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Research article

The norm of pre-Schwarzian derivative on subclasses of bi-univalent functions

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Abstract: In the present paper, we give the best estimates for the norm of pre-Schwarzian derivatives $\|T_f(z)\| = \sup_{|z|<1}(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right|$ for subclasses of bi-univalent functions.

Keywords: bi-univalent functions; bi-starlike functions; subordination; pre-Schwarzian derivatives
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1. Introduction and definitions

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. An analytic function in a domain D is said to be univalent in D if it does not take the same value twice i.e, $f(z_1) \neq f(z_2)$ for all pairs of distinct points z_1 and z_2 in D.

The Koebe one-quarter theorem et al [3] ensures that the image of Δ under every univalent function $f \in \mathcal{A}$ contains the disk with the center at origin and of the radius $1/4$. Thus, every univalent function $f \in \mathcal{A}$ has an inverse $f^{-1} : f(\Delta) \to \Delta$, satisfying $f^{-1}(f(z)) = z,\ (z \in \Delta)$ and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right).$$

Moreover, it is easy to see that the inverse function has the series expansion of the form

$$f^{-1}(w) = w - a_2 w^2 + (2a_2{}^2 - a_3)w^3 - (5a_2{}^3 - 5a_2 a_3 + a_4)w^4 + \ldots;\ w \in \Delta,$$

which implies that f^{-1} is analytic. The derivative of f^{-1} (see pp. 1038 [4]) is given by

$$\frac{d}{dw}\left(f^{-1}(w)\right) = \frac{1}{f'(z)}.$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. We denote the class of bi-univalent functions by σ.(see [2])
The function f in class $\mathcal{A}$ is said to be starlike of order α where $0 \leq \alpha < 1$ in Δ if it satisfies the condition

$$Re\{\frac{zf'(z)}{f(z)}\} > \alpha,$$

where $z \in \Delta$. We denote the class of starlike functions of order α by $S^*(\alpha)$. The function f of the form (1) is said to be bi-starlike function of order α where $0 \leq \alpha < 1$ if each of the following conditions are satisfied

$$Re\{\frac{zf'(z)}{f(z)}\} > \alpha$$

and

$$Re\{\frac{wg'(w)}{g(w)}\} > \alpha,$$

where $f \in \sigma$, $g = f^{-1}$ and $w = f(z)$. We denote the class of bi-starlike functions of order α by $S^*_\sigma(\alpha)$ (see [5]). If f and g are analytic functions in Δ, we say that f is subordinate to g, written as $f \prec g$, if there exists a Schwarz function w analytic in Δ, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \Delta)$, such that $f(z) = g(w(z))$. In particular, when g is univalent then the above definition reduces to $f(0) = 0$ and $f(\Delta) \subseteq g(\Delta)$.
The pre-Schwarzian derivative of f is denoted by

$$T_f(z) = \frac{f''(z)}{f'(z)}$$

and its norm is given by

$$\|T_f\| = \sup_{|z|<1}(1 - |z|^2)\left|\frac{f''(z)}{f'(z)}\right|.$$

This norm have a significant meaning in the theory of Teichmuller spaces. For a univalent function f it is well known that $\|T_f\| < 6$. This is the best possible estimation.
Defining two subclasses for bi-univalent functions as follows

Definition 1.1. *A function f given by* (1) *is said to be in the class $S^*_\sigma[A, B]$, if the following conditions are satisfied*

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz},$$
$$\frac{w\,g'(w)}{g(w)} \prec \frac{1 + Aw}{1 + Bw},$$

where $f \in \sigma$, $g = f^{-1}$, $w = f(z)$, $w \in \Delta$ and $-1 \leq B < A \leq 1$.

Remark 1.1. *If we take $A = (1 - 2\alpha)$ and $B = -1$ in the above Definition* 1.1 *where $0 \leq \alpha < 1$, the class becomes $S^*_\sigma[(1 - 2\alpha), -1] \equiv S^*_\sigma(\alpha)$.*

Remark 1.2. *If we take* $A = 1$ *and* $B = -1$ *in the above Definition* 1.1*, the class becomes* $S^*_\sigma[1, -1] \equiv S^*_\sigma$.

Definition 1.2. *A function* f *given by* (1) *is said to be in the class* $V^*_\sigma[A, B]$, *if the following conditions are satisfied*

$$\left(\frac{z}{f(z)}\right)^2 f'(z) \prec \frac{1 + Az}{1 + Bz},$$

$$\left(\frac{w}{g(w)}\right)^2 g'(w) \prec \frac{1 + Aw}{1 + Bw},$$

where $f \in \sigma$, $g = f^{-1}$, $w = f(z)$, $w \in \Delta$ *and* $-1 \leq B < A \leq 1$.

