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Simple-minded systems, configurations and mutations for representation-finite self-injective algebras ☆

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ABSTRACT

Simple-minded systems of objects in a stable module category are defined by common properties with the set of simple modules, whose images under stable equivalences do form simple-minded systems. Over a representation-finite self-injective algebra, it is shown that all simple-minded systems are images of simple modules under stable equivalences of Morita type, and that all simple-minded systems can be lifted to Nakayama-stable simple-minded collections in the derived category. In particular, all simple-minded systems can be obtained algorithmically using mutations.

1. Introduction

Module categories contain two kinds of especially important objects: From *simple* modules other objects can be produced by iteratively forming extensions. From *projective* modules other objects can be produced by considering presentations or resolutions. Moreover, by Morita theory, projective objects control equivalences of module categories. The role of projective modules can in derived categories be taken over by appropriate generalisations ("projective-minded" objects satisfying certain homological conditions) such as tilting complexes, which still control equivalences of such categories. In stable categories, no substitutes of projective objects are known and stable equivalences are, in general, not known to be controlled by particular

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objects. It is not even known whether equivalences of stable module categories of finite dimensional algebras preserve the number of non-projective simple modules (up to isomorphism); the Auslander–Reiten conjecture – which appears to be wide open – predicts a positive answer to this question.

The images of simple modules under a stable equivalence do keep some of the properties of simple objects such as their endomorphism ring being a skew-field and every non-zero homomorphism between them being an isomorphism. Moreover, they still generate the stable category. Such systems of objects in a stable module category have been called simple-minded systems in [16]. Analogous systems of objects in a derived module category (defined in a slightly different way) have been called cohomologically Schurian collections in [3] and simple-minded collections in [17].

Any information on simple-minded systems for an algebra can help to describe the still rather mysterious stable module category and in particular equivalences between stable categories. The following two problems appear to be crucial:

The simple-image problem. Is every simple-minded system the image of the set of simples of some algebra under some stable equivalence?

The liftability problem. Is there a connection between the simple-minded systems in the stable category of a self-injective algebra and the simple-minded collections in its derived module category? More precisely, are the simple-minded systems images of simple-minded collections under the quotient functor from the derived to the stable category?

Note that when the algebra is self-injective, its stable module category is a quotient of its derived module category.

On a numerical level, a positive answer to the question if all simple-minded systems of an algebra have the same cardinality implies validity of the Auslander–Reiten conjecture. The information we are looking for is stronger and is part of an attempt to better understand the structure of stable categories and stable equivalences.

Expecting positive answers to these questions appears to be rather optimistic. In this article we do, however, provide positive answers to both problems for the class of representation-finite self-injective algebras, which includes for instance all the blocks of cyclic defect of group algebras of finite groups over fields of arbitrary characteristic.

Before stating our main result, Theorem A, we remark that Riedtmann et al. essentially answered the simple-image problem for standard representation-finite self-injective algebras in the 1980's, using the notion of configurations instead of simple-minded systems, and that Asashiba and Dugas have recently resolved the liftability problem in this case (see Section 3 and 4 for details). So essentially the new part of the following theorem is the case of the non-standard representation-finite self-injective algebras.

Theorem A. *(See Theorem 4.1 and Corollary 4.2.) Let A be self-injective of finite representation type over an algebraically closed field. Then every simple-minded system is the image of simples under a stable equivalence of Morita type that lifts to a derived equivalence.*

A main tool for answering the simple-image problem is a combinatorial description of simple-minded systems over a representation-finite self-injective algebra A: there is a bijection between simple-minded systems in $\underline{\mathrm{mod}}\, A$ and Riedtmann's configurations in the stable AR-quiver of A. Again this result was already known by Riedtmann et al. for the *standard* representation-finite self-injective algebras (see Section 3 for the details). Using covering theory we can solve this problem both in the standard case and in the non-standard case simultaneously (see Proposition 3.6).

Note that all stable equivalences between standard algebras are liftable stable equivalences of Morita type, that is, they can be lifted to standard derived equivalences. This yields an unexpected property of simple-minded systems in this case; they are all Nakayama-stable. This stability appears to be a crucial property that is potentially useful in other situations, too. Moreover, combining mutation theory of simple-minded systems and the identification with configurations, we can show that some stable self-equivalences of the non-standard representation-finite self-injective algebras are liftable. This provides the main step in complementing the known cases for Theorem A (see Lemma 4.10). In fact, the same proof also gives an alternative proof for Dugas' result [15] on the liftability of stable equivalences between particular representation-finite self-injective algebras.

Simple-minded systems may be compared with other concepts that arise for instance in cluster theory or in the emerging generalisation of tilting to silting. These concepts also come with a theory of mutation. Therefore, it makes sense to ask for the phenomena which replicate in different situations. In this context, we will prove the following result, that is formally independent of simple-minded systems, but intrinsically related to our approach:

Theorem B. *(See Theorem 5.5.) Let A be a self-injective algebra of finite representation type over an algebraically closed field. Then the homotopy category $K^b(\mathrm{proj}A)$ is strongly tilting connected.*

Combining this with other results, we show an analogous result for the stable module category. In particular, we get that all simple-minded systems in this case can be obtained by iterative left irreducible mutations starting from simple modules (see Proposition 5.8).

The proofs use a variety of rather strong results and methods from the literature, including covering theory, Riedtmann's description of configurations of representation finite self-injective algebras, Asashiba's classification results on stable and derived equivalences, Asashiba's and Dugas' results on liftability of stable equivalences, and various mutation theories.

This article is organised as follows. Section 2 contains some general statements on sms's over self-injective algebras: their connection with smc's; the relationship between the orbits of sms's under stable Picard group and the Morita equivalence classes of stably equivalent algebras. We shall formulate the basic problems about sms: simple-image problem and liftability problem.

From Section 3, we restrict our discussion to representation-finite self-injective algebras over an algebraically closed field. Section 3 gives the correspondence between configurations and sms's. As a consequence, we can solve the simple-image problem of sms's for representation-finite self-injective algebras.

Based on the results in previous sections and on a lifting theorem for stable equivalences between representation-finite self-injective algebras, we give the proof of Theorem A in Section 4.

In Section 5 we discuss some aspects of the various mutations of different objects: tilting complex, smc, and sms. We will show that the sms's of a representation-finite self-injective algebra can be obtained by iterative mutations. As a by-product of our point of view we obtain Theorem B.

2. Statement of problems, and their motivations

Let k be a field and A a finite dimensional self-injective k-algebra.

We denote by $\mathrm{mod}\,A$ the category of all finitely generated left A-modules, by $\mathrm{mod}_{\mathcal{P}}A$ the full subcategory of $\mathrm{mod}\,A$ whose objects have no non-zero projective direct summand, and by $\underline{\mathrm{mod}}\,A$ the stable category of $\mathrm{mod}\,A$ modulo projective modules. Let $\mathcal{S}$ be a class of A-modules. The full subcategory $\langle\mathcal{S}\rangle$ of $\mathrm{mod}\,A$ is the additive closure of $\mathcal{S}$. Denote by $\langle\mathcal{S}\rangle * \langle\mathcal{S}'\rangle$ the class of indecomposable A-modules Y such that there is a short exact sequence $0 \to X \to Y \oplus P \to Z \to 0$ with $X \in \langle\mathcal{S}\rangle$, $Z \in \langle\mathcal{S}'\rangle$, and P projective. Define $\langle\mathcal{S}\rangle_1 := \langle\mathcal{S}\rangle$ and $\langle\mathcal{S}\rangle_n := \langle\langle\mathcal{S}\rangle_{n-1} * \langle\mathcal{S}\rangle\rangle$ for $n > 1$.

To study sms's over A, without loss of generality, we may assume the following throughout the article: A is indecomposable non-simple and contains no nodes (see [16]). We can then simplify the definition of sms from [16] as follows.

Definition 2.1. (See [16].) Let A be as above. A class of objects $\mathcal{S}$ in $\operatorname{mod}_{\mathcal{P}} A$ is called a simple-minded system (sms) over A if the following conditions are satisfied:

(1) (orthogonality condition) For any $S, T \in \mathcal{S}$, $\underline{\operatorname{Hom}}_A(S,T) = \begin{cases} 0 & (S \neq T), \\ \text{division ring} & (S = T). \end{cases}$
(2) (generating condition) For each indecomposable non-projective A-module X, there exists some natural number n (depending on X) such that $X \in \langle \mathcal{S} \rangle_n$.

It has been shown in [16] that each sms has finite cardinality and the sms's are invariant under stable equivalence, i.e. the image of an sms under a stable equivalence is also an sms. Note that the set of simple A-modules clearly forms an sms. We are going to present two fundamental problems, as noted in the introduction, on the study of sms, and we provide motivations for them. To state these problems, we first introduce a special class of stable equivalences — stable equivalences of Morita type, which occur frequently in representation theory of finite groups, and more generally, in representation theory of finite dimensional algebras (see, for example, [10,25,18,19,21]).

Let A and B be two algebras. Following Broué [10], we say that there is a *stable equivalence of Morita type* (StM) $\phi : \underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, B$ if there are two left–right projective bimodules ${}_AM_B$ and ${}_BN_A$ such that the following two conditions are satisfied:

(1) ${}_AM \otimes_B N_A \simeq {}_AA_A \oplus {}_AP_A$, ${}_BN \otimes_A M_B \simeq {}_BB_B \oplus {}_BQ_B$, where ${}_AP_A$ and ${}_BQ_B$ are some projective bimodules;
(2) ϕ is a stable equivalence which lifts to the functor $N \otimes_A -$, that is, the diagram

$$\begin{tikzcd} \operatorname{mod} A \arrow[r, "N\otimes_A -"] \arrow[d, "\pi_A"'] & \operatorname{mod} B \arrow[d, "\pi_B"] \\ \underline{\operatorname{mod}}\, A \arrow[r, "\phi"] & \underline{\operatorname{mod}}\, B \end{tikzcd}$$

commutes up to natural isomorphism, where π_A and π_B are the natural quotient functors.

The first one is the *simple-image problem*:

Problem 2.2 *(Simple-image problem).*

(1) Given an sms $\mathcal{S}$ of A, is this the image of the simple modules under a stable equivalence? (When this is true, we say $\mathcal{S}$ is a simple-image sms, or shorter, it is simple-image.)
(2) Is every sms of A simple-image?
(3) Given an sms $\mathcal{S}$ of A, is this the image of the simple modules under a stable equivalence of Morita type? (When this is true, we say $\mathcal{S}$ is a strong simple-image sms, or shorter, it is strong simple-image.)
(4) Is every sms of A strong simple-image?

