Bounds on entanglement in qudit subsystems

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Quantum information is a rapidly developing field exploiting the peculiar entanglement properties of quantum states. Its applications can be broadly divided into two general types: quantum computation [1, 2], and quantum communication [3]. Quantum communication (including quantum cryptography and quantum teleportation) can be framed in terms of repeated use of pairs of entangled qubits – the entanglement properties of two qubits have been well characterised, and a number of analytical measures of entanglement are known [4–6]. However, for multi-qubit systems, which are essential for quantum computation, few entanglement measures can be calculated even for pure states. Despite these difficulties, arrays of qubits have been the focus of recent attention [7–12], though often only pairwise entanglement has been considered in such systems. The more general question of entanglement in multi-qudit systems (d–dimensional quantum particles) is also important since most real systems (*e. g.*, atoms) have more than two states.

A system of N qudits has a Hilbert space of size d^N. We will investigate entanglement in such systems by sampling randomly from the set of all pure N–qudit states and considering their average, or typical properties. This will give us useful information about the majority of states in Hilbert space. Having chosen a random pure state of N qudits, we can divide it into two subsystems by partial tracing. A subsystem of size m is obtained by partial tracing over the other $N-m$ qudits. This partitioning could represent the system of interest plus the environment, for example, or two different parts of one system. It is well-known that the two subsystems have the same entropy, $S_{m,N} = -\sum_j \lambda_j \log \lambda_j$, where the λ_j are the eigenvalues of the density matrix of the subsystem. The value of $S_{m,N}$ shows how entangled the two subsystems are with each other. If $S_{m,N} = 0$, there is no entanglement, but in general $S_{m,N} > 0$ and the subsystems are entangled. For a general pure quantum system divided into two parts with Hilbert spaces of dimensions M and K, with $M < K$, Page [13] conjectured (later proved [14, 15]) that on average (taken over the set of all pure states of MK qudits sampled according to the natural, unitarily invariant, Fubini–Study measure, induced by the Haar measure on the unitary group)

$$\langle S_{M,MK} \rangle = \sum_{j=K+1}^{MK} \frac{1}{j} - \frac{M-1}{2K} \simeq \ln M - \frac{M}{2K}, \tag{1}$$

where the approximation holds for $0 \ll M < K$. In our case, $M = d^m$ and $K = d^{N-m}$. On average, the smaller subsystem has nearly maximal entropy, showing that the two sybsystems are highly entangled with each other, a typical pure state is highly entangled, see also [16]. The value of $S_{m,N}$ provides a good measure of entanglement between the two subsystems, but says little about entanglement between the qudits within a single subsystem. It would be useful to know more about how the entanglement is distributed within the N–qudit system.

In recent work by Kendon *et. al.* [17], for the qubit case $d = 2$, it was shown numerically that entanglement in a mixed m–qubit subsystem of a typical N–qubit pure state falls off sharply towards zero with increasing N. For example, consider pairwise entanglement in pure states of N qubits randomly chosen with respect to the natural measure on the space of pure states. Pairwise entanglement can be measured in a mixed state using the Wootters concurrence formula [4]. For $N = 4$, numerical results showed 24% of the possible pairs have no entanglement, for $N = 5$, more than 80% have no entanglement and for $N = 6$, close to 100% of the pairs have no entanglement. The N qubit pure state is itself highly entangled, thus for $N > 5$ the entanglement must be concentrated in groups of qubits larger than two. This pattern was found to be repeated for subsystems of 3, 4 and 5 qubits, except that a simpler test for entanglement was used, a partial transpose [18] on the density matrix of the subsystem. This test does not distinguish bound entangled states [19, 20] from separable states, but since bound entanglement is relatively rare [21]), and for most purposes it is the free entanglement that is useful, this still provides a useful characterisation of entanglement.

These results were qualitatively explained in ref. [17] as follows. There is at least a ball of finite measure of separable states surrounding the maximally mixed state in the induced measure of these density matrices [5]. As N is increased for fixed m, the average entropy $\langle S_{m,N}\rangle$ becomes more nearly maximal, *i. e.*, the subsystem of size m becomes nearly maximally mixed, and at some finite N it will lie inside the ball of separable states. In [17] this finite value of N was only determined numerically for qubits ($d = 2$) for modest values of $m \leq 5$ and $N \leq 13$. In this article we derive a general analytical formula for an upper bound on N at which this mixed m–qudit subsystem changes from being entangled to being approximately disentangled (positive partial transpose).

