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BSDEs with general filtration driven by Lévy processes, and an application in stochastic controllability

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ABSTRACT

In this paper, we introduce a weak version of the strong solution (the adapted solution used in Pardoux and Peng (1990) [2]), i.e., the transposition solution, to the backward stochastic differential equation (BSDE) with general filtration and random jumps, and study the corresponding well-posedness. The main tools that we employ are the Riesz representation theorem and the Banach fixed point theorem, without using the martingale representation theorem. As an application, we give a definition of controllability to the stochastic linear control system in the sense of the transposition solution and provide a Kalman-type rank condition to guarantee this property.

1. Introduction

Given $T > 0$, let $(\Omega, \mathscr{F}, \mathscr{F}_t, P; t \geq 0)$ be a complete filtration space and $\mathbb{F} = \{\mathscr{F}_t; t \geq 0\}$ be a filtration satisfying the usual conditions. On the above filtration space there exist two mutually independent stochastic processes:

(i) a d-dimensional Brownian motion $\{W(t); t \geq 0\}$;

(ii) a Poisson random measure N on $R^+ \times E$, where $E = R^l - \{0\}$ with the Borel σ-field $\mathscr{B}(E)$. λ is the intensity (Lévy measure) of N with the property that

$$\int_E (1 \wedge |z|^2)\lambda(dz) < \infty$$

and μ is the compensator of N, $\mu(dt, dz) = dt\lambda(dz)$. Then $\{\widetilde{N}((0, t] \times A) = (N - \mu)((0, t] \times A), \mathscr{F}_t; t \geq 0\}$ is a compensated Poisson process which is a càdlàg martingale for all $A \in \mathscr{B}(E)$ satisfying $\lambda(A) < \infty$.

Throughout this paper, the filtration $\mathbb{F}$ is not necessarily the natural filtration generated by the Brownian motion and the Poisson random measure.

For simplicity, we consider only the case $d = l = 1$ in this paper; the general cases can be treated by a similar method. For any $n \geq 1$, denote by $|x|$ and $\langle x, y\rangle$ the Euclidean norm and the inner product of $x, y \in \mathbf{R}^n$ respectively. For any square integrable martingale $\{M(t), \mathscr{F}_t; t \geq 0\}$, write $[M]_t$ and $\langle M\rangle_t$ for its quadratic variation and predictable quadratic variation at time t, respectively. Also, we introduce the following classes of processes which will be used in the sequel.

- $L^2_{\mathscr{F}_t}(\Omega; \mathbf{R}^n)$ is the space of all $\mathscr{F}_t$-measurable and $\mathbf{R}^n$-valued random variables ξ satisfying $|\xi|^2_{L^2_{\mathscr{F}_t}(\Omega;\mathbf{R}^n)} = \mathbb{E}|\xi|^2 < \infty$.
- $L^2_{P,\mathbb{F}}(\Omega; L^2(0, T; \mathbf{R}^n))$ is the space of all $\mathbb{F}$-predictable stochastic processes K satisfying $|K|^2_{L^2_{P,\mathbb{F}}(\Omega;L^2(0,T;\mathbf{R}^n))} = \mathbb{E}\Big(\int_0^T \int_E |K(t, z)|^2 \lambda(dz)dt\Big) < \infty$.
- $L^2_{\mathbb{F}}(\Omega; D([0, T]; \mathbf{R}^n))$ is the space of all $\mathbb{F}$-adapted càdlàg stochastic processes X satisfying $|X|^2_{L^2_{\mathbb{F}}(\Omega;D([0,T];\mathbf{R}^n))} = \mathbb{E}(\sup_{t\in[0,T]} |X(t)|^2) < \infty$.
- For any $p, q \geq 1$, $L^p_{\mathbb{F}}(\Omega; L^q(0, T; \mathbf{R}^n))$ denotes the space of all $\mathbb{F}$-adapted processes Y satisfying $|Y|_{L^p_{\mathbb{F}}(\Omega;L^q(0,T;\mathbf{R}^n))} = \Big(\mathbb{E}\Big(\int_0^T |Y(t)|^q dt\Big)^{p/q}\Big)^{1/p} < \infty$.

We consider the following BSDE in $[0, T]$

$$\begin{cases} dy(t) = f(t, y(t), Y(t), K(t, \cdot))dt + Y(t)dW(t) \\ \qquad\qquad + \displaystyle\int_E K(t, z)\widetilde{N}(dt, dz) \\ y(T) = y_T, \end{cases} \tag{1.1}$$

where f satisfies $f(\cdot, 0, 0, 0) \in L^2_{\mathbb{F}}(\Omega; L^1(0, T; \mathbf{R}^n))$, and there exist $g \in L^1(0, T)$, $h \in L^2(0, T)$, such that

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$$|f(t,y,Y,K)-f(t,\widetilde{y},\widetilde{Y},\widetilde{K})| \leq g(t)|y-\widetilde{y}|+h(t)(|Y-\widetilde{Y}|+|K-\widetilde{K}|_{L^2(E,\mathscr{B}(E),\lambda;\mathbf{R}^n)}) \quad \text{a.e. } t\in[0,T], \text{ a.s.}, \tag{1.2}$$

for any $y,\widetilde{y},Y,\widetilde{Y}\in\mathbf{R}^n$, $K,\widetilde{K}\in L^2(E,\mathscr{B}(E),\lambda;\mathbf{R}^n)$.

Linear BSDEs were first introduced by Bismut in [1] as the equations for the conjugate variable in the stochastic version of the Pontryagin maximum principle. Non-linear BSDEs were first studied by Pardoux and Peng in [2]. General BSDEs with jumps were met in [3,4]. The main theorems in [2–4] for the existence and uniqueness of the strong solutions depend on the martingale representation theorems for the natural filtration spaces, i.e., $\{\mathscr{F}_t;t\geq 0\}=\{\mathscr{F}_t^W;t\geq 0\}$ (generated by $\{W(t);t\geq 0\}$ and augmented by all the P-null sets) in [2] while $\{\mathscr{F}_t;t\geq 0\}=\{\mathscr{F}_t^{W,N};t\geq 0\}$ in [3,4].

