The weak global dimension of Gaussian rings

Eman Alhassan

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Abstract

A Prüfer domain is defined as an integral domain for which each nonzero finitely generated ideal is invertible or, equivalently, projective. In general, if R is a domain, we have the following equivalent statements:

(i) R is semihereditary (see §2.6 for the definition).

(ii) w.gl.dim$(R) \leq 1$ (see §2.5 for the definition).

(iii) $R_{\mathfrak{m}}$ is a chain domain for every $\mathfrak{m} \in \mathrm{Max}(R)$ (see SS1.2,, 1.3 for definitions).

(iv) R is a Gaussian domain (see §3.1 for the definition).

(v) R is a Prüfer domain.

The definition of each class of ring featured in (i)–(v) can be extended to commutative rings which are not necessarily integral domains, i.e., which may have zero-divisors. However, there are examples showing that no two of (i)–(v) are equivalent in this more general setting. In fact we have strict implications

R is semihereditary $\Rightarrow$ w.gl.dim$(R) \leq 1 \Rightarrow R_{\mathfrak{m}}$ is a chain ring for every maximal ideal $\mathfrak{M}$ of $R \Rightarrow R$ is a Gaussian ring $\Rightarrow$ (v) R is a Prüfer ring.

We concentrate our studies on the weak global dimension of Gaussian rings. The authors of [BaGl] proposed a conjecture that there were only three possibilities for the weak global dimension of a Gaussian ring, namely 0, 1 or ∞. We follow the authors of [DT] by referring to this as the *Bazzoni–Glaz Conjecture*. In [Gl2, Theorem 2.2], it is shown that if the Gaussian ring R is a reduced ring (i.e., R is a ring for which the zero element

is the only nilpotent element) then R has weak global dimension at most 1, verifying the Conjecture in this case. The case of non-reduced Gaussian rings is given a great deal of attention in the 2011 preprint [DT]. The authors of [DT] prove the Conjecture is true using a number of concepts from homological algebra. We give details for some of these results and refer to articles and books for others. The proof in [DT] is quite long, involving several reduction steps to reach the final outcome.

In the Chapter 1 we introduce some well-established results from the ideal theory of commutative rings. For the most part, we will skip the proofs of these results and give references for them to the reader. This chapter is very important for the following chapters. We then look at some homological algebra definitions, results and methods in Chapter 2. In particular, this chapter will look at the weak global dimension of a ring R. The third chapter concentrates on the ideal structure of Gaussian rings, especially local Gaussian rings, detailing general properties of their internal structure. Finally, in Chapter 4, we give a detailed proof of the Bazzoni–Glaz Conjecture. This final lengthy chapter considers an all-inclusive number of cases of Gaussian rings and shows that the conjecture holds for every case.

To my mother who was like the guiding light of a candle,

a constant unwavering flame of love and support.

To my father who is leading me to show my best.

To my siblings and friends.

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Contents

Chapter 1

Ideal–Theoretic Preliminaries.

The aim of Chapter 1 is to give some definitions and state (often without proof) some crucial well-established results from the ideal theory of commutative rings which we will use later. When we omit a proof, at least one textbook reference will be given for it.

In what follows, R denotes a commutative, associative ring with nonzero identity 1 (unless otherwise specified).

1.1 $\mathrm{Spec}(R)$, $\mathrm{Nil}(R)$, and reduced rings.

Definition 1.1. The set of all prime ideals of a ring R is called the (**prime**) **spectrum** of R and is denoted by $\mathrm{Spec}(R)$.

The set of all maximal ideals of R is called the **maximal spectrum** of R and is denoted by $\mathrm{Max}(R)$.

A prime ideal $\mathfrak{P}$ of R is called **minimal** if it does not properly contain any other prime ideal of R.[1] The set of all minimal prime ideals of R is called the **minimal prime spectrum** of R and is denoted by $\mathrm{Min}(R)$.

[1]We use the notation $\mathfrak{P}$ if the prime ideal is minimal and P for an unspecified prime ideal.

Our first lemma is proved using a straightforward Zorn's Lemma argument on the partially ordered set $(\mathrm{Spec}(R), \supseteq)$. (See, for example, [LM, Lemma 2.15].)

Lemma 1.2. *Every prime ideal of the ring R contains a minimal prime ideal.*

Definition 1.3. An element a of a ring R is said to be **nilpotent** if there exists a natural number n such that $a^n = 0$. Similarly, an ideal I of R is called **nilpotent** if there exists $n \in \mathbb{N}$ such that $I^n = 0$ or, equivalently, there exists $n \in \mathbb{N}$ such that $x_1 x_2 \cdots x_n = 0$ for every sequence $x_1, x_2, \ldots, x_n$ of elements in I.

Definition 1.4. The set of all nilpotent elements of a ring R is called the **nilradical** of R and we denote it by $\mathrm{Nil}(R)$.

The following is a well-known result from commutative ring theory, but we sketch its proof.

Proposition 1.5. *For any ring R we have $\mathrm{Nil}(R) = \cap_{P \in \mathrm{Spec}(R)} P$, i.e., $\mathrm{Nil}(R)$ is the intersection of all the prime ideals of R.*

Proof. (Sketch.) Let a be a nonzero element of $\mathrm{Nil}(R)$ and suppose by way of contradiction that there is a prime ideal P of R for which $a \notin P$. Let n be the smallest positive integer for which $a^n = 0$. Then, since $n > 1$, we have $a \cdot a^{n-1} = a^n \in P$ so, since P is prime either $a \in P$ (a contradiction) or $a^{n-1} \in P$. Now consider $a^{n-1} = a \cdot a^{n-2} \in P$ to get $a^{n-2} \in P$ and continue in this way to finally get $a \in P$. Thus $\mathrm{Nil}(R) \subseteq \cap_{P \in \mathrm{Spec}(R)} P$.

For the reverse inclusion, assume to the contrary that $a \in \cap_{P \in \mathrm{Spec}(R)} P$ but $a \notin \mathrm{Nil}(R)$. Then $S = \{1, a^n : n \in \mathbb{N}\}$ is a multiplicatively closed subset of R. By a Zorn's Lemma argument, there is an ideal P of R maximal with respect to the property of being disjoint from S. Then one can show that P is a prime ideal. Since $a \notin P$, we have a contradiction. Hence $\cap_{P \in \mathrm{Spec}(R)} P \subseteq \mathrm{Nil}(R)$, as required. $\square$

Combining Lemma 1.2 with Proposition 1.5 gives

Corollary 1.6. *For any ring R we have* $\mathrm{Nil}(R) = \cap_{\mathfrak{P}\in \mathrm{Min}(R)}\mathfrak{P}$.

Definition 1.7. A ring is called **reduced** if 0 is its only nilpotent element.

Note that any integral domain R is reduced. (If $r \in R$ with $r \neq 0$ then $r^2 \neq 0$ and, proceeding inductively, $r^n \neq 0$ for any $n \in \mathbb{N}$.)

Definition 1.8. An ideal I of the ring R is called **nil** if every element of I is nilpotent.

Notice that we can deduce from these definitions that every nilpotent ideal is nil. However the converse is not always true. (Let p be a prime number. Consider $R = \prod_{n\geq 2} \mathbb{Z}_{p^n}$ with ideal $I = \oplus_{n\geq 2} J_n$ where J_n is the unique maximal ideal of $\mathbb{Z}_{p^n}$.)

1.2 Local rings and localisation.

Definition 1.9. A ring R is called **local** if it has exactly one maximal ideal.[2]

We follow common practice by saying that $(R, \mathfrak{m})$ is a local ring when R has $\mathfrak{m}$ as its unique maximal ideal.

The following characterization of local rings is well-known. (See for example [AF, Bl].)

Lemma 1.10. *The following statements are equivalent for a ring R.*

(i) *R is a local ring.*

(ii) *For any $r \in R$, either r is a unit or $1 - r$ is a unit.*

(iii) *The set of non-units of R is closed under addition.*

(iv) *The set of non-units of R is an ideal of R.*

The next two lemmas show how any local ring can be used to manufacture more local rings. The proof of the first lemma is easy, so omitted.

[2] Some authors use *quasi-local* instead of *local*, reserving *local ring* to mean a *Noetherian ring with exactly one maximal ideal.*

Lemma 1.11. *If I is a proper ideal of the local ring $(R, \mathfrak{m})$, then the factor ring R/I is also local, with unique maximal ideal $\mathfrak{m}/I$.*

Lemma 1.12. *Let $(R, \mathfrak{m})$ be a local ring, x an indeterminate, and $n \in \mathbb{N}$. Then $R[x]/(x^n)$, the polynomial ring $R[x]$ factored by its ideal generated by x^n, is local, with $\{f(x) + (x^n) : f(x) \in R[x], f(0) \in \mathfrak{m}\}$ as its unique maximal ideal.*

Proof. Let $f(x) + (x^n)$ and $g(x) + (x^n)$ be elements of $R[x]/(x^n)$. Then we may write $f(x)+(x^n) = a_0+a_1x+\cdots+a_{n-1}x^{n-1}+(x^n)$, $g(x)+(x^n) = b_0+b_1x+\cdots+b_{n-1}x^{n-1}+(x^n)$ where each $a_i, b_i \in R$. Then $f(x) + (x^n)$ is a unit in $R[x]/(x^n)$ with inverse $g(x) + (x^n)$ if and only if

$$a_0b_0 + (a_0b_1 + a_1b_0)x + \cdots + \sum_{i\geq 0, j\geq 0, i+j=n-1} a_ib_jx^{n-1} + (x^n) = 1 + (x^n).$$

Comparing coefficients of each power of x on both sides of this equation gives

$$a_0b_0 = 1, a_0b_1 + a_1b_0 = 0 = a_0b_2 + a_1b_1 + a_2b_0 = \cdots = \sum_{i\geq 0, j\geq 0, i+j=n-1} a_ib_j$$

and so $a_0 \notin \mathfrak{m}$ with $b_0 = a_0^{-1}$, $b_1 = -a_0^{-1}(a_1b_0) = -(a_0^{-1})^2a_1$, $b_2 = -a_0^{-1}(a_1b_1 + a_2b_0) = -a_0^{-1}(-(a_0^{-1})^2a_1^2 + a_0^{-1}a_2)$ and, continuing in this way, each b_i can be expressed using the a_j terms. This shows that $a_0 \notin \mathfrak{m}$ is necessary and sufficient for $f(x) + (x^n)$ to be a unit and so $f(0) \in \mathfrak{m}$ characterizes non-units in $R[x]/(x^n)$. It is then clear that the sum of any two non-units in $R[x]/(x^n)$ is also a non-unit and so $R[x]/(x^n)$ is local, by Lemma 1.10. □

Definition 1.13. An element $a \in R$ is said to be a **proper zero-divisor** if $a \neq 0$ and $ab = 0$ for some nonzero element b in R. On the other hand, an element $r \in R$ is called **regular** if $r \neq 0$ and r is not a proper zero-divisor, i.e., given $s \in R$ with $sr = 0$ then $s = 0$.

Definition 1.14. A **multiplicatively closed subset** (m.c.s. for short) S of a ring R is a subset of R which contains 1 but not 0 and is closed under multiplication of its elements.

Examples 1.15. (1) It is easy to see that the set S of all regular elements in R is an m.c.s. of R.

(2) Given any prime ideal P of R, the set $S = R \setminus P$ is an m.c.s. since:

(i) $0_R \notin S$ since $0_R \in P$.

(ii) $1_R \in S$ since $1_R \notin P$ because $P \neq R$.

(iii) If $s, t \in S$ then $s \notin P$ and $t \notin P$ so $st \notin P$ since P is a prime ideal and therefore $st \in S$.

Definition 1.16. Let S be a multiplicatively closed subset of the ring R. We define an equivalence relation $\sim$ on $R \times S$ by setting $(r_1, s_1) \sim (r_2, s_2)$ if there exists $t \in S$ such that $t(r_1 s_2 - r_2 s_1) = 0$. Denote the equivalence class of (r, s) by $\frac{r}{s}$ and the set of all equivalence classes of $R \times S$ with respect to $\sim$ by R_S. One can make the set R_S into a commutative ring called the **quotient ring of** R **with respect to** S with identity $\frac{1}{1}$ and zero $\frac{0}{1}$ by defining addition and multiplication as follows:

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r_1 s_2 + r_2 s_1}{s_1 s_2}, \quad \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2}$$

for all $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in R_S$. For detailed verification that R_S is indeed a commutative ring, we refer to [LM, Chapter III].

Examples 1.17. (1) Let P be a prime ideal of the commutative ring R and $S = R \setminus P$. Then we can form the quotient ring of R with respect to S. In this case, the special notation R_P is used instead of $R_{R \setminus P}$ and R_P is called the **localisation** of R at P .

(2) Since any maximal ideal M of a commutative ring R is prime, we can form the multiplicatively closed subset $S = R \setminus M$ and then the quotient ring R_M as a special case of example (1).

(3) Let S be the multiplicatively closed subset of all regular elements of the ring R. Then the ring R_S is denoted by $Q(R)$ and called the **total quotient ring** or **total ring of quotients** of R. Thus

$$Q(R) = \left\{ \frac{r}{s} : \ r, s \in R, \ s \text{ is regular} \right\}.$$

Note that if R is an integral domain, then $Q(R)$ is simply the quotient field of R.

Lemma 1.18. *Let P be a prime ideal of the ring R. Then the localisation R_P of R at P is a local ring with maximal ideal PR_P.*

Proof. Any element in R_P is of the form $\frac{x}{s}$, where $x \in R$, and $s \in R \setminus P$. If $\frac{x}{s} \notin PR_P$ then $x \notin P$ and so $x \in R \setminus P$. Thus, $\frac{s}{x}$ is defined in R_P and it is straightforward to show that it is the inverse of $\frac{x}{s}$. Now suppose that $\frac{y}{t} \in PR_P$. Then we have $\frac{y}{t} = \frac{z}{u}$ for some $z \in R$, $u \in R \setminus P$. If $\frac{z}{u}$ is a unit, then there exists $\frac{w}{v}$ with $w \in R$ and $v \in R \setminus P$ such that $\frac{z}{u} \cdot \frac{w}{v} = 1$. Hence, there is an element $q \in R \setminus P$ such that $qzw = quv$. However, $quv \in R \setminus P$ and $qzw \in P$, a contradiction. This shows that the set of units in R_P is $R_P \setminus PR_P$.

Now we show that the set of non-units is closed under addition. To this end take $\frac{p_1}{s_1}, \frac{p_2}{s_2} \in PR_P$. Then $\frac{p_1}{s_1} + \frac{p_2}{s_2} = \frac{p_1 s_2 + p_2 s_1}{s_1 s_2} \in PR_P$ since $p_1 s_2 + p_2 s_1 \in P$. □

Notes.

(1) For any m.c.s. S of R, we can consider R as a subring of R_S, identifying it with

$$\left\{ \frac{r}{1} : \ r \in R \right\} \subseteq R_S.$$

(2) It can be easily shown that R_S is an R-module with module multiplication given by $r\frac{r_1}{s_1} = \frac{rr_1}{s_1}$ for every $r, r_1 \in R$ and $s_1 \in S$.

Definition 1.19. Let S be an m.c.s. of the commutative ring R. Then every R-module M gives rise to an R_S-module M_S as follows. We define an equivalence relation on $M \times S$ given by $(m_1, s_1) \sim (m_2, s_2)$ if there is an element $t \in S$ such that $ts_1m_2 = ts_2m_1$. The equivalence class represented by (m, s) is denoted by $\frac{m}{s}$ and we set

$$M_S = \left\{ \frac{m}{s} : m \in M, s \in S \right\},$$

called the **extension** or **localisation** of the R-module M at S. This becomes an R_S module with addition and module multiplication given by:

$$\frac{m_1}{s_1} + \frac{m_2}{s_2} = \frac{s_1 m_1 + s_1 m_2}{s_1 s_2} \text{ and } \frac{r_1}{s_1}\frac{m_1}{s_2} = \frac{r_1 m_1}{s_1 s_2},$$

where $m_1, m_2 \in M$, $s_1, s_2 \in S$, and $r_1 \in R$.

In particular, if I is an ideal of R, then regarding I as an R-module, we get an ideal I_S of the quotient ring R_S.

Notes.

(1) It should be noted that given any ideal A of the ring R_S there exists an ideal I of R such that $A = I_S$. In other words, every ideal of R_S is an extension of some ideal I of R. In fact $I = \left\{ r \in R : \frac{r}{1} \in A \right\}$.

(2) If Q is an ideal of the ring R and P is a prime ideal of R then the notation $R_P Q$ is used for the extension of Q instead of Q_P. In particular, this avoids the notation P_P.

Definition 1.20. For any ideals I, J of the ring R, one can define $I : J$, the **residual** of I by J, which is also an ideal of R, given by:

$$I : J = \{r \in R : rJ \subseteq I\}.$$

The results from Theorem 1.21 to Theorem 1.27 record some useful results for rings of quotients. Details can be found in [LM, Chapter III].

Theorem 1.21. *Let S be a multiplicatively closed subset of the ring R. If A, B are ideals of R then:*

(i) $(A + B)_S = A_S + B_S$.

(ii) $(AB)_S = A_S B_S$.

(iii) $(A \cap B)_S = A_S \cap B_S$.

(iv) $(A : B)_S = A_S : B_S$ *if B is finitely generated.*

Theorem 1.22. *Let S be an m.c.s. of the ring R and I be an ideal of R. Then $I_S = R_S$ if and only if $I \cap S \neq \emptyset$.*

The next theorem gives useful information on the ideal P_S when $P \in \mathrm{Spec}(R)$.

Theorem 1.23. *Let S be an m.c.s. of the ring R and P be a prime ideal of R. Then the ideal P_S of R_S is a prime ideal if $P \cap S = \emptyset$; otherwise, $P_S = R_S$.*

Proposition 1.24 (The Local-Global Principle). *Let A, B be ideals of the ring R. Then $A = B$ if and only if $A_M = B_M$ for every maximal ideal M of R, where A_M and B_M are the extensions of A and B, respectively, at the maximal ideal M.*

Regarding R as a subring of R_S, we now define an operation which undoes the extension of an ideal I of R.

Lemma 1.25. *Let S be an m.c.s. of the ring R and J be an ideal of R_S. Take $\hat{J} = J \cap R$. Then $\hat{J}$ is an ideal of R with $\hat{J}_S = J$. The ideal $\hat{J}$ is called the* **contraction** *of the ideal J.*

The next two results shows that contractions get along well with prime ideals.

Theorem 1.26. *Let S be an m.c.s. of the ring R and Q be a prime ideal of R_S. Then the contraction $\hat{Q}$ of Q is a prime ideal of R.*

Theorem 1.27. *Let S be an m.c.s. of the ring R and $\mathcal{P}$ be the collection of all prime ideals of R having empty intersection with S. Let $\mathcal{P}_S$ be the collection of all prime ideals of the ring R_S. Then the mapping $\phi : \mathcal{P} \longrightarrow \mathcal{P}_S$, given by $\phi(P) = P_S$ for all $P \in \mathcal{P}$, is both onto and one-to-one. The inverse of ϕ is given by $Q \mapsto \hat{Q}$ for all $Q \in P_S$.*

Lemma 1.28. *If $\mathfrak{P}$ is a minimal prime ideal of the ring R then the ideal $\mathfrak{P}R_{\mathfrak{P}}$ is the only prime ideal of the localisation $R_{\mathfrak{P}}$.*

Proof. Suppose $\overline{\mathfrak{Q}}$ is another prime ideal of $R_{\mathfrak{P}}$. Then either (i) $\overline{\mathfrak{Q}} \not\subseteq \mathfrak{P}R_{\mathfrak{P}}$ or (ii) $\overline{\mathfrak{Q}} \subsetneq \mathfrak{P}R_{\mathfrak{P}}$. If (i) holds, then there exists $q \in \mathfrak{Q} \setminus \mathfrak{P}$ and $s \in R \setminus \mathfrak{P}$ such that $\frac{q}{s} \in \overline{\mathfrak{Q}} \setminus \mathfrak{P}R_{\mathfrak{P}}$. This gives $\frac{s}{q} \in R_{\mathfrak{P}}$ and so $\frac{1}{1} = \frac{s}{q}\frac{q}{s} \in \overline{\mathfrak{Q}}$ and so $\overline{\mathfrak{Q}} = R_{\mathfrak{P}}$, a contradiction. If (ii) holds, let $\mathfrak{Q} = \{q \in R : \frac{q}{1} \in \overline{\mathfrak{Q}}\}$. It is straightforward to see that $\mathfrak{Q}$ is a prime ideal of R contained in $\mathfrak{P}$ with $\mathfrak{Q}R_{\mathfrak{P}} = \overline{\mathfrak{Q}}$. Since $\mathfrak{P}$ is minimal, we have $\mathfrak{Q} = \mathfrak{P}$. This gives $\overline{\mathfrak{Q}} = \mathfrak{P}R_{\mathfrak{P}}$, again a contradiction. Thus $\mathfrak{P}R_{\mathfrak{P}}$ is the only prime ideal of $R_{\mathfrak{P}}$. □

Definition 1.29. Let M be a module over the ring R and X be a nonempty subset of M. Then the **annihilator** of X is defined to be

$$\operatorname{ann}_R(X) = \{r \in R : rx = 0 \text{ for all } x \in X\}.$$

When there is no risk of confusion, we simply write $\operatorname{ann}(X)$. It is straightforward to see that, if X is an ideal of R, then $\operatorname{ann}(X)$ is also an ideal of R.

When $X = \{x\}$, a singleton subset of M, we simplify the annihilator notation to $\operatorname{ann}_R(x)$ or $\operatorname{ann}(x)$. Each $\operatorname{ann}(x)$ is an ideal of R.

Lemma 1.30. *Let M be a module over the ring R and $a \in M$. Then, for any m.c.s. of R, in M_S we have*

$$\operatorname{ann}_{R_S}\left(\frac{a}{1}\right) = (\operatorname{ann}_R(a))_S.$$

Proof. If $x \in (\operatorname{ann}_R(a))_S$ then $x = \frac{b}{s}$, where $b \in R$ with $ba = 0$ and $s \in S$ and so $x\frac{a}{1} = \frac{b}{s} \cdot \frac{a}{1} = \frac{ba}{s} = 0$, i.e., $x \in \operatorname{ann}_{R_S}\left(\frac{a}{1}\right)$. Thus $(\operatorname{ann}_R(a))_S \subseteq \operatorname{ann}_{R_S}\left(\frac{a}{1}\right)$.

Conversely, if $y \in \operatorname{ann}_{R_S}\left(\frac{a}{1}\right)$, then $y = \frac{c}{t}$, where $c \in R$, $t \in S$, and $\frac{c}{t} \cdot \frac{a}{1} = \frac{0}{1}$. This implies that $cau = 0$ for some $u \in S$. Thus $y = \frac{c}{t} = \frac{cu}{tu} \in (\operatorname{ann}_R(a))_S$ since $cu \in \operatorname{ann}_R(a)$ and $tu \in S$ and this proves the equality. □

Lemma 1.31. *Let R be a local ring and S be an m.c.s. of R. Then* $(\operatorname{Nil}(R))_S = \operatorname{Nil}(R_S)$.

Proof. To simplify the notation set $A = (\operatorname{Nil}(R))_S$ and $B = \operatorname{Nil}(R_S)$. If $x \in A$ then $x = \frac{r}{s}$ for some $r \in R$, $s \in S$ and $r^n = 0$ for some $n \geq 1$. This gives $x^n = \frac{r^n}{s^n} = \frac{0}{s^n} = 0$. Hence

$x \in \text{Nil}(R_S) = B$ and therefore $A \subseteq B$. Conversely, if $y \in B$ we have $y = \dfrac{r}{s}$ for some $r \in R$, $s \in S$ and $y^n = 0$ for some $n \in \mathbb{N}$. Then $\dfrac{r^n}{s^n} = \dfrac{0}{1}$ and so $tr^n = 0$ for some $t \in S$. Thus $(tr)^n = 0$, giving $tr \in \text{Nil}(R)$. Hence $y = \dfrac{tr}{ts} \in (\text{Nil}(R))_S = A$, as required. □

The following result is immediate from Lemma 1.31.

Corollary 1.32. *Let R be a reduced ring. Then the localisation R_P is also reduced for any prime ideal P of R.*

Definition 1.33. Let S be an m.c.s. of the ring R and $\psi_S : R \to R_S$ be the natural ring homomorphism given by $r \mapsto \dfrac{r}{1}$ for all $r \in R$. We define

$$O_S = \ker(\psi_S) = \{r \in R : ru = 0 \ \text{ for some } u \in S\}.$$

In the special case of $S = R \setminus P$ where P is a prime ideal of R, we denote O_S by O_P.

Lemma 1.34. *Let S be an m.c.s. of the ring R. Then O_S is an ideal of R and there is a ring monomorphism $\phi_S : R/O_S \longrightarrow R_S$.*

Proof. To show that O_S is an ideal of R, let $r, s \in O_S$. Then there exists $u, v \in S$ such that $ur = 0 = vs$. Since S is an m.c.s. we get $uv \in S$. Moreover, since $uv(r - s) = 0$, we get $r - s \in O_S$ and this shows that O_S is closed under subtraction in R.

Now let $r \in O_S$ and $t \in R$. Then taking $u \in S$ with $ru = 0$, we get $u(tr) = 0$. Therefore $tr \in O_S$. Hence O_S is an ideal of R.

To show the existence of the ring monomorphism we define $\phi_S : R/O_S \longrightarrow R_S$ by $\phi_S(r + O_S) = \dfrac{r}{1}$. Then ϕ_S is well-defined since if $r + O_S = s + O_S$ then $r - s \in O_S$ and so $u(r - s) = 0$ for some $u \in S$. Then $ur = us$ and $\phi_S(r + O_S) = \dfrac{r}{1} = \dfrac{ur}{u} = \dfrac{us}{u} = \dfrac{s}{1} = \phi_S(s + O_S)$.

Furthermore, ϕ_S is an R-homomorphism since for $r + O_S, s + O_S \in R/O_S$ and $t \in R$ we have:

(i) $\phi_S[(r + O_S) + (s + O_S)] = \phi_S(r + s + O_S) = \dfrac{r + s}{1} = \dfrac{r}{1} + \dfrac{s}{1} = \phi_S(r + O_S) + \phi_S(s + O_S)$.

(ii) $t \cdot \phi_S(r + O_S) = t\dfrac{r}{1} = \dfrac{tr}{1} = \phi_S(tr + O_S)$.

Now suppose $\phi_S(r+O_S) = \dfrac{0}{1}$. Then $\dfrac{r}{1} = \dfrac{0}{1}$ in R_S. Thus there exists $u \in S$ such that $ur = 0$. This gives $r \in O_S$ and so $r + O_S = 0$. Hence ϕ_S is an R-monomorphism. □

The following lemma from [Gl1, Lemma 3.3.4] is an example of a result that is proved using the localisation process although the result itself has no mention of the process.

Lemma 1.35. *Let R be a reduced ring and $P \in \mathrm{Spec}(R)$. Then $P \in \mathrm{Min}(R)$ if and only if $\mathrm{ann}(x) \nsubseteq P$ for all $x \in P$. In this case $P = O_P$.*

Proof. Suppose $\mathrm{ann}(x) \nsubseteq P$ for every $x \in P$. We want to show that P is a minimal prime ideal of R. To this end, suppose there exists $Q \in \mathrm{Spec}(R)$ with $Q \subseteq P$. If $Q \neq P$, then there exists $x \in P \setminus Q$. Then $\mathrm{ann}(x) \nsubseteq P$ and so there exists $r \in \mathrm{ann}(x)$ such that $r \notin P$. It follows that $rx = 0 \in Q$. However this can not be the case since $x \notin Q$ and $r \notin Q$ because $Q \subseteq P$ and $r \notin P$. This contradiction gives $P = Q$. Hence $P \in \mathrm{Min}(R)$.

Conversely, suppose $P \in \mathrm{Min}(R)$ and let $x \in P$. Notice that, by Lemma 1.28, the ideal PR_P is the only prime ideal in the localisation R_P and so $\mathrm{Nil}(R_P) = PR_P$. Thus, by Proposition 1.5, PR_P is the set of nilpotent elements of R_P. Since $x \in P$, we have $\dfrac{x}{1} \in PR_P$ and so $\left(\dfrac{x}{1}\right)^n = \dfrac{0}{1}$ for some $n \in \mathbb{N}$. Then there exists $u \in R \setminus P$ such that $ux^n = 0$. This gives $(ux)^n = 0$ in R and so $ux = 0$ since R is reduced. Hence $u \in \mathrm{ann}(x)$ and since $u \in R \setminus P$ we obtain $\mathrm{ann}(x) \nsubseteq P$.

To finish the proof, we show that in this case we have $P = O_P$. Let $x \in P$. Then $\mathrm{ann}(x) \nsubseteq P$, so $xu = 0$ for some $u \notin P$. This implies that $x \in O_P$ and so $P \subseteq O_P$. If conversely $x \in O_P$, then $xu = 0$ for some $u \notin P$. Thus, since P is prime, we must have $x \in P$. Hence $O_P \subseteq P$ which gives the equality. □

1.3 Chain rings and arithmetical rings.

Definition 1.36. Let I and J be ideals of the ring R. If either $I \subseteq J$ or $J \subseteq I$ then I and J are said to be **comparable**.

If I and J are comparable for all ideals I and J of R then R is called a **chain ring** or **valuation ring**.

The proof of the next result is straightforward. (See for example [LM, Proposition 5.2].)

Proposition 1.37. *Let R be a ring.*

(a) *The following statements are equivalent.*

(i) *R is a chain ring.*

(ii) *If $I = Ra$ and $J = Rb$ are any two principal ideals of R, then I and J are comparable.*

(iii) *Given any $a, b \in R$, there is an $r \in R$ such that either $a = rb$ or $b = ra$.*

(b) *If R is a chain ring then*

(i) *every finitely generated ideal of R is principal,*

(ii) *R is a local ring,*

(iii) *R has a unique minimal prime ideal, $\mathfrak{P}$ say, and*

(iv) $\mathrm{Nil}(R) = \mathfrak{P}$.

The next definition is due to Fuchs [F].

Definition 1.38. A ring R is called **arithmetical** if

$$I \cap (J + K) = (I \cap J) + (I \cap K) \text{ for all ideals } I, J \text{ and } K \text{ of } R.$$

Definition 1.39. Let I be an ideal of the ring R. If I_M is a principal ideal in the localisation R_M for every maximal ideal M of R, then I is said to be **locally principal**.

The next result is due to Jensen [J1, J2]. (See also [LM, Exercise 6.18].)

Proposition 1.40. *The following statements are equivalent for a ring R.*

(i) *R is an arithmetical ring.*

(ii) *$I + (J \cap K) = (I + J) \cap (I + K)$ for all ideals I, J and K of R.*

(iii) *R_M is a chain ring for every maximal ideal M of R.*

(iv) *R_P is a chain ring for every $P \in \mathrm{Spec}(R)$.*

(v) *Every finitely generated ideal of R is locally principal.*

1.4 Idempotent elements and idempotent ideals.

Definition 1.41. An element e of the ring R is said to be **idempotent** if $e^2 = e$.

Similarly, an ideal I of R is **idempotent** if $I^2 = I$, i.e., given any $x \in I$, there are $y_1, \ldots, y_n, z_1, \ldots, z_n \in I$ such that $x = y_1 z_1 + \cdots + y_n z_n$.

Notes. (1) If e is an idempotent of a ring R then $1 - e$ is also an idempotent since $(1-e)^2 = 1 + e^2 - 2e = 1 + e - 2e = 1 - e$. Moreover, $(1-e)e = e - e^2 = e - e = 0$.
(2) In a local ring R, 0 and 1 are the only idempotent elements. To see this, note first that, in any ring R, if e is both a unit and an idempotent then $e = 1$ since $e = 1e = e^{-1}ee = e^{-1}e = 1$. Since our R is local, given any $e \in R$, by Lemma 1.10 either e or $1 - e$ is a unit so, if e is an idempotent, either $e = 1$ or $1 - e = 1$. Thus $e = 1$ or 0. □

The following theorem is a special case of [KOTS, Theorem 2.17].

Theorem 1.42. *Let I be a finitely generated ideal of the ring R. If I is an idempotent ideal, then $I = Re$ for some idempotent element $e \in R$.*

Proof. We suppose that I is generated by $a_1, a_2, \ldots, a_n \in R$ and use induction on n.

If $n = 1$ then $I = Ra_1$, where $a_1 \in R$. Then, since I is an idempotent, we have $I = I^2 = Ra_1^2$. Thus $a_1 = ra_1^2$ for some $r \in R$. Then $a_1(1 - ra_1) = 0$ and so $I(1 - ra_1) = Ra_1(1 - ra_1) = 0$. In particular, $ra_1(1 - ra_1) = 0$ and so $(ra_1)^2 = ra_1$, i.e., ra_1 is an idempotent element. Moreover $I = Rra_1$ since $Rra_1 \subseteq I$ and $I = Ra_1 = Ra_1^2 = R(ra_1)^2 \subseteq Rra_1$. This establishes the result for the case when $n = 1$.

Now suppose the result holds for all rings R and all idempotent ideals of R generated by $n - 1$ elements, for some $n \geq 2$. Let $I = Ra_1 + Ra_2 + \cdots + Ra_n$ with $I^2 = I$. Consider the factor ideal I/Ra_1 in the factor ring R/Ra_1. Then I/Ra_1 is generated by the $n - 1$ elements $a_2 + Ra_1, a_3 + Ra_1, \ldots, a_n + Ra_1$ and $(I/Ra_1)^2 = I^2/Ra_1 = I/Ra_1$, i.e., I/Ra_1 is idempotent. By the induction hypothesis, I/Ra_1 is generated by an idempotent element in R/Ra_1, say $I/Ra_1 = (R/Ra_1)(e + Ra_1)$, where $(e + Ra_1)^2 = e + Ra_1$. Note that $e \in I$ and $e^2 - e \in Ra_1$. Moreover $I = Ra_1 + Re$ since:

(i) $Ra_1 + Re \subseteq I$ because $a_1, e \in I$ and

(ii) $I \subseteq Ra_1 + Re$ because if $x \in I$ then $x + Ra_1 \in I/Ra_1 = (R/Ra_1)(e + Ra_1)$. Hence $x + Ra_1 = (r + Ra_1)(e + Ra_1) = re + Ra_1$ for some $r \in R$. Thus $x - re \in Ra_1$ and so $x \in Ra_1 + Re$.

Now $a_1 \in I = I^2 = (Ra_1 + Re)^2 = Ra_1^2 + Ra_1e + Re^2$, so $a_1(1 - e) \in (Ra_1^2 + Ra_1e + Re^2)(1 - e) \subseteq Ra_1^2(1 - e) + Ra_1e(1 - e) + Re^2(1 - e) \subseteq Ra_1^2 + Ra_1e$, using $e - e^2 \in Ra_1$. Again using $e - e^2 \in Ra_1$, we now get $a_1(1 - e)^2 \in Ra_1^2$. Hence there exists $t \in R$ such that $a_1(1 - e)^2 = a_1^2t$ and so

$$a_1((1 - e)^2 - ta_1) = 0 \qquad (*).$$

However, $I(1-e) = (Ra_1+Re)(1-e) \subseteq Ra_1(1-e) \subseteq Ra_1$ (since $e-e^2 \in Ra_1$). Thus, setting $f = 1-(1-e)[(1-e)^2-a_1t]$, we get $I(1-f) = I(1-e)[(1-e)^2-a_1t] \subseteq Ra_1((1-e)^2-a_1t)$ since $I(1 - e) \subseteq Ra_1$. Then by $(*)$ we get $I(1 - f) = 0$. Moreover, f is an element of I since $f = 1 - (1 - e)[(1 - e)^2 - a_1t] = 3e - 3e^2 + e^3 - (1 - e)a_1t \in Ra_1 + Re = I$. Since

$I(1-f)=0$, given any $x \in I$ we have $x(1-f)=0$ and so $x = xf$. From this it follows that $I = Rf$ and, since $f(1-f)=0$, f is an idempotent. □

The next lemma appears as [FS, Chapter II, Lemma 1.3 (c)].

Lemma 1.43. *Let R be a chain ring and let I be a proper ideal of R. Then either I is nilpotent or $P = \cap_{n\in\mathbb{N}} I^n$ is a prime ideal.*

Proof. Suppose I is not a nilpotent ideal. Let $x, y \in R \setminus P$. Then there exist $m, n \in \mathbb{N}$ such that $x \notin I^m$ and $y \notin I^n$. In particular, $Rx \nsubseteq I^m$ and $Ry \nsubseteq I^n$. Since R is a chain ring, we must then have $I^m \subset Rx$ and $I^n \subset Ry$. This gives $I^{m+n} \subseteq Rxy$.

We aim to prove that $xy \notin P$ in order to show that P is prime. To this end we show that $Rxy \neq I^{m+n+1}$. Suppose to the contrary that $Rxy \subseteq I^{m+n+1}$. Since $I^{m+n} \subseteq Rxy$, we get $I^{m+n+1} \subseteq I^{m+n} \subseteq Rxy \subseteq I^{m+n+1}$ and so $I^{m+n+1} = I^{m+n} = Rxy$. Notice that since $I^{m+n+1} = I^{m+n}$ we have $I^{m+n} = I^{m+n+1} = I^{m+n+2} = \cdots = I^{2(m+n)} = (I^{m+n})^2$ and so $I^{m+n} = Rxy$ is a finitely generated idempotent ideal. Then, by Theorem 1.42, $I^{m+n} = Rxy = Re$ for some idempotent element in R. Because R is a chain ring, Note (2) on page 13 shows that e is either 1 or 0. If $e = 0$ then $I^{m+n} = Rxy = 0$, contradicting the assumption that I is not nilpotent. If otherwise, $e = 1$ then $I^{m+n} = Rxy = R$ and therefore I is not proper, another contradiction. Hence $xy \notin P$ as required. □

The following result is stated on [FS, page 357].

Lemma 1.44. *Let R be a chain ring. Then the ideal* $\mathrm{Nil}(R)$ *is either idempotent or nilpotent.*

Proof. Let $N = \mathrm{Nil}(R)$. By Lemma 1.43, either N is nilpotent or $J = \cap_{n\in\mathbb{N}} N^n$ is a prime ideal. If N is neither idempotent nor nilpotent, then $N \neq N^2$ and J is prime. Then $N^2 \subsetneq N$ so, since $J \subseteq N^2 \subsetneq N$, the prime ideal J is properly contained in the unique minimal prime ideal N. This contradiction forces N to be either idempotent or nilpotent. □

Chapter 2

Flat Modules and Weak Global Dimension.

2.1 Flat modules.

We begin with a brief explanation of the tensor product which plays a major rule in defining flat modules.

Definition 2.1. Let M and N be two modules over the ring R, and let A be an additive abelian group.

