CHIN. PHYS. LETT. Vol. 26, No. 10 (2009) 100501

# Chaos in the Fractional Order Generalized Lorenz Canonical Form *

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(Received 6 May 2009)

*A fractional-order generalized Lorenz system is constructed and numerically investigated. Chaotic behavior existing in the fractional-order generalized Lorenz system is found. The numerical simulations and interesting figures are performed.*

*PACS: 05.45.−a, 05.45.Xt*

The theory of derivatives of the fractional order, i.e., non-integer order, goes back to Leibniz's note in his list to L'Hopital, dated 30 September 1695, in which the meaning of derivative of the order one half was discussed.[1] For three centuries the theory of fractional derivatives has been developed mainly as a pure theoretical field of mathematics useful only for mathematicians. Nearly 30 years ago, the paradigm began to shift from pure mathematical formulations to applications in various fields. During the last decade, fractional calculus has been applied to almost every field of science, engineering, and mathematics. Some of the areas where fractional calculus has made a profound impact include viscoelasticity and rheology, electrical engineering, electrochemistry, biology, biophysics and bioengineering, signal and image processing, mechanics, mechatronics, physics, and control theory.[2−5] It is necessary to point that many physicists and mathematicians focus on studying the chaotic dynamics of fractional nonlinear systems, naturally, based on important and famous chaotic fractional systems, such as the Lorenz system, the corresponding chaotic fractional systems are constructed and investigated.[6−11]

Recently, Celikovsky and Chen[12] presented a new generalized Lorenz canonical form (GLCF), such a canonical representation has only one parameter, satisfying $\tau > -1$ for the generalized Lorenz system (GLS) while for $\tau \le -1$ it is either equivalent to the so-called hyperbolic-type GLS[13] or to the Shimizu-Morioka model.[14,15] Notably, the Lorenz system[16−18] satisfy $\tau > 0$; the Lü system,[19,20] $\tau = 0$; the Chen system,[21] $-1 < \tau < 0$.

In this Letter, the fractional-order GLCF are presented. The fractional-order GLCF includes: the fractional-order classic Lorenz system;[8] the fractional-order Lü system;[9] the fractional-order Chen system;[10] the fractional-order Shimizu-Morioka system or the fractional-order Liu-Liu-Liu-Liu system;[11] the fractional-order hyperbolic-type GLS. The fractional-order GLCF are discredited successfully and numerically investigated. The results obtained show that the chaotic dynamics in the fractional-order GLCF and the chaos corresponding to the above- mentioned fractional-order systems[8−11] exist by choosing parameters. It is very interesting that chaos exists in the fractional-order GLCF with order lower than 3. The numerical simulations and interesting figures are performed.

There are several definitions of fractional derivatives. Two commonly used definitions for the general fractional differintegral are the Grunwald definition and the Riemann-Liouville definition, and the best known is the Rieman-Liouville definition. However, in this letter, we would prefer Caputo derivative to the Riemann-Liouville one since the former is more popular in real applications. In real applications, the Caputo derivative is more popular since the unhomogenous initial conditions are permitted if such conditions are necessary. Furthermore, these initial values are prone to measure since they all have idiographic meanings.[3] The Caputo derivative definition is given by

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha - n)} \int_a^t \frac{f^n(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \\ (m-1 < \alpha \le m). $$

The Adams-Bashforth-Moulton predictor-corrector scheme is used for numerical solutions of the fractional derivatives. The details regarding the algorithms of the scheme are available in Ref. [4].

The fractional order GLCF and its discrete form are presented as follows. As is known, the GLCF is described by

$$ \dot{X} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} X + cX \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & \tau & 0 \end{pmatrix} X, \\ \lambda_1 > 0, \quad \lambda_{2,3} < 0, \tag{1} $$

*Supported by the National Natural Science Foundation of China under Grant Nos 10735030 and 90718041, Shanghai Leading Academic Discipline Project under Grant No B412, Program for Changjiang Scholars and Innovative Research Team in University (IRT0734) and K. C. Wong Magna Fund in Ningbo University.

