K_b, K_\sigma, K_f, K_g > 0$ such that

$$\left.\begin{array}{rcl} \|b(t, x, y)\|^2_{\mathbb{X}} &\leq& K_b[1 + \|x\|^2_{\mathbb{X}} + \|y\|^2_{\mathbb{X}}], \\ \|\sigma(t, x, y)\|^2_{L_Q} &\leq& K_\sigma[1 + \|x\|^2_{\mathbb{X}} + \|y\|^2_{\mathbb{X}}], \\ \|f(t, s, x)\|^2_{\mathbb{X}} &\leq& K_f[1 + \|x\|^2_{\mathbb{X}}], \\ \|g(t, s, x)\|^2_{\mathbb{X}} &\leq& K_g[1 + \|x\|^2_{\mathbb{X}}]. \end{array}\right\} \tag{4}$$

Let us first quote some basic definitions from fractional calculus. For $\alpha, \beta > 0$, with $n - 1 < \alpha < n$, $n - 1 < \beta < n$ and $n \in \mathbb{N}$, D is the usual differential operator and suppose $f \in L_1(\mathbb{R}_+)$, $\mathbb{R}_+ = [0, \infty)$.

(i) Caputo Fractional Derivative:
The Riemann Liouville fractional integral of a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1} f(s)\mathrm{d}s,$$

and the Caputo derivative of f is ${}^CD^\alpha f(t) = I^{n-\alpha} f^{(n)}(t)$, that is,

$${}^CD^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)}\int_0^t (t - s)^{n-\alpha-1} f^{(n)}(s)\mathrm{d}s,$$

where the function $f(t)$ has absolutely continuous derivatives up to order $n - 1$.

(ii) Mittag-Leffler Operator Function: Two parameter family of Mittag-Leffler operator functions is defined as

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(k\alpha + \beta)}, \alpha, \beta > 0.$$

Here A is the bounded linear operator. In particular, for $\beta = 1$, the one parameter Mittag-Leffler operator function is

$$E_\alpha(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(k\alpha + 1)}.$$

The Mittag-Leffler functions are in fact generalizations of the exponential function and are applicable in varied situations involving fractional derivatives, see for example [9]. Assume the following boundedness on the Mittag-Leffler operator functions with one and two parameters:

$$M_1 = \sup_{t\in J} \left\| E_\alpha(At^\alpha) \right\|_{L(\mathbb{X})}, \quad M_2 = \sup_{t\in J} \left\| E_{\alpha,\alpha}(At^\alpha) \right\|_{L(\mathbb{X})}. \tag{5}$$

In order to find the solution representation, we need the following hypothesis and make use of the Lemma that follows.

(H1) The operator $A \in L(\mathbb{X})$ commutes with the fractional integral operator I^α on $\mathbb{X}$ and $\|A\|^2_{L(\mathbb{X})} < \frac{(2\alpha-1)(\Gamma(\alpha))^2}{T^{2\alpha}}$.

Lemma 2.1. *[17] Suppose that A is a linear bounded operator defined on $\mathbb{X}$ (more generally, $\mathbb{X}$ may be a Banach space) and assume that $\|A\|_{L(\mathbb{X})} < 1$. Then $(I - A)^{-1}$ is linear and bounded. Also*

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

The convergence of the above series is in the operator norm and $\|(I - A)^{-1}\|_{L(\mathbb{X})} \le (1 - \|A\|_{L(\mathbb{X})})^{-1}$.

We next show that $\|I^\alpha A\|_{L(\mathbb{X})} < 1$ and, by the Lemma, we obtain $(I - I^\alpha A)^{-1}$ is bounded and linear. Let $X \in \mathbb{X}$; then by (H1), we have

$$\begin{aligned}
\mathbb{E}\Big[\|(I^\alpha A)X\|^2_{\mathbb{C}(J;\mathbb{X})}\Big] &\le \frac{T}{(\Gamma(\alpha))^2}\mathbb{E}\Big[\sup_{t\in J}\int_0^t (t-s)^{2\alpha-2}\|AX(s)\|^2_{\mathbb{X}}\mathrm{d}s\Big] \\
&\le \frac{T^{2\alpha}}{(2\alpha-1)(\Gamma(\alpha))^2}\mathbb{E}\Big[\sup_{t\in J}\|AX(t)\|^2_{\mathbb{X}}\Big] < \mathbb{E}\|X\|^2_{\mathbb{C}(J;\mathbb{X})},
\end{aligned}$$

hence yielding the desired inequality. On the other hand, defining the random differential operator

$$\mathrm{d}F(t, X(t)) := b\left(t, X(t), \int_0^t f(t, s, X(s))\mathrm{d}s\right)\mathrm{d}t + \sigma\left(t, X(t), \int_0^t g(t, s, X(s))\mathrm{d}s\right)\mathrm{d}W(t)$$

and operating by I^α on both sides of (2), we have

$$\begin{aligned}
X(t) &= X_0 + I^\alpha AX(t) + I^\alpha \frac{\mathrm{d}F(t, X(t))}{\mathrm{d}t}, \\
X(t) &= (I - I^\alpha A)^{-1}\left(X_0 + I^\alpha \frac{\mathrm{d}F(t, X(t))}{\mathrm{d}t}\right).
\end{aligned}$$

Therefore, using Lemma 2.1 and the fact that I^α commutes with A, we obtain (see [3, 20])

$$X(t) = \sum_{k=0}^{\infty}(I^\alpha A)^k\left(X_0 + I^\alpha \frac{\mathrm{d}F(t, X(t))}{\mathrm{d}t}\right)$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} I^{k\alpha} A^k X_0 + I^{k\alpha} A^k I^{\alpha} \frac{\mathrm{d}F(t,X(t))}{\mathrm{d}t} \\
&= \sum_{k=0}^{\infty} I^{k\alpha} A^k X_0 + I^{k\alpha+\alpha} A^k \frac{\mathrm{d}F(t,X(t))}{\mathrm{d}t} \\
&= \sum_{k=0}^{\infty} \frac{A^k t^{\alpha k}}{\Gamma(k\alpha+1)} X_0 + \int_0^t (t-s)^{\alpha-1} \left(\sum_{k=0}^{\infty} \frac{A^k (t-s)^{\alpha k}}{\Gamma(k\alpha+\alpha)} \right) \mathrm{d}F(s,X(s)), \\
&= E_\alpha(At^\alpha) X_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) \mathrm{d}F(s,X(s)).
\end{aligned}
$$

Thus we obtain the solution representation of (2) as

$$
\begin{aligned}
X(t) &= E_\alpha(At^\alpha) X_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) b\left(s, X(s), \int_0^s f(s,\tau,X(\tau))\mathrm{d}\tau\right) \mathrm{d}s \\
&\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) \sigma\left(s, X(s), \int_0^s g(s,\tau,X(\tau))\mathrm{d}\tau\right) \mathrm{d}W(s). \qquad (6)
\end{aligned}
$$

We now present some basic definitions and results from large deviation theory. For this, let $\{X^\epsilon\}$ be a family of random variables defined on the space $\mathbb{X}$ and taking values in a Polish space $\mathcal{Z}$ (i.e., a complete separable metric space $\mathcal{Z}$).