A function $b : M \times N \to A$ is called R-**bilinear**[1] if for all $m, m_1, m_2 \in M$, all $n, n_1, n_2 \in N$, and all $r \in R$, the following hold:

(i) $b(m_1 + m_2, n) = b(m_1, n) + b(m_2, n)$.

(ii) $b(m, n_1 + n_2) = b(m, n_1) + b(m, n_2)$.

(iii) $b(mr, n) = b(m, rn)$.

A **tensor product** of M and N over R is a pair (V, t), where V is an additive abelian group and $t : M \times N \to V$ is an R-bilinear map, satisfying the following universal mapping

[1] An R-bilinear map is also called R-balanced by some authors.

property:

For any R-bilinear map $b : M \times N \to A$ that maps to an abelian group A, there exists a unique abelian group homomorphism $h : V \to A$ such that $ht = b$, i.e., the following diagram commutes

$$\begin{tikzcd} & V \arrow[dd, dotted, "h"] \\ M \times N \arrow[ur, "t"] \arrow[dr, "b"'] & \\ & A \end{tikzcd}$$

It can be shown that a tensor product for any pair of R-modules M and N always exists and is unique up to isomorphism. Because of this uniqueness we denote it by $M \otimes N$ or $M \otimes_R N$, read as M tensor N. (See §2.3 [Bl] and §2.2 [Osb]).

Let $t : M \times N \longrightarrow M \otimes N$ be the bilinear map associated with the tensor product. Given $(m, n) \in M \times N$, we denote $t((m, n))$ by $m \otimes n$. The elements $m \otimes n$, $(m \in M, n \in N)$ generate the group $M \otimes N$ (but in general, an element of $M \otimes N$ is a sum of elements $m \otimes n$ and need not be expressible as a single term $m \otimes n$). It follows from the definition that we have the following properties:

$$\begin{aligned} &(m_1 + m_2) \otimes n = (m_1 \otimes n) + (m_2 \otimes n), \\ &m \otimes (n_1 + n_2) = (m \otimes n_1) + (m \otimes n_2), \\ &mr \otimes n = m \otimes rn, \end{aligned}$$

in $M \otimes N$. The last of these properties enables the abelian group $M \otimes N$ to become an R-module with module multiplication given by $r(m \otimes n) = mr \otimes n (= m \otimes rn)$ for all $r \in R, m \in M, n \in N$and extending this multiplication distibutively to sums $m_1 \otimes n_1 + \cdots m_k \otimes n_k$.

The uniqueness of the tensor product is useful in proving some important results as the following examples show. Some of these results are given without proof noting that proofs can be found in [Bl, Propositions 2.3.4, 2.3.6 and Problems 2.3.2, 2.3.6] and [Osb, Proposition 2.2].

Lemma 2.2. *Let M, N be modules over the ring R, and I, J be ideals of R. Then the following hold:*

(i) $M \otimes_R N \simeq N \otimes_R M$.

(ii) $M \otimes_R (R/I) \simeq M/IM$.

(iii) $M \otimes_R R \simeq M$.

(iv) $(R/I) \otimes_R (R/J) \simeq R/(I+J)$.

(v) *If I and J are comaximal, i.e., $I + J = R$ then* $(R/I) \otimes_R (R/J) \simeq 0$.

(vi) *If $\{M_\lambda : \lambda \in \Lambda\}$ is a family of R-modules with $M = \oplus_{\lambda\in\Lambda} M_\lambda$ then* $M \otimes N = \oplus_{\lambda\in\Lambda}(M_\lambda \otimes N)$.

Notice that (iii) and (iv) are special cases of (ii) and (v) is a special case of (iv).

Examples 2.3. (1) Let A and B be the $\mathbb{Z}$-modules $\mathbb{Z}_2$ and $\mathbb{Z}_3$ respectively (the abelian groups of order 2 and 3 respectively). Then $A = \mathbb{Z}/2\mathbb{Z}$, $B = \mathbb{Z}/3\mathbb{Z}$ and so, by Lemma 2.2, $A \otimes_{\mathbb{Z}} A = A$, $B \otimes_{\mathbb{Z}} B = B$, and $A \otimes_{\mathbb{Z}} B = \mathbb{Z}/(2\mathbb{Z} + 3\mathbb{Z}) = 0$.

(2) Consider a generating element $m \otimes n$ of $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$, where m, n are both nonzero. Then we may write $m = \frac{a}{b} + \mathbb{Z}$, $n = \frac{c}{d} + \mathbb{Z}$, where a, b, c, d are nonzero integers. This gives

$$\begin{aligned} m \otimes n &= \left(\frac{a}{b} + \mathbb{Z}\right) \otimes \left(\frac{c}{d} + \mathbb{Z}\right) = \left(\frac{ad}{bd} + \mathbb{Z}\right) \otimes \left(\frac{c}{d} + \mathbb{Z}\right) = \left(\frac{a}{bd}\frac{d}{1} + \mathbb{Z}\right) \otimes \left(\frac{c}{d} + \mathbb{Z}\right) \\ &= \left(\frac{a}{bd} + \mathbb{Z}\right) \otimes \left(\frac{d}{1}\frac{c}{d} + \mathbb{Z}\right) = \left(\frac{a}{bd} + \mathbb{Z}\right) \otimes \left(\frac{c}{1} + \mathbb{Z}\right) = \left(\frac{a}{bd} + \mathbb{Z}\right) \otimes 0 = 0. \end{aligned}$$

Therefore $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$.

(3) We show that $\mathbb{Q} \otimes \mathbb{Q}/\mathbb{Z}$ is also 0. Let $m \otimes n$ be a generating element of $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$. Then we may write $m \otimes n = p \otimes (q + \mathbb{Z})$, where $p, q \in \mathbb{Q}$, say $p = \frac{r}{s}$, $q = \frac{t}{u}$ where r, s, t, u are nonzero integers. This gives

$$m \otimes n = \frac{r}{s} \otimes \left(\frac{t}{u} + \mathbb{Z}\right) = \frac{ru}{su} \otimes \left(\frac{t}{u} + \mathbb{Z}\right) = \frac{r}{su} \otimes \left(\frac{tu}{u} + \mathbb{Z}\right) = \frac{r}{su} \otimes (t + \mathbb{Z}) = \frac{r}{su} \otimes 0 = 0.$$

This shows that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$.

Given an R-homomorphism $f : A \longrightarrow B$, for any R-module M we have an R-homomorphism $f \otimes 1_M : A \otimes M \longrightarrow B \otimes M$ defined on the generators of $A \otimes M$ by $(f \otimes 1_M)(a \otimes m) = f(a) \otimes m$, for all $a \in A$, $m \in M$. If $g : B \longrightarrow C$ is another R-homomorphism then, clearly, the composition $(g \otimes 1_M)(f \otimes 1_M)$ is simply $gf \otimes 1_M$. We may also define $1_M \otimes f : M \otimes A \longrightarrow M \otimes B$ similarly but, since $M \otimes N \simeq N \otimes M$ for all R-modules M, N, we may regard $1_M \otimes f$ as another way of writing $f \otimes 1_M$.

Definition 2.4. Given an R-module A and an index set I, we use $A^{(I)}$ to denote the module $\oplus_{i \in I} A_i$ where $A_i = A$ for each $i \in I$.

It follows from Lemma 2.2 that if K is an ideal of R and I is an index set then

$$R/K \otimes R^{(I)} \simeq (R/K \otimes R)^{(I)} \simeq (R/K)^{(I)}.$$

This prepares us for the next lemma.

Lemma 2.5. *Let I and K be index sets and $f : R^{(I)} \longrightarrow R^{(K)}$ be an R-homomorphism. Let a be an element of R, J be an ideal of R and let $\phi_I : Ra/Ja \otimes R^{(I)} \longrightarrow (Ra/Ja)^{(I)}$, $\phi_K : Ra/Ja \otimes R^{(K)} \longrightarrow (Ra/Ja)^{(K)}$ be the natural isomorphisms. This induces the following commutative diagram*

$$\begin{array}{ccc} Ra/Ja \otimes R^{(I)} & \xrightarrow{1 \otimes f} & Ra/Ja \otimes R^{(K)} \\ \phi_I \downarrow & & \downarrow \phi_K \\ (Ra/Ja)^{(I)} & \xrightarrow{\hat{f}} & (Ra/Ja)^{(K)} \end{array}$$

where 1 is the identity map on Ra/Ja and $\hat{f} = \phi_K (1 \otimes f) \phi_I^{-1}$. If $\overline{w} = (w_i a + Ja)_{i \in I} \in \ker(\hat{f})$, where $w_i \in R$ for each $i \in I$, then taking $w = (w_i a)_{i \in I}$ we have $f(w) = f((w_i a)_{i \in I}) \in Ja^{(K)}$.

Proof. Since Ra/Ja is generated by $a + Ja$, each element of $Ra/Ja \otimes R^{(I)}$ can be written as $a + Ja \otimes (r_i)_{i \in I}$ and similarly for elements of $Ra/Ja \otimes R^{(K)}$.

Let $\{e_i : i \in I\}$ and $\{e_k : k \in K\}$ be the natural bases for $R^{(I)}$ and $R^{(K)}$ respectively. Suppose that, for each $i \in I$, we have $f(e_i) = \sum_{k \in K} r_{ik} \overline{e}_k$. Then, given $u = (u_i)_{i \in I} \in R^{(I)}$, we have $f(u) = f(\sum_{i \in I} u_i e_i) = \sum_{i \in I} \sum_{k \in K} u_i r_{ik} \overline{e}_k = (\sum_{i \in I} u_i r_{ik})_{k \in K}$.

This gives

$$(1 \otimes f)((a + Ja) \otimes (u_i)_{i \in I}) = (a + Ja) \otimes f((u_i)_{i \in I}) = (a + Ja) \otimes (\textstyle\sum_{i \in I} u_i r_{ik})_{k \in K}$$

and so, since $\phi_K((a+Ja)\otimes(r_k)_{k\in K}) = (r_k a+Ja)_{k\in K}$, $\phi_I^{-1}((r_i a+Ja)_{i\in I}) = (a+Ja)\otimes(r_i)_{i\in I}$, and $\hat{f} = \phi_K(1 \otimes f)\phi_I^{-1}$, we have

$$\begin{aligned} 0 = \hat{f}(\overline{w}) \;=\; \hat{f}((w_i a + Ja)_{i \in I}) &= \phi_K(1 \otimes f)((a + Ja) \otimes (w_i)_{i \in I}) \\ &= \phi_K((a + Ja) \otimes (\textstyle\sum_{i \in I} w_i r_{ik})_{k \in K}) \\ &= (\textstyle\sum_{i \in I} w_i a r_{ik} + Ja)_{k \in K}. \end{aligned}$$

This gives $\sum_{i\in I} w_i a r_{ik} \in Ja$ for each $k \in K$ and so $f(w) = (\sum_{i\in I} w_i a r_{ik})_{k\in K} \in Ja^{(K)}$, as required. □

Definition 2.6. Let A, B and C be R-modules and let $f : A \to B$ and $g : B \to C$ be R-homomorphisms. Then the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is called **exact (at B)** if $\ker g = \operatorname{im} f$.

More generally, a sequence

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1}$$

of R-modules and R-homomorphisms is called **exact** if it is exact at each A_i, i.e., if

$$A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} A_{i+2}$$

is exact for $i = 0, \ldots, n-1$, i.e., if $\ker f_{i+1} = \operatorname{im} f_i$ for $i = 0, \ldots, n-1$.

Definition 2.7. An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called a **short exact sequence**, abbreviated to **s.e.s.**

It is straightforward to see that the sequence is an s.e.s. if and only if f is a monomorphism, g is an epimorphism and $\ker g = \operatorname{im} f$.

It can be shown that the tensor product is **right exact**, i.e., tensoring any s.e.s. $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ by an R-module M gives the exact sequence

$$A \otimes M \xrightarrow{f \otimes 1_M} B \otimes M \xrightarrow{g \otimes 1_M} C \otimes M \longrightarrow 0$$

(see [Bl, Proposition 3.3.4]). On the other hand, the tensor product need not be **left exact**, i.e., if we tensor the s.e.s. $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ by an R-module M then

$$0 \longrightarrow A \otimes M \xrightarrow{f \otimes 1_M} B \otimes M \xrightarrow{g \otimes 1_M} C \otimes M$$

is not necessarily exact.

Example 2.8. Consider the following s.e.s. of $\mathbb{Z}$-modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \longrightarrow 0\,.$$

Tensoring this sequence with $M = \mathbb{Q}/\mathbb{Z}$ and using Lemma 2.2 (iii) and Examples 2.3 (2) and (3) gives the following sequence

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{i \otimes 1_{\mathbb{Q}/\mathbb{Z}}} 0 \xrightarrow{\pi \otimes 1_{\mathbb{Q}/\mathbb{Z}}} 0 \longrightarrow 0$$

which is certainly not left exact.

We now make a very important definition.

Definition 2.9. A module M over the ring R is called a **flat module** if the tensor product by M is exact, i.e., not only right exact but also left exact, i.e., tensoring any s.e.s. $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ by M gives the exact sequence

$$0 \longrightarrow A \otimes M \xrightarrow{f \otimes 1_M} B \otimes M \xrightarrow{g \otimes 1_M} C \otimes M \longrightarrow 0\,.$$

The next proposition characterises flat modules (see [Bl, Proposition 5.3.7]).

Proposition 2.10. *The following statements are equivalent for an R-module M:*

(i) *M is a flat module.*

(ii) *The sequence $0 \longrightarrow A \otimes M \xrightarrow{i \otimes 1_M} R \otimes M \simeq M$ is exact for any ideal A of R, where $i : A \longrightarrow R$ is the inclusion map.*

(ii)' *For every ideal A the homomorphism $M \otimes A \to MA$, given by $m \otimes x \mapsto mx$ for all $m \in M$ and $x \in A$ is an isomorphism.*

(iii) *The sequence $0 \to M \otimes A \to M \otimes R \simeq M$ is exact for any finitely generated ideal A of R.*

(iii)' *For every finitely generated ideal A, the homomorphism $M \otimes A \to MA$, given by $m \otimes x \mapsto mx$ for all $m \in M$ and $x \in A$ is an isomorphism.*

The following proposition appears as [Bl, Proposition 5.3.9] and will be used in several proofs later on.

Proposition 2.11. *Let $0 \longrightarrow K \xrightarrow{i} F \xrightarrow{f} M \longrightarrow 0$ be an s.e.s. of R-homomorphisms where F is a flat module over the ring R. Then the following statements are equivalent.*

(i) *M is a flat R-module.*

(ii) *$K \cap FA = KA$ for any ideal A of R.*

(iii) *$K \cap FA = KA$ for any finitely generated ideal A of R.*

The following important results about flat modules can be found in [Bl, Corollary 5.3.4], [Bl, Proposition 5.3.5], and [Bl, Corollary 5.3.6].

Lemma 2.12. *The direct sum $\oplus_{i\in I} M_i$ of a family of R-modules $\{M_i : i \in I\}$ is flat if and only if every M_i is flat.*

Lemma 2.13. *Let M be a module over the ring R.*

(i) *If M is free, then M is flat.*

(ii) *If M is projective, then M is flat.*

The following well–known result is essential for later arguments. (See [LM, Theorem 3.3], [Osb, Proposition 5.15], and [R, Theorem 3.73].)

Proposition 2.14. *Let S be an m.c.s. of the ring R. Then R_S is a flat R-module under the R-module multiplication given by $a\dfrac{r}{s} = \dfrac{ar}{s}$ for all $a, r \in R$ and all $s \in S$.*

Corollary 2.15. *Let R be any commutative ring. Then*

(i) *R's total ring of quotients $Q(R)$ is a flat R-module and*

(ii) *the localisation R_P of R at any $P \in \mathrm{Spec}(R)$ is a flat R-module.*

The next lemma is a special case of [LM, Exercise 1.9 (c)] and of [R, Exercise 3.38]. Its proof is straightforward, given that $(A \otimes_R B) \otimes_R C$ and $A \otimes_R (B \otimes_R C)$ are isomorphic as R-modules for any three R-modules A, B and C. Its corollary is [Osb, Exercise 5.8].

Lemma 2.16. *If M and N are two flat modules over the ring R, then the R-module $M \otimes N$ is also flat.*

Corollary 2.17. *If M is a flat module over the ring R and S is an m.c.s. of R then $M_S = M \otimes R_S$ is both a flat R-module and a flat R_S-module since R_S is flat.*

The (right R-module version of the) following characterization appears as [Bl, Proposition 5.3.10] and [AF, Lemma 19.19].

Proposition 2.18. *An R-module M is flat if and only if, given $r_1a_1 + \cdots + r_na_n = 0$, where $r_i \in R$ and $a_i \in M$, then, for some $t \in \mathbb{N}$ there exist an $n \times t$ matrix (r_{ij}), where each $r_{ij} \in R$, and $b_1, \dots, b_t \in M$ such that*

(1) $\sum_{i=1}^{n} r_i r_{ij} = 0$ *for each $j = 1, \dots, t$, and*

(2) $\sum_{j=1}^{t} r_{ij} b_j = a_i$ *for each $i = 1, \dots, n$.*

We note that equations (1) and (2) of Proposition 2.18 can be written in matrix form as

$$(1) \quad \begin{bmatrix} r_1 & \cdots & r_n \end{bmatrix} \begin{bmatrix} r_{11} & \cdots & r_{1t} \\ \vdots & \ddots & \vdots \\ r_{n1} & \cdots & r_{nt} \end{bmatrix} = 0 \qquad (2) \quad \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} r_{11} & \cdots & r_{1t} \\ \vdots & \ddots & \vdots \\ r_{n1} & \cdots & r_{nt} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_t \end{bmatrix}$$

The next lemma, given as [Gl1, Lemma 4.2.1], first appeared as [E, Proposition 9]. It is also part (2) of [Lam, Theorem 4.38], with a different proof, involving Nakayama's

Lemma. Yet another proof, involving projective covers, is given in Hannah's MSc thesis [H], where it appears as Theorem 5.6.

Lemma 2.19. *Let* $(R, \mathfrak{m})$ *be a local ring. Then every finitely generated flat* R*-module* M *is free.*

Proof. Since M is finitely generated, it has a minimal generating set, i.e., a set of generators of the form $A = \{a_1, \dots, a_k\}$, where no proper subset of A generates I. We will show that M is free by showing that A is a basis for M.

Suppose A is not a basis. Then A is not linearly independent and so we have equations of the form

$$r_1 a_1 + \cdots + r_l a_l = 0$$

where $a_1, \dots, a_l$ come from the minimal generating set of I and $r_1, \dots, r_l$ are elements of R not all zero.

We must show that such an equation does not exist. Reordering the a_i if necessary and omitting terms $r_i a_i$ where $r_i = 0$, we may assume that each $r_1, \dots, r_l$ are all nonzero. From these modified equations, choose one with l as small as possible, say $l = n$ and then we get

$$r_1 a_1 + \cdots + r_n a_n = 0,$$

where $r_1, \dots, r_n$ are nonzero elements of R, and $a_1, \dots, a_n$ are part of the minimal generating set $\{a_1, \dots, a_n, a_{n+1}, \dots, a_k\}$. By Proposition 2.18, there exist $b_1, \dots, b_t \in M$ and an $n \times t$ matrix (r_{ij}) where each r_{ij} is in R such that

(1) $\sum_{i=1}^n r_{ij} a_i = 0$ for $j = 1, \dots, t$,

(2) $\sum_{j=1}^t r_{ij} b_j = a_i$ for $i = 1, \dots, n$.

Since $b_j \in M$ for each $j = 1, \dots, t$, there are $s_{jg} \in R$ such that

$$b_j = \sum_{g=1}^n s_{jg} a_g \text{ and so } a_i = \sum_{j=1}^t \sum_{g=1}^n r_{ij} s_{jg} a_g.$$

In particular

$$(*)\quad a_n = q_1a_1 + q_2a_2 + \cdots + q_na_n \text{ where } q_g = \sum_{j=1}^{t} r_{nj}s_{jg} \text{ for each } g = 1, 2, \ldots, n.$$

If $q_n \in \mathfrak{m}$, then, since R is local, $1 - q_n$ is a unit by Lemma 1.10 and so from equation $(*)$ we get $a_n = (1 - q_n)^{-1}(q_1a_1 + q_2a_2 + \cdots + q_{n-1}a_{n-1})$, contradicting the minimality of the generating set A. Thus $q_n = \sum_{j=1}^{t} r_{nj}s_{jn} \notin \mathfrak{m}$ and so there is at least one index j for which r_{nj} and s_{jn} are both units. If $n > 1$, for this index j equation (1) gives $\sum_{i=1}^{n} r_{ij}a_i = 0$ and so $a_n = (-r_{nj}^{-1})\sum_{i=1}^{n-1} r_{ij}a_i$, contradicting the minimality of A again. Thus $n = 1$ and this time equation (1) gives $r_{1j}a_1 = 0$ for all j and so, since r_{1j} is a unit for some j, we get $a_1 = 0$, again a contradiction. Thus A is a basis for M and M is free. $\square$

2.2 Direct limits.

Definition 2.20. Let I be a set with a partial ordering $\leq$ with the property that if $x, y \in I$ then there exists $z \in I$ such that $x \leq z$ and $y \leq z$. We say that I is a **directed set**.

Definition 2.21. A **direct system of modules** $(M_i, f_{ij})_I$ over a fixed ring R consists of:

(i) A family $\{M_i : i \in I\}$ of R-modules, where the index set I is directed.

(ii) A family $\{f_{ij} : M_i \longrightarrow M_j;\ i, j \in I \text{ and } i \leq j\}$ of R-homomorphisms such that, for each $i \in I$, we have $f_{ii} : M_i \longrightarrow M_i$ is the identity map and whenever $i \leq j \leq k$ in I, we have $f_{jk}f_{ij} = f_{ik}$, i.e., the following diagram is commutative

$$\begin{tikzcd} M_i \arrow[rr, "f_{ij}"] \arrow[dr, "f_{ik}"'] & & M_j \arrow[dl, "f_{jk}"] \\ & M_k & \end{tikzcd}$$

A **direct system of homomorphisms** $(M_i, u_i)_I$ from a direct system of modules $(M_i, f_{ij})_I$ to an R-module L is defined to be a family $\{u_i : M_i \longrightarrow L\}$ of R-homomorphisms such that the following diagram commutes for every $i, j \in I$ with $i \leq j$.

$$
\begin{tikzcd}
M_i \arrow[rr, "f_{ij}"] \arrow[dr, "u_i"'] & & M_j \arrow[dl, "u_j"] \\
& L &
\end{tikzcd}
$$

Definition 2.22. Let $(M_i, f_{ij})_I$ be a direct system of modules over I. A direct system of homomorphisms $\{f_i : M_i \longrightarrow M\}$ for some R-module M is said to be a **direct limit** of $(M_i, f_{ij})_I$ if, given any direct system of R-homomorphisms $\{u_i : M_i \longrightarrow L\}$ for some R-module L, then there exists a unique R-homomorphism $u : M \longrightarrow L$ such that the following diagram commutes for any $i, j \in I$ with $i \leq j$

$$
\begin{tikzcd}
M_i \arrow[rr, "f_{ij}"] \arrow[dr, "f_i"] \arrow[ddr, bend right, "u_i"'] & & M_j \arrow[dl, "f_j"'] \arrow[ddl, bend left, "u_j"] \\
& M \arrow[d, dotted, "u"] & \\
& L &
\end{tikzcd}
$$

In this case the R-module M is unique up to isomorphism and we say that the system $(M, f_i)_I$ is *the* direct limit of the direct system $(M_i, f_ij)_I$ and denote it by $\varinjlim_{i \in I}(M_i, f_{ij})$. This is often abbreviated by saying that the direct limit of the system is M. Every direct system of modules and homomorphisms $(M_i, f_{ij})_I$ has a direct limit $\varinjlim_{i \in I}(M_i, f_{ij})$. (See for example [Osb, Proposition 8.5].)

Examples 2.23. (1) Let A be an R-module and consider the set $\{A_i : i \in I\}$ of all finitely generated submodules of A. Define $\leq$ on the index set I by setting $i \leq j$ if and only if $A_i \subseteq A_j$. Then I is a directed set and $\{A_i : i \in I\}$ is a direct family of modules with $f_{ij} : A_i \longrightarrow A_j$ defined to be the inclusion map. Moreover, it is straightforward to show that the direct system $(A, \phi_i)_I$ is its direct limit, where $\phi_i : A_i \longrightarrow A$ is the inclusion map.
(2) Let $\{A_i : i \in I\}$ be any family of R-modules and let $\leq$ be the trivial ordering on I, i.e., for any $i, j \in I$, $i \leq j$ if and only if $i = j$. Then the maps f_{ij} are only defined when $i = j$ and in this case f_{ij} is the identity map on A_i. Thus we may form the direct system $(A_i, f_{ij})_I$ and it is straightforward to show that this has direct limit given by the direct

system $(\oplus_{i\in I}A_i, \phi_i)_I$, where $\phi_j : A_j \longrightarrow \oplus_{i\in I}A_i$ is the natural injection mapping.

Our next lemma can be described as saying that the direct limit commutes with the tensor product. See [Osb, Corollary 8.8] for its proof.

Lemma 2.24. *Let A be an R-module and let $(B_i, f_{ij})_I$ be a direct system of R-modules. Then $((A \otimes B_i), g_{ij})_I$ is a direct system, where $g_{ij} = 1_A \otimes f_{ij} : A \otimes B_i \longrightarrow A \otimes B_j$ and $\varinjlim_{i\in I}((A \otimes B_i), g_{ij})$ is isomorphic to $A \otimes (\varinjlim_{i\in 0I} B_i, f_{ij})$.*

We refer the reader to [Osb, Corollary 8.11] or [R, Theorem 3.47] for a proof of the next result.

Theorem 2.25. *If $(A_i, f_{ij})_I$ is a direct system of flat R-modules, then the direct limit of the system is also a flat R-module.*

The next corollary is an immediate consequence of Theorem 2.25 and Example 2.23 (1).

Corollary 2.26. *Let M be a module over the ring R. Then M is flat if each of its finitely generated submodules is flat.*

We will use the following lemma several times in the following chapters.

Lemma 2.27. *Let A and B be two ideals of the ring R. Let $\mathcal{F}(B) = \{J_\lambda : \lambda \in \Lambda\}$ be the set of finitely generated ideals of R contained in B. Then the index set Λ is a directed set if we define $\leq$ on Λ by $\alpha \leq \beta$ if $J_\alpha \subseteq J_\beta$, noting that for any $\alpha, \beta \in \Lambda$, $J_\gamma = J_\alpha + J_\beta$ gives $\alpha \leq \gamma$ and $\beta \leq \gamma$. When $\alpha, \beta \in \Lambda$ with $\alpha \leq \beta$, we define $f_{\alpha\beta} : R/(A + J_\alpha) \longrightarrow R/(A + J_\beta)$ to be the natural monomorphism given by $r + A + J_\alpha \mapsto r + A + J_\beta$ for all $r \in R$. Then $\{\mathcal{F}(B), f_{\alpha\beta}\}_\Lambda$ is a direct family of modules with $R/(A + B) = \varinjlim_{\alpha\in\Lambda}(R/(A + J_\alpha), f_{\alpha\beta})$.*

Proof. If $\alpha, \beta \in \Lambda$ with $\alpha \leq \beta$, we get the following commutative diagram

$$\begin{tikzcd} R/(A+J_\alpha) \arrow[rr, "f_{\alpha\beta}"] \arrow[dr, "f_\alpha"'] & & R/(A+J_\beta) \arrow[dl, "f_\beta"] \\ & R/(A+B) & \end{tikzcd}$$

where $f_\alpha : R/(A+J_\alpha) \longrightarrow R/(A+B)$ is defined by $r + A + J_\alpha \mapsto r + A + B$. Also, given an R-module M and a set of R-homomorphisms $\{g_\alpha : R/(A+d) \longrightarrow M\}$ such that $g_\beta f_{\alpha\beta} = f_\alpha$ then we construct the following commutative diagram

$$\begin{tikzcd}
R/(A+J_\alpha) \arrow[rr, "f_{\alpha\beta}"] \arrow[dr, "f_\alpha"] \arrow[ddr, bend right, "g_\alpha"'] & & R/(A+J_\beta) \arrow[dl, "f_\beta"'] \arrow[ddl, bend left, "g_\beta"] \\
& R/(A+B) \arrow[d, dotted, "g"] & \\
& M &
\end{tikzcd}$$

where $g : R/(A+B) \longrightarrow M$ is given by $g(r+A+B) = g_\alpha(r+A+J_\alpha)$ for every $\alpha \in \Lambda$. Observe that g is well-defined since if $\alpha, \beta \in \Lambda$ then there exists $\gamma \in \Lambda$ such that $J_\alpha, J_\beta \subseteq J_\gamma$ and $J\gamma \in \chi$. Then $g_\alpha f_{\alpha\gamma} = g_\gamma$ and $g_\gamma f_{\beta\gamma} = g_\beta$ and so

$$\begin{aligned}
g_\alpha(r+A+J_\alpha) &= g_\gamma(f_{\alpha\gamma}(r+A+J_\alpha)) \\
&= g_\gamma(r+A+J_\gamma) \\
&= g_\gamma f_{\beta\gamma}(r+A+J_\beta) \\
&= g_\beta(r+A+J_\beta)
\end{aligned}$$

which implies that the definition of g does not depend on $\alpha \in \Lambda$ and so g is well-defined. Moreover, $g(f_\alpha(r+A+J_\alpha)) = g(r+A+B) = g_\alpha(r+A+J_\alpha)$ and therefore $gf_\alpha = g_\alpha$. If $h : R/(A+B) \longrightarrow M$ is another R-homomorphism such that $hf_\alpha = g_\alpha$ then $h = g$ since

$$\begin{aligned}
h(r+A+B) &= f_\alpha(r+A+J_\alpha) \\
&= g_\alpha(r+A+J_\alpha) \\
&= g(r+A+B).
\end{aligned}$$

□

2.3 Flat ideals.

We will use the following lemma to prove a theorem due to Endo [E] which shows the connection between flat ideals and chain domains.

Lemma 2.28. *Let I be a nonzero ideal of the ring R. Then I is a free R-module if and* *only if $I = Ra$ for some regular element of R.*

Proof. If $I = Ra$ where a is a regular element of R then it is straightforward to show that I is free with basis $\{a\}$.

Conversely suppose that I is free, say with basis $\{r_j : r_j \in R,\ j \in \Lambda\}$. If there is more than one index in Λ, choose $j_1, j_2 \in \Lambda$ with $j_1 \neq j_2$. Then r_{j_1}, r_{j_2} are regular elements of R and so, since R is commutative, $r_i r_j = r_j r_i$ is a nonzero element of $Rr_{j_1} \cap Rr_{j_2}$. This contradicts the linear independence of the basis $\{r_j : r_j \in R, j \in \Lambda\}$. Thus the index set Λ is a singleton and so $I = Ra$ where $\{a\}$ is a basis for I (and consequently a is a regular element of R). $\square$

The following result appears as Theorem 10 in [E], where it is noted that the necessity was proved earlier in [CE, VI, 2.9]

Theorem 2.29. *Let $(R, \mathfrak{m})$ be a local ring. Then R is a chain domain if and only if every ideal of R is flat.*

Proof. Suppose R is a chain domain and I is an ideal of R. Then, by Example 2.23 (1), $I = \varinjlim J_\lambda$, where the J_λ's are the finitely generated ideals contained in I. Since R is a chain domain, each J_λ is principal, say Ra_λ, so is either 0 or free with $\{a_\lambda\}$ as a basis. Hence, by Theorem 2.25, $I = \varinjlim J_\lambda$ is flat.

Conversely, suppose each ideal of R is flat. Let I be a nonzero finitely generated ideal of R. Then, by Lemma 2.19, I is free and so, by Lemma 2.28, I is of the form Ra with basis $\{a\}$ where a is a regular element of R. In particular if $b, c \in R$ and $I = Rb + Rc$ then $Rb + Rc = Ra$ for some regular a. Then $a = rb + sc$, $b = ua$, and $c = va$ for some $r, s, u, v \in R$. This gives $b = urb + usc$ and so $(1 - ur)b = usc$. If either u or r is in $\mathfrak{m}$, then $1 - ur$ is a unit and so we obtain $b = usc$. This implies that $Rb \subseteq Rc$. On the other hand, if $u, r \notin \mathfrak{m}$, then u and r are units and so $c = va = vu^{-1}b$ giving $Rc \subseteq Rb$. Hence R is a chain ring. Also, if $I = Rd$ for the nonzero element $d \in R$, then by Lemmas 2.19

and 2.28, I is free with a basis $\{a\}$ where a is regular. Since $a = rd$ for some $r \in R$, we have $\mathrm{ann}(d) = 0$. Since d was an arbitrary nonzero element in R, it follows that R is a domain. □

2.4 von Neumann regular rings.

Definition 2.30. A ring R (not necessarily commutative) is called **von Neumann regular** or simply **regular** if given any $a \in R$, there exists $x \in R$ such that $axa = a$ (i.e., if R is commutative then $a = a^2x$, equivalently $Ra = Ra^2$).

Examples 2.31. (1) Let X be any non-empty set and $\mathcal{P}(X)$ denote the set of all subsets of X. Then $\mathcal{P}(X)$ becomes a ring by defining $A+B = (A \setminus B) \cup (B \setminus A)$ and $A \cdot B = A \cap B$ for any $A, B \subseteq X$. (The zero of this ring is the empty subset $\emptyset$ and X is the multiplicative identity.) It is a von Neumann regular ring since, for any subset A of X, we have $AXA = A \cap X \cap A = A$.

(2) Clearly, every field F is regular (taking $x = a^{-1}$ when a is a nonzero element of F to get $a = axa$). However this is the only time that an integral domain is regular: if a is a nonzero element of the integral domain R with $a = axa$ for some $x \in R$, canceling a from both sides of the equation gives $1 = xa$ showing that a is a unit with inverse x.

(3) Straightforward arguments show that if R is a regular ring then so is every factor ring R/I of R and localization R_S (where I is a proper ideal of R and S is an m.c.s. of R).

(4) Let $\{R_i : i \in I\}$ be a family of rings with index set I and let R be their direct product $\Pi_{i \in I} R_i$. Then it's easy to see that R is regular if and only if each R_i is regular.

The second part of Example (3) has an important converse:

Lemma 2.32. *The ring R is regular if R_M is von Neumann regular for all $M \in \mathrm{Max}(R)$.*

Proof. Let $a \in R, A = Ra$ and $B = Ra^2$. Then our assumption on R gives $A_M = R_M \dfrac{a}{1} = R_M \dfrac{a^2}{1} = B_M$ for each $M \in$ max. Then, by Lemma 1.24, we get $A = B$, i.e. $Ra = Ra^2$. □

The proof of the next result can be found on [Bl, pages 163–164].

Proposition 2.33. (1) *The ring R is regular if and only if every principal ideal of R is generated by an idempotent.*

(2) *If R is a regular ring then any finitely generated ideal of R is principal.*

Now we show how flatness can characterise regularity.

Proposition 2.34. *The following statements are equivalent for a ring R.*

(i) *R is a regular ring.*

(ii) *Every R-module is flat.*

(iii) *Every cyclic R-module is flat.*

Proof. (i) $\Rightarrow$ (ii). Let R be a regular ring and A be a finitely generated ideal of R. Then, by Proposition 2.33 (1), (2), we can set $A = Re$ where e is an idempotent in R and then, setting $B = R(1-e)$ gives the direct sum $R = A \oplus B$. Let $i : A \to R$ be the inclusion map and $\pi : R \to B$ be the natural projection given by the direct sum. Then $\pi i = 1_A$. Now let M be any R-module. Then we have the following sequence

$$0 \longrightarrow M \otimes A \xrightarrow{1_M \otimes i} M \otimes R.$$

Since $\pi i = 1_A$ we get$(1_M \otimes \pi)(1_M \otimes i) = 1_M \otimes 1_A = 1_{M \otimes IA}$, showing that $1_M \otimes i$ has a left inverse and so is a monomorphism. Thus, by Proposition 2.10, M is flat.

(ii) $\Rightarrow$ (iii) is immediate.

(iii) $\Rightarrow$ (i). Given $a \in R$, we can form the s.e.s.

$$0 \longrightarrow Ra \longrightarrow R \longrightarrow R/Ra \longrightarrow 0.$$

Since Ra and R/Ra are both cyclic, by hypothesis they are both flat. Thus, since R is flat, Proposition 2.11 shows that $Ra \cap Ra = RaRa$, i.e., $Ra = Ra^2$. Hence R is regular. $\square$

2.5 Weak global dimension and Tor_n.

In this section we begin with a quick introduction to weak dimension and the functor Tor. The reader can find more details in Bland [Bl] and Osborne [Osb]. We first give some preparatory definitions and lemmas.

Definition 2.35. Let M be a (left) R-module. A **flat** (respectively, **projective**) **resolution** of M is an exact sequence

$$\cdots \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} M \longrightarrow 0$$

of R-modules and R-homomorphisms where each A_n is flat (respectively, projective). We denote this resolution by $\langle A_n, d_n \rangle$. If there is an $n \geq 0$ for which $A_n \neq 0$ but $A_t = 0$ for all $t > n$, then we say that the resolution has **length** n. In this case, we write the resolution as

$$0 \longrightarrow A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} M \longrightarrow 0.$$

We now show that every R-module has a flat resolution.

For any R-module M we can find a free R-module A_0 together with an epimorphism $f_0 : A_0 \to M$. Since free modules are projective and projective modules are flat, we have the following exact sequence where A_0 is flat:

$$A_0 \xrightarrow{f_0} M \longrightarrow 0.$$

Then we can extend this to an s.e.s. as follows:

$$\begin{tikzcd}
& & A_0 \arrow[r, "f_0"] & M \arrow[r] & 0 \\
& \ker(f_0) \arrow[ur, "i_0"] & & & (*) \\
0 \arrow[ur] & & & &
\end{tikzcd}$$

where i_0 is inclusion. If $\ker(f_0) = 0$, then $0 \longrightarrow A_0 \xrightarrow{f_0} M \longrightarrow 0$ is a flat resolution of M of length 0. If $\ker(f_0)$ is nonzero but flat, then $(*)$ is a flat resolution of M of length 1. If $\ker(f_0)$ is not flat, we can repeat the s.e.s. construction using $\ker(f_0)$ instead of M with a flat R-module A_1 and an epimorphism $f_1 : A_1 \to \ker(f_0)$, to give

$$\begin{tikzcd}
0 \arrow[dr] & & & & & & \\
& \ker(f_1) \arrow[dr, "i_1"] & & & & & \\
& & A_1 \arrow[rr, "g_1 = i_0 f_1"] \arrow[dr, "f_1"] & & A_0 \arrow[r, "g_0 = f_0"] & M \arrow[r] & 0 \\
& & & \ker(f_0) \arrow[ur, "i_0"] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

where i_1 is inclusion. Note that (i) we've renamed f_0 as g_0 and set $g_1 = i_0 f_1$, (ii) $\mathrm{im}(i_1) = \ker(f_1) = \ker(g_1)$ since i_0 is a monomorphism ($g_1(x) = 0$ gives $i_0 f_1(x) = 0$ so $f_1(x) = 0$) and (iii) $\mathrm{im}(g_1) = \ker(f_0)$ and so the sequence

$$0 \longrightarrow \ker(f_1) \xrightarrow{i_1} A_1 \xrightarrow{g_1} A_0 \xrightarrow{g_0} M \longrightarrow 0 \qquad (**)$$

is exact. If $\ker(f_1) = 0$, then $0 \longrightarrow A_1 \xrightarrow{g_1} A_0 \xrightarrow{g_0} M \longrightarrow 0$ is a flat resolution of M of length 1. If $\ker(f_1)$ is nonzero but flat, then $(**)$ is a flat resolution of M of length 2. If $\ker(f_1)$ is not flat, repeat the s.e.s. construction using $\ker(f_1)$, instead of M, with a flat module A_2, an epimorphism $f_2 : A_2 \to \ker(f_1)$, and i_2 is inclusion, as follows.