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where $X = [x, y, z]^T$, $c = [1, -1, 0]$, with parameter $\tau \in R$.

That is,

$$\begin{aligned}\dot{x} &= \lambda_1 x - z(x-y),\\ \dot{y} &= \lambda_2 y - z(x-y),\\ \dot{z} &= \lambda_3 z + (x+\tau y)(x-y).\end{aligned} \tag{2}$$

Now, we construct the fractional order GLCF, which is described by

$$\begin{aligned}\frac{d^\alpha x}{dt^\alpha} &= \lambda_1 x - z(x-y),\\ \frac{d^\alpha y}{dt^\alpha} &= \lambda_2 y - z(x-y),\\ \frac{d^\alpha z}{dt^\alpha} &= \lambda_3 z + (x+\tau y)(x-y),\end{aligned} \tag{3}$$

where $a$ is the fractional order, $0 < \alpha \le 1$.

According to the predictor-corrector scheme presented in Ref. [4], the following differential equation

$${}_a^C D_t^\alpha f(t) = f(t, y(t)), \quad 0 \le t \le T,$$

$$y^k(0) = y_0^k,\ k = 1, 2, \ldots, m-1,\ (m-1 < \alpha \le m),$$

can be discretized as follows:

$$h = \frac{T}{N},\ t_k = kh,\ k = 0, 1, \ldots, N,\ m = [\alpha],$$

$$\begin{aligned}y_h(t_{n+1}) = &\sum_{k=0}^{m-1} \frac{{t_{n+1}}^k {y_0}^k}{k!} + \frac{h^\alpha f(t_{n+1}, p_{n+1})}{\Gamma(\alpha+2)}\\ &+ \frac{h^\alpha (n^{\alpha+1} - (n-\alpha) j^\alpha) f(0, y_0)}{\Gamma(\alpha+2)},\\ &+ \frac{h^\alpha \sum_{k=1}^{n} a_{n+1-k} f(t_k, y_k)}{\Gamma(\alpha+2)},\end{aligned}$$

where

$$p_{n+1} = \sum_{k=0}^{m-1} \frac{{t_{n+1}}^k {y_0}^k}{k!} + \frac{h^\alpha \sum_{k=0}^{n} b_{n-k+1} f(t_k, y_k)}{\Gamma(\alpha+1)},$$

$$a_k = (k+1)^{\alpha+1} - 2\, k^{\alpha+1} + (k-1)^{\alpha+1},\ b_k = k^\alpha - (k-1)^\alpha.$$

The error estimate is $e = \max|x(t_j) - x_h(t_j)| = O(h^\rho)$, $(j = 0, 1, \ldots, N)$, where $\rho = \min(2, 1+\alpha)$.

Applying the above scheme, Eq. (3) can be discretized as follows:

$$\begin{aligned}x_{n+1} = &\sum_{k=0}^{m-1} \frac{{t_{n+1}}^k x_k}{k!}\\ &+ \frac{h^\alpha f_1(t_{n+1}, p_{1,n+1}, p_{2,n+1}, p_{3,n+1})}{\Gamma(\alpha+2)}\\ &+ \frac{h^\alpha \left[(n^{\alpha+1} - (n-\alpha) j^\alpha) f_1(0, x_0, y_0, z_0)\right]}{\Gamma(\alpha+2)}\\ &+ \frac{h^\alpha \left[\sum_{k=1}^{j-1} a_{n+1-k} f_1(t_k, x_k, y_k, z_k)\right]}{\Gamma(\alpha+2)},\end{aligned}$$

$$\begin{aligned}y_{n+1} = &\sum_{k=0}^{m-1} \frac{{t_{n+1}}^k x_k}{k!}\\ &+ \frac{h^\alpha f_2(t_{n+1}, p_{1,n+1}, p_{2,n+1}, p_{3,n+1})}{\Gamma(\alpha+2)}\\ &+ \frac{h^\alpha \left[(n^{\alpha+1} - (n-\alpha) j^\alpha) f_2(0, x_0, y_0, z_0)\right]}{\Gamma(\alpha+2)}\\ &+ \frac{h^\alpha \left[\sum_{k=1}^{j-1} a_{n+1-k} f_2(t_k, x_k, y_k, z_k)\right]}{\Gamma(\alpha+2)},\end{aligned}$$