BSDEs with enlarged filtration in financial market were considered in [5] and the references therein: the ordinary agent has the natural information flow $\{\mathscr{F}_t;t\geq 0\}$, while an insider possesses from the beginning additional information (some random variable G) and therefore has the enlarged filtration $\{\mathscr{F}_t\vee\sigma(G);t\geq 0\}$. Under some hypothesis on G, the martingale representation theorem on $\{\mathscr{F}_t\vee\sigma(G);t\geq 0\}$ still holds.

BSDEs driven by general martingales $M=\{M(t),\mathscr{F}_t;t\geq 0\}$ were considered in [6–8]. The authors in [6,7] considered the general filtrations but they decomposed the processes space into two orthogonal subspaces, one of which has the martingale representation property. The author in [8] assumed that M enjoys the predictable representation property, i.e., for any square integrable martingale $L=\{L(t),\mathscr{F}_t;t\geq 0\}$ with zero initial value, there exists a predictable process H such that $L(t)=\int_0^t H(s)dM(s)$. Hence their method still depends on the representation property of the martingale.

In [9], the authors discussed BSDEs driven by Brownian motions with general filtration using a new approach. They defined the transposition solution of Eq. (1.1), and obtained the corresponding well-posedness and the comparison theorem. The main novelty of their method is that they did not need the martingale representation theorem.

Before 1990, there are a few works about the controllability of stochastic linear control systems (SLCSs), say [10,11] and so on. As an application of BSDEs, Peng in [12] defined the controllability of SLCSs driven by Brownian motions and obtained the Kalman rank condition to guarantee this property. His result was based on the following fact: the role of control in the SLCSs is similar to the second component of the solution for BSDEs, hence he studied the controllability of SLCSs by means of BSDEs.

In this paper, we study the BSDE (1.1) with jumps and obtain its well-posedness in the sense of the transposition solution. As an application, we consider the controllability, in the transposition sense, for SLCSs with jumps and establish the Kalman-type rank condition in this situation.

2. Preliminaries

For any $t\in[0,T]$ we introduce some processes spaces

$$\begin{aligned}
\mathcal{K}^D[t,T] &= L^2_{\mathbb{F}}(\Omega;D([t,T];\mathbf{R}^n))\times L^2_{\mathbb{F}}(\Omega;L^2(t,T;\mathbf{R}^n)) \\
&\quad\times L^2_{P,\mathbb{F}}(\Omega;L^2(t,T;\mathbf{R}^n)),\\
\mathcal{K}^\infty[t,T] &= L^2_{\mathbb{F}}(\Omega;L^\infty(t,T;\mathbf{R}^n))\times L^2_{\mathbb{F}}(\Omega;L^2(t,T;\mathbf{R}^n)) \\
&\quad\times L^2_{P,\mathbb{F}}(\Omega;L^2(t,T;\mathbf{R}^n)),\\
\mathcal{K}^1[t,T] &= L^2_{\mathbb{F}}(\Omega;L^1(t,T;\mathbf{R}^n))\times L^2_{\mathbb{F}}(\Omega;L^2(t,T;\mathbf{R}^n)) \\
&\quad\times L^2_{P,\mathbb{F}}(\Omega;L^2(t,T;\mathbf{R}^n))\times L^2_{\mathscr{F}_t}(\Omega;\mathbf{R}^n),
\end{aligned}$$

with the canonical norms.

Let us consider the following auxiliary forward stochastic differential equation (FSDE) in $[t,T]$

$$\begin{cases} dx(\tau)=u(\tau)d\tau+v(\tau)dW(\tau)+\displaystyle\int_E w(\tau,z)\widetilde{N}(d\tau,dz)\\ x(t)=\eta, \end{cases} \tag{2.1}$$

where $(u,v,w,\eta)\in\mathcal{K}^1[t,T]$.

It is clear that Eq. (2.1) admits a unique strong solution $x\in L^2_{\mathbb{F}}(\Omega;D([t,T];\mathbf{R}^n))$. If Eq. (1.1) admits a strong solution (y,Y,K), then by applying Itô's formula to $\langle x(t),y(t)\rangle$, we obtain that

$$\begin{aligned}
&\mathbb{E}\langle x(T),y(T)\rangle-\mathbb{E}\langle\eta,y(t)\rangle\\
&\quad=\mathbb{E}\int_t^T\langle x(\tau),f(\tau,y(\tau),Y(\tau),K(\tau))\rangle d\tau\\
&\qquad+\mathbb{E}\int_t^T\langle u(\tau),y(\tau)\rangle d\tau+\mathbb{E}\int_t^T\langle v(\tau),Y(\tau)\rangle d\tau\\
&\qquad+\mathbb{E}\int_t^T\int_E\langle w(\tau,z),K(\tau,z)\rangle\lambda(dz)d\tau.
\end{aligned}\tag{2.2}$$

This reminds us to introduce a weak version of the strong solution to Eq. (1.1) as follows.

Definition 2.1. We call $(y,Y,K)\in\mathcal{K}^D[0,T]$ to be a transposition solution of Eq. (1.1) if for any $t\in[0,T]$ and $(u,v,w,\eta)\in\mathcal{K}^1[t,T]$, (2.2) holds, where x is the strong solution of Eq. (2.1).

Next, we list three lemmas which will be used in proving the well-posedness of linear non-homonomous BSDEs. The first one is obtained by virtue of a similar approach in [13] and the second one is from [9], hence we omit the proofs.

Lemma 2.1. *For any $r\in[1,\infty)$, it holds that*

$$L^2_{\mathbb{F}}(\Omega;L^r(0,T;\mathbf{R}^n))^*=L^2_{\mathbb{F}}(\Omega;L^{r'}(0,T;\mathbf{R}^n)),$$
$$L^2_{P,\mathbb{F}}(\Omega;L^r(0,T;\mathbf{R}^n))^*=L^2_{P,\mathbb{F}}(\Omega;L^{r'}(0,T;\mathbf{R}^n)),$$

where $r'=r/(r-1)$ if $r>1$; $r'=\infty$ if $r=1$.