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & & & \\
 & & & \ker(f_1) \arrow[ur] \arrow[dr, "i_1"] & & & & & \\
 & & A_2 \arrow[ur] \arrow[rr, "g_2=i_1f_2"] & & A_1 \arrow[rr, "g_1=i_0f_1"] \arrow[dr, "f_1"] & & A_0 \arrow[r, "f_0"] & M \arrow[r] & 0 \\
 & \ker(f_2) \arrow[ur, "i_2"] & & & & \ker(f_0) \arrow[ur, "i_0"] \arrow[dr] & & & \\
0 \arrow[ur] & & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Following the argument above, if $\ker(f_2)$ is zero or flat we get a flat resolution of M of length 2 or 3, respectively. We continue in this way, either getting a flat resolution of finite length (when we reach a flat $\ker(f_i)$) at which case we stop, or, if none of the $\ker(f_i)$ are zero, we can keep doing this construction, in which case we get an infinite sequence where $\ker(g_n)$ is not flat for any $n \geq 0$:

$$\cdots \longrightarrow A_n \xrightarrow{g_n} A_{n-1} \xrightarrow{g_{n-1}} A_{n-2} \xrightarrow{g_{n-2}} \cdots \longrightarrow A_1 \xrightarrow{g_1} A_0 \xrightarrow{g_0} M \longrightarrow 0\,.$$

This last case does happen, as the following simple example shows.

Example 2.36. Let R be the chain ring $\mathbb{Z}_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$. Let I be the only proper nonzero ideal of R, namely its principal ideal $R\overline{2} = \{\overline{0}, \overline{2}\}$. Then I is not a flat R-module since otherwise, by Lemma 2.19, it would be free. Now let $A_0 = R$ and $g_0 : A_0 \to I$ be the epimorphism defined by $g_0(x) = \overline{2}x$ for all $x \in R$. Then $\ker(g_0) = I$ which is not flat. Now we may take $A_1 = R$ and $g_1 : A_1 \to A_0$ be g_0 so that $\mathrm{im}(g_1) = \ker(g_0) = I$ and $\ker(g_1) = I$. Repeating this process we get the following infinite flat resolution of I:

$$\cdots \longrightarrow R \xrightarrow{g_n} R \xrightarrow{g_{n-1}} R \xrightarrow{g_{n-2}} \cdots \longrightarrow R \xrightarrow{g_1} R \xrightarrow{g_0} I \longrightarrow 0\,.$$

Although we've used projective, indeed free, modules A_0 above to start a flat resolution of any module M, there are cases where the start can be made with a flat, but not projective, module:

Examples 2.37. (1) Trivially, if M is a flat module which is not projective, taking $A_0 = M$ gives the flat resolution $0 \longrightarrow A_0 \xrightarrow{1_M} M \longrightarrow 0$.

(2) Less trivially, given any integral domain R which is not a field, then its quotient field Q is a flat R-module which is not projective (see Example 2.8 for the case $R = \mathbb{Z}$) while Q/R is not a flat R-module (see Corollary 2.15 (i)). Then the following sequence, in which i is inclusion and π is the natural epimorphism, gives a flat resolution of Q/R which is not a projective resolution:

$$0 \longrightarrow R \xrightarrow{i} Q \xrightarrow{\pi} Q/R \longrightarrow 0 .$$

Definition 2.38. Let M be an R-module. The **weak dimension** of M, denoted by w.dim(M), is the smallest integer n (if such exists) for which there is a flat resolution of M of length n.

$$0 \longrightarrow A_n \longrightarrow A_{n-1} \longrightarrow \cdots \longrightarrow A_1 \longrightarrow A_0 \longrightarrow M \longrightarrow 0 .$$

If M does not have a finite flat resolution, we set w.dim$(M) = \infty$.

Examples 2.39. (1) Trivially, w.dim$(M) = 0$ if and only if M is flat.

(2) Let M be the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$. Since $\mathbb{Z}$ and $\mathbb{Q}$ are both flat $\mathbb{Z}$-modules (see Example 2.39 (2)), we get w.dim$(M) \leq 1$. Furthermore M is not flat - see Example 2.3 (2). Thus we get w.dim$(M) = 1$.

(3) As may be expected from Example 2.36, it can be shown that for the ideal $I = \{\overline{0}, \overline{2}\}$ of the ring $\mathbb{Z}_4$ we have w.dim$(I) = \infty$.

Definition 2.40. A sequence

$$\mathbf{a} = \langle A_n, a_n \rangle \quad = \quad \cdots \xrightarrow{a_{n+1}} A_n \xrightarrow{a_n} A_{n-1} \xrightarrow{a_{n-1}} \cdots \xrightarrow{a_2} A_1 \xrightarrow{a_1} A_0$$

of R-modules and R-homomorphisms is called a **complex** or **zero sequence** if $a_n a_{n+1} = 0$, i.e. $\mathrm{im}(a_{n+1}) \subseteq \ker(a_n)$, for each $n \geq 1$. The n **th homology group** of the complex is defined to be the factor $H_n = \ker(a_n)/\,\mathrm{im}(a_{n+1})$.

The next proof follows arguments given on page 40 of Osborne [Osb].

Lemma 2.41. *In the following diagram, the two rows* $\mathbf{b} = \langle B_n, b_n \rangle$ *and* $\mathbf{c} = \langle C_n, c_n \rangle$ *are complexes and* $\mathbf{f} = \{f_n : n \geq 1\}$ *is a set of R-homomorphisms making the diagram commute, i.e.,* $c_n f_n = f_{n-1} b_n$ *for all* $n \geq 1$.

$$\begin{tikzcd}
\cdots \arrow[r, "b_{n+2}"] & B_{n+1} \arrow[r, "b_{n+1}"] \arrow[d, "f_{n+1}"'] & B_n \arrow[r, "b_n"] \arrow[d, "f_n"'] & B_{n-1} \arrow[r, "b_{n-1}"] \arrow[d, "f_{n-1}"'] & \cdots \arrow[r, "b_2"] & B_1 \arrow[r, "b_1"] \arrow[d, "f_1"'] & B_0 \arrow[d, "f_0"'] \\
\cdots \arrow[r, "c_{n+2}"] & C_{n+1} \arrow[r, "c_{n+1}"] & C_n \arrow[r, "c_n"] & C_{n-1} \arrow[r, "c_{n-1}"] & \cdots \arrow[r, "c_2"] & C_1 \arrow[r, "c_1"] & C_0
\end{tikzcd}$$

For each $n \geq 1$, *let* H_n^b *and* H_n^c *respectively denote the* n *th homology groups of the two complexes, i.e.,* $H_n^b = \ker(b_n)/\operatorname{im}(b_{n+1})$ *and* $H_n^c = \ker(c_n)/\operatorname{im}(c_{n+1})$. *Then there is an R-homomorphism* $f_n^* : H_n^b \to H_n^c$ *given by* $f_n^*(x + \operatorname{im}(b_{n+1})) = f_n(x) + \operatorname{im}(c_{n+1})$ *for each* $x \in \ker(b_n)$.

Proof. First we show that f_n^* is well-defined. If $x + \operatorname{im}(b_{n+1}) = y + \operatorname{im}(b_{n+1})$ (where $x, y \in \ker(b_n)$) then $x - y \in \operatorname{im}(b_{n+1})$, say $x - y = b_{n+1}(z)$, where $z \in B_{n+1}$. Then $f_n(x - y) = f_n(b_{n+1}(z)) = c_{n+1}(f_{n+1}(z))$ (the last equality from the commutativity). This then gives $f_n(x) - f_n(y) \in \operatorname{im}(c_{n+1})$ and so $f_n^*(x + \operatorname{im}(b_{n+1})) = f_n(x) + \operatorname{im}(c_{n+1}) = f_n(y) + \operatorname{im}(c_{n+1}) = f_n^*(y + \operatorname{im}(b_{n+1}))$, as required.

Next we show that the image of f_n^* is indeed in H_n^c, i.e., that $f_n(x) \in \ker(c_n)$ for each $x \in \ker(b_n)$ (from the definition of f_n^*). To see this, simply note that $0 = f_{n+1}(0) = f_{n+1}(b_n(x)) = c_n(f_n(x))$. It's straightforward to see that f_n^* is an R-homomorphism. □

Definition 2.42. Let $\mathbf{b} = \langle B_n, b_n \rangle$, $\mathbf{c} = \langle C_n, c_n \rangle$ be complexes and $\mathbf{f} = \{f_n : B_n \to C_n\}_{n \geq 0}$ and $\mathbf{g} = \{g_n : B_n \to C_n\}_{n \geq 0}$ be two families of R-homomorphisms for which $f_{n-1}b_n = c_n f_n$ and $g_{n-1}b_n = c_n g_n$ for all $n \geq 1$. Then we say that $\mathbf{f}$ and $\mathbf{g}$ are **homotopic** if there exists a family of R-homomorphisms $\mathbf{d} = \{d_n : B_n \to C_{n+1}\}_{n \geq 0}$ such that

$$f_n - g_n = c_{n+1}d_n + d_{n-1}b_n \text{ for each } n \geq 0, \text{ taking } d_{-1} = 0.$$

In this case, $\mathbf{d}$ is called a **homotopy** from $\mathbf{f}$ to $\mathbf{g}$.

Corollary 2.43. *Let* $\mathbf{b}$, $\mathbf{c}$, $\mathbf{f}$, *and* $\mathbf{g}$ *with homotopy* $\mathbf{d}$ *be as in the statement of Definition 2.42. Let* H_n^b *and* H_n^c *be the* n *th homology groups of the complexes* $\mathbf{b}$ *and* $\mathbf{c}$ *respectively. Let* $\mathbf{f}^* = \{f_n^* : n \in \mathbb{N}\}$ *and* $\mathbf{g}^* = \{g_n^* : n \in \mathbb{N}\}$ *be the set of homomorphisms* $f_n^*, g_n^* : H_n^b \to H_n^c$ *induced by* $\mathbf{f}$ *and* $\mathbf{g}$ *as determined in 2.41. Then* $\mathbf{f}^* = \mathbf{g}^*$ *i.e.* $f_n^* = g_n^*$ *for all* $n \in \mathbb{N}$.

Proof. The following diagram, featured on the front cover of [Bl], may be helpful.

$$\begin{tikzcd}
\cdots \arrow[r, "b_{n+2}"] & B_{n+1} \arrow[r, "b_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & B_n \arrow[r, "b_n"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dl, dotted, "d_n"'] & B_{n-1} \arrow[r, "b_{n-1}"] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dl, dotted, "d_{n-1}"'] & \cdots \\
\cdots \arrow[r, "c_{n+2}"'] & C_{n+1} \arrow[r, "c_{n+1}"'] & C_n \arrow[r, "c_n"'] & C_{n-1} \arrow[r, "c_{n-1}"'] & \cdots
\end{tikzcd}$$

For each $n \geq 0$, by Lemma 2.41 there is an R-homomorphism $f_n^* : H_n^b \to H_n^c$ given by $f_n^*(x + \mathrm{im}(b_{n+1})) = f_n(x) + \mathrm{im}(c_{n+1})$ for each $x \in \ker(b_n)$. By assumption, $f_n - g_n = c_{n+1}d_n + d_{n-1}b_n$ and so, if $x + \mathrm{im}(b_{n+1}) \in H_n^b$, with $x \in \ker(b_n)$, then $f_n^*(x + \mathrm{im}(b_{n+1})) = (g_n(x) + c_{n+1}d_n(x) + d_{n-1}b_n(x)) + \mathrm{im}(c_{n+1}) = g_n(x) + \mathrm{im}(c_{n+1}) = g_n^*(x + \mathrm{im}(b_{n+1}))$, as required. $\square$

The next result is [Osb, Proposition 3.1]. We prove it for the reader's convenience.

Proposition 2.44. *Let* B *and* C *be* R*-modules and* $f : B \to C$ *be an* R*-homomorphism. Let* $\mathbf{b} = \langle B_n, b_n \rangle$ *and* $\mathbf{c} = \langle C_n, c_n \rangle$ *be projective resolutions of* B *and* C *respectively. Then there is a family* $\mathbf{f} = \{f_n : B_n \to C_n\}_{n \geq 0}$ *of* R*-homomorphisms making the following diagram commute*

$$\begin{tikzcd}
\cdots \arrow[r, "b_{n+2}"] & B_{n+1} \arrow[r, "b_{n+1}"] \arrow[d, dotted, "f_{n+1}"'] & B_n \arrow[r, "b_n"] \arrow[d, dotted, "f_n"'] & B_{n-1} \arrow[r, "b_{n-1}"] \arrow[d, dotted, "f_{n-1}"'] & \cdots \arrow[r, "b_2"] & B_1 \arrow[r, "b_1"] \arrow[d, dotted, "f_1"'] & B_0 \arrow[r, "b_0"] \arrow[d, dotted, "f_0"'] & B \arrow[r] \arrow[d, "f"'] & 0 \\
\cdots \arrow[r, "c_{n+2}"] & C_{n+1} \arrow[r, "c_{n+1}"] & C_n \arrow[r, "c_n"] & C_{n-1} \arrow[r, "c_{n-1}"] & \cdots \arrow[r, "c_2"] & C_1 \arrow[r, "c_1"] & C_0 \arrow[r, "c_0"] & C \arrow[r] & 0
\end{tikzcd}$$

i.e., $fb_0 = c_0 f_0$ *and* $f_{n-1}b_n = c_n f_n$ *for all* $n \geq 1$. *Moreover, if there is another family* $\mathbf{g} = \{g_n : B_n \to C_n\}$ *of* R*-homomorphisms with this commuting property then there is a homotopy* $\mathbf{d}$ *from* $\mathbf{f}$ *to* $\mathbf{g}$.

Proof. We define f_n inductively, starting at $n = 0$. Since B_0 is projective and c_0 is an epimorphism, there is a homomorphism $f_0 : B_0 \to C_0$ making the following diagram commute, as required.

$$\begin{tikzcd} & B_0 \arrow[d, "b_0"] \arrow[dl, dotted, "f_0"'] & \\ & B \arrow[d, "f"] & \\ C_0 \arrow[r, "c_0"] & C \arrow[r] & 0 \end{tikzcd}$$

Now suppose that for some n, the homomorphisms $f_0, \ldots, f_n$ have been constructed meeting the commutativity requirements. If $x \in \operatorname{im}(b_{n+1})$ then $b_n(x) = 0$ and so $c_n f_n(x) = f_{n-1}(b_n(x)) = 0$. Thus $f_n(\operatorname{im}(b_{n+1})) \subseteq \ker(c_n) = \operatorname{im}(c_{n+1})$. This gives the following commutative diagram where the row is exact and f_{n+1} is induced by the projectivity of B_{n+1}.

$$\begin{tikzcd} & B_{n+1} \arrow[d, "b_{n+1}"] \arrow[ddl, dotted, "f_{n+1}"'] & \\ & \operatorname{im}(b_{n+1}) \arrow[d, "f_n"] & \\ C_{n+1} \arrow[r, "c_{n+1}"] & \operatorname{im}(c_{n+1}) \arrow[r] & 0 \end{tikzcd}$$

as required.

Now assume there is another family $\mathbf{g} = \{g_n : B_n \to C_n\}$ of R-homomorphisms with the commuting property, i.e., $fb_0 = c_0 g_0$ and $g_{n-1} b_n = c_n g_n$ for all $n \geq 1$. Starting at $n = 0$, note first that $c_0 g_0 = f b_0 = c_0 f_0$ and so $\operatorname{im}(f_0 - g_0) \subseteq \ker(c_0) = \operatorname{im}(c_1)$. This gives the following commutative diagram exact row and d_0 induced by the projectivity of B_0:

$$\begin{tikzcd} & B_0 \arrow[d, "f_0 - g_0"] \arrow[dl, dotted, "d_0"'] & \\ C_1 \arrow[r, "c_1"] & \operatorname{im}(c_1) \arrow[r] & 0 \end{tikzcd}$$

as required, taking $d_{-1} = 0$.

Now assume recursively that we are given the required $d_0, d_1, \ldots, d_n$ for some $n \geq 0$.

Then $f_n - g_n = c_{n+1}d_n + d_{n-1}b_n$ and so

$$\begin{aligned} c_{n+1}(f_{n+1} - g_{n+1} - d_n b_{n+1}) &= c_{n+1}f_{n+1} - c_{n+1}g_{n+1} - c_{n+1}d_n b_{n+1} \\ = f_n b_{n+1} - g_n b_{n+1} - c_{n+1}d_n b_{n+1} &= (f_n - g_n - c_{n+1}d_n)b_{n+1} \\ = d_{n-1}b_n b_{n+1} &= 0. \end{aligned}$$

This gives $\operatorname{im}(f_{n+1} - g_{n+1} - d_n b_{n+1}) \subseteq \ker(c_{n+1}) = \operatorname{im}(c_{n+2})$ and so we get the following commutative diagram with exact row and d_{n+1} induced by the projectivity of B_{n+1}:

$$\begin{tikzcd} & B_{n+1} \arrow[dl, dotted, "d_{n+1}"'] \arrow[d, "f_{n+1}-g_{n+1}-d_n b_{n+1}"] & \\ C_{n+2} \arrow[r, "c_{n+2}"] & \operatorname{im}(c_{n+2}) \arrow[r] & 0 \end{tikzcd}$$

as required. $\square$

We now define Tor.

Definition 2.45. Let X and M be R-modules and let

$$\cdots \xrightarrow{a_{n+1}} A_n \xrightarrow{a_n} A_{n-1} \xrightarrow{a_{n-1}} \cdots \xrightarrow{a_2} A_1 \xrightarrow{a_1} A_0 \xrightarrow{a_0} M \longrightarrow 0.$$

be a projective resolution of M. Although this resolution is an exact complex, tensoring with X gives the following complex which will not be exact in general:

$$\cdots \xrightarrow{a_{n+1}\otimes 1_X} A_nX \xrightarrow{a_n\otimes 1_X} A_{n-1}\otimes X \xrightarrow{a_{n-1}\otimes 1_X} \cdots \xrightarrow{a_2\otimes 1_X} A_1\otimes X \xrightarrow{a_1\otimes 1_X} A_0\otimes X \xrightarrow{a_0\otimes 1_X} 0. \quad (\dagger)$$

Then we define $\operatorname{Tor}_R(M, X)$ to be the nth homology group of the complex $(\dagger)$, i.e., $\operatorname{Tor}_n(M, X) = \ker(a_n \otimes 1_X)/\operatorname{im}(a_{n+1} \otimes 1_X)$.

Of course, this definition uses a particular projective resolution of M. We now indicate how the definition is, up to isomorphism, independent of the choice of projective resolution of M.

Let $f : M \to N$ be an R-homomorphism and let

$$\cdots \xrightarrow{e_{n+1}} B_n \xrightarrow{e_n} B_{n-1} \xrightarrow{e_{n-1}} \cdots \xrightarrow{e_2} B_1 \xrightarrow{e_1} B_0 \xrightarrow{e_0} N \longrightarrow 0.$$

be a projective resolution of N. Then, by Proposition 2.44, there is a family of R-

homomorphisms $\mathbf{f} = \{f_n : A_n \to B_n\}_{n\geq 0}$ making the following diagram commute

$$\begin{tikzcd}
\cdots \arrow[r, "a_{n+2}"] & A_{n+1} \arrow[r, "a_{n+1}"] \arrow[d, dotted, "f_{n+1}"'] & A_n \arrow[r, "b_n"] \arrow[d, dotted, "f_n"'] & A_{n-1} \arrow[r, "a_{n-1}"] \arrow[d, dotted, "f_{n-1}"'] & \cdots \arrow[r, "a_2"] & A_1 \arrow[r, "a_1"] \arrow[d, dotted, "f_1"'] & A_0 \arrow[r, "a_0"] \arrow[d, dotted, "f_0"'] & M \arrow[r] \arrow[d, "f"'] & 0 \\
\cdots \arrow[r, "b_{n+2}"] & B_{n+1} \arrow[r, "b_{n+1}"] & B_n \arrow[r, "b_n"] & B_{n-1} \arrow[r, "b_{n-1}"] & \cdots \arrow[r, "b_2"] & B_1 \arrow[r, "b_1"] & B_0 \arrow[r, "b_0"] & N \arrow[r] & 0.
\end{tikzcd}$$

Moreover, if there is another family $\mathbf{g} = \{g_n : A_n \to B_n\}$ of R-homomorphisms with this commuting property then there is a homotopy $\mathbf{d} = \{d_n\}_{n\geq 0}$ from $\mathbf{f}$ to $\mathbf{g}$. Now apply $-\otimes X$ to the diagram to get the following commutative diagram where both rows are complexes:

$$\begin{tikzcd}
\cdots \arrow[r, "a_{n+1}\otimes 1_X"] & A_n \otimes X \arrow[r, "a_n\otimes 1_X"] \arrow[d, dotted, "f_n\otimes 1_X"'] & A_{n-1} \otimes X \arrow[r, "a_{n-1}\otimes 1_X"] \arrow[d, dotted, "f_{n-1}\otimes 1_X"'] & \cdots \arrow[r, "a_2\otimes 1_X"] & A_1 \otimes X \arrow[r, "a_1\otimes 1_X"] \arrow[d, dotted, "f_1\otimes 1_X"'] & A_0 \otimes X \arrow[r, "a_0\otimes 1_X"] \arrow[d, dotted, "f_0\otimes 1_X"'] & M \otimes X \arrow[r] \arrow[d, "f\otimes 1_X"'] & 0 \\
\cdots \arrow[r, "b_{n+1}\otimes 1_X"] & B_n \otimes X \arrow[r, "b_n\otimes 1_X"] & B_{n-1} \otimes X \arrow[r, "b_{n-1}\otimes 1_X"] & \cdots \arrow[r, "b_2\otimes 1_X"] & B_1 \otimes X \arrow[r, "b_1\otimes 1_X"] & B_0 \otimes X \arrow[r, "b_0\otimes 1_X"] & N \otimes X \arrow[r] & 0.
\end{tikzcd}$$

It is straightforward to show that $\mathbf{d}\otimes X = \{d_n \otimes 1_X\}_{n\geq 0}$ is a homotopy from $\mathbf{f}\otimes X = \{f_n \otimes 1_X\}_{n\geq 0}$ to $\mathbf{g}\otimes X = \{g_n \otimes 1_X\}_{n\geq 0}$. In particular, if f is the identity homomorphism $1_M : M \to M$ then, using Lemma 2.41 and Corollary 2.43, one can show that, for all $n \geq 0$, $\ker(a_n \otimes 1_X)/\operatorname{im}(a_{n+1} \otimes 1_X) \simeq \ker(b_n \otimes 1_X)/\operatorname{im}(b_{n+1} \otimes 1_X)$, as desired.

We now record for future use some important properties of Tor_n. For the details see [Osb, Proposition 3.16, Theorem 8.10] and [N, Chapter 8, Theorem 7].

Lemma 2.46. *Let R be a ring and A, B be R-modules. Then the following hold:*

(i) $\operatorname{Tor}_n(A, B) \simeq \operatorname{Tor}_n(B, A)$.

(ii) *Let $(M_\alpha, F_{\alpha\beta})_\Lambda$ be a direct system of R-modules. Then $(\operatorname{Tor}_n(A, M_\alpha); f^{(n)}_{\alpha\beta})_\Lambda$ is a direct system of R-modules, where the mappings $f^{(n)}_{\alpha\beta}$ are given by $\operatorname{Tor}_n(A, f_{\alpha\beta}) : \operatorname{Tor}_n(A, M_\alpha) \longrightarrow \operatorname{Tor}_n(A, M_\beta)$. Then $\varinjlim(\operatorname{Tor}_n(A, M_\alpha)) = \operatorname{Tor}_n(A, \varinjlim(M_\alpha))$.*

(iii) *For any m.c.s. S of R, $(\operatorname{Tor}^R_n(A, B))_S \simeq \operatorname{Tor}^{R_S}_n(A_S, B_S)$.*

Definition 2.47. Suppose that the following sequence is a flat (or projective) resolution of the R-module M

$$\cdots \xrightarrow{\delta_{n+1}} F_n \xrightarrow{\delta_n} F_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} M \longrightarrow 0.$$

Set $K_0 = M$ and $K_n = \ker(\delta_{n-1})$ for each $n \in \mathbb{N}$. Then K_n is called the n **th kernel** of the resolution.

Note that if K_n is flat (or projective), we get a new flat (or projective) resolution of M, namely

$$0 \longrightarrow K_n \xrightarrow{i_n} F_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} M \longrightarrow 0,$$

where i_n is the inclusion map.

This leads to the following important result, which appears with proof as Proposition 4.5 in Osborne [Osb]. See also [Bl, Proposition 12.2.4] and [R, Theorem 9.13].

Theorem 2.48 (Flat Dimension Theorem). *Let M be an R-module. Then the following statements are equivalent*:

(i) $\mathrm{w.dim}_R(M) \leq n$.

(ii) $\mathrm{Tor}_{n+1}(R/I, M) = 0$ *for all finitely generated (right) ideals.*

(iii) $\mathrm{Tor}_{n+1}(X, M) = 0$ *for all R-modules X.*

(iv) *The n th kernel of any flat resolution of M is flat.*

(v) *There is a flat resolution of M with flat n th kernel.*

(vi) *There is a flat resolution $\langle F_K, \delta_k\rangle$ of M with $F_k = 0$ for all $k > n$.*

We now record how a fixed module, the tensor product and a short exact sequence give rise to a long exact sequence. This appears in [Osb] as Theorem 3.4.

Theorem 2.49 (Long exact sequence for Tor). *Let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of R-modules and M also be an R-module. Then there is a long exact sequence

$$\cdots \xrightarrow{g_{n+1}} \mathrm{Tor}_{n+1}(C,M) \xrightarrow{\delta_{n+1}} \mathrm{Tor}_n(A,M) \xrightarrow{f_n} \mathrm{Tor}_n(B,M) \xrightarrow{g_n} \mathrm{Tor}_n(C,M)$$

$$[\text{continued}] \xrightarrow{\delta_n} \mathrm{Tor}_{n-1}(A,M) \xrightarrow{f_{n-1}} \mathrm{Tor}_{n-1}(B,M) \xrightarrow{g_{n-1}} \mathrm{Tor}_{n-1}(C,M) \xrightarrow{\delta_{n-1}}$$

$$\cdots\cdots\cdots \xrightarrow{\delta_2} \mathrm{Tor}_1(A,M) \xrightarrow{f_1} \mathrm{Tor}_1(B,M) \xrightarrow{g_1} \mathrm{Tor}_1(C,M)$$

$$[\text{continued}] \xrightarrow{\delta_1} A\otimes M \xrightarrow{f\otimes 1_M} B\otimes M \xrightarrow{g\otimes 1_M} C\otimes M \longrightarrow 0.$$

Definition 2.50. Let R be any (commutative) ring. The **weak global dimension** of R, denoted by w.gl.dim(R), is defined as follows:

$$\text{w.gl.dim}(R) = \sup\{\text{w.dim}(M) : M \text{ is an } R\text{-module}\}.$$

As a consequence of Theorem 2.48, we have the next lemma.

Lemma 2.51. *Let R be a ring with* w.gl.dim$(R) \leq 1$*. Then every ideal I of R is flat.*

Proof. Since $\sup\{\text{w.dim}(M) : M \text{ is an } R\text{-module}\} = \text{w.gl.dim}(R) \leq 1$, it follows that w.dim$(R/I)$ is at most 1. Thus, since R is flat, in the s.e.s.

$$0 \longrightarrow I \xrightarrow{i} R \xrightarrow{\pi} R/I \longrightarrow 0$$

the kernel of the natural epimorphism π (namely I) must be flat. □

The next result appears as [Gl1, Theorem 1.3.13].

Theorem 2.52. *Let R be a ring, M be an R-module and S be an m.c.s. of R. Then* $\text{w.dim}_{R_S}(M_S) \leq \text{w.dim}_R(M)$*. In particular it follows that* $\text{w.gl.dim}(R_S) \leq \text{w.gl.dim}(R)$*.*

The following theorem states part (1) of [Gl1, Theorem 1.3.14] and is to be used in showing several results later in the coming sections.

Theorem 2.53. *Let M be a module over the ring R. Then*

$$\text{w.dim}_R(M) = \sup\{\text{w.dim}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}) : \mathfrak{m} \in \mathrm{Max}(R)\}.$$

Consequently,

$$\text{w.gl.dim}(R) = \sup\{\text{w.gl.dim}(R_{\mathfrak{m}}) : \mathfrak{m} \in \text{Max}(R)\}.$$

Our next proposition appears as [Bl, Problem 12.2.8].

Proposition 2.54. *Let $\{A_i : i \in I\}$ be any family of modules over the ring R. Then*

$$\text{w.dim}_R(\oplus_{i\in I} A_i) = \sup\{\text{w.dim}_R\, A_i, i \in I\}.$$

The next result shows how a particular subclass of cyclic modules determines the weak global dimension of a ring. Osborne [Osb, Proposition 4.13] refers to it as the *Weak Dimension Theorem.* See also [Bl, Proposition 12.2.6] and [R, Theorem 9.19].

Theorem 2.55. *For any ring R we have*

$$\text{w.gl.dim}(R) = \sup\{\text{w.dim}_R(R/I) : I \textit{ is a finitely generated ideal of } R\}.$$

This powerful result gives the converse to Lemma 2.51:

Corollary 2.56. *If all the (finitely generated) ideals of the ring R are flat, then* w.gl.dim$(R) \leq$ 1*.*

Proof. Given any (finitely generated) ideal of R, the s.e.s.

$$0 \longrightarrow I \xrightarrow{\;i\;} R \xrightarrow{\;\pi\;} R/I \longrightarrow 0\ ,$$

where i is the inclusion map and π the natural projection, is a flat resolution of R/I and so $\text{w.dim}_R(R/I) \leq 1$. Now apply Theorem 2.55. □

The following corollary is Lemma 3.2 of [DT].

Corollary 2.57. *Let M, A and B be modules over the ring R. Suppose that* $\text{w.dim}(M) = n$ *and $f : A \to B$ is a monomorphism, i.e., with π as the natural epimorphism,*

$$0 \longrightarrow A \xrightarrow{\;f\;} B \xrightarrow{\;\pi\;} B/f(A) \longrightarrow 0$$

is a short exact sequence. Then the map $f_n : \mathrm{Tor}_n(A, M) \to \mathrm{Tor}_n(B, M)$ *(given in the long exact sequence for* Tor *induced by this s.e.s. and* M*) is also a monomorphism.*

Proof. Since $\mathrm{w.dim}(M) = n$, it follows from Theorem 2.48 that $\mathrm{Tor}_{n+1}(B/f(A), M) = 0$ and so the result follows from the exactness of the portion

$$\mathrm{Tor}_{n+1}(B/f(A), M) \xrightarrow{\delta_{n+1}} \mathrm{Tor}_n(A, M) \xrightarrow{f_n} \mathrm{Tor}_n(B, M)$$

of the Tor long exact sequence. □

We use the next two results in Chapter 4.

Lemma 2.58. *Consider the s.e.s. of* R*-modules*

$$0 \longrightarrow A \xrightarrow{\phi} F \xrightarrow{\delta} B \longrightarrow 0\,,$$

where F *is flat but* B *is not flat. Then* $\mathrm{w.dim}(B) = \mathrm{w.dim}(A) + 1$*.*

Proof. To prove this result we use the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & & & \\
 & & & \ker(\pi_1) \arrow[ur] \arrow[dr, "i_1"] & & & & & \\
 & & F_2 \arrow[ur] \arrow[rr, dotted, "\delta_2"] & & F_1 \arrow[rr, dotted, "\delta_1"] \arrow[dr, "\pi_1"] & & F \arrow[r, "\delta"] & B \arrow[r] & 0 \\
 & \ker(\pi_2) \arrow[ur, "i_2"] & & & & A \arrow[ur, "i_0"'] \arrow[dr] & & & \\
0 \arrow[ur] & & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

which gives flat resolutions for A and B respectively

$$F_n \xrightarrow{\delta_n} F_{n-1} \xrightarrow{\delta_{n-1}} \ldots \longrightarrow F_1 \xrightarrow{\delta_1} F \xrightarrow{\delta} B \longrightarrow 0\,,$$

$$F_n \xrightarrow{\delta_n} F_{n-1} \xrightarrow{\delta_{n-1}} \ldots \longrightarrow F_1 \xrightarrow{\pi_1} A \longrightarrow 0\,.$$

This clearly shows that $\mathrm{w.dim}(B) = \mathrm{w.dim}(A) + 1$ since the flat resolution of B has one step more than the free resolution of A. □

The next lemma is [BaGl, Lemma 6.2].

Lemma 2.59. *If $(R, \mathfrak{m})$ is a local ring with $\mathfrak{m}$ nonzero (so $(R, \mathfrak{m})$ is not a field) then*

$$\text{w.dim}_R(R/\mathfrak{m}) = \text{w.dim}_R(\mathfrak{m}) + 1.$$

Proof. Consider the s.e.s.

$$0 \longrightarrow \mathfrak{m} \xrightarrow{i} R \xrightarrow{\pi} R/\mathfrak{m} \longrightarrow 0\ .$$

Then we either have $R/\mathfrak{m}$ is flat or not flat. If it is not flat then by Lemma 2.58 we have $\text{w.dim}_R(R/\mathfrak{m}) = \text{w.dim}_R(\mathfrak{m}) + 1$ as required. If otherwise $R/\mathfrak{m}$ is flat, then by Proposition 2.11 (1$\Rightarrow$ 2) we obtain $\mathfrak{m} \cap RA = \mathfrak{m}A$ for every ideal A of R. In particular if we take any nonzero element $a \in \mathfrak{m}$ (possible since $\mathfrak{m} \neq 0$) then $\mathfrak{m} \cap Ra = \mathfrak{m}Ra$, in other words, $Ra = \mathfrak{m}a$. In particular, $a \in \mathfrak{m}a$. Thus $a = ma$ for some $m \in \mathfrak{m}$. Hence $(1 - m)a = 0$, but $1 - m$ is a unit and so $a = 0$, a contradiction. □

Remark 2.60. For a local ring $(R, \mathfrak{m})$ and an R-module B, if $\mathfrak{m}B = 0$ then we can turn B into a vector space over the field $R/\mathfrak{m}$, i.e., an $(R/\mathfrak{m})$-module, defining the module multiplication to be $(r + \mathfrak{m})b = rb + \mathfrak{m}$ for $b \in B$ and $r \in R$. Thus B can be regarded as the direct sum of copies of $R/\mathfrak{m}$.

This remark is easily shown and is important for the proof of the next result which appears as [BaGl, Proposition 6.3].

Proposition 2.61. *Suppose $(R, \mathfrak{m})$ is a local ring whose maximal ideal $\mathfrak{m}$ is nonzero nilpotent. Then* $\text{w.dim}(\mathfrak{m}) = \infty$*.*

Proof. Let n be the nilpotency index of $\mathfrak{m}$, i.e., $n \geq 2$ and $\mathfrak{m}^n = 0$, but $\mathfrak{m}^{n-1} \neq 0$. We show that for all $1 \leq k \leq n$, $\text{w.dim}(\mathfrak{m}^{n-k}) = \text{w.dim}(\mathfrak{m}) + 1$. Taking $k = n - 1$ then gives $\text{w.dim}(\mathfrak{m}) = \text{w.dim}(\mathfrak{m}) + 1$ which implies that $\text{w.dim}(\mathfrak{m}) = \infty$ as required.

We begin with $k = 1$. Then $\mathfrak{m}\mathfrak{m}^{n-1} = 0$ so, by Remark 2.60, $\mathfrak{m}^{n-1}$ is a nonzero direct sum of copies of $R/\mathfrak{m}$, say $(R/\mathfrak{m})^{(I)}$, where I is an index set. Then we obtain

$$\begin{aligned}\text{w.dim}_R(\mathfrak{m}^{n-1}) &= \text{w.dim}_R((R/\mathfrak{m})^{(I)}) \\ &= \text{w.dim}_R(R/\mathfrak{m}) \text{ (by Proposition 2.54)} \\ &= \text{w.dim}_R(\mathfrak{m}) + 1 \text{ (by Lemma 2.59)},\end{aligned}$$

establishing the first case. Now let $h \in \{1, \ldots, n-1\}$ be the maximum integer k for which $\text{w.dim}(\mathfrak{m}^{n-k}) = \text{w.dim}(\mathfrak{m}) + 1$ for all $k \leq h$. We complete the proof by showing that $h = n-1$. Suppose to the contrary that $h \neq n-1$. We have the following s.e.s.

$$0 \longrightarrow \mathfrak{m}^{n-h} \xrightarrow{i} \mathfrak{m}^{n-(h+1)} \xrightarrow{\pi} \frac{\mathfrak{m}^{n-(h+1)}}{\mathfrak{m}^{n-h}} \longrightarrow 0 \quad (*)$$

where i is inclusion and π is the natural epimorphism. Letting $A = \dfrac{\mathfrak{m}^{n-(h+1)}}{\mathfrak{m}^{n-h}}$ gives $\mathfrak{m}A = 0$, as before, A is a module over the field $R/\mathfrak{m}$ and $\text{w.dim}(A) = \text{w.dim}(\mathfrak{m}) + 1$. By assumption $\text{w.dim}(\mathfrak{m}^{n-h}) = \text{w.dim}(\mathfrak{m}) + 1$. Tensoring $(*)$ by an arbitrary R-module X gives the long exact sequence for Tor:

$$\begin{array}{c}\text{Tor}_{n+1}(\mathfrak{m}^{n-h}, X) \longrightarrow \text{Tor}_{n+1}(\mathfrak{m}^{n-(h+1)}, X) \longrightarrow \text{Tor}_{n+1}(A, X) \longrightarrow \\[2em] \text{Tor}_{n}(\mathfrak{m}^{n-h}, X) \longrightarrow \text{Tor}_{n}(\mathfrak{m}^{n-(h+1)}, X) \longrightarrow \text{Tor}_{n}(A, X) \longrightarrow \cdots \\[2em] \mathfrak{m}^{n-h} \otimes X \longrightarrow \mathfrak{m}^{n-(h+1)} \otimes X \longrightarrow A \otimes X \longrightarrow 0\end{array}$$

If $\text{w.dim}(\mathfrak{m}) = t < \infty$, then $\text{w.dim}(A) = \text{w.dim}(\mathfrak{m}^{n-h}) = t+1$. This implies that $\text{Tor}_{t+2}(A, X) = 0 = \text{Tor}_{t+2}(\mathfrak{m}^{n-h}, X)$. Thus, since we have the exact sequence

$$\text{Tor}_{t+2}(\mathfrak{m}^{n-h}, X) \longrightarrow \text{Tor}_{t+2}(\mathfrak{m}^{n-(h+1)}, X) \longrightarrow \text{Tor}_{t+2}(A, X)$$

as a part of the long exact sequence for Tor, we then get $\text{Tor}_{t+2}(\mathfrak{m}^{n-(h+1)}, X) = 0$. Observe that $\text{w.dim}(\mathfrak{m}^{n-(h+1)}) \geq t+1$ since otherwise we would have an exact sequence of the form

$$\text{Tor}_{t+2}(A, X) \longrightarrow \text{Tor}_{t+1}(\mathfrak{m}^{n-h}, X) \longrightarrow \text{Tor}_{t+1}(\mathfrak{m}^{n-(h+1)}, X),$$

where $\text{Tor}_{t+2}(A, X) = 0$ as proved above and $\text{Tor}_{t+1}(\mathfrak{m}^{n-(h+1)}, X) = 0$ by assumption, and so $\text{Tor}_{t+1}(\mathfrak{m}^{n-h}, X) = 0$, for every R-module X. This shows that $\text{w.dim}(\mathfrak{m}^{n-h}) \leq t$, a contradiction. Thus w.dim($\mathfrak{m}^{n-(h+1)}$) $= t+1$ which contradicts the definition of h. This contradiction completes the proof that w.dim($\mathfrak{m}$) $= \infty$. $\square$

2.6 Semi-hereditary rings and Prüfer domains.

Definition 2.62. Let R be an integral domain with quotient field Q. A **fractional ideal** of R is an R-submodule F of Q for which there exists a nonzero $r \in R$ such that $rF \subseteq R$

Note that every ideal I of R in the usual sense is a fractional ideal since $1_R I \subseteq R$.