(3) and (4) are the strong versions of (1) and (2), respectively. Our aim is to solve the strong simple-image problem in the case of representation-finite self-injective algebras over algebraically closed fields.

In [16], a weaker version of sms has been introduced, and it has been shown that when A is representation-finite self-injective, the following system is sufficient (hence equivalent) for defining an sms.

Definition 2.3. (See [16].) Let A be as in Definition 2.1. A class of objects $\mathcal{S}$ in $\operatorname{mod}_{\mathcal{P}} A$ is called a weakly simple-minded system (wsms) if the following two conditions are satisfied:

(1) (orthogonality condition) For any $S, T \in \mathcal{S}$, $\underline{\mathrm{Hom}}_A(S,T) = \begin{cases} 0 & (S \neq T), \\ \text{division ring} & (S=T). \end{cases}$
(2) (weak generating condition) For any indecomposable non-projective A-module X, there exists some $S \in \mathcal{S}$ (depends on X) such that $\underline{\mathrm{Hom}}_A(X,S) \neq 0$.

A similar concept used for derived module categories is the simple-minded collection (smc) of [17], which coincides with the cohomologically Schurian collection of Al-Nofayee [3].

Definition 2.4. (See [17].) A collection $X_1, \cdots, X_r$ of objects in a triangulated category $\mathcal{T}$ is simple-minded if for $i, j = 1, \cdots, r$, the following conditions are satisfied:

(1) (orthogonality) $\mathrm{Hom}(X_i, X_j) = \begin{cases} \text{division ring} & \text{if } i=j, \\ 0 & \text{otherwise;} \end{cases}$
(2) (generating) $\mathcal{T} = \mathit{thick}(X_1 \oplus \cdots \oplus X_r)$;
(3) (silting/tilting) $\mathrm{Hom}(X_i, X_j[m]) = 0$ for any $m < 0$.

For any (finite dimensional) k-algebra A, the simple A-modules form a simple-minded collection of the bounded derived category $D^b(\mathrm{mod}\, A)$. Simple-minded collections appeared already in the work of Rickard [26], who constructed tilting complexes inducing equivalences of derived categories that send a simple-minded collection for a symmetric algebra to the simple modules of another symmetric algebra. Al-Nofayee [3] generalised Rickard's work to self-injective algebras, requiring an smc to satisfy the following Nakayama-stability condition. Recall that for a self-injective algebra A, the Nakayama functor $\nu_A = \mathrm{Hom}_k(A, k) \otimes_A - : \mathrm{mod}\, A \to \mathrm{mod}\, A$ is an exact self-equivalence and therefore induces a self-equivalence of $D^b(\mathrm{mod}\, A)$ which will also be denoted by ν_A. By Rickard [25], if $\phi : D^b(\mathrm{mod}\, A) \to D^b(\mathrm{mod}\, B)$ is a derived equivalence between two self-injective algebras A and B, then $\phi\nu_A(X) \simeq \nu_B\phi(X)$ for any object $X \in D^b(\mathrm{mod}\, A)$. We shall say an smc $X_1, \cdots, X_r$ of $D^b(\mathrm{mod}\, A)$ is *Nakayama-stable* if the Nakayama functor ν_A permutes $X_1, \cdots, X_r$. In particular, any derived equivalence $\phi : D^b(\mathrm{mod}\, A) \to D^b(\mathrm{mod}\, B)$ sends simple modules to a Nakayama-stable smc.

We will frequently use the following two well-known results of Rickard and Linckelmann. The former says that for a self-injective A, the embedding functor $\mathrm{mod}\, A \to D^b(\mathrm{mod}\, A)$ induces an equivalence $\underline{\mathrm{mod}}\, A \to D^b(\mathrm{mod}\, A)/K^b(\mathrm{proj}\, A)$. So there is a natural quotient functor $\eta_A : D^b(\mathrm{mod}\, A) \to \underline{\mathrm{mod}}\, A$ of triangulated categories. A standard derived equivalence between two self-injective algebras induces an StM (here a *standard derived equivalence* means that it is isomorphic to the functor given by tensoring with a two-sided tilting complex, see [24,25,5] for more details). Linckelmann [18] showed that an StM between two self-injective algebras lifts to a Morita equivalence if and only if it sends simple modules to simple modules.

We then have the following observation.

Proposition 2.5. *Let A be a self-injective algebra. Then every Nakayama-stable smc of $D^b(\mathrm{mod}\, A)$ determines an sms of $\underline{\mathrm{mod}}\, A$ under the natural functor $\eta_A : D^b(\mathrm{mod}\, A) \to \underline{\mathrm{mod}}\, A$. Conversely, if $\mathcal{S}$ is a simple-image sms of $\underline{\mathrm{mod}}\, A$ under a stable equivalence $\phi : \underline{\mathrm{mod}}\, B \to \underline{\mathrm{mod}}\, A$, and if ϕ can be lifted to a derived equivalence, then $\mathcal{S}$ lifts to a Nakayama-stable smc of $D^b(\mathrm{mod}\, A)$.*

Proof. This is straightforward by results of Al-Nofayee. □

The second fundamental problem asks how an sms is related to Nakayama-stable smc:

Problem 2.6 *(The liftability problem).* Is a given sms $\mathcal{S}$ of $\underline{\mathrm{mod}}\, A$ isomorphic to the image of a Nakayama-stable smc of $D^b(\mathrm{mod}\, A)$ under η_A?

According to Proposition 2.5, given a simple-image sms $\mathcal{S}$ of $\underline{\mathrm{mod}}\, A$ under an StM $\phi : \underline{\mathrm{mod}}\, B \to \underline{\mathrm{mod}}\, A$, if ϕ can be lifted to a derived equivalence, then $\mathcal{S}$ is isomorphic to the image of a Nakayama-stable smc of $D^b(\mathrm{mod}\, A)$ under η_A. In such situation, we simply say $\mathcal{S}$ is a *liftable simple-image* sms.

Next we recall the notion of stable Picard group from [19,5]. Let A be an algebra. The more conventional notion of *Picard group* $\mathrm{Pic}(A)$ of A is defined to be the set of natural isomorphism classes of Morita self-equivalences over A. The set $\mathrm{StPic}(A)$ of natural isomorphism classes $[\phi]$ of StM $\phi : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, A$ form a group under the composition of functors, which is called the *stable Picard group* of A. Notice that the definitions for stable Picard group used by Linckelmann [19] and by Asashiba [5] are different even in the case of representation-finite self-injective algebras. Linckelmann used the isomorphism classes of bimodules which define StM, while Asashiba used the isomorphism classes of *all* stable self-equivalences. We use the one closer to Linckelmann's version of stable Picard group in the propositions to follow. In Section 4 we will specify the link between the two versions when A is representation-finite. Similarly we define the *derived Picard group* $\mathrm{DPic}(A)$ of A as the set of natural isomorphism classes of standard derived self-equivalences of the bounded derived category $D^b(\mathrm{mod}\, A)$. Clearly each Morita equivalence: $\mathrm{mod}\, A \to \mathrm{mod}\, A$ induces an StM: $\underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, A$. We denote the image of the canonical homomorphism $\mathrm{Pic}(A) \to \mathrm{StPic}(A)$ by $\mathrm{Pic}'(A)$. Note that two non-isomorphic bimodules may induce isomorphic StM, which is the reason why we use $\mathrm{Pic}'(A)$ here. This distinction will become important in Section 4.

Let A be an algebra. In the following, we will identify two sms's of A, $\mathcal{S}_1 = \{X_1, \cdots, X_r\}$ and $\mathcal{S}_2 = \{X'_1, \cdots, X'_s\}$, if $r = s$ and $X_i \simeq X'_i$ for all $1 \le i \le r$ up to a permutation. We use the same convention for smc's. We use calligraphic font (e.g. $\mathcal{S}$) and bold font (e.g. $\mathbf{S}$) for sms's and smc's respectively to distinguish the two. Now we fix some notations:

$\mathcal{S}_A = \{$isomorphism classes of simple A-modules$\}$;

$\mathrm{StMAlg}(A) = \{$the Morita equivalence classes of algebras which are StM to $A\}$;

$\mathrm{sms}(A)/\mathrm{StPic}(A) = \{$the orbits of sms's of $\underline{\mathrm{mod}}\, A$ under $\mathrm{StPic}(A)\}$;

$\mathrm{smc}(A)/\mathrm{DPic}(A) = \{$the orbits of Nakayama-stable smc's of $D^b(\mathrm{mod}\, A)$ under $\mathrm{DPic}(A)\}$.

Proposition 2.7. *Let A be a self-injective algebra. Let* $\mathrm{StMAlg}(A)$ *and* $\mathrm{sms}(A)/\mathrm{StPic}(A)$ *be as above. Then:*

(1) *There is a well-defined map from* $\mathrm{StMAlg}(A)$ *to* $\mathrm{sms}(A)/\mathrm{StPic}(A)$.
(2) *This map is injective. It is a bijection if and only if every sms of A is simple-image of Morita type.*

Proof. This is straightforward by Linckelmann's results in [18, Theorem 2.1]. □

Remark 2.8. (1) We will see in Section 4 that the above map is a bijection in case that A is a representation-finite self-injective algebra.

(2) We do not know whether there is an example with a non-bijective map. Note that the algebra A in Example 3.5 of [16] is in fact not a counterexample to the strong simple-image problem (despite a misleading formulation in [16]), where A is given by the following regular representation

$$A = \begin{matrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{matrix} \oplus \begin{matrix} 2 \\ 1 \\ 2 \\ 1 \\ 2 \end{matrix} .$$

There is an StM from A to the following Brauer tree algebra B such that the sms $\mathcal{S}_2 = \{1, \begin{smallmatrix} 2 \\ 1 \\ 2 \end{smallmatrix}\}$ over A is mapped to simple B-modules:

$$B = \begin{array}{c}1\\2\\1\end{array} \oplus \begin{array}{ccc} & 2 & \\ 1 & & 2\\ & 2 & \end{array}.$$

(3) This proposition is true for any finite dimensional algebra once we replace the simple modules by non-projective simple modules in the argument, due to Linckelmann's results in [18, Theorem 2.1] being valid for general finite dimensional algebras (see [20] and [17, Section 4]).