Definition 2.1. *(Rate Function). A function $\mathcal{I} : \mathcal{Z} \to [0,\infty]$ is called a rate function if $\mathcal{I}$ is lower semicontinuous. A rate function $\mathcal{I}$ is called a good rate function if for each $N < \infty$, the level set $K_N = \{f \in \mathcal{Z} : \mathcal{I}(f) \le N\}$ is compact in $\mathcal{Z}$.*

Definition 2.2. *(Large Deviation Principle). Let $\mathcal{I}$ be a rate function on $\mathcal{Z}$. We say the family $\{X^\epsilon\}$ satisfies the large deviation principle with rate function $\mathcal{I}$ if the following two conditions hold:*
(i) L*arge deviation upper bound. For each closed subset F of $\mathcal{Z}$,*

$$
\limsup_{\epsilon\to 0} \epsilon \log \mathbb{P}(X^\epsilon \in F) \le -\mathcal{I}(F).
$$

(ii) L*arge deviation lower bound.For each open subset G of $\mathcal{Z}$,*

$$
\liminf_{\epsilon\to 0} \epsilon \log \mathbb{P}(X^\epsilon \in G) \ge -\mathcal{I}(G).
$$

Definition 2.3. *(Laplace Principle). Let $\mathcal{I}$ be a rate function on $\mathcal{Z}$. We say the family $\{X^\epsilon\}$ satisfies the Laplace principle with rate function $\mathcal{I}$ if for all real-valued bounded continuous functions h defined on $\mathcal{Z}$,*

$$
\lim_{\epsilon\to 0} \epsilon \log \mathbf{E}\left\{ \exp\left[-\frac{1}{\epsilon} h(X^\epsilon) \right] \right\} = -\inf_{f\in\mathcal{Z}} \{h(f) + \mathcal{I}(f)\}.
$$

One of the main results of the theory of large deviations is the equivalence between the Laplace principle and the large deviation principle when the underlying space is Polish. For a proof we refer the reader to Theorem 1.2.1 and Theorem 1.2.3 in [11].

Theorem 2.1. *The family $\{X^\epsilon\}$ satisfies the Laplace principle with good rate function $\mathcal{I}$ on a Polish space $\mathcal{Z}$ if and only if $\{X^\epsilon\}$ satisfies the large deviation principle with the same rate function $\mathcal{I}$.*

3. Large Deviation Principle

In this section, we consider the stochastic fractional integrodifferential equation (2) with the random noise term being perturbed by a small parameter $\epsilon > 0$ in the form

$$\left.\begin{array}{rcl} {}^{C}D^{\alpha}X^{\epsilon}(t) & = & AX^{\epsilon}(t) + b\left(t, X^{\epsilon}(t), \int_0^t f(t,s,X^{\epsilon}(s))\mathrm{d}s\right) \\ & & + \sqrt{\epsilon}\,\sigma\left(t, X^{\epsilon}(t), \int_0^t g(t,s,X^{\epsilon}(s))\mathrm{d}s\right)\frac{\mathrm{d}W(t)}{\mathrm{d}t},\ t \in (0,T], \\ X^{\epsilon}(0) & = & X_0. \end{array}\right\} \tag{7}$$

Let $\mathcal{G}^{\epsilon} : \mathbb{C}(J : \mathbb{H}) \to \mathcal{Z}$ be a measurable map defined by $\mathcal{G}^{\epsilon}(W(\cdot)) := X^{\epsilon}(\cdot)$, where X^{ϵ} is the solution of the above equation (7). We implement the variational representation developed by Budhiraja and Dupuis to study the large deviation principle for the solution processes $\{X^{\epsilon}\}$. Let

$$\mathcal{A} = \left\{ v : v \text{ is } \mathcal{H}_0 \text{ - valued } \mathcal{F}_t \text{ - predictable process and } \int_0^T \|v(s,\omega)\|_0^2\,\mathrm{d}s < \infty \text{ a.s.} \right\},$$

$$S_N = \left\{ v \in L^2(J;\mathcal{H}_0) : \int_0^T \|v(s)\|_0^2\,\mathrm{d}s \le N \right\},$$

where $L^2(J;\mathcal{H}_0)$ is the space of all $\mathcal{H}_0$ -valued square integrable functions on J. Then S_N endowed with the weak topology in $L^2(J;\mathcal{H}_0)$ is a compact Polish space (see [10]). Let us also define

$$\mathcal{A}_N = \{ v \in \mathcal{A} : v(\omega) \in S_N\ \mathbb{P} - a.s\}.$$

We now state the variational representation developed by Budhiraja and Dupuis [7, Theorem 4.4] that provides sufficient conditions under which Laplace principle (equivalently, large deviation principle) holds for the family $\{X^{\epsilon}\}$:

Proposition 3.1. *Suppose that there exists a measurable map $\mathcal{G}^0 : \mathbb{C}(J : \mathbb{H}) \to \mathcal{Z}$ such that the following hold:*

(i) Let $\{v^{\epsilon} : \epsilon > 0\} \subset \mathcal{A}_N$ for some $N < \infty$. Let v^{ϵ} converge in distribution as S_N-valued random elements to v. Then $\mathcal{G}^{\epsilon}\left(W(\cdot) + \frac{1}{\sqrt{\epsilon}}\int_0^{\cdot} v^{\epsilon}(s)\,\mathrm{d}s\right)$ converges in distribution to $\mathcal{G}^0\left(\int_0^{\cdot} v(s)\,\mathrm{d}s\right)$.
(ii) For every $N < \infty$, the set

$$K_N := \left\{ \mathcal{G}^0\left(\int_0^{\cdot} v(s)\,\mathrm{d}s \right) : v \in S_N \right\}$$

is a compact subset of $\mathcal{Z}$.

For each $h \in \mathcal{Z}$, define

$$\mathcal{I}(h) := \inf_{\left\{ v \in \mathbb{L}^2(J:\mathcal{H}_0) : h = \mathcal{G}^0\left(\int_0^{\cdot} v(s)\,\mathrm{d}s\right)\right\}} \left\{ \frac{1}{2}\int_0^T \|v(s)\|_0^2\,\mathrm{d}s \right\}, \tag{8}$$

where the infimum over an empty set is taken as ∞. Then the family $\{X^{\epsilon} : \epsilon > 0\} = \mathcal{G}^{\epsilon}(W(\cdot))$ satisfies the Laplace principle in $\mathcal{Z}$ with the rate function $\mathcal{I}$ given by (8).

In Proposition 3.1, (ii) is a compactness criterion and it is to be noticed that it has a coincidence with the fact that the level set for a good rate function is compact. Thanks to the variational representation prescribed by Budhiraja and Dupuis, the study of large deviation principle for any stochastic differential equation can now be simplified to the problem of identifying Borel measurable function $\mathcal{G}^0$ so that the hypothesis in the above proposition is satisfied.

Consider the controlled equation associated to (7) with control $v \in S_N$.

$$\left.\begin{aligned} {}^C D^\alpha X_v(t) &= AX_v(t) + b\left(t, X_v(t), \int_0^t f(t,s,X_v(s))\mathrm{d}s\right) \\ &\quad + \sigma\left(t, X_v(t), \int_0^t g(t,s,X_v(s))\mathrm{d}s\right) v(t),\ t \in (0,T], \\ X_v(0) &= X_0, \end{aligned}\right\} \tag{9}$$

and let $X_v(t)$ denote the solution of the equation (9). The main result in this chapter is the following Freidlin-Wentzell type theorem:

Theorem 3.1. *With the assumption (H1) on the bounded linear operator A, the family $\{X^\epsilon(t)\}$ of solutions of (7) satisfies the large deviation principle (equivalently, Laplace principle) in $C(J;\mathbb{X})$ with the good rate function*

$$\mathcal{I}(h) := \inf\left\{\frac{1}{2}\int_0^T \|v(t)\|_0^2\,\mathrm{d}t; X_v = h\right\}, \tag{10}$$

where $v \in L^2(J;\mathcal{H}_0)$ and X_v denotes the solution of the control equation (9) with the convention that the infimum of an empty set is infinity.