For any ideal I of R we define

$$I^{-1} = \{q \in Q : qI \subseteq R\}.$$

Note that I^{-1} is a fractional ideal of R. Also, clearly $I^{-1}I = \{\sum_{i=1}^{n} q_i x_i : q_i \in I^{-1}, x_i \in I, n \in \mathbb{N}\} \subseteq R$. If $I^{-1}I = R$ then I is called an **invertible** ideal.

The following information on invertible ideals can be found in [LM, Chapter VI, §1], [R, page 125].

Proposition 2.63. *Let R be an integral domain and I be a nonzero ideal of R.*

(a) *I is invertible if and only if it is projective as an R-module.*

(b) *If I is invertible then it is finitely generated.*

(c) *If $I = Ra$ where a is a nonzero element of R, then I is invertible with $I^{-1} = \{b/a \in Q : b \in R\}$.*

Definition 2.64. A ring R is said to be **semi-hereditary** if each of its finitely generated ideals is projective.

The next proposition is essentially due to Endo [E]. We follow the argument given in [I] in the proof of part (iii).

Proposition 2.65. (i) *If R is a regular ring then R is semi-hereditary.*

(ii) *If R is a semi-hereditary ring, then* $\mathrm{w.gl.dim}(R) \leq 1$.

(iii) *If R is a semi-hereditary ring, then its total quotient ring, $Q(R)$, is regular.*

Proof. (i) If R is regular, then by Lemma 2.33 every finitely generated ideal I of R is principal and generated by an idempotent, i.e., $I = Re$ for some idempotent $e \in R$. Then, since $R = Re \oplus R(1-e)$, I is projective, as required.

(ii) Let I be an ideal of R. Then, by Example 2.23 (1), I is the direct limit of the directed system $(J_\lambda, f_{\lambda,\mu})_{\lambda\in\Lambda}$ where each J_λ is a finitely generated ideal of R contained in I and $f_{\lambda,\mu} : J_\lambda \to J_\mu$ is the inclusion map when $J_\lambda \subseteq J_\mu$. Since each J_λ is projective (and so flat), I is also flat by Theorem 2.25. Since every ideal of R is flat, $\mathrm{w.gl.dim}(R) \leq 1$ by Corollary 2.56.

(iii) Let $a \in R$. By assumption, Ra is a projective R-module and so, if in the following diagram we start with 1_{Ra} and $g : R \to Ra$ given by $g(x) = ax$ for all $x \in R$, there is an R-homomorphism $f : Ra \to R$ which gives commutativity:

$$\begin{tikzcd} & Ra \arrow[dl, dotted, "f"'] \arrow[d, "1_{Ra}"] & \\ R \arrow[r, "g"] & Ra \arrow[r] & 0. \end{tikzcd}$$

Then $gf = 1_{Ra}$ and so $gf(a) = a$, i.e., $af(a) = a$. Now let $c = 1 - f(a) + a$. This gives $ac = a - af(a) + a^2 = a^2$. Now suppose that $d \in R$ with $cd = 0$. Then $a^2d = acd = 0$. Moreover, since $af(a) = a$, we get $ad = af(a)d = f(a^2d) = 0$. This gives $d = 1d = (c + f(a) - a)d = cd = 0$. We've now shown that if $cd = 0$ for some $d \in R$ then $d = 0$. Thus c is a regular element of R. If b is any regular element of R we have $b/c \in Q(R)$ and $a/b = ac/bc = a^2/bc = a^2b/b^2c = (a/b)(b/c)(a/b)$. Since every element in $Q(R)$ can be written as a/b for some $a \in R$ and some regular element $b \in R$, it follows that $Q(R)$ is regular. □

In the forthcoming Example 2.75, we will use Proposition 2.65 (iii) to show that the converse to part (i) of the Proposition is not true in general.

Definition 2.66. A ring R is said to be **PF** if every principal ideal of R is flat.

The next theorem we establish is given as [Gl1, Theorem 4.2.2].

Theorem 2.67. *Let R be a ring. Then the following statements are equivalent:*

(1) *R is a PF ring.*

(2) *For any prime ideal P, the localisation R_P is an integral domain.*

(3) *For any maximal ideal M, the localisation R_M is an integral domain.*

(4) *R is reduced and any maximal ideal M of R contains exactly one minimal prime ideal $\mathfrak{P}$. In particular, $\mathfrak{P} = O_M$ and $R_{\mathfrak{P}} = Q(R_M)$, the quotient field of R_M.*

Proof. (1) $\Rightarrow$ (2). Let $\frac{a}{s}, \frac{b}{t} \in R_P$ for some $a, b \in R$, $s, t \notin P$. Suppose that $\frac{a}{s} \cdot \frac{b}{t} = 0$ but $\frac{a}{s} \neq 0$. We need to show that $\frac{b}{t} = 0$, in order to show that R_P is an integral domain.

Let $I = R_P\frac{a}{s}$. Notice that $0 \neq I = R_P\frac{a}{s} = R_P \otimes Rr$ for some $r \neq 0$. Since R_P is flat and Rr is also flat by assumption, by Corollary 2.17, I is a flat R_P-module. Then, by Lemma 2.19, I is a free R_P-module, say with basis $\{x_\lambda : \lambda \in \Lambda\}$. For each $\lambda \in \Lambda$, we have $x_\lambda = \frac{r_\lambda}{s_\lambda}\frac{a}{s}$ for some $r_\lambda \in R$ and $s_\lambda \in R \setminus P$. Then $x_\lambda \frac{b}{t} = \frac{r_\lambda}{s_\lambda}\frac{a}{s} \cdot \frac{b}{t} = 0$. This is impossible if $\frac{b}{t} \neq 0$ since $\operatorname{ann}(x_\lambda) = 0$. Hence $\frac{b}{t} = 0$, as required.

(2) $\Rightarrow$ (3). This is clear since every maximal ideal is prime.

(3) $\Rightarrow$ (4). Let $M \in \operatorname{Max}(R)$ and $\mathfrak{P} \in \operatorname{Min}(R)$ with $\mathfrak{P} \subseteq M$. Then $O_M \subseteq \mathfrak{P}$ since if $r \in O_M$ then $ru = 0$ for some $u \in R \setminus \mathfrak{P}$, giving $ru = 0 \in \mathfrak{P}$ with $u \notin \mathfrak{P}$, so $r \in \mathfrak{P}$ since $\mathfrak{P}$ is prime. Now, since $O_M = \ker(\phi_M)$ where $\phi_M : R \to R_M$ is the natural ring homomorphism, we have R/O_M isomorphic to $\operatorname{im}(\phi_M)$ which is a subring of R_M. Then, since R_M is an integral domain by (3), it follows that R/O_M is an integral domain and therefore O_M is a prime ideal. Consequently, O_M is the unique minimal prime ideal contained in M since it is contained in every minimal prime ideal contained in M. Thus, we obtain $\operatorname{Min}(R) = \{O_M : M \in \operatorname{Max}(R)\}$.

By Corollary 1.6, $\mathrm{Nil}(R) = \cap\{\mathfrak{P} : \mathfrak{P} \in \mathrm{Min}(R)\}$ so to show that R is reduced we need to show $\cap\{O_M : M \in \mathrm{Max}(R)\} = 0$. If $x \in \cap\{O_M : M \in \mathrm{Max}(R)\}$, then for any $M \in \mathrm{Max}(R)$ there exists $u_M \in R \setminus M$ such that $u_M x = 0$. Thus $\mathrm{ann}(x) \nsubseteq M$ for every maximal ideal M of R. This shows that $\mathrm{ann}(x) = R$ and so $x = 0$. Hence $\cap\{O_M : M \in \mathrm{Max}(R)\} = 0$ and so R is reduced.

For any $\mathfrak{P} \in \mathrm{Min}(R)$, there exists a maximal ideal M such that $\mathfrak{P} \subseteq M$ and so $\mathfrak{P} = O_{\mathfrak{P}}$ by Lemma 1.35. Notice that $\mathfrak{P}R_{\mathfrak{P}} = 0$ since if $\frac{r}{s} \in \mathfrak{P}R_{\mathfrak{P}}$ then we have $s \notin \mathfrak{P}$ and $r \in \mathfrak{P} = O_{\mathfrak{P}}$ and so there exists $u \notin \mathfrak{P}$ such that $ur = 0$ and this gives $\frac{r}{s} = \frac{ur}{us} = \frac{0}{us} = 0$. Since, by Lemma 1.28, $\mathfrak{P}R_{\mathfrak{P}}$ is the only prime (and so only maximal) ideal of $R_{\mathfrak{P}}$ and $\mathfrak{P}R_{\mathfrak{P}} = 0$ it follows that $R_{\mathfrak{P}}$ is a field.

We finally show that $R_{\mathfrak{P}}$ is (isomorphic to) the quotient field $Q(R_M)$ of the localisation R_M. The elements of R_M are fractions of the form r/u, where $r \in R$ and $u \in R \setminus M$. So the elements of $Q(R_M)$ are of the form $\dfrac{r/u}{s/t}$, where $r, s \in R$ and $u, t \in R \setminus M$ and $s \neq 0$. Then $\dfrac{r/u}{s/t} = \dfrac{rt/ut}{su/ut}$. We now show that $su \notin O_M$. If otherwise then there is a $c \notin M$ such that $suc = 0$. Then, in the integral domain, R_M we have

$$(s/1) \cdot (1/t) = s/t = suc/tuc = 0/tuc = 0$$

and so either $s/1 = 0$ or $1/t = 0$, which is impossible since $s \neq 0$. Thus $su \notin O_M$.

Using this we define $\alpha : Q(R_M) \to R_{\mathfrak{P}}$ by $\dfrac{r/u}{s/t} = \dfrac{rt/ut}{su/ut} \xmapsto{\alpha} rt/su$, $rt/su \in R_{\mathfrak{P}} = R_{O_M}$. To see that α is well defined, let $\dfrac{r_1/u_1}{s_1/t_1} = \dfrac{r_2/u_2}{s_2/t_2}$ in $Q(R_M)$. Then $r_1s_2/u_1t_2 = r_2s_1/u_2t_1$ in R_M. Thus there exists $q \in R\setminus M$ such that $q(r_1s_2u_2t_1) = q(u_1t_2r_2s_1)$. However, this also gives $r_1t_1/s_1u_1 = r_2t_2/s_2u_2$ which implies that $\alpha\left(\dfrac{r_1/u_1}{s_1/t_1}\right) = \alpha\left(\dfrac{r_2/u_2}{s_2/t_2}\right)$. Moreover α is a ring isomorphism since, for all $r_1, r_2 \in R, s_1, s_2 \in R \setminus \{0\}, t_1, t_2, u_1, u_2 \in R \setminus M$,

(i) $\alpha\left(\dfrac{r_1/u_1}{s_1/t_1} \cdot \dfrac{r_2/u_2}{s_2/t_2}\right) = \alpha\left(\dfrac{r_1r_2/u_1u_2}{s_1s_2/t_1t_2}\right) = \dfrac{t_1t_2r_1r_2}{s_1s_2u_1u_2} = \dfrac{t_1r_1}{s_1u_1} \cdot \dfrac{t_2r_2}{s_2u_2} = \alpha\left(\dfrac{r_1/u_1}{s_1/t_1}\right) \cdot \alpha\left(\dfrac{r_2/u_2}{s_2/t_2}\right)$,

(ii) $\alpha\left(\dfrac{r_1/u_1}{s_1/t_1}+\dfrac{r_2/u_2}{s_2/t_2}\right)=\alpha\left(\dfrac{r_1s_2/u_1t_2+r_2s_1/u_2t_1}{s_1s_2/t_1t_2}\right)=\left(\dfrac{(r_1s_2u_2t_1+r_2s_1u_1t_2)/u_1t_2u_2t_1}{s_1s_2/t_1t_2}\right)=$ $\dfrac{(t_1t_2)(r_1s_2u_2t_1+r_2s_1u_1t_2)}{(s_1s_2)(u_1t_2u_2t_1)}=\dfrac{r_1s_2u_2t_1+r_2s_1u_1t_2}{s_1s_2u_1u_2}=\dfrac{t_1r_1}{s_1u_1}+\dfrac{t_2r_2}{s_2u_2}=\alpha\left(\dfrac{r_1/u_1}{s_1/t_1}\right)+\alpha\left(\dfrac{r_2/u_2}{s_2/t_2}\right)$,

(iii) α is a one-to-one mapping since, if $r\in R, s\in R\setminus\{0\}, t,u\in R\setminus M$ with $\alpha\left(\dfrac{r/u}{s/t}\right)=0$, then $tr/su=0$ in $R_{\mathfrak{P}}=R_{O_M}$ and so there exists $q\in R\setminus O_M$ such that $qtr=0$. Hence $\dfrac{r/u}{s/t}=\dfrac{qtr/qtu}{s/t}=\dfrac{0/qtu}{su/tu}=0$, and

(iv) α is an onto mapping since if $\dfrac{a}{b}\in R_{O_M}$ (with $a\in R$ and $b\in R\setminus O_M$), then $\dfrac{a/1}{b/1}\in Q(R_M)$ with $\alpha\left(\dfrac{a/1}{b/1}\right)=\dfrac{a}{b}$.

(4) $\Rightarrow$ (1). Let $M\in \mathrm{Max}(R)$. Note first that given $r\in O_M$ and $\dfrac{a}{s}\in R_M$ where $a,s\in R$, $s\notin M$, then, since there exists $u\notin M$ such that $ru=0$, we have $r\dfrac{a}{s}=\dfrac{r}{1}\cdot\dfrac{ua}{us}=\dfrac{rua}{us}=0$. Thus $O_MR_M=0$. Now suppose that $\dfrac{a}{s},\dfrac{b}{t}\in R_M$ with $\dfrac{a}{s}\cdot\dfrac{b}{t}=\dfrac{0}{1}$, where $a,b,s,t\in R$; $s,t\notin M$. Then there exists $u\notin M$ such that $uab=0$. This gives $ab\in O_M$ and so, since O_M is a prime ideal by (4), we have either $a\in O_M$ or $b\in O_M$. Thus there exists $v\notin M$ such that either $av=0$ or $bv=0$. Then either $\dfrac{a}{s}=\dfrac{av}{sv}=\dfrac{0}{1}$ or $\dfrac{b}{t}=\dfrac{bv}{tv}=\dfrac{0}{1}$. This has shown that R_M is an integral domain.

Now let $a\in R$. Then we have the following s.e.s. of R-modules

$$0\longrightarrow \mathrm{ann}(a)\xrightarrow{i} R\xrightarrow{f} Ra\longrightarrow 0 \quad (*),$$

where $f(r)=ra$ for every $r\in R$ and i is the inclusion map. Tensoring $(*)$ with the flat R-module R_M, i.e., localising $(*)$ at M, gives the the s.e.s. of R_M-modules

$$0\longrightarrow (\mathrm{ann}(a))_M\xrightarrow{i\otimes R_M} R_M\xrightarrow{f\otimes R_M}(Ra)_M\longrightarrow 0\,.$$

However, $(Ra)_M=\left\{\dfrac{ra}{s}: r\in R, s\notin M\right\}=\left\{\dfrac{r}{s}\cdot\dfrac{a}{1}: r\in R, s\notin M\right\}=R_M\dfrac{a}{1}$. Moreover, by Lemma 1.30, $(\mathrm{ann}_R(a))_M=\mathrm{ann}_{R_M}\left(\dfrac{a}{1}\right)$ and so we have the s.e.s.

$$0\longrightarrow \mathrm{ann}\left(\frac{a}{1}\right)\xrightarrow{i\otimes R_M} R_M\xrightarrow{f\otimes R_M} R_M\frac{a}{1}\longrightarrow 0 \quad (**).$$

However, since R_M is an integral domain, we have

$$\operatorname{ann}\left(\frac{a}{1}\right) = \left\{\frac{b}{s} : \frac{b}{s}\cdot\frac{a}{1} = \frac{0}{1},\ b, s \in R, s \notin M\right\} = \begin{cases} 0 & \text{if} \quad \text{(i)} \quad \dfrac{a}{1} \neq 0, \\ R_M & \text{if} \quad \text{(ii)} \quad \dfrac{a}{1} = 0. \end{cases}$$

If (i) holds, then $R_M\left(\frac{a}{1}\right) \simeq R_M$ as R_M-modules and so $R_M\left(\frac{a}{1}\right)$ is a flat R_M-module. On the other hand, if (ii) holds, then $R_M\left(\frac{a}{1}\right) = 0$, also a flat R_M-module. Thus in either case $\text{w.dim}_{R_M}(R_M\left(\frac{a}{1}\right)) = 0$, i.e., $\text{w.dim}_{R_M}((Ra)_M) = 0$ for every $M \in \text{Max}(R)$. Then by Theorem 2.53, $\text{w.dim}_R(Ra) = 0$. Thus Ra is a flat R-module and so R is a PF ring. □

The following result, due to Jensen [J2], pinpoints an exact relationship between arithmetical rings and rings of weak global dimension at most 1.

Theorem 2.68. *For any ring R, we have* $\text{w.gl.dim}(R) \leq 1$ *if and only if R is reduced and arithmetical.*

Proof. If $\text{w.gl.dim}(R) \leq 1$, R is a PF ring by Lemma 2.51 and so, by Theorem 2.67, R is reduced. To show that R is arithmetical, by Proposition 1.40, it suffices to show that the localization ring R_M is a chain ring for any maximal ideal M of R. Since $\text{w.gl.dim}(R) \leq 1$ we have $\text{w.gl.dim}(R_M) \leq 1$, by Theorem 2.53, and so, by Lemma 2.51, every ideal of R_M is flat. Then, by Theorem 2.29, R_M is a chain ring as required.

Conversely assume that R is reduced and arithmetical. Then, for any maximal ideal M of R, the localization R_M is a reduced chain ring by Corollary 1.32 and Proposition 1.40. Let a, b be nonzero elements of R_M. Since R_M is a chain ring, by Proposition 1.37 we may suppose that $a = bx$ for some $x \in R_M$. Then $abx = a^2 \neq 0$ since R_M is reduced, and so $ab \neq 0$. This has shown that R_M is an integral domain. Now let J be a nonzero finitely generated ideal of R_M. Since R_M is a chain ring, J is principal (see Proposition 1.37), say $J = R_M c$, where c is a nonzero element of the integral domain R_M. Then J is a free, so flat, R_M-module (with basis $\{c\}$) and so, by Corollary 2.56, $\text{w.gl.dim}(R_M) \leq 1$. Applying Theorem 2.53 finishes the proof. □

The reduced condition in Theorem 2.68 is not redundant. To see this, the easiest counterexample is the chain ring $R = \mathbb{Z}_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$. Since $\overline{2}^2 = \overline{0}$, R is not reduced and so $\text{w.gl.dim}(R) > 1$, by Theorem 2.68. (In fact $\text{w.gl.dim}(R) = \infty$, by Example 2.36.)

We restrict the next two definitions to integral domains.

Definition 2.69. An ideal A of an integral domain R is called a **cancellation ideal** if, whenever $AB = AC$ where B, C are nonzero ideals of R then $B = C$.

For example, every invertible ideal A is a cancellation ideal since if B, C are ideals such that $AB = AC$ then $A^{-1}AB = A^{-1}AC$ and so, since $AA^{-1} = R$, we get $B = C$.

Definition 2.70. Let R be an integral domain with quotient field Q. An element $q \in Q$ is said to be **integral over** R if there is a monic polynomial $f(x) \in R[x]$ such that $f(q) = 0$, i.e., there exist $a_0, a_1, \ldots, a_{n-1} \in R$ such that

$$a_0 + a_1 q + a_2 q^2 + \cdots + a_{n-1} q^{n-1} + q^n = 0.$$

Note that any element $r \in R$ is integral over R, since $a_0 + a_1 r = 0$ on taking $a_0 = -r, a_1 = 1$.

If the elements of R are the only elements of Q which are integral over R, then R is said to be **integrally closed**.

Our next result is Proposition 3.9 of [BuS].

Proposition 2.71. *Let R be an integral domain with the property that, for any nonzero $r \in R$ and nonzero finitely generated ideal I of R, if $I(r) = I^2$, then $I \subseteq (r)$. Then R is integrally closed.*

Proof. Let Q denote the quotient field of R and q be an element of Q which is integral over R. Then q is a root of a monic polynomial over R, say of degree $n+1$, where $n \geq 0$. Let $F = (1, q, q^2, \ldots, q^n)$ be the R-submodule of Q generated by $\{q^i\}_{i=0}^n$. Then $F^2 = F$ (where, using the multiplication of Q, $F^2 = \{\sum_{j=1}^t y_j z_j : y_j, z_j \in F, t \in \mathbb{N}\}$) since any $w \in F$ can be written as $1_R w$ and, for each $s, t \in \mathbb{N}$, we have $q^s q^t \in F$ since q^n is a linear combination of $1, q, q^2, \ldots, q^{n-1}$. Let $q = a/b$ where $a, b \in R$ and $b \neq 0$. Then, setting $d = b^n$, we can write $F = E(1/d)$ where E is the nonzero ideal of R given by $E = (b^n, b^{n-1}, \ldots, ba^{n-1}, a^n)$ and $(1/d)$ is the R-submodule of Q generated by $1/d$. Then, since $F = F^2$, we get $E(1/d) = E^2(1/d^2)$ which, on multiplication by d^2, gives $E(d) = E^2$. Our hypothesis then gives $E \subseteq (d)$. In particular, $ab^{n-1} \in (d) = (b^n)$ and so $q = a/b = ab^{n-1}/b^n \in R$. Thus R is integrally closed. □

We now present a result of Butts and Smith [BuS].

Proposition 2.72. *Let R be an integrally closed domain and $a, b \in R$ with $a \neq 0$ such that $a^{n-1}b \in (a^n, b^n)$ for some $n \geq 2$. Then (a, b) is invertible.*

Proof. We use induction on n. If $n = 2$ then $ab \in (a^2, b^2)$ and so $ab = xa^2 + yb^2$ for some $x, y \in R$. Multiplying by y/a^2 we get $(yb/a)^2 - (yb/a) + xy = 0$, an equation in the quotient field Q of R. Thus yb/a is integral over R and so, since R is integrally closed, we have $yb/a \in R$, say $yb/a = z$. Then $az = yb$. Now note that

$$\begin{aligned}
(a,b)(y, 1-z) &= (ay, by, a(1-z), b(1-z)) \\
&= (ay, az, a(1-z), b(1-z)) \text{ (since } by = az) \\
&= (ay, az, a(1-z), xa) \text{ (since } b(1-z) = b - \frac{yb^2}{a} = b - \frac{(ab - xa^2)}{a} = xa). \\
&= (a).
\end{aligned}$$

Then, since the principal ideal (a) is invertible, it follows that (a, b) is also invertible (with inverse $(1/a)(y, 1-z)$), thereby establishing the case of $n = 2$.

Now assume that the Proposition is true for some $n - 1$ where $n \geq 3$. Further assume that $a, b \in R$ with $a^{n-1}b \in (a^n, b^n)$, say $a^{n-1}b = xa^n + yb^n$ where $x, y \in R$. Then, multiplying by y^{n-1}/a^n gives

$$y^{n-1}b/a = y^{n-1}x + y^n b^n/a^n \text{ and so } (yb/a)^n + y^{n-2}(yb/a) - y^{n-1}x = 0.$$

This shows that yb/a is integral over R and so, by assumption, is in R, say $yb/a = z \in R$. Now note that $a^{n-1}b = xa^n + azb^{n-1}$ and so cancelling the nonzero a gives

$$a^{n-2}b = xa^{n-1} + zb^{n-1} \in (a^{n-1}, b^{n-1}).$$

Using our induction hypothesis then shows that (a, b) is invertible, as required. □

Definition 2.73. An integral domain R is called a **Prüfer domain** if each of its nonzero finitely generated ideals is invertible.

The following information on Prüfer domains can be found in [LM, Theorem 6.6], [R, Chapter 4], or Gilmer [Gi2]. The equivalence of (i) and (ii) is due to Krull [K]. The equivalence of (ii) and (iii) is due to Prüfer [P]. Since ideals are invertible precisely when they are projective (by Proposition 2.63 (a)), (iv) and (v) are equivalent to (ii) and (iii). We give details of (vii) ⇒ (viii) ⇒ (ix) ⇒ (iii).

Proposition 2.74. *Let R be an integral domain. The following statements are equivalent.*

(i) *R is an arithmetical domain.*

(ii) *R is a Prüfer domain.*

(iii) *For any nonzero $a, b \in R$, the 2-generated ideal (a, b) is invertible.*

(iv) *For any nonzero $a, b \in R$, the 2-generated ideal (a, b) is projective.*

(v) *R is a semi-hereditary domain.*

(vi) $\text{w.gl.dim}(R) \leq 1$.

(vii) *Every finitely generated ideal of R is a cancellation ideal.*

(viii) *R is integrally closed and, for some $n \in \mathbb{N}$, $(a, b)^n = (a^n, b^n)$ for all $a, b \in R$.*

(ix) *R is integrally closed and, for some $n \in \mathbb{N}$, $a^{n-1}b \in (a^n, b^n)$ for each $a, b \in R$.*

Proof. (vii) $\Rightarrow$ (viii). If R satisfies (vii), then R is integrally closed by Proposition 2.71. Moreover, since

$$(a,b)^3 = (a^3, a^2b, ab^2, b^3) = (a,b)(a^2,b^2),$$

cancelling (a,b) gives $(a,b)^2 = (a^2,b^2)$ and so R satisfies (viii).

(viii) $\Rightarrow$ (ix) is clear.

(ix) $\Rightarrow$ (iii). Let a,b be nonzero elements of R. If R satisfies (ix), it follows from Proposition 2.72 that the ideal (a,b) is invertible, as required. $\square$

We note that, with regard to conditions (viii) and (ix) of Theorem 2.74 and in reply to a question of H. Butts, J. Ohm [Oh] has provided an example of an integral domain R which is not integrally closed but $(a,b)^2 = (a^2,b^2)$ for all $a,b \in R$.

While an integral domain R is semi-hereditary if and only if it has weak global dimension at most 1, this is not true in general for all commutative rings, as the following example from [Cl1] shows.

Example 2.75. Let $\mathbb{Q}$ denote the field of rational numbers and let S denote the direct product $\prod_{i=1}^{\infty} S_i$ where, for each positive integer i, $S_i = \mathbb{Q}[x]$, the ring of polynomials in one variable, x, over $\mathbb{Q}$. Then each element of S may be regarded as a sequence $(f_1, f_2, f_3, \ldots)$ where $f_i \in \mathbb{Q}[x]$ for $i = 1, 2, 3, \ldots$. Let R denote the subring of S generated by the sequence $(x, 0, x, 0, \ldots)$ and all sequences of the form $(f_1, f_2, \ldots, f_n, q, q, q, \ldots)$ where $f_1, f_2, \ldots, f_n \in \mathbb{Q}[x]$ and $q \in \mathbb{Q}$. We shall show that w.gl.dim$(R) = 1$ but R is not semi-hereditary.

The general form of an element of R is

$$a = (f_1, f_2, f_3, \ldots, f_n, q, q + g_1(x), q, q + g_2(x), q, q + g_3(x), \ldots)$$

where n is an odd integer, $f_1, f_2, \ldots, f_n \in \mathbb{Q}[x], q \in \mathbb{Q}$, and, for each positive integer $i, g_i(x)$ is an element of $\mathbb{Q}[x]\,x$. Suppose such an element a is nilpotent. Then for some positive integer s we have

$$a^s = (f_1^s, f_2^s, \ldots, f_n^s, q^s, (q + g_1(x))^s, q^s, (q + g_2(x))^s, q^s, \ldots) = 0.$$

This implies that $0 = f_1^s, f_2^s, \ldots, f_n^s, q^s$ and $(q + g_i(x))^s = 0$ for each positive integer i. Thus, since $f_1, f_2, \ldots, f_n, q$, and $q + g_1(x)$ are all elements of the integral domain $\mathbb{Q}[x]$, we have $a = 0$. This shows that R is a reduced ring.

It is easily seen that any maximal ideal of R is generated by a sequence of the form $(1, 1, \ldots, 1, f, 1, 1, \ldots)$, where the unit element occupies every position except one which is occupied by a non-constant irreducible polynomial f in $\mathbb{Q}[x]$. Let M_1 denote the maximal ideal of R generated by $(f, 1, 1, \ldots)$, where f is a non-constant irreducible polynomial. Let y, z be any two nonzero elements of the local ring R_{M_1}. Then we may write $y = \dfrac{a}{b}$ and $z = \dfrac{c}{d}$ where a, b, c, d are nonzero elements of R and $b, d \notin M_1$. We will show that either $y \in R_{M_1} z$ or $z \in R_{M_1} y$. To do this, we construct two sequences $m = (m_1, m_2, \ldots) \in R$ and $n = (n_1, n_2, \ldots) \in R \setminus M_1$ such that either $y = \dfrac{m}{n} z$ or $z = \dfrac{m}{n} y$, i.e., $\dfrac{c}{d} = \dfrac{ma}{nb}$ or $\dfrac{a}{b} = \dfrac{mc}{nd}$.

Denote the elements occupying the first positions of a, b, c, d by a_1, b_1, c_1, d_1, respectively. Then $b_1, d_1 \notin \mathbb{Q}[x]\, f$. If either $a_1 = 0$ or $c_1 = 0$, we define $m_1 = 0$ and $n_1 = 1$. Assuming that a_1 and c_1 are both nonzero, let k_1 denote the least common multiple of the two elements $a_1 d_1$ and $b_1 c_1$.[2] Then there exist elements m_1, n_1 of $\mathbb{Q}[x]$ such that $m_1(a_1 d_1) = k_1 = n_1(b_1 c_1)$.

Next note that, since $\mathbb{Q}[x]$ is a unique factorization domain, m_1 and n_1 can not both have f in their irreducible factorization, since otherwise the least common multiple property of k_1 is contradicted. Suppose $n_1 \notin \mathbb{Q}[x]\, f$.

Now, for $i = 2, 3, \ldots$ let a_i, b_i, c_i, d_i denote the i th. elements of the sequences a, b, c, d, respectively, and let k_i denote the least common multiple of $a_i d_i$ and $b_i c_i$ in $\mathbb{Q}[x]$ if a_i, b_i are both nonzero and otherwise we take k_i to be zero. Then, as for $i = 1$, for $i = 2, 3, \ldots$ there exist elements m_i, n_i of $\mathbb{Q}[x]$ such that $m_i(a_i d_i) = k_i = n_i(b_i c_i)$.

[2]Since $\mathbb{Q}[x]$ is a principal ideal domain, given any two elements $v, w \in \mathbb{Q}[x]$, there is an element $z \in \mathbb{Q}[x]$ such that $\mathbb{Q}[x]v \cap \mathbb{Q}[x]w = \mathbb{Q}[x]z$. The element z is then a **least common multiple** of v and w and it is unique up to unit multiples.

Let n, m denote the sequences $(n_1, n_2, \ldots)$ and $(m_1, m_2, \ldots)$ respectively. Then it is easily checked that $n, m \in R$, $\frac{m}{n} \in R_{M_1}$, and $\frac{m}{n} \cdot \frac{a}{b} = \frac{c}{d}$. On the other hand, if $m_1 \notin \mathbb{Q}[x] f$ then the previous argument will give $\frac{n}{m} \in R_{M_1}$ and $\frac{n}{m} \cdot \frac{c}{d} = \frac{a}{b}$. It follows that the ideals of R_{M_1} are linearly ordered. By similar arguments it may be proved that for any maximal ideal M of R the ideals of the local ring R_M are linearly ordered. Thus R is arithmetical and so, since it is also reduced, we have w.gl.dim$(R) \leq 1$ by Theorem 2.68.

We will now show that the total ring of quotients of R is not a von Neumann regular ring. Let $\frac{a}{b}$ be an element of Q, the total ring of quotients of R. Then b is not a zero-divisor of R and so it is a sequence of nonzero elements. Moreover there exists an odd integer n such that the nth elements a_n, b_n of a, b respectively, are of the form $a_n = q + f(x), b_n = p + g(x)$ where $p, q \in \mathbb{Q}$ and $f(x), g(x) \in x \cdot \mathbb{Q}[x]$. Now $\frac{(x, 0, x, 0, x, 0, \ldots)}{(1, 1, 1, \ldots)}$ is an element of Q. Moreover denoting this element by w, if $w^2 \frac{a}{b} = w$ then $b \cdot (x, 0, x, 0, \ldots) = a(x^2, 0, x^2, 0, \ldots)$. Thus, on comparing the nth elements of these sequences, we have $(q + f(x)) \cdot x^2 = (p + g(x)) \cdot x$. This implies that $p = 0$ and so b is a zero-divisor.

This is in contradiction to the definition of b. Thus there exists no element $\frac{a}{b}$ of Q such that $w^2 \frac{a}{b} = w$. Hence Q is not von Neumann regular. By Proposition 2.65 (iii), R is not semi-hereditary.

Thus w.gl.dim$(R) = 1$ and R is not semi-hereditary, as required.

Chapter 3

The Ideal Structure of Gaussian Rings.

3.1 General properties of Gaussian rings.

In this chapter, we continue dealing with a commutative ring R.

Definition 3.1. Let $R[x]$ be the ring of polynomials of the ring R in the indeterminate x and let $f(x) \in R[x]$ with $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, where $a_0, a_1, \ldots, a_n \in R$. The **content** of f, denoted by $c(f)$, is the ideal of R generated by the set of its coefficients $\{a_0, a_1, \ldots, a_n\}$. In this case, we usually write $c(f) = (a_0, a_1, \ldots, a_n)$.

In the next theorem we introduce some important properties of the content.

Theorem 3.2. *Let $f, g \in R[x]$, $r \in R$. Then*

(1) $c(f + g) \subseteq c(f) + c(g)$,

(2) $c(fg) \subseteq c(f)c(g)$, *and*

(3) $c(rf) = rc(f)$.

Proof. (1) Suppose $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$. Then $(f+g)(x) = \sum_i^{\max(m,n)}(a_i + b_i)x^i$ so $c(f+g)$ is the ideal generated by $a_i + b_i$ for $i = 0, 1, \ldots, \max(m,n)$. Since $a_i \in c(f)$ and $b_i \in c(g)$ for each i, we have $\sum r_i(a_i + b_i) = \sum r_ia_i + \sum r_ib_i$ (where $r_i \in R$) so $c(f+g) \subseteq c(f) + c(g)$ as required.

(2) Take f and g as above. Then $fg = a_0b_0 + (a_1b_0 + a_0b_1)x + \cdots + (\sum_{i+j=k} a_ib_j)x^k + \cdots + a_mb_nx^{m+n}$. Thus $c(fg)$ is generated by $c_k = \sum_{i+j=k} a_ib_j$ for $k = 0, 1, \ldots, m+n$. Each c_k is in $c(f)c(g)$ since $a_i \in c(f)$ and $b_j \in c(g)$. Hence it follows that $c(fg) \subseteq c(f)c(g)$.

(3) Consider f as above and let $r \in R$. Then $c(rf) = c(r(a_0 + a_1x + a_2x^2 + \cdots + a_mx^m)) = c(ra_0 + ra_1x + ra_2x^2 + \cdots + ra_mx^m) = (ra_0, ra_1, ra_2, \ldots, ra_m) = r(a_0, a_1, a_2, \ldots, a_m) = rc(f)$. □

Note. Clearly $c(f) = c(x^tf)$, for all $t \in \mathbb{N}$.

The following theorem is known as the *Dedekind-Mertens Lemma.*.[1]

Theorem 3.3. *Let f, g be nonzero polynomials in $R[x]$ with $\deg(g(x)) = n$. Then $c(f)^{n+1}c(g) = c(f)^nc(fg)$.*

Proof. By Theorem 3.2 (2), $c(fg) \subseteq c(f)c(g)$ and this gives $c(f)^nc(fg) \subseteq c(f)^nc(f)c(g) = c(f)^{n+1}c(g)$. This proves the reverse inclusion. However the other inclusion is not as easy to prove. We use induction on m and n, where $m = \deg(f(x))$.

First suppose that f is a monomial, i.e., $f(x) = a_mx^m$, where a_m is a nonzero element of R. Then $c(f) = Ra_m$ and so $c(f)^{n+1}c(g) = (Ra_m)^{n+1}c(g) = (Ra_m)^nRa_mc(g)$. Thus we have $c(f)^{n+1}c(g) = (Ra_m)^{n+1}c(g) = (Ra_m)^nRa_mc(g) = (Ra_m)^na_mc(g) = (Ra_m)^nc(a_mg)$ by Theorem 3.2 (3). Notice that $(fg)(x) = a_mg(x)x^m$ and from the note above we have $c(g(x)x) = c(g(x))$ and so $c(fg) = c(a_mg(x)x) = a_mc(g(x)x^m) = a_mc(g(x))$. Then $c(f)^{n+1}c(g) = (Ra_m)^nc(a_mg) = c(f)^nc(fg)$, as required.