$$\begin{aligned}z_{n+1} = &\sum_{k=0}^{m-1} \frac{{t_{n+1}}^k x_k}{k!}\\ &+ \frac{h^\alpha f_3(t_{n+1}, p_{1,n+1}, p_{2,n+1}, p_{3,n+1})}{\Gamma(\alpha+2)}\\ &+ \frac{h^\alpha \left[(n^{\alpha+1} - (n-\alpha) j^\alpha) f_3(0, x_0, y_0, z_0)\right]}{\Gamma(\alpha+2)}\\ &+ \frac{h^\alpha \left[\sum_{k=1}^{j-1} a_{n+1-k} f_3(t_k, x_k, y_k, z_k)\right]}{\Gamma(\alpha+2)},\end{aligned} \tag{4}$$

where

$$\begin{aligned}f_1(t, x, y, z) &= \lambda_1 x - z(x-y),\\ f_2(t, x, y, z) &= \lambda_2 y - z(x-y),\\ f_3(t, x, y, z) &= \lambda_3 z + (x+\tau y)(x-y),\end{aligned}$$

$$p_{i,n+1} = \sum_{k=0}^{m-1} \frac{{t_{n+1}}^k {y_0}^k}{k!} + \frac{h^\alpha \sum_{k=0}^{n} b_{n-k+1} f_i(t_k, y_k)}{\Gamma(\alpha+1)}.$$

Applying the derived discrete scheme and with the help of Maple, we find that chaos does exist in the fractional-order GLCF. For convenience, the parameters are chosen to be $\lambda_1 = -3$, $\lambda_2 = -5$, $\lambda_3 = -1$, with an initial state $x_0 = 0$, $y_0 = 2$, $z_0 = 2$. In order to obtain different fraction-order chaos systems, we only need to choose different values of $\tau$.

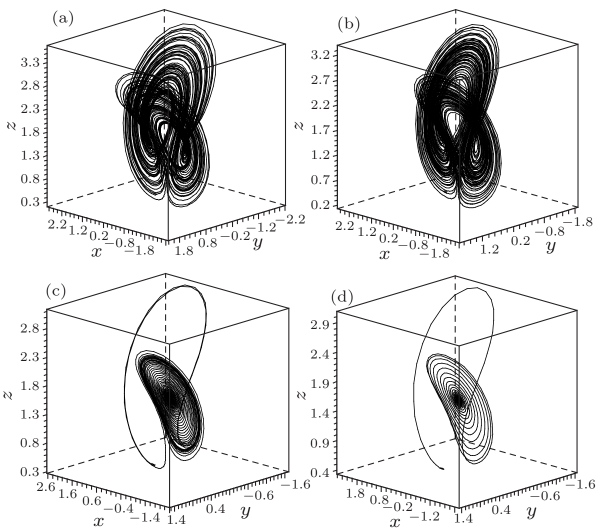


**Fig. 1.** The fractional-order classic Lorenz system under conditions $\alpha = 1$, $\alpha = 0.95$, $\alpha = 0.92$, $\alpha = 0.91$.

*Case* 1. The fractional-order classic Lorenz system. When $\tau > 0$, the equivalent systems of the GLCF is the classic Lorenz system. Set $\tau = 0.2$ in Eq. (3), we obtain the fractional-order classic Lorenz system. We find that the chaos does exist when $\alpha \in (0.92, 1]$. For example, when $\alpha = 1$, 0.95, 0.92 and 0.91, we find that chaos exists in the fractional order system and the phase portraits are shown in Fig. 1.