In order to prove the theorem, the main work is to verify the sufficient conditions in Proposition 3.1. Initially we formulate the following perturbed controlled stochastic equation corresponding to (7):

$$\left.\begin{aligned} {}^C D^\alpha X_v^\epsilon(t) &= AX_v^\epsilon(t) + b\left(t, X_v^\epsilon(t), \int_0^t f(t,s,X_v^\epsilon(s))\mathrm{d}s\right) + \sigma\left(t, X_v^\epsilon(t), \int_0^t g(t,s,X_v^\epsilon(s))\mathrm{d}s\right) v(t) \\ &\quad + \sqrt{\epsilon}\sigma\left(t, X_v^\epsilon(t), \int_0^t g(t,s,X_v^\epsilon(s))\mathrm{d}s\right)\frac{\mathrm{d}W(t)}{\mathrm{d}t},\ t \in (0,T], \\ X_v^\epsilon(0) &= X_0. \end{aligned}\right\} \tag{11}$$

The solution representation is given by

$$\begin{aligned} X_v^\epsilon(t) &= E_\alpha(At^\alpha)X_0 + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha) b\left(s, X_v^\epsilon(s), \int_0^s f(s,\tau,X_v^\epsilon(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)\sigma\left(s, X_v^\epsilon(s), \int_0^s g(s,\tau,X_v^\epsilon(\tau))\mathrm{d}\tau\right) v(s)\mathrm{d}s \\ &\quad + \sqrt{\epsilon}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)\sigma\left(s, X_v^\epsilon(s), \int_0^s g(s,\tau,X_v^\epsilon(\tau))\mathrm{d}\tau\right)\mathrm{d}W(s). \end{aligned} \tag{12}$$

Before proceeding further analysis, we show that the solution $X_v^\epsilon(t)$ obeys the following energy estimate:

Theorem 3.2. *The solution $X_v^\epsilon(t)$ of (11) is bounded in the space $\mathbb{L}^2(\Omega;\mathbb{C}(J;\mathbb{X}))$, that is, there exists a positive constant $K > 0$ such that*

$$\mathbb{E}\left[\sup_{t\in J}\|X_v^\epsilon(t)\|_{\mathbb{X}}^2\right] \le K. \tag{13}$$

Proof. First we define the stopping time $\tau_N := \inf\left\{t : \left\|X_v^\epsilon(t)\right\|^2 \geq N\right\}$. And, for any $t \in [0, T \wedge \tau_N]$, consider the solution representation of (11) given by (12), take $\|\cdot\|_{\mathbb{X}}^2$ on both sides and use the algebraic identity $(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2)$ to get

$$\left\|X_v^\epsilon(t)\right\|_{\mathbb{X}}^2 \leq 4\left\|E_\alpha(At^\alpha)\right\|_{L(\mathbb{X})}^2 \|X_0\|_{\mathbb{X}}^2 + 4\left\|\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) b\left(s, X_v^\epsilon(s), \int_0^s f(s,\tau,X_v^\epsilon(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right\|_{\mathbb{X}}^2$$
$$+ 4\left\|\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)\sigma\left(s, X_v^\epsilon(s), \int_0^s g(s,\tau,X_v^\epsilon(\tau))\mathrm{d}\tau\right)v(s)\mathrm{d}s\right\|_{\mathbb{X}}^2$$
$$+ 4\,\epsilon\left\|\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)\sigma\left(s, X_v^\epsilon(s), \int_0^s g(s,\tau,X_v^\epsilon(\tau))\mathrm{d}\tau\right)\mathrm{d}W(s)\right\|_{\mathbb{X}}^2.$$

Using the Holder inequality and the bounds on $\left\|E_\alpha(\cdot)\right\|_{L(\mathbb{X})}$ and $\left\|E_{\alpha,\alpha}(\cdot)\right\|_{L(\mathbb{X})}$ given by (5), we obtain the estimate

$$\begin{aligned}\left\|X_v^\epsilon(t)\right\|_{\mathbb{X}}^2 &\leq 4\,M_1^2\|X_0\|_{\mathbb{X}}^2 + 4M_2^2\int_0^t (t-s)^{2\alpha-2}\mathrm{d}s\int_0^t \left\|b\left(s, X_v^\epsilon(s), \int_0^s f(s,\tau,X_v^\epsilon(\tau))\mathrm{d}\tau\right)\right\|_{\mathbb{X}}^2\mathrm{d}s \\ &\quad + 4\,M_2^2\int_0^t (t-s)^{2\alpha-2}\left\|\sigma\left(s, X_v^\epsilon(s), \int_0^s g(s,\tau,X_v^\epsilon(\tau))\mathrm{d}\tau\right)\right\|_{L_Q}^2\mathrm{d}s\ \int_0^t \|v(s)\|_0^2\mathrm{d}s \\ &\quad + 4\,\epsilon\left\|\int_0^t (t-s)^{2\alpha-2}E_{\alpha,\alpha}(A(t-s)^\alpha)\sigma\left(s, X_v^\epsilon(s), \int_0^s g(s,\tau,X_v^\epsilon(\tau))\mathrm{d}\tau\right)\mathrm{d}W(s)\right\|_{\mathbb{X}}^2.\end{aligned}$$

Now using the linear growth property of 'b' and 'σ' given by (3) results in

$$\begin{aligned}\left\|X_v^\epsilon(t)\right\|_{\mathbb{X}}^2 &\leq 4\,M_1^2\,\|X_0\|_{\mathbb{X}}^2 + 4K_b\,M_2^2\,\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t\left[1+\left\|X_v^\epsilon(s)\right\|_{\mathbb{X}}^2 + \left\|\int_0^s f(s,\tau,X_v^\epsilon(\tau))\mathrm{d}\tau\right\|_{\mathbb{X}}^2\right]\mathrm{d}s \\ &\quad + 4\,K_\sigma M_2^2\,N\int_0^t (t-s)^{2\alpha-2}\left[1+\left\|X_v^\epsilon(s)\right\|_{\mathbb{X}}^2 + \left\|\int_0^s g(s,\tau,X_v^\epsilon(\tau))\mathrm{d}\tau\right\|_{\mathbb{X}}^2\right]\mathrm{d}s \\ &\quad + 4\,\epsilon\left\|\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)\sigma\left(s, X_v^\epsilon(s), \int_0^s g(s,\tau,X_v^\epsilon(\tau))\mathrm{d}\tau\right)\mathrm{d}W(s)\right\|_{\mathbb{X}}^2.\end{aligned}$$