[1] According to Heinzer and Huneke [HeHu], Dedekind [De] proved a weaker version of this Lemma in 1892 for rings of algebraic integers, while in the same year Mertens [Me] proved the Lemma for rings of characteristic 0 and $n = \deg(g)^2$. Then Prüfer [P] reproved the result in 1932 with $n = \deg(g)$. The first mention of the name "Dedekind–Mertens Lemma" appears to be made by Krull in [K2].

Now suppose g is a monomial, i.e., $g(x) = b_n x^n$ with b_n a nonzero element of R. Then $c(g) = Rb_n$ and so $c(f)^{n+1}c(g) = c(f)^{n+1}Rb_n = c(f)^n c(f)b_n = c(f)^n c(b_n f) = c(f)^n c(fg)$ since $(fg)(x) = (gf)(x) = b_n f(x)x^n$.

In particular we have now proved the equality if either f or g is a constant polynomial, i.e., either $\deg(f) = 0$ or $\deg(g) = 0$.

For our induction argument we now make the following two assumptions:

(A) For n fixed, $c(f)^{n+1}c(g) \subseteq c(f)^n c(fg)$ for all f with $\deg(f) < m$.

(B) For m fixed, $c(f)^{n+1}c(g) \subseteq c(f)^n c(fg)$ for all g with $\deg(g) < n$.

This double induction is described by the following diagram.

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (0,n) \arrow[d, "n \text{ is fixed; assume that } (r,n) \text{ true for } 0 \le r < m"] \\
(m,0) \arrow[r, "m \text{ is fixed; assume that } (m,s) \text{ true for } 0 \le s < n"'] & (m,n)
\end{tikzcd}$$

From earlier we may assume that f, g are not monomials. Now define :

$$f_1 = f - a_m x^m.$$

$$g_1 = g - b_n x^n.$$

$$h = fg = \sum_{k=0}^{m+n} c_k x^k, \qquad \text{where } c_k = \sum_{i+j=k} a_i b_j.$$

$$h_1 = f_1 g = \sum_{k=0}^{m+n-1} c_k^{(1)} x^k, \qquad \text{where } c_k^{(1)} = \begin{cases} c_k & \text{if } 0 \le k \le m-1, \\ c_k - b_{k-m}a_m & \text{if } m \le k \le m+n-1. \end{cases}$$

$$h_2 = fg_1 = \sum_{k=0}^{m+n-1} c_k^{(2)} x^k, \qquad \text{where } c_k^{(2)} = \begin{cases} c_k & \text{if } 0 \le k \le n-1, \\ c_k - a_{k-n}b_n & \text{if } n \le k \le m+n-1. \end{cases}$$

Then the content of h_1 is the ideal generated by the set $\{c_k^{(1)} : k = 0, \ldots, m+n-1\}$, i.e.,

$$\begin{aligned}
c(h_1) &= \textstyle\sum_{k=0}^{m+n-1} Rc_k^{(1)} \\
&= \textstyle\sum_{k=0}^{m-1} Rc_k + \sum_{k=m}^{m+n-1} R(c_k - b_{k-m}a_m) \\
&\subseteq Rc_0 + \cdots + Rc_{m+n-1} + \textstyle\sum_{k=m}^{m+n-1} Rb_{k-m}a_m \\
&= Rc_0 + \cdots + Rc_{m+n-1} + a_m(\textstyle\sum_{k=m}^{m+n-1} Rb_{k-m}) \\
&= Rc_0 + \cdots + Rc_{m+n-1} + a_m(Rb_0 + \cdots + Rb_{n-1}) \\
&= c(h) + a_m c(g_1).
\end{aligned}$$

Similarly we can show that $c(h_2) = c(fg_1) \subseteq c(h) + b_n c(f_1)$.

Notice that since $a_0, a_1, \ldots, a_m$ generate $c(f)$ and $b_0, b_1, \ldots, b_n$ generate $c(g)$, the ideal $c(f)^{n+1}c(g)$ is generated by elements of the form

$$\alpha = a_0^{t_0} a_1^{t_1} \ldots a_m^{t_m} b_j,$$

where $t_i \geq 0$ for $1 \leq i \leq m$ and $\sum_{i=1}^{m} t_i = n+1$. (To clarify this we give the following example: suppose $I = Ra_1 + Ra_2 = \{r_1a_1 + r_2a_2 : r_1, r_2 \in R\}$. Then $I^2 = \{\sum xy : x, y \in I\}$, where $x = r_1a_1 + r_2a_2$ and $y = s_1a_1 + s_2a_2$ for some r_1, r_2, s_1, and s_2 elements of R. Hence $xy = (r_1s_1)a_1^2 + (r_1s_2)a_1a_2 + (r_2s_1)a_2a_1 + (r_2s_2)a_2^2$.)

It is now sufficient to show that each of these elements is in $c(f)^n c(fg)$, in order to show that $f(c)^{n+1}c(g) \subseteq c(f)^n c(fg)$. Here, there are three different cases:

(1) First, suppose that $t_m \neq 0$ and $j = n$. Then

$$\begin{aligned}
\alpha &= a_0^{t_0} \ldots a_m^{t_m - 1} a_m b_n \quad \text{(note that } t_m - 1 \text{ is defined because } t_m \neq 0) \\
&= a_0^{t_0} \ldots a_m^{t_m - 1} c_{m+n} \in c(f)^n c(fg), \quad \text{as required,}
\end{aligned}$$

since $a_0^{t_0} a_1^{t_1} \ldots a_m^{t_m - 1} \in c(f)^n$ and $c_{m+n} \in c(fg)$.

(We illustrate this with the following example: suppose $n + 1 = 5$, $n = \deg g = 4$ and $\deg f = 6$. Then

$$\begin{aligned}
\alpha &= a_0 a_5^2 a_6^2 b_4 \in c(f)^5 \\
&= (a_0 a_5^2 a_6)(a_6 b_4) \in c(f)^4 c(fg).)
\end{aligned}$$

(2) Now suppose that $t_m \neq 0$ and $j < n$. Then $\alpha = \bar{\alpha} a_m b_j$, where $j < n$ and $\bar{\alpha} =$

$a_0^{t_0} \dots a_m^{t_m-1} \in c(f)^n$, so, since $b_j \in c(g_1)$, we get $\alpha \in c(f)^n a_m c(g_1)$.

(3) Finally, suppose that $t_m = 0$. Then $\alpha = a_0^{t_0} \dots a_{m-1}^{t_{m-1}} b_j \in c(f_1)^{n+1} c(g)$ since $a_0^{t_0} \dots a_{m-1}^{t_{m-1}} \in c(f_1)^{n+1}$ because $\sum_{i=1}^{m-1} t_i = n+1$.

Thus taking the three cases into account, we have

$$c(f)^{n+1}c(g) \subseteq c(f)^n c(fg) + c(f)^n a_m c(g_1) + c(f_1)^{n+1} c(g) \qquad (*).$$

But by assumption (A), since $\deg f_1 < m$ we get $c(f_1)^{n+1}c(g) \subseteq c(f_1)^n c(f_1 g)$. Moreover $c(fg_1) = c(h_1) \subseteq c(fg) + a_m c(g_1)$ as shown above. So we obtain

$$\begin{aligned} c(f_1)^{n+1}c(g) &\subseteq c(f_1)^n c(f_1 g) \quad \text{(by assumption (A))} \\ &\subseteq c(f_1)^n [c(fg) + a_m c(g_1)] \quad \text{(as proved above)} \\ &\subseteq c(f_1)^n c(fg) + c(f_1)^n a_m c(g_1) \\ &\subseteq c(f)^n c(fg) + c(f)^n a_m c(g_1). \end{aligned}$$

Then, since the third term on the right hand side of $(*)$ is contained in the sum of the first and the second terms, $(*)$ becomes

$$c(f)^{n+1}c(g) \subseteq c(f)^n c(fg) + c(f)^n a_m c(g_1) \qquad (**).$$

Let $\deg(g_1) = t$. By assumption (B), since $t < n$, we have $c(f)^{t+1}c(g_1) \subseteq c(f)^t c(fg_1)$. Since $t \leq n-1$, we also have $c(f)^n c(g_1) \subseteq c(f)^{n-1} c(fg_1)$. But we have proved earlier that

$$c(fg_1) = c(h_2) \subseteq c(fg) + b_n c(f_1) \qquad (\dagger).$$

Thus

$$\begin{aligned} c(f)^n a_m c(g_1) &\subseteq c(f)^{n-1} a_m c(fg_1) \\ &\subseteq c(f)^{n-1} a_m c(fg) + c(f)^{n-1} a_m b_n c(f_1) \quad \text{(by } (\dagger)\text{)} \\ &\subseteq c(f)^n c(fg) + c(f)^n c(fg) \quad \text{(since } a_m \in c(f) \text{ and } b_n \in c(g)\text{)} \\ &= c(f)^n c(fg). \end{aligned}$$

This has shown that the last term of $(**)$ is contained in its predecessor and so we have the inclusion $c(f)^{n+1}c(g) \subseteq c(f)^n c(fg)$. Thus, by Theorem 3.2 (3), $c(f)^{n+1}c(g) = c(f)^n c(fg)$ for all $f(x), g(x) \in R[x]$. $\square$

Corollary 3.4. *Let R be an integral domain and $f \in R[x]$. If $c(f)$ is an invertible ideal then $c(f)c(g) = c(fg)$ for all $g(x) \in R[x]$. In particular, if $c(f)$ is a nonzero principal ideal then $c(f)c(g) = c(fg)$ for all $g(x) \in R[x]$.*

Proof. Let J be the inverse of $c(f)$. Given $g(x) \in R[x]$ with $\deg g = n$, we have $c(f)^{n+1}c(g) = c(f)^n c(fg)$ by Theorem 3.3. Then, since $Jc(f) = R$, we get $J^n c(f)^{n+1}c(g) = J^n c(f)^n c(fg)$ giving $c(f)c(g) = c(f)c(g)$.

If $c(f)$ is a nonzero principal ideal, say $c(f) = Ra$ where a is a nonzero element of R, then $c(f)$ is invertible with inverse $J = R(1/a)$. Thus the second statement of the Corollary follows from the first. □

Definition 3.5. A polynomial $f(x) \in R[x]$ is said to be a **Gaussian polynomial** if $c(fg) = c(f)c(g)$ for every polynomial $g(x) \in R[x]$. If every polynomial of the ring $R[x]$ is Gaussian then we call R a **Gaussian ring**.

Examples 3.6. (1) It follows from Corollary 3.4 that any principal ideal domain (PID) is a Gaussian ring and, from Lemma 1.37, any chain domain is also Gaussian.

(2) More generally, from Corollary 3.4 and Definition 2.73 we see that any Prüfer domain is Gaussian. (We will see below, in Corollary 3.11, that, conversely, if R is a Gaussian domain then R is a Prüfer domain.)

(3) Here we present an example of an integral domain which is not Gaussian. (Later, Theorem 3.13 will be used to give examples of local rings which are not Gaussian.)

Let R be the subring of the complex number field $\mathbb{C}$ given by

$$R = \mathbb{Z} + 2i\mathbb{Z} = \{a + 2ib; a, b \in \mathbb{Z}, i^2 = -1\}.$$

Let $f = g = 2i + 2x \in R[x]$. Then $fg = -4 + 8ix + 4x^2$, so $c(fg) = R4 + R8i + R4 = R4 = \{4a + 8bi : a, b \in \mathbb{Z}\}$. On the other hand $c(f) = c(g) = R2i + R2$ and so $c(f)c(g) = R(-4) + R4i + R4 = R4 + R4i$. It follows that $4i \in c(f)c(g)$ but $4i \notin c(fg)$. Thus f is not Gaussian and so R is not a Gaussian ring.

(4) This example shows that the Gaussian ring property does not transfer to polynomial rings in general. Let R be the ring of integers $\mathbb{Z}$, a Gaussian ring by (1) above. Let $S = R[y]$, the ring of polynomials in the indeterminate y over R. Let $f, g \in S[x]$ be given by $f(x) = 2+yx, g(x) = y+2x$. Then $fg = 2y+4x+y^2x+2yx^2 = 2y+(4+y^2)x+2yx^2$, so that $c(fg) = R2y+R(4+y^2)+R2y = R2y+R(4+y^2)$. On the other hand, $c(f) = R2+Ry = c(g)$ and so $c(f)c(g) = (R2+Ry)^2 = R4+R2y+Ry^2$. Since $4 \in c(f)c(g)$ but $4 \notin c(fg)$, $R = \mathbb{Z}[x]$ is not Gaussian. $\square$

We now show that the Gaussian ring property is stable under localization. Given an m.c.s. S of R and any $f(x) \in R[x]$, we may also regard $f(x)$ as an element of $R_S[x]$. We use $c_R(f)$ to denote the content of $f(x)$ in R and $c_{R_S}(f)$ as the content of $f(x)$ in R_S. Then it is straightforward to show that $c_{R_S}(f) = (c_R(f))_S$.

Lemma 3.7. *Let S be an m.c.s. of the ring R. If R is a Gaussian ring then so is R_S.*

Proof. Let $f(x) = p_0 + p_1x + \cdots + p_mx^m$, $g(x) = q_0 + q_1x + \cdots + q_nx^n$ be two polynomials in $R_S[x]$. For each $i \in \{0, 1, \ldots, m\}$ and $j \in \{0, 1, \ldots, n\}$ we have $p_i = a_i/s_i$ and $q_j = b_j/t_j$ where $a_i, b_j \in R$ and $s_i, t_j \in S$. Setting $d = s_0s_1 \cdots s_m$ and $e = t_0t_1 \cdots t_n$ gives $d, e \in S$ and $dp_i, eq_j \in R$ for each i, j and so df, eg are elements of $R[x]$. Since R is Gaussian, $c_R(df)c_R(eg) = c_R(dfeg)$. Now $c_{R_S}(f)c_{R_S}(g) = c_{R_S}(df/d)c_{R_S}(eg/e) = 1/de(c_R(df))_S(c_{R_S}(eg))_S$ by Theorem 3.2 (3). By the remark above, this gives $c_{R_S}(f)c_{R_S}(g) = 1/de(c_R(df)c_R(eg))_S$, and so, since R is Gaussian, $c_{R_S}(f)c_{R_S}(g) = 1/de(c_{R_S}(dfeg))$. Using Theorem 3.2 again, since $1/de \in R_S$, we get $c_{R_S}(f)c_{R_S}(g) = c_{R_S}(fg)$ in R_S as required. $\square$

Next we show that the Gaussian property is inherited by factor rings.

Lemma 3.8. *If I is an ideal of the Gaussian ring R, the factor ring R/I is also Gaussian.*

Proof. Let $\overline{f}, \overline{g} \in R/I[x]$, say $\overline{f}(x) = (r_0 + I) + (r_1 + I)x + \cdots + (r_m + I)x^m$ and $\overline{g}(x) = (s_0 + I) + (s_1 + I)x + \cdots + (s_n + I)x^n$ where each r_i and each s_j are elements of R. Let

$f, g \in R[x]$ be defined as $f(x) = r_0 + r_1 + x + \cdots + r_m x^m$ and $g(x) = s_0 + s_1 x + \cdots + s_n x^n$. Then

$$\begin{aligned}
& c(\overline{f})c(\overline{g}) \\
= \;& (R/I(r_0 + I) + R/I(r_1 + I) + \cdots + R/I(r_m + I)) \cdot (R/I(s_0 + I) + R/I(s_1 + I) + \cdots \\
& \qquad + R/I(s_n + I)) \\
= \;& R/I(r_0 s_0 + I) + R/I(r_0 s_1 + I) + \cdots + R/I(r_0 s_n + I) + R/I(r_1 s_0 + I) + \cdots \\
& \qquad + R/I(r_m s_{n-1} + I) + R/I(r_m s_n + I) \\
= \;& R/I((r_0 s_0, r_0 s_1, \ldots, r_0 s_n, r_1 s_0, \ldots, r_m s_{n-1}, r_m s_n) + I) \\
= \;& R/I(c(f)c(g) + I) \\
= \;& R/I(c(fg) + I) \quad \text{since, by assumption, } R \text{ is Gaussian} \\
= \;& R/I(r_0 s_0 + I) + R/I(r_0 s_1 + r_1 s_0 + I) + \cdots + R/I(r_m s_{n-1} + r_{m-1} s_n + I) \\
& \qquad + R/I(r_m s_n + I) \\
= \;& c(\overline{f}\overline{g})
\end{aligned}$$

as required. □

The following useful result appears in Tsang's dissertation [T].

Proposition 3.9. *Let R be a commutative ring. The following statements are equivalent.*

(i) *R is a Gaussian ring.*

(ii) *R_M is a Gaussian ring for every maximal ideal M of R.*

(iii) *R_P is a Gaussian ring for every $P \in \mathrm{Spec}(R)$.*

Proof. (i) ⇒ (iii). Assume that R is Gaussian. Then R_P is Gaussian by taking $S = R \setminus P$ in Lemma 3.7.

(iii) ⇒ (ii). Since every maximal ideal is prime, this is immediate.

(ii) $\Rightarrow$ (i). Suppose R_M is Gaussian for every $M \in \mathrm{Max}(R)$. Let $f, g \in R[x]$ with $f(x) = a_0 + a_1x + \cdots + a_mx^m$ and $g(x) = b_0 + b_1x + \cdots + b_nx^n$. Then, for any $M \in \mathrm{Max}(R)$, $f_M, g_M \in R_M[x]$, where $f_M(x) = \frac{a_0}{1} + \frac{a_1}{1}x + \cdots + \frac{a_m}{1}x^m$ and $g_M(x) = \frac{b_0}{1} + \frac{b_1}{1}x + \cdots + \frac{b_n}{1}x^n$ and so $c(f_M)c(g_M) = c(f_Mg_M)$ because R_M is Gaussian. Next note that if $I = c(f) = (a_0, \ldots, a_m) = I$ then $c(f_M) = \left(\frac{a_0}{1}, \ldots, \frac{a_m}{1}\right) = I_M$ and similarly if $J = c(g) = (b_0, \ldots, b_n)$ then $c(g_M) = \left(\frac{b_0}{1}, \ldots, \frac{b_n}{1}\right) = J_M$. Thus, using Theorem 1.21 (ii), we get $(c(f)c(g))_M = (c(f))_M(c(g))_M = c(f_M)c(g_M) = c(f_Mg_M) = c((fg)_M) = (c(fg))_M$. Since this holds for any $M \in \mathrm{Max}(R)$, by Theorem 1.24, we obtain $c(f)c(g) = c(fg)$. Thus R is Gaussian. $\square$

Our goal now is to show that an integral domain is Gaussian if and only if it is Prüfer.

Lemma 3.10. *Let R be an integral domain and let $f \in R[x]$. If $c(f)$ is a cancellation ideal then f is Gaussian.*

Proof. By the Dedekind–Mertens Lemma, if $g \in R[x]$ with $\deg(g) = n$, we have $c(f)^{n+1}c(g) = c(f)^nc(fg)$. Since $c(f)$ is a cancellation ideal so is any power of $c(f)$. In particular $c(f)^n$ is a cancellation ideal and so $c(f)^{n+1}c(g) = c(f)^nc(fg)$ gives $c(f)c(g) = c(fg)$. $\square$

We now present the main result of this section. It first appeared in Tsang's unpublished 1965 PhD thesis [T] but was later proved independently by Gilmer, appearing in his 1967 publication [Gi1].

Theorem 3.11. *An integral domain R is Prüfer if and only if R is Gaussian.*

Proof. First suppose that R is a Prüfer domain. Then, as noted in Example 3.6 (2), R is Gaussian.

Conversely, suppose R is Gaussian. We first show that R is integrally closed. Let Q denote the quotient field of R and $q \in Q$ be an integral element over R. We want to show that $q \in R$. Since q is integral, there exists a monic polynomial $f(x) \in R[x]$, say $f(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + x^n$, such that $f(q) = 0$. Then there exists $g(x) \in Q[x]$

such that $f_Q(x) = (x-q)g(x)$. Notice that this implies that $g(x)$ is also monic because the highest power of $(x-q)g(x)$ comes from the term xx^{n-1}, say $1 \cdot b_{n-1}$, where b_{n-1} is the coefficient of the variable with the highest power in $g(x)$ and so, since f is monic, we get $1 = 1 \cdot b_{n-1} = b_{n-1}$.

Since both f and g are monic, we have $c(f) = R$ and $c(g) = Q$. Also, since R is Gaussian so too is Q, by Lemma 3.7, and so we also have $c(f_Q) = c((x-q)g(x)) = c(x-q)c(g(x)) = (R+Rq)c(g)$. Thus $R = (R+Rq)c(g)$. Since g is monic, $R \subseteq c(g)$ and hence $1 \in c(g)$. So $q = q1 \in (R+Rq)c(g) = R$ and therefore $q \in R$ as required.

To finish the proof we need to show that if $a, b \in R$ then $(Ra+Rb)^2 = Ra^2+Rb^2$. Let $f, g \in R[x]$ be defined by $f(x) = a-bx$, $g(x) = a+bx$. Then we get $(fg)(x) = a^2-b^2x$. Thus $c(f) = Ra+Rb = c(g)$, $c(fg) = Ra^2+Rb^2$. Since R is Gaussian we have $c(fg) = c(f)c(g)$ and so $(a,b)^2 = (a^2, b^2)$. Then R is Prüfer by Proposition 2.74, (viii) $\Rightarrow$ (ii). $\square$

3.2 Properties of local Gaussian rings.

The following lemma can be found in [Gl2, Lemma 2.1].

Lemma 3.12. *Let R be a local Gaussian ring and let I be an ideal of R generated by the elements $a_1, a_2, \ldots, a_n$. Then $I^2 = (a_i^2)$ for some $i \in \{1, 2, \ldots, n\}$.*

Proof. Notice first that the case of $n = 1$ trivially holds. Suppose then that $n = 2$. For simplicity, suppose $I = (a, b)$. Let $f(x), g(x) \in R[x]$ be defined as $f(x) = ax+b$, $g(x) = ax-b$. Then $c(f)c(g) = c(fg)$ and so $(a,b)(a,b) = (a^2, b^2)$. Let $h(x) = bx+a \in R[x]$. Then $c(f)c(h) = c(fh)$, so $(a,b)^2 = (ab, a^2+b^2)$. Hence $(ab, a^2+b^2) = (a^2, b^2)$. In particular, $a^2 \in (ab, a^2+b^2)$ and so

$$a^2 = rab + s(a^2+b^2) \qquad (*)$$

for some $r, s \in R$. Then $(1-s)a^2 = rab + sb^2$. Since R is local, by Lemma 1.10 either $1-s$

is a unit or s is a unit. If $1-s$ is a unit we get $a^2 = vrab + vsb^2$, where $v = (1-s)^{-1}$. So if we can show that $ab \in (b^2)$ it will follow that $(a,b)^2 = (b^2)$ (since $(a,b)^2 = (a^2, ab, b^2)$ and $a^2 \in (ab, b^2)$). Now let $k(x) = (a - vrb)x - b \in R[x]$ so that $c(k) = (a - vrb, b)$. Since $c(f)c(k) = c(fk)$ we get

$$\begin{aligned}(a,b)(a-vrb,b) &= (a^2 - avrb, -vrb^2, -b^2)\\ &= (a^2 - avrb, -b^2) = (vrab + vsb^2 - avrb, -b^2)\\ &= (vsb^2, -b^2) = (b^2)\end{aligned}$$

So $(a,b)(a-vrb,b) = (b^2)$ but $ab \in (a,b)(a-vrb,b)$ and so $ab \in (b^2)$ as required.

Suppose otherwise that $1-s$ is not a unit. Then s is a unit and so equation $(*)$ gives $wa^2 = wrab + (a^2 + b^2)$, where $w = s^{-1}$. Hence, $w(a^2 - rab) - a^2 = b^2$ and thus $b^2 \in (a^2, ab)$. Then $(a,b)^2 = (a^2, ab, b^2)$ will be equal to (a^2) if we can show that $ab \in (a^2)$. Let $l(x) = (b - da)x - a$, where $d \in R$. Then $c(l) = (b - da, a)$. Let $p(x) = bx - a$ so that $c(p) = (a,b)$ Then $lp(x) = (b^2 - dab)x^2 - (-ab + da^2 + ab)x + a^2 = (b^2 - dab)x^2 - da^2x + a^2$. Since $c(l)c(p) = c(lp)$ we then obtain $(b-da, a)(a,b) = (b^2 - dab, -da^2, a^2) = (b^2 - dab, a^2) = (-a^2 + w(a^2 - rab) - dab, a^2)$ since $b^2 = w(a^2 - rab) - a^2$. Now choose $d = -wr$. This gives $(b - da, a)(a,b) = (-a^2 + wa^2 - wrab + wrab, a^2) = (-a^2 + wa^2, a^2) = (a^2)$. So in particular, $ab \in (a^2)$ and this shows that the result holds in the case of $n = 2$.

Now suppose that $I = (a_1, a_2, \dots, a_n)$, $n \geq 3$. From the argument above, given $a_i, a_j \in R$, either $a_ia_j \in (a_i^2)$ or $a_ia_j \in (a_j^2)$. Notice that I^2 is generated by $\{a_ia_j : i,j \in \{1,2,\dots,n\}\}$. Then we can reduce this to $\{a_i^2 : i \in \{1,2,\dots,n\}\}$ using the last notice. Then,

$$I^2 = Ra_1^2 + \cdots + Ra_{n-1}^2 + Ra_n^2 = (\text{by the case of } n=2) \begin{cases} Ra_1^2 + \cdots + Ra_{n-2}^2 + Ra_{n-1}^2, \\ \text{or} \\ Ra_1^2 + \cdots + Ra_{n-2}^2 + Ra_n^2. \end{cases}$$

Proceeding in this way we get $I^2 = (a_i^2)$ for some $i \in \{1,2,\dots,n\}$. $\square$

The following can be found in [Luc] as Theorem 3.5. (The necessity appears in Tsang's thesis [T]. The sufficiency was first given by Lucas in [Luc] together with a shorter proof of the necessity which we reproduce here.)

Theorem 3.13. *If R is a local ring then R is Gaussian if and only if the following two conditions hold:*

(i) *For any $a, b \in R$, $(a,b)^2 = (a^2)$ or $(a,b)^2 = (b^2)$.*

(ii) *If $a, b \in R$ and $b^2 \in (a^2)$ and $ab = 0$, then $b^2 = 0$.*

Proof. Suppose R is Gaussian and let $f(x) = a+bx$, $g(x) = b+ax$. Then $fg(x) = ab+(a^2+b^2)x + abx^2$ and so $(a,b)^2 = (a^2, ab, b^2) = (ab, a^2+b^2)$. In particular, $a^2 = rab + s(a^2+b^2)$ for some $r, s \in R$. Since R is local, by Lemma 1.10, either s is a unit or $1-s$ is a unit. Supposing that s is a unit we obtain $b^2 \in (a^2, ab)$. If otherwise, $1-s$ is a unit, then $a^2 \in (b^2, ab)$. Assume, without loss of generality, that the first case holds. Then $(a,b)^2 = (a^2, ab)$ and so $b^2 = ta^2 + uab$ for some $t, u \in R$. Let $h(x) = (ua-b)+ax$. This gives $gh(x) = uab - b^2 + (ab + ua^2 - ab)x + a^2x^2 = uab - b^2 + ua^2x + a^2x^2$ and therefore $c(gh) = (uab - b^2, ua^2, a^2) = (a^2)$ since $b^2 - uab = ta^2$. Since R is Gaussian we obtain $c(gh) = (a^2) = (ua^2 - ab, a^2, uab - b^2, ab) = c(g)c(h)$. Hence $ab \in (a^2)$ and so $b^2 \in (a^2)$. This shows that $(a,b)^2 = (a^2)$. Now, suppose furthermore that $ab = 0$. Observe that since $(a,b)^2 = (a^2)$, we have $b^2 = va^2$ for some $v \in R$. Let $p(x) = bx - va$, then $pg(x) = abx^2 + (b^2 - va^2)x - vab$ and so $c(pg) = (ab, b^2 - va^2, -vab) = (0)$. Using the Gaussian property of R again implies that $c(p)c(g) = (a,b)(b, -va) = (ab, -va^2, b^2, -vab) = (b^2) = 0 = c(pg)$ as required.

To show the converse, suppose that R is a local ring satisfying conditions (i) and (ii). We first observe that if $I = (a_0, a_1, \ldots, a_n)$ and if we choose $i, j \in \{0, 1, \ldots, n\}$ with $i \neq j$ then by (i) $(a_i, b_j)^2 = (a_i)^2$ or $(a_i, b_j)^2 = (a_j)^2$. We can suppose that $(a_i, b_j)^2 = (a_i)^2$. Then $a_ia_j \in (a_i^2)$. Therefore, I^2 which is generated by sums of elements of the form a_ia_j must be written as $I^2 = (a_0^2, a_1^2, \ldots, a_n^2) = (a_i^2)$. Suppose without loss of generality that $I^2 = (a_0^2)$. We now show that $I = (a_0, b_1, \dots, b_n)$, where each $b_i \in (I \cap \operatorname{ann}(I))$. Since $I^2 = (a_0^2)$, then for each $i \geq 1$, we have $a_0 a_i = r_i a_0^2$ for some $r_i \in R$. Let $b_i = a_i - r_i a_0 \in I$. Clearly, $(a_0, b_1, \dots b_n) \subseteq I$ and it can be shown that $I \subseteq (a_0, b_1, \dots b_n)$ since if $c = s_0 a_0 + s_1 a_1 + \cdots + s_n a_n$ then $c = r a_0 + s_1 b_1 + \cdots + s_n b_n$, where $r = s_0 + r_1 s_1 + \cdots + s_n b_n$ and therefore $I = (a_0, b_1, \dots, b_n)$. Moreover, $a_0 b_i = a_0 a_i - r_i a_0^2 = a_0 a_i - a_0 a_i = 0$. Note that $(a_0, b_i)^2 = (a_0^2)$ or (b_i^2). However, $b_i^2 = (a_i - r_i a_0)^2 = a_i^2 - 2 r_i a_0 a_i - a_0^2 \in I^2 = (a_0^2)$ which implies that in either cases we have $(a_0, b_i)^2 = (a_0^2)$ and so (ii) gives that $b_i^2 = 0$. Now consider $(b_i, b_j)^2 = (b_i^2)$ or (b_j^2) which is 0 in both cases. Hence $b_i b_j = 0$ and since $b_i a_0 = 0$ we get $b_i \in \operatorname{ann}(I)$. Thus $I = (a_0, b_1, \dots, b_n)$, where each $b_i \in I \cap \operatorname{ann}(I)$ as wanted.

To show that R is Gaussian consider $f(x) = \sum_{i=0}^m f_i x^i$, $g(x) = \sum_{j=0}^n g_j x^j$. Let I be the ideal $c(f) + c(g)$. Then $I = (f_0, \dots, f_m, g_0, \dots, g_n)$ and by above we may rewrite $I = (f_p, s_0, \dots, \hat{s}_p, \dots, s_m, v_0, \dots, v_n)$, where $f_i = r_i f_p + s_i$, $g_j = t_j f_p + v_j$ with $s_i, v_j \in (I \cap \operatorname{ann}(I))$ and $\hat{s}_p$ means excluding s_p from the generators. We also have $f_p = r_p f_p + s_p$, where $r_p = 1$ and $s_p = 0$. Then for any i, j, $f_i g_j = (r_i f_p + s_i)(t_j f_p + v_j) = r_i t_j f_p^2$. Then $c(f)c(g)$ is generated by the elements $f_i g_j$ which are multiples of f_p. Thus

$$
\begin{aligned}
c(f)c(g) &= c(\textstyle\sum_{i=0}^m r_i f_p x^i) c(\sum_{j=0}^n t_j f_p x^j) \\
&= f_p^2 c(r) c(t), \qquad \text{(where } r = \textstyle\sum_{i=0}^m r_i x^i \text{ and } t = \sum_{j=0}^n t_j x^j.\text{)} \\
&= f_p^2 c(t), \qquad \text{(since } c(r) = R \text{ because it contains } r_p = 1.\text{)} \\
&= f_p^2 (t_0, \dots, t_n) \\
&= f_p (g_0 - v_0, \dots, g_n - v_n) \\
&= f_p (g_0, \dots, g_n) \qquad \text{(since } v_j \in \operatorname{ann}(I).\text{)}
\end{aligned}
$$

Note that $c(fg)$ is generated by $f_i g_j = f_p^2 r_i t_j$. Taking $i = p$ we obtain $f_p g_j = f_p^2 t_j$ to be one of the generators for $c(fg)$ and every other generator will be in the form $f_i g_j = r_i (f_p^2 t_j) = r_i f_p g_j$, a multiple of the generator $f_p g_j$ and so generates $c(fg)$. Thus all the $f_i g_j$'s generate $c(fg)$, which implies the equality $c(fg) = c(f)c(g)$. □

The next theorem builds on the characterization of local Gaussian rings given in

Theorem 3.13. It appears as [BaGl, Theorem 2.2]. The equivalence of statements (a), (b) and (b') appears in Tsang's dissertation [T]. In the words of [BaGl], (c) is "just a reformulation" of (b'), but we give the details.

Theorem 3.14. *Let $(R, \mathfrak{m})$ be a local ring. Then the following statements are equivalent.*

(a) *R is Gaussian.*

(b) *For any finitely generated ideal I of R, the R-module $I/(I \cap \operatorname{ann}(I))$ is cyclic.*

(b') *For any ideal I of R generated by two elements, the R-module $I/(I \cap \operatorname{ann}(I))$ is cyclic.*

(c) *For every $a, b \in R$, there is an element $d \in \operatorname{ann}((a, b))$ such that either $(a, b) = (a, d)$ or $(a, b) = (b, d)$. Moreover, one can choose d so that $b \in d + aR$ (or $a \in d + bR$).*

(d) *For every $a, b \in R$, the following statements hold in R:*

  (i) $(a, b)^2 = (a^2)$ *or* $(a, b)^2 = (b^2)$

  (ii) *If* $(a, b)^2 = (a^2)$ *and* $ab = 0$, *then* $b^2 = 0$.

Proof. (a) $\Leftrightarrow$ (d). This was shown in Theorem 3.13.

(d) $\Rightarrow$ (b). Let $I = (a_0, a_1, \ldots, a_n)$ be an ideal of R. Then, following the same argument of Theorem 3.13, we can rewrite I to be $I = (a_0, b_1, \ldots, b_n)$, where each $b_i \in (I \cap \operatorname{ann}(I))$. Thus $I/(I \cap \operatorname{ann}(I))$ is a finitely generated ideal with generators $a_0 + (I \cap \operatorname{ann}(I)), b_1 + (I \cap \operatorname{ann}(I)), \cdots + b_n + (I \cap \operatorname{ann}(I))$. However $b_i + (I \cap \operatorname{ann}(I)) = 0$ for each $1 \leq i \leq n$. Hence $I/(I \cap \operatorname{ann}(I)) = Ra_0 + (I \cap \operatorname{ann}(I))$, i.e., a cyclic module.

(b) $\Rightarrow$ (b') is immediate since (b') is a special case of (b).

(b') $\Rightarrow$ (c). Let $a, b \in R$ and let $I = (a, b)$. For brevity, let $A = \operatorname{ann}(I)$. By the Second Isomorphism Theorem, $I/(I \cap A) \simeq (I + A)/A$. Thus, by (b'), there is an element $g \in I$ such that $g + A$ generates $(I + A)/A$. Then there exist $r, s, u, v \in R$ such that

(i) $g = ra + sb$,

(ii) $a + A = u(g + A)$ and so $a = ug + e$ for some $e \in A$,

(iii) $b + A = v(g + A)$ and so $b = vg + f$ for some $f \in A$.

Substituting (ii) and (iii) into (i) gives

$$g = rug + re + svg + sf \quad \text{and so} \quad (1 - ru - sv)g = re + sf.$$

Now, since R is local, either $1-ru-sv$ is a unit or $ru+sv$ is a unit. If $1-ru-sv$ is a unit, with inverse w say, then $g = w(1-ru-sv)g = wre+wsf \in A$ which gives $(I+A)/A = 0$. In this case, (c) holds trivially. Thus we now suppose that $ru+sv$ is a unit. Then, since R is local, either ru or sv is a unit. Without loss of generality, we assume that ru is a unit. Then u is also a unit and so, from (i), $R(a + A) = Ru(g + A) = R(g + A)$. Using (ii), we now get

$$R(b + A) = Rv(g + A) \subseteq R(g + A) = R(a + A)$$

and so $b + A = ta + A$ for some $t \in R$. Thus $b = d + ta$ for some $d \in A$. Then $Ra + Rb = Ra + R(ta + d) = Ra + Rd$, as required.

$(c) \Rightarrow (d)$. Let $a, b \in R$ and let $I = (a, b)$. Then, by (c), there is an element $d \in (I \cap \operatorname{ann}(I))$ such that either $I = (a, d)$ or $I = (b, d)$. Furthermore, we can choose d so that either $b = d + ra$ for some $r \in R$ or, respectively, $a = d + sb$ for some $s \in R$. Suppose, without loss of generality that $I = (a, d)$ and $b = d + ra$. Then $I^2 = (a, d)^2 = (a^2, 2ad, d^2) = (a^2)$, since $ad = d^2 = 0$. This has established (d)(i).

Next suppose that $ab = 0$ (and we have $I^2 = (a, d)^2$). Then, since we have $b = d+ra$ for some $r \in R$, we have $b^2 = (d + ra)b = db + rab = 0 + 0 = 0$, establishing (d)(ii). □

Next we examine the relationship between arithmetical rings and Gaussian rings. (We know from Theorem 3.11 that these two classes of rings coincide if we restrict the rings to integral domains.)

Theorem 3.15. *Every arithmetical ring is Gaussian.*

Proof. We first show that any chain ring R is Gaussian. Let $a, b \in R$. Since R is a chain ring, it is local and we have either $a \subseteq Rb$ or $Rb \subseteq Ra$. Choose $Rb \subseteq Ra$. Then $(a,b)^2 = (a^2)$ since $b = ra$ for some $r \in R$ and this gives $ab = ra^2$ and $b^2 = r^2a^2$. Moreover, if $ab = 0$ then $b^2 = rab = 0$. This shows that R satisfies statement (d) of Theorem 3.14 and so R is Gaussian.

Now suppose R is any arithmetical ring. Then, by Proposition 1.40, R_M is a chain ring for every $M \in \mathrm{Max}(R)$. Thus, by the previous paragraph, R_M is Gaussian for every $M \in \mathrm{Max}(R)$ and so R is Gaussian by Proposition 3.9. $\square$

The following is [T, Theorem 6.1].