Lemma 2.2. *For any $T>0$, assume that $p\in(1,\infty]$,*

$$q=\begin{cases}\dfrac{p}{p-1}, & p\in(1,\infty)\\ 1, & p=\infty,\end{cases}$$

$f_1\in L^p_{\mathbb{F}}(0,T;L^2(\Omega;\mathbf{R}^n))$ and $f_2\in L^q_{\mathbb{F}}(0,T;L^2(\Omega;\mathbf{R}^n))$. Then

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}\mathbb{E}\langle f_1(t),f_2(\tau)\rangle d\tau=\mathbb{E}\langle f_1(t),f_2(t)\rangle,$$

a.e. $t\in[0,T)$.

Lemma 2.3. *For any $t\in[0,T]$, $(u,v,w,\eta)\in\mathcal{K}^1[t,T]$, the solution $x\in L^2_{\mathbb{F}}(\Omega;D([t,T];\mathbf{R}^n))$ of Eq. (2.1) satisfies*

$$|x|_{L^2_{\mathbb{F}}(\Omega;D([t,T];\mathbf{R}^n))}\leq C|(u,v,w,\eta)|_{\mathcal{K}^1[t,T]},$$

where C is a constant depending only on T.

Proof. It is clear that for any $t\in[0,T]$, $(u,v,w,\eta)\in\mathcal{K}^1[t,T]$, the FSDE (2.1) admits a strong solution

$$x(s)=\eta+\int_t^s u(\tau)d\tau+\int_t^s v(\tau)dW(\tau)+\int_t^s\int_E w(\tau,z)\widetilde{N}(d\tau,dz),$$

and $x \in L^2_{\mathbb{F}}(\Omega; D([t,T]; \mathbf{R}^n))$. By the definition of $L^2_{\mathbb{F}}(\Omega; D([t,T]; \mathbf{R}^n))$, the Hölder inequality and the Burkholder–Davis–Gundy inequality,

$$
\begin{aligned}
|x|^2_{L^2_{\mathbb{F}}(\Omega;D([t,T];\mathbf{R}^n))} &= \mathbb{E}\sup_{t\le s\le T}|x(s)|^2 \\
&\le 4\mathbb{E}\sup_{t\le s\le T}\left(|\eta|^2 + \left|\int_t^s u(\tau)d\tau\right|^2 + \left|\int_t^s v(\tau)dW(\tau)\right|^2 \right.\\
&\quad \left. + \left|\int_t^s\int_E w(\tau,z)\widetilde{N}(d\tau,dz)\right|^2\right) \\
&\le 4\left(\mathbb{E}|\eta|^2 + \mathbb{E}\left(\int_t^T |u(\tau)|d\tau\right)^2\right) + 4C_1\mathbb{E}\Big[\int_t^\cdot v(\tau)dW(\tau)\Big]_T \\
&\quad + 4C_2\mathbb{E}\Big[\int_t^\cdot\int_E w(\tau,z)\widetilde{N}(d\tau,dz)\Big]_T \\
&= 4\left(\mathbb{E}|\eta|^2 + \mathbb{E}\left(\int_t^T |u(\tau)|d\tau\right)^2\right) + 4C_1\mathbb{E}\Big\langle\int_t^\cdot v(\tau)dW(\tau)\Big\rangle_T \\
&\quad + 4C_2\mathbb{E}\Big\langle\int_t^\cdot\int_E w(\tau,z)\widetilde{N}(d\tau,dz)\Big\rangle_T \\
&\le (4+4C_1+4C_2)|(u,v,w,\eta)|^2_{\mathcal{K}^1[t,T]},
\end{aligned}
$$

where C_1, C_2 are constants. □

3. Well-posedness of BSDEs with jumps

In this section, we establish the well-posedness of linear nonhomonomous BSDEs first, then by the Banach fixed point theorem we obtain the well-posedness result for Eq. (1.1).

Theorem 3.1. *For any $f(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^1(0,T;\mathbf{R}^n))$ and any $y_T \in L^2_{\mathscr{F}_T}(\Omega; \mathbf{R}^n)$, the following BSDE in $[0,T]$*

$$
\begin{cases}
dy(t) = f(t)dt + Y(t)dW(t) + \displaystyle\int_E K(t,z)\widetilde{N}(dt,dz) \\
y(T) = y_T,
\end{cases}
$$

admits a unique transposition solution $(y,Y,K) \in \mathcal{K}^D[0,T]$. Furthermore, for any $t \in [0,T]$,

$$|(y,Y,K)|_{\mathcal{K}^D[0,T]} \le C(|f|_{L^2_{\mathbb{F}}(\Omega;L^1(t,T;\mathbf{R}^n))} + |y_T|_{L^2_{\mathscr{F}_T}(\Omega;\mathbf{R}^n)}),$$

where C is a constant depending only on T.

Since the proof of this theorem is similar to that of Theorem 3.1 in [9], we give below only a sketch.

Sketch Proof of Theorem 3.1. *Step* 1. For any $t \in [0,T]$, we define a linear functional S on $\mathcal{K}^1[t,T]$ as follows

$$S(u,v,w,\eta) = \mathbb{E}\langle x(T), y_T\rangle - \mathbb{E}\int_t^T \langle x(\tau), f(\tau)\rangle d\tau,$$

for any $(u,v,w,\eta) \in \mathcal{K}^1[t,T]$, where $x \in L^2_{\mathbb{F}}(\Omega; D([t,T]; \mathbf{R}^n))$ solves Eq. (2.1). By Lemma 2.3, S is bounded. Thus, by Lemma 2.1 and the Riesz representation theorem, there exist $(y^t, Y^t, K^t) \in \mathcal{K}^\infty[t,T]$ and $\varsigma^t \in L^2_{\mathscr{F}_t}(\Omega, \mathbf{R}^n)$ such that

$$
\begin{aligned}
&\mathbb{E}\langle x(T), y_T\rangle - \mathbb{E}\int_t^T \langle x(\tau), f(\tau)\rangle d\tau \\
&= \mathbb{E}\int_t^T \langle u(\tau), y^t(\tau)\rangle d\tau + \mathbb{E}\int_t^T \langle v(\tau), Y^t(\tau)\rangle d\tau \\
&\quad + \mathbb{E}\int_t^T\int_E \langle w(\tau,z), K^t(\tau,z)\rangle \lambda(dz)d\tau + \mathbb{E}\langle \eta, \varsigma^t\rangle.
\end{aligned}
$$