(4) Uniqueness is false if we replace StM by general stable equivalence, even the involved algebras are indecomposable and have no nodes. For example, let A be a k-algebra without oriented cycles in its ordinary quiver and $DA = \mathrm{Hom}_k(A,k)$. Using a 2-cocycle $\alpha : A \times A \to DA$ one can construct the Hochschild extension algebra $A \ltimes_\alpha DA$. When $\alpha = 0$, this is just the trivial extension algebra $A \ltimes DA$. Yamagata showed that $A \ltimes_\alpha DA$ and $A \ltimes DA$ are related by a socle equivalence which naturally induces a stable equivalence. This stable equivalence maps simples to simples. However, when k is not algebraically closed, there exists some $A \ltimes_\alpha DA$ which is indecomposable and self-injective, but not symmetric (see [23]). In this case, $A \ltimes_\alpha DA$ and $A \ltimes DA$ are not Morita equivalent since $A \ltimes DA$ is symmetric.

Proposition 2.9. *Let A be a self-injective algebra. Let* $\mathrm{smc}(A)/\mathrm{DPic}(A)$ *and* $\mathrm{sms}(A)/\mathrm{StPic}(A)$ *be as above. Then there is an injective map from* $\mathrm{smc}(A)/\mathrm{DPic}(A)$ *to* $\mathrm{sms}(A)/\mathrm{StPic}(A)$*. This map is a bijection if every sms $\mathcal{S}$ of A is a liftable simple-image.*

Proof. This is straightforward by Linckelmann's results in [18, Theorem 2.1]. $\square$

Remark 2.10. (1) We will see in Section 4 that every sms of A is a liftable simple-image in case that A is a representation-finite self-injective algebra, and therefore the above map is a bijection in this case.

(2) The map in Proposition 2.9 could be a bijection without every sms of A being liftable. We are grateful to the referee for pointing this out and for suggesting the following example. Let P be a finite p-group and $A = kP$ be the group algebra, where k is a field of characteristic p. Since A is local, $\mathrm{smc}(A)/\mathrm{DPic}(A)$ will be trivial (the only tilting complexes are isomorphic to $A[n]$ for an integer n). Since A is a p-group algebra any sms consists of endotrivial modules by [11]. A single endotrivial module M is already an sms as it is the image of the trivial module k under the stable equivalence $M \otimes_k - : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, A$ of Morita type. So an sms is the same thing as an endotrivial module here, and all such modules are in the same $\mathrm{StPic}(A)$-orbit. However, if there is an endotrivial module not of the form $\Omega^n(k)$, then it is an sms of A that is not liftable.

(3) In the representation-infinite case, there exists simple-image sms $\mathcal{S}$ of Morita type under a non-liftable StM. For example, let A and B be the principal blocks of the Suzuki group $S_z(8)$ and of the normaliser of a Sylow 2-subgroup of $S_z(8)$ over a field k of characteristic 2. Then A and B are stably equivalent of Morita type, say under ϕ, but not derived equivalent by [10]. Obviously, $\mathcal{S} = \phi(\mathcal{S}_B)$ is a simple-image sms of Morita type over A. If there is another algebra C so that $\psi : \underline{\mathrm{mod}}\, C \to \underline{\mathrm{mod}}\, A$ is a stable equivalence sending $\mathcal{S}_C$ to $\mathcal{S}$ and ψ liftable, then $\phi^{-1}\psi(\mathcal{S}_C) = \mathcal{S}_B$. By Linckelmann's results in [18, Theorem 2.1], C and B are Morita equivalent, as A and C are derived equivalent. This implies that A and B also are derived equivalent, which is a contradiction. Therefore, we have an example of a simple-image sms of Morita type which is *never* liftable.

3. Sms's and configurations

In this section we are going to address the simple-image problem. Following Asashiba [4], we abbreviate (indecomposable, basic) representation-finite self-injective algebra over an algebraically closed field k (not isomorphic to the underlying field k) by RFS algebra.

Theorem 3.1. *Let A be an RFS algebra over an algebraically closed field, and $\mathcal{S}$ an sms of A. Then there is an RFS algebra B and a stable equivalence from $\underline{\mathrm{mod}}\, B$ to $\underline{\mathrm{mod}}\, A$ such that the set of simple B-modules is mapped to $\mathcal{S}$ under the stable equivalence, i.e. $\mathcal{S}$ is a simple-image sms.*

Remark 3.2.

(1) We will see in Section 4 that, for an RFS algebra A, all sms's of A are in fact simple-image of Morita type.
(2) The classification theorem of RFS algebras, first proved in the 1980's, does already imply implicitly that B is determined uniquely up to Morita equivalence.

The main tools in proving Theorem 3.1 come from Riedtmann's work on RFS algebras and their AR-quivers, and from Asashiba's stable equivalence classification of RFS algebras. We use standard definitions of AR theory without explanations; see [6–8] for details. In the following we recall the definitions of configurations and combinatorial configurations, and see how these notions are translated into the setting of sms's. Throughout this section Q denotes a Dynkin quiver of type A_n, D_n, E_6, E_7 or E_8; and $\mathbb{Z}Q$ is the corresponding translation quiver with translation denoted as τ. For a translation quiver Γ, we let $k(\Gamma)$ be its mesh category, that is, the path category whose objects are the vertices of Γ; morphisms are generated by arrows of Γ quotiented out by the mesh relations. Riedtmann showed in [27] that for an RFS algebra over an algebraically closed field, the stable AR-quiver is of the form $\mathbb{Z}Q/\Pi$ for some admissible group Π. Consequently we say such algebra is of tree class Q and has admissible group Π. Note that we always assume the RFS algebras considered to be indecomposable, basic and not isomorphic to the underlying field k.

Definition 3.3. (See [9].) A configuration of $\mathbb{Z}Q$ is a subset $\mathcal{C}$ of vertices of $\mathbb{Z}Q$ such that the quiver $\mathbb{Z}Q_{\mathcal{C}}$ is a representable translation quiver. $\mathbb{Z}Q_{\mathcal{C}}$ is constructed by adding one vertex c^* for each $c \in \mathcal{C}$ on $\mathbb{Z}Q$; adding arrows $c \to c^* \to \tau^{-1}c$; and letting the translation of c^* be undefined.

Here, the following notation is used: A translation quiver is representable if and only if its mesh category is an Auslander category. We do not go through the technicalities of these definitions; the reader can bear in mind that the mesh category of the Auslander–Reiten quiver (or its universal cover) of a representation-finite algebra is an Auslander category (see [8]). The idea is that for Π-stable configuration $\mathcal{C}$, $\mathbb{Z}Q/\Pi$ is the stable AR-quiver of an RFS algebra and $\mathbb{Z}Q_{\mathcal{C}}/\Pi$ is the AR-quiver of the algebra, where the extra (projective) vertices c^* are the vertices representing the (isoclasses of) indecomposable projective modules of the algebra. In particular, the set $\{\mathrm{rad}(P) \mid P \text{ an (isoclass of) indecomposable projective}\}$ of an RFS algebra is a configuration.

Definition 3.4. (See [28].) Let Δ be a stable translation quiver. A combinatorial configuration $\mathcal{C}$ is a set of vertices of Δ which satisfy the following conditions:

(1) For any $e, f \in \mathcal{C}$, $\mathrm{Hom}_{k(\Delta)}(e,f) = \begin{cases} 0 & (e \neq f), \\ k & (e=f). \end{cases}$
(2) For any $e \in \Delta_0$, there exists some $f \in \mathcal{C}$ such that $\mathrm{Hom}_{k(\Delta)}(e,f) \neq 0$.

We also note the following fact in [28, Proposition 2.3]: if $\pi : \Delta \to \Gamma$ is a covering, then $\mathcal{C}$ is a combinatorial configuration of Γ if and only if $\pi^{-1}\mathcal{C}$ is a combinatorial configuration of Δ. When applied to the universal cover of stable AR-quiver of RFS algebra A, this translates to the following statement: $\mathcal{C}$ is a combinatorial configuration of the stable AR-quiver $\mathbb{Z}Q/\Pi$ if and only if $\pi^{-1}\mathcal{C}$ is a Π-stable combinatorial configuration of the universal cover $\mathbb{Z}Q$.

Combinatorial configurations have been defined by Riedtmann when studying self-injective algebras [28]. At first this is a generalisation of configuration. It is often easier to study and compute than a configuration as it suffices to look 'combinatorially' at sectional paths of the translation quiver $\mathbb{Z}Q$ rather than checking whether $k(\mathbb{Z}Q_{\mathcal{C}})$ can be realised as an Auslander category. Therefore, it is interesting to know if these two concepts coincide. In the case of RFS algebras, this is true. As mentioned in the sketch previously, a configuration represents a set $\{\mathrm{rad}(P) \mid P \text{ an (isoclass of) indecomposable projective}\}$. Applying the inverse Heller operator Ω^{-1}, which is an auto-equivalence of the stable category of an RFS algebra, the above set is mapped to the set of simples of the RFS algebra. Indeed, in [28,29,9] it has been shown that Π-stable configuration of $\mathbb{Z}Q$ and combinatorial configuration of $\mathbb{Z}Q/\Pi$ do coincide. Thus in the following, for an RFS algebra A, we can identify the configurations and combinatorial configurations of the stable AR-quiver ${}_s\Gamma_A$.

In [28,29,9], it was also shown that the isoclasses of Π-stable $\mathbb{Z}Q$ configurations (two configurations $\mathcal{C}$ and $\mathcal{C}'$ of $\mathbb{Z}Q$ are called isomorphic if $\mathcal{C}$ is mapped onto $\mathcal{C}'$ under an automorphism of $\mathbb{Z}Q$) correspond bijectively to isoclasses of RFS algebras of tree class Q with admissible group Π, except in the case of $Q = D_{3m}$ with underlying field having characteristic 2. In such a case, each configuration corresponds to two (isoclasses of) RFS algebras; both are symmetric algebras, one of which is standard, while the other one is non-standard. Here, a representation-finite k-algebra A is called *standard* if $k(\Gamma_A)$ is equivalent to $\mathrm{ind}\, A$, where Γ_A is the AR-quiver of A and $\mathrm{ind}\, A$ is the full subcategory of $\mathrm{mod}\, A$ whose objects are specific representatives of the isoclasses of indecomposable modules. This implies that any other standard RFS algebras with AR-quiver isomorphic to Γ_A are isomorphic to A. Non-standard algebras are algebras which are not standard. The non-standard algebras also have been studied by Waschbüsch in [31]. Note that when A is standard, then $k({}_s\Gamma_A) \simeq \underline{\mathrm{ind}}\, A$, where $\underline{\mathrm{ind}}\, A$ is the full subcategory of $\underline{\mathrm{mod}}\, A$ whose objects are objects in $\mathrm{ind}\, A$. In this case, it immediately follows that combinatorial configurations of $k({}_s\Gamma_A)$ correspond exactly to (weakly) sms's of $\underline{\mathrm{mod}}\, A$. While in case that A is non-standard, $k_s\Gamma_A/J \simeq \underline{\mathrm{ind}}\, A$, where $k_s\Gamma_A$ is the path category of ${}_s\Gamma_A$ and the ideal J is defined by some modified mesh relations (see [30,4]).