Theorem 3.16. *If $(R, \mathfrak{m})$ is a local Gaussian ring, then* $\mathrm{Spec}(R)$ *is linearly ordered under inclusion.*

Proof. Let $P, Q \in \mathrm{Spec}(R)$ and suppose by way of contradiction that P and Q are not comparable. Then there exists $a \in P \setminus Q$ and there is $b \in Q \setminus P$. By Theorem 3.14, $(a,b)^2 = (a^2)$ or $(a,b)^2 = (b^2)$. Suppose without loss of generality that $(a,b)^2 = (a^2)$, then $b^2 \in (a^2) \subseteq P$. However P is prime and therefore $b \in P$, a contradiction and so P and Q are comparable as required. $\square$

The following corollary is immediate from Theorem 3.16.

Corollary 3.17. *If $(R, \mathfrak{m})$ is a local Gaussian ring then* $\mathrm{Nil}(R)$ *is the unique minimal prime ideal of R.*

The following theorem, appearing as [BaGl, Theorem 4.1], is crucial in proving some important later results.

Theorem 3.18. *Let $(R, \mathfrak{m})$ be a local ring, define $D = \{x \in R : x^2 = 0\}$, and consider the following conditions:*

(1) *D is an ideal of R, $D^2 = 0$, and the factor ring R/D is a chain ring.*

(2) *Given any $a \in R$, we have $aD \subseteq a(Ra \cap D)$.*

(3) *Given any $a \in R$, $\operatorname{ann}(a)$ and D are comparable and $D \subseteq Ra + \operatorname{ann}(a)$.*

Then R is a Gaussian ring if and only if (1) *and* (2) *hold or* (1) *and* (3) *hold.*

Proof. We first show that (1) + (3) $\Rightarrow$ (2). If $\operatorname{ann}(a)$ and D are comparable, then either (i) $D \subseteq \operatorname{ann}(a)$ or (ii) $\operatorname{ann}(a) \subseteq D$.

If (i) holds then for every $d \in D$ we have $ad = 0$, i.e., $aD = 0$, and so (2) holds trivially. Now suppose (ii) holds. By (3), for every $d \in D$ we have $d = ra + x$ for some $r, x \in R$ with $ax = 0$. Then, by (1), D is an ideal of R and so we have $ra = d - x \in D$ since $d \in D$ and $x \in \operatorname{ann}(a) \subseteq D$. This gives $ad = a(ra + x) = ara$. Setting $y = ra$ gives $y \in Ra \cap D$ and $ad = ay \in a(Ra \cap D)$, as required.

We now show that a Gaussian ring R satisfies conditions (1), (2) and (3).

Condition (1). Let $x, y \in D$. Since $x^2 = 0 = y^2$, we get $(x^2) = 0 = (y^2)$. Hence, $(x, y)^2 = 0$ by Theorem 3.14(d). Thus $(x + y)^2 = 0 = (x - y)^2$ and $(rx)^2 = 0$ for every $x, y \in D$ and for every $r \in R$. Therefore $x - y \in D$ and $rx \in D$. Thus D is an ideal of R. Moreover $D^2 = 0$ since given any $x, y \in D$, we have $xy \in (x, y)^2 = 0$ and so $xy = 0$.

To show that R/D is a chain ring we show that, for any $a, b \in R$, the principal factor ideals $(a + D)$, $(b + D)$ are comparable in R/D. Given that R is Gaussian, by Theorem 3.14(c), there exists an element $d \in \operatorname{ann}_R((a, b))$ such that either $(a, b) = (a, d)$ or $(a, b) = (b, d)$. Notice that the ideal $(a+D, b+D)$ in R/D can be written as $\dfrac{(a) + (b) + D}{D} = \dfrac{(a, b) + D}{D}$. Without loss of generality, by above we can take $(a, b) = (a, d)$ which gives $(a + D, b + D) = \dfrac{(a, d) + D}{D}$. Since $d \in \operatorname{ann}((a, b)) = \operatorname{ann}((a, d))$ we have $d^2 = 0$ and so $d \in D$. Therefore $\dfrac{(a, d) + D}{D} = \{ra + sd + D : r, s \in R\} = \{ra + D : r \in R\} =$

$\dfrac{(a)+D}{D} = (a+D)$. Since we have now proved that $(a+D, b+D) = (a+D)$ this gives $(b+D) \subseteq (a+D)$. Thus R satisfies condition (1).

Condition (2).[2] Let $a \in R$. If $ad = 0$ for every $d \in D$, then clearly $aD \subseteq a(Ra \cap D)$. Consequently, we may now suppose that $ad \neq 0$ for some $d \in D$. Since R satisfies condition (1), we have $D^2 = 0$ and so $a \notin D$. By Theorem 3.14(c), there exists c in the annihilator of (a, d) such that $(a, d) = (a, c)$ or $(a, d) = (c, d)$. In either case we get $c \in (a, d)$. In particular $c^2 = 0 = ac$ and therefore $c \in D$. Hence we obtain $(c, d) \subseteq D$. In fact, $(a, d) \neq (c, d)$ since $(c, d) \subseteq D$ and $a \notin D$ and so we must have $(a, d) = (a, c)$. Furthermore, by the last part of Theorem 3.14 (c), we can choose c such that $d = c + ar$ for some $r \in R$. Then $ad = ac + a^2r = a^2r$ since $c \in \operatorname{ann}((a, d))$. Setting $y = ar$ gives $y \in Ra$ and $y \in D$ since $y = ar = d - c \in D$. Hence, $ad = ay \in a(Ra \cap D)$. Thus $aD \subseteq a(Ra \cap D)$ as required.

Condition (3). Let $a \in R$. If $a^2 = 0$ then $a \in \operatorname{ann}(a)$ and $a \in D$ and so $aD \subseteq D^2 = 0$ since R satisfies condition (1). Hence $D \subseteq \operatorname{ann}(a)$ and so we also have $D \subseteq Ra + \operatorname{ann}(a)$. Thus (3) holds in this case and so we may now assume that $a^2 \neq 0$.

Let $c \in \operatorname{ann}(a)$. Then, by Theorem 3.14 (d) (i), either (i) $(a, c)^2 = (c)^2$ or (ii) $(a, c)^2 = (a)^2$. If (i) holds, by Theorem 3.14 (d) (ii) we get $a^2 = 0$, since $ac = 0$, in contradiction to our assumption. Thus (ii) holds and so, since $ac = 0$, using Theorem 3.14 (d) (ii) again we get $c^2 = 0$ and so $c \in D$. This has shown that $\operatorname{ann}(a) \subseteq D$.

We now show that $D \subseteq Ra + \operatorname{ann}(a)$. Let $d \in D$. If $ad = 0$ then $d \in \operatorname{ann}(a)$. So $d \in Ra + \operatorname{ann}(a)$ as required. If $ad \neq 0$, then, since R satisfies condition (2), $ad \in a(Ra \cap D)$, i.e., there exists $r \in R$ such that $ad = ara$ and $ra \in D$. Then $a(d - ra) = 0$ and so $d - ra \in \operatorname{ann}(a)$. Thus $d = ra + x$, where $x \in \operatorname{ann}(a)$ and so $d \in Ra + \operatorname{ann}(a)$, again as required. This has shown that $D \subseteq Ra + \operatorname{ann}(a)$. Thus condition (3) holds.

To show the converse and complete the proof, it is enough to show that a ring R satisfying (1) and (2) is Gaussian (since (3) $\Rightarrow$ (2)). Moreover, by the equivalence of (a) and

[2]Since (3) $\Rightarrow$ (2), this paragraph could be omitted.

(c) in Theorem 3.14, it is enough to show that given $a, b \in R$, there exists $c \in \operatorname{ann}((a, b))$ such that $(a, b) = (a, c)$ or $(a, b) = (b, c)$ and c can be chosen such that $b \in c + Ra$ or $a \in c + Rb$. By (1), the factor ring R/D is a chain ring and so either $(a + D) \subseteq (b + D)$ or $(b + D) \subseteq (a + D)$. Suppose without loss of generality that $(b + D) \subseteq (a + D)$. Then $b + D \in (a + D)$ and therefore there exists $r \in R$ and $d \in D$ such that $b = ra + d$. This gives $b \in (a, d)$ and so $(a, b) \subseteq (a, d)$. Moreover $d \in (a, b)$ and therefore $(a, d) \subseteq (a, b)$. Hence $(a, b) = (a, d)$.

Now note that, by (2), $ad \in a(Ra \cap D)$, say $ad = a(ay)$, where $ay \in D$. Then $a(d - ay) = 0$ and $(d - ay)^2 = 0$ since $d - ay \in D$ and $D^2 = 0$ from (1). Let $c = d - ay$. Then $c \in (a, d)$ and $d \in (a, c)$. Thus $(a, c) = (a, d) = (a, b)$. Moreover $ac = a(d - ay) = 0$ and $cd = (d - ay)d \in D^2 = 0$. Therefore $c \in \operatorname{ann}((a, d)) = \operatorname{ann}((a, b))$. Finally, notice that choosing $c = d - ay$ implies that $c = b - ra - ay$. Thus $b = c + (r + y)a$ and so $b \in c + Ra$, finishing the proof. $\square$

We can now bring our material in §2.5 back into the picture with the following corollary, first proved by Tsang in her unpublished 1965 doctoral dissertation [T] and then independently in 1967 by Gilmer in [Gi1]. (See also [Gi2, page 347].)

Corollary 3.19. *Let R be an integral domain.*

(i) *If R is local, R is a Gaussian ring if and only if R is a chain ring.*

(ii) *More generally, if R is not necessarily local, then R is a Gaussian ring if and only if R is a Prüfer domain.*

Proof. (i). Since R is an integral domain, the ideal D of Theorem 3.18 is zero. From this it follows that R satisfies conditions (2) and (3) of the Theorem while condition (1) holds exactly when R is a chain ring. Thus the Theorem in this case gives (i).

(ii). By Proposition 3.9, R is Gaussian if and only if R_M is Gaussian for each $M \in \operatorname{Max}(R)$. Since each R_M is also an integral domain, it follows from (i) that R is

Gaussian if and only if R_M is a chain ring for each $M \in \mathrm{Max}(R)$. Then, by Proposition 1.40, R is Gaussian if and only if R is arithmetical and so, by Proposition 2.74, if and only if R is a Prüfer domain. $\square$

We now prepare for an example, [BaGl, Example 4.2], which shows that condition (1) of Theorem 3.18 by itself is not enough for a ring R to be Gaussian. We will use Lemmas 1.12 and 1.11.

Example 3.20. Let K be a field and X, Y be two commuting indeterminates. Let I be the ideal generated by X^3, Y^2, and X^2Y. Take S to be the factor ring $K[X,Y]/I$ and denote the images of X and Y in S by x and y respectively. Since K is local, the ring $R = K[X]/(X^3)$ is local, by Lemma 1.12, and so, again by Lemma 1.12, the ring $T = K[X,Y]/(X^3,Y^2) = R[Y]/(Y^2)$ is also local. Since $S = T/(X^2Y)$, it then follows from Lemma 1.11 that S is local.

We show that S satisfies condition (1) of Theorem 3.18.

The elements of S can be written as $s = a_0 + a_1x + a_2x^2 + by + cxy$ for some $a_0, a_1, a_2, b, c \in K$. Moreover, for $\bar{s} = \bar{a}_0 + \bar{a}_1x + \bar{a}_2x^2 + \bar{b}y + \bar{c}xy$ in S we have $s\bar{s} = a_0\bar{a}_0 + (a_0\bar{a}_1 + a_1\bar{a}_0)x + (a_0\bar{a}_2 + a_1\bar{a}_1 + a_2\bar{a}_0)x^2 + (a_0\bar{b} + b\bar{a}_0)y + (a_0\bar{c} + a_1\bar{b} + b\bar{a}_1 + c\bar{a}_0)xy$. Hence

$$s^2 = a_0^2 + (a_0a_1 + a_1a_0)x + (a_0a_2 + a_1^2 + a_2a_0)x^2 + (a_0b + ba_0)y + (a_0c + a_1b + ba_1 + ca_0)xy = 0$$

if and only if

$$\text{(i) } a_0^2 = 0, \text{ (ii) } 2a_0a_1 = 0, \text{ (iii) } 2a_0a_2 + a_1^2 = 0, \text{ (iv) } 2a_0b = 0, \text{ (v) } 2a_0c + 2a_1b = 0$$

if and only if $a_0 = 0$ (from (i)) and $a_1 = 0$ (from (iii)). Then in S we have

$$D = \{s \in S : s^2 = 0\} = \{a_2x^2 + by + cxy : a_2, b, c \in K\} = (X^2, Y)/I.$$

In particular, D is an ideal of S. Moreover $D^2 = 0$ since $(a_2x^2 + by + cxy)(\bar{a}_2x^2 + \bar{b}y + \bar{c}xy)$ has only powers of x higher than 2 or powers of y higher than 1, except for the term

$(a_2\bar{b} + b\bar{a}_2^2)x^2y$ which is 0 by our definition of I.

Now we look at the factor ring $S/D = S/(x^2, y)$. Denoting the element $x + D$ in S/D by $\bar{x}$, we can write $S/D = \{a_0 + a_1\bar{x} : a_0, a_1 \in K\}$ and then the product of two elements in S/D is given by $(a_0 + a_1\bar{x})(b_0 + b_1\bar{x}) = a_0b_0 + (a_0b_1 + a_1b_0)\bar{x}$. It is now straightforward to see that S/D has only one proper nonzero ideal, namely $\{a_1\bar{x} : a_1 \in K\}$ (the principal ideal generated by any element of the form $a\bar{x}$ where $a \in K$ and $a \neq 0$). In particular, S/D is a chain ring.

Thus S satisfies condition (1) of Theorem 3.18.

However we now show that the local ring S is not Gaussian by showing that it fails condition (d)(i) of Theorem 3.14. To fit the notation of condition (d)(i), take $a = x$, $b = y$. Then $(a, b) = (x, y)$ and so $(a, b)^2 = (x^2, xy, y^2) = (x^2, xy) = \{a_2x^2 + cxy : a_2, c \in K\}$. On the other hand, $(a^2) = (x^2) = \{a_2x^2 : a_2 \in K\} \neq (a, b)^2$ and $(b^2) = (y^2) = 0 \neq (a, b)^2$. Thus (d)(i) fails and so S is not Gaussian. □

In all of the following we let $D = \{x \in R : x^2 = 0\}$.

We now introduce an important result, appearing as [BaGl, Lemma 4.3]. Given $a \in R$ and an ideal I of R, we use $[I : a]$ to denote $\{r \in R : ra \in I\}$.

Lemma 3.21. *Every local Gaussian ring* $(R, \mathfrak{m})$ *satisfies the following:*

(1) *If* $a \in \mathfrak{m} \setminus D$*, then* $\operatorname{ann}(a) \subseteq D$.

(2) *If the maximal ideal* $\mathfrak{m}$ *is nil, then* $D \subsetneq [D : a]$ *for any* $a \in \mathfrak{m} \setminus D$.

(3) *If the maximal ideal* $\mathfrak{m}$ *is nil and* $D\mathfrak{m} = 0$*, then* $\mathfrak{m}^4 = 0$.

Proof. (1) Let $b \in \operatorname{ann}(a)$, i.e., $ba = 0$. Since $a \notin D$, we have $a^2 \neq 0$. Then Theorem 3.14(d)(i) gives $(a, b)^2 = (a^2)$ or $(a, b)^2 = (b^2)$. If $(a, b)^2 = (b^2)$ then, since $ab = 0$, Theorem 3.14(d)(ii) implies that $a^2 = 0$, a contradiction. Thus $(a, b)^2 = (a^2)$, and so, since $ab = 0$, Theorem 3.14(d)(ii) gives $b^2 = 0$. Thus $b \in D$ and so $\operatorname{ann}(a) \subseteq D$.

(2) Let $a \in \mathfrak{m} \setminus D$. Since $\mathfrak{m}$ is nil, we have $a^k = 0$ for some $k \in \mathbb{N}$ with $k \geq 3$ because $a \notin D$. Choose t to be the smallest power of a such that $a^t \in D$ and $a^t \neq 0$. (Note that such a power always exists since if $a^{2n} = 0$ for some smallest even power, then $a^n \in D$ but $a^n \neq 0$ while if $a^{2n+1} = 0$ for some smallest odd power, $a^{2(n+1)} = aa^{2n+1} = 0$ and so $a^{n+1} \in D$.) Note that $t \geq 2$. Then $a^{t-1} \in [D : a]$ and $a^{t-1} \notin D$. However, D is contained in $[D : a]$ since, by Theorem 3.18(1), D is an ideal of R and so for every $d \in D$ we have $ad \in D$. Hence $D \subsetneq [D : a]$ as required.

(3) Let $a \in \mathfrak{m} \setminus D$. By hypothesis $Da = 0$, so $D \subseteq \operatorname{ann}(a)$. Furthermore, by (1), we also have $\operatorname{ann}(a) \subseteq D$ and so $\operatorname{ann}(a) = D$.

Since $\mathfrak{m}$ is nil, $a^n = 0$ for some $n \in \mathbb{N}$. As in the proof of (2), $n \geq 3$ and take t to be the minimum positive integer such that $a^t \in D$ and $a^t \neq 0$. Then $t \geq 2$ and $a^{t-1} \notin D$. Then, by the previous paragraph, $\operatorname{ann}(a^{t-1}) = D$. Also, $a^t\mathfrak{m} \subseteq D\mathfrak{m} = 0$ and so $\mathfrak{m} \subseteq \operatorname{ann}(a^t)$. Since $a^t \neq 0$ we also have $\operatorname{ann}(a^t) \subseteq \mathfrak{m}$ and so $\operatorname{ann}(a^t) = \mathfrak{m}$. This gives

$$\mathfrak{m} = \operatorname{ann}(a^t) = \{r \in R : ra^t = 0\} = \{r \in R : ra \in \operatorname{ann}(a^{t-1})\} = [\operatorname{ann}(a^{t-1}) : a] = [D : a].$$

Thus, for every $x \in \mathfrak{m}$ we have $ax \in D$ for all $a \in \mathfrak{m} \setminus D$ and so $a\mathfrak{m} \subseteq D$ when $a \in \mathfrak{m} \setminus D$. Moreover if $b \in D$, then $b\mathfrak{m} \subseteq D$ since D is an ideal of R by Theorem 3.18(1). Hence $c\mathfrak{m} \subseteq D$ for all $c \in \mathfrak{m}$. Then, given $p, q, r, s \in \mathfrak{m}$, we have $pq, rs \in D$ and so by Theorem 3.18(1), $pqrs = 0$. It follows that $\mathfrak{m}^4 = 0$ as wanted. $\square$

In [DT, §5], [BaGl, Theorem 6.4] is restated and proved with an extra condition in the hypothesis. An example is also given in [DT] to show the extra condition can not be dropped from the proof of Theorem 6.4 in BaGl. We will detail this example below. First we introduce a lemma from [DT].

Lemma 3.22. *Let the ring R be local Gaussian with nilradical $\operatorname{Nil}(R) \neq D$. Then the maximal ideal of the localisation $R_{\operatorname{Nil}(R)}$ is nonzero.*

Proof. Corollary 3.17 shows that $\operatorname{Nil}(R)$ is a prime ideal and so we may form the localisation

$R_{\mathrm{Nil}(R)}$. Let $N = \mathrm{Nil}(R)R_{\mathrm{Nil}(R)}$. Then, since $\mathrm{Nil}(R)$ is the only minimal prime ideal of R, it follows that N is the only prime ideal (and so the only maximal ideal) of $R_{\mathrm{Nil}(R)}$. We wish to show that $N \neq 0$.

Suppose to the contrary that $N = 0$. Since $\mathrm{Nil}(R) \neq D$, there exists an element $x \in \mathrm{Nil}(R) \setminus D$ such that $\frac{x}{1} = \frac{0}{1}$ in N and so there is $y \in R \setminus \mathrm{Nil}(R)$ such that $xy = 0$. Since R is local Gaussian, Theorem 3.14(d)(i) shows that $(Rx+Ry)^2 = Rx^2$ or $(Rx+Ry)^2 = Ry^2$. Moreover since $xy = 0$, by Theorem 3.14(d)(ii), either $Rx^2 = 0$ or $Ry^2 = 0$. The former gives $x \in D$, a contradiction. The latter gives $y \in \mathrm{Nil}(R)$ since $y^2 \in Ry^2 = 0$, a contradiction since $y \in R \setminus \mathrm{Nil}(R)$. Hence $N \neq 0$ as required. $\square$

Now for the promised example. It is a local Gaussian ring $(R, \mathfrak{m})$ with nonzero nilpotent $\mathrm{Nil}(R)$ but the maximal ideal of $R_{\mathrm{Nil}(R)} = 0$. It appears as [DT, Example 5.3] showing that the proof of [BaGl, Theorem 6.4] is incomplete.

Example 3.23. Set $T = K[X,Y]/(XY, Y^2)$ where K is a field and X, Y are two commuting indeterminates. If we set I to be the ideal (XY, Y^2) of $K[X,Y]$, a typical element in T is of the form

$$\bar{a} + b\bar{Y} + c_1\bar{X} + c_2\bar{X}^2 + \cdots + c_n\bar{X}^n,$$

where $\bar{a} = a + I$, $\bar{X} = X + I$, $\bar{Y} = Y + I$, and $a, b, c_1, \ldots, c_n \in K$. Note that $\bar{a} = 0$ if and only if $a \in I$, but I contains no constant apart from 0 and so $a = 0$. Let $S = \{\bar{a} + b\bar{Y} + c_1\bar{X} + \cdots + c_n\bar{X}^n : a \neq 0\}$. It is not difficult to see that S is an m.c.s. of $T = K[X,Y]/I$. Set $R = T_S$ and write x for $\frac{\bar{X}}{1}$ and y for $\frac{\bar{Y}}{1}$ in R. The elements of R are of the form $\frac{t}{s}$, where $t \in T$, $s \in S$, i.e., $\frac{\bar{a}_1 + b_1\bar{Y} + \bar{X}f_1(\bar{X})}{\bar{a}_2 + b_2\bar{Y} + \bar{X}f_2(\bar{X})}$ for some polynomials f_1, f_2 and $\bar{a}_2 \neq 0$.

If $\bar{a}_1 \neq 0$ then $t \in S$ and so $\frac{t}{s}$ is a unit. Conversely, if $\frac{t}{s}$ is a unit, say with inverse $\frac{v}{w}$, then there exists $u \in S$ such that $utv = usw$. However $usw \in S$ and this forces t to be in S. It follows that the non-units of R have zero constant term in their numerator and so the sum of two non-units is again a non-unit. Thus R is local with maximal ideal $\mathfrak{m}$ consisting of elements of the form $\dfrac{b\bar{Y}+\bar{X}f(\bar{X})}{s}$ for some $s \in S$, $b \in K$, and polynomial $f \in K[\bar{X}]$ and from this it is clear to see that $\mathfrak{m}$ is generated in R by $\{x, y\}$.

Let $c, d \in \mathfrak{m}$, say $c = \frac{c_0}{s_0}y + \frac{c_1}{s_1}x + \cdots + \frac{c_m}{s_m}x^m$ and $d = \frac{d_0}{t_0}y + \frac{d_1}{t_1}x + \cdots + \frac{d_n}{t_n}x^n$, where $m, n \in \mathbb{N}$, $c_i, d_j \in K$ and $s_i, t_j \in S$ for $0 \le i \le m$ and $0 \le j \le n$. If $c_i = 0$ for each $i \ge 1$ then $c = \frac{c_0}{s_0}y$ and so since $y^2 = 0 = xy$, we get $c^2 = 0$ and $(c, d)^2 = (c^2, cd, d^2) = (d^2)$ and so conditions (i) and (ii) of Theorem 3.14 (d) are satisfied. Similarly, these conditions hold if $d_j = 0$ for each $j \ge 1$ and c arbitrary. Therefore we may assume that there exist $c_i \ne 0$ and $d_j \ne 0$ for some $i, j \ge 1$. Choose i, j such that these are the smallest indices giving $c_i \ne 0$, $d_j \ne 0$ respectively. We can then rewrite c and d as

$$c = b_1 y + x^i(c_i + c_{i+1}x + \cdots + c_n x^{n-i}) = b_1 y + x^i u_1,$$
$$d = b_2 y + x^j(d_j + d_{j+1}x + \cdots + d_m x^{m-j}) = b_2 y + x^j u_2,$$

where u_1, u_2 are units. Moreover $(c, d)^2 = (c^2, d^2, cd)$ with $c^2 = (b_1 y + x^i u_1)^2 = x^{2i}u_1^2$, $d^2 = (b_2 y + x^j u_2)^2 = x^{2j}u_2^2$ and $cd = x^{i+j}u_1u_2$. Hence $(c, d)^2 = (x^{2i}u_1^2, x^{2j}u_2^2, x^{i+j}u_1u_2) = (x^{2i}, x^{2j}, x^{i+j}) = (x^t)$, where $t = \min(2i, 2j, i+j)$. We may assume without loss of generality that $i \le j$. Then $(c, d)^2 = (x^i)^2 = (c^2)$. This shows that condition (i) of Theorem 3.14 (d) holds. If $cd = 0$ then $x^{i+j} = 0$ and so $d^2 = x^{2j}u_2^2 = x^{i+j}x^{j-i}u_2^2 = 0x^{i-j}u_2^2 = 0$. Thus part (ii) of Theorem 3.14 holds so R is a local Gaussian ring.

We now determine $\mathrm{Nil}(R)$. Let $c = by + xf(x) \in \mathrm{Nil}(R)$, where $xf(x) = c_1x + c_2x^2 + \cdots + c_nx^n$. Then $c^t = 0$ for some $t \ge 1$. However, since $y^2 = 0 = xy$, $c^t = b^ty^t + x^tf(x)^t$ and so $c^t = 0$ if and only if $by = 0$ and $f(x) = 0$ or $t \ge 2$ and $f(x) = 0$. This shows that $c^t = 0$ if and only if $c \in (y)$. Thus $\mathrm{Nil}(R) = (y)$ and its nilpotency degree is 2 (in other words $\mathrm{Nil}(R) = D$). Since R is local with maximal ideal $\mathfrak{m}$, $R_{\mathfrak{m}} = R$. Moreover $R_{\mathrm{Nil}(R)} = \left\{\frac{r}{s} : r \in R, s \notin \mathrm{Nil}(R)\right\}$ has a unique prime ideal $\mathrm{Nil}(R)R_{\mathrm{Nil}(R)}$ (see Lemma 1.28) and $\mathrm{Nil}(R)R_{\mathrm{Nil}(R)}$ is generated by $\frac{y}{1}$. However, since $x \notin \mathrm{Nil}(R)$ we have $\frac{1}{1} = \frac{x}{x}$ so $\frac{y}{1} = \frac{x}{x}\frac{y}{1} = \frac{0}{x} = 0$. Thus $\mathrm{Nil}(R)R_{\mathrm{Nil}(R)} = 0$, i.e., $R_{\mathrm{Nil}(R)}$ is a field. This shows that the condition $\mathrm{Nil}(R) \neq D$ can not be removed from the hypothesis of Lemma 3.22 and likewise for Theorem 4.16.

3.3 Local Gaussian rings $(R, \mathfrak{m})$ with $\mathrm{Nil}(R) = \mathfrak{m}$.

The next lemma is from [BaGl, Lemma 4.4].

Lemma 3.24. *Let $(R, \mathfrak{m})$ be a local Gaussian ring with $\mathfrak{m} = \mathrm{Nil}(R)$ (so that $\mathfrak{m}$ is the only prime ideal of R). If $\mathfrak{m}$ is not nilpotent then $\mathfrak{m} = \mathfrak{m}^2 + D$ and $\mathfrak{m}^2 = \mathfrak{m}^3$.*

Proof. By Theorem 3.18, the factor ring R/D is a chain ring. Now if $(\mathfrak{m}/D)^t = 0$ for some $t \in \mathbb{N}$ then, given $m_1, m_2, \dots, m_t \in \mathfrak{m}$ and setting $x = m_1 m_2 \cdots m_t$, we have $x + D = (m_1 + D)(m_2 + D) \cdots (m_t + D) = 0$ and so $x \in D$. This gives $\mathfrak{m}^t \subseteq D$ and so, since $D^2 = 0$, we then get $\mathfrak{m}^{2t} = 0$, a contradiction to our assumption that $\mathfrak{m}$ is not nilpotent. Thus the maximal ideal $\mathfrak{m}/D$ of R/D is also not nilpotent.

Then since $\mathfrak{m}/D = \mathrm{Nil}(R)/D = \mathrm{Nil}(R/D)$, the nilradical of R/D, by Lemma 1.44 it follows that $\mathfrak{m}/D$ is an idempotent ideal of R/D, i.e., $(\mathfrak{m}/D)^2 = \mathfrak{m}/D$. Then, for any $m \in \mathfrak{m}$ we have $m + D \in (\mathfrak{m}/D)^2$ and so, for finitely many $m_i, m_j \in \mathfrak{m}$, we get $m + D = \sum (m_i + D)(m_j + D) = (\sum m_i m_j) + D$. Thus $m = \sum m_i m_j + d$ for some $d \in D$. This gives

$$\mathfrak{m} = \mathfrak{m}^2 + D. \qquad (*)$$

Now, multiplying $(*)$ by $\mathfrak{m}$ gives

$$\mathfrak{m}^2 = \mathfrak{m}(\mathfrak{m}^2 + D) = \mathfrak{m}^3 + \mathfrak{m}D. \qquad (**)$$

Moreover, multiplying $(*)$ by D gives $\mathfrak{m}D = \mathfrak{m}^2 D + D^2 = \mathfrak{m}^2 D$ since $D^2 = 0$ by Theorem 3.18. This gives $\mathfrak{m}D \subseteq \mathfrak{m}^3$ since $D \subseteq \mathfrak{m}$. Applying this to $(**)$ we get $\mathfrak{m}^2 = \mathfrak{m}^3$ as required. $\square$

The next lemma is from [BaGl, Lemma 4.5].

Lemma 3.25. *Let* $(R, \mathfrak{m})$ *be a local Gaussian ring with* $\mathfrak{m} = \mathrm{Nil}(R)$*. If* $\mathfrak{m}$ *is not nilpotent, there exists an element* $d \in D$ *such that* $D \subsetneq \mathrm{ann}(d) \subsetneq \mathfrak{m}$*.*

Proof. By Lemma 3.21(3), $D\mathfrak{m} \neq 0$ and so there exists $d_1 \in D$ such that $d_1\mathfrak{m} \neq 0$. However, $\mathrm{ann}(d_1)$ is a set of non-units and so a subset of $\mathfrak{m}$ and so $\mathrm{ann}(d_1) \subsetneq M$. Let $a \in \mathfrak{m} \setminus \mathrm{ann}(d_1)$. Then $0 \neq ad_1$ and we know that $ad_1 \in D$ since D is an ideal of R. Moreover, $a \notin D$ since if otherwise, we have $ad_1 \in D^2 = 0$ and so $ad_1 = 0$, a contradiction. Notice that $\mathrm{ann}(ad_1) = \{r \in R : rad_1 = 0\} = \{r \in R : ra \in \mathrm{ann}(d_1)\} = [\mathrm{ann}(d_1) : a] \supseteq [D : a]$ since $D^2 = 0$ by Theorem 3.18. Since $\mathfrak{m}$ is a nil ideal and $a \in \mathfrak{m} \setminus D$, by Lemma 3.21(2), we have $D \subsetneq [D : a]$ and so $D \subsetneq \mathrm{ann}(ad_1)$.

To finish the proof we show that $\mathrm{ann}(ad_1) \subsetneq \mathfrak{m}$. We know that $\mathrm{ann}(ad_1)$ is an ideal of non-units and so contained in the unique maximal ideal $\mathfrak{m}$. Thus it is enough to show that $\mathrm{ann}(ad_1) \neq \mathfrak{m}$. Suppose, by way of contradiction, that $\mathrm{ann}(ad_1) = \mathfrak{m}$ for every $a \in \mathfrak{m}$, $a \notin \mathrm{ann}(d_1)$. Let $\overline{R}$ be the local ring $R/\mathrm{ann}(d_1)$, with unique maximal ideal $\mathfrak{m}/\mathrm{ann}(d_1) = \overline{\mathfrak{m}}$, say. If $\bar{a} = a + \mathrm{ann}(d_1) \in \overline{R}$ with $a \in \mathfrak{m}$, $\bar{a} \neq 0$ (i.e., $a \notin \mathrm{ann}(d_1)$), we get

$$\begin{aligned}
\mathrm{ann}_{\overline{R}}(\bar{a}) &= \{\bar{r} = r + \mathrm{ann}(d_1) \in \overline{R} : (r + \mathrm{ann}(d_1))(a + \mathrm{ann}(d_1)) = 0\} \\
&= \{\bar{r} = r + \mathrm{ann}(d_1) \in \overline{R} : r \in R, ra \in \mathrm{ann}(d_1)\} \\
&= \{\bar{r} = r + \mathrm{ann}(d_1) \in \overline{R} : r \in R, rad_1 = 0\} \\
&= \{\bar{r} = r + \mathrm{ann}(d_1) \in \overline{R} : r \in R, r \in \mathrm{ann}(ad_1)\} \\
&= \{\bar{r} = r + \mathrm{ann}(d_1) \in \overline{R} : r \in \mathfrak{m}\} \\
&= \overline{\mathfrak{m}}.
\end{aligned}$$

In particular, $\overline{\mathfrak{m}}^2 = 0$. Thus $\mathfrak{m}^2 \subseteq \mathrm{ann}(d_1)$ and so $\mathfrak{m}^2 + D \subseteq \mathrm{ann}(d_1) + D = \mathrm{ann}(d_1)$ (since $D \subseteq \mathrm{ann}(d_1)$ because $d_1 \in D$ and $D^2 = 0$). Therefore $\mathfrak{m} \subseteq \mathrm{ann}(d_1)$ since $\mathfrak{m}^2 + D = \mathfrak{m}$ by Lemma 3.24, a contradiction since $\mathrm{ann}(d_1) \subsetneq \mathfrak{m}$ as shown in the first part of the proof. Hence there exists an element $a \in \mathfrak{m} \setminus \mathrm{ann}(d_1)$ such that $\mathrm{ann}(ad_1) \subsetneq \mathfrak{m}$. Then, taking $d = ad_1$, the result is proved. □

Definition 3.26. In the ring R, an ideal I of R, with $\operatorname{ann}(I) = A$, is said to satisfy the **double annihilator property** if $\operatorname{ann}(A) = I$, i.e., $\operatorname{ann}(\operatorname{ann}(I)) = I$.

The following two lemmas are from section 4 of [DT].

Lemma 3.27. *Let $(R, \mathfrak{m})$ be a local chain ring with $\operatorname{Nil}(R) = \mathfrak{m}$. Then any principal ideal of $(R, \mathfrak{m})$ satisfies the double annihilator property. In other words, for any $x \in R$, if $(\operatorname{ann}(x) =) \operatorname{ann}(Rx) = I$, then we have $\operatorname{ann}(I) = Rx$.*

Proof. It is always the case that $Rx \subseteq \operatorname{ann}(I) \, (= \operatorname{ann}(\operatorname{ann}(x)))$ since if $a \in Rx$ then $ab = 0$ for any $b \in I$ and so $a \in \operatorname{ann}(I)$. Now to show the opposite containment, suppose by way of contradiction, that there is $c \in \operatorname{ann}(I)$ such that $c \notin Rx$. Then $Rc \nsubseteq Rx$. However, since R is a chain ring, we must now have $Rx \subset Rc$. Hence $x = \lambda c$ for some $\lambda \in \mathfrak{m}$. Because $x \neq 0$ we have $\lambda \notin I$ since otherwise we get $x = \lambda c = 0$. Therefore $R\lambda \nsubseteq I$ and so the chain condition gives $I \subset R\lambda$. We now show that $I \subseteq (\lambda^k)$ for all $k \in \mathbb{N}$ using induction. Note that the above shows that this holds for $k = 1$. Suppose inductively that $I \subseteq (\lambda^k)$ for some $k \in \mathbb{N}$. Take $y \in I$ and note that $I \subseteq (\lambda)$ implies that there is an $r \in \mathfrak{m}$ such that $y = r\lambda$. (Notice that $r \in \mathfrak{m}$ since otherwise r is a unit and so we get the contradiction $\lambda = r^{-1}y \in I$). Observe that $r \in I$ since $0 = cy = cr\lambda = r\lambda c = rx$. Using our induction hypothesis, we obtain $r = s\lambda^k$ for some $s \in R$. Thus $y = r\lambda = s\lambda^{k+1} \in (\lambda^{k+1})$. Hence $I \subseteq (\lambda^{k+1})$ and so $I \subseteq (\lambda^k)$ for all $k \in \mathbb{N}$. However, since $\lambda \in \mathfrak{m}$, by our hypothesis λ is nilpotent and so $\lambda^n = 0$ for some $n \in \mathbb{N}$. Hence $I = 0$, a contradiction, so $\operatorname{ann}(I) \subseteq Rx$ which gives the equality desired. □

Recall that the nilpotency degree of an element $r \in R$ is the smallest $n \in \mathbb{N}$ such that $r^n = 0$. We denote this n by $\deg(r)$.

Lemma 3.28. *Let $(R, \mathfrak{m})$ be a local Gaussian ring with the property that $\operatorname{Nil}(R) = \mathfrak{m}$ and let a be an element of $\mathfrak{m}$. Suppose that $\mathfrak{m}$ is not nilpotent, i.e., there is no $n \in \mathbb{N}$ such that $\mathfrak{m}^n = 0$. Then there is an element $z \in \mathfrak{m}$ such that $\deg(z) > \deg(a)$.*

Proof. Let $\deg(a) = k$. Assume by way of contradiction that $\deg(z) \leq k$ for every $z \in \mathfrak{m}$. Let $z_1, z_2, \ldots, z_k \in \mathfrak{m}$ and let I denote the ideal generated by these z_i. By Theorem 3.14, $I/(I \cap \operatorname{ann}(I))$ is a cyclic R-module and so there exists $z \in I$ such that $z_i = r_i z + d_i$, where $r_i \in R$ and $d_i \in I \cap \operatorname{ann}(I) \subseteq \operatorname{ann}(z_j)$ for every $j \in \{1, \ldots, k\}$. Then we have the product $z_1 z_2 \ldots z_k = \prod_{i=1}^{k}(r_i z + d_i)$. Expanding this product gives terms involving z and some d_i except for the term $d_1 \ldots d_k$. Since $d_i z = 0$ and $d_i d_j = 0$ for all i and j we get $z_1 z_2 \ldots z_k = 0$. This shows that $\mathfrak{m}^k = 0$ and so nilpotent, giving us our contradiction. □

The next lemma appears as [DT, Lemma 6.3]. It will be used in proving coming results.