Remark 1.3. *If we take* $A = (1 - 2\alpha)$ *and* $B = -1$ *in the above Definition* 1.2 *where* $0 \leq \alpha < 1$*, the class becomes* $V^*_\sigma[(1 - 2\alpha), -1] \equiv V^*_\sigma(\alpha)$.

Remark 1.4. *If we take* $A = 1$ *and* $B = -1$ *in the above Definition* 1.2*, the class becomes* $V^*_\sigma[1, -1] \equiv V^*_\sigma$.

In this paper, we shall give the best norm estimation for the classes $S^*_\sigma[A, B]$ and $V^*_\sigma[A, B]$.

2. Main result

Theorem 2.1. *Let the function* f *given by* (1) *be in the class* $f \in S^*_\sigma[A, B]$*, then*

$$\|T_f\| \leq \min\left\{\frac{2(A - B)(A + 2)}{(A + 1)}, \frac{2(A - B)|A|}{(A + 1)}\right\}.$$

Proof. Since $f \in S^*_\sigma[A, B]$, let us assume that

$$h(z) = \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} = p(z).$$

Using the definition of subordination, there exists a Schwarz function $\phi : \Delta \to \Delta$ with $\phi(0) = 0$ and $|\phi(z)| < 1$, such that

$$h(z) = p\, o\, \phi(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)}.$$

Hence, $h(z)$ becomes

$$h(z) = \frac{zf'(z)}{f(z)} = \frac{1 + A\phi(z)}{1 + B\phi(z)}. \tag{2.1}$$

By logarithmic differentiation of (3), we get

$$\frac{1}{z} + \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)} = \frac{A}{(1 + A\phi(z))} - \frac{B}{(1 + B\phi(z))}.$$

Above equation gives us the pre-Schwarzian derivative of f, i.e,

$$T_f(z) = \frac{f''(z)}{f'(z)} = \frac{(A^2 - AB)\phi(z) + 2(A - B)}{(1 + A\phi(z))(1 + B\phi(z))},$$

Setting $\phi(z) = id_{\Delta}$ (as ϕ belongs to the class of Schwarz functions and $\phi(z) \prec z \ on\ \Delta$) and rearranging the terms, we get

$$\left(1-|z|^2\right)\left|T_f(z)\right| = \left(1-|z|^2\right)\left|\frac{\left(A^2-AB\right)z+2(A-B)}{(1+Az)(1+Bz)}\right|.$$

Taking the supremum value both sides in the unit disc, the above equation becomes

$$\sup_{|z|<1}\left(1-|z|^2\right)\left|T_f(z)\right| \le \sup_{|z|<1}\left(1-|z|^2\right)\left[\frac{\left(A^2-AB\right)|z|+2(A-B)}{(1+A|z|)(1+B|z|)}\right].$$

As $-1 \le B$, we get $(1-|z|) \le (1+B|z|)$, therefore the above inequality becomes

$$\sup_{|z|<1}\left(1-|z|^2\right)\left|T_f(z)\right| \le \sup_{|z|<1}(1+|z|)\left[\frac{\left(A^2-AB\right)|z|+2(A-B)}{(1+A|z|)}\right].$$

The above inequality gives us the norm of pre-Schwarzian derivative of f, denoted by $\|T_f\|$. To estimate the upper bound of $\|T_f\|$ in the unit disc Δ, z must lead to 1 and therefore

$$\lim_{z\to 1}(1+|z|)\left[\frac{\left(A^2-AB\right)|z|+2(A-B)}{(1+A|z|)}\right] = \frac{2(A-B)(A+2)}{(A+1)}.$$

Finally we get

$$\left\|T_f\right\| \le \frac{2(A-B)(A+2)}{(A+1)}. \tag{2.2}$$

For the second part of the proof, let us assume that

$$k(z) = \frac{w\,g'(w)}{g(w)} \prec \frac{1+Aw}{1+Bw} = p(w)$$

where $z = f^{-1}(w) = g(w)$. By definition of subordination, there exists a Schwarz function $\phi : \Delta \to \Delta$ with $\phi(0) = 0$ and $|\phi(z)| < 1$, such that

$$k(z) = p\ o\ \phi(z) = \frac{1+A\phi(z)}{1+B\phi(z)}.$$

Since $f \in \sigma$, both f and f^{-1} are analytic and univalent in Δ. The derivative of f^{-1} is given by

$$\frac{d(f^{-1}(w))}{w} = \frac{1}{f'(z)}.$$

Therefore, $k(z)$ can be expressed as

$$\frac{wg'(w)}{g(w)} = k(z) = \frac{f(z)}{zf'(z)} = \frac{1+A\phi(z)}{1+B\phi(z)}. \tag{2.3}$$

