Interference channel capacity region for randomized fixed-composition codes

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Abstract:
The random coding error exponents are studied [5], [6] for the finite alphabet interference channel (IFC) with two transmitter receiver pairs. The code words are uniform on a fixed-composition set and the decoding is optimum, as opposed to decoding based on interference cancellation, and decoding that considers the interference as additional noises. In this paper we further study the error exponents of randomized fixed-composition coding, some simple lower bounds are derived for universal decoding rules. Furthermore, we give a complete characterization of the capacity region of this coding scheme that is first proposed in [5] and [6]. It is shown that even with a sophisticated time-sharing scheme among randomized fixed-composition codes, the capacity region of the randomized fixed-composition coding is not bigger than the known Han-Kobayashi capacity region first appeared in [12]. This suggests that the average behavior of random codes are not sufficient to get new capacity regions.

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Abstract

The randomized fixed-composition with optimal decoding error exponents are studied [7], [8] for the finite alphabet interference channel (IFC) with two transmitter-receiver pairs. In this paper we investigate the capacity region of the randomized fixed-composition coding scheme. A complete characterization of the capacity region of the said coding scheme is given. The inner bound is derived by showing the existence of a positive error exponent within the capacity region. A simple universal decoding rule is given. The tight outer bound is derived by extending a technique first developed in [6] for single input output channels to interference channels. It is shown that even with a sophisticated time-sharing scheme among randomized fixed-composition codes, the capacity region of the randomized fixed-composition coding is not bigger than the known Han-Kobayashi [15] capacity region. This suggests that the average behavior of random codes are not sufficient to get new capacity regions.

I. Introduction

In [15], the capacity region of interference channel is studied for both discrete and Gaussian cases. In this paper we study the discrete interference channels $W_{Z|X,Y}$ and $\tilde{W}_{\tilde{Z}|X,Y}$ with two pairs of encoders and decoders as shown in Figure 1. The two channel inputs are $x^n \in \mathcal{X}^n$ and $y^n \in \mathcal{Y}^n$, outputs are $z^n \in \mathcal{Z}^n$ and $\tilde{z}^n \in \tilde{\mathcal{Z}}^n$ respectively, where $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ are finite sets. We study the basic interference channel where each encoder only has a private message to the correspondent decoder.

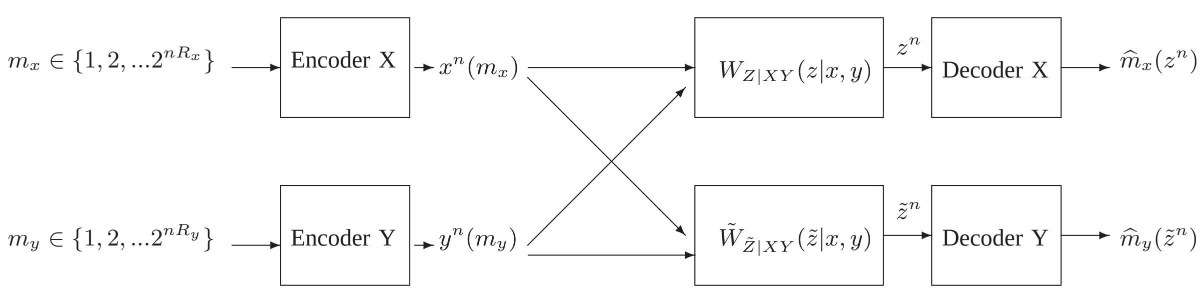

Fig. 1. A discrete memoryless interference channel of two users

Some recent progress on the capacity region for Gaussian interference channels is reported in [9], however, the capacity regions for general interference channels are unknown. We focus our investigation on the capacity region for a specific coding scheme: randomized fixed-composition codes while the error probability is defined as the average error over all code book with a certain composition (type).

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Fixed-composition coding is a useful coding scheme in the investigation of both upper [10] and lower bounds of channel coding error exponents [4] for point to point channel and [14], [13] for multiple access (MAC) channels. Recently in [7] and [8], randomized fixed-composition codes are used to derive a lower bound on the error exponent for discrete interference channels. A lower bound on the maximum-likelihood decoding error exponent is derived, this is a new attempt in investigating the error exponents for interference channels. The unanswered question is the capacity region of such coding schemes.

In this paper, we give a complete characterization of the interference channel capacity region for randomized fixed-composition codes. To prove the achievability of the capacity region, we prove the positivity everywhere in the capacity region of a universal decoding error exponent. This error exponent is derived by the method of types [3], in particular the universal decoding scheme used for multiple-access channels [14]. A better error exponent can be achieved by using the more complicated universal decoding rules developed in [13]. But since they both have the same achievable capacity region, we use the simpler scheme in [14]. To prove the the converse, that the achievable region matches the outer bound, we extend the technique in [6] for point to point channels to interference channels by using the known capacity region results for multiple-access channels. The result reveals the intimate relations between interference channels and multiple-access channels. With the capacity region for fixed-composition code established, it is evident that this capacity region is a subset of the Han-Kobayashi region [15].

The technical proof of this paper is focused on the average behavior of fixed-composition code books. However this fundamental setup can be generalized in the following three directions.

- It is obvious that there exists a code book that its decoding error is no bigger than the average decoding error over all code books. Hence the achievability results in this paper guarantees the existence of a of deterministic coding scheme with at least the same error exponents and capacity region. More discussions are in Section II-E.
- The focus of this paper is on the fixed-composition codes with a composition P, where P is a distribution on the input alphabet. This code book generation is different from the non-fixed-composition random coding [12] according to distribution P. It is well known in the literature that the fixed-composition code gives better error exponent result in low rate regime for point to point channels [4] and multiple-access channels [14], [13]. It is the same case for interference channels and hence the capacity region result in this paper applies to the non-fixed-composition random codes.
- Time-sharing is a key element in achieving capacity regions for multi-terminal channels [2]. For instance, for multiple-access channels, simple time-sharing among operational rate pairs gives the entire capacity region. We show that the our fixed composition codes can be used to build a time-sharing capacity region for interference channel. More interestingly, we show that the simple time-sharing technique that gives the entire capacity region for multiple-access channels is not enough to get the largest capacity region, a more sophisticated time-sharing scheme is needed. Detailed discussions are in Section IV.

The outline of the paper is as follows. In Section II we first formally define randomized fixed-composition codes and its capacity region and then in Section II-C we present the main result of this paper: the interference channel capacity region for randomized fixed-composition code in Theorem 1. The proof is later shown in Section III with more details in the appendix. Finally in Section IV, we argue that due to the non-convexity of the randomized fixed-composition coding, a more sophisticated time-sharing scheme is needed. This shows the necessity of studying the geometry of the code-books for interference channels.

II. RANDOMIZED FIXED-COMPOSITION CODE AND ITS CAPACITY REGION

We first review the definition of randomized fixed-composition code that is studied intensively in previous works. Then the definition of the interference channel capacity region for such codes is introduced. Then we give the main result of this paper: the complete characterization of the capacity region for randomized fixed-composition codes.

A. Randomized fixed-composition codes

A randomized fixed-composition code is a uniform distribution on the code books in which every codeword is from the type set with the fixed composition (type).

First we introduce the notion of type set [2]. A type set $\mathcal{T}^n(P)$ is a set of all the strings $x^n \in \mathcal{X}^n$ with the same type P where P is a probability distribution [2]. A sequence of type sets $\mathcal{T}^n \subseteq \mathcal{X}^n$ has composition P_X if the types of $\mathcal{T}^n$ converges to P_X, i.e. $\lim_{n\to\infty} \frac{N(a|\mathcal{T}^n)}{n} = P_X(a)$ for all $a \in \mathcal{X}$ that $P_X(a) > 0$ and $N(a|\mathcal{T}^n) = 0$ for all $a \in \mathcal{X}$ that $P_X(a) = 0$, where $N(a|\mathcal{T}^n)$ is the number of occurrence of a in type $\mathcal{T}^n$. We ignore the nuisance of the integer effect and assume that $nP_X(a)$ is an integer for all $a \in \mathcal{X}$ and nR_x and nR_y are also integers. This is indeed a reasonable assumption since we study long block length n and all the information theoretic quantities studied in this paper are continuous on the code compositions and rates. We simply denote by $\mathcal{T}^n(P_X)$ the length-n type set which has "asymptotic" type P_X, later in the appendix we abuse the notations by simply writing $x^n \in P_X$ instead of $x^n \in \mathcal{T}^n(P_X)$. Obviously, there are $|\mathcal{T}^n(P_X)|^{2^{nR_x}}$ many code books with fixed-composition P_X and rate R_x

In this paper, we study the randomized fixed-composition codes, where each code book with all codewords from the fixed composition being chosen with the same probability. Equivalently, over all these code books, a code word for message i is uniformly i.i.d distributed on the type set $\mathcal{T}^n(P_X)$. A formal definition is as follows.

Definition 1: Randomized fixed-composition codes: for a probability distribution P_X on $\mathcal{X}$, a rate R_x randomized fixed-composition-P_X encoder picks a code book with the following probability, for any fixed-composition-P_X code book $\theta^n = (\theta^n(1), \theta^n(2), ..., \theta(2^{nR_x}))$, where $\theta^n(i) \in \mathcal{T}^n(P_X)$, $i = 1, 2, ..., 2^{nR_x}$, and $\theta^n(i)$ and $\theta^n(j)$ may not be different for $i \neq j$, the code book θ_n is chosen, i.e. $x^n(i) = \theta^n(i), \quad i = 1, 2, ..., 2^{nR_x}$, with probability

$$\left(\frac{1}{|\mathcal{T}^n(P_X)|}\right)^{2^{nR_x}}$$

In other words, the choice of the code book is a random variable c_X uniformly distributed on the index set of all the possible code books with fixed-composition P_X: $\{1, 2, 3, ..., |\mathcal{T}^n(P_X)|^{2^{nR_x}}\}$, while c_X is shared between the encoder X and the decoders X and Y.

The key property of the randomized fixed-composition code is that for any message subset $\{i_1, i_2, ...i_l\} \subseteq \{1, 2, ..., 2^{nR_x}\}$, the code words for these messages are identical independently distributed on the type set of $\mathcal{T}^n(P_X)$.