We emphasise that we are interested in average properties of all possible pure states. It is easy to construct states with entanglement properties that lie outside our bounds. The so-called W states (symmetric superposition of one qubit in state $|1\rangle$ with the rest in state $|0\rangle$) have only pairwise entanglement [12] for any value of N, but such states are of small measure and do not contribute significantly to the average properties of pure states sampled from the whole of Hilbert space.

To derive our bounds, we will first examine how mixed a typical m–qudit state ρ_m might be. A convenient measure is the *inverse participation ratio* (IPR) [22] defined by

$$R(\rho_m) := 1/[\mathrm{tr}(\rho_m^2)]. \tag{2}$$

Pure states are defined to have $R(\rho_m) = 1$ while mixed states have $1 < R(\rho_m) \leq d^m$. We will also refer to $\mathrm{tr}(\rho_m^2) \equiv 1/R$ as the *purity* of the mixed state ρ_m. Thus larger R means more mixed, while larger purity means more nearly pure. It has been shown that all states ρ_m (within the d^m dimensional Hilbert space) for which

$$R \geq R_{\mathrm{PPT}} = d^m - 1 \tag{3}$$

have a positive partial transpose (PPT) [5] and thus are either separable, or have only bound entanglement. Although this result was originally established for bipartite systems ($m = 2$), an analogous statement may be obtained for the multiparticle case as follows. An m–particle system can be split into two parts containing j and $m - j$ particles respectively. The partial transpose is applied to the j particle part of the system producing $\rho_m^{T_j} := (\mathbf{1}_L \otimes T_j)\rho_m$. Here T_j denotes the transpose operation in the d^j dimensional subspace, while the dimension of the subspace that has not been transposed is $L = d^{m-j}$. We say that the m–qudit state is PPT if $\rho_m^{T_j}$ is positive for all possible values of j (it is suffucient to take $1 \leq j \leq m/2$) and choices of j particles, *i. e.*, with respect to *all* of its possible splitting into two subsystems. A simplified proof of the existence of the maximal ball of the PPT states for the bipartite systems sketched in [23] is based on the algebraic lemma of Mehta [24], which may also be used without change to obtain the result (3) for the multipartite case.

Consider random states drawn according to the natural measure on the space of pure states in a MK dimensional Hilbert space. In refs. [25–27] it was shown that the average IPR of a density matrix of size M obtained by partial tracing with respect to the K–dimensional subsystem is equal to $\langle R \rangle = (MK+1)/(M+K)$. This result does not depend on the particular way in which the MK–dimensional Hilbert space is composed, but only on the initial and the final dimensionality of the spaces. Thus it also holds in the problem we are analyzing here of pure states of N–qudits eventually reduced to m–qudits. In this case $M = d^m$ and $K = d^{N-m}$ so that the average IPR of the system reads

$$\langle R \rangle_{N,m} = \frac{d^N + 1}{d^m + d^{N-m}} \tag{4}$$

If the number of qubits is halved to $m = N/2$ by partial tracing, then the dimensions of both subspaces are equal, $M = K = d^m$, and the induced measure is equal to the Hilbert-Schmidt measure, which covers uniformly the entire $d^{2m} - 1$ dimensional body of mixed quantum states [26]. For example, in the case $N = 2$ qubits and $m = 1$, the measure induced by partial tracing covers uniformly the entire Bloch ball, the 3-dimensional space of one-qubit mixed quantum states. On the other hand, if the number of qubits left, $m > N/2$ then the measure induced by partial tracing covers only the boundary of the set of mixed states, the subspace of the states of non-maximal rank. In the opposite case, $m < N/2$, the probability of obtaining mixed states of high purity is negligible. Notice that $\langle R \rangle_{N,m} = \langle R \rangle_{N,N-m}$, but the probability of the states ρ_m and $\rho_{(N-m)}$ being PPT is *not* the same since $R_{\text{PPT}} = d^m - 1$ for ρ_m and $d^{(N-m)} - 1$ for $\rho_{(N-m)}$. Similarly, the smaller subsystem has nearly maximal entropy, see (1), but the larger one does not, though both subsystems have the same entropy.