Step 2. Choosing appropriate $(u,v,w,\eta) \in \mathcal{K}^1[\cdot,T]$, we conclude that for any t_1 and t_2 satisfying $0 \le t_2 \le t_1 \le T$, it holds that

$$
\begin{aligned}
&(y^{t_2}(\tau,\omega), Y^{t_2}(\tau,\omega), K^{t_2}(\tau,\cdot,\omega)) \\
&\quad = (y^{t_1}(\tau,\omega), Y^{t_1}(\tau,\omega), K^{t_1}(\tau,\cdot,\omega)), \\
&\quad \text{a.e. } (\tau,\omega) \in [t_1,T]\times\Omega.
\end{aligned}
$$

Put $y(t,\omega) = y^0(t,\omega), Y(t,\omega) = Y^0(t,\omega), K(t,\cdot,\omega) = K^0(t,\cdot,\omega)$.

Step 3. One can show that

$$X(t) := \varsigma^t - \int_0^t f(s)ds, \quad t \in [0,T]$$

is a $\mathbb{F}$-martingale, thus $\{\varsigma^t; 0 \le t \le T\}$ has a càdlàg modification, i.e. $\varsigma \in L^2_{\mathbb{F}}(\Omega; D([0,T]; \mathbf{R}^n))$.

Step 4. Using Lemma 2.2, for a.e. $t \in [0,T]$, we can show that

$$\varsigma^t = y(t) \quad \text{a.s.}$$

This completes the proof. □

Now by virtue of Theorem 3.1 and the Banach fixed point theorem, we get the well-posedness result for the general semilinear BSDE (1.1).

Theorem 3.2. *Under the assumption (1.2), for any $y_T \in L^2_{\mathscr{F}_T}(\Omega; \mathbf{R}^n)$, Eq. (1.1) admits a unique transposition solution $(y,Y,K) \in \mathcal{K}^D[0,T]$. Furthermore, there is a constant C, depending only on g, h and T, such that*

$$
\begin{aligned}
&|(y,Y,K)|_{\mathcal{K}^D[0,T]} \\
&\quad \le C\Big(|f(\cdot,0,0,0)|_{L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbf{R}^n))} + |y_T|_{L^2_{\mathscr{F}_T}(\Omega;\mathbf{R}^n)}\Big).
\end{aligned} \tag{3.1}
$$

Proof. It is easy to show the uniqueness of the transposition solution from (3.1). Hence we prove only the existence and the estimate (3.1).

Given a triplet $(\bar{y}, \bar{Y}, \bar{K}) \in \mathcal{K}^D[T_1,T]$, $T_1 \in [0,T]$, by Theorem 3.1, we obtain a unique transposition solution $(y,Y,K) \in \mathcal{K}^D[T_1,T]$ of the following equation in $[T_1,T]$

$$
\begin{cases}
dy(t) = f(t,\bar{y}(t),\bar{Y}(t),\bar{K}(t))dt + Y(t)dW(t) \\
\qquad + \displaystyle\int_E K(t,z)\widetilde{N}(dt,dz) \\
y(T) = y_T.
\end{cases}
$$

This defines a map F from $\mathcal{K}^D[T_1,T]$ into itself by $F(\bar{y},\bar{Y},\bar{K}) = (y,Y,K)$. To complete the proof, it suffices to show that F is contractive. For this purpose, let $(\bar{y}_i,\bar{Y}_i,\bar{K}_i) \in \mathcal{K}^D[T_1,T]$ and $(y_i,Y_i,K_i) = F(\bar{y}_i,\bar{Y}_i,\bar{K}_i)$ for $i = 1,2$. Hence $\Delta y = y_1 - y_2$, $\Delta Y = Y_1 - Y_2$, $\Delta K = K_1 - K_2$ satisfies the following BSDE in $[T_1,T]$

$$
\begin{cases}
d\Delta y(t) = \Delta f(t)dt + \Delta Y(t)dW(t) + \displaystyle\int_E \Delta K(t,z)\widetilde{N}(dt,dz) \\
\Delta y(T) = 0,
\end{cases}
$$

where $\Delta f(t) = f(t,\bar{y}_1,\bar{Y}_1,\bar{K}_1) - f(t,\bar{y}_2,\bar{Y}_2,\bar{K}_2)$.

Set $\Delta\bar{y} = \bar{y}_1 - \bar{y}_2$, $\Delta\bar{Y} = \bar{Y}_1 - \bar{Y}_2$, $\Delta\bar{K} = \bar{K}_1 - \bar{K}_2$. Since

$$
\begin{aligned}
|\Delta f|^2_{L^2_{\mathbb{F}}(\Omega;L^1(T_1,T;\mathbf{R}^n))} &= \mathbb{E}\Big(\int_{T_1}^T |\Delta f|dt\Big)^2 \\
&\le 3\bigg(\mathbb{E}\Big(\int_{T_1}^T g(t)|\Delta\bar{y}(t)|dt\Big)^2 + \mathbb{E}\Big(\int_{T_1}^T h(t)|\Delta\bar{Y}(t)|dt\Big)^2 \\
&\quad + \mathbb{E}\Big(\int_{T_1}^T h(t)\sqrt{\int_E |\Delta\bar{K}(t,z)|^2\lambda(dz)}dt\Big)^2\bigg)
\end{aligned}
$$