For randomized fixed-composition codes, the average error probability $P^n_{e(x)}(R_x, R_y, P_X, P_Y)$ for X is the expectation of decoding error over all message, code books and channel behaviors.

$$\begin{aligned} P^n_{e(x)}(R_x, R_y, P_X, P_Y) \quad &= \left(\frac{1}{|\mathcal{T}^n(P_X)|}\right)^{2^{nR_x}} \left(\frac{1}{|\mathcal{T}^n(P_Y)|}\right)^{2^{nR_y}} \\ \sum_{c_X}\sum_{c_Y}\frac{1}{2^{nR_x}} \quad &\sum_{m_x}\frac{1}{2^{nR_y}}\sum_{m_y}\sum_{z^n} W_{Z|XY}(z^n|x^n(m_x), y^n(m_y))1(\widehat{m}_x(z^n) \neq m_x) \end{aligned} \tag{1}$$

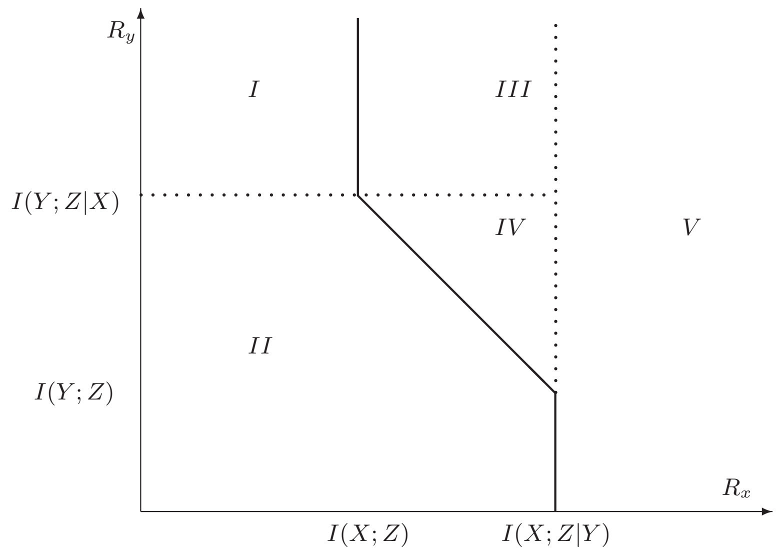

Fig. 2. Randomized fixed-composition capacity region $\mathcal{R}_x(P_X, P_Y)$ for X, the achievable region is the union of Region I and II.

where $x^n(m_x)$ is the code word of message m_x in code book c_X, similarly for $y^n(m_y)$, $\widehat{m}_x(z^n)$ is the decision made by the decoder knowing the code books c_X and c_Y.

B. Randomized fixed-composition coding capacity for interference channels

Given the definitions of randomized fixed-composition coding and the average error probability in (1) for such codes, we can formally define the capacity region for such codes.

Definition 2: Capacity region for randomized fixed-composition codes: for a fixed-composition P_X and P_Y, a rate pair (R_x, R_y) is said to be achievable for X, if for all $\delta > 0$, there exists $N_\delta < \infty$, s.t. for all $n > N_\delta$,

$$P^n_{e(x)}(R_x, R_y, P_X, P_Y) < \delta \tag{2}$$

We denote by $\mathcal{R}_x(P_X, P_Y)$ the closure of the union of the all achievable rate pairs. Similarly we denote by $\mathcal{R}_y(P_X, P_Y)$ the achievable region for Y, and $\mathcal{R}_{xy}(P_X, P_Y)$ for (X, Y) where both decoding errors are small. Obviously

$$\mathcal{R}_{xy}(P_X, P_Y) = \mathcal{R}_x(P_X, P_Y) \bigcap \mathcal{R}_y(P_X, P_Y). \tag{3}$$

We only need to focus our investigation on $\mathcal{R}_x(P_X, P_Y)$, then by the obvious symmetry, both $\mathcal{R}_y(P_X, P_Y)$ and $\mathcal{R}_{xy}(P_X, P_Y)$ follow.

C. Capacity region of the fixed-composition code, $\mathcal{R}_x(P_X, P_Y)$, for X

The main result of this paper is the complete characterization of the randomized fixed-composition capacity region $\mathcal{R}_x(P_X, P_Y)$ for X, as illustrated in (3), by symmetry, $\mathcal{R}_{xy}(P_X, P_Y)$ follows.

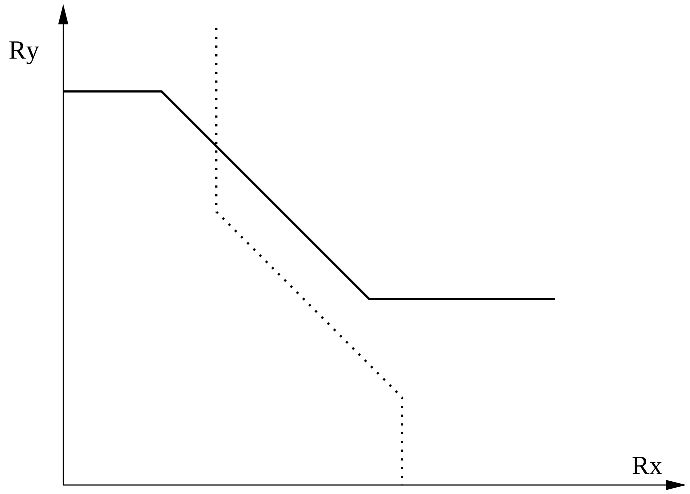

Fig. 3. A typical randomized fixed-composition capacity region $\mathcal{R}_{xy}(P_X, P_Y) = \mathcal{R}_x(P_X, P_Y) \cap \mathcal{R}_y(P_X, P_Y)$ is the intersection of the dotted line and the solid lines, this capacity region is not necessarily convex.

Theorem 1: Interference channel capacity region $\mathcal{R}_x(P_X, P_Y)$ for randomized fixed-composition codes with compositions P_X and P_Y:

$$\begin{aligned} \mathcal{R}_x(P_X, P_Y) \;\; &= \;\; \{(R_x, R_y) : 0 \le R_x < I(X;Z), 0 \le R_y\} \quad \bigcup \\ & \quad\;\; \{(R_x, R_y) : 0 \le R_x < I(X;Z|Y), R_x + R_y < I(X,Y;Z)\} \end{aligned} \tag{4}$$

where the random variables in (4), $(X, Y, Z) \sim P_X P_Y W_{Z|X,Y}$. The region $\mathcal{R}_x(P_X, P_Y)$ is illustrated in Figure 2.

The achievable part of the theorem states that: for a rate pair $(R_x, R_y) \in \mathcal{R}_x(P_X, P_Y)$, the union of Region I and II in Figure 2, for all $\delta > 0$, there exists $N_\delta < \infty$, s.t. for all $n > N_\delta$, the average error probability (1) for the randomized code from compositions P_X and P_Y is smaller than δ for X:

$$P^n_{e(x)}(R_x, R_y, P_X, P_Y) < \delta$$

for some decoding rule. Region II is also the multiple-access capacity region for fixed-composition codes (P_X, P_Y) for channel $W_{Z|XY}$.

The converse of the theorem states that for any rate pair (R_x, R_y) outside of $\mathcal{R}_x(P_X, P_Y)$, that is region III, IV and IV in Figure 2, there exists $\delta > 0$, such that for all n,

$$P^n_{e(x)}(R_x, R_y, P_X, P_Y) > \delta$$

no matter what decoding rule is used. Note that the definition of the error probability $P^n_{e(x)}(R_x, R_y, P_X, P_Y)$ defined in (1)

The proof of Theorem 1 is in Section III.

D. Necessities of more sophisticated time-sharing schemes

In the achievability part of Theorem 1, we prove that the average error probability for X is arbitrarily small for a randomized fixed-composition code if the rate pair (R_x, R_y) is inside the capacity region $\mathcal{R}_x(P_X, P_Y)$. For interference channels, it is obvious that the rate region for both X and Y is:

$$\mathcal{R}_{xy}(P_X, P_Y) = \mathcal{R}_x(P_X, P_Y) \cap \mathcal{R}_y(P_X, P_Y), \tag{5}$$

where $\mathcal{R}_y(P_X, P_Y)$ is defined in the same manner as $\mathcal{R}_x(P_X, P_Y)$ but the channel is $\tilde{W}_{\tilde{Z}|XY}$ instead of $W_{Z|XY}$ as shown in Figure 1. A typical capacity region $\mathcal{R}_{xy}(P_X, P_Y)$ is shown in Figure 3. It is not necessarily convex.

However, by a simple time-sharing between different rate pairs for the same composition, we can convexify the capacity region. Then the convex hull of the union of all such capacity regions of different compositions gives a bigger convex achievable capacity region. This capacity region of the interference channel is

$$CONVEX\left(\bigcup_{P_X,P_Y} \mathcal{R}_{xy}(P_X, P_Y)\right).$$

It is tempting to claim that the above convex capacity region is the largest one can get by time-sharing the "basic" fixed-composition codes as multiple-access channels shown in [2]. However, as will be discussed later in Section IV, it is not the case. A more sophisticated time-sharing gives a bigger capacity region.

This is an important difference between interference channel coding and multiple-access channel coding because the fixed-composition capacity region is convex for the latter and hence the simple time-sharing gives the biggest capacity region [2]. Time-sharing capacity is detailed in Section IV.

E. Existence of a good code for an interference channel

In this paper we focus our study on the average (over all messages) error probability over all code books with the same composition. For a rate pair (R_x, R_y), if the average error probability for X is smaller than δ, then obviously there exists a code book such that the error probability is smaller than δ for X. This should be clear from the definition of error probability $P^n_{e(x)}(R_x, R_y, P_X, P_Y)$ in (1). In the following example, we illustrate that this is also the case for decoding error for both X and Y. We claim without proof that this is also true for "uniform" time-sharing coding schemes later discussed in Section IV. The existence of a code book that achieves the error exponents in the achievability part of the proof of Theorem 1 can also be shown. The proof is similar to that in [12] and Exercise 30 (b) on page 198 [5].

Similar to the error probability for X defined in (1), we define the average joint error probability for X and Y as

$$\begin{aligned} P^n_{e(xy)}(R_x, R_y, P_X, P_Y) &= \left(\frac{1}{|\mathcal{T}^n(P_X)|}\right)^{2^{nR_x}} \left(\frac{1}{|\mathcal{T}^n(P_Y)|}\right)^{2^{nR_y}} \sum_{c_X}\sum_{c_Y} \frac{1}{2^{nR_x}} \sum_{m_x} \frac{1}{2^{nR_y}} \sum_{m_y} \\ &\quad \{ \sum_{z^n} W_{Z|XY}(z^n|x^n(m_x), y^n(m_y)) 1(\widehat{m}_x(z^n) \neq m_x) \\ &\quad + \sum_{\tilde{z}^n} \tilde{W}_{\tilde{Z}|XY}(\tilde{z}^n|x^n(m_x), y^n(m_y)) 1(\widehat{m}_y(\tilde{z}^n) \neq m_y) \} \end{aligned} \tag{6}$$

For a rate pair $(R_x, R_y) \in \mathcal{R}_{xy}(P_X, P_Y) = \mathcal{R}_x(P_X, P_Y) \bigcap \mathcal{R}_y(P_X, P_Y)$. We know that for all $\delta > 0$, there exists $N_\delta < \infty$, s.t. for all $n > N_\delta$, the average error probability is smaller than δ for user X and user Y:
$P^n_{e(x)}(R_x, R_y, P_X, P_Y) < \delta$ and $P^n_{e(y)}(R_x, R_y, P_X, P_Y) < \delta$. It is easy to see that the average joint error probability for user X and Y can be bounded by:

$$\begin{aligned} P^n_{e(xy)}(R_x, R_y, P_X, P_Y) &= P^n_{e(x)}(R_x, R_y, P_X, P_Y) + P^n_{e(y)}(R_x, R_y, P_X, P_Y) \\ &\leq 2\delta \end{aligned} \tag{7}$$

From (6), we know that $P^n_{e(xy)}(R_x, R_y, P_X, P_Y)$ is the average error probability of *all* (P_X, P_Y)-fixed-composition codes. Together with (7), we know that there exists at least *one* code book such that the error probability is no bigger than 2δ.