Combining both eqs. (3) and (4) allows us to calculate an upper bound on the value of N, above which we expect almost all subsystems of m qudits will have a positive partial transpose. We find $\langle R \rangle_{N,m} \geq R_{\text{PPT}}$ when

$$d^N \geq d^{3m}(1 - d^{-m} - d^{-2m}). \tag{5}$$

Taking the logarithm (base d) gives us an estimate of how many qudits the initial pure state should contain, such that after the reduction to m–qudits the obtained mixed state is PPT on average,

$$N_{\text{PPT}} \geq 3m + \log_d(1 - d^{-m} - d^{-2m}). \tag{6}$$

The $\log_d$ term is always negative becoming rapidly smaller for either $m > 2$ or $d > 2$. Hence for systems initially consisting of

$$N \geq 3m \tag{7}$$

qudits we expect the subsystems of size m to be PPT mixed states with a considerable probability. For the simplest case of $m = d = 2$ the smallest integer number larger than the right hand side of (6) gives $N_{\text{PPT}} = \text{int}[6 - \log_2(16/11)] + 1 = 6$. For a fixed number N this establishes an upper bound on m below which subsystems of this size have on average no useful entanglement in an N–qudit pure state.

To estimate a lower bound on the transition from entangled to PPT subsystems, we need another relationship between N and m, this time with the condition ensuring the subsystems are definitely entangled rather than definitely PPT. There is a one parameter family of mixed states (known as the generalised Werner states) containing m qudits defined as a mixture of the maximally entangled pure state $|\Psi\rangle$ and the maximally mixed state I_{d^m},

$$\rho_\varepsilon = \frac{(1-\varepsilon)}{d^m} I_{d^m} + \varepsilon |\Psi\rangle\langle\Psi| \tag{8}$$

where

$$|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle_1 \otimes \ldots \otimes |i\rangle_m \tag{9}$$

and ε is a real parameter between 0 and 1 specifying the proportions of the mixture. There is no loss of generality here in choosing this parametrisation as we are simply constructing a set of minimally entangled states to obtain a bound. These states have been shown to be entangled for [28–30]

$$\varepsilon > \varepsilon_{ent} = \frac{1}{d^{m-1}+1}. \tag{10}$$

and strictly separable for $\varepsilon \leq \varepsilon_{ent}$ (there are no bound entangled states in this family). This boundary defines a set of entangled states of positive measure as can be seen as follows. The states ρ_ε lie on a line that is on an axis of rotational symmetry in the set of mixed states. This line intersects the set of separable states at $\varepsilon = \varepsilon_{ent}$. The set of separable states is convex, so beyond the hyperplane normal to the ρ_ε line there exists a set of entangled states of a positive measure.

It is straight forward to show that the inverse participation ratio for ρ_ε at this boundary point ($\varepsilon = \varepsilon_{ent}$) is given by

$$R_{ent} = \frac{(d^m + d)^2}{d^m + d^2 + 2d}. \tag{11}$$

Thus there exists a positive probability to encounter entangled states with $R < R_{ent}$. Note that this argument is weaker than (3), since there are also separable states with $R < R_{ent}$, *e. g.*, pure product states with $R = 1$.

Equating $\langle R\rangle_{N,m}$ from eq. (4) with R_{ent} from (11) allows us to establish an estimate for the entangled side of the transition between entangled and PPT states. It is easily shown that $\langle R\rangle_{N,m} \leq R_{ent}$ when

$$N_{ent} \leq 3m - 2 + \log_d[1 + (2d+1)d^{-m} + (d+2)d^{1-2m}]. \tag{12}$$

For all d and m, the $\log_d$ term is positive and tending to zero for increasing d and m. Hence for all finite N with

$$N \leq 3m - 2 \tag{13}$$

we expect the probability of finding the subsystem of size m entangled (not PPT), to be positive. Looking at the smallest case, $d = m = 2$, we have $N_{ent} = \mathrm{int}[4 + \log_2(11/4)] = 5$. Let us emphasize again, that this is an estimation only; for this (or lower) values of N the probability of finding entangled subsystems of size m is positive, but it does not rule out the existance of substantial numbers of PPT subsystems of size m, so the transition might occur effectively for even smaller values of N.