Taking logarithmic differentiation of (5), we get

$$\frac{f'(z)}{f(z)} - \frac{1}{z} - \frac{f''(z)}{f'(z)} = \frac{A\phi'(z)}{(1+A\phi(z))} - \frac{B\phi'(z)}{(1+B\phi(z))}.$$

Setting $\phi(z) = id_{\Delta}$ (as ϕ belongs to the class of Schwarz functions and $\phi(z) \prec z$ *on* Δ), we have

$$\frac{f''(z)}{f'(z)} = \frac{Az(A-B)}{(1+Az)(1+Bz)}.$$

Following the previous steps and using $(1-|z|) \leq (1+B|z|)$, we get

$$\sup_{|z|<1}\left(1-|z|^2\right)\left|T_f(z)\right| \leq \sup_{|z|<1}(1+|z|)\left[\frac{|A|(A-B)|z|}{(1+A|z|)}\right].$$

For upper bound of $\|T_f\|$, z must lead to 1, i.e,

$$\lim_{z\to 1}(1+|z|)\frac{|A|(A-B)|z|}{(1+A|z|)} = \frac{2(A-B)|A|}{(A+1)}.$$

Therefore

$$\left\|T_f\right\| \leq \frac{2(A-B)|A|}{(A+1)}. \tag{2.4}$$

Combining (4) and (6), the proof is complete. □

Let $A = (1-2\alpha)$ and $B = -1$ in the above theorem where $0 \leq \alpha < 1$, the class becomes $S^*_\sigma[(1-2\alpha), -1] \equiv S^*_\sigma(\alpha)$.

Corollary 2.1. *If $f \in S^*_\sigma(\alpha)$, then $\|T_f\| \leq min\{6-4\alpha, |2-4\alpha|\}$.*

If f is analytic and locally univalent in Δ such that $f \in S^*(\alpha)$, where $0 \leq \alpha < 1$ then $\|T_f\| \leq (6-4\alpha)$, which is due to Yamashita [1]. The above corollary generalizes the result for bi-univalent functions.

Let $A = 1$ and $B = -1$ in the above theorem, the class becomes $S^*_\sigma[1, -1] \equiv S^*_\sigma$.

Corollary 2.2. *If $f \in S^*_\sigma$, then $\left\|T_f\right\| \leq 6$.*

The above corollary generalizes the norm estimation for bi-univalent functions.

Theorem 2.2. *Let the function $f(z)$ given by* (1) *be in the class $V^*_\sigma[A, B]$, then*

$$\left\|T_f\right\| \leq \min\left\{\frac{2(3+2A)(A-B)}{(A+1)}, \frac{2(A-B)(1+2A)}{(A+1)}\right\}.$$

Proof. Since $f \in V^*_\sigma[A, B]$, let us assume that

$$k(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) \prec \frac{1+Az}{1+Bz} = p(z).$$

Therefore, there exists a Schwarz function $\phi : \Delta \to \Delta$ with $\phi(0) = 0$ and $|\phi(z)| < 1$ such that

$$k(z) = p\, o\, \phi(z) = \frac{1+A\phi(z)}{1+B\phi(z)}.$$

Therefore, $k(z)$ can be expressed as

$$k(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) = \frac{1+A\phi(z)}{1+B\phi(z)}. \tag{2.5}$$

By logarithmic differentiation on (7), we get

$$\frac{2}{z}-\frac{2f'(z)}{f(z)}+\frac{f''(z)}{f'(z)}=\frac{A\phi'(z)}{(1+A\phi(z))}-\frac{B\phi'(z)}{(1+B\phi(z))},$$

$$\frac{f''(z)}{f'(z)}=\frac{2f'(z)}{f(z)}-\frac{2}{z}+\frac{A\phi'(z)}{(1+A\phi(z))}-\frac{B\phi'(z)}{(1+B\phi(z))}.$$

Hence, the pre-Schwarzian derivative of f becomes

$$T_f(z)=\frac{f''(z)}{f'(z)}=\frac{2(1+A\phi(z))}{z(1+B\phi(z))}-\frac{2}{z}+\frac{A\phi'(z)}{(1+A\phi(z))}-\frac{B\phi'(z)}{(1+B\phi(z))}.$$