Note, the converse of the randomized coding does not guarantee that there is not a single good fixed-composition code book. The converse claims that, the average (over all code books with the composition) decoding error probability does not converge to zero if the rate pair is outside the capacity region in Theorem 1.

III. Proof of Theorem 1

There are two parts of the theorem, achievability and converse. The achievability part is proved by applying the classical method of types in point to point channel coding and MAC channel coding for randomized fixed-composition code. The converse is proved by extending the technique first developed in [6] for point to point channels to interference channels.

A. Achievability

We show that in the interior of the capacity region, i.e. the union of Region I and II in Figure 2, a positive error exponent is achieved by applying the randomized fixed-composition coding defined in Definition 1. In Sections III-A.1 and III-A.2, we describe the universal decoding rules for Region II and I respectively. We then present the error exponent results in Lemma 1 in Section III-A.3 and Lemma 2 in Section III-A.4 that covers Region II and I respectively. Then in Lemma 3 in Section III-A.5, we show that these error exponents are positive in the interior of the capacity region $\mathcal{R}_x(P_X, P_Y)$ and hence conclude the proof of the achievability part in Theorem 1.

1) Decoding rule in Region II: In Region II, we show that decoder X can decode both message m_x and m_y with small error probabilities. This is essentially a multiple-access channel coding problem. We use the technique developed in [5] to derive the positive error exponents that parallel to those in [14]. The decoder is a simple maximum mutual information[1] decoder [5]. This decoding rule is universal in the sense that the decoder does not need to know the multiple access channel $W_{Z|XY}$. We describe the decoding rule here, the estimate of the joint message is the message pair such that the input to the channel $W_{Z|XY}$ and the output of the channel have the maximal empirical mutual information. i.e.:

$$(\widehat{m}_x(z^n), \widehat{m}_y(z^n)) = \underset{i\in\{1,2,\ldots,2^{nR_x}\}, j\in\{1,2,\ldots,2^{nR_y}\}}{\arg\max} I(z^n; x^n(i), y^n(j)) \tag{8}$$

where z^n is the channel output and $x^n(i)$ and $y^n(j)$ are the channel inputs for message i and j respectively. $I(z^n; x^n, y^n)$ is the empirical mutual information between z^n and (x^n, y^n), the point to point maximal mutual mutual information decoding is studied in [5].

If there is a tie, the decoder can choose an arbitrary winner or simply declare error. In Lemma 1, we show that by using the randomized fixed-composition encoding and the maximal mutual information decoding, a non-negative error exponent is achieved in Region II.

[1] A more sophisticated decoding rule based on minimum conditional entropy decoding for multiple-access channel is developed in [13], it is shown that this decoding rule achieves a bigger error exponent in low rate regime. The goal of this paper is, however, not to derive the tightest lower bound on the error exponent. We only need a coding scheme to achieve positive error exponent in the capacity region in Theorem 1. Hence we use the simpler decoding rule here.

2) Decoding rule in Region I: In Region I, decoder X only estimates m_x by treating the input of encoder Y as a source of random noises. This is essentially a point to point channel coding problem. The channel itself has memory since the input of encoder Y is not memoryless. Similar to the multiple access channel coding problem studied in Region II, we use a maximal mutual information decoding rule:

$$\widehat{m}_x(z^n) = \arg\max_{i\in\{1,2,\ldots,2^{nR_x}\}} I(z^n; x^n(i)) \tag{9}$$

In Lemma 2, we show that by using the randomized fixed-composition encoding and the maximal mutual information decoding, a non-negative error exponent is achieved in Region I.

3) Lower bound on the error exponent in Region II:

Lemma 1: (Region II) Multiple-access channel error exponents (joint error probability). For the randomized coding scheme described in Definition 1, and the decoding rule described in (8), the decoding error probability averaged over all messages, code books and channel behaviors is upper bounded by an exponential term:

$$\begin{aligned}
& \Pr((m_x, m_y) \neq (\widehat{m}_x, \widehat{m}_y)) \\
= & \left(\frac{1}{|\mathcal{T}^n(P_X)|}\right)^{2^{nR_x}} \left(\frac{1}{|\mathcal{T}^n(P_Y)|}\right)^{2^{nR_y}} \quad (10)\\
& \sum_{c_X}\sum_{c_Y} \frac{1}{2^{nR_x}} \sum_{m_x} \frac{1}{2^{nR_y}} \sum_{m_y}\sum_{z^n} W_{Z|XY}(z^n|x^n(m_x), y^n(m_y)) 1\left((\widehat{m}_x(z^n), \widehat{m}_y(z^n)) \neq (m_x, m_y)\right) \\
\leq & 2^{-n(E-\epsilon_n)}. \quad (11)
\end{aligned}$$

ϵ_n converges to zero as n goes to infinity, and $E = \min\{E_{xy}, E_{x|y}, E_{y|x}\},$ where

$$\begin{aligned}
E_{xy} &= \min_{Q_{XYZ}:Q_X=P_X, Q_Y=P_Y} D(Q_{Z|XY}\|W|Q_{XY}) + D(Q_{XY}\|P_X \times P_Y) + |I_Q(X,Y;Z) - R_x - R_y|^+ \\
E_{x|y} &= \min_{Q_{XYZ}:Q_X=P_X, Q_Y=P_Y} D(Q_{Z|XY}\|W|Q_{XY}) + D(Q_{XY}\|P_X \times P_Y) + |I_Q(X;Z|Y) - R_x|^+ \\
E_{y|x} &= \min_{Q_{XYZ}:Q_X=P_X, Q_Y=P_Y} D(Q_{Z|XY}\|W|Q_{XY}) + D(Q_{XY}\|P_X \times P_Y) + |I_Q(Y;Z|X) - R_y|^+
\end{aligned}$$

where $|t|^+ = \max\{0, t\}$ and the random variables $(X, Y, Z) \sim Q_{XYZ}$ in $I_Q(X;Z|Y), I_Q(Y;Z|X)$ and $I_Q(X,Y;Z)$.

Remark 1: it is easy to verify that $D(Q_{Z|XY}\|W|Q_{XY}) + D(Q_{XY}\|P_X \times P_Y) = D(Q_{XYZ}\|P_X \times P_Y \times W)$, so the expressions for the error exponents can be further simplified. We use the expressions similar to those in [14] because they are more intuitive.

Remark 2: The proof parallels that in [14] which is in turn an extension to the point to point channel coding problem studied in [5]. The method of types is the main tool for the proofs. The difference is that we need to show the lower bound to the average error probability instead of showing the existence of a good code book in [14]. Without giving details, we follow Gallager's proof in [12] and claim the existence of a good code with the same error exponent as that in [14] as a simple corollary of Lemma 1.

Proof: First we have an obvious upper bound on the error probability

$$\begin{aligned}
& \Pr((m_x, m_y) \neq (\widehat{m}_x, \widehat{m}_y)) \\
= & \Pr(m_x \neq \widehat{m}_x, m_y \neq \widehat{m}_y) + \Pr(m_x \neq \widehat{m}_x, m_y = \widehat{m}_y) + \Pr(m_x = \widehat{m}_x, m_y \neq \widehat{m}_y) \\
\leq & \Pr(m_x \neq \widehat{m}_x, m_y \neq \widehat{m}_y) + \Pr(m_x \neq \widehat{m}_x | m_y = \widehat{m}_y) + \Pr(m_y \neq \widehat{m}_y | m_x = \widehat{m}_x)) \quad (12)
\end{aligned}$$

The inequality (12) follows the equality $P(A,B) = P(A|B)P(B) \leq P(A|B)$. Now we upper bound each individual error probability in (12) respectively by exponentials of n. We only need to show that

$$\Pr(m_x \neq \widehat{m}_x, m_y \neq \widehat{m}_y) \leq 2^{-n(E_{xy}-\epsilon_n)}, \tag{13}$$

$$\Pr(m_x \neq \widehat{m}_x | m_y = \widehat{m}_y) \leq 2^{-n(E_{x|y}-\epsilon_n)}, \tag{14}$$

and $$\Pr(m_y \neq \widehat{m}_y | m_x = \widehat{m}_x) \leq 2^{-n(E_{y|x}-\epsilon_n)}. \tag{15}$$

We prove (13) and (14), (15) follows (14) by symmetry. The proofs are in Appendix A, where a standard method of type argument is used. □

4) Lower bound on the error exponent in Region I:

Lemma 2: (Region I) point to point channel coding error exponent (decoding X only). For the randomized coding scheme described in Definition 1, and the decoding rule described in (9), the decoding error probability averaged over all messages, code books and channel behaviors is upper bounded by an exponential term:

$$\begin{aligned}\Pr(m_x \neq \widehat{m}_x) &= \left(\frac{1}{|\mathcal{T}^n(P_X)|}\right)^{2^{nR_x}} \left(\frac{1}{|\mathcal{T}^n(P_Y)|}\right)^{2^{nR_y}} \\ &\quad \sum_{c_X}\sum_{c_Y}\frac{1}{2^{nR_x}}\sum_{m_x}\frac{1}{2^{nR_y}}\sum_{m_y}\sum_{z^n} W_{Z|XY}(z^n|x^n(m_x), y^n(m_y))1\left(\widehat{m}_x(z^n) \neq m_x\right) \\ &\leq 2^{-n(E_x-\epsilon_n)}. \end{aligned} \tag{16}$$

ϵ_n converges to zero as n goes to infinity, and

$$E_x = \min_{Q_{XYZ}: Q_X = P_X, Q_Y = P_Y} D(Q_{Z|XY}\|W|Q_{XY}) + D(Q_{XY}\|P_X \times P_Y) + |I_Q(X;Z) - R_x|^+$$

Proof: We give a unified proof for (13), (14) and (16) in Appendix A. □

With Lemma 1 and Lemma 2, we know that some non-negative error exponents can be achieved for the randomized (P_X, P_Y) fixed-composition code if the rate pair $(R_x, R_y) \in \mathcal{R}_x(P_X, P_Y)$. This is because both Kullback-Leibler divergence and $|\cdot|^+$ are always non-negative. Now we only need to show the positiveness of those error exponents when the rate pair is in the interior of $\mathcal{R}_x(P_X, P_Y)$.