Using Holder's inequality for the integrands $\left\|\int_0^s f(s,\tau,X_v^\epsilon(\tau))\mathrm{d}\tau\right\|_{\mathbb{X}}^2$ and $\left\|\int_0^s g(s,\tau,X_v^\epsilon(\tau))\right\|_{\mathbb{X}}^2$ and also making use of the linear growth property of 'f' and 'g' given by (4), we get, on simplifying,

$$\begin{aligned}\left\|X_v^\epsilon(t)\right\|_{\mathbb{X}}^2 &\leq 4\,M_1^2\|X_0\|_{\mathbb{X}}^2 + 4K_b\,M_2^2\,\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t\left[1+\left\|X_v^\epsilon(s)\right\|_{\mathbb{X}}^2 + K_f T\int_0^t\left[1+\left\|X_v^\epsilon(\tau)\right\|_{\mathbb{X}}^2\right]\mathrm{d}\tau\right]\mathrm{d}s \\ &\quad + 4\,K_\sigma M_2^2\,N\int_0^t (t-s)^{2\alpha-2}\left[1+\left\|X_v^\epsilon(s)\right\|_{\mathbb{X}}^2 + K_g T\int_0^t\left[1+\left\|X_v^\epsilon(\tau)\right\|_{\mathbb{X}}^2\right]\mathrm{d}\tau\right]\mathrm{d}s \\ &\quad + 4\,\epsilon\left\|\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)\sigma\left(s, X_v^\epsilon(s), \int_0^s g(s,\tau,X_v^\epsilon(\tau))\mathrm{d}\tau\right)\mathrm{d}W(s)\right\|_{\mathbb{X}}^2. \end{aligned} \quad (14)$$

The stochastic integral term can be estimated by means of the Burkholder-Davis-Gundy inequality as

$$\mathbb{E}\left\{\sup_{0\leq t\leq T\wedge\tau_N}\left[\left\|\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)\,\sigma\left(s, X_v^\epsilon(s), \int_0^s g(s,\tau,X_v^\epsilon(\tau))\mathrm{d}\tau\right)\mathrm{d}W(s)\right\|_{\mathbb{X}}^2\right]\right\}$$

$$
\begin{aligned}
&\leq M_2^2 \int_0^T (T-s)^{2\alpha-2} \left\| \sigma\left(s, X_v^\epsilon(s), \int_0^s g(s,\tau,X_v^\epsilon(\tau))\mathrm{d}\tau\right)\right\|_{L_Q}^2 \mathrm{d}s \\
&\leq K_\sigma M_2^2 \int_0^T (T-s)^{2\alpha-2}\left[1+\left\|X_v^\epsilon(s)\right\|_{\mathbb{X}}^2 + K_g T\int_0^T \left[1+\left\|X_v^\epsilon(\tau)\right\|_{\mathbb{X}}^2\right]\mathrm{d}\tau\right]\mathrm{d}s \\
&\leq K_\sigma M_2^2 \int_0^T (T-s)^{2\alpha-2}\left[1+\left\|X_v^\epsilon(s)\right\|_{\mathbb{X}}^2\right]\mathrm{d}s + K_\sigma M_2^2 K_g T\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^T \left[1+\left\|X_v^\epsilon(s)\right\|_{\mathbb{X}}^2\right]\mathrm{d}s.
\end{aligned}
$$

Hence (14) becomes, after taking supremum and expectation on both sides and simplifying,

$$
\begin{aligned}
\mathbb{E}\Big[\sup_{0\leq t\leq T\wedge\tau_N}\left\|X_v^\epsilon(t)\right\|_{\mathbb{X}}^2\Big] \;\leq\; & 4\,M_1^2\mathbb{E}\|X_0\|_{\mathbb{X}}^2 + 4K_b\,M_2^2\,(1+K_fT^2)\frac{T^{2\alpha-1}}{2\alpha-1}\mathbb{E}\int_0^T\left[1+\left\|X_v^\epsilon(s)\right\|_{\mathbb{X}}^2\right]\mathrm{d}s \\
&+ 4\,K_\sigma M_2^2\,(N+\epsilon)\,\mathbb{E}\int_0^T (T-s)^{2\alpha-2}\left[1+\left\|X_v^\epsilon(s)\right\|_{\mathbb{X}}^2\right]\mathrm{d}s \\
&+ 4\,K_\sigma M_2^2 K_g T\frac{T^{2\alpha-1}}{2\alpha-1}\,(N+\epsilon)\,\mathbb{E}\int_0^T\left[1+\left\|X_v^\epsilon(s)\right\|_{\mathbb{X}}^2\right]\mathrm{d}s.
\end{aligned}
$$

Further simplifying and applying the well known Gronwall inequality, we end up with

$$
\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_N}\left\|X_v^\epsilon(t)\right\|_{\mathbb{X}}^2\right] \leq \left(4\,M_1^2\mathbb{E}\|X_0\|_{\mathbb{X}}^2 + C_T\right)e^{C_T} = K, \tag{15}
$$

where $C_T = 4M_2^2\frac{T^{2\alpha-1}}{2\alpha-1}\left[K_b\left(1+K_fT^2\right)T + K_\sigma\left(1+K_gT\right)(N+\epsilon)\right]$. Observe that $T\wedge\tau_N \to T$ as $N\to\infty$, hence resulting in (13). □

Lemma 3.1 (Compactness)**.** *Define $\mathcal{G}^0 : C(J;\mathbb{H}) \to C(J;\mathbb{X})$ by*

$$
\mathcal{G}^0(h) := \begin{cases} X_v, & \text{if } h = \int_0^{\cdot} v(s)\mathrm{d}s \text{ for some } v\in S_N, \\ 0\;, & \text{otherwise.}\end{cases}
$$

Then, for each $N<\infty$, the set

$$
K_N = \left\{\mathcal{G}^0\left(\int_0^{\cdot} v(s)\mathrm{d}s\right) : v\in S_N\right\}
$$

is a compact subset of $C(J;\mathbb{X})$.

Proof. Let $\{v_n\}$ be a sequence of controls from S_N that converge weakly to v in $\mathbb{L}^2(J;\mathcal{H}_0)$ and let $X_{v_n}(t)$ denote the solution of (9) with control v replaced by v_n. Take $Y_n(t) = X_{v_n}(t) - X_v(t)$. Then the equation corresponding to $Y_n(t)$ would be

$$
\left.\begin{aligned}
{}^C D^\alpha Y_n(t) &= AY_n(t) + b\left(t, X_{v_n}(t), \int_0^t f(t,s,X_{v_n}(s))\mathrm{d}s\right) - b\left(t, X_v(t), \int_0^t f(t,s,X_v(s))\mathrm{d}s\right) \\
&\quad + \sigma\left(t, X_{v_n}(t), \int_0^t g(t,s,X_{v_n}(s))\mathrm{d}s\right)v_n(t) - \sigma\left(t, X_v(t), \int_0^t g(t,s,X_v(s))\mathrm{d}s\right)v(t), \\
Y_n(0) &= 0.
\end{aligned}\right\} \tag{16}
$$