Let us fix the initial size of the pure states of N qudits and decrease the size of the final system of m qudits. For $m = N$ the probability of finding a separable pure state is equal to zero. As m is reduced, the states obtained by partial tracing over $N - m$ qudits become increasingly mixed, and the probability of finding PPT states increases. Putting both bounds together we may characterize quantitatively a transition region

$$3m - 2 \lesssim N \leq 3m, \tag{14}$$

which does not depend on the qudit size d, in which we estimate that PPT subsystems come to dominate over entangled subsystems. (Note that the inequalities have reversed because we are

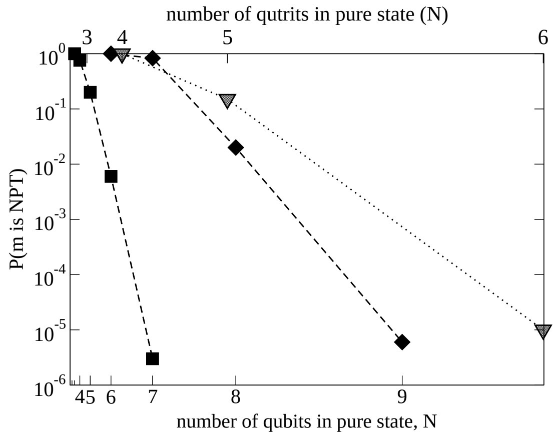

FIG. 1: Probability of finding that a subsystem of m qudits is entangled in random pure states sampled uniformly over the Haar measure for qubits for $m = 2$ (■) and $m = 3$ (filled ⋄), and qutrits for $m = 2$ (grey ▽).

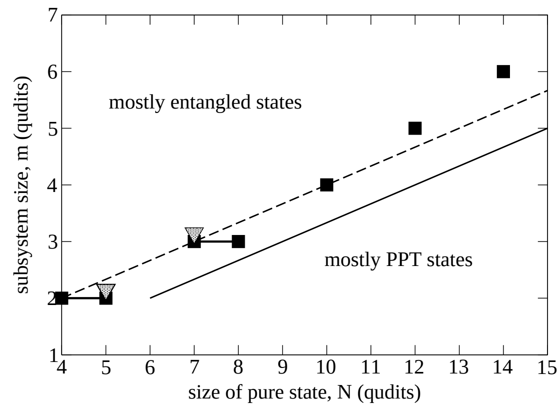

FIG. 2: Numerical results showing the transition from entangled to PPT subsystems. For each m, points are shown for which $P_{\rm PPT} \in (1\%, 99\%)$ for qubits (■) and qutrits (grey ▽). The solid/dashed lines represent upper/lower estimations, eq. (14).

identifying the region where the ratio of PPT to NPT (negative partial transpose) subsystems of size m is of order unity, rather than the regions where one case dominates over the other.) These are surprisingly tight estimates of the boundary between entangled and PPT states in terms of the number of qudits in the state. However, it must be remembered that the number of qudits is a logarithmic measure of the size of the Hilbert space. The addition of one qudit enlarges the Hilbert space by a factor of d.

We have checked the quality of the estimation (14) numerically by generating random pure states in the d^N dimensional Hilbert space, tracing over $N - m$ qudits, and analyzing whether the remaining mixed state, with a d^m–dimensional Hilbert space, has a positive or negative partial transpose (PPT or NPT). In fig. 1 the probability $P(\rho_m \text{is NPT})$ of finding an entangled subsystem of size $m = 2$ is shown for qubits ($d = 2$) and qutrits ($d = 3$), and $m = 3$ for qubits. The axes are scaled to show that the probability falls very rapidly with increasing N, the numerical results are consistent with $P(\rho_m \text{is NPT}) \propto \exp(-d^N)$, *i. e.*, decreasing exponentially in the size of the Hilbert space d^N. This supports out assertion that exceptions to our upper bound (7) do not contribute significantly to the average properties.

In fig. 2 we represent the transition region in the N–m plane for qubits up to $m = 6$, $N = 15$ and qutrits up to $m = 3$, $N = 8$. In general, the results obtained are in a fair agreement with the bounds (14) for small m. On the other hand, for larger m these estimations become not precise enough: as anticipated, the transition entanglement–separability at fixed m begins for N smaller than $N_{\rm ent} \approx 3m - 2$ and is completed by the number of qudits smaller than at $N_{\rm PPT} \approx 3m$.