5) Positiveness of the error exponents:

Lemma 3: For rate pairs (R_x, R_y) in the interior of $\mathcal{R}_x(P_X, P_Y)$ defined in Theorem 1:

$$\max\{\min\{E_{xy}, E_{x|y}, E_{y|x}\}, E_x\} > 0.$$

More specifically, we show two things. First, if $R_x < I(X, Z)$, where $(X, Z) \sim P_X \times P_Y \times W_{Z|XY}$, then $E_x > 0$. This covers Region I. Secondly, if $R_x < I(X, Z|Y)$, $R_y < I(Y, Z|X)$ and $R_x + R_y < I(X, Y; Z)$, where $(X, Y, Z) \sim P_X \times P_Y \times W_{Z|XY}$, then $\min\{E_{xy}, E_{x|y}, E_{y|x}\} > 0$, this covers Region II.

Proof: First, suppose that for some $R_x < I(X, Z)$, $E_x \leq 0$. Since both Kullback-Leibler divergence and $|\cdot|^+$ are non-negative functions, we must have $E_x = 0$ and hence there exists a distribution Q_{XYZ}, s.t. $Q_X = P_X$, $Q_Y = P_Y$ and all the individual non-negative functions are zero:

$$\begin{aligned} D(Q_{XY}\|P_X \times P_Y) &= 0 \\ D(Q_{Z|XY}\|W|Q_{XY}) &= 0 \\ |I_Q(X;Z) - R_x|^+ &= 0 \end{aligned}$$

The first equation tells us that $Q_{XY} = P_X \times P_Y$. Then the second equation becomes $D(Q_{Z|XY} \| W | P_X \times P_Y) = 0$, this means that $Q_{Z|XY} \times P_X \times P_Y = W \times P_X \times P_Y$, so $I_Q(X;Z) = I(X;Z)$ where the random variables $(X,Y,Z) \sim P_X \times P_Y \times W_{Z|XY}$ in $I(X;Z)$. Now the third equation becomes $|I(X;Z) - R_x|^+ = 0$ which is equivalent to $I(X;Z) \le R_x$, this is a contradiction to the fact that $R_x < I(X,Z)$.

Secondly, suppose that for some rate pair (R_x, R_y) in Region II, i.e. $R_x < I(X,Z|Y)$, $R_y < I(Y,Z|X)$ and $R_x + R_y < I(X,Y;Z)$ and $\min\{E_{xy}, E_{x|y}, E_{y|x}\} \le 0$, then $\min\{E_{xy} = 0$ or $E_{x|y} = 0$ or $E_{y|x}\} = 0$. Following exactly the same argument as that in the first part of the proof of Lemma 3, we can get contradictions with the fact that the rate pair (R_x, R_y) is in the interior of Region II. □

From the above three lemmas, we conclude that the error probability for decoding message X is upper bounded by $2^{-n(E-\epsilon_n)}$ for all $(R_x, R_y) \in \mathcal{R}_x(P_X, P_Y)$, where $E > 0$ and $\lim\limits_{n\to\infty} \epsilon_n = 0$. Hence the error probability converges to zero exponentially fast for large n. This concludes the achievability part of the proof for Theorem 1.

B. Converse

We show that the average decoding error of Decoder X does not converge to zero with increasing n if the rate pair (R_x, R_y) is outside the capacity region $\mathcal{R}_x(P_X, P_Y)$ shown in Figure 2. There are three parts of the proof for Regions V, IV and III respectively.

1) Region V: First, we show that in Region V the average error probability does not converge to zero as block length goes to infinity. This is proved by using a modified version of the reliability function for rate higher than the channel capacity [6].

Lemma 4: Region V, the average error probability for X does not converge to 0 with block length n if $R_x > I(X;Z|Y)$, where $(X,Y,Z) \sim P_X \times P_Y \times W_{Z|XY}$.

Proof: It is enough to show the case where there is only one message for Y and encoder Y sends a code word y^n with composition P_Y. The code book for encoder X is still uniformly generated among all the fixed-composition-P_X code books. In the rest of the proof, we investigate the typical behavior of the codewords x^n and modify the Lemma 3 and Lemma 5 from [6] to show that

$$\Pr(\widehat{m}_x \neq m_x) = P^n_{e(x)}(R_x, R_y, P_X, P_Y) > \frac{1}{2} \tag{17}$$

for large n. The details of the proof are in Appendix B. □

2) Region IV: The more complicated case is in Region IV. We show that the decoding error probability for user X does not converge to zero with block length n. The proof is by contradiction. The idea is to construct a decoder that decodes both message m_x and message m_y correctly with high probability, if the decoding error for m_x converges to zero. Then again by using a modified proof used in proving the reliability function for rate higher than channel capacity in [6], we get a contradiction.

Lemma 5: Region IV, the average error probability for X does not converge to 0 with block length n if $R_x < I(X;Z|Y)$, $R_y < I(Y;Z|X)$ and $R_x + R_y > I(X,Y;Z)$ where $(X,Y,Z) \sim P_X \times P_Y \times W_{Z|XY}$.

Proof: Suppose that

$$\Pr(\widehat{m}_x \neq m_x) = P^n_{e(x)}(R_x, R_y, P_X, P_Y) \le \delta_n \tag{18}$$

where δ_n goes to zero with n. Let decoder X decode m_y by the same decoding rule devised in (8):

$$\widehat{m}_y(z^n) = \mathop{\arg\max}_{j \in \{1,2,\ldots,2^{nR_y}\}} I(z^n; x^n(\widehat{m}_x(z^n)), y^n(j)). \tag{19}$$

The decoding error for either message at decoder X is now:

$$\begin{aligned}\Pr((\widehat{m}_x,\widehat{m}_y)\neq(m_x,m_y)) &= \Pr(\widehat{m}_x\neq m_x)+\Pr(\widehat{m}_x=m_x,\widehat{m}_y\neq m_y)\\ &\leq \Pr(\widehat{m}_x\neq m_x)+\Pr(\widehat{m}_y\neq m_y|\widehat{m}_x=m_x)\end{aligned} \tag{20}$$

Given $\widehat{m}_x=m_x$, (19) becomes

$$\widehat{m}_y(z^n)=\underset{j\in\{1,2,\ldots,2^{nR_y}\}}{\arg\max}\ I(z^n;x^n(m_x),y^n(j)). \tag{21}$$

So the second term in the RHS of (20), $\Pr(\widehat{m}_y\neq m_y|\widehat{m}_x=m_x)$, can be upper bounded as shown in (14). Substitute the upper bounds (14) and (18) into (20), we have:

$$\Pr((\widehat{m}_x,\widehat{m}_y)\neq(m_x,m_y))\leq\delta_n+2^{-n(E_{y|x}-\epsilon_n)} \tag{22}$$

This upper bound (22) converges to 0 as n goes to infinity. However in Appendix B, we show that

$$P^n_{e(xy)}(R_x,R_y,P_X,P_Y)=\Pr((\widehat{m}_x,\widehat{m}_y)\neq(m_x,m_y))>\frac{1}{2} \tag{23}$$

This is contradicted to (22). $\square$

3) Region III: This is a corollary of Lemma 5. This is intuitively obvious since for each rate pair (R_x,R_y) in Region III, we can find a rate pair (R_x,R'_y) in Region IV such that $R_y>R'_y$. We construct a contradiction as follows. For a (R_x,R_y) decoder, we can construct a new decoder for (R_x,R'_y) where $R'_y<R_y$, by revealing a random selection of a (R_x,R_y) code book that is the superset of the (R_x,R'_y) code book to the (R_x,R_y) decoder and accept the estimate of the (R_x,R_y) decoder as the estimate for the (R_x,R'_y) decoder. If the average error probability is small for the (R_x,R_y) code books, the average error probability is small for this particular (R_x,R'_y) decoder as well, this is a contradiction to Lemma 5. Hence the decoding error for encoder X does not converge to 0 with n if the rate pair (R_x,R_y) is in Region III. $\square$

This concludes the converse part of the proof for Theorem 1.

IV. Discussions on Time-sharing

The main result of this paper is the randomized fixed-composition coding capacity region for X that is $\mathcal{R}_x(P_X,P_Y)$ shown in Figure 2. So obviously, the interference channel capacity region, where decoding errors for both X and Y are small, is the intersection of $\mathcal{R}_x(P_X,P_Y)$ and $\mathcal{R}_y(P_X,P_Y)$ where $\mathcal{R}_y(P_X,P_Y)$ is defined in the similar way but with channel $\tilde{W}_{\tilde{Z}|XY}$ instead of $W_{Z|XY}$. The intersected region defined in (5), $\mathcal{R}_{xy}(P_X,P_Y)$, is in general non-convex as shown in Figure 3. Similar to multiple-access channels capacity region, studied in Chapter 15.3 [2], we use this capacity region $\mathcal{R}_{xy}(P_X,P_Y)$ as the building blocks to generate larger capacity regions.

A. A digression to MAC channel capacity region

Before giving the time-sharing results for interference channels and show why the simple time-sharing idea works for MAC channels but not for interference channels, we first look at $\mathcal{R}_x(P_X,P_Y)$ in Figure 2. Region II is obviously the multiple access channel $W_{Z|XY}$ region achieved by input composition (P_X,P_Y) at the two encoders, denoted by $\mathcal{R}^{mac}_{xy}(P_X\times P_Y)$. In [2], the full description of the MAC channel capacity region is given in two different manners:

$$CONVEX\left(\bigcup_{P_X,P_Y}\mathcal{R}^{mac}_{xy}(P_X\times P_Y)\right)=CLOSURE\left(\bigcup_{P_U,P_{X|U},P_{Y|U}}\mathcal{R}^{mac}_{xy}(P_{X|U}\times P_{Y|U}\times P_U)\right)$$

where $R_{xy}^{mac}(P_{X|U} \times P_{Y|U} \times P_U) = \{(R_x, R_y) : R_x \leq I(X;Z|Y,U), R_y \leq I(Y;Z|X,U), R_x + R_y \leq I(X,Y;Z|U)\}$ and U is the time-sharing auxiliary random variable and $|U| = 4$.

The LHS of (24) is the convex hull of all the fixed-composition MAC channel capacity regions. The RHS of (24) is the closure (without convexification) of all the time-sharing MAC capacity regions.The equivalence in (24) is non-trivial, it is not a consequence of the tightness of the achievable region. It hinges on the convexity of the "basic" capacity regions $\mathcal{R}_{xy}^{mac}(P_X, P_Y)$. As will be shown in Section IV-C, this is not the case for interference channels, i.e. (24) does not hold anymore.