The solution representation is

$$
\begin{aligned}
Y_n(t) = & \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha}) \left[b\left(s, X_{v_n}(s), \int_0^s f(s,\tau,X_{v_n}(\tau))\mathrm{d}\tau\right) \right. \\
& \left. -b\left(s, X_v(s), \int_0^s f(s,\tau,X_v(\tau))\mathrm{d}\tau\right)\right] \mathrm{d}s \\
& + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha}) \left[\sigma\left(s, X_{v_n}(s), \int_0^s g(s,\tau,X_{v_n}(\tau))\mathrm{d}\tau\right) v_n(s) \right. \\
& \left. -\sigma\left(s, X_v(s), \int_0^s g(s,\tau,X_v(\tau))\mathrm{d}\tau\right) v(s)\right] \mathrm{d}s \\
=: & \, I_1(t) + I_2(t) + I_3(t),
\end{aligned} \tag{17}
$$

where

$$
\begin{aligned}
I_1(t) := & \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha}) \left[b\left(s, X_{v_n}(s), \int_0^s f(s,\tau,X_{v_n}(\tau))\mathrm{d}\tau\right) \right. \\
& \left. -b\left(s, X_v(s), \int_0^s f(s,\tau,X_v(\tau))\mathrm{d}\tau\right)\right] \mathrm{d}s,
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
I_2(t) := & \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha}) \left[\sigma\left(s, X_{v_n}(s), \int_0^s g(s,\tau,X_{v_n}(\tau))\mathrm{d}\tau\right) \right. \\
& \left. -\sigma\left(s, X_v(s), \int_0^s g(s,\tau,X_v(\tau))\mathrm{d}\tau\right)\right] v_n(s)\mathrm{d}s,
\end{aligned} \tag{19}
$$

$$
I_3(t) := \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha})\, \sigma\left(s, X_v(s), \int_0^s g(s,\tau,X_v(\tau))\mathrm{d}\tau\right) (v_n(s)-v(s))\mathrm{d}s. \tag{20}
$$

First consider the integral $I_1(t)$ and taking $\|\cdot\|_{\mathbb{X}}$ on both sides, we get

$$
\begin{aligned}
\left\|I_1(t)\right\|_{\mathbb{X}} \leq & \int_0^t (t-s)^{\alpha-1} \|E_{\alpha,\alpha}(A(t-s)^{\alpha})\|_{L(\mathbb{X})} \left\| b\left(s, X_{v_n}(s), \int_0^s f(s,\tau,X_{v_n}(\tau))\mathrm{d}\tau\right) \right. \\
& \left. - b\left(s, X_v(s), \int_0^s f(s,\tau,X_v(\tau))\mathrm{d}\tau\right)\right\|_{\mathbb{X}} \mathrm{d}s.
\end{aligned}
$$

Using the boundedness of $\|E_{\alpha,\alpha}(\cdot)\|$ given by (5) and Lipschitz continuity of 'b' from (3), we obtain

$$
\left\|I_1(t)\right\|_{\mathbb{X}} \leq L_b M_2 \int_0^t (t-s)^{\alpha-1}\left[\left\|Y_n(s)\right\|_{\mathbb{X}} + \int_0^s \left\|f(s,\tau,X_{v_n}(\tau)) - f(s,\tau,X_v(\tau))\right\|_{\mathbb{X}} \mathrm{d}\tau\right]\mathrm{d}s.
$$

Using the Lipschitz continuity of 'f', we get subsequently

$$
\begin{aligned}
\left\|I_1(t)\right\|_{\mathbb{X}} & \leq L_b\, M_2 \int_0^t (t-s)^{\alpha-1}\left[\left\|Y_n(s)\right\|_{\mathbb{X}} + L_f \int_0^s \left\|Y_n(\tau)\right\|_{\mathbb{X}} \mathrm{d}\tau\right]\mathrm{d}s \\
& \leq L_b\, M_2 \int_0^t (t-s)^{\alpha-1}\left\|Y_n(s)\right\|_{\mathbb{X}} \mathrm{d}s + L_b L_f M_2 \int_0^t (t-s)^{\alpha-1} \int_0^t \left\|Y_n(\tau)\right\|_{\mathbb{X}} \mathrm{d}\tau\, \mathrm{d}s \\
& = L_b\, M_2 \int_0^t (t-s)^{\alpha-1}\left\|Y_n(s)\right\|_{\mathbb{X}} \mathrm{d}s + L_b L_f M_2 \frac{T^{\alpha}}{\alpha} \int_0^t \left\|Y_n(s)\right\|_{\mathbb{X}} \mathrm{d}s.
\end{aligned} \tag{21}
$$

In a similar way, consider the integral $I_2(t)$ and estimating using the boundedness of $\|E_{\alpha,\alpha}(\cdot)\|$ and Lipschitz continuity of 'σ', we get

$$\left\|I_2(t)\right\|_{\mathbb{X}} \leq L_\sigma M_2 \int_0^t (t-s)^{\alpha-1}\Big[\left\|Y_n(s)\right\|_{\mathbb{X}} + \int_0^s \left\|g(s,\tau,X_{v_n}(\tau)) - g(s,\tau,X_v(\tau))\right\|_{\mathbb{X}}\, \mathrm{d}\tau\Big]\|v_n(s)\|_0\, \mathrm{d}s.$$

Now, using the Lipschitz continuity of 'g', we obtain

$$\begin{aligned}\left\|I_2(t)\right\|_{\mathbb{X}} \leq{}& L_\sigma\, M_2 \int_0^t (t-s)^{\alpha-1}\left\|Y_n(s)\right\|_{\mathbb{X}}\|v_n(s)\|_0\, \mathrm{d}s \\ &+ L_\sigma L_g M_2 \int_0^t \left\|Y_n(\tau)\right\|_{\mathbb{X}}\mathrm{d}\tau \int_0^t (t-s)^{\alpha-1}\|v_n(s)\|_0\, \mathrm{d}s. \qquad (22)\end{aligned}$$

For the third integral, applying Holder's inequality, one gets

$$\begin{aligned}\left\|I_3(t)\right\|_{\mathbb{X}} \leq{}& \int_0^t (t-s)^{\alpha-1}\, \left\|E_{\alpha,\alpha}(A(t-s)^\alpha)\right\|_{L(\mathbb{X})}\left\|\sigma\left(s, X_v(s), \int_0^s g(s,\tau,X_v(\tau))\,\mathrm{d}\tau\right)(v_n(s)-v(s))\right\|_{\mathbb{X}} \mathrm{d}s \\ \leq{}& M_2\left(\int_0^t (t-s)^{2\alpha-2}\,\mathrm{d}s\right)^{1/2}\left(\int_0^t \left\|\sigma\left(s, X_v(s), \int_0^s g(s,\tau,X_v(\tau))\,\mathrm{d}\tau\right)(v_n(s)-v(s))\right\|_{\mathbb{X}}^2 \mathrm{d}s\right)^{1/2} \\ \leq{}& M_2 T_\alpha\left(\int_0^t \left\|\sigma\left(s, X_v(s), \int_0^s g(s,\tau,X_v(\tau))\,\mathrm{d}\tau\right)(v_n(s)-v(s))\right\|_{\mathbb{X}}^2 \mathrm{d}s\right)^{1/2}, \qquad (23)\end{aligned}$$

where $T_\alpha = \frac{T^{\alpha-1/2}}{\sqrt{2\alpha-1}}$. Now (17) becomes, after substituting (21) - (23) and applying Gronwall's inequality,

$$\begin{aligned}\left\|Y_n(t)\right\|_{\mathbb{X}} \leq{}& M_2 T_\alpha\left(\int_0^t \left\|\sigma\left(s, X_v(s), \int_0^s g(s,\tau,X_v(\tau))\,\mathrm{d}\tau\right)(v_n(s)-v(s))\right\|_{\mathbb{X}}^2 \mathrm{d}s\right)^{1/2} \\ &\times \exp\Big\{L_b M_2 \int_0^t (t-s)^{\alpha-1}\mathrm{d}s + L_b L_f M_2 \frac{T^{\alpha+1}}{\alpha} + L_\sigma M_2 \int_0^t (t-s)^{\alpha-1}\|v_n(s)\|_0 \mathrm{d}s \\ &+\ L_\sigma L_g M_2 T \int_0^t (t-s)^{\alpha-1}\|v_n(s)\|_0\, \mathrm{d}s\Big\}.\end{aligned}$$