Lemma 3.29. *Let $(R, \mathfrak{m})$ be a local Gaussian ring. Let M be a module over R and let $x \in M$. Then $I = \{a \in \mathfrak{m} : a^q x = 0$ for some $q \geq 1\}$ is a prime ideal of R.*

Proof. Let $J = \{a \in \mathfrak{m} : ax = 0\}$. Then $J = \mathfrak{m} \cap \operatorname{ann}(x)$ is a proper ideal of R and $J \subseteq I$. We now show that I/J is the nilradical of R/J. Let $a + J$ be a nilpotent element of R/J. Then $a^q + J = 0$ for some $q \geq 1$. Thus $a^q \in J$ (i.e., $a^q x = 0$) and so $a \in I$. Therefore $a + J \in I/J$ and so $\operatorname{Nil}(R/J) \subseteq I/J$. Conversely, if $a \in I$ then $a^q x = 0$ for some $q \geq 1$ and so in R/J we have $(a+J)^q = a^q + J = 0$ since $a^q x = 0$, i.e., $a^q \in J$ and so $I/J = \operatorname{Nil}(R/J)$.

Since R is local Gaussian, so is R/J by Lemmas 1.11 and 3.8 and so its nilradical I/J is a prime ideal. Note that $R/I \simeq \dfrac{R/J}{I/J}$, and I/J is prime so $\dfrac{R/J}{I/J}$ is an integral domain and therefore R/I is also an integral domain. Thus I is a prime ideal of R. □

Chapter 4

The Bazzoni–Glaz Conjecture

This final chapter studies the following conjecture proposed in [BaGl].

The Bazzoni–Glaz Conjecture. If R is a Gaussian ring then $\text{w.gl.dim}(R) = 0$, 1, or ∞.

We detail the positive response to the Conjecture given first for reduced rings by Glaz in [Gl2] and then for non-reduced rings partly by Bazzoni and Glaz in [BaGl] and, particulary, by Donadze and Thomas in [DT].

4.1 Gaussian rings R with $\text{w.gl.dim}(R) \leq 1$.

The result immediately below verifies the Bazzoni–Glaz Conjecture for Gaussian rings which are either PF or, equivalently, reduced. It is due to Glaz, given as [Gl2, Theorem 2.2], and characterizes the rings R for which $\text{w.gl.dim}(R) \leq 1$.

Theorem 4.1. *Let R be a ring. Then the following statements are equivalent:*

(1) $\text{w.gl.dim}(R) \leq 1$.

(2) *The ring R is Gaussian and PF.*

(3) *The ring R is Gaussian and reduced.*

Proof. (1) $\Rightarrow$ (2). Since $\text{w.gl.dim}(R) \leq 1$, every ideal I of R is flat by Lemma 2.51. In particular R is a PF ring and so, by Theorem 2.67, the localisation R_M is an integral domain for every $M \in \text{Max}(R)$. Furthermore, by Theorem 2.53, hypothesis (1) gives $\text{w.gl.dim}(R_M) \leq 1$. Then, by Theorem 2.29, R_M is a chain domain and so a Gaussian domain by Corollary 3.19. Consequently, R is Gaussian by Proposition 3.9.

(2) $\Rightarrow$ (3). This follows from Theorem 2.67.

(3) $\Rightarrow$ (1). Since R is reduced, so is R_P for any $P \in \text{Spec}(R)$ by Corollary 1.32. Then, by Lemma 1.18 and Proposition 3.9, R_P is local, reduced and Gaussian. Now let $a, b \in R_P$ with $ab = 0$. By Theorem 3.14(d)(i), $(a,b)^2 = (a^2)$ or $(a,b)^2 = (b^2)$, say the former holds. Then by Theorem 3.14(d)(ii), we get $b^2 = 0$ and so $b = 0$ since R_P is reduced. Hence R_P is an integral domain. Since R_P is also Gaussian, it is a chain domain by Corollary 3.19. Then, by Theorem 2.29, every ideal of R_P is flat. In particular $\text{w.dim}_{R_P}(R_P/I) \leq 1$ for each ideal I of R_P. Hence, by Theorem 2.55, $\text{w.gl.dim}(R_P) \leq 1$ for all $P \in \text{Spec}(R)$. Thus, by Theorem 2.53, $\text{w.gl.dim}(R) \leq 1$. $\square$

4.2 Gaussian rings R with $\text{w.gl.dim}(R) \geq 1$.

The following result appears as [DT, Lemma 3.1].

Lemma 4.2. *Let R be a local Gaussian ring, I be an ideal of R, and M be an R-module such that* $\text{w.dim}(M) = n$. *If* $\text{Tor}_n(R/I, M) \neq 0$, *then there is an ideal J of R such that* $\text{Tor}_n(R/J, M) \neq 0$ *and* $I + D \subseteq J$.

Proof. We show that if $x_1, x_2, \ldots, x_m \in D$, then the natural projection map $\pi : R/I \longrightarrow R/(I + Rx_1 + \cdots + Rx_m)$ gives the following inclusion:

$$\text{Tor}_n(R/I, M) \hookrightarrow \text{Tor}_n(R/(I + Rx_1 + \cdots + Rx_m), M)$$

Set $I = I_0$ and inductively define $I_p = I_{p-1} + Rx_p$, for $1 \leq p \leq m$. Then we have the following s.e.s., (where i is the inclusion map and π is the natural projection):

$$0 \longrightarrow I_p/I_{p-1} \xrightarrow{i} R/I_{p-1} \xrightarrow{\pi} R/I_p \longrightarrow 0 \quad (*),$$

Now let $f : R \longrightarrow I_P/I_{p-1}$ be given by $f(r) = rx_p + I_{p-1}$. This is an R-homomorphism with $\ker(f) = \{r \in R : rx_p \in I_{p-1}\}$. Then the First Isomorphism Theorem gives $R/\ker(f) \simeq \mathrm{im}(f) = I_p/I_{p-1}$. Notice that $\mathrm{ann}(x_p) = \{r \in R : rx_p = 0\} \subseteq \ker(f)$. Moreover, since $x_p \in D$ then by Lemma 3.21 we obtain $I_{p-1} + D \subseteq I_{p-1} + \mathrm{ann}(x_p) \subseteq \ker(f)$. Now if $\mathrm{Tor}_n(I_p/I_{p-1}, M) \neq 0$for some $1 \leq p \leq m$, then taking $J = \ker(f)$ we have $\mathrm{Tor}_n(R/J, M) \simeq \mathrm{Tor}_n(I_p/I_{p-1}, M) \neq 0$ and so, since $I + D \subseteq I_{p-1} + D \subseteq J$, the result is proved in this case.

If otherwise, $\mathrm{Tor}_n(I_p/I_{p-1}, M) = 0$ for all $1 \leq p \leq m$, then tensoring $(*)$ by M gives the following long exact sequence for Tor:

$$\begin{array}{ccccccccc} 0 = & \mathrm{Tor}_n(I_p/I_{p-1}, M) \rightarrow & \mathrm{Tor}_n(R/I_{p-1}, M) \rightarrow & \mathrm{Tor}_n(R/I_p, M) \rightarrow & \cdots \\ \cdots \longrightarrow & \mathrm{Tor}_1(R/I_p, M) \longrightarrow & I_p/I_{p-1} \otimes M \longrightarrow & R/I_{p-1} \otimes M \longrightarrow & R/I_p \otimes M \rightarrow & 0. \end{array}$$

This gives a monomorphism from $\mathrm{Tor}_n(R/I_{p-1}, M)$ to $\mathrm{Tor}_n(R/I_p, M)$. Thus we have the following sequence of inclusions

$$\mathrm{Tor}_n(R/I_0, M) \hookrightarrow \mathrm{Tor}_n(R/I_1, M) \hookrightarrow \cdots \hookrightarrow \mathrm{Tor}_n(R/I_m, M) \quad (**),$$

where $I_m = I_0 + \sum_{j=1}^{m} Rx_j$. Now let $\mathcal{X}$ be the collection of all finitely generated ideals J in R such that $J \subseteq D$. For each $J_p, J_q \in \mathcal{X}$ with $J_p \subseteq J_q$, define $\pi_{pq} : R/(I+J_p) \longrightarrow R/(I+J_q)$ to be the natural epimorphism. Taking $A = I$ and $B = D$ in Lemma 2.27, shows that $\{R/(I+J) : J \in \mathcal{X}\}$ is a directed set with $\varinjlim_{J\in\mathcal{X}}(R/(I+J)) = R/(I+D)$. By Lemma 2.46 (ii), we have $\varinjlim_{J\in\mathcal{X}}(\mathrm{Tor}_n(R/(I+J), M)) = \mathrm{Tor}_n(R/(I+D), M)$. Combining this with $(**)$ we get the inclusion $\mathrm{Tor}_n(R/I, M) \longrightarrow \mathrm{Tor}_n(R/(I+D))$. Since $\mathrm{Tor}_n(R/I, M) \neq 0$, we obtain $\mathrm{Tor}_n(R/(I+D)) \neq 0$ as required. □

The next result is Lemma 3.3 of [DT].

Lemma 4.3. *Let $(R, \mathfrak{m})$ be a local Gaussian ring and M be an R-module. If* $\operatorname{w.dim}_R(M) = n \geq 1$ *then* $\operatorname{Tor}_n(R/D, M) = 0$.

Proof. Most of the following proof shows that, more generally, $\operatorname{Tor}_n(R/J, M) = 0$ for any ideal J of R such that $D \subseteq J \subseteq \mathfrak{m}$, provided an annihilator condition is satisfied. We will need $J = D$ only in the last paragraph and, in Lemma 4.8, we will take J to be $\mathfrak{m}$.

Thus let J be a proper ideal of R containing D and suppose (to the contrary) that $\operatorname{Tor}_n(R/J, M) \neq 0$. Let

$$\cdots \xrightarrow{\delta_{n+2}} R^{(I_{n+1})} \xrightarrow{\delta_{n+1}} R^{(I_n)} \xrightarrow{\delta_n} R^{(I_{n-1})} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_1} R^{(I_0)} \xrightarrow{\delta_0} M \longrightarrow 0 \quad (*)$$

be a free resolution of M, where the I_t are index sets for each $t \geq 0$. Then, by assumption, $\operatorname{Tor}_n(R/J, M) = \ker(\overline{\delta}_n)/\operatorname{im}(\overline{\delta}_{n+1}) \neq 0$, where the homomorphism $\overline{\delta}_t$ is given by $1_{R/J} \otimes \delta_t : R/J \otimes R^{(I_t)} \longrightarrow R/J \otimes R^{(I_{t-1})}$ obtained by tensoring the resolution above by R/J. Since $R/J \otimes R^{(I_t)} \simeq (R/J)^{(I_t)}$, we will regard $\overline{\delta}_t$ as a homomorphism from $(R/J)^{(I_t)}$ to $(R/J)^{(I_{t-1})}$ and, as such, its action is given by

$$\overline{\delta}_t((w_i + J)_{i \in I_t}) = (\textstyle\sum_{i \in I_t} w_i r_{ij} + J)_{j \in I_{t-1}}$$

where for each $i \in I_t$, we have $\delta_t(e_i) = \sum_{j \in I_{t-1}} r_{ij} e_j$ and $\{e_i : i \in I_t\}$ and $\{e_j : j \in I_{t-1}\}$ are the natural bases for $R^{(I_t)}$ and $R^{(I_{t-1})}$ respectively (see the proof of Lemma 2.5).

Our assumption thus provides an element $\overline{w} = (w_i + J)_{i \in I_n} \in \ker(\overline{\delta}_n$ which is not in $\operatorname{im}(\overline{\delta_{n+1}})$. By Lemma 2.5, if we set $w = (w_i)_{i \in I_n}$ then $\delta_n(w) = (\sum_{i \in I_n} w_i r_{ij})_{j \in I_{n-1}} \in J^{(I_{n-1})}$. Let $w_1, \ldots, w_m \in J$ be the finitely many nonzero entries of $\delta_n(w)$. The proof now splits into two cases.

Case 1. In this case we assume there is an $a \in \mathfrak{m} \setminus D$ such that $aw_j = 0$ for each $j \in \{1, \ldots, m\}$.

Define $f : R/J \longrightarrow Ra/Ja$ by $f(r + J) = ra + Ja$ for all $r \in R$. Then f is well-defined since if $r, s \in R$ with $r + J = s + J$ then $r - s \in J$ so $ra - sa \in Ja$. Now, by Lemma 3.21 (1), $\operatorname{ann}(a) \subseteq D$. Thus, if $f(r+J) = 0$, i.e., $ra \in aJ$, say $ra = xa$ where $x \in J$, then $r - x \in \operatorname{ann}(a) \subseteq D \subseteq J$, giving $r \in J$. This has shown that f is a monomorphism. It is easily seen to be onto and so it is an isomorphism. Now set ϕ_t to be the induced isomorphism $f^{(I_t)} : (R/J)^{(I_t)} \longrightarrow (Ra/Ja)^{(I_t)}$ for each $t \geq 0$. Then we have the following diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\overline{\delta}_{n+1}"] & (R/J)^{(I_n)} \arrow[r, "\overline{\delta}_n"] \arrow[d, "\phi_n"'] & (R/J)^{(I_{n-1})} \arrow[r, "\overline{\delta}_{n-1}"] \arrow[d, "\phi_{n-1}"'] & \cdots \arrow[r, "\overline{\delta}_2"] & (R/J)^{(I_1)} \arrow[r, "\overline{\delta}_1"] \arrow[d, "\phi_1"'] & (R/J)^{(I_0)} \arrow[d, "\phi_0"'] \\
\cdots \arrow[r, "\overline{\delta'}_{n+1}"] & (Ra/Ja)^{(I_n)} \arrow[r, "\overline{\delta'}_n"] & (Ra/Ja)^{(I_{n-1})} \arrow[r, "\overline{\delta'}_{n-1}"] & \cdots \arrow[r, "\overline{\delta'}_2"] & (Ra/Ja)^{(I_1)} \arrow[r, "\overline{\delta'}_1"] & (Ra/Ja)^{(I_0)}
\end{tikzcd}$$

where the top row is the complex obtained by tensoring the free resolution $(*)$ by R/J and the bottom is obtained by tensoring $(*)$ by Ra/Ja. In particular, from the proof of Lemma 2.5, the homomorphisms $\overline{\delta'}_t$ are given by

$$\overline{\delta'}_t((s_i a + Ja)_{i \in I_t}) = (\textstyle\sum_{i \in I_t} s_i r_{ij} a + Ja)_{j \in I_{t-1}}.$$

We show that the diagram is commutative. Thus, for $t \geq 1$, we must show $\phi_{t-1}\overline{\delta}_t = \overline{\delta'}_t \phi_t$. To this end, let $(s_i + J)_{i \in I_t} \in (R/J)^{(I_t)}$. Then, since $\delta_t(e_i) = \sum_{j \in I_{t-1}} r_{ij} e_j$ and $\{e_i : i \in I_t\}$ and $\{e_j : j \in I_{t-1}\}$ are the natural bases for $R^{(I_t)}$ and $R^{(I_{t-1})}$ respectively, we have

$$\phi_{t-1}(\overline{\delta}_t((s_i + J)_{i \in I_t})) = \phi_{t-1}((\textstyle\sum_{i \in I_t} s_i r_{ij} + J)_{j \in I_{t-1}}) = (f(\sum_{i \in I_t} s_i r_{ij} + J))_{j \in I_{t-1}}$$

$$= (\textstyle\sum_{i \in I_t} s_i r_{ij} a + Ja)_{j \in I_{t-1}} = \overline{\delta'}_t((s_i a + Da)_{i \in I_t}) = \overline{\delta'}_t(\phi_t((s_i + J)_{i \in I_t})),$$

as required.

Since $\mathrm{w.dim}_R(M) = n \geq 1$ and $f : R/J \to Ra/Ja$ is a monomorphism, it follows from Lemma 2.57 that there is a monomorphism $\phi_n^* : \mathrm{Tor}_n(R/J, M) \to \mathrm{Tor}_n(Ra/Ja, M)$ given by $\phi_n^*((r_i + J)_{i\in I_n} + \mathrm{im}(\overline{\delta}_{n+1})) = \phi_n((r_i + J)_{i\in I_n}) + \mathrm{im}(\overline{\delta'}_{n+1})$. In particular, $\phi_n^*(\overline{w} + \mathrm{im}(\overline{\delta}_{n+1})) \neq 0$. However, $\phi_n^*((w_i + J)_{i\in I_n} + \mathrm{im}(\overline{\delta}_{n+1})) = \phi_n((w_i + J)_{i\in I_n}) + \mathrm{im}(\overline{\delta'}_{n+1}) = (w_i a + Ja)_{i\in I_n} + \mathrm{im}(\overline{\delta'}_{n+1}) = 0$ (since $w_i a = 0$ for each $i = 1, \ldots, m$). This contradiction shows that $\mathrm{Tor}_n(R/J, M) = 0$ in this case.

Case 2. Here we assume that, in contrast to Case 1, given any $a \in R \setminus D$ we have $aw_j \neq 0$ for some $j \in \{1, \ldots, m\}$.

We define $g : R/D \to R^{(m)}$ by setting $g(r + D) = r(w_1, \ldots, w_m)$ for all $r \in R$. It is easy to see that g is a homomorphism. Moreover, given any $a \in R \setminus D$, since $a(w_1, \ldots, w_m) \neq 0$, we see that g is a monomorphism. Then, by Lemma 2.57, the induced map $g_n : \mathrm{Tor}_n(R/D, M) \to \mathrm{Tor}_n(R^{(m)}, M)$ is also a monomorphism. However, $\mathrm{Tor}_n(R^{(m)}, M) \simeq \oplus_{\lambda=1}^m \mathrm{Tor}_n(R, M) \simeq M^m \neq 0$. Thus g_n can not be a monomorphism. This contradiction completes the proof. $\square$

Corollary 4.4. *Let $(R, \mathfrak{m})$ be a local Gaussian ring and M be an R-module. If* $\mathrm{w.dim}_R(M) = n \geq 1$ *then* $\mathrm{Tor}_k(R/D, M) = 0$ *for all $k \geq n$.*

Proof. Since $\mathrm{w.dim}(M) = n$ we have $\mathrm{Tor}_k(X, M) = 0$ for all $k > n$ and all R-modules X. Lemma 4.3 gives $\mathrm{Tor}_n(R/D, M) = 0$. $\square$

The following lemma appears as [DT, Lemma 3.4].

Lemma 4.5. *Let $(R, \mathfrak{m})$ be a local Gaussian ring and let M be an R-module with* $\mathrm{w.dim}_R(M) = n > 1$. *Then there is an element $a \in \mathfrak{m} \setminus D$ such that* $\mathrm{Tor}_n(R/(D + Ra), M) \neq 0$.

Before proving this lemma, we mention the following note.

Note. Observe that Theorem 2.48 implies that $\mathrm{w.dim}(M) = n$ if and only if $\mathrm{Tor}_{n+1}(R/I, M) = 0$ for all ideals I of R and there exists an ideal X of R such that $\mathrm{Tor}_n(R/X, M) \neq 0$.

Proof. Since $\mathrm{w.dim}(M) = n$ there exists an ideal I of R such that $\mathrm{Tor}_n(R/I, M) \neq 0$. By Lemma 4.2, we may assume that $D \subseteq I$. Since $D \neq I$ by Lemma 4.3, we obtain $D \subset I$. Let $\mathcal{J} = \{J_\lambda : J_\lambda$ is an ideal of R such that $D \subseteq J_\lambda \subseteq I$ and J_λ/D is finitely generated$\}$. Then, given λ and μ such that $J_\lambda \subseteq J_\mu$, we define $f_{\lambda\mu} : R/J_\lambda \longrightarrow R/J_\mu$ by setting $f_{\lambda\mu}(r + J_\lambda) = r + J_\mu$ for all $r \in R$. This makes $\mathcal{J}$ into a direct system and $\varinjlim_{\mathcal{J}} R/J_\lambda = R/I$. Then $\mathrm{Tor}_n(R/I, M) = \mathrm{Tor}_n(\varinjlim_{\mathcal{J}} R/J_\lambda, M) = \varinjlim_{\mathcal{J}}(\mathrm{Tor}_n(R/J_\lambda, M))$ and so $\mathrm{Tor}_n(R/J_\lambda, M) \neq 0$ for some ideal J_λ such that $D \subseteq J_\lambda \subseteq I$ and J_λ/D is finitely generated, say J_λ/D is generated by $a_1 + D, \ldots, a_m + D$. Then $J_\lambda = Ra_1 + \cdots + Ra_m + D$. Note that, since R is local Gaussian, R/D is a chain ring and therefore $Ra_1 + \cdots + Ra_m + D = Ra + D$, where $a = a_i$ for some $i \in \{1, 2, \ldots, m\}$. This gives $0 \neq \mathrm{Tor}_n(R/J_\lambda, M) = \mathrm{Tor}_n(R/(Ra + D), M)$.

Observe that $a \notin D$ since otherwise $Ra + D = D$, contradicting $\mathrm{Tor}_n(R/D, M) = 0$. Also $a \in \mathfrak{m}$ since otherwise a is a unit and so $Ra = R$ which implies that $\mathrm{Tor}_n(R/(Ra + D), M) = \mathrm{Tor}_n(0, M) = 0$, again a contradiction. $\square$

We now give Lemma 4.2 of [DT].

Lemma 4.6. *Let $(R, \mathfrak{m})$ be a local Gaussian ring for which* $\mathrm{Nil}(R) = \mathfrak{m}$ *and let M be a module over R. Let R' and $\mathfrak{m}'$ denote R/D and $\mathfrak{m}/D$ respectively. Suppose that* $\mathrm{w.dim}_R(M) = n \geq 1$. *Then R satisfies the following statements:*

(i) *There exists an element $x \in \mathfrak{m}' \setminus 0$ such that* $\mathrm{Tor}_n(R'/R'x, M) \neq 0$.

(ii) *For all $z \in \mathfrak{m}' \setminus 0$, if J is an ideal of R' such that $z \in J$ and $J \neq R'z$ then the natural projection $R'/R'z \longrightarrow R'/J$ induces the zero mapping from* $\mathrm{Tor}_n(R'/R'z, M)$ *to* $\mathrm{Tor}_n(R'/J, M)$.

(iii) *For every nonzero element z in $\mathfrak{m}'$,*$\mathrm{Tor}_n(R'/R'z, M)$ *is non trivial.*

Proof. Recall that R/D is a chain ring and so R' is also Gaussian.

(i) If $x \in \mathfrak{m}'/0$, say $x = a + D$ for some $a \in \mathfrak{m}$, then $R'x = (R/D)(a + D) = \dfrac{Ra + D}{D}$ and so $R'/R'x \simeq \dfrac{R/D}{(Ra + D)/D} \simeq R/(Ra + D)$. Then, by Lemma 4.5, there is an element

$a \in \mathfrak{m} \setminus D$ such that $\mathrm{Tor}_n(R/(Ra+D), M) \neq 0$. It follows from above that taking $x = a+D$ ($x \neq 0$ since $a \notin D$) we obtain $\mathrm{Tor}_n(R'/R'x, M) \neq 0$.

(ii) By Lemma 3.27, if we set $I = \mathrm{ann}(z)$ then $\mathrm{ann}(I) = R'z$. This gives $\mathrm{ann}(I) \subseteq J$ but $\mathrm{ann}(I) \neq J$ and so $\mathrm{ann}(I) \subset J$. Since $\mathrm{ann}(I) = \bigcap\{\mathrm{ann}(x) : x \in I\}$, this shows that there exists an element $y \in J$ such that $\mathrm{ann}(y) \subseteq J$ but $\mathrm{ann}(y) \neq J$. We have the inclusions $R'z \subseteq \mathrm{ann}(y) \subsetneq J$ and so we have the natural epimorphism

$$R'/R'z \xrightarrow{g} R'/\mathrm{ann}(y) \xrightarrow{f} R'/J \qquad (*).$$

Note that $R'/\mathrm{ann}(y) \simeq R'y$. Now taking $\phi : R'y \longrightarrow R'$ to be the inclusion map, Corollary 2.57 shows that ϕ induces a monomorphism $\phi_* : \mathrm{Tor}_n(R'y, M) \longrightarrow \mathrm{Tor}_n(R', M)$. However by Lemma 4.3 $\mathrm{Tor}_n(R', M) = \mathrm{Tor}_n(R/D, M) = 0$. Hence $\mathrm{Tor}_n(R'y, M) = 0$. Now the composition $(*)$ induces the composition

$$\mathrm{Tor}_n(R'/R'z, M) \xrightarrow{g} \mathrm{Tor}_n(R'/\mathrm{ann}(y), M) \xrightarrow{f} \mathrm{Tor}_n(R'/J, M)\ .$$

Observe that for any composition of homomorphisms $A \xrightarrow{g} B \xrightarrow{f} C$, if g is the 0 map then so too is fg. Thus, since $\mathrm{Tor}_n(R'/\mathrm{ann}(y), M) = 0$, the induced homomorphsm $\mathrm{Tor}_n(R'/R'z, M) \longrightarrow \mathrm{Tor}_n(R'/J, M)$ is the 0 mapping.

(iii) By (i) there exists $x \in \mathfrak{m}' \setminus 0$ such that $\mathrm{Tor}_n(R'/R'x, M) \neq 0$. Let $z \in \mathfrak{m}'$ with $z \neq 0$. Since R' is a chain ring, either $z \in R'x$ or $x \in R'z$.

Case 1. If $R'x = R'z$, we finish by taking $J = R'z$.

Case 2. Suppose $z \in R'x$ and z is not a unit multiple of x, i.e., $R'z \subsetneq R'x'$. Then $z = ax$ for some $a \in \mathfrak{m}'$. Define $\alpha : R'/R'x \longrightarrow R'/R'z$ by $\alpha(r' + R'x) = ar' + R'z$ for all $r \in R$. It is easily checked that α is a well-defined ring homomorphism. Notice that since $ax = z \neq 0$, $x \notin \mathrm{ann}(a)$. Thus, since R' is a chain ring, we obtain $\mathrm{ann}(a) \subset R'x$. If $\alpha(r' + R'x) = 0$, we have $ar' + R'z = 0$. Thus $ar' \in R'z$ and therefore $ar' = s'ax$ for some $s' \in R'$. This gives $a(r' - s'x) = 0$, i.e., $r' - s'x \in \mathrm{ann}(a) \subset R'x$, and so $r' - s'x = t'x$ for some $t' \in R'$. Hence $r' = s'x + t'x \in R'x$ and so $r' + R'x = 0$. This shows that α is a monomorphism. Now using Corollary 2.57, the monomorphism α induces a monomorphism $\alpha_* : \mathrm{Tor}_n(R'/R'x, M) \longrightarrow \mathrm{Tor}_n(R'/R'z, M)$. Because $\mathrm{Tor}_n(R'/R'x, M) \neq 0$, it follows that $\mathrm{Tor}_n(R'/R'z, M) \neq 0$.

Case 3: Now suppose that $x \in R'z$ and $R'z \neq R'x$. There exists $b \in \mathfrak{m}'$ such that $x = bz$. Define $\sigma : R'/R'z \longrightarrow R'/R'x$ by $\sigma(r' + R'z) = br' + R'x$. Similar to the first case we show that σ is a monomorphism. Note first that $bz = x \neq 0$ and so $z \notin \mathrm{ann}(b)$. Since ideals in the chain ring R' are linearly ordered under inclusion we get $\mathrm{ann}(b) \subset R'z$. Thus if $\sigma(r' + R'z) = 0$ we obtain $br' + R'x = 0$ and so $br' \in R'x$. Say $br' = u'x = u'bz$ where $u' \in R'$. Then $b(r' - u'z) = 0$. In other words, $r' - u'z \in \mathrm{ann}(b) \subset R'z$. Hence $r' - u'z = v'z$ for some element u' in R'. Thus $r' = u'z + v'z \in R'z$ and so $r' + R'z = 0$. This has shown that σ is a monomorphism. Notice that $R'x \subset R'b$ since $x = bz \in R'b$ and $R'x \neq R'b$. Form the s.e.s.

$$0 \longrightarrow R'/R'z \xrightarrow{\sigma} R'/R'x \xrightarrow{\tau} R'/R'b \longrightarrow 0\,,$$

where τ is the natural epimorphism. By item (ii), the map τ induces the trivial mapping $0 : \mathrm{Tor}_n(R'/R'x, M) \longrightarrow \mathrm{Tor}_n(R'/R'b, M)$ and so the mapping $\mathrm{Tor}_n(R'/R'z, M) \longrightarrow \mathrm{Tor}_n(R'/R'x, M)$ induced by σ is an epimorphism. Since we have $\mathrm{Tor}_n(R'/R'x, M) \neq 0$ we get $\mathrm{Tor}_n(R'/R'z, M) \neq 0$ as required. $\square$

4.3 Local Gaussian rings $(R, \mathfrak{m})$ where all elements of $\mathfrak{m}$ are zero-divisors.

Suppose $(R, \mathfrak{m})$ is a local Gaussian ring such that every element of $\mathfrak{m}$ is a zero-divisor, i.e., for every $x \in \mathfrak{m}$ there exists a nonzero element $r_x \in R$ such that $xr_x = 0$. Since R is local, it follows that the elements of R are either units or zero-divisors It is well–known (and straightforward to show) that such a ring R is (isomorphic to) its own total ring of quotients $Q(R)$. The next lemma from [DT, Lemma 3.5] is a significant result for such a local Gaussian ring.

Lemma 4.7. *Let $(R, \mathfrak{m})$ be a local Gaussian ring in which every element of $\mathfrak{m}$ is a zero-divisor. Then, given nonzero elements $\lambda_1, \lambda_2, \dots, \lambda_n$ in $\mathfrak{m}$, there exists $0 \neq a \in \mathfrak{m}$ such that a annihilates every λ_i, where $i \in \{1, 2, \dots, n\}$.*

Proof. To prove this lemma, there are two different cases to be considered:

Case 1: Suppose $\lambda_1, \lambda_2, \dots, \lambda_n \in D$. Then we can take a to be any λ_i since $D^2 = 0$ by Theorem 3.18.

Case 2: For $i \in \{1, 2, \dots, n\}$, suppose not every λ_i is in D, i.e., there is an element $\lambda_j \in \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ such that $\lambda_j \notin D$, say $\lambda_j = \lambda_1$. Then $\operatorname{ann}(\lambda_1) \subseteq D$ by part (1) of Lemma 3.21. Take $I \subseteq \mathfrak{m}$ to be the ideal generated by $\lambda_1, \lambda_2, \dots, \lambda_n$. This gives $\operatorname{ann}(I) \subseteq \operatorname{ann}(\lambda_1) \subseteq D$. Let $J = I \cap \operatorname{ann}(I)$. Then by Theorem 3.14, there exists $\lambda \in R$ such that $I/J = R(\lambda + J)$. In particular, $\lambda_i + J = r_i\lambda + J$ for some $r_i \in R$, for all $i = 1, 2, \dots, n$. Thus $\lambda_i = r_i\lambda + d_i$, where $d_i \in J$. Also, $d_i \in D$ since $d_i \in J \subseteq \operatorname{ann}(I) \subseteq D$. Note that $\lambda_1 \notin D$ since if otherwise we obtain $\lambda_1 = r_1\lambda + d_1 \in D$ in contradiction to our assumption that $\lambda_1 \notin D$. Moreover, because $\lambda + J \in I/J$, we get $\lambda \in I$ and so $\lambda = \sum_{i=1}^{n} r_i\lambda_i \in \mathfrak{m}$ where each $r_i \in R$. This implies that $\lambda \in \mathfrak{m} \setminus D$. Because λ is a zero-divisor, we can take a nonzero element $d \in \operatorname{ann}(\lambda)$ and by Lemma 3.21(1) we get $d \in D$. Then, by multiplying the expression $\lambda_i = r_i\lambda + d_i$ by d, we get $d\lambda_i = dr_i\lambda + dd_i = 0$ since $d \in \operatorname{ann}(\lambda)$ and $dd_i \in D^2 = 0$ by Theorem 3.18. Therefore $d\lambda_i = 0$ for every $i = 1, 2, \dots, n$ as required. $\square$

If we choose $(R, \mathfrak{m})$ to be a local Gaussian ring which is its own total ring of quotients, i.e., each element of $\mathfrak{m}$ is a zero-divisor, the result of Lemma 4.3 holds for R/D replaced by $R/\mathfrak{m}$ as shown in the next lemma from [DT, Lemma 3.6]

Lemma 4.8. *Let $(R, \mathfrak{m})$ be a local Gaussian ring such that $Q(R) = R$ and let M be an R-module. Suppose that* $\text{w.dim}(M) = n \geq 1$. *Then* $\operatorname{Tor}_n(R/\mathfrak{m}, M) = 0$.

Proof. By Lemma 4.7 there is a nonzero element $a \in \mathfrak{m}$ such that $a\lambda_i = 0$ for any $\lambda_i \in \mathfrak{m}$ with $i = 1, 2, \ldots, n$. Then, taking $J = \mathfrak{m}$, case 1 of the proof of Lemma 4.3 gives the desired result. □

The following result, appearing as [DT, Proposition 3.7] is crucial in showing some important results in the sequel.

Proposition 4.9. *Let $(R, \mathfrak{m})$ be a local Gaussian ring. If $\mathfrak{m} \neq 0$ and each of its elements is a zero-divisor, then* $\text{w.gl.dim}(R) \geq 3$.

Proof. We shorten the proof in [DT] a little bit.

If $\mathfrak{m} = D$, since $D^2 = 0$ by Theorem 3.18, $\mathfrak{m}$ is nilpotent and so by Proposition 2.61 $\text{w.dim}(\mathfrak{m}) = \infty$ and so $\text{w.gl.dim}(R) = \infty$. Thus we now suppose that $\mathfrak{m} \neq D$. If $x \in \mathfrak{m} \setminus D$, then we have the following exact sequence

$$0 \longrightarrow \operatorname{ann}(x) \longrightarrow R \xrightarrow{x_m} R \xrightarrow{\pi} R/Rx \longrightarrow 0\,, \qquad (*)$$

where x_m is multiplication by x and π is the natural epimorphism. By way of contradiction, assume that $\text{w.gl.dim}(R) \leq 2$. Then $\text{w.dim}(R/Rx) \leq 2$ and so, by Theorem 2.48, the sequence $(*)$ shows that $\operatorname{ann}(x)$ must be flat. To reach a contradiction we show that $\operatorname{ann}(x)$ is not flat. By Proposition 2.10 ((i) $\Rightarrow$ (ii)') if $\operatorname{ann}(x)$ is flat then the mapping $\phi : \operatorname{ann}(x) \otimes I \longrightarrow \operatorname{ann}(x)I$ is an isomorphism for every ideal I of R. Take $I = Rx$. Then $\operatorname{ann}(x)I = 0$ and therefore $\operatorname{im}(\phi) = 0$ and this implies that $\operatorname{ann}(x) \otimes Rx = 0$ since ϕ is a monomorphism. Thus, to get our contradiction, it is enough to show that $\operatorname{ann}(x) \otimes Rx \neq 0$. Since $Rx \simeq R/\operatorname{ann}(x)$, by Lemma 2.2 (ii), we obtain $\operatorname{ann}(x) \otimes Rx \simeq \operatorname{ann}(x) \otimes R/\operatorname{ann}(x) \simeq \dfrac{\operatorname{ann}(x)}{(\operatorname{ann}(x))^2}$. Note that, by Lemma 3.21 (1), we have $\operatorname{ann}(x) \subseteq D$ because $x \in \mathfrak{m} \setminus D$. This implies that $(\operatorname{ann}(x))^2 \subseteq D^2 = 0$ and so $\dfrac{\operatorname{ann}(x)}{(\operatorname{ann}(x))^2} \simeq \operatorname{ann}(x)$. So we have $\mathrm{ann}(x) \otimes Rx \simeq \mathrm{ann}(x)$ and so $\mathrm{ann}(x) \otimes Rx \neq 0$ since $\mathrm{ann}(x) \neq 0$ (this is because of our assumption that $x \in \mathfrak{m}$ and so a zero-divisor). $\square$

In some of the next results we concentrate on local Gaussian rings R which satisfies the following property, appearing as [DT, Property 3.8].

Property 4.10. For every $x \in D \setminus 0$, the annihilator of x is not cyclic modulo D, i.e., there is no element $a \in \mathrm{ann}(x)$ such that $\mathrm{ann}(x)/D = R(a+D)/D$, i.e., there is no element $a \in \mathrm{ann}(x)$ such that $\mathrm{ann}(x) = Ra + D$.

Before we state the following lemmas from [DT, Lemma 3.9 and Lemma 3.10] we make an important remark regarding the chain ring R/D.

Remark 4.11. Consider the chain ring R/D and two principal ideals in R/D, say $\dfrac{Ra+D}{D}$ and $\dfrac{Rb+D}{D}$. Since R/D is a chain ring $\dfrac{Ra+D}{D}$ and $\dfrac{Rb+D}{D}$ are comparable. From this it follows that the ideals $Ra + D$ and $Rb + D$ are comparable in the ring R.

Lemma 4.12. *Let $(R, \mathfrak{m})$ be a local Gaussian for which all the elements of $\mathfrak{m}$ are zero-divisors. If R satisfies Property 4.10 and $\mathfrak{m} \neq D$, then*

(i) $\mathfrak{m} = \mathfrak{m}^2 + D$*,*

(ii) *if $J \subseteq \mathfrak{m}$ is a finitely generated ideal of R, then $J^2 \subseteq Rx^2$ for some $x \in \mathfrak{m}$ with the property that $x^2 \notin D$, and*

(iii) *the ideal $\mathfrak{m}^2$ is flat.*

Proof. (i) Clearly $\mathfrak{m}^2 + D \subseteq \mathfrak{m}$ and so we only have to show that if $a \in \mathfrak{m}$ then $a \in \mathfrak{m}^2 + D$. We may obviously assume that $a \in \mathfrak{m} \setminus D$ since the result holds trivially if $a \in D$. Then, by part (1) of Lemma 3.21, $\mathrm{ann}(a) \subseteq D$. Moreover, since $a \neq 0$ and by our hypothesis on $\mathfrak{m}$, we obtain $ann(a) \neq 0$. Then there exists an $x \in D \setminus 0$ such that $ax = 0$. By Property 4.10, $\mathrm{ann}(x) \neq Ra + D$. However since $ax = 0$ and $x \in D$, we have $(ra + d)x = rax + dx = 0$ for any $r \in R$ and $d \in D$ and so $Ra + D \subseteq \mathrm{ann}(x)$. Hence there is an element $b \in \mathrm{ann}(x)$ such that $b \notin Ra + D$. Note that Theorem 3.18 tells us that R/D is a chain ring. Therefore since $Rb + D \nsubseteq Ra + D$, by Remark 4.11, we must then have $Ra + D \subseteq Rb + D$. Hence $a = rb + d$ for some $r \in R$ and $d \in D$. If r is a unit we get $b = r^{-1}(a - d) \in Ra + D$, contradicting $b \notin Ra + D$. Thus r must be in $\mathfrak{m}$. Moreover, $b \in \mathfrak{m}$ since $b \in \operatorname{ann}(x) \subseteq \mathfrak{m}$ and so $rb \in \mathfrak{m}^2$. Thus $a = rb + d \in \mathfrak{m}^2 + D$. This gives $\mathfrak{m} \subseteq \mathfrak{m}^2 + D$ which proves the equality desired.