$$\leq 3\left(\left(\int_{T_1}^{T} g(t)dt\right)^2 + \int_{T_1}^{T} h^2(t)dt\right)\left(\mathbb{E}\sup_{t\in[T_1,T]}|\Delta\bar{y}(t)|^2 + \mathbb{E}\int_{T_1}^{T}|\Delta\bar{Y}(t)|^2dt + \mathbb{E}\int_{T_1}^{T}\int_E|\Delta\bar{K}(t,z)|^2\lambda(dz)dt\right)$$
$$= 3\left(\left(\int_{T_1}^{T} g(t)dt\right)^2 + \int_{T_1}^{T} h^2(t)dt\right)|(\Delta\bar{y}, \Delta\bar{Y}, \Delta\bar{K})|^2_{\mathcal{K}^D[T_1,T]}$$
$$< \infty,$$

it follows that $\Delta f \in L^2_{\mathbb{F}}(\Omega; L^1(T_1, T; \mathbf{R}^n))$. By means of Theorem 3.1, we have the following estimate

$$|F(\Delta\bar{y}, \Delta\bar{Y}, \Delta\bar{K})|^2_{\mathcal{K}^D[T_1,T]} = |(\Delta y, \Delta Y, \Delta K)|^2_{\mathcal{K}^D[T_1,T]}$$
$$\leq C^2|\Delta f|^2_{L^2_{\mathbb{F}}(\Omega;L^1(T_1,T;\mathbf{R}^n))}$$
$$\leq 3C^2\left(\left(\int_{T_1}^{T} g(t)dt\right)^2 + \int_{T_1}^{T} h^2(t)dt\right)|(\Delta\bar{y}, \Delta\bar{Y}, \Delta\bar{K})|^2_{\mathcal{K}^D[T_1,T]}.$$

Choose T_1 such that $3C^2\left(\left(\int_{T_1}^T g(t)dt\right)^2+\int_{T_1}^T h^2(t)dt\right) < 1$, hence F is contractive. By the Banach fixed point theorem, F has a fixed point $(y, Y, K) \in \mathcal{K}^D[T_1, T]$ which is a transposition solution of the following equation in $[T_1, T]$

$$\begin{cases} dy(t) = f(t, y(t), Y(t), K(t,\cdot))dt + Y(t)dW(t) \\ \qquad\qquad + \int_E K(t,z)\widetilde{N}(dt, dz) \\ y(T) = y_T. \end{cases}$$

Applying Theorem 3.1, we find that

$$|(y, Y, K)|_{\mathcal{K}^D[T_1,T]}$$
$$\leq C\left(|f(\cdot, y(\cdot), Y(\cdot), K(\cdot))|_{L^2_{\mathbb{F}}(\Omega;L^1(T_1,T;\mathbf{R}^n))} + |y_T|_{L^2_{\mathscr{F}_T}(\Omega;\mathbf{R}^n)}\right)$$
$$\leq C\left(\sqrt{3\left(\left(\int_{T_1}^{T} g(t)dt\right)^2 + \int_{T_1}^{T} h^2(t)dt\right)}|(y, Y, K)|_{\mathcal{K}^D[T_1,T]} + |f(\cdot, 0, 0, 0)|_{L^2_{\mathbb{F}}(\Omega;L^1(T_1,T;\mathbf{R}^n))} + |y_T|_{L^2_{\mathscr{F}_T}(\Omega;\mathbf{R}^n)}\right). \tag{3.2}$$

Noting that $C\sqrt{3\left(\left(\int_{T_1}^T g(t)dt\right)^2 + \int_{T_1}^T h^2(t)dt\right)} < 1$, by (3.2), we have

$$|(y, Y, K)|_{\mathcal{K}^D[T_1,T]}$$
$$\leq C\left(|f(\cdot, 0, 0, 0)|_{L^2_{\mathbb{F}}(\Omega;L^1(T_1,T;\mathbf{R}^n))} + |y_T|_{L^2_{\mathscr{F}_T}(\Omega;\mathbf{R}^n)}\right). \tag{3.3}$$

Repeating the above argument by finite steps, we obtain a transposition solution of (1.1) in $[0, T]$. The desired estimate (3.1) follows from (3.3). □

Remark 3.1. If the filtration $\{\mathscr{F}_t; t \geq 0\} = \{\mathscr{F}_t^{W,N}; t \geq 0\}$, from [3], the BSDE (1.1) admits a unique strong solution (in the classical sense) which is also a transposition solution. From the uniqueness of the transposition solution, we see that the transposition solution coincides with the strong solution when the filtration is natural.

Remark 3.2. If $\xi \in L^2_{\mathscr{F}_T^{W,N}}(\Omega; \mathbf{R}^n)$ and f is a deterministic function, similar to the method in [6], the transposition solution $(y, Y, K) \in \mathcal{K}^D[0, T]$ is the strong solution of the BSDE (1.1).

We give an example indicating that the transposition solution is a weak version of the strong solution of BSDEs.

Example 3.1. Suppose that $W_1 = \{W_1(t), \mathscr{F}_t; t \geq 0\}$ and $W_2 = \{W_2(t), \mathscr{F}_t; t \geq 0\}$ are two 1-dimensional mutually independent Brownian motions on $(\Omega, \mathscr{F}, P)$ and the filtration $\{\mathscr{F}_t; t \geq 0\} = \{\mathscr{F}_t^{W_1,W_2}; t \geq 0\}$. Consider the following BSDE on $(\Omega, \mathscr{F}, \mathscr{F}_t, P; t \geq 0)$ in $[0, T]$

$$\begin{cases} dy(t) = Y_1(t)dW_1(t) \\ y(T) = W_2(T). \end{cases} \tag{3.4}$$

Also, we introduce a BSDE on $(\Omega, \mathscr{F}, \mathscr{F}_t, P; t \geq 0)$ in $[0, T]$ as follows

$$\begin{cases} dy(t) = Y_1(t)dW_1(t) + Y_2(t)dW_2(t) \\ y(T) = W_2(T). \end{cases} \tag{3.5}$$

It is easy to see that $(W_2, (0, 1))$ is the unique strong solution of (3.5), and therefore $(W_2, 0)$ is the transposition solution of (3.4), but (3.4) admits no strong solution.