B. Simple time-sharing capacity region and error exponent

The simple idea of time-sharing is well studied for multi-user channel coding, broadcast channel coding. Whenever there are two operational points $(R_x^1, R_y^1), (R_x^2, R_y^2)$, while there exist two coding schemes to achieve small error probability at each operational point, one can use λn amount of channel uses at (R_x^1, R_y^1) with coding scheme 1 and $(1-\lambda)n$ amount of channel uses at (R_x^2, R_y^2) with coding scheme 2. The rate of this coding scheme is $(\alpha R_x^1 + (1-\alpha)R_x^2, \alpha R_y^1 + (1-\alpha)R_y^2)$ and the error probability is still small[2] (no bigger than the sum of two small error probabilities). This idea is easily generalized to more than 2 operational points.

This simple time sharing idea works perfectly for MAC channel coding as shown in (24). The whole capacity region can be described as time sharing among fixed-composition codes where the fixed-composition codes are building blocks. If we extend this idea to interference channel, we have the following simple time sharing region as discussed in Section II-D:

$$CONVEX\left(\bigcup_{P_X,P_Y} \mathcal{R}_{xy}(P_X, P_Y)\right) = CONVEX\left(\bigcup_{P_X,P_Y} \mathcal{R}_x(P_X, P_Y) \bigcap R_y(P_X, P_Y)\right). \tag{24}$$

We shall soon see in the next section that this result can be improved.

C. Beyond simple time-sharing: "Uniform" time-sharing

In this section we give a time-sharing coding scheme that was first developed by Gallager [11] and later further studied for universal decoding by Pokorny and Wallmeier [14] to get better error exponents for MAC channels. This type of "uniform" time-sharing schemes not only achieves better error exponents, more importantly, we show that this achieve **bigger** capacity region than the simple time-sharing scheme does for interference channels! Unlike the multiple-access channels where the simple time-sharing achieves the whole capacity region, this is unique to the interference channels, due to the fact that the capacity region is the convex hull of the intersections of pairs of non-convex regions (convex or not is not the issue here, the real difference is the intersection operation).

The organization of this section parallel to that for the fixed-composition. We first introduce the "uniform" time-sharing coding scheme, then give the achievable error exponents and lastly drive the achievable rate region for such coding schemes. The proofs are omitted since they are similar to those for the randomized fixed-composition codes.

Definition 3: "Uniform" time-sharing codes: for a probability distribution P_U on $\mathcal{U}$, where $\mathcal{U} = \{u_1, u_2, ..., u_K\}$ with $\sum_{i=1}^K P_U(u_i) = 1$, and a pair of conditional independent distributions $P_{X|U}, P_{Y|U}$. We define the two codeword sets[3] as

$$X_c(n) = \{x^n : x_1^{nP_U(u1)} \in P_{X|u_1}, x_{nP_U(u_1)+1}^{n(P_U(u_1)+P_U(u_2))} \in P_{X|u_2}, ..., x_{n(1-P_U(u_1))}^n \in P_{X|u_L}\}$$

[2]The error exponent is, however, at most half of the individual error exponent.

[3]Again, we ignore the nuisance of the non-integers here.

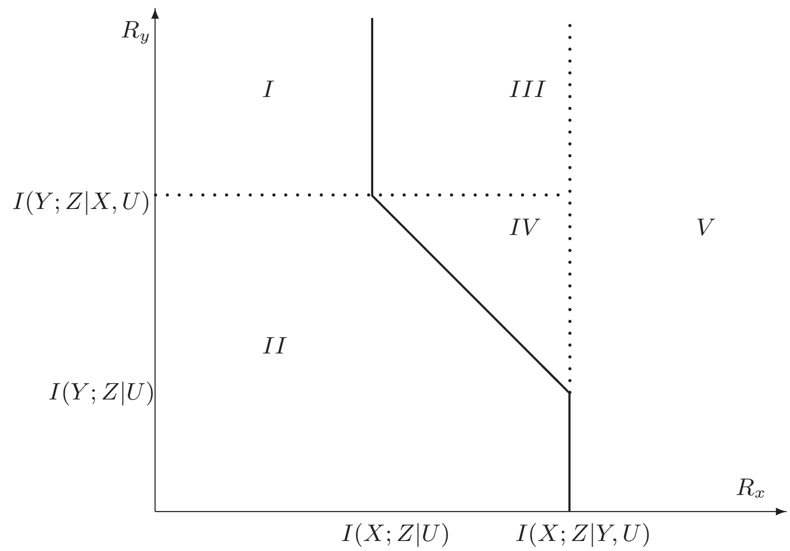

Fig. 4. "Uniform" time-sharing capacity region $\mathcal{R}_x(P_U P_{X|U} P_{Y_U})$ for X, the achievable region is the union of Region I and II. This region is very similar to that for fixed-composition coding shown in Figure 2, only difference is now there is an auxiliary time-sharing random variable U.

i.e. the i'th chunk of the codeword x^n with length $nP_U(u_i)$ has composition $P_{X|u_i}$, and similarly

$$Y_c(n) = \{y^n : y_1^{nP_U(u1)} \in P_{Y|u_1}, y_{nP_U(u_1)+1}^{n(P_U(u_1)+P_U(u_2))} \in P_{Y|u_2}, ..., y_{n(1-P_U(u_1))}^{n} \in P_{Y|u_L}\}.$$

A "uniform" time-sharing code $(R_x, R_y, P_U P_{X|U} P_{Y|U})$ encoder picks a code book with the following probability: for any message $m_x \in \{1, 2, ..., 2^{nR_x}\}$, the code word $x^n(m_x)$ is uniformly distributed in $X_c(n)$, similarly for encoder Y.

After the code book is randomly generated and revealed to the decoder, the decoder uses a maximum mutual information decoding rule. Similar to the fixed-composition coding, the decoder needs to either decode both message X and Y jointly or simply treats Y as noise and decode X only, depending on where the rate pairs are in Region I or II, as shown in Figure 4. The error probability we investigate is again the average error probability over all messages and code books.

Theorem 2: Interference channel capacity region $\mathcal{R}_x(P_U P_{X|U} P_{Y|U})$ for "uniform" time-sharing codes with composition $P_U P_{X|U} P_{Y|U}$:

$$\begin{aligned}\mathcal{R}_x(P_U P_{X|U} P_{Y|U}) \;=\; & \{(R_x, R_y) : 0 \le R_x < I(X;Z|U), 0 \le R_y\} \quad \bigcup \\ & \{(R_x, R_y) : 0 \le R_x < I(X;Z|Y,U), R_x + R_y < I(X,Y;Z|U)\}\end{aligned} \tag{25}$$

where the random variables in (25), $(U, X, Y, Z) \sim P_U P_{X|U} P_{Y|U} W_{Z|X,Y}$. And the interference capacity region for $P_U P_{X|U} P_{Y|U}$ is

$$\mathcal{R}_{xy}(P_U P_{X|U} P_{Y|U}) = \mathcal{R}_x(P_U P_{X|U} P_{Y|U}) \bigcap \mathcal{R}_y(P_U P_{X|U} P_{Y|U}) \tag{26}$$

The rate region defined in (25) itself does not give any new X-capacity regions for X, since both Region I and II in Figure 4 can be achieved by simple time-sharing of Region I and II repectively in (4). But for the interference channel capacity, we argue in the next section that this coding scheme gives a strictly bigger capacity region than that given by the simple time-sharing of fixed-composition codes in (24).

The proof of Theorem 2 is similar to that of Theorem 1. We omit the details here. We only point out that the achievability part is proved by deriving a positive error exponent for rate pair in the interior of the capacity region defined in Theorem 2. As shown in [14] and also detailed in this paper for the randomized coding, the error exponents in Region II of in Figure 4 is:

$$E = \min\{E_{xy}, E_{x|y}, E_{y|x}\}, \text{ where}$$

$$\begin{aligned}
E_{xy} &= \min_{Q_{XYZ|U}:Q_{X|U}=P_{X|U},Q_{Y|U}=P_{Y|U}} \\
&\quad D(Q_{Z|XY}\|W|Q_{XYU}) + D(Q_{XY|U}\|P_{X|U}\times P_{Y|U}|U) + |I_Q(X,Y;Z) - R_x - R_y|^+ \\
E_{x|y} &= \min_{Q_{XYZ|U}:Q_{X|U}=P_{X|U},Q_{Y|U}=P_{Y|U}} \\
&\quad D(Q_{Z|XY}\|W|Q_{XYU}) + D(Q_{XY|U}\|P_{X|U}\times P_{Y|U}|U) + |I_Q(X;Z|Y,U) - R_x|^+ \\
E_{y|x} &= \min_{Q_{XYZ|U}:Q_{X|U}=P_{X|U},Q_{Y|U}=P_{Y|U}} \\
&\quad D(Q_{Z|XY}\|W|Q_{XYU}) + D(Q_{XY|U}\|P_{X|U}\times P_{Y|U}|U) + |I_Q(Y;Z|X,U) - R_y|^+
\end{aligned}$$

This is the error exponents in Lemma 1 with a conditional auxiliary random variable U.

The error exponent in Region I is

$$\begin{aligned}
E_x = &\min_{Q_{XYZ|U}:Q_{X|U}=P_{X|U},Q_{Y|U}=P_{Y|U}} \\
&D(Q_{Z|XY}\|W|Q_{XYU}) + D(Q_{XY|U}\|P_{X|U}\times P_{Y|U}|U) + |I_Q(X;Z|U) - R_x|^+
\end{aligned}$$

D. Why the "uniform" time sharing is needed?

It is obvious that the "uniform" time-sharing fixed-composition coding gives a bigger error exponent than the simple time-sharing coding does. More interestingly, we argue that it gives a bigger interference channel capacity region. First we write down the interference channel capacity region generated from the basic "uniform" time-sharing fixed-composition codes:

$$CONVEX\left(\bigcup_{P_{X|U}P_{Y|U}P_U} \mathcal{R}_{xy}(P_U P_{X|U} P_{Y|U})\right). \tag{27}$$

where $\mathcal{R}_{xy}(P_U P_{X|U} P_{Y|U})$ is defined in (26) and $CONVEX(A)$ is the convex hull (simple time sharing) of set A.

U is a time-sharing auxiliary random variable. Unlike the MAC coding problem, where simple time-sharing of fixed-composition codes achieve the full capacity region, it is not guaranteed for interference channels. The reason is the intersection operator in the basic building blocks in (5) and (26) respectively, i.e. the interference nature of the problem[4].

Obviously the rate region by simple time sharing of fixed composition code in (24) is a subset of simple time sharing of the "uniform" time sharing capacity region (27). In the following example, we illustrate why (27) is bigger than (24).

[4]To understand why intersection is the difference but not the non-convexity, we consider four convex sets: A_1, A_2, B_1, B_2. We show that $CONVEX(A_1\bigcap B_1, A_2\bigcap B_2)$ can be strictly smaller than $CONVEX(A_1, A_2)\bigcap CONVEX(B_1, B_2)$. Let $A_1 = B_2 \subset B_1 = A_2$, then $CONVEX(A_1\bigcap B_1, A_2\bigcap B_2) = A_1$ is strictly smaller than $CONVEX(A_1, A_2)\bigcap CONVEX(B_1, B_2) = A_2$. This shows why uniform time-sharing gives bigger capacity region.