One interesting phenomenon is that combinatorial configurations of $k({}_s\Gamma_A)$ also correspond to sms's of $\underline{\mathrm{mod}}\, A$ in the non-standard case. In fact, using covering theory, we can prove this fact both in the standard case and in the non-standard case simultaneously. We begin with recalling some results from [8,9,27,28,30].

Definition 3.5. (See [27,28].) Let $\pi : \Delta \to \Gamma$ be a covering where Γ is the AR-quiver (or stable AR-quiver) of A. A k-linear functor $F : k(\Delta) \to \mathrm{ind}\, A$ (or $\underline{\mathrm{ind}}\, A$) is said to be well-behaved if and only if

(1) For any $e \in \Delta_0$ with $\pi e = e_i$, we have $Fe = M_i$ where M_i is the indecomposable A-module corresponding to e_i;
(2) For any $e \xrightarrow{\alpha} f$ in Δ_1, $F\alpha$ is an irreducible map.

By [8, Example 3.1b], for any RFS algebra A (whenever A is standard or non-standard), there is a well-behaved functor $F : k(\widetilde{\Gamma}_A) \to \mathrm{ind}\, A$ such that F coincides with π on objects, where $\pi : \widetilde{\Gamma}_A \to \Gamma_A$ is the universal covering of the AR-quiver Γ_A. By [27, Section 2.3], a well-behaved functor is a covering functor and therefore there is a bijection

$$\bigoplus_{Fh=Ff} \mathrm{Hom}_{k(\widetilde{\Gamma}_A)}(e,h) \simeq \mathrm{Hom}_A(Fe, Ff)$$

for any $e, f, h \in (\widetilde{\Gamma}_A)_0$. Since an irreducible morphism between non-projective indecomposable remains irreducible under the restriction $\mathrm{ind}\, A \to \underline{\mathrm{ind}}\, A$, the well-behaved functor $F : k(\widetilde{\Gamma}_A) \to \mathrm{ind}\, A$ restricts to a well-behaved functor $\overline{F} : k({}_s\widetilde{\Gamma}_A) \to \underline{\mathrm{ind}}\, A$, where ${}_s\widetilde{\Gamma}_A$ is the stable part of the translation quiver $\widetilde{\Gamma}_A$. Note that the restriction $\pi : {}_s\widetilde{\Gamma}_A \to {}_s\Gamma_A$ is also a covering of the stable AR-quiver ${}_s\Gamma_A$. It follows that there are bijections:

$$\bigoplus_{Fh=Ff} \mathrm{Hom}_{k({}_s\widetilde{\Gamma}_A)}(e,h) \simeq \underline{\mathrm{Hom}}_A(Fe,Ff);$$

$$\bigoplus_{\pi h=\pi f} \mathrm{Hom}_{k({}_s\widetilde{\Gamma}_A)}(e,h) \simeq \mathrm{Hom}_{k({}_s\Gamma_A)}(\pi e,\pi f).$$

This implies:

Proposition 3.6. *Let A be an RFS algebra over an algebraically closed field. Then there is a bijection:*

$$\{\text{Configurations of } {}_s\Gamma_A\} \leftrightarrow \{\text{sms's of } \underline{\mathrm{mod}}\, A\}$$
$$\mathcal{C} \mapsto \overline{F}\pi^{-1}(\mathcal{C})$$

where π^{-1} denotes the inverse of the restriction map, and $\overline{F}: k({}_s\widetilde{\Gamma}_A) \to \underline{\mathrm{ind}}\, A$ is the well-behaved functor.

Remark 3.7. (1) This proposition shows that all sms's of an RFS algebra A can be determined from the stable AR-quiver ${}_s\Gamma_A$, even in non-standard case.

(2) This proposition also shows that ${}_s\Gamma_A$ determines sms(B) for all indecomposable self-injective algebra B such that ${}_s\Gamma_B \simeq {}_s\Gamma_A$. In fact, such phenomenon also appears in the following tame case: There is an infinite series of 4-dimensional weakly symmetric local algebras $k\langle x,y\rangle/\langle xy - qyx\rangle$ for $q \in k^\times$ which have isomorphic stable AR-quivers, and are not stably equivalent to each other. Their respective sms's are located in the same positions in the stable AR-quivers of these algebras, namely, each sms contains exactly one indecomposable module lying in the unique stable AR-component $\mathbb{Z}\tilde{A}_1$ (see the paragraphs before [16, Cor. 3.3]). It would be interesting to know whether we can "locate" sms's using just the stable AR-quiver in general.

Now we recall briefly Asashiba's stable equivalence classification of RFS algebras. First we need to define the type of an RFS algebra A. If A is as above, by a theorem of Riedtmann [27], Π has the form $\langle\zeta\tau^{-r}\rangle$ where ζ is some automorphism of Q and τ is the translation. We also recall the Coxeter numbers of $Q = A_n, D_n, E_6, E_7, E_8$ are $h_Q = n+1, 2n-2, 12, 18, 30$ respectively. The frequency of A is defined to be $f_A = r/(h_Q - 1)$ and the torsion order t_A of A is defined as the order of ζ. The type of A is defined as the triple (Q, f_A, t_A). Note that the number of isoclasses of simple A-modules is equal to nf_A.

Theorem 3.8. *(See [4,5].) Let A and B be RFS k-algebras for k algebraically closed.*

(1) *If A is standard and B is non-standard, then A and B are not stably equivalent, and hence not derived equivalent.*
(2) *If both A and B are standard, or both non-standard, the following are equivalent:*
  (a) *A, B are derived equivalent;*
  (b) *A, B are stably equivalent of Morita type;*
  (c) *A, B are stably equivalent;*
  (d) *A, B have the same stable AR-quiver;*
  (e) *A, B have the same type.*
(3) *The types of standard RFS algebras are the following:*
  (a) $\{(A_n, s/n, 1) \mid n, s \in \mathbb{N}\}$,
  (b) $\{(A_{2p+1}, s, 2) \mid p, s \in \mathbb{N}\}$,
  (c) $\{(D_n, s, 1) \mid n, s \in \mathbb{N}, n \geq 4\}$,
  (d) $\{(D_{3m}, s/3, 1) \mid m, s \in \mathbb{N}, m \geq 2, 3 \nmid s\}$,
  (e) $\{(D_n, s, 2) \mid n, s \in \mathbb{N}, n \geq 4\}$,

(f) $\{(D_4, s, 3) \mid s \in \mathbb{N}\}$,
(g) $\{(E_n, s, 1) \mid n = 6, 7, 8; s \in \mathbb{N}\}$,
(h) $\{(E_6, s, 2) \mid s \in \mathbb{N}\}$.
Non-standard RFS algebras are of type $(D_{3m}, 1/3, 1)$ *for some* $m \geq 2$.

Remark 3.9. (1) By the classification of RFS algebras of Riedtmann et al., for a fixed standard (resp. non-standard) algebra A, the isoclasses of configurations on ${}_s\Gamma_A$ are in bijection with the isoclasses of standard (resp. non-standard) RFS algebras B with ${}_s\Gamma_B \simeq {}_s\Gamma_A$. Combining this fact with (1) and (2) of the above theorem we get a bijection between the set $\mathrm{Conf}({}_s\Gamma_A)/\mathrm{Aut}({}_s\Gamma_A)$ and the set of Morita equivalence classes of algebras stably equivalent (of Morita type) to A.

(2) The RFS types which correspond to symmetric algebras are $\{(A_n, s/n, 1) \mid s \in \mathbb{N}, s \mid n\}$, $\{(D_{3m}, 1/3, 1) \mid m \geq 2\}$, $\{(D_n, 1, 1) \mid n \in \mathbb{N}, n \geq 4\}$ and $\{(E_n, 1, 1) \mid n = 6, 7, 8\}$.

Proof of Theorem 3.1. Let $\mathcal{S}$ be an sms of A. Then, by Proposition 3.6, $\mathcal{S}$ corresponds to a configuration $\mathcal{C}$ in the stable AR-quiver ${}_s\Gamma_A$. This configuration $\mathcal{C}$ represents the set $\{\mathrm{rad}(P) \mid P$ an (isoclass of) indecomposable projective B-module$\}$ for some RFS algebra B with ${}_s\Gamma_B \simeq {}_s\Gamma_A$. It follows from Theorem 3.8 that B is stably equivalent to A (say, via ϕ) and they are both standard/non-standard. Since for any pair of isomorphic configurations on ${}_s\Gamma_A$, the associated automorphism induces self-equivalence on category $k({}_s\Gamma_A)$ or $k_s\Gamma_A/J$, we can take ϕ as a stable equivalence sending $\{\mathrm{rad}(P)\}$ to $\mathcal{S}$. In particular, $\phi\Omega_B : \underline{\mathrm{mod}}\, B \to \underline{\mathrm{mod}}\, A$ is a stable equivalence sending simple B-modules to $\mathcal{S}$. $\square$

As a by-product of using configurations, we can pick out the RFS algebras for which the transitivity problem raised in [16] has a positive answer. That is, we can decide whether given two sms's of an algebra there always is a stable self-equivalence sending the first sms to the second one.

Proposition 3.10. *If A is an RFS algebra in the following list, then for any pair of sms's $\mathcal{S}$, $\mathcal{S}'$ of A, there is a stable self-equivalence $\phi : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, A$ such that $\phi(\mathcal{S}) = \mathcal{S}'$. The list consists of* $\{(A_2, s/2, 1) \mid s \geq 1\}$, $\{(A_n, s/n, 1) \mid n \geq 1, \gcd(s, n) = 1\}$, $\{(A_3, s, 2) \mid s \geq 1\}$, $\{(D_6, s/3, 1) \mid s \geq 1, 3 \nmid s\}$, $\{(D_4, s, 3) \mid s \geq 1\}$.