Setting $\phi(z)=id_\Delta$, we get

$$T_f(z)=\frac{f''(z)}{f'(z)}=\frac{B(A-B)+2Az(A-B)}{(1+Az)(1+Bz)}=\frac{(A-B)(3+2Az)}{(1+Az)(1+Bz)}.$$

Therefore

$$\sup_{|z|<1}\left(1-|z|^2\right)\left|T_f(z)\right|\leq\sup_{|z|<1}(1+|z|)\left[\frac{(3+2A|z|)(A-B)}{(1+A|z|)}\right].$$

Again, to estimate the upper bound of $\|T_f\|$, z must lead to 1 and hence we get

$$\lim_{z\to1}(1+|z|)\frac{(A-B)(3+2A|z|)}{(1+A|z|)}=\frac{2(A-B)(3+2A)}{(A+1)}.$$

Finally

$$\left\|T_f\right\|\leq\frac{2(A-B)(3+2A)}{(A+1)}. \tag{2.6}$$

For the second part of the proof, its given $f\in V_\sigma^*[A,\ B]$ and therefore we assume

$$k(z)=\left(\frac{w}{g(w)}\right)^2g'(w)\prec\frac{1+Aw}{1+Bw}=p(w)$$

where $w=f(z)$, $g=f^{-1}$ and w $\in\Delta$. Since $f\in\sigma$ (as explained in the second part of previous theorem) we see,

$$g'=\frac{d}{dw}\left(f^{-1}(w)\right)=\frac{1}{f'(z)}.$$

Using above equation, $k(z)$ can be expressed as

$$k(z)=\left(\frac{f(z)}{z}\right)^2\frac{1}{f'(z)}=\frac{1+A\phi(z)}{1+B\phi(z)}.$$

By logarithmic differentiation of above equation, we get

$$\frac{f''(z)}{f'(z)}=\frac{2(1+A\phi(z))}{z(1+B\phi(z))}-\frac{2}{z}-\frac{A\phi'(z)}{(1+A\phi(z))}+\frac{B\phi'(z)}{(1+B\phi(z))}.$$

Setting $\phi(z) = id_{\Delta}$, the pre-Schwarzian derivative of f becomes

$$T_f(z) = \frac{f''(z)}{f'(z)} = \frac{(A-B)(1+2Az)}{(1+Az)(1+Bz)}$$

Following the similar steps as in the first part of this theorem, we get

$$\|T_f\| \leq \sup_{|z|<1} (1+|z|) \frac{(A-B)(1+2A|z|)}{(1+A|z|)}.$$

Finally,

$$\left\|T_f\right\| \leq \frac{2(A-B)(1+2A)}{(A+1)}. \tag{2.7}$$

Combining (8) and (9), the proof is complete. □

Let $A = (1-2\alpha)$ and $B = -1$ in the above theorem where $0 \leq \alpha < 1$, the class becomes $V_\sigma^*[(1-2\alpha), -1] \equiv V_\sigma^*(\alpha)$.

Corollary 2.3. *If $f \in V_\sigma^*(\alpha)$, then $\|T_f\| \leq min\{10-8\alpha, 6-8\alpha\}$.*

The above corollary deduces to the exact same norm estimation for analytic and bi-univalent functions in Δ which lies in a similar class denoted by $V_\sigma^*(\alpha)$ and is studied by Rahmatan [4].
Let $A = 1$ and $B = -1$ in the above theorem, the class becomes $V_\sigma^*[1, -1] \equiv V_\sigma^*$.

Corollary 2.4. *If $f \in V_\sigma^*$, then $\left\|T_f\right\| \leq 6$.*

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Conflict of Interest

Authors declare that there is no conflict of interest.

References

1. S. Yamashita, *Norm estimates for function starlike or convex of order alpha*, Hokkaido Math. J., **28** (1999), 217–230.
2. D. A. Brannan and T. S. Taha, *On some classes of bi-univalent functions*, Studia Univ. Babeş-Bolyai Math., **31** (1986), 70–77.
3. P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, 259, Springer, New York, 1983.
4. H. Rahmatan, Sh. Najafzadeh and A. Ebadian, *The norm of pre-Schwarzian derivatives on bi-univalent functions of order α*, B. Iran. Math. Soc., **43** (2017), 1037–1043.

5. T. S. Taha, *Topics in Univalent Function Theory*, Ph.D. Thesis, University of London, 1981.