4. Application: controllability in the transposition sense

In this section, we choose the intensity λ satisfying $\lambda(dz) = \gamma\delta_1(dz)$, where γ is a positive constant and $\delta_1(\cdot)$ is a Dirac measure. Then $N(t) = \int_0^t\int_E N(dt, dz)$ is a Poisson process with intensity γ. In this case, the Itô integral $\int_0^T\int_E K(t)d\widetilde{N}(dt, dz)$ degenerates to $\int_0^T K(t)d\widetilde{N}(t)$, for any $K(\cdot) \in L^2_{P,\mathbb{F}}(\Omega; L^2(0, T; \mathbf{R}^n))$. We consider the following stochastic linear control systems with jumps on the filtration $\{\mathscr{F}_t; t \geq 0\}$ in $[0, T]$:

$$dy(t) = (Fy(t) + G_1u(t) + G_2v(t))dt + H_1u(t)dW(t) + H_2v(t)d\widetilde{N}(t), \tag{4.1}$$

where $y \in \mathbf{R}^n$, $F \in \mathbf{R}^{n\times n}$, G_1, $H_1 \in \mathbf{R}^{n\times m}$, G_2, $H_2 \in \mathbf{R}^{n\times k}$; $u \in \mathcal{U}_0$, $v \in \mathcal{V}_0$ and for any $t \in [0, T]$, $\mathcal{U}_t$, $\mathcal{V}_t$ is defined by $\mathcal{U}_t = L^2_{\mathbb{F}}\left(\Omega; L^2(t, T; \mathbf{R}^m)\right)$, $\mathcal{V}_t = L^2_{P,\mathbb{F}}\left(\Omega; L^2(t, T; \mathbf{R}^k)\right)$.

In this part, first, we give a definition of the controllability for SLCSs with jumps on general filtrations and by the result in the last section, we establish the corresponding Kalman rank condition. Also, we compare our controllability to that in the classical sense.

Definition 4.1. System (4.1) is said to be terminal-controllable (in the sense of transposition), if for any $\xi \in L^2_{\mathscr{F}_T}(\Omega, \mathbf{R}^n)$, (4.1) with terminal condition $y(T) = \xi$ admits a transposition solution (y, u, v).

Definition 4.2. System (4.1) is said to be controllable (in the sense of transposition), if for any $y_0 \in \mathbf{R}^n$, $\xi \in L^2_{\mathscr{F}_T}(\Omega, \mathbf{R}^n)$, (4.1) with terminal condition $y(T) = \xi$ admits a transposition solution (y, u, v) satisfying $y(0) = y_0$.

Now we list the corresponding definition of controllability in the classical sense, so we can compare the differences of them later.

Definition 4.3. System (4.1) is said to be terminal-controllable in the classical sense, if for any $\xi \in L^2_{\mathscr{F}_T}(\Omega, \mathbf{R}^n)$, (4.1) with terminal condition $y(T) = \xi$ admits a strong solution (y, u, v).

Definition 4.4. System (4.1) is called controllable in the classical sense, if for any $y_0 \in \mathbf{R}^n$ and $\xi \in L^2_{\mathscr{F}_T}(\Omega, \mathbf{R}^n)$, (4.1) with terminal condition $y(T) = \xi$ admits a strong solution (y, u, v) satisfying $y(0) = y_0$.

Definition 4.5. We say that S_c^T is the controllable subspace in the classical sense if for any $y_0 \in \mathbf{R}^n$ and $\xi \in S_c^T$, the system (4.1) is controllable in the classical sense.

Theorem 4.1. *System (4.1) is terminal-controllable in the transposition sense if and only if Rank(H_1) = Rank$(H_2) = n$.*

Proof. We borrow some idea from [12]. First, we use the contradiction argument to prove the "only if" part. If Rank$(H_1) < n$, then there exists $b \in \mathbf{R}^n$, $|b| = 1$, such that for any $z \in \mathbf{R}^m$, $b^* H_1 z = 0$. Consider the following FSDE in $[t, T]$,

$$\begin{cases} dx(\tau) = b dW(\tau) \\ x(t) = 0. \end{cases}$$

Obviously, $x(\tau) = b(W(\tau) - W(t))$ is the strong solution of this equation. Set $\xi = bW(T)$. Since the SLCS (4.1) is terminal-controllable in the transposition sense, by the definition we obtain that

$$\begin{aligned} T - t &\leq \mathbb{E}\int_t^T |W(\tau) - W(t)||Fy(\tau) + G_1 u(\tau) + G_2 v(\tau)| d\tau \\ &\leq (T-t)/\sqrt{2}\Big(\mathbb{E}\int_t^T |Fy(\tau) + G_1 u(\tau) + G_2 v(\tau)|^2 d\tau\Big)^{1/2}. \end{aligned}$$

This contradicts the fact $(y, u, v) \in L^2_{\mathbb{F}}(\Omega; D([t, T]; \mathbf{R}^n)) \times \mathcal{U}_t \times \mathcal{V}_t$. Hence Rank$(H_1) = n$. Similarly, Rank$(H_2) = n$.

Next, we prove the "if" part. If Rank$(H_1) = n$, Rank$(H_2) = n$, then there exist two matrices $M_1 \in \mathbf{R}^{m\times m}$ and $M_2 \in \mathbf{R}^{k\times k}$, such that

$$H_1 M_1 = (I_n, 0), \qquad H_2 M_2 = (I_n, 0).$$

Set

$$M_1^{-1} u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}, \qquad M_2^{-1} v = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix},$$

$$G_1 M_1 = (B_1, \widetilde{B}_1), \qquad G_2 M_2 = (B_2, \widetilde{B}_2),$$

then the SLCS (4.1) is equivalent to

$$\begin{aligned} dy(t) = &(Fy(t) + B_1 u^1(t) + B_2 v^1(t) + \widetilde{B}_1 u^2(t) \\ &+ \widetilde{B}_2 v^2(t))dt + u^1(t) dW(t) + v^1(t) d\widetilde{N}(t). \end{aligned} \tag{4.2}$$