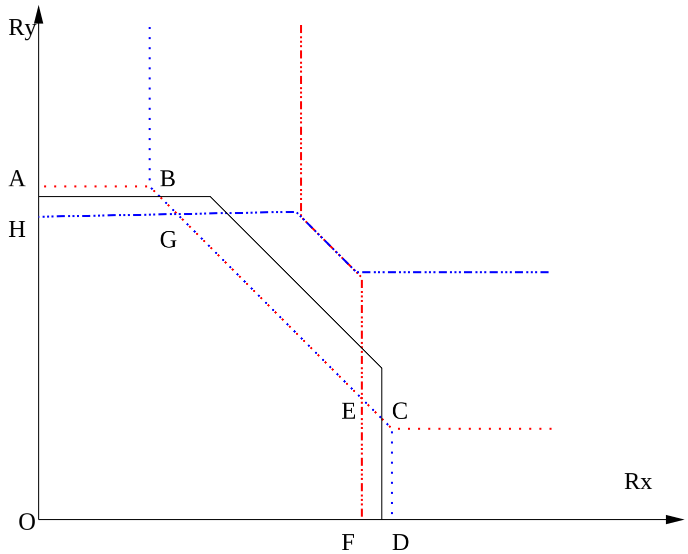

Fig. 5. Simple timesharing of fixed-composition capacity $ABCDO$ VS time-sharing fixed composition capacity(0.5) (the black pentagon)

Example: Suppose we have a symmetric interference channel, i.e. $\mathcal{R}_x(P_X, P_Y) = \mathcal{R}_y^T(P_Y, P_X)$ for all P_X, P_Y where ^{T} is the transpose operation. The comparison of simple timesharing capacity region and the more sophisticated time-sharing fixed-composition capacity region are illustrated by a toy example in Figure 5.

For a distribution (P_X, P_Y), the achievable region for the fixed-composition code is illustrated in Figure 5, $\mathcal{R}_x(P_X, P_Y)$ and $\mathcal{R}_y(P_X, P_Y)$ respectively, these are bounded by the red dotted lines and red dash-dotted lines respectively, so the interference capacity region $\mathcal{R}_{xy}(P_X, P_Y)$ is bounded by the pentagon $ABEFO$. By symmetry, $\mathcal{R}_x(P_Y, P_X)$ and $\mathcal{R}_y(P_X, P_Y)$ are bounded by the blue dotted lines and blue dash-dotted lines respectively, the capacity region $\mathcal{R}_x y(P_Y, P_X)$ is bounded by the pentagon $HGCDO$. So the convex hull of these two regions is $ABCDO$.

Now consider the following timesharing fixed-composition coding $P_{X|U}P_{Y|U}P_U$ where $\mathcal{U} = \{0, 1\}$, $P_U(0) = P_U(1) = 0.5$ and $P_{X|0} = P_{Y|1} = P_X$, $P_{X|1} = P_{Y|0} = P_Y$. The interference capacity region is obviously bounded by the black pentagon in Figure 5. This toy example shows why (27) is bigger than (24).

V. Future directions

The most interesting question about interference channel is the geometry of the two code books. For point to point channel coding, the code words in the optimal code book is uniformly distributed on a sphere of the optimal compositions and the optimal composition achieves the capacity. For MAC channels, a simple time-sharing among different fixed-composition codes is sufficient and necessary to achieve the whole capacity region, meanwhile for each fixed-composition codes, the codewords are uniformly distributed. However as illustrated in Section IV, a more interesting "uniform" time sharing is needed. So what is time sharing? Both simple time sharing and "uniform" time sharing change the shape of the code books, however, in different ways. Simple time sharing "glue" segments of code words together due to the independence of the coding in different segments of the channel uses, meanwhile for "uniform" time sharing, code words still have equal distances between one another. Better understanding of the shape of code books may help us understand the interference channels. Also in this paper, we give our first attempt at giving an outer bound of the interference channel capacity region. We only manage to give a tight outer bound to the time-sharing fixed-composition code. An important future direction is to categorize the coding schemes for interference channels and more outer bound result may follow. This is in contrast to the traditional outer bound derivations [1] where genie is used.

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APPENDIX

A. Proof of (13), (14) and (16)

We give a unified proof in lower bounding the error probability for randomized fixed-composition coding, where the error probabilities in (13), (14) and (16) are taken over all messages, code books and channel behaviors. We examine the object function to be minimized in (13), (14) and (16).

First, the *common* part of the three error exponents E_{xy}, $E_{x|y}$ and E_x: $D(Q_{Z|XY}\|W|Q_{XY})+D(Q_{XY}\|P_X\times P_Y)$. $D(Q_{XY}\|P_X\times P_Y)$ is the logarithm of the inverse of the probability that type Q_{XY} is the empirical distribution of the code pair $x^n(1), y^n(1)$ individually generated from fixed-compositions P_X and P_Y. $D(Q_{Z|XY}\|W|Q_{XY})$ is logarithm of the inverse of the conditional probability that the input to the channel W is Q_{XY}, while the empirical type of the input/output is $Q_{XYZ}=Q_{XY}\times Q_{Z|XY}$.

Secondly for the individual part of the error exponents in (13), (14) and (16): $|I_Q(X,Y;Z)-R_x-R_y|^+$, $|I_Q(X;Z|Y)-R_x|^+$ and $|I_Q(X;Z)-R_x|^+$ respectively, each one is the logarithm of the inverse of an upper bound on the probability that there exists another message (pair) with higher mutual information with the channel output, while the channel inputs/ouput has type Q_{XYZ}. This is derived by a union bound argument. We now give the details of the proofs.

1) Proof of (13): Because of the symmetry of the code book selection, we can fix the message pair $(m_x, m_y) = (1,1)$ and write the error probability (13) in the following way:

$$\begin{aligned}
&\Pr(m_x \neq \widehat{m}_x, m_y \neq \widehat{m}_y) \\
&= \left(\frac{1}{|\mathcal{T}^n(P_X)|}\right)^{2^{nR_x}} \left(\frac{1}{|\mathcal{T}^n(P_Y)|}\right)^{2^{nR_y}} \sum_{c_X}\sum_{c_Y} \qquad (28)\\
&\quad \frac{1}{2^{nR_x}}\sum_{m_x}\frac{1}{2^{nR_y}}\sum_{m_y}\sum_{z^n} W_{Z|XY}(z^n|x^n(m_x), y^n(m_y))1(\widehat{m}_x(z^n)\neq m_x, \widehat{m}_y(z^n)\neq m_y)\\
&= \left(\frac{1}{|\mathcal{T}^n(P_X)|}\right)^{2^{nR_x}} \left(\frac{1}{|\mathcal{T}^n(P_Y)|}\right)^{2^{nR_y}}\\
&\quad \sum_{c_X}\sum_{c_Y}\sum_{z^n} W_{Z|XY}(z^n|x^n(1), y^n(1))1(\widehat{m}_x(z^n)\neq 1, \widehat{m}_y(z^n)\neq 1)\\
&= \sum_{Q_{XY}:Q_X=P_X, Q_Y=P_Y} \Big\{ \Pr\big((x^n(1), y^n(1))\in Q_{XY}\big) \sum_{Q_{Z|XY}} \Pr(z^n|(x^n(1), y^n(1))\in Q_{Z|XY})\\
&\quad \Pr(\widehat{m}_x(z^n)\neq 1, \widehat{m}_y(z^n)\neq 1)\Big\} \qquad (29)\\
&\leq \sum_{Q_{XY}:Q_X=P_X, Q_Y=P_Y} \Big\{ \Pr\big((x^n(1), y^n(1))\in Q_{XY}\big) \sum_{Q_{Z|XY}} \Pr(z^n|(x^n(1), y^n(1))\in Q_{Z|XY})\\
&\quad \min\{1, \sum_{i=2}^{2^{nR_x}}\sum_{j=2}^{2^{nR_y}} \Pr\left(I(z^n; x^n(1), y^n(1)) \leq I(z^n; x^n(i), y^n(j)) | (x^n(1), y^n(1), z^n)\in Q_{XYZ})\right)\}\Big\}\\
&\leq |\mathcal{T}^n_{XYZ}| \max_{Q_{XYZ}:Q_X=P_X, Q_Y=P_Y} \Pr\big((x^n(1), y^n(1))\in Q_{XY}\big) \Pr(z^n|(x^n(1), y^n(1))\in Q_{Z|XY}) \qquad (30)\\
&\quad \min\{1, \sum_{i=2}^{2^{nR_x}}\sum_{j=2}^{2^{nR_y}} \Pr\left(I(z^n; x^n(1), y^n(1)) \leq I(z^n; x^n(i), y^n(j)) | (x^n(1), y^n(1), z^n)\in Q_{XYZ})\right)\}
\end{aligned}$$

(28) and (29) are two different interpretations of the same error probability. In (28), we first randomly pick a fixed-composition code book pair c_X and c_Y, then sum over the all probabilities that the output of the channel causes a decoding error for the chosen code book pair. (29) is an equivalent interpretation of the above error probability because the codewords for each message is independently generated. We interpret (29) as follows, we first randomly pick a codeword pair for message 1 in X and message 1 in Y, then the codeword pair is transmitted to through the channel. Then we randomly generate the rest of the code book and investigate the probability that other message pairs maximize the mutual information with the channel output. We upper bound the four terms in (30) individually in (31), (32), (33) and (34).

First, the number of type sets of length n:

$$|\mathcal{T}^n_{XYZ}| \leq (n+1)^{|\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}|} = 2^{n(\frac{\log(n+1)}{n}|\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}|)} = 2^{na_n}. \qquad (31)$$

Secondly, for any Q_{XY}, s.t. $Q_X = P_X$ and $Q_Y = P_Y$, from the method of types [2] and [3], we know that $2^{n(H(P_Y)-\frac{\log n}{n}|\mathcal{Y}|)} \leq |\mathcal{P}_Y| \leq 2^{nH(P_Y)}$, similar bounds applies to $|\mathcal{P}_X|$. And for a fixed X-sequence, $x^n(1)\in P_X = Q_X$, we have $2^{n(H(Q_{Y|X})-\frac{\log n}{n}|\mathcal{X}Y|)} \leq |\{y^n\in\mathcal{Y}^n : (x^n(1), y^n)\in Q_{XY}\}| \leq 2^{nH(Q_{Y|X})}$.