*Case* 2. The fractional-order Lü system. When $\tau = 0$, the equivalent systems of the GLCF is the Lü system. Set $\tau = 0$ in Eq. (3), we obtain the fractional-order Lü system. We find that the chaos does exist when $\alpha \in (0.90, 1]$. For example, when $\alpha = 1$, 0.93, 0.90 and 0.89, we find that chaos exists in the fractional order system and the phase portraits are shown in Fig. 2.

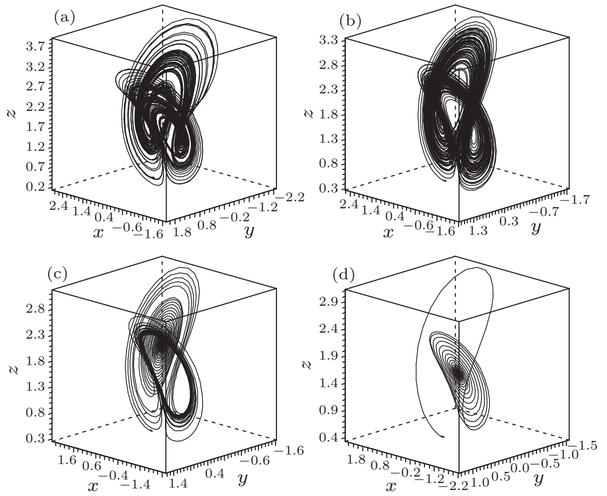


**Fig. 2.** The fractional-order Lü system with conditions: $\alpha = 1$, $\alpha = 0.93$, $\alpha = 0.90$, $\alpha = 0.89$.

*Case* 3. The fractional-order Chen system. When $-1 < \tau < 0$, the equivalent systems of the GLCF is the Chen system. Set $\tau = -0.6$ in Eq. (3), we obtain the fractional- order Chen system. We find that the chaos does exist when $\alpha \in (0.86, 1]$. For example, when $\alpha = 1$, 0.87, 0.86 and 0.85, we find that chaos exists in the fractional order system and the phase portraits are shown in Fig. 3.

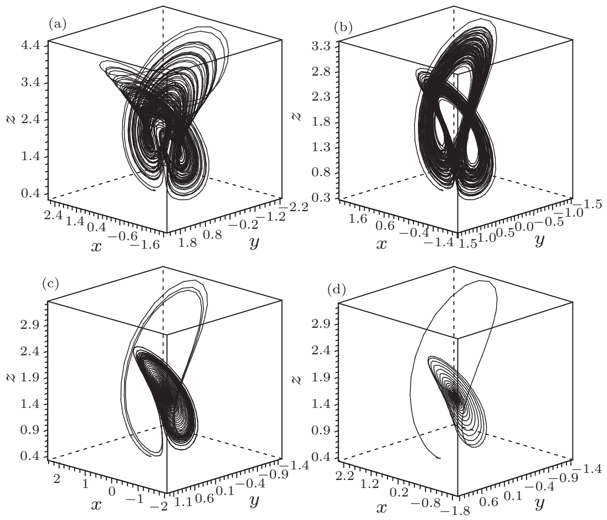


**Fig. 3.** The fractional-order Chen system with conditions: $\alpha = 1$, $\alpha = 0.87$, $\alpha = 0.86$, $\alpha = 0.85$.

*Case* 4. The fractional-order Shimizu-Morioka system or the fractional-order Liu-Liu-Liu-Liu system. When $\tau = -1$, the equivalent systems of the GLCF was the Shimizu-Morioka model or the Liu-Liu-Liu-Liu mode. Set $\tau = -1$ in Eq. (3), we obtain the fractional-order Shimizu-Morioka system or the fractional-order Liu-Liu-Liu-Liu system. We find that the chaos does exist when $\alpha \in (0.85, 1]$. For example, when $\alpha = 1$, 0.85 and 0.84, we find that chaos exists in the fractional order system and the phase portraits are shown in Fig. 4.