By Theorem 3.2, for any $t \in [0, T]$, $u^2 \in L^2_{\mathbb{F}}(\Omega; L^2(t, T; \mathbf{R}^{m-n}))$, $v^2 \in L^2_{\mathcal{P},\mathbb{F}}(\Omega; L^2(t, T; \mathbf{R}^{k-n}))$, $\xi \in L^2_{\mathcal{F}_t}(\Omega; \mathbf{R}^n)$, $(g, h, l, \eta) \in \mathcal{K}^1[t, T]$, there exists a $(y, u^1, v^1) \in \mathcal{K}^D[t, T]$ satisfying

$$\begin{aligned} &\mathbb{E}\langle x(T), \xi\rangle - \mathbb{E}\langle \eta, y(t)\rangle \\ &= \mathbb{E}\int_t^T \langle x(\tau), Fy(\tau) + B_1 u^1(\tau) + \widetilde{B}_1 u^2(\tau) \\ &\quad + B_2 v^1(\tau) + \widetilde{B}_2 v^2(\tau)\rangle d\tau + \mathbb{E}\int_t^T \langle g(\tau), y(\tau)\rangle d\tau \\ &\quad + \mathbb{E}\int_t^T \langle h(\tau), u^1(\tau)\rangle d\tau + \gamma\mathbb{E}\int_t^T \langle l(\tau), v^1(\tau)\rangle d\tau. \end{aligned}$$

Thus the SLCS (4.2) is terminal-controllable in the transposition sense. By the equivalence of (4.1) and (4.2), we get the terminal controllability (in the transposition sense) of (4.1). This completes the proof. □

Remark 4.1. By this theorem, if an SLCS is terminal-controllable in the transposition sense, then the dimension of the control is not less than that of the state, i.e. $k \geq n$ and $m \geq n$. This is different from the deterministic linear control system.

Noting the equivalence of (4.1) and (4.2), from now on, we consider only the SLCS (4.2).

Consider the following BSDE in $[0, T]$

$$\begin{cases} dy(t) = (Fy(t) + B_1 u^1(t) + B_2 v^1(t) + \widetilde{B}_1 u^2(t) \\ \qquad + \widetilde{B}_2 v^2(t))dt + u^1(t) dW(t) + v^1(t) d\widetilde{N}(t) \\ y(T) = 0. \end{cases} \tag{4.3}$$

By Theorem 3.2, for any $(u^2, v^2) \in \mathcal{H} := L^2_{\mathbb{F}}(\Omega, L^2([0, T]; \mathbf{R}^{m-n})) \times L^2_{\mathcal{P},\mathbb{F}}(\Omega, L^2([0, T]; \mathbf{R}^{k-n}))$, the BSDE (4.3) admits a unique transposition solution $\big(y^{(u^2,v^2)}, u^{1(u^2,v^2)}, v^{1(u^2,v^2)}\big) \in \mathcal{K}^D[0, T]$.

Theorem 4.2.

Span$\{y^{(u^2,v^2)}(0) : (u^2, v^2) \in \mathcal{H}\}$ = *Span*$\{F, B_1, B_2; \widetilde{B}_1, \widetilde{B}_2\}$,

where $\{F, B_1, B_2; \widetilde{B}_1, \widetilde{B}_2\}$ *is a matrix with infinite columns:*

$$\begin{aligned} &\{F, B_1, B_2; \widetilde{B}_1, \widetilde{B}_2\} \\ &= \Big[\widetilde{B}_1, \widetilde{B}_2, F\widetilde{B}_1, B_1\widetilde{B}_1, B_2\widetilde{B}_1, F\widetilde{B}_2, B_1\widetilde{B}_2, B_2\widetilde{B}_2, \\ &\quad F^2\widetilde{B}_1, FB_1\widetilde{B}_1, FB_2\widetilde{B}_1, B_1^2\widetilde{B}_1, B_1B_2\widetilde{B}_1, B_2^2\widetilde{B}_1, \\ &\quad F^2\widetilde{B}_2, FB_1\widetilde{B}_2, FB_2\widetilde{B}_2, B_1^2\widetilde{B}_2, B_1B_2\widetilde{B}_2, B_2^2\widetilde{B}_2, \ldots\Big]. \end{aligned}$$

Proof. We divide the proof into two steps.

Step 1. If there exists a $\beta \in \mathbf{R}^n$ such that

$$\beta y^{(u^2,v^2)}(0) = 0, \quad \forall (u^2, v^2) \in \mathcal{H},$$

consider the following FSDE in $[0, T]$,

$$\begin{cases} dx(t) = -F^T x(t) dt - B_1^T x(t) dW(t) \\ \qquad -1/\gamma B_2^T x(t) d\widetilde{N}(t) \\ x(0) = \beta. \end{cases} \tag{4.4}$$

By the definition of the transposition solution, it follows that, for any $(u^2, v^2) \in \mathcal{H}$,

$$\mathbb{E}\int_0^T \langle \widetilde{B}_1^T x(t), u^2(t)\rangle + \langle \widetilde{B}_2^T x(t), v^2(t)\rangle dt = 0.$$

Since $x(\cdot)$ is càdlàg, we deduce that

$$\widetilde{B}_1^T x(t) = \widetilde{B}_2^T x(t) = 0, \quad \forall t \in [0, T].$$

Using (4.4) again, we deduce that

$$\begin{aligned} 0 &= \widetilde{B}_1^T \beta - \widetilde{B}_1^T x(t) \\ &= \int_0^t \widetilde{B}_1^T F^T x(s) ds + \int_0^t \widetilde{B}_1^T B_1^T x(s) dW(s) \\ &\quad + 1/\gamma \int_0^t \widetilde{B}_1^T B_2^T x(s) d\widetilde{N}(s), \ \forall t \in [0, T]. \end{aligned}$$

Hence,

$$\widetilde{B}_1^T F^T x(t) = 0, \qquad \widetilde{B}_1^T B_1^T x(t) = 0,$$
$$\widetilde{B}_1^T B_2^T x(t) = 0, \quad \forall t \in [0, T].$$

Similarly,

$$\widetilde{B}_2^T F^T x(t) = 0, \qquad \widetilde{B}_2^T B_1^T x(t) = 0,$$
$$\widetilde{B}_2^T B_2^T x(t) = 0, \quad \forall t \in [0, T].$$