$x^n(1)$ and $y^n(1)$ are independently distributed in type set P_X and P_Y. Hence,

$$\Pr\left((x^n(1), y^n(1)) \in Q_{XY}\right) = \frac{|\{y^n \in \mathcal{Y}^n : (x^n(1), y^n) \in Q_{XY}\}|}{|P_Y|} \leq 2^{n(H(Q_{Y|X}) - H(Q_Y) + \frac{\log n}{n}|\mathcal{X}|)}$$

Notice that $H(Q_{Y|X}) - H(Q_Y) = -D(Q_{XY}\|Q_X \times Q_Y) = -D(Q_{XY}\|P_X \times P_Y)$ and let $b_n = \frac{\log n}{n}|\mathcal{X}|$, we have:

$$\Pr\left((x^n(1), y^n(1)) \in Q_{XY}\right) \leq 2^{-n(D(Q_{XY}\|P_X \times P_Y) - b_n)} \tag{32}$$

Thirdly, For $(x^n(1), y^n(1)) \in Q_{XY}$, for any empirical channel behavior $Q_{Z|XY}$:

$$\begin{aligned}
\Pr(z^n|(x^n(1), y^n(1)) \in Q_{Z|XY}) &= |\{z^n : (x^n(1), y^n(1), z^n) \in Q_{XYZ}\}| W_{Z|XY}(Q_{Z|XY}) \\
&\leq 2^{nH(Q_{Z|XY})} \times 2^{n(-D(Q_{Z|XY}\|W|Q_{XY}) - H(Q_{Z|XY}))} \\
&= 2^{-nD(Q_{Z|XY}\|W|Q_{XY})}
\end{aligned} \tag{33}$$

Finally, for $(x^n(1), y^n(1), z^n) \in Q_{XYZ}$, we investigate the probability that there exists (i, j), $i \neq 1, j \neq 1$, s.t. the mutual information between $(x^n(i), y^n(j))$ and z^n is at least as much as the mutual information between $(x^n(1), y^n(1))$ and z^n. For all $i \neq 1$, the codeword $x^n(i)$ is uniformly distributed on the fixed-composition set P_X, same for Y. Given $(x^n(1), y^n(1), z^n) \in Q_{XYZ}$, we have $I(z^n; x^n(1), y^n(1)) = I_Q(Z; X, Y)$, so:

$$\begin{aligned}
&\min\{1, \sum_{i=2}^{2^{nR_x}} \sum_{j=2}^{2^{nR_y}} \Pr\left(I(z^n; x^n(1), y^n(1)) \leq I(z^n; x^n(i), y^n(j)) | (x^n(1), y^n(1), z^n) \in Q_{XYZ}\right)\} \\
&\leq \min\{1, 2^{n(R_x+R_y)} \sum_{V_{XYZ}: V_X = Q_X, V_Y = Q_Y, V_Z = Q_Z, I_Q(Z;X,Y) \leq I_V(Z;X,Y)} \Pr(((x^n(i), y^n(j), z^n) \in V_{XYZ} | z^n \in Q_Z)\} \\
&= \min\{1, 2^{n(R_x+R_y)} \sum_{V_{XYZ}: V_X = Q_X, V_Y = Q_Y, V_Z = Q_Z, I_Q(Z;X,Y) \leq I_V(Z;X,Y)} \frac{|\{(x^n, y^n) \in P_X \times P_Y : (x^n, y^n, z^n) \in V_{XYZ}\}|}{|\{x^n : x^n \in P_X\}||\{y^n : y^n \in P_Y\}|}\} \\
&\leq \min\{1, 2^{n(R_x+R_y)} \sum_{V_{XYZ}: V_X = Q_X, V_Y = Q_Y, V_Z = Q_Z, I_Q(Z;X,Y) \leq I_V(Z;X,Y)} 2^{n(H_V(X,Y|Z) - H_V(X) - H_V(Y) + \frac{\log n(|\mathcal{X}|+|\mathcal{Y}|)}{n})}\} \\
&\leq \min\{1, 2^{n(R_x+R_y)} \sum_{V_{XYZ}: V_X = Q_X, V_Y = Q_Y, V_Z = Q_Z, I_Q(Z;X,Y) \leq I_V(Z;X,Y)} 2^{n(H_V(X,Y|Z) - H_V(X,Y) + \frac{\log n(|\mathcal{X}|+|\mathcal{Y}|)}{n})}\} \\
&= \min\{1, 2^{n(R_x+R_y)} \sum_{V_{XYZ}: V_X = Q_X, V_Y = Q_Y, V_Z = Q_Z, I_Q(Z;X,Y) \leq I_V(Z;X,Y)} 2^{n(-I_V(X,Y;Z) + \frac{\log n(|\mathcal{X}|+|\mathcal{Y}|)}{n})}\} \\
&\leq \min\{1, 2^{n(R_x+R_y)} n^{|\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}|} 2^{n(-I_Q(X,Y;Z) + \frac{\log n(|\mathcal{X}|+|\mathcal{Y}|)}{n})}\} \\
&= 2^{-n(|I_Q(X,Y;Z) - R_x - R_y|^+ - c_n)}
\end{aligned} \tag{34}$$

Substituting (31), (32), (33) and (34) in (30), and noticing that a_n b_n and c_n converges to zero when n goes to infinity, (13) is proved.

2) Sketch of the proof of (14) and (16): (14) and (16) can be proved by following the same argument in proving (13). Similar to how we upper bound the LHS of (13) in (30), we upper bound the LHS of (14) by:

$$\begin{aligned}
&\Pr(m_x \neq \widehat{m}_x | m_y = \widehat{m}_y) \\
&\quad \leq \quad |\mathcal{T}^n_{XYZ}| \max_{Q_{XYZ}: Q_X = P_X, Q_Y = P_Y} \Pr\left((x^n(1), y^n(1)) \in Q_{XY}\right) \Pr(z^n | (x^n(1), y^n(1)) \in Q_{Z|XY}) \\
&\qquad\qquad \min\{1, \sum_{i=2}^{2^{nR_x}} \Pr\left(I(z^n; x^n(1), y^n(1)) \leq I(z^n; x^n(i), y^n(1)) | (x^n(1), y^n(1), z^n) \in Q_{XYZ})\right)\}. \qquad (35)
\end{aligned}$$

and the LHS of (16) by

$$\begin{aligned}
&\Pr(m_x \neq \widehat{m}_x) \\
&\quad \leq \quad |\mathcal{T}^n_{XYZ}| \max_{Q_{XYZ}: Q_X = P_X, Q_Y = P_Y} \Pr\left((x^n(1), y^n(1)) \in Q_{XY}\right) \Pr(z^n | (x^n(1), y^n(1)) \in Q_{Z|XY}) \\
&\qquad\qquad \min\{1, \sum_{i=2}^{2^{nR_x}} \Pr\left(I(z^n; x^n(1)) \leq I(z^n; x^n(i)) | (x^n(1), y^n(1), z^n) \in Q_{XYZ})\right)\}. \qquad (36)
\end{aligned}$$

The common parts (the three terms on the first line) in (35) and (36) are upper bounded the same way as those in (31) (32) and (33) for (30). The individual part (the $\min\{1, \cdot\}$ term on the second line) of (35) and (36) are upper bounded by a similar argument for upper bounding the individual part of (30) shown in (33). We omit the details here. □

B. Proof of (17) and (23)

We give a constant lower bound, $\frac{1}{2}$, on the error probabilities $\Pr(\widehat{m}_x \neq m_x)$ and $\Pr((\widehat{m}_x, \widehat{m}_y) \neq (m_x, m_y))$ in (17) and (23) respectively. The technical details of lower bounding $\Pr(\widehat{m}_x \neq m_x)$ is carried out in Appendix B.1. We extend the two very technical Lemmas 5 and 3 from [6] into Lemmas 6 and 7 respectively, where Lemma 7 is used to prove Lemma 6. The proof of lower bounding $\Pr((\widehat{m}_x, \widehat{m}_y) \neq (m_x, m_y))$ is similar, we only give the necessary definition of jointly good code books in Appendix B.2.

The difference between the setups in this paper and that in [6] is that we are dealing with an interference channel instead of a memoryless channel in [6]. Hence a notion of the conditionally typical code book in the proof of (17) and jointly typical code book in the proof of (23) is necessary in the proofs.

1) Proof of (17): we give an upper bound of the *correct decoding probability* $\Pr(\widehat{m}_x = m_x) = 1 - \Pr(\widehat{m}_x \neq m_x)$ and hence prove the lower bound on $\Pr(\widehat{m}_x \neq m_x)$ in (17) .

$$\begin{aligned}
\Pr(\widehat{m}_x = m_x) &= P^n_{e(x)}(R_x, R_y, P_X, P_Y) \\
&= \left(\frac{1}{|\mathcal{T}^n(P_X)|}\right)^{2^{nR_x}} \sum_{c_X} \frac{1}{2^{nR_x}} \sum_{m_x} \sum_{z^n} W_{Z|XY}(z^n | x^n(m_x), y^n) 1(\widehat{m}_x(z^n) = m_x)
\end{aligned}$$

The codewords $x^n(m_x)$ is uniformly distributed on the type set P_X, so the probability that the joint type of $(x^n(m_x), y^n)$ is close to $P_X \times P_Y$ with high probability [2], i.e. for all $\sigma > 0$, for large n,

$$\Pr(D((x^n(m_x), y^n) \| P_X \times P_Y)) > \sigma) < \sigma. \qquad (37)$$

We denote by $T_\sigma(y^n) = \{x^n : D((x^n, y^n) \| P_X \times P_Y)) \leq \sigma\}$, the typical set conditional on y^n. We say a code book c_X is good conditional on y^n if

$$|c_X \bigcap T^C_\sigma(y^n)| \leq \frac{|c_X|}{4} \qquad (38)$$

where $|c_X| = 2^{nR_x}$. The set of all good code books is denoted by G, at most 4σ of the code books are not in G because of (37). For a good code book c_X, we use the technique from [6] to upper bound the correct probability for the good code book c_X.

$$\begin{aligned}\Pr(\widehat{m}_x = m_x) &\le \frac{|c_X \bigcap T_\sigma^C(y^n)|}{|c_X|} + \frac{1}{|c_X|} \sum_{i: x^n(i) \in T_\sigma(y^n)} \Pr(i = \widehat{m}_x(z^n)) \\ &\le \frac{1}{4} + \frac{1}{|c_X|} \sum_{i: x^n(i) \in T_\sigma(y^n)} \Pr(i = \widehat{m}_x(z^n)) \\ &\le \frac{1}{4} + 2^{-n(E-\epsilon_n)} \end{aligned} \tag{39}$$

where ϵ_n goes to zero with n, and

$$E = \min_{Q_{XYZ}: D(Q_{XY} \| P_X \times P_Y) < \sigma} D(Q_{Z|XY} \| W_{Z|XY} | Q_{XY}) + |R_x - I_Q(X; Z|Y)|^+$$

where (39) is proved by Lemma 6 which is an extension of Lemma 5 in [6] from memoryless to conditional on y^n.