Proof. A is an RFS algebra satisfying the condition stated if and only if the set of its sms's modulo the action of stable self-equivalences (i.e. the set of orbits of sms's under stable self-equivalences) is of size 1. Every stable self-equivalence induces an automorphism of the stable AR-quiver ${}_s\Gamma_A = \mathbb{Z}Q/\Pi$ of A. Conversely, any automorphism of ${}_s\Gamma_A$ induces a self-equivalence of $k({}_s\Gamma_A)$ or of $k_s\Gamma_A/J$, depending on A being standard or not. Hence it induces stable self-equivalences of $\underline{\mathrm{ind}}\, A$, and consequently of $\underline{\mathrm{mod}}\, A$. Therefore, identifying an sms with a configuration using Proposition 3.6, the algebras A we are looking for are those whose set $\mathrm{Conf}({}_s\Gamma_A)/\mathrm{Aut}({}_s\Gamma_A)$ has just one element (cf. Remark 3.9 (1)). Here $\mathrm{Conf}({}_s\Gamma_A)$ is the set of configurations of ${}_s\Gamma_A$. We now look at the number of $\mathrm{Aut}({}_s\Gamma_A)$-orbits case by case.

For E_n cases, one can count explicitly from the list of configurations in [9] that the number of $\mathrm{Aut}({}_s\Gamma_A)$-orbits is always greater than 1.

Now consider class $(A_n, s/n, 1)$, ${}_s\Gamma_A = \mathbb{Z}A_n/\langle\tau^s\rangle$. Note that configurations of $\mathbb{Z}A_n$ are $\tau^{n\mathbb{Z}}$-stable, so any configuration of $(A_n, s/n, 1)$ is $\tau^{d\mathbb{Z}}$-stable with $d = \gcd(s, n)$. Let $s = ld$ and $n = md$. The above implies configurations of $(A_n, l/m, 1)$ are the same as configurations of $(A_n, 1/m, 1)$. But the number of the configurations of $(A_n, 1/m, 1)$ is equal to the number of Brauer trees with d edges and multiplicity m, which is equal to 1 if and only if the pair $(d, m) = (2, 1)$ or $d = 1$. Therefore, $(d, m) = (2, 1)$ gives $\{(A_2, 1, 1)\}$, and $d = 1$ yields the family $\{(A_m, 1/m, 1)\}$.

Let $n = 2p + 1$. For the class $(A_n, s, 2)$, ${}_s\Gamma_A = \mathbb{Z}A_n/\langle\zeta\tau^{sn}\rangle$. A configuration of $(A_n, s, 2)$ is $\tau^{n\mathbb{Z}}$-stable as it is also a configuration of $\mathbb{Z}A_n$. So we only need to consider the case $s = 1$. Recall from [30, Lemma 2.5] that there is a map which takes configurations of $\mathbb{Z}A_n$ to configurations of $\mathbb{Z}A_{n+1}$, so the numbers of orbits of $(A_n, 1, 2)$-configurations form an increasing sequence. Therefore, we can just count the orbits explicitly. $(A_3, 1, 2)$ has one orbit of configurations given by the representative $\{(0, 1), (1, 2), (2, 3)\}$, whereas $(A_5, 1, 2)$ has two orbits. This completes the A_n cases.

Note that configuration of $\mathbb{Z}D_n$ is $\tau^{(2n-3)\mathbb{Z}}$-stable, so similar to A_n case we can reduce to the cases $(D_n, 1, 1)$, $(D_n, 1, 2)$, $(D_4, 1, 3)$, and $(D_{3m}, 1/3, 1)$. We make full use of the main theorem in [29] combining with our result in the A_n cases. Part (a) of the theorem implies that $(D_n, 1, 1)$ and $(D_n, 1, 2)$ with $n \geq 5$ all have more than one orbits. Part (c) of the theorem implies that $(D_4, 1, 1)$ and $(D_4, 1, 2)$ have two orbits, with representatives $\{(0, 1), (1, 1), (3, 3), (3, 4)\}$ and $\{(0, 2), (3, 3), (3, 4), (4, 1)\}$. Since the latter is the only orbit which is stable under the order 3 automorphism of $\mathbb{Z}D_4$, implying $\{(D_4, s, 3) \mid s \geq 1\}$ is on our required list. Finally, for $(D_{3m}, 1/3, 1)$ case, we use the description of this class of algebras from [31], which says that such class of algebra can be constructed via Brauer tree with m edges and multiplicity 1 with a chosen extremal vertex. Therefore, the only m with a single isomorphism class of stably equivalent algebra is when $m = 2$, hence giving us $\{(D_6, s/3, 1) \mid s \geq 1, 3 \nmid s\}$. □

4. Sms's and Nakayama-stable smc's

Our aim in this section is to prove that for an RFS algebra A, every sms of A lifts to a Nakayama-stable smc of $D^b(\operatorname{mod} A)$, i.e. all sms of A are liftable simple-image. We first state the results and some consequences; the second part of this section then provides the proof of the following result:

Theorem 4.1. *Let A be an RFS k-algebra over k algebraically closed. Then every sms $\mathcal{S}$ of A is simple-image of Morita type under a liftable StM.*

Corollary 4.2. *Let A be an RFS algebra over k algebraically closed. Then every sms $\mathcal{S}$ of A lifts to a Nakayama-stable smc of $D^b(\operatorname{mod} A)$. In particular, the map from $\operatorname{smc}(A)/\operatorname{DPic}(A)$ to $\operatorname{sms}(A)/\operatorname{StPic}(A)$ in Proposition 2.9 is a bijection.*

Proof. By Proposition 2.9, it is enough to show that every sms $\mathcal{S}$ of $\underline{\operatorname{mod}} A$ is a liftable simple-image: There exists an algebra B and an StM $\bar{\phi} : \underline{\operatorname{mod}} B \to \underline{\operatorname{mod}} A$ such that $\bar{\phi}$ sends simple B-modules onto $\mathcal{S}$ and that $\bar{\phi}$ lifts to a derived equivalence $\phi : D^b(\operatorname{mod} B) \to D^b(\operatorname{mod} A)$. But this follows from Theorem 3.1 and Theorem 4.1. □

Corollary 4.3. *Let A be an RFS algebra. The map $\operatorname{StMAlg}(A) \to \operatorname{sms}(A)/\operatorname{StPic}(A)$ constructed in Theorem 2.7 is a bijection. In particular, the number of Morita equivalence classes of algebras which are StM to A is the same as the number of the orbits of sms's of $\underline{\operatorname{mod}} A$ under the action of the stable Picard group of A.*

Combining Theorem 4.1 with Proposition 2.9 implies the following result which was not expected from the definition of sms's.

Corollary 4.4. *Let A be an RFS algebra over k algebraically closed. Then every sms $X_1, \cdots, X_r$ over A is Nakayama-stable, that is, the Nakayama functor ν_A permutes $X_1, \cdots, X_r$.*

Proof. An sms $\mathcal{S} = \{X_1, \cdots, X_r\}$ over an RFS algebra A can be lifted to a Nakayama-stable smc of $D^b(\operatorname{mod} A)$. □

In [16, Section 6], the following question has been posed: Is the cardinality of each sms over an Artin algebra A equal to the number of non-isomorphic non-projective simple A-modules? A positive answer of this question implies the Auslander–Reiten conjecture for any stable equivalence related to A. We answer this question positively for RFS algebras.

Corollary 4.5. *Let A be an RFS algebra over k algebraically closed. Then the cardinality of each sms over A is equal to the number of non-isomorphic simple A-modules.*

Proof. By Corollary 4.2, every sms $\mathcal{S}$ of $\underline{\mathrm{mod}}\, A$ lifts to a Nakayama-stable smc of $D^b(\mathrm{mod}\, A)$, and the cardinality of a Nakayama-stable smc must be equal to the number of (isoclasses of) simple modules by Rickard's or Al-Nofayee's result (cf. the proof of Theorem 2.9).

Alternatively, using Proposition 3.6, all sms's of A correspond to configurations, which are all finite and have the same cardinality, equal to the number of isoclasses of simple A-modules (cf. [9]). □

Validity of the Auslander–Reiten conjecture in this case first has been shown in [9]. By results of Martinez-Villa [22] the conjecture is valid for all representation finite algebras.

The proof of Theorem 4.1 adopts the idea from our alternative proof of Dugas' liftability result [15, Section 5], which uses the mutation theory of sms's. The definition of mutation we use in this article is a variation of Dugas's original one by shifting the objects by $\Omega^{\pm 1}$, so that the mutations "align" with the mutation for smc defined in [17] (see [13] Remark under Definition 4.1, [17] and Section 5 for more details). We restrict to the stable category of a self-injective algebra, although the original definition works for more general triangulated categories. For the definitions of left/right approximations see for example [1,2,17,14].

Definition 4.6. (See [13, Definition 4.1 and Remark].) Let A be a finite-dimensional self-injective algebra and $\mathcal{S} = \{X_1, \dots, X_r\}$ an sms of A. Suppose that $\mathcal{X} \subseteq \mathcal{S}$ is a Nakayama-stable subset: $\nu_A(\mathcal{X}) = \mathcal{X}$. Denote by $\mathcal{F}(\mathcal{X})$ the smallest extension-closed subcategory of $\underline{\mathrm{mod}}\, A$ containing $\mathcal{X}$. The left mutation of the sms $\mathcal{S}$ with respect to $\mathcal{X}$ is the set $\mu^+_{\mathcal{X}}(\mathcal{S}) = \{Y_1, \dots, Y_r\}$ such that

(1) $Y_j = \Omega^{-1}(X_j)$, if $X_j \in \mathcal{X}$
(2) Otherwise, Y_j is defined by the following distinguished triangle

$$\Omega(X_j) \to X \to Y_j,$$

where the first map is a minimal left $\mathcal{F}(\mathcal{X})$-approximation of $\Omega(X_j)$.

The right mutation $\mu^-_{\mathcal{X}}(\mathcal{S})$ of $\mathcal{S}$ is defined similarly.

It has been shown in [13] that the above defined sets $\mu^+_{\mathcal{X}}(\mathcal{S})$ and $\mu^-_{\mathcal{X}}(\mathcal{S})$ are again sms's. This definition works for all self-injective algebras as long as $\nu(\mathcal{X}) = \mathcal{X}$, which is automatically true for weakly symmetric algebras. Mutation of sms is designed to keep track of the images of simple modules (which form an sms) under (liftable) StM. It is interesting to ask if all sms's can be obtained just by mutations; this will be considered in Section 5.