Applying Holder's inequality to the last two integral terms on the exponential index, one gets

$$\begin{aligned}\left\|Y_n(t)\right\|_{\mathbb{X}} \leq{}& M_2 T_\alpha\left[\int_0^t \left\|\sigma\left(s, X_v(s), \int_0^s g(s,\tau,X_v(\tau))\,\mathrm{d}\tau\right)(v_n(s)-v(s))\right\|_{\mathbb{X}}^2 \mathrm{d}s\right]^{1/2} \\ &\times \exp\Big\{L_b M_2 \frac{T^\alpha}{\alpha} + L_b L_f M_2 \frac{T^{\alpha+1}}{\alpha} \\ &+\ (L_\sigma M_2 + L_\sigma L_g M_2 T)\left(\int_0^t (t-s)^{2\alpha-2}\mathrm{d}s\right)^{1/2}\left(\int_0^t \|v_n(s)\|_0^2 \mathrm{d}s\right)^{1/2}\Big\}.\end{aligned}$$

On simplifying and taking supremum over $t \in J$, we get

$$\sup_{t\in J}\left\|Y_n(t)\right\|_{\mathbb{X}} \leq M_2 T_\alpha\left[\int_0^T \left\|\sigma\left(s, X_v(s), \int_0^s g(s,\tau,X_v(\tau))\,\mathrm{d}\tau\right)(v_n(s)-v(s))\right\|_{\mathbb{X}}^2 \mathrm{d}s\right]^{1/2}$$

$$\times \exp\left\{L_b M_2 \frac{T^\alpha}{\alpha}(1 + L_f T) + L_\sigma M_2 \frac{T^{2\alpha-1}}{2\alpha - 1}(1 + L_g T)\sqrt{N}\right\}. \tag{24}$$

Since $v_n \rightharpoonup v$ weakly in $\mathbb{L}^2(J; \mathcal{H}_0)$ and σ is a Hilbert-Schmidt operator and hence compact, we have that $\sigma v_n \to \sigma v$ strongly in $\mathbb{L}^2(J; \mathbb{X})$ and so $Y_n = X_{v_n} - X_v \to 0$ in $\mathbb{C}(J; \mathbb{X})$, thereby proving the compactness. ◻

Lemma 3.2 (Weak Convergence). *Let $\{v^\epsilon : \epsilon > 0\} \subset \mathcal{A}_N$ for some $N < \infty$. Assume that v^ϵ converge to v in distribution as S_N-valued random elements; then*

$$\mathcal{G}^\epsilon\left(W(\cdot) + \frac{1}{\sqrt{\epsilon}} \int_0^{\cdot} v^\epsilon(s)\mathrm{d}s\right) \to \mathcal{G}^0\left(\int_0^{\cdot} v(s)\mathrm{d}s\right)$$

in distribution as $\epsilon \to 0$.

Proof. Consider the nonlinear stochastic fractional integrodifferential equation (11) with control $v^\epsilon \in \mathbb{L}^2(J; \mathcal{H}_0)$ and let the solution be denoted by $X^\epsilon_{v^\epsilon}(t)$. Take $Y^\epsilon(t) = X^\epsilon_{v^\epsilon}(t) - X_v(t)$. Then

$$\begin{aligned}
Y^\epsilon(t) = {} & \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)\left[b\left(s, X^\epsilon_{v^\epsilon}(s), \int_0^s f(s,\tau, X^\epsilon_{v^\epsilon}(\tau))\mathrm{d}\tau\right)\right. \\
& \left. - b\left(s, X_v(s), \int_0^s f(s,\tau, X_v(\tau))\mathrm{d}\tau\right)\right]\mathrm{d}s \\
& + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)\left[\sigma\left(s, X^\epsilon_{v^\epsilon}(s), \int_0^s g(s,\tau, X^\epsilon_{v^\epsilon}(\tau))\mathrm{d}\tau\right)\right. \\
& \left. - \sigma\left(s, X_v(s), \int_0^s g(s,\tau, X_v(\tau))\mathrm{d}\tau\right)\right] v^\epsilon(s)\mathrm{d}s \\
& + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)\sigma\left(s, X_v(s), \int_0^s g(s,\tau, X_v(\tau))\mathrm{d}\tau\right)(v^\epsilon(s) - v(s))\mathrm{d}s \\
& + \sqrt{\epsilon}\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)\sigma\left(s, X^\epsilon_{v^\epsilon}(s), \int_0^s g(s,\tau, X_v(\tau))\mathrm{d}\tau\right)\mathrm{d}W(s).
\end{aligned}$$

Taking $\|\cdot\|^2$ on both sides and using the algebraic inequality $(a+b+c+d)^2 \le 4(a^2+b^2+c^2+d^2)$, we obtain

$$\left\|Y^\epsilon(t)\right\|^2_{\mathbb{X}} \le \mathcal{I}_1(t) + \mathcal{I}_2(t) + \mathcal{I}_3(t) + \mathcal{I}_4(t), \tag{25}$$

where

$$\begin{aligned}
\mathcal{I}_1(t) := {} & 4\left\|\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)\left[b\left(s, X^\epsilon_{v^\epsilon}(s), \int_0^s f(s,\tau, X^\epsilon_{v^\epsilon}(\tau))\mathrm{d}\tau\right)\right.\right. \\
& \left.\left. - b\left(s, X_v(s), \int_0^s f(s,\tau, X_v(\tau))\mathrm{d}\tau\right)\right]\mathrm{d}s\right\|^2_{\mathbb{X}}, \qquad (26) \\
\mathcal{I}_2(t) := {} & 4\left\|\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)\left[\sigma\left(s, X^\epsilon_{v^\epsilon}(s), \int_0^s g(s,\tau, X^\epsilon_{v^\epsilon}(\tau))\mathrm{d}\tau\right)\right.\right.
\end{aligned}$$

$$
- \sigma\left(s, X_v(s), \int_0^s g(s,\tau,X_v(\tau))\mathrm{d}\tau\right)\Bigg] v^\epsilon(s)\mathrm{d}s\Bigg\|_{\mathbb{X}}^2, \tag{27}
$$

$$
\mathcal{I}_3(t) := 4\left\| \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)\sigma\left(s, X_v(s), \int_0^s g(s,\tau,X_v(\tau))\mathrm{d}\tau\right)(v^\epsilon(s)-v(s))\mathrm{d}s\right\|_{\mathbb{X}}^2, \tag{28}
$$

$$
\mathcal{I}_4(t) := 4\,\epsilon\left\| \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)\sigma\left(s, X^\epsilon_{v^\epsilon}(s), \int_0^s g(s,\tau,X_v(\tau))\mathrm{d}\tau\right)\mathrm{d}W(s)\right\|_{\mathbb{X}}^2. \tag{29}
$$