Repeating the above process, we conclude that

$$\beta \perp \text{Span}\{F, B_1, B_2; \widetilde{B}_1, \widetilde{B}_2\}.$$

Step 2. Conversely, if $\beta \perp \text{Span}\{F, B_1, B_2; \widetilde{B}_1, \widetilde{B}_2\}$, for any $t \in [0, T]$, set

$$\begin{aligned} x^0(t) &= \beta, \\ x^{k+1}(t) &= -\int_0^t F^T x^k(t) dt - \int_0^t B_1^T x^k(t) dW(t) \\ &\quad -1/\gamma \int_0^t B_2^T x^k(t) d\widetilde{N}(t), \ k \geq 0. \end{aligned}$$

By the proof of existence of the strong solution for a FSDE as in [14], we know that

$$\lim_{k\to\infty} x^k = x \quad \text{in } L^2_{\mathbb{F}}(\Omega; L^2(0, T; \mathbf{R}^n))$$

and x satisfies (4.4). Besides this, by

$$\widetilde{B}_1^T x^k(t) = \widetilde{B}_2^T x^k(t) = 0, \quad \forall k \geq 0, \forall t \in [0,T],$$

we obtain that

$$\widetilde{B}_1^T x(t) = \widetilde{B}_2^T x(t) = 0, \quad \forall t \in [0,T]. \tag{4.5}$$

Choosing the FSDE (4.4), by the definition of the transposition solution, we get

$$\begin{aligned}&\langle x(0), y^{u^2,v^2}(0)\rangle \\ &\quad = -\mathbb{E}\int_0^T \langle x(t), \widetilde{B}_1 u^2(t) + \widetilde{B}_2 v^2(t)\rangle dt.\end{aligned} \tag{4.6}$$

By (4.5) and (4.6), we can get

$$\beta \perp \{y^{(u^2,v^2)}(0) : (u^2,v^2) \in \mathcal{H}\}.$$

This completes the proof. $\square$

The following theorem provides an algebra criterion for the controllability of (4.2) in the transposition sense.

Theorem 4.3. *The SLCS (4.2) is controllable in the transposition sense if and only if*

$$Rank\{F, B_1, B_2; \widetilde{B}_1, \widetilde{B}_2\} = n. \tag{4.7}$$

Proof. The controllability of the SLCS (4.2) in the transposition sense is equivalent to that for any $\xi \in L^2_{\mathcal{F}_T}(\Omega, \mathbf{R}^n)$ and $x \in \mathbf{R}^n$, there exists $\left(u^2, v^2\right) \in \mathcal{H}$, such that the following BSDE in $[0,T]$

$$\begin{cases} dy(t) = (Fy(t) + B_1 u^1(t) + B_2 v^1(t) + \widetilde{B}_1 u^2(t) \\ \qquad + \widetilde{B}_2 v^2(t))dt + u^1(t)dW(t) + v^1(t)d\widetilde{N}(t) \\ y(T) = \xi, \end{cases} \tag{4.8}$$

is solvable in the transposition sense and $y^{u^2,v^2}(0) = x$.

By Theorem 3.2, the following BSDE is uniquely solvable in the transposition sense in $[0,T]$

$$\begin{cases} dy_1(t) = (Fy_1(t) + B_1 u^1(t) + B_2 v^1(t))dt \\ \qquad + u^1(t)dW(t) + v^1(t)d\widetilde{N}(t) \\ y_1(T) = \xi. \end{cases}$$

From this, the solvability of (4.8) is equivalent to the following BSDE

$$\begin{cases} dy_2(t) = (Fy_2(t) + B_1 u^1(t) + B_2 v^1(t) + \widetilde{B}_1 u^2(t) \\ \qquad + \widetilde{B}_2 v^2(t))dt + u^1(t)dW(t) + v^1(t)d\widetilde{N}(t) \\ y_2(T) = 0 \end{cases} \tag{4.9}$$

which is solvable in the transposition sense and $y_2^{(u^2,v^2)}(0) = x - y_1^{(u^2,v^2)}(0)$.

From the arbitrariness of ξ and x, by Theorem 4.2, the solvability of (4.9) in the transposition sense is equivalent to $\{y^{(u^2,v^2)}(0) : (u^2,v^2) \in \mathcal{H}\} = \mathbf{R}^n$, which is equivalent to Rank $\{F, B_1, B_2; \widetilde{B}_1, \widetilde{B}_2\} = n$. $\square$

Remark 4.2. If the SLCS (4.2) is controllable in the transposition sense, then by Remark 3.2,

$$S_c^T = L^2_{\mathcal{F}_T^{W,N}}.$$

Remark 4.3. When the filtration $\{\mathcal{F}_t; t \geq 0\}$ is the natural filtration $\{\mathcal{F}_t^{W,N}; t \geq 0\}$, by Remark 3.1, the controllability of (4.2) in the transposition sense is equivalent to its controllability in the classical sense. Hence (4.7) is also a sufficient and necessary condition for the controllability of (4.2) in the classical sense.

Remark 4.4. For the following special case of (4.1)

$$dy(t) = (Fy(t) + G_1 u(t))dt + H_1 u(t)dW(t) \tag{4.10}$$

driven only by Brownian motion, by a similar method, the terminal controllability of (4.10) in the transposition case is equivalent to $\mathrm{Rank}(H_1) = n$. In this case (4.10) can be reduced to

$$dy(t) = (Fy(t) + B_1 u^1(t) + \widetilde{B}_1 u^2(t))dt + u^1(t)dW(t) \tag{4.11}$$

and on the filtration $\{\mathcal{F}_t; t \geq 0\}$, (4.11) is controllable in the transposition sense if and only if

$$\mathrm{Rank}\{F, B_1; \widetilde{B}_1\} = n. \tag{4.12}$$

If the filtration $\{\mathcal{F}_t; t \geq 0\}$ is the natural filtration $\{\mathcal{F}_t^W; t \geq 0\}$, by Remark 4.3, the sufficient and necessary condition for the controllability of (4.11) in the classical sense is (4.12), which coincides with Theorem 3.2 in [12].

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