Example 4.7. Let A be a symmetric Nakayama algebra with 4 simples and Loewy length 5. The canonical sms is the set of simple A-modules $\{1, 2, 3, 4\}$. The left mutation of $\mathcal{S}$ at $\mathcal{X} = \{2, 3\}$ is

$$\mu^+_{\mathcal{X}}(\{1,2,3,4\}) = \left\{ \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}, \begin{smallmatrix} 2 \\ 3 \\ 4 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 3 \\ 4 \\ 1 \\ 2 \end{smallmatrix}, 4 \right\}.$$

Proof of Theorem 4.1. It will be subdivided in a first part dealing with standard RFS algebras, and a second part dealing with the non-standard case.

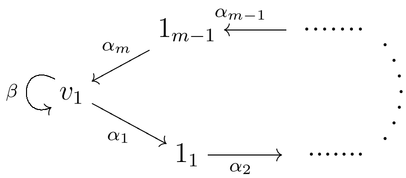


Fig. 1. $Q(D_{3m}, 1/3)$.

The standard case.

For standard RFS algebras, Asashiba [5] and Dugas [15, Section 5] already solved this problem. For convenience of the reader, we give a brief account for the main steps. We first recall Asashiba's description of stable Picard groups for standard RFS algebras.

Theorem 4.8. *(See [5].) Let A be a standard RFS algebra. If A is not of type $(D_{3m}, s/3, 1)$ with $m \geq 2, 3 \nmid s$, then* $\mathrm{StPic}(A) = \mathrm{Pic}'(A)\langle[\Omega_A]\rangle$. *If A is of type $(D_{3m}, s/3, 1)$ with $m \geq 2, 3 \nmid s$, then*

$$\mathrm{StPic}(A) = \big(\mathrm{Pic}'(A)\big\langle[\Omega_A]\big\rangle\big) \cup \big(\mathrm{Pic}'(A)\big\langle[\Omega_A]\big\rangle\big)[H],$$

where H is a stable self-equivalence of A induced from an automorphism of the quiver D_{3m} by swapping the two high vertices; it satisfies $[H]^2 \in \mathrm{Pic}'(A)$.

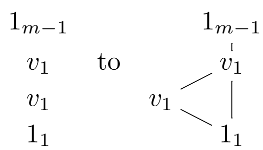
Remark 4.9. See [30] and [9] for an explanation of the concept of high vertices. Note that the stable Picard group here, by definition, contains *all* stable self-equivalences, rather than as usual only the stable self-equivalences of Morita type. Nevertheless, such different choice of the stable Picard group does not matter here, as all elements are in fact liftable (see below), hence of Morita type.

Clearly elements in $\mathrm{Pic}'(A)$ and the Heller functor Ω_A can be lifted to a standard derived equivalence. In [15], Dugas used mutation theory to prove that H is also liftable. On the level of configurations, when combining liftability of $\mathrm{StPic}(A)$ with Asashiba's construction of derived equivalences in [4], this means that every automorphism on ${}_s\Gamma_A$ sending a configuration to another can be realised by a liftable StM. Statement of Theorem 4.1 in the standard case now follows.

The non-standard case.

Now we prove Theorem 4.1 in the non-standard case. Let A be a non-standard RFS algebra of type $(D_{3m}, 1/3, 1)$ and let A_s be its standard counterpart, that is, the standard RFS algebra such that $\mathcal{S}_{A_s}$ and $\mathcal{S}_A$ are the same set when regarded as a $\tau^{(2m-1)\mathbb{Z}}$-stable configuration of $\mathbb{Z}D_{3m}$. First we recall some facts:

(1) (standard-non-standard correspondence): There is a bijection $\mathrm{ind}(A) \leftrightarrow \mathrm{ind}(A_s)$ between the set of indecomposable objects and irreducible morphisms, which is compatible with the position on the stable AR-quiver $\Gamma = \mathbb{Z}D_{3m}/\langle\tau^{2m-1}\rangle$. In particular, when A is the representative of non-standard RFS algebra, whose quiver is given in Fig. 1, then Waschbüsch [31] described the AR-quiver of A using that of A_s by replacing every part of the Loewy diagram:



The position of the indecomposable modules is presented in [31].

(2) There is one-to-one correspondence between the following three sets:

$$\mathrm{sms}(A) \leftrightarrow \mathrm{Conf}(\Gamma) \leftrightarrow \mathrm{sms}(A_s)$$

where the first is the set of sms's of A, the second is the set of configurations of Γ, and the third is the set of sms's of A_s.

(3) If B is another non-standard RFS algebra of type $(D_{3m}, 1/3, 1)$, then there is a liftable StM $\phi : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, B$ (see Theorem 3.8).

Therefore, by (3), we can assume A is the representative of the class of algebras of type $(D_{3m}, 1/3, 1)$, whose quiver is also given in Fig. 1 and the path relations can be found in [31] where A is denoted as $B(T_m, S, 1)$.

Lemma 4.10. *There is a (standard) derived self-equivalence of A which restricts to H on the $\underline{\mathrm{mod}}\, A$, with the same effect on the objects as the functor H in Theorem 4.8 on $\underline{\mathrm{mod}}\, A_s$ under the standard–non-standard correspondence.*

Proof. Consider the set $\mathcal{S}_A$ of simple A-modules, which corresponds to the configuration $\mathcal{C} = \{(0, 3m), (2m-1-j, 1) \mid j = 1, \dots, m-1\}$. More specifically, the vertex $(0, 3m)$ corresponds to the simple A-modules which can be identified with v_1 on $Q(D_{3m}, 1/3)$, whereas the vertices $(2m-1-j, 0)$ correspond to the simple A-modules 1_j. Perform sms mutation at 1_1, then we obtain

$$\mu^+_{1_1}(\mathcal{S}_A) = \left\{ \begin{smallmatrix} v_1 \\ 1_1 \end{smallmatrix},\ \begin{smallmatrix} 1_1 \\ \vdots \\ 1_{m-1} \\ v_1 \\ v_1 \end{smallmatrix},\ 1_j \mid j = 2, \dots, m-1 \right\}.$$

One checks the position of indecomposable A-modules from [31], which gives $(2m-2, 3m-1)$ for the first indecomposable, and $(m-1, 1)$ for the second one (which is $\Omega^{-1}(1_1)$). Let $\mathcal{S} = \tau^{-1}\mu^+_{1_1}(\mathcal{S}_A)$, then the configuration corresponding to $\mathcal{S}$ is the same as applying H (regarded as automorphism of ${}_s\Gamma_A$) on $\mathcal{C}$.

Combining with Dugas' result [13, Section 5], we obtain a derived equivalence $\phi : D^b(\mathrm{mod}\, B) \to D^b(\mathrm{mod}\, A)$, which restricts to a stable equivalence $\overline{\phi}$ sending $\mathcal{S}_B$ to $\mathcal{S}$. In particular, vertices in ${}_s\Gamma_B$ corresponding to $\mathcal{S}_B$ lies in $H\mathcal{C}$. But $H\mathcal{C}$ and $\mathcal{C}$ are isomorphic configurations so by Remark 3.9 B is isomorphic to A. The statement follows by taking $H = \overline{\phi}$. $\square$

Remark 4.11. The same proof works for the standard RFS algebra A_s. In fact, this proof can be extended to showing the liftability of H for the standard RFS algebras of type $(D_{3m}, s/3, 1)$ with $3 \nmid s \geq 2$ and $m \geq 2$ by covering theory. We show the data that are changed in such a proof, and leave the details as an exercise:

(1) The quiver of A is shown in Fig. 2 (cf. [5, Appendix 2]),
(2) v_1 is to be replaced by v_i's with $i \in \{1, \dots, s\}$ (vertices appearing in the inner cycle $\beta_s \dots \beta_1$),
(3) $1_1, \dots, 1_{m-1}$ will be replaced by i_j with $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, m-1\}$ (vertices on the path $\alpha^{(i)}_{m-1} \cdots \alpha^{(i)}_2$),
(4) configuration $\mathcal{C}$ (corresponding to $\mathcal{S}_A$) replaced by $\{((2m-1)i, 3m), ((2m-1)i-j, 1) \mid i = 1, \dots, s; j = 1, \dots, m-1\}$

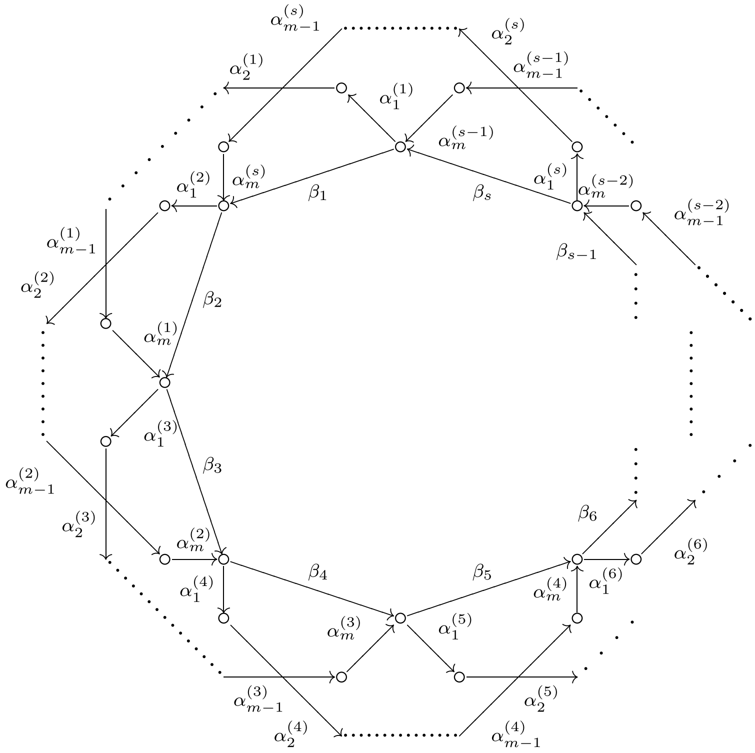


Fig. 2. $(Q(D_{3m}, s/3), s \geq 2)$.