First consider the integral $\mathcal{I}_1(t)$ and applying Holder's inequality along with the bound for $\left\|E_{\alpha,\alpha}(\cdot)\right\|_{L(\mathbb{X})}$ given by (5) and the Lipschitz continuity of 'b' given by (3), one gets

$$
\begin{aligned}
\mathcal{I}_1(t) &\leq 4\int_0^t (t-s)^{2\alpha-2}\left\|E_{\alpha,\alpha}(A(t-s)^\alpha)\right\|_{L(\mathbb{X})}^2 \mathrm{d}s \\
&\quad \times \int_0^t \left\|b\left(s, X^\epsilon_{v^\epsilon}(s), \int_0^s f(s,\tau,X^\epsilon_{v^\epsilon}(\tau))\mathrm{d}\tau\right) - b\left(s, X_v(s), \int_0^s f(s,\tau,X_v(\tau))\mathrm{d}\tau\right)\right\|_{\mathbb{X}}^2 \mathrm{d}s \\
&\leq 4L_b^2M_2^2\int_0^t (t-s)^{2\alpha-2}\mathrm{d}s\int_0^t\left[\left\|Y^\epsilon(s)\right\|_{\mathbb{X}} + \int_0^s \left\|f(s,\tau,X^\epsilon_{v^\epsilon}(\tau)) - f(s,\tau,X_v(\tau))\right\|\mathrm{d}\tau\right]^2\mathrm{d}s.
\end{aligned}
$$

Using the algebraic identity $(a+b)^2 \leq 2(a^2+b^2)$ and Holder's inequality to the last integral term on the right hand side and then using the Lipschitz continuity of 'f', we obtain simultaneously

$$
\begin{aligned}
\mathcal{I}_1(t) &\leq 8L_b^2\,M_2^2\,\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t\left[\left\|Y^\epsilon(s)\right\|_{\mathbb{X}}^2 + T\int_0^s\left\|f(s,\tau,X^\epsilon_{v^\epsilon}(\tau)) - f(s,\tau,X_v(\tau))\right\|_{\mathbb{X}}^2\right]\mathrm{d}s \\
&\leq 8L_b^2\,M_2^2\,\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t\left[\left\|Y^\epsilon(s)\right\|_{\mathbb{X}}^2 + L_f^2T\int_0^s\left\|Y^\epsilon(\tau)\right\|_{\mathbb{X}}^2\mathrm{d}\tau\right]\mathrm{d}s.
\end{aligned}
$$

On simplifying, the integral $\mathcal{I}_1(t)$ can be estimated as

$$
\mathcal{I}_1(t) \leq\; 8L_b^2\,M_2^2\,(1+L_f^2\,T^2)\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t\left\|Y^\epsilon(s)\right\|_{\mathbb{X}}^2\mathrm{d}s. \tag{30}
$$

Similarly consider the integral $\mathcal{I}_2(t)$, apply Holder's inequality followed by the bound for $\|E_{\alpha,\alpha}(\cdot)\|_{\mathbb{X}}$ and the Lipschitz continuity of 'σ' to get

$$
\begin{aligned}
\mathcal{I}_2(t) &\leq 4\int_0^t (t-s)^{2\alpha-2}\left\|E_{\alpha,\alpha}(A(t-s)^\alpha)\right\|_{L(\mathbb{X})}^2 \mathrm{d}s \\
&\quad \times \int_0^t \left\|\sigma\left(s, X^\epsilon_{v^\epsilon}(s), \int_0^s g(s,\tau,X^\epsilon_{v^\epsilon}(\tau))\mathrm{d}\tau\right) - \sigma\left(s, X_v(s), \int_0^s g(s,\tau,X_v(\tau))\mathrm{d}\tau\right)\right\|_{L_Q}^2 \|v^\epsilon(s)\|_0^2\mathrm{d}s \\
&\leq 8L_\sigma^2\,M_2^2\,\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t\left[\left\|Y^\epsilon(s)\right\|_{\mathbb{X}}^2 + L_g^2T\int_0^s\left\|Y^\epsilon(\tau)\right\|_{\mathbb{X}}^2\mathrm{d}\tau\right]\left\|v^\epsilon(s)\right\|_0^2\mathrm{d}s.
\end{aligned}
$$

On further simplifying and making use of the fact that the control variable $v \in S_N$, we obtain

$$
\mathcal{I}_2(t) \leq 8L_\sigma^2\,M_2^2\,\frac{T^{2\alpha-1}}{2\alpha-1}\Big[\int_0^t\left\|Y^\epsilon(s)\right\|_{\mathbb{X}}^2\left\|v^\epsilon(s)\right\|_0^2\mathrm{d}s + L_g^2\,N\,T\int_0^t\left\|Y^\epsilon(\tau)\right\|_{\mathbb{X}}^2\mathrm{d}\tau\Big]. \tag{31}
$$

Now consider the integral $\mathcal{I}_3(t)$, apply Holder's inequality and the bound for $\left\|E_{\alpha,\alpha}(\cdot)\right\|_{L(\mathbb{X})}$ to obtain

$$\mathcal{I}_3(t) \leq 4M_2^2 \frac{T^{2\alpha-1}}{2\alpha-1} \int_0^t \left\| \sigma\left(s, X_v(s), \int_0^s g(s,\tau,X_v(\tau))\mathrm{d}\tau\right)(v^\epsilon(s)-v(s)) \right\|_{\mathbb{X}}^2 \mathrm{d}s. \tag{32}$$

Finally consider the stochastic integral $\mathcal{I}_4(t)$ and taking supremum and then taking expectation on both sides and making use of the Burkholder-Davis-Gundy inequality, we get

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in J} \mathcal{I}_4(t)\right] &= 4\,\epsilon\,\mathbb{E}\left\{\sup_{t\in J}\left\| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)\sigma\left(s, X_{v^\epsilon}^\epsilon(s), \int_0^s g(s,\tau,X_v(\tau))\mathrm{d}\tau\right)\mathrm{d}W(s)\right\|_{\mathbb{X}}^2\right\} \\
&\leq 4\epsilon M_2^2 \mathbb{E}\int_0^T (T-s)^{2\alpha-2}\left\|\sigma\left(s, X_{v^\epsilon}^\epsilon(s), \int_0^s g(s,\tau,X_{v^\epsilon}^\epsilon(\tau))\,\mathrm{d}\tau\right)\right\|_{L_Q}^2 \mathrm{d}s.
\end{aligned}$$

Using the linear growth property of 'σ' and 'g' and simplifying, we get

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in J} \mathcal{I}_4(t)\right] &\leq 4\epsilon K_\sigma M_2^2 \mathbb{E}\int_0^T (T-s)^{2\alpha-2}\left[1 + \|X_{v^\epsilon}^\epsilon(s)\|_{\mathbb{X}}^2 + K_g T\int_0^s (1+\|X_{v^\epsilon}^\epsilon(\tau)\|_{\mathbb{X}}^2)\mathrm{d}\tau\right]\mathrm{d}s \\
&\leq 4\epsilon K_\sigma M_2^2\left[\mathbb{E}\int_0^T (T-s)^{2\alpha-2}[1+\left\|X_{v^\epsilon}^\epsilon(s)\right\|_{\mathbb{X}}^2]\mathrm{d}s + K_g T\frac{T^{2\alpha-1}}{2\alpha-1}\mathbb{E}\int_0^T [1+\left\|X_{v^\epsilon}^\epsilon(s)\right\|_{\mathbb{X}}^2]\mathrm{d}s\right] \\
&\leq 4\epsilon K_\sigma M_2^2\frac{T^{2\alpha-1}}{2\alpha-1}(1+K_gT^2)\left\{1+\mathbb{E}\left[\sup_{t\in J}\left\|X_{v^\epsilon}^\epsilon(t)\right\|_{\mathbb{X}}^2\right]\right\}.
\end{aligned} \tag{33}$$

With all these estimates on the integrals $\mathcal{I}_i(t), i = 1,2,3,4$, given by (30) - (33), equation (25) becomes, after taking supremum over $t \in J$ and then taking expectation,