Following the argument in Lemma 3, it is easy to see that $E > 0$ for $R_x > I(X; Z|Y)$ and small σ, where $(X, Y, Z) \sim W_{Z|XY} \times P_X \times P_Y$. Now we have

$$\begin{aligned}\Pr(\widehat{m}_x = m_x) &= \left(\frac{1}{|\mathcal{T}^n(P_X)|}\right)^{2^{nR_x}} \Big(\sum_{c_X \in G} \Pr(\widehat{m}_x = m_x) + \sum_{c_X \in G^C} \Pr(\widehat{m}_x = m_x)\Big) \\ &\le \frac{1}{4} + 2^{-n(E-\epsilon_n)} + 4\sigma \end{aligned} \tag{40}$$

Let σ be small enough and let n goes to infinity, so $\Pr(\widehat{m}_x \neq m_x) = 1 - \Pr(\widehat{m}_x = m_x) \ge \frac{1}{2}$. (17) is proved. $\square$

The following two Lemmas 6 and 7 are extensions of Lemma 5 and 3 in [6] respectively. They contain the technical details in the proof of (39).

Lemma 6: Extension of Lemma 5 in [6] from memoryless to conditional on y^n, for a good code book $c_X \in G$ defined in (38). Recall that $|c_X \bigcap T_\sigma(y^n)| \ge \frac{3|c_X|}{4} = \frac{3}{4} \times 2^{nR_x}$, then for any decoding rule (previously known as $\widehat{m}_x$) $\phi : \mathcal{Z}^n \to \{1, 2, ..., 2^{nR_x}\}$,

$$\frac{1}{|c_X|} \sum_{i: x^n(i) \in T_\sigma(y^n)} \Pr(i = \phi(z^n)) \le 2^{-n(E-\epsilon_n)} \tag{41}$$

where $E = \min_{Q_{XYZ}: D(Q_{XY} \| P_X \times P_Y) < \sigma} D(Q_{Z|XY} \| W_{Z|XY} | Q_{XY}) + |R_x - I_Q(X; Z|Y)|^+$

and $\epsilon_n = \epsilon(|\mathcal{X}|, |\mathcal{Y}|, |\mathcal{Z}|, n)$ which converges to zero as n goes to infinity.

Proof: We write $M = \{i \in \{1, 2, ..., 2^{nR_x}\} : x^n(i) \in T_\sigma(y^n)\}$ then we know that from the definition of a good code book: $\frac{3}{4} \times 2^{nR_x} \le |M| \le 2^{nR_x} = |c_X|$. Notice that

$$\Pr(i = \phi(z^n)) = \sum_{z^n \in \phi^{-1}(i)} W_{Z|XY}(z^n | x^n(i), y^n) = W_{Z|XY}(\phi^{-1}(i) | x^n(i), y^n) \tag{42}$$

We rewrite the LHS of (41):

$$
\begin{aligned}
&= 2^{-nR_x} \sum_{i:x^n(i)\in T_\sigma(y^n)} W_{Z|XY}(\phi^{-1}(i)|x^n(i),y^n) \\
&= 2^{-nR_x} \sum_{Q_{XY}:D(Q_{XY}\|P_X\times P_Y)<\sigma} \left(\sum_{i:(x^n(i),y^n)\in Q_{XY}} W_{Z|XY}(\phi^{-1}(i)|x^n(i),y^n) \right) \\
&\leq (n+1)^{|\mathcal{X}||\mathcal{Y}|} \max_{Q_{XY}:D(Q_{XY}\|P_X\times P_Y)<\sigma} \left(2^{-nR_x} \sum_{i:(x^n(i),y^n)\in Q_{XY}} W_{Z|XY}(\phi^{-1}(i)|x^n(i),y^n) \right) \\
&= (n+1)^{|\mathcal{X}||\mathcal{Y}|} \max_{Q_{XY}:D(Q_{XY}\|P_X\times P_Y)<\sigma} \\
&\quad \left(2^{-nR_x} \sum_{i:(x^n(i),y^n)\in Q_{XY}} \sum_{Q_{Z|XY}} W_{Z|XY}(\phi^{-1}(i) \bigcap Q_{Z|XY}(x^n(i),y^n)|x^n(i),y^n) \right) \\
&\leq (n+1)^{|\mathcal{X}||\mathcal{Y}|+|\mathcal{X}||\mathcal{Y}||\mathcal{Z}|} \max_{Q_{XYZ}:D(Q_{XY}\|P_X\times P_Y)<\sigma} \\
&\quad \left(2^{-nR_x} \sum_{i:(x^n(i),y^n)\in Q_{XY}} W_{Z|XY}(\phi^{-1}(i) \bigcap Q_{Z|XY}(x^n(i),y^n)|x^n(i),y^n) \right) \\
&\leq 2^{n\epsilon_n(1)} \max_{Q_{XYZ}:D(Q_{XY}\|P_X\times P_Y)<\sigma} \\
&\quad \left(2^{-nR_x} \sum_{i:(x^n(i),y^n)\in Q_{XY}} W_{Z|XY}(Q_{Z|XY}(x^n(i),y^n)|x^n(i),y^n) \frac{|Q_{Z|XY}(x^n(i),y^n)\bigcap \phi^{-1}(i)|}{|Q_{Z|XY}(x^n(i),y^n)|} \right) \\
&\leq 2^{n\epsilon_n(1)} \max_{Q_{XYZ}:D(Q_{XY}\|P_X\times P_Y)<\sigma} \\
&\quad \left(2^{-nD(Q_{Z|XY}\|W_{Z|XY}|Q_{XY})} 2^{-nR_x} \sum_{i:(x^n(i),y^n)\in Q_{XY}} \frac{|Q_{Z|XY}(x^n(i),y^n)\bigcap \phi^{-1}(i)|}{|Q_{Z|XY}(x^n(i),y^n)|} \right) \\
&\leq 2^{n\epsilon_n(1)} \max_{Q_{XYZ}:D(Q_{XY}\|P_X\times P_Y)<\sigma} \left(2^{-nD(Q_{Z|XY}\|W_{Z|XY}|Q_{XY})} 2^{-n|R-I_Q(X;Z|Y)-\epsilon_n(2)|^+} \right) && (43) \\
&= 2^{-n(E-\epsilon_n)} && (44)
\end{aligned}
$$

where (43) follows Lemma 7. The rest are obvious by the method of types. □

Lemma 7: Extension of Lemma 3 in [6] from memoryless to conditional on y^n, for any $R \geq R_x > 0$, for any coding system $X(y^n)$ with joint input distribution $(x^n(i), y^n) \in Q_{XY}$, $i = 1, 2, ...2^{nR_x}$, and decoding rule $\phi : \mathcal{Z}^n \to \{1, 2, ..., 2^{nR_x}\}$, let $Q_{Z|XY}(x^n(i), y^n) = \{z^n : (x^n(i), y^n, z^n) \in Q_{XYZ}\}$ (this is the V-shell notation T_V used in [6]), we have:

$$
\frac{1}{2^{nR}} \sum_{i=1}^{2^{nR_x}} \frac{|Q_{Z|XY}(x^n(i),y^n)\bigcap \phi^{-1}(i)|}{|Q_{Z|XY}(x^n(i),y^n)|} \leq 2^{-n|R-I_Q(X;Z|Y)-\epsilon_n|^+} \tag{45}
$$

where $\epsilon_n = \epsilon(n, |\mathcal{X}|, |\mathcal{Y}|, |\mathcal{Z}|)$ converges to zero as n goes to infinity.

Proof: Write $Q_{Z|Y}(y^n) = \{z^n : (y^n, z^n) \in Q_{ZY}\}$. By the method of types [3], we know that

$$(n+1)^{-|\mathcal{Z}|}2^{nH_Q(Z|XY)} \leq |Q_{Z|XY}(x^n(i), y^n)| \leq 2^{nH_Q(Z|XY)}$$

$$\text{and } (n+1)^{-|\mathcal{Z}|}2^{nH_Q(Z|Y)} \leq |Q_{Z|Y}(y^n)| \leq 2^{nH_Q(Z|Y)}.$$

So the LHS of (45) is upper bounded by

$$\begin{aligned}
& \frac{1}{2^{nR}} \sum_{i=1}^{2^{nR_x}} \frac{|Q_{Z|XY}(x^n(i), y^n) \bigcap \phi^{-1}(i)|}{|Q_{Z|XY}(x^n(i), y^n)|} \\
\leq \; & (n+1)^{|\mathcal{Z}|}2^{-nH_Q(Z|XY)}2^{-nR} \sum_{i=1}^{2^{nR_x}} |Q_{Z|XY}(x^n(i), y^n) \bigcap \phi^{-1}(i)| \\
\leq \; & (n+1)^{|\mathcal{Z}|}2^{-nH_Q(Z|XY)}2^{-nR}|Q_{Z|Y}(y^n)| \qquad (46) \\
\leq \; & (n+1)^{|\mathcal{Z}|}2^{-nH_Q(Z|XY)}2^{-nR}(n+1)^{|\mathcal{Z}|}2^{nH_Q(Z|Y)} \\
= \; & 2^{-n(R - I_Q(X;Z|Y) - \epsilon_n)} \qquad (47)
\end{aligned}$$

(46) is true because $Q_{Z|XY}(x^n(i), y^n) \bigcap \phi^{-1}(i)$, $i = 1, 2, ..., 2^{nR_x}$ are disjoint and $\bigcup_i Q_{Z|XY}(x^n(i), y^n) \subseteq Q_{Z|Y}(y^n)$. Now notice that the LHS of (45) is at most $2^{n(R_x - R)} \leq 1$, hence the LHS of (45) is no bigger than 1. This together with (47), Lemma 7 is proved. □

2) Proof of (23): The proof is similar to that of (17). The difference is that we need the notion of jointlyg good code books. A code book pair (c_X, c_Y) is good if

$$|c_X \times c_Y \bigcap T_\sigma^C| \leq \frac{|c_X||c_Y|}{4} \qquad (48)$$

where the joint typical set $T_\sigma = \{(x^n, y^n) : D((x^n, y^n)\|P_X \times P_Y) < \sigma\}$. The rest of the proof are similar to that in the proof for (17). We conclude that

$$\Pr((\widehat{m}_x, \widehat{m}_y) = (m_x, m_y)) \quad \leq \quad \frac{1}{4} + 2^{-n(E - \epsilon_n)} + 4\sigma \qquad (49)$$

where $E = \min_{Q_{XYZ}: D(Q_{XY}\|P_X \times P_Y) < \sigma} D(Q_{Z|XY}\|W_{Z|XY}|Q_{XY}) + |R - I_Q(X, Y; Z)|^+ > 0$, for $R_x + R_y > I(X, Y; Z)$.

Again, we need to use a modified version of Lemma 5 and 3 from [6] to prove (49). The proof is extremely similar to those in Lemma 7 and 6. We omit the details here. □