(5) mutation to be performed with respect to $\mathcal{X} = \{1_1, \ldots, s_1\}$, which results in the mutated sms:

$$\mu^+_{\mathcal{X}}(\mathcal{S}_A) = \left\{ \begin{smallmatrix} v_i \\ i_1 \end{smallmatrix},\ \begin{smallmatrix} i_1 \\ \vdots \\ i_{m-1} \\ v_{i+2} \\ v_{i+3} \end{smallmatrix},\ 1_j \mid i = 1, \ldots, s; j = 2, \ldots, m-1 \right\}.$$

This gives an alternative proof to Dugas liftability result in [15]; it also avoid the calculation of the algebra B_s, an advantage of regarding sms's as configurations.

Lemma 4.12. *Every stable self-equivalence $\phi_s \in \mathrm{StPic}(A_s)$ has a non-standard counterpart $\phi \in \mathrm{StPic}(A)$ such that, if ϕ_s maps the set $\mathcal{S}_{A_s}$ of simple A_s-modules to $\mathcal{S}_s$, then $\phi(\mathcal{S}_A) = \mathcal{S}$ where $\mathcal{S}$ corresponds to $\mathcal{S}_s$ in the above correspondence. Moreover, ϕ is a liftable StM.*

Proof. By Asashiba's description, $\mathrm{StPic}(A_s) = \mathrm{Pic}'(A_s)\langle\Omega\rangle[H]$. If $\phi_s \in \mathrm{Pic}'(A_s)$, then it must permute the $m-1$ simple modules on the mouth of the stable tube and fixes the remaining one in a high vertex. It follows from the description of the stable AR-quiver of A_s that ϕ_s fixes $\mathcal{S}_{A_s}$ and induces the identity map $\mathrm{Conf}(\Gamma) \to \mathrm{Conf}(\Gamma)$. Therefore we can simply pick the (liftable StM) identity functor for ϕ. If $\phi_s = \Omega^n_{A_s}$ for some $n \in \mathbb{Z}$, then by standard–non-standard correspondence, picking ϕ to be the Heller shift Ω^n_A of A will do the trick. This is obviously a liftable StM. The case $\phi_s = H$ follows from the previous Lemma 4.10. □

Now let $\mathcal{S}$ be an sms of A. We know by Theorem 3.1 that $\mathcal{S}$ is simple-image, so there is some stable equivalence $\psi : \underline{\mathrm{mod}}\, B \to \underline{\mathrm{mod}}\, A$ with $\psi(\mathcal{S}_B) = \mathcal{S}$. By the above fact (3), there is a liftable StM $\phi_1 : \underline{\mathrm{mod}}\, B \to \underline{\mathrm{mod}}\, A$. Let $\mathcal{S}' = \phi_1(\mathcal{S}_B)$. If $\mathcal{S}' = \mathcal{S}$, then we are done. Otherwise, their corresponding sms's $\mathcal{S}_s$ and $\mathcal{S}'_s$ of A_s are also not equal. But they belong to the same $\mathrm{StPic}(A_s)$-orbit, since ϕ_1 induces an automorphism on the stable AR-quiver of A or A_s, so there is some stable equivalence $\phi_s : \underline{\mathrm{mod}}\, A_s \to \underline{\mathrm{mod}}\, A_s$ sending $\mathcal{S}'_s$ to $\mathcal{S}_s$.

This gives a liftable StM $\phi_2 : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, A$ by Lemma 4.12, and it maps $\mathcal{S}'$ to $\mathcal{S}$. Now we have a liftable StM $\phi = \phi_2\phi_1 : \underline{\mathrm{mod}}\, B \to \underline{\mathrm{mod}}\, A$ with $\phi(\mathcal{S}_B) = \mathcal{S}$.

This finishes the proof of Theorem 4.1. □

5. Sms's and mutations

In this section, we discuss connections with mutations and with tilting quivers and how to use these concepts for sms. A main result is Theorem 5.5, which states that the homotopy category $\mathcal{T} = K^b(\mathrm{proj}\, A)$ is strongly tilting-connected when A is an RFS algebra. This result is formally independent of sms, but it fits well with the point of view taken in this paper.

The first connection we consider here comes from the aforementioned result of Dugas [13], which opens up a new and efficient way to study (and compute) simple-image sms's of Morita type and their liftability, as demonstrated in the previous section.

We have seen how mutation of sms and Nakayama-stable smc are connected. We remind the reader of the main result of [17], which in particular gives a bijection between smc and silting objects as well as compatibility of the respective mutations. Since we have already established a connection between sms and smc, we can now exploit the connection with silting/tilting objects.

First we briefly recall some information on silting theory developed by Aihara and Iyama [2]. Throughout this section, A is an indecomposable non-simple self-injective algebra over an algebraically closed field. We use $\mathcal{T}$ to denote the (triangulated) homotopy category $K^b(\mathrm{proj}\, A)$ of bounded complexes of projective A-modules; the suspension functor in this category is denoted by [1], and by $[n]$ we mean $[1]^n$.

Definition 5.1. (See [2].)

(1) Let $T \in \mathcal{T}$. Then T is a silting (resp. tilting) object if:
 (a) $\mathrm{Hom}_{\mathcal{T}}(T, T[i]) = 0$ for any $i > 0$ (resp. $i \neq 0$)
 (b) The smallest thick subcategory of $\mathcal{T}$ containing T is $\mathcal{T}$ itself.

(2) Let $T = X_1 \oplus \cdots \oplus X_r$ be a silting object (where each X_i is indecomposable) and $\mathcal{X} \subset \{1, \ldots, r\}$. A left silting mutation of T with respect to $\mathcal{X}$, denoted by $\mu^+_{\mathcal{X}}(T) = Y_1 \oplus \cdots \oplus Y_r$ satisfies by definition that the indecomposable summands Y_i are given as follows:
 (a) $Y_i = X_i$ for $i \notin \mathcal{X}$
 (b) For $i \in \mathcal{X}$:

$$Y_j := \mathrm{cone}\left(\text{minimal left add}\left(\bigoplus_{i\notin\mathcal{X}} X_i\right)\text{-approximation of } X_j\right)$$

A right silting mutation $\mu^-_{\mathcal{X}}$ is defined similarly using right approximation. A silting mutation is said to be irreducible if $\mathcal{X} = \{i\}$ for some i.

Note that tilting objects in $\mathcal{T}$ (i.e. one-sided tilting complexes) are exactly the silting objects that are stable under Nakayama functor (see, for example, [1, Theorem A.4]). As we have hinted throughout the whole article, Nakayama-stability plays a vital role in the study of sms's, at least for sms's which are liftable and simple-image of Morita type. For convenience, we denote the Nakayama functor $\nu = \nu_A$ when the algebra A under consideration is clear, and we assume every tilting object is basic, i.e. its indecomposable summands are pairwise non-isomorphic.

Lemma 5.2. *Let A, $\mathcal{T}$ be as above and $\mathcal{C}$ a full subcategory of $\mathcal{T}$ with $\nu\mathcal{C} = \mathcal{C}$. If $X \in \mathcal{T}$ and $f : X \to Y$ is a (minimal) left $\mathcal{C}$-approximation of X, then $\nu_A(f) : \nu X \to \nu Y$ is a (minimal) left $\mathcal{C}$-approximation of νX. In particular, if $\nu X = X$, then $\nu Y = Y$.*

Proof. This is just a routine check. $\square$

By this lemma, a mutation of a tilting object (i.e. a Nakayama-stable silting object) is a tilting object if we mutate at a Nakayama-stable summand. For convenience, we say that a Nakayama-stable mutation of a tilting complex is a tilting mutation. An irreducible silting mutation mutates with respect to an indecomposable summand. By thinking of this as mutating with respect to a "minimal" Nakayama-stable summand, we can make sense of "irreducibility" for tilting mutation for general self-injective algebras (rather than just weakly symmetric algebras).

Definition 5.3. (Compare to [1].)

(1) Let $T = T_1 \oplus \cdots \oplus T_r$ be a basic tilting object in $\mathcal{T} = K^b(\operatorname{proj} A)$. If X is a Nakayama-stable summand of T such that for any Nakayama-stable summand Y of X, we have $Y = X$, then we call X a minimal Nakayama-stable summand. A (left) tilting mutation $\mu_X^+(T)$ is said to be irreducible if X is minimal. Similarly for right tilting mutation $\mu_X^-(T)$.
(2) Let T, U be basic tilting objects in $\mathcal{T}$. We say that U is tilting-connected (respectively, left tilting-connected) to T if U can be obtained from T by iterative irreducible (respectively, left) tilting mutations.
(3) $\mathcal{T}$ is tilting-connected if all its basic tilting objects are tilting-connected to each other. We say that $\mathcal{T}$ is strongly tilting-connected if for any basic tilting objects T, U with $\operatorname{Hom}_{\mathcal{T}}(T, U[i]) = 0$ for all $i > 0$, U is left tilting-connected to T.

Remark 5.4.

(1) Note that the irreducible tilting mutation just defined is different from an irreducible silting mutation when A is self-injective non-weakly symmetric, even though it is itself a silting mutation as well. We will emphasise irreducible *tilting* mutation throughout to distinguish between our notion and irreducible silting mutation.
(2) We can define the analogous notion of (left or right) irreducible sms mutation similar to irreducible tilting mutation above. More precisely, for an sms $\mathcal{S} = \{X_1, \ldots, X_r\}$ as in Definition 4.6, its irreducible mutation means that we mutate at a Nakayama-stable subset $\mathcal{X} = \{X_{i_1}, \ldots, X_{i_m}\}$ which is minimal in the obvious sense.
(3) For any tilting complex T, there exists a tilting complex P (e.g. $A[l]$ for $l \gg 0$) such that $\operatorname{Hom}_{\mathcal{T}}(T, P[i]) = 0$ for all $i > 0$.
(4) Strongly tilting-connected implies tilting-connected. This follows from (3) and the fact that left and right mutations are inverse operations to each other, i.e. $\mu_Y^- \mu_X^+(T) = T = \mu_Z^+ \mu_X^-(T)$ where $T = X \oplus M$, $\mu_X^+(T) = Y \oplus M$, and $\mu_X^-(T) = Z \oplus M$.

We can now reformulate a question asked in [2] and [1, Question 3.2]: Is $\mathcal{T} = K^b(\operatorname{proj} A)$ tilting-connected for self-injective algebra A? By reproving the Nakayama-stable analogue of the results in [2] and [1], we can answer this question positively for RFS algebras A. These proofs are not directly related to the simple-minded theories and are really about modifying the proofs of Aihara and of Aihara and Iyama in an appropriate way.