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in J}\left\|Y^\epsilon(t)\right\|_{\mathbb{X}}^2\right] &\leq 8L_b^2 M_2^2 (1+L_f^2T^2)\frac{T^{2\alpha-1}}{2\alpha-1}\mathbb{E}\int_0^T \left\|Y^\epsilon(s)\right\|_{\mathbb{X}}^2 \mathrm{d}s \\
&\quad + 8L_\sigma^2 M_2^2 \frac{T^{2\alpha-1}}{2\alpha-1}\left[\mathbb{E}\int_0^T \left\|Y^\epsilon(s)\right\|_{\mathbb{X}}^2\left\|v^\epsilon(s)\right\|_0^2\mathrm{d}s + L_g^2 N T\mathbb{E}\int_0^T \left\|Y^\epsilon(\tau)\right\|_{\mathbb{X}}^2\mathrm{d}\tau\right] \\
&\quad + 4M_2^2\frac{T^{2\alpha-1}}{2\alpha-1}\mathbb{E}\int_0^T\left\|\sigma\left(s, X_v(s), \int_0^s g(s,\tau,X_v(\tau))\mathrm{d}\tau\right)(v^\epsilon(s)-v(s))\right\|_{\mathbb{X}}^2\mathrm{d}s \\
&\quad + 4\epsilon K_\sigma M_2^2\frac{T^{2\alpha-1}}{2\alpha-1}(1+K_gT^2)\left\{1+\mathbb{E}\left[\sup_{t\in J}\left\|X_{v^\epsilon}^\epsilon(t)\right\|_{\mathbb{X}}^2\right]\right\}.
\end{aligned}$$

Applying Gronwall's inequality and further simplifying, we end up with

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in J}\left\|Y^\epsilon(t)\right\|_{\mathbb{X}}^2\right] &\leq \left\{4M_2^2\frac{T^{2\alpha-1}}{2\alpha-1}\mathbb{E}\int_0^T\left\|\sigma\left(s, X_v(s), \int_0^s g(s,\tau,X_v(\tau))\,\mathrm{d}\tau\right)(v^\epsilon(s)-v(s))\right\|_{\mathbb{X}}^2\mathrm{d}s\right. \\
&\quad \left. + 4\,\epsilon K_\sigma M_2^2\frac{T^{2\alpha-1}}{2\alpha-1}(1+K_gT^2)\left(1+\mathbb{E}\left[\sup_{t\in J}\left\|X_{v^\epsilon}^\epsilon(t)\right\|_{\mathbb{X}}^2\right]\right)\right\} \\
&\quad \times\exp\left(8M_2^2\frac{T^{2\alpha-1}}{2\alpha-1}\left[L_b^2(1+L_f^2T^2)T + L_\sigma^2(1+L_g^2T^2)N\right]\right).
\end{aligned} \tag{34}$$

Since σ is a Hilbert-Schmidt operator and hence compact and since $v^\epsilon \rightharpoonup v$ weakly in $\mathbb{L}^2(J;\mathcal{H}_0)$ as $\epsilon \to 0$, we have that $\sigma v^\epsilon \to \sigma v$ strongly in $\mathbb{L}^2(J;\mathbb{X})$ and so $Y^\epsilon = X_{v^\epsilon}^\epsilon - X_v \to 0$ in probability in the space $\mathbb{L}^2(\Omega;\mathbb{C}(J;\mathbb{X}))$. Since convergence in probability always implies convergence in expectation, we have finally proved the required weak convergence criterion. $\square$

4. Examples

Example 4.1. Consider the stochastic fractional integrodifferential equation with additive noise given by

$$\left.\begin{array}{rcl} {}^C D^\alpha X(t) &=& \int_0^t X(s)\,\mathrm{d}s + \sin(X(t)) + \int_0^t \sqrt{1+X^2(s)}\,\mathrm{d}s + \sqrt{\epsilon}\,\frac{\mathrm{d}W(t)}{\mathrm{d}t},\ t \in (0,T], \\ X(0) &=& X_0, \end{array}\right\} \tag{35}$$

with $X_0 \in \mathbb{R}$ and $1/2 < \alpha \leq 1$. The corresponding controlled equation with control $v \in L^2(0,T;\mathbb{R})$ takes the form

$$\begin{array}{rcl} {}^C D^\alpha X_v(t) &=& \displaystyle\int_0^t X_v(s)\mathrm{d}s + \sin(X_v(t)) + \int_0^t \sqrt{1+X_v^2(s)}\mathrm{d}s + v(t), t \in (0,T], \\ X_v(0) &=& X_0. \end{array}$$

It is observed that if there exists a unique solution $X_v(\cdot)$ for the above mentioned equation, then the control $v \in L^2([0,T],\mathbb{R})$ with which the unique solution X_v is attained is also unique and hence the rate function $\mathcal{I} : C([0,T];\mathbb{R}) \to [0,\infty]$ is given explicitly by

$$\mathcal{I}(\phi) = \frac{1}{2}\int_0^T \left| {}^C D^\alpha \phi - \sin\phi - \int_0^t \left(\phi(s) + \sqrt{1+\phi^2(s)}\right)\mathrm{d}s \right|^2 \mathrm{d}t, \tag{36}$$

if ϕ satisfies (35) for appropriate control v, and ∞ otherwise.

Example 4.2. As an example for (7) with multiplicative type noise, consider the following stochastic equation:

$$\left.\begin{array}{rcl} {}^C D^{3/4} X(t) &=& \beta\int_0^t \left[X(s) + \exp\left(\frac{1}{1+X^2(s)}\right)\right]\mathrm{d}s + \sqrt{\epsilon}\eta\int_0^t X(s)\,\mathrm{d}s\,\frac{\mathrm{d}W(t)}{\mathrm{d}t},\ t \in (0,1], \\ X(0) &=& 1, \end{array}\right\} \tag{37}$$

where $\eta, \beta > 0$ are positive constants. Then the rate function $\mathcal{I} : C([0,1];\mathbb{R}) \to [0,\infty]$ is given by

$$\mathcal{I}(\phi) = \inf\left\{\frac{1}{2}\int_0^1 |v(t)|^2\mathrm{d}t : v \in L^2([0,1],\mathbb{R}) \text{ such that } X_v = \phi\right\}, \tag{38}$$

where $\inf\emptyset = \infty$ and X_v is the unique solution of

$$\begin{array}{rcl} X_v(t) &=& 1 + \dfrac{\beta}{\Gamma\left(\frac{3}{4}\right)}\displaystyle\int_0^t \frac{1}{(t-s)^{1/4}}\int_0^s \left[X_v(r) + \exp\left(\frac{1}{1+X_v^2(r)}\right)\right]\mathrm{d}r\,\mathrm{d}s \\ && + \dfrac{\eta}{\Gamma\left(\frac{3}{4}\right)}\displaystyle\int_0^t \frac{v(s)}{(t-s)^{1/4}}\int_0^s X_v(r)\,\mathrm{d}r\,\mathrm{d}s, t \in [0,1]. \end{array} \tag{39}$$

It is evident from (38) that estimating the rate function $\mathcal{I}(\phi)$ is a problem of finding the minimal cost $\frac{1}{2}\int_0^1 |v(t)|^2\mathrm{d}t$, out of all the controls v that steers the desired solution $\phi = X_v$ from (39).

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Conflict of Interest

All authors declare that there is no conflict of interest.

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