In the above analysis we investigated the question of whether a random pure state has subsystems (obtained by partial tracing) with negative partial transpose. Estimating the probability of such an event we did not discuss the complementary question, *to what extent* the positivity is violated. This may be measured quantitatively by the *negativity*, $t_k = |\rho^{T_k}| - 1$ where $|.|$ denotes the trace norm, *i. e.*, the sum of the absolute values of all eigenvalues. By definition, for a PPT m-qudit system $t_k(\rho) = 0$ for $k = 1, \ldots m - 1$. For $m = 2$ there exists only one possible way of performing the partial transposition and one recovers $t = t_1$, the negativity defined originally for bipartite systems [5, 21], and recently shown to satisfy the properties of an entanglement monotone [31]. To investigate the relation between the degree of mixedness (IPR) and entanglement, it is instructive to plot the numerical data obtained for random states in the (t, R) plane.

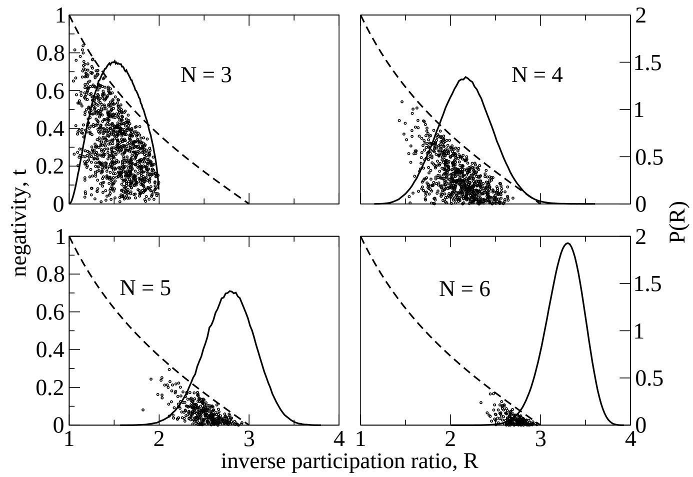

FIG. 3: Negativity t as a function of the inverse participation ratio R for mixed states of $m = 2$ qubits obtained by partial tracing of random pure states of $N = 3$, 4, 5 and 6 qubits. Dashed line denotes upper bound (15), solid line shows the numerically obtained probability distribution $P(R)$.

Results are shown in fig. 3 obtained for $m = 2$ qubits, while the initial number of qubits N ranged from 3 to 6. With increase of N the probability distribution $P(R)$ shifts toward larger vales of R, in agreement with the analytical result (4) for the expectation value $\langle R \rangle$. Simultaneously, the mean negativity decreases, and the probability of finding NPT states displays a transition from unity to zero as $\langle R \rangle$ sweeps between the values given by eqs. (3) and (11), which coincide in this case, $R_{ent} = R_{\rm PPT} = 3$. The shape of $P(R)$ also confirms that $\langle R \rangle$ does provide a good estimate of the actual value of R for a typical state (a distribution with two peaks would not, for example). For any fixed value of R the maximal negativity

$$t_{max}(R) = \frac{1}{2}\left(\sqrt{12/R - 3} - 1\right) \tag{15}$$

is obtained for a family of states which includes the Werner states ρ_ϵ, as well as the maximally entangled mixed states (MEM states) introduced in [32]. Keeping the purity fixed, these MEM states are maximally entangled with respect to concurrence (tangle or relative entropy) but with respect to negativity they are equally entangled as the Werner states. (see also discussion in [33, 34].)

Our results emphasise the importance of multipartite entanglement in quantum information processing. The types of states necessary for exploring the main bulk of Hilbert space with quantum computers are highly entangled, but the entanglement will not be evident if only a small subset of the qudits are examined. A detailed discussion of the role of entanglement in quantum computation using pure states is provided by Jozsa and Linden in [35]. Our results suggest additionally that the type of entanglement in a quantum computer must be highly multipartite, involving at least a third of the qudits in order to access a set of non-classical states of significantly larger measure than the set of classical states in its own Hilbert space. Pairwise entanglement will not be enough to harness the power of quantum computation.

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