Theorem 5.5. *Let A be an RFS algebra. Then the homotopy category $\mathcal{T} = K^b(\operatorname{proj} A)$ is strongly tilting-connected.*

The proof will occupy a separate subsection below.

Recall the silting quiver as defined in [2] and [1]. Again we can define a "Nakayama-stable version" and the sms's version of this combinatorial gadget.

Definition 5.6. (Compare to [1,2].) Let A be a self-injective algebra.

(1) Let $\mathrm{tilt}(A)$ be the class of all tilting objects in $\mathcal{T} = K^b(\mathrm{proj}\, A)$ up to homotopy equivalence. The tilting quiver of $\mathcal{T}$ is a quiver $Q_{\mathrm{tilt}}(A)$ such that the set of vertices is the class of basic tilting objects of $\mathcal{T}$; and for T, U tilting objects, $T \to U$ is an arrow in the quiver if U is an irreducible left tilting mutation of T.
(2) Let $\mathrm{sms}(A)$ denote the class of all sms's of A. The mutation quiver of $\mathrm{sms}(A)$ is a quiver $Q_{\mathrm{sms}}(A)$ such that the set of vertices is $\mathrm{sms}(A)$; and for two sms's $\mathcal{S}$, $\mathcal{S}'$, $\mathcal{S} \to \mathcal{S}'$ is an arrow in the quiver if $\mathcal{S}'$ is an irreducible left mutation of $\mathcal{S}$.

Remark 5.7.

(1) Long before the work of [2], the term tilting quiver has been used for a graph whose vertices are tilting modules over a finite dimensional algebra. The tilting quiver here is a specialisation of the silting quiver of [2], whose vertices are objects in a triangulated category.
(2) Combinatorially (i.e. ignoring the "labeling" of the vertices), $Q_{\mathrm{tilt}}(A) = Q_{\mathrm{tilt}}(B)$ (respectively $Q_{\mathrm{sms}}(A) = Q_{\mathrm{sms}}(B)$) if A and B are derived (resp. stably) equivalent.

Proposition 5.8. *Suppose A is an RFS algebra. Then there is a surjective quiver morphism $Q_{\mathrm{tilt}}(A) \to Q_{\mathrm{sms}}(A)$. In particular, every sms of A can be obtained by iterative left irreducible mutation starting from the simple A-modules.*

Proof. Define a map

$$\{\text{tilting complexes of } A\} \to \mathrm{sms}(A)$$
$$T \mapsto \phi^{-1}(\mathcal{S}_B)$$

where ϕ is the induced stable equivalence of Morita type given by restricting the derived equivalence associated to T. This induces a quiver morphism as correspondence between tilting complexes and Nakayama-stable smc's respect mutation [17], and restricting simple-image smc's to sms's also respects mutation by the proof of [13, Proposition 5.3]. Now surjectivity on the set of vertices follows from the proof of Theorem 4.1, which asserts that every sms of A is liftable simple-image. For the last statement, let $\mathcal{S}$ be an sms of A, then $\mathcal{S}$ is liftable to a Nakayama-stable smc $\mathbf{S}$, which corresponds to a tilting object T. By Theorem 5.5, we can obtain T by iterative tilting mutations starting from A. The bijection in [17] then implies that $\mathbf{S}$ can be obtained by iterative smc mutations starting from simple A-modules. The statement now follows from the surjective quiver morphism. □

Since the sms's of an RFS algebra are in general not acted upon transitively by the stable Picard group, this result shows that a mutation of sms's usually cannot be realised by a stable self-equivalence.

This result can also be compared with Proposition 2.9, where we formed the quotient of the class of all smc's (respectively sms's) by the derived (respectively stable) Picard group, obtaining an injection regardless of representation-finiteness. On the other hand, these quivers visualise how we can "track" simple-image sms's of Morita type, and they contain more structure than the sets considered in Proposition 2.9. Yet it is still unclear how these links between smc's (hence tilting complexes) and sms's can be used to extract information about derived and/or stable Picard groups.

Another connection of this kind, with two-term tilting complexes, will be discussed in [12].

5.1. Proof of Theorem 5.5 à la Aihara

We use the notation $\mathcal{T} = K^b(\operatorname{proj} A)$ with A an RFS algebra over a field. The term tilting object refers to objects in $\mathcal{T}$, that is, to complexes. Recall the following notation from [2] and [1].

Definition 5.9. Let T, U be tilting objects of $\mathcal{T}$, write $T \geq U$ if $\operatorname{Hom}_{\mathcal{T}}(T, U[i]) = 0$ for all $i > 0$.

Note this defines a partial order on the class of silting (and hence, tilting) objects of $\mathcal{T}$. Applying Lemma 5.2 to [2, Prop. 2.24] yields:

Proposition 5.10. *Let T be a tilting complex of a self-injective algebra, and $U_0 \cong \nu U_0 \in \mathcal{T}$ such that $T \geq U_0$, then there are triangles*

$$U_1 \xrightarrow{g_1} T_0 \xrightarrow{f_0} U_0 \longrightarrow U_1[1],$$
$$\cdots,$$
$$U_\ell \xrightarrow{g_\ell} T_{\ell-1} \xrightarrow{f_{\ell-1}} U_{\ell-1} \longrightarrow U_\ell[1],$$
$$0 \xrightarrow{g_{\ell+1}} T_\ell \xrightarrow{f_\ell} U_\ell \longrightarrow 0,$$

for some $\ell \geq 0$ such that f_i is a minimal right $\operatorname{add} T$*-approximation, g_{i+1} belongs to the Jacobson radical $J_{\mathcal{T}}$, $\nu U_i = U_i$ and $\nu T_i = T_i$, for any $0 \leq i \leq \ell$.*

Proof. The only difference of the proof here and the one in [2] is to use Lemma 5.2 on the triangles in the proof. More precisely, following the proof in [2] we have a triangle $U_1 \xrightarrow{g_1} T_0 \xrightarrow{f_0} U_0 \to U_1[1]$ with f_0 a minimal right $\operatorname{add} T$-approximation of U_0. Apply the Nakayama functor to this triangle yields another triangle

$$\nu U_1 \xrightarrow{\nu g_1} \nu T_0 \xrightarrow{\nu f_0} \nu U_0 \nu \longrightarrow U_1[1],$$

where $\nu T_0 \cong T_0$ and νf_0 is a minimal right $\operatorname{add} T$-approximation by Lemma 5.2. Let $\theta : \nu U_0 \to U_0$ be an isomorphism. Then both f_0 and $\theta \circ \nu f_0$ are minimal right $\operatorname{add} T$-approximation of U_0. As $\theta \circ \nu f_0$ is a right $\operatorname{add} T$-approximation, there is $\phi : \nu T_0 \to T_0$ with $f_0 \circ \phi = \theta \circ \nu f_0$. Minimality of $\theta \circ \nu f_0$ implies that ϕ is an isomorphism. We then obtain a morphism of triangles:

$$\begin{tikzcd}
\nu U_1 \arrow[r, "\nu g_1"] \arrow[d, dotted] & \nu T_0 \arrow[r, "\nu f_0"] \arrow[d, "\phi", "\cong"'] & \nu U_0 \arrow[r] \arrow[d, "\theta", "\cong"'] & \nu U_1[1] \arrow[d, dotted] \\
U_1 \arrow[r, "g_1"] & T_0 \arrow[r, "f_0"] & U_0 \arrow[r] & \nu U_1[1].
\end{tikzcd}$$

By the axioms of triangulated category, $\nu U_1 \cong U_1$. Now the proof continues as in [2]. $\square$

This can be used to deduce the Nakayama-stable analogue of [2, Theorem 2.35, Prop. 2.36]:

Theorem 5.11. *Let T, U be tilting objects of a self-injective algebra. Then*

(1) *If $T > U$, then there exists an irreducible left tilting mutation P of T such that $T > P \geq U$.*

(2) *The following are equivalent:*
   (a) *U is an irreducible left tilting mutation of T;*
   (b) *T is an irreducible right tilting mutation of U;*
   (c) *$T > U$ and there is no P tilting such that $T > P > U$.*

Proof. Proof of (1) is the same as the proof of [1, Prop. 2.12], except that now we take a minimal ν-stable summand of T_ℓ instead of an indecomposable summand. Proof of (2) is the same as the proof of [2, Theorem 2.35], without any change. □

We modify the proof of Aihara in [1] to show that any tilting object of an RFS algebra can be obtained through iterative irreducible tilting mutation.

The proof of Theorem 5.5 is based on the following key proposition:

Proposition 5.12. *(See [1, Prop. 5.1].) $\mathcal{T}$ is tilting-connected if, for any algebra B derived equivalent to A, the following conditions are satisfied:*

(A1) *Let T be a basic tilting object in $K^b(\operatorname{proj} B)$ with $B[-1] \geq T \geq B$. Then T is tilting-connected to both $B[-1]$ and B.*

(A2) *Let P be a basic tilting object in $K^b(\operatorname{proj} B)$ with $B[-\ell] \geq P \geq B$ for a positive integer ℓ. Then there exists a basic tilting object T in $K^b(\operatorname{proj} B)$ satisfying $B[-1] \geq T \geq B$ such that $T[-\ell+1] \geq P \geq T$.*

Since we are only interested in tilting-connectedness rather than silting-connectedness, the original condition (A3), which says that any silting object is connected to a tilting object, is discarded.

(A2) is known to be true from [1, Lemma 5.4]. Therefore, what is left is to look carefully at the arguments and results that are used by Aihara in the proof of (A1).

Lemma 5.13. *(See [1, Lemma 5.3].) Condition (A1) holds for all RFS algebras A.*

Proof. The original proof relies on [1, Prop. 2.9] and [1, Theorem 3.5]. Proposition 2.9 is true regardless of what kind of algebra A is. We are left to show the analogue of [1, Theorem 3.5] is true, i.e. if there exist only finitely many tilting objects P such that $T \geq P \geq U$ for any basic tilting objects T, U in $\mathcal{T}$, then U is left tilting-connected to T.

Looking at the proof of [1, Theorem 3.5], it depends on [1, Theorem 2.17] and [2, Theorem 2.35, Proposition 2.36]. The proof of [1, Theorem 2.17] can be translated word-by-word in our setting by replacing the relying proposition to Theorem 5.11; the analogue of [2, Theorem 2.35, Proposition 2.36] is just Theorem 5.11 as mentioned. □

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