*Case* 5. The fractional-order hyperbolic-type generalized Lorenz system. When $\tau < -1$, the equivalent systems of the GLCF is the hyperbolic-type generalized Lorenz systems. Set $\tau = -2$ in Eq. (3), we obtain the fractional-order hyperbolic-type generalized Lorenz systems. We find that the chaos does exist when $\alpha \in (0.82, 1]$. For example, when $\alpha = 1$, 0.82 and 0.81, we find that chaos exists in the fractional order system and the phase portraits are shown in Fig. 5.

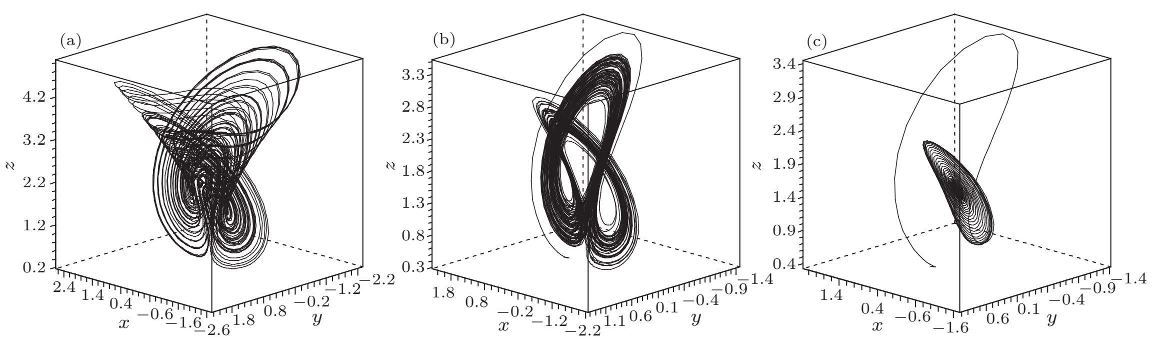


**Fig. 4.** The fractional-order Shimizu-Morioka system or the fractional-order Liu-Liu-Liu-Liu system under conditions $\alpha = 1$, $\alpha = 0.85$, $\alpha = 0.84$.

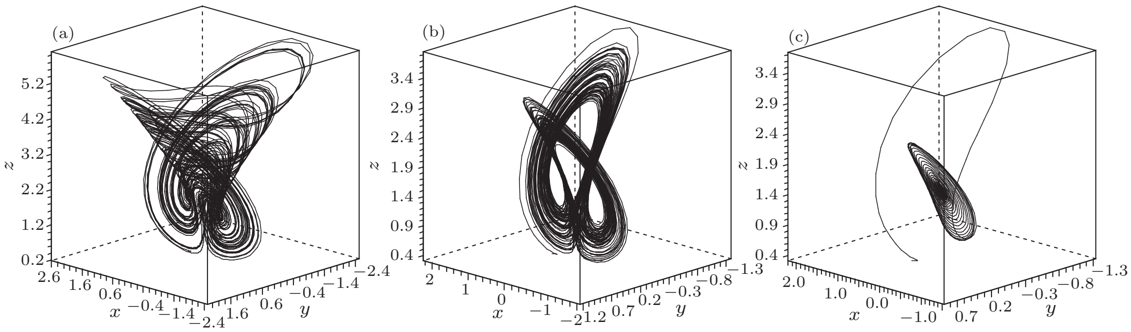

**Fig. 5.** The fractional-order hyperbolic-type generalized Lorenz system under conditions $\alpha = 1$, $\alpha = 0.82$, $\alpha = 0.81$.

In summary, a fractional order GLCF has been presented and studied. We find that chaos exists in the fractional order unified system with an order lower than 3. The numerical simulations and interesting figures are performed. Because synchronization of chaotic fractional differential systems has potential applications in secure communication and control processing, this studying field starts to attract increasing attention. In Ref. [22], we investigate the generalized Q-S synchronization[23] between the GLCF and the Rössler system, the more general controller is obtained. By choosing different parameters in the generalized controller obtained here, without much extra effort, we can obtain the controller of synchronization of the corresponding system respectively. Similarly, we will study the analysis property and the synchronization of the fractional order GLCF.

We would like to thank the two anonymous referees for their valuable suggestions.

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