(ii) We first show that if $x^2 \in D$ for all $x \in \mathfrak{m}$ then $\mathfrak{m}^2 \subseteq D$. Let $z \in \mathfrak{m}^2$. Then for some $n \geq 1$ we have $z = \sum_{i=1}^{n} x_i y_i$, where $x_i, y_i \in \mathfrak{m}$ for each i. Now, by Theorem 3.14 (d), $(Rx_i + Ry_i)^2 = Rx_i^2$ or $(Rx_i + Ry_i)^2 = Ry_i^2$. Thus $x_i y_i = rx_i^2$ or $x_i y_i = sy_i^2$ for some $r, s \in R$. In either case we get $x_i y_i \in D$ since $x_i^2, y_i^2 \in D$ by assumption. Since D is an ideal by Theorem 3.18, we get $z = \sum_{i=1}^{n} x_i y_i \in D$. Hence $\mathfrak{m}^2 \subseteq D$ as expected. Now, using item (i), we obtain $\mathfrak{m} = D$ in contradiction to our hypothesis. Thus there exists $x \in \mathfrak{m}$ such that $x^2 \notin D$.

By Lemma 3.12 , if J is a finitely generated ideal of R then $J^2 = Ry^2$ for some $y \in J$. We are finished if $y^2 \notin D$. If otherwise $y^2 \in D$ then, from above, we may pick an element $x \in \mathfrak{m}$ such that $x^2 \notin D$. It follows from Theorem 3.14 (d) that $(Rx + Ry)^2 = Rx^2$ or $(Rx + Ry)^2 = Ry^2$. Since $Ry^2 \subseteq D$ but $Rx^2 \nsubseteq D$ we must have $(Rx + Ry)^2 = Rx^2$ (since otherwise $x^2 = ry^2 \in D$). Then $y^2 \in Rx^2$ and so $J^2 = Ry^2 \subseteq Rx^2$ with $x^2 \notin D$ as wanted.

(iii) By Proposition 2.10, to show that $\mathfrak{m}^2$ is flat, it suffices to show that if I is any proper ideal of R, the induced map $f : I \otimes \mathfrak{m}^2 \longrightarrow \mathfrak{m}^2$ determined by $x \otimes y \mapsto xy$, where $x \in I$ and $y \in \mathfrak{m}^2$, is a monomorphism. Let $w \in \ker(f)$. Then $w = \sum_{i=1}^{n} z_i \otimes x_i y_i$, where $z_i \in I$, $x_i, y_i \in \mathfrak{m}$. Let J be the ideal generated by $x_1, \ldots, x_n, y_1, \ldots, y_n$. By part (ii), there exists $x \in \mathfrak{m}$ with $x^2 \notin D$ and $J^2 \subseteq Rx^2$ and so for all $i = 1, \ldots, n$ we have $x_i y_i = r_i x^2$ for some $r_i \in R$. Then $w = \sum_{i=1}^{n} z_i \otimes x_i y_i = \sum_{i=1}^{n} z_i \otimes r_i x^2 = \sum_{i=1}^{n} z_i r_i \otimes x^2 = z \otimes x^2$, where $z = \sum_{i=1}^{n} z_i r_i$. We are finished if we can show that $z \otimes x^2 = 0$. If $z = 0$ we're done, so suppose that $z \neq 0$. Since $0 = f(z \otimes x^2) = zx^2$ we have $z \in \operatorname{ann}(x^2)$ and so, since

$x^2 \in \mathfrak{m} \setminus D$, $z \in D$ by Lemma 3.21. This leads us to consider the following two cases:

Case 1. Suppose $z \in \operatorname{ann}(x)$, i.e., $zx = 0$. Then we also have $x \in \operatorname{ann}(z)$ and so, since $z \in D \setminus 0$, Property 4.10 shows that $\operatorname{ann}(z) \neq Rx + D$. Now because $Rx + D \subsetneq \operatorname{ann}(z)$, there exists an element y such that $y \in \operatorname{ann}(z)$ but $y \notin Rx + D$ and so $Ry + D \nsubseteq Rx + D$. Then, using Remark 4.11, we get $Rx + D \subsetneq Ry + D$. Hence $x = cy + d_1$, for some $c \in R$, $d_1 \in D$. Now $c \in \mathfrak{m}$ since if not then $y = c^{-1}(x - d_1) \in Rx + D$, a contradiction. Then

$$w = z \otimes x^2 = z \otimes (cy + d_1)^2 = z \otimes (c^2y^2 + 2cyd_1 + d_1^2) = z \otimes c^2y^2 + z \otimes 2cyd_1 + z \otimes d_1^2.$$

However $z \otimes d_1^2 = z \otimes 0 = 0$ since $d_1 \in D$. Also, $z \otimes 2cyd_1 = zy \otimes 2cd_1 = 0$ since $zy = 0$. Similarly, $z \otimes c^2y^2 = zy^2 \otimes c^2 = 0$ and therefore $w = z \otimes x^2 = 0$ as desired.

Case 2. Suppose $zx \neq 0$. Since $z \in D$ and $x^2 \in \mathfrak{m}$ we have $zx \in D \setminus 0$ and $zx^2 = 0$ and so $x \in \operatorname{ann}(zx)$. We aim to show that $w = z \otimes x^2 = 0$ in this case also. Note first that since $zx \in D \setminus 0$, by Property 4.10 we have $\operatorname{ann}(zx) \neq Rx + D$. Now note that $Rx + D \subseteq \operatorname{ann}(zx)$ and so there exists $h \in \operatorname{ann}(zx)$ but $h \notin Rx + D$. Thus $Rh + D \nsubseteq Rx + D$ which implies that $Rx + D \subseteq Rh + D$ using Remark 4.11. Then $x = sh + d_2$ for some $s \in R$, $d_2 \in D$. Again, note that $s \in \mathfrak{m}$ since if not then $h = s^{-1}(x - d_2) \in Rx + D$, a contradiction. This gives

$$\begin{aligned}
w &= z \otimes x^2 \\
&= z \otimes (sh + d_2)^2 \\
&= z \otimes (s^2h^2 + 2shd_2 + d_2^2) \\
&= z \otimes (s^2h^2 + 2shd_2) \qquad (\text{since } d_2^2 \in D^2 = 0) \\
&= z \otimes s^2h^2 + z \otimes 2shd_2 \\
&= z \otimes s^2h^2 + zd_2 \otimes 2sh \\
&= z \otimes s^2h^2 + 0 \otimes 2sh \qquad (\text{since } zd_2 \in D^2 = 0) \\
&= z \otimes s^2h^2.
\end{aligned}$$

By (i), we can write $s = b + d$, where $b \in \mathfrak{m}^2$ and $d \in D$. Therefore

$$
\begin{aligned}
w &= z \otimes sh^2(b+d) = z \otimes sh^2 b + z \otimes sh^2 d \\
&= zsh^2 \otimes b + zd \otimes sh^2 \\
&= zsh^2 \otimes b + 0 \otimes sh^2 \qquad (\text{since } zd \in D^2 = 0) \\
&= z(sh)h \otimes b = z(x - d_2)h \otimes b \\
&= (zxh - zd_2 h) \otimes b \\
&= 0 \otimes b = 0 \qquad (\text{because } h \in \operatorname{ann}(zx) \text{ and } zd_2 \in D^2 = 0).
\end{aligned}
$$

Thus the result holds in this case also. □

The next lemma is [DT, Lemma 3.10].

Lemma 4.13. *Let $(R, \mathfrak{m})$ be a local Gaussian ring in which every element of $\mathfrak{m}$ is a zero-divisor, i.e., $R = Q(R)$. Suppose that R satisfies Property 4.10, $\mathfrak{m} \neq D$ and* $\text{w.gl.dim}(R) < \infty$. *Then $\mathfrak{m}$ is a flat R-module.*

Proof. If $\mathfrak{m} = \mathfrak{m}^2$ then, by Lemma 4.12 (i), $\mathfrak{m}$ is flat as required. Now suppose that $\mathfrak{m} \neq \mathfrak{m}^2$ and consider $\text{w.dim}(R/\mathfrak{m}) = n < \infty$. If $R/\mathfrak{m}$ is flat, then, by Proposition 2.11, $\mathfrak{m} \cap RA = \mathfrak{m}A$ for any ideal A of R. In particular, taking $A = \mathfrak{m}$ gives $\mathfrak{m} = \mathfrak{m}^2$, contradicting our assumption. Thus $R/\mathfrak{m}$ is not flat and so $\text{w.dim}(R/\mathfrak{m}) > 0$. We now show that $\text{w.dim}(R/\mathfrak{m}) = 1$. By way of contradiction assume that $\text{w.dim}(R/\mathfrak{m}) \geq 2$. Then there exists an R-module M such that $\text{Tor}_n(R/\mathfrak{m}, M) \neq 0$. By Lemma 4.12 (iii) we have $\mathfrak{m}^2$ is flat. Therefore, $\text{Tor}_k(\mathfrak{m}^2, M) = 0$ for all $k \geq 1$. Now tensor the s.e.s. $0 \longrightarrow \mathfrak{m}^2 \longrightarrow R \longrightarrow R/\mathfrak{m}^2 \longrightarrow 0$ by M to get the following segment of the long exact sequence for Tor

$$\text{Tor}_n(R, M) \longrightarrow \text{Tor}_n(R/\mathfrak{m}^2, M) \longrightarrow \text{Tor}_{n-1}(R/\mathfrak{m}^2, M).$$

Observe that $\text{Tor}_n(R, M) = 0$ since R is free and $\text{Tor}_{n-1}(\mathfrak{m}^2, M) = 0$ since $\mathfrak{m}^2$ is flat. Hence we get $\text{Tor}_n(R/\mathfrak{m}^2, M) = 0$.

Now consider the s.e.s.

$$0 \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{i} R/\mathfrak{m}^2 \xrightarrow{\pi} R/\mathfrak{m} \longrightarrow 0 \qquad (*),$$

where i is the inclusion map and $\pi(r+\mathfrak{m}^2)=r+\mathfrak{m}$ for all $r\in R$. Tensor $(*)$ by M to get a long exact sequence for Tor with the following exact segment

$$\mathrm{Tor}_{n+1}(R/\mathfrak{m},M)\longrightarrow \mathrm{Tor}_n(\mathfrak{m}/\mathfrak{m}^2,M)\longrightarrow \mathrm{Tor}_n(R/\mathfrak{m}^2,M).$$

Notice that $\mathrm{Tor}_{n+1}(R/\mathfrak{m},M)=0$ since $\mathrm{w.dim}(R/\mathfrak{m})=n$ and $\mathrm{Tor}_n(R/\mathfrak{m}^2,M)=0$ by above. This implies that $\mathrm{Tor}_n(\mathfrak{m}/\mathfrak{m}^2,M)=0$. Moreover, because $\mathfrak{m}/\mathfrak{m}^2$ is annihilated by $\mathfrak{m}$, i.e., $(\mathfrak{m}/\mathfrak{m}^2)\mathfrak{m}=0$, by Remark 2.60 we may regard $\mathfrak{m}/\mathfrak{m}^2$ as a module over the field $R/\mathfrak{m}$ and so can be written as $(R/\mathfrak{m})^{(I)}$ for some index set I. Then $(\mathrm{Tor}_n(R/\mathfrak{m},M))^{(I)}\simeq \mathrm{Tor}_n((R/\mathfrak{m})^{(I)},M)=\mathrm{Tor}_n(\mathfrak{m}/\mathfrak{m}^2,M)=0$. Hence $\mathrm{Tor}_n(R/\mathfrak{m},M)=0$, a contradiction and so $\mathrm{w.dim}(R/\mathfrak{m})=1$. Then we have a flat resolution

$$F_1\xrightarrow{\ i\ } R\xrightarrow{\ \pi_0\ } R/\mathfrak{m}\longrightarrow 0$$

and so by [Bl], $\ker(\pi_0)=\mathfrak{m}$ is flat. $\square$

4.4 Local Gaussian rings R with non-nilpotent $\mathrm{Nil}(R)$.

We begin this section with Lemma 4.4 of [DT].

Lemma 4.14. *Let $(R,\mathfrak{m})$ be a local Gaussian ring with $\mathrm{Nil}(R)=\mathfrak{m}$ and let M be an R-module with $\mathrm{w.dim}(M)=n\geq 1$. Set $R'=R/D$ and $\mathfrak{m}'=\mathfrak{m}/D$. If $\mathfrak{m}$ is not nilpotent then $\mathrm{Tor}_n(R'/a\mathfrak{m}',M)=0$ for every nonzero $a\in\mathfrak{m}'$.*

Proof. Lemma 4.3 gives $\mathrm{Tor}_n(R/D,M)=0$. If $a\mathfrak{m}'=0$ then $R'/a\mathfrak{m}'=R'=R/D$ and so $\mathrm{Tor}_n(R'/a\mathfrak{m}',M)=\mathrm{Tor}_n(R/D,M)=0$. Thus we now assume that $a\mathfrak{m}'\neq 0$. We first show that $\mathfrak{m}'$ is not nilpotent. If otherwise then $(\mathfrak{m}/D)^t=(\mathfrak{m}')^t=0$ for some $t\in\mathbb{N}$ and so, given $m_1,m_2,\ldots,m_t\in\mathfrak{m}$, we have $0=(m_1+D)(m_2+D)\cdots(m_t+D)=m_1m_2\cdots m_t+D$. Then $m_1m_2\cdots m_t\in D$. Thus, because $D^2=0$ we have $(m_1m_2\cdots m_t)(m_{t+1}m_{t+2}\cdots m_{2t})=0$ for any $m_1,\ldots,m_{2t}\in\mathfrak{m}$. This gives $\mathfrak{m}^{2t}=0$ contradicting our assumption that $\mathfrak{m}$ is not nilpotent. Hence $\mathfrak{m}'$ is not nilpotent.

Next we show that the ideal $a\mathfrak{m}'$ of R' is not finitely generated. Suppose to the contrary that $a\mathfrak{m}'$ is finitely generated. Then, since $R' = R/D$ is a chain ring, $a\mathfrak{m}'$ is a principal ideal of R', say $a\mathfrak{m}' = R'b$, where $b \in R'$. Since $b \in a\mathfrak{m}'$, we have $b = ac$ for some $c \in \mathfrak{m}'$ and so $a\mathfrak{m}' = R'ac$.

By Lemma 3.28, there exists $z \in \mathfrak{m}'$ such that $\deg(z) > \deg(c)$. Since R' is a chain ring, $R'c \subset R'z$ (since otherwise we would have $z \in R'c$ and this gives $\deg(z) \leq \deg(c)$). Say $c = uz$, where $u \in R'$. Note that since $\deg(z) > \deg(c)$, u is not a unit and so $u \in \mathfrak{m}'$. Then $0 \neq ac = auz$ shows that $az \neq 0$. Moreover, since R is local, $1 - ur$ is a unit for every $r \in R'$. Then $az - acr = az - auzr = az(1 - ur) \neq 0$. This implies that $az \neq acr$ for any $r \in R'$ and therefore $az \notin R'ac$. However, $az \in a\mathfrak{m}' = R'ac$, a contradiction. Thus $a\mathfrak{m}'$ is not finitely generated.

Now let $\mathcal{X}$ denote the direct system given by the nonzero finitely generated ideals I of R' contained in $a\mathfrak{m}'$ and the associated inclusion maps. Then $R'/a\mathfrak{m}' = \varinjlim_{I \in \mathcal{X}} R'/I$ and so $\mathrm{Tor}_n(R'/a\mathfrak{m}', M) = \varinjlim_{I \in \mathcal{X}}(\mathrm{Tor}_n(R'/I, M))$. Given $I \in \mathcal{X}$, since R' is a chain ring, $I = R'x$ for some $x \in a\mathfrak{m}'$. Thus $R'x \neq a\mathfrak{m}'$ since $a\mathfrak{m}'$ is not finitely generated. By Lemma 4.6 (ii), with $J = a\mathfrak{m}'$ and $z = x$ we have $\pi : R'/R'x \longrightarrow R'/a\mathfrak{m}'$ induces the zero mapping $\mathrm{Tor}_n(R'/R'x, M) \longrightarrow \mathrm{Tor}_n(R'/a\mathfrak{m}', M) = \varinjlim_{I \in \mathcal{X}}(\mathrm{Tor}_n(R'/I, M))$. Notice that this will give the following commutative diagram whenever $\phi_{\alpha\beta} : I_\alpha \longrightarrow I_\beta$ is a homomorphism in our direct system $\mathcal{X}$.

$$\begin{tikzcd}
\mathrm{Tor}_n(R'/I_\alpha, M) \arrow[rr, "\phi_{\alpha\beta}"] \arrow[dr, "0"'] & & \mathrm{Tor}_n(R'/I_\beta, M) \arrow[dl, "0"] \\
& \mathrm{Tor}_n(R'/a\mathfrak{m}', M) &
\end{tikzcd}$$

Since $\mathrm{Tor}_n(R'/a\mathfrak{m}', M) = \varinjlim_{I \in \mathcal{X}}(\mathrm{Tor}_n(R'/I, M))$ we also have the following commutative diagram:

$$\begin{tikzcd}
\operatorname{Tor}_n(R'/I_\alpha, M) \arrow[rr, "\phi_{\alpha\beta,M}"] \arrow[dr, "\phi_\alpha"'] & & \operatorname{Tor}_n(R'/I_\beta, M) \arrow[dl, "\phi_\beta"] \\
& \operatorname{Tor}_n(R'/a\mathfrak{m}', M) &
\end{tikzcd}$$

Then there exists a unique homomorphism $\phi : \operatorname{Tor}_n(R'/a\mathfrak{m}', M) \longrightarrow \operatorname{Tor}_n(R'/a\mathfrak{m}', M)$ such that the following diagram commutes.

$$\begin{tikzcd}
\operatorname{Tor}_n(R'/I_\alpha, M) \arrow[rr, "\phi_{\alpha\beta}"] \arrow[dr, "\phi_\alpha"'] \arrow[ddr, bend right, "0"'] & & \operatorname{Tor}_n(R'/I_\beta, M) \arrow[dl, "\phi_\beta"] \arrow[ddl, bend left, "0"] \\
& \operatorname{Tor}_n(R'/a\mathfrak{m}', M) \arrow[d, dotted, "\phi"] & \\
& \operatorname{Tor}_n(R'/a\mathfrak{m}', M) &
\end{tikzcd}$$

Note that taking ϕ to be the zero or the identity map on $\operatorname{Tor}_n(R'/a\mathfrak{m}', M)$ keeps the commutativity of the diagram and so we get $1_{\operatorname{Tor}_n(R'/a\mathfrak{m}', M)} = 0$. Hence we obtain $0 = \operatorname{Tor}_n(R'/a\mathfrak{m}', M) = \varinjlim_{I\in\mathcal{X}}(\operatorname{Tor}_n(R'/I, M))$. $\square$

The following theorem is the major result of section 4 in [DT].

Theorem 4.15. *Let* $(R, \mathfrak{m})$ *be a local Gaussian ring with* $\operatorname{Nil}(R) = \mathfrak{m}$. *If* $\mathfrak{m}$ *is not nilpoent then* $\operatorname{w.gl.dim}(R) = \infty$.

Proof. Suppose to the contrary that $\operatorname{w.gl.dim}(R) = n < \infty$. Then by Proposition 4.9 $n \geq 3$. Let M be an R- module with $\operatorname{w.dim}_R(M) = n$. Now the proof splits into two cases.

Case 1: The ring R does not satisfy Property 4.10. Then there exists $x \in D \setminus 0$ such that $\operatorname{ann}(x) = Ra + D$ for some $a \in \mathfrak{m} \setminus D$. Then $Rx \simeq R/\operatorname{ann}(x) = R/(Ra + D)$. By Corollary 2.57, the inclusion $Rx \hookrightarrow R$ induces a monomorphism $\operatorname{Tor}_n(Rx, M) \hookrightarrow \operatorname{Tor}_n(R, M)$ and so we have a monomorphism $\operatorname{Tor}_n(R/(Ra+D), M) \longrightarrow \operatorname{Tor}_n(R, M) = 0$. Thus $\operatorname{Tor}_n(R/(Ra+D), M) = 0$. Now note that $\dfrac{R'}{R'(a+D)} = \dfrac{R/D}{(Ra+D)/D} \simeq R/(Ra+D)$ and so $\operatorname{Tor}_n(R'/R'(a+D), M) = 0$. However since $a + D \in \mathfrak{m}' \setminus 0$, Lemma 4.6 (iii) gives a contradiction.

Case 2: The ring R satisfies Property 4.10. Choose $a \in \mathfrak{m} \setminus D$ (noting that such an element exists since otherwise we get $\mathfrak{m} = D$ and so $\mathfrak{m}^2 = D^2 = 0$, contradicting our non-nilpotency hyypothesis). Then we have the following s.e.s. where $a + D$, i is inclusion and π is the projection map.

$$0 \longrightarrow \frac{R'\bar{a}}{\mathfrak{m}'\bar{a}} \xrightarrow{\ i\ } \frac{R'}{\mathfrak{m}'\bar{a}} \xrightarrow{\ \pi\ } \frac{R'}{R'\bar{a}} \longrightarrow 0$$

Tensoring this s.e.s. with M induces a long exact sequence for Tor with the segment

$$\mathrm{Tor}_n(R'/\mathfrak{m}'\bar{a}, M) \longrightarrow \mathrm{Tor}_n(R'/R'\bar{a}, M) \longrightarrow \mathrm{Tor}_{n-1}(R'\bar{a}/\mathfrak{m}'\bar{a}, M) \quad (*).$$

However $\mathrm{Tor}_n(R'/\mathfrak{m}'\bar{a}, M) = 0$ by Lemma 4.14. Now define $f : R/\mathfrak{m} \longrightarrow R'\bar{a}/\mathfrak{m}'\bar{a}$ by $f(r + \mathfrak{m}) = \bar{r}\bar{a} + \mathfrak{m}'\bar{a}$ (where $\bar{x} = x + D$ for all $x \in R$). Then f is well-defined since if $r + \mathfrak{m} = s + \mathfrak{m}$ then $r - s \in \mathfrak{m}$ so $\bar{r} - \bar{s} \in \mathfrak{m}'$ and therefore $\bar{r}\bar{a} - \bar{s}\bar{a} \in \mathfrak{m}'\bar{a}$. Hence $f(r + \mathfrak{m}) = \bar{r}\bar{a} + \mathfrak{m}'\bar{a} = \bar{s}\bar{a} + \mathfrak{m}'\bar{a} = f(s + \mathfrak{m})$. It is easily seen that f is a homomorphism. Moreover, f is an isomorphism because:

(i) f is one-to-one since if $f(r + \mathfrak{m}) = 0$ then $\bar{r}\bar{a} + \mathfrak{m}'\bar{a} = 0$. Thus $\bar{r}\bar{a} \in \mathfrak{m}'\bar{a}$, say $\bar{r}\bar{a} = \bar{m}\bar{a}$, where $m \in \mathfrak{m}$. Then $(\bar{r} - \bar{m})\bar{a} = 0$ and so $(r - m)a \in D$. If $r \notin \mathfrak{m}$ then r is a unit, say with inverse t. Then $t(r - m)a \in D$ and so $(tr - tm)a \in D$ which implies that $(1 - tm)a \in D$. Note that $1 - tm$ is a unit since $tm \in \mathfrak{m}$ and this gives $a \in D$, a contradiction. Thus $r \in \mathfrak{m}$ as required.

(ii) f is clearly onto since if $\bar{r}\bar{a} + \mathfrak{m}'\bar{a}$, where $r \in R$, we have $f(r + \mathfrak{m}) = \bar{r}\bar{a} + \mathfrak{m}'\bar{a}$.

Hence $R/\mathfrak{m} \simeq R'\bar{a}/\mathfrak{m}'\bar{a}$ and so $\mathrm{Tor}_{n-1}(R'\bar{a}/\mathfrak{m}'\bar{a}, M) \simeq \mathrm{Tor}_{n-1}(R/\mathfrak{m}, M)$ which is 0 since tensoring the s.e.s.

$$0 \longrightarrow \mathfrak{m} \longrightarrow R \longrightarrow R/\mathfrak{m} \longrightarrow 0$$

with M gives a long exact sequence for Tor with the following segment

$$\mathrm{Tor}_{n-1}(R, M) \longrightarrow \mathrm{Tor}_{n-1}(R/\mathfrak{m}, M) \longrightarrow \mathrm{Tor}_{n-2}(\mathfrak{m}, M).$$

Since R is flat and, by Lemma 4.13 so is $\mathfrak{m}$, we obtain $\mathrm{Tor}_{n-1}(R, M) = 0 = \mathrm{Tor}_{n-2}(\mathfrak{m}, M)$. Hence $\mathrm{Tor}_{n-1}(R/\mathfrak{m}, M) = 0$ and so the second term of $(*)$, $\mathrm{Tor}_n(R'/R'\bar{a}, M)$, is 0. Again, part (iii) of Lemma 4.6 gives a contradiction, and so $\mathrm{w.gl.dim}(R) = \infty$. □

4.5 Local Gaussian rings R with $\mathrm{Nil}(R)^2 \neq 0$.

We begin this short section with Theorem 5.2 of [DT]. It is a modification of Theorem 6.4 of [BaGl] which did not have the nilpotency index condition. However [DT] spotted a flaw in the proof of Theorem 6.4, giving an example to show the result is false without the condition, leading to the following amendment.

Theorem 4.16. *Let R be a Gaussian ring with a maximal ideal $\mathfrak{m}$ such that the nilradical $\mathrm{Nil}(R_\mathfrak{m})$ of the localisation $R_\mathfrak{m}$ has nilpotency index n at least 3. Then $\mathrm{w.gl.dim}(R) = \infty$.*

Proof. Let $\mathfrak{m}$ be as stated. Since $R_\mathfrak{m}$ is local Gaussian, $\mathrm{Nil}(R_\mathfrak{m})$ is its unique minimal prime ideal. By Lemma 1.31, $\mathrm{Nil}(R_\mathfrak{m}) = (\mathrm{Nil}(R))_\mathfrak{m}$. Since $\mathrm{Nil}(R_\mathfrak{m})$ is prime so is $\mathrm{Nil}(R)$, by Theorem 1.27. Localising R at $\mathrm{Nil}(R)$, the ring $R_{\mathrm{Nil}(R)}$ is local Gaussian and has a unique prime (so maximal) ideal, namely $R_{\mathrm{Nil}(R)}\,\mathrm{Nil}(R)$ $(= \mathrm{Nil}(R_{\mathrm{Nil}(R)}))$, by Lemmas 3.7 and 1.28.

Now, given $r/s \in R_{\mathrm{Nil}(R)}\,\mathrm{Nil}(R)$ where $r \in \mathrm{Nil}(R)$ and $s \in R \setminus \mathrm{Nil}(R)$, we have $r/1 \in \mathrm{Nil}(R_\mathfrak{m})$ and so $r^n/1 = 0$ in $R_\mathfrak{m}$. Thus $r^n t = 0$ for some $t \in R \setminus \mathfrak{m}$ and so in $R_{\mathrm{Nil}(R)}$ we have $(r/s)^n = r^n/s^n = r^n t/s^n t = 0$. This has shown that $R_{\mathrm{Nil}(R)}\,\mathrm{Nil}(R)$ is also nilpotent (of nilpotency index at most n).

Since the nilpotency index of $\mathrm{Nil}(R_\mathfrak{m})$ is at least 3, there is an element $a/u \in \mathrm{Nil}(R_\mathfrak{m})$ with $(a/u)^2 \neq 0$, where $a \in \mathrm{Nil}(R)$ and $u \in R \setminus \mathfrak{m}$. Then $a^2 \neq 0$ and the element $a/1$ in $R_{\mathrm{Nil}(R)}$ is in $\mathrm{Nil}(R_{\mathrm{Nil}(R)})$ since $(a/1)^n = (au/u)^n = (a/u)^n(u/1)^n = 0$. Moreover, if $a/1 = 0$ in $R_{\mathrm{Nil}(R)}$ then $ab = 0$ for some $b \in R \setminus \mathrm{Nil}(R)$ and so, since R is local Gaussian, either $a^2 = 0$ or $b^2 = 0$ by Theorem 3.13, a contradiction. It follows that $a/1$ is a nonzero element in the maximal ideal $R_{\mathrm{Nil}(R)}\,\mathrm{Nil}(R)$ of $R_{\mathrm{Nil}(R)}$. Then, by Proposition 2.61, $\mathrm{w.gl.dim}(R_{\mathrm{Nil}(R)}) = \infty$. Thus, since $\mathrm{w.gl.dim}(R) = \sup\{\mathrm{w.gl.dim}(R_P) : P \in \mathrm{Spec}(R)\}$, the result is proved. $\square$

The following is Theorem 5.4 of [DT].

Theorem 4.17. *Let R be a local Gaussian ring with $\mathrm{Nil}(R) \neq 0$, i.e., R is non-reduced, and $\mathrm{Nil}(R) \neq D$. Then $\mathrm{w.gl.dim}(R) = \infty$.*

Proof. By Corollary 3.17, $\mathrm{Nil}(R)$ is the uniquet prime ideal of R and so, since $\mathrm{w.gl.dim}(R) \geq \mathrm{w.gl.dim}(R_S)$ for any m.c.s. S of R, it is sufficient to show that $\mathrm{w.gl.dim}(R_{\mathrm{Nil}(R)}) = \infty$.

The maximal ideal of $R_{\mathrm{Nil}(R)}$ is the only prime ideal of $R_{\mathrm{Nil}(R)}$ and so is its nilradical. Denote this ideal by N. By Lemma 3.22, $N \neq 0$. If N is nilpotent then, by Proposition 2.61, $\mathrm{w.dim}_{R_{\mathrm{Nil}(R)}}(N) = \infty$ and so $\mathrm{w.gl.dim}(R_{\mathcal{N}}) = \infty$ which proves the result in this case. If otherwise N is not nilpotent then Theorem 4.15 shows that $\mathrm{w.gl.dim}(R_{\mathrm{Nil}(R)}) = \infty$ as required. $\square$

4.6 Local Gaussian rings R with nonzero $\mathrm{Nil}(R)$ but $\mathrm{Nil}(R)^2 = 0$.

We saw in Theorem 4.1 that the Bazzoni–Glaz conjecture holds if R is a reduced Gaussian ring. Since $\mathrm{w.gl.dim}(R) = \sup\{\mathrm{w.gl.dim}(R_P) : P \in \mathrm{Max}(R)\}$ by Theorem 2.53, if we can show that if R is any Gaussian ring R which is not reduced then $\mathrm{w.gl.dim}(R_M) = \infty$ for some maximal ideal M then $\mathrm{w.gl.dim}(R) = \infty$ and this will complete the proof of the conjecture. Furthermore, Theorem 4.17 shows the conjecture to be true if R is non-reduced local with $\mathrm{Nil}(R) \neq D$.

Thus we assume henceforth that R is local Gaussian with $\mathrm{Nil}(R) \neq 0$ but $\mathrm{Nil}(R)^2 = 0$.

Recall that $\mathrm{Nil}(R)$ is the unique minimal prime ideal in a local Gaussian ring R. If we let S be the set of regular elements then, by Lemma 1.31, $(\mathrm{Nil}(R))_S = \mathrm{Nil}(R_S)$.

From this it follows that the total ring of quotients $Q(R)$ also has nonzero $\mathrm{Nil}(Q(R))$ but $(\mathrm{Nil}(Q(R)))^2 = 0$. Furthermore, $Q(R)$ is also local Gaussian and, since $\mathrm{w.gl.dim}(R_S) \leq \mathrm{w.gl.dim}(R)$, we may assume that, in addition to the conditions imposed on R above, R is its own total ring of quotients, i.e., the maximal ideal $\mathfrak{m}$ consists of zero-divisors. Moreover, by Proposition 2.61, if $\mathfrak{m}$ is nilpotent then $\mathrm{w.dim}(\mathfrak{m}) = \infty$ and so we can ignore this case.

Thus, unless otherwise stated, **we henceforth assume that** $(R, \mathfrak{m})$ **is a local Gaussian ring, the elements of** $\mathfrak{m}$ **are zero-divisors,** $\mathfrak{m} \neq D$**, and** $\mathrm{Nil}(R) = D \neq 0$.

Before proceeding with the theory, we give an example of a ring satisfying our assumptions. It is a variation of an example in [Cl2] (see also [NY, page 123]).

Example 4.18. Let A be $\mathbb{Z}_{(2)}$, the ring of integers localized at its prime ideal (2), i.e., $A = \{x/y : x, y \in \mathbb{Z}, y \text{ odd}\}$. Then A is a chain domain with ideals given by the strictly descending chain

$$A \supset 2A \supset 4A \supset 8A \supset \cdots \supset 2^n A \supset \cdots \supset 0.$$

Now let M be the A-module $\mathbb{Q}/A$, where $\mathbb{Q}$ is the field of all rational numbers and the module multiplication is given by $x/y(p/q + A) = xp/yq + A$ for all $x, y, p, q \in \mathbb{Z}$ with y odd and $q \neq 0$. The A-submodules of M form the strictly ascending chain:

$$0 \subset A(1/2)/A \subset A(1/4)/A \subset A(1/8)/A \subset \cdots \subset A(1/2^n)/A \subset \cdots \subset M$$

where $A(1/2^n) = \{a2^{-n} : a \in A\}$. Given $n \in \mathbb{N}$, let B be the A-module M^n, the direct sum of n copies of M. Now form the ring R as the set of upper triangular 2×2 matrices

$$\left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a \in A, b \in B \right\}, \text{multiplication given by } \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \begin{bmatrix} c & d \\ 0 & c \end{bmatrix} = \begin{bmatrix} ac & ad + bc \\ 0 & ac \end{bmatrix}$$

for all $a, c \in A, b, d \in B$. To save room, we write a typical matrix in R as an ordered pair

(a, b) where $a \in A, b \in B$ so that multiplication in R is given by $(a, b)(c, d) = (ac, ad + bc)$ for all $a, c \in A, b, d \in B$. It is straightforward to show that $\mathrm{Nil}(R) = \{(0, b) : b \in B\}$ (the set of strictly upper triangular matrices), $\mathrm{Nil}(R)^2 = 0$ and R is local with unique maximal ideal $\mathfrak{m} = \{(a, b); a \in 2A, b \in B\} \neq \mathrm{Nil}(R)$. Moreover, every $r = (a, b) \in \mathfrak{m}$ is a zero-divisor since if $a = 0$ then r is nilpotent while if $a \neq 0$ then $a = 2^t x/y$ for some $t \geq 1$ and odd $x, y \in \mathbb{Z}$ and so taking $d = (d_1, \ldots, d_n) \in B$ with $d_i = 2^{-t} + A \in M$ for each i gives $(a, b)(0, d) = (0, ad) = (0, (2^t x/y)(2^{-t} + A) = (0, x/y + A) = 0$.

Given a nonzero $a \in A$, we have $a = 2^s u/v$ where $s \geq 0$ and u, v are odd integers. Let $d = (d_1, 0, 0, \ldots, 0) \in B$ with $d_1 = 2^{-t}x/y + A$ where $t \geq 0$ and x, y are odd integers. Then $(0, d) = (0, (2^{-s-t}yv/xu + A, 0, 0, \ldots, 0))(a, b)$ for any $b \in B$ and so $(0, d) \in R(a, b)$. An obvious extension of this argument shows that $(0, d) \in R(a, b)$ *for all* $d, b \in B$ and so $R(a, b) = \{(\lambda a, d) : \lambda \in A, d \in B\} = R(a, 0)$.

Now we show that R is Gaussian using Theorem 3.13. Let $p = (a, b), q = (c, d) \in R$. Then $p^2 = (a^2, 2ab), q^2 = (c^2, 2cd)$. If $a = 0 = c$ then $p^2 = q^2 = pq = 0$ so $(p, q)^2 = 0 = (p^2)$ as required. If $a \neq 0$ but $c = 0$ then the previous paragraph shows that $q \in Rp$ and so $pq, q^2 \in (p^2)$ and then $(p, q)^2 = (p^2)$. Similarly, if $a = 0$ but $c \neq 0$ then $(p, q)^2 = (q^2)$. If a and c are both nonzero, let $a = 2^s u/v, c = 2^t x/y$ where $s, t \geq 0$ and u, v, x, y are odd integers. Without loss of generality, $s \leq t$. Then, since $A2^{2t} \subseteq A2^{s+t} \subseteq A2^{2s}$ in A, we get

$$(p, q)^2 = (p^2, pq, q^2) = R(a^2, 0) + R(ac, 0) + R(c^2, 0) = R(2^s, 0) = (p^2).$$

This has verified that R satisfied condition (i) of Theorem 3.13.

For condition (ii), let $p = (a, b), q = (c, d)$ be as before and suppose that $(p, q)^2 = (q)^2$ and $pq = 0$. We wish to show that $p^2 = 0$. If $a = 0$ this is immediate. If $a \neq 0$, then, since $pq = 0$, we must have $c = 0$ and so $q^2 = (0, d)^2 = 0$. Since $p^2 \in (q^2)$ we get $0 = p^2 = (a^2, 2ab)$, a contradiction since $a^2 \neq 0$. Thus R satisfies condition (ii) of Theorem

3.13 and so R is Gaussian.

If $n = 1$, i.e., $B = M$, then the ideals of R contained in $\mathrm{Nil}(R)$ are precisely the A-submodules of M. Thus, from above, R is a chain ring with ideal lattice given by

$$0 \subset (0, A(2^{-1})/A) \subset \cdots \subset (0, A(2^{-n})/A) \subset \cdots \subset (0, M) \subset \cdots$$
$$\cdots \subset (2^n A, M) \subset \cdots \subset (2A, M) \subset R.$$

On the other hand, if $n \geq 2$ then R is not a chain ring. For example, if $n = 2$ then $B = M \oplus M$ and if we take $I_1 = \{(a, (b_1, 0)) : a \in A, a = 0, (b_1, 0) \in M \oplus M\}$ and $I_2 = \{(a, (0, b_2)) : a \in A, a = 0, (0, b_2) \in M \oplus M\}$, then I_1 and I_2 are ideals of R which are not comparable.

The following lemmas can be found in [DT, Lemma 6.1, Lemma 6.2 and Lemma 6.4]

Lemma 4.19. *Suppose* $\mathrm{w.gl.dim}(R) = n < \infty$. *Then* $\mathrm{w.dim}_R(R/D) = n - 1$.

Proof. Since $\mathrm{w.gl.dim}(R) = n$, there is an R-module M with $\mathrm{w.dim}_R(M) = n$. By Lemma 4.5 and since $\mathfrak{m} \neq D$, there exists an element $a \in \mathfrak{m} \backslash D$ such that $\mathrm{Tor}_n(R/(Ra+D), M) \neq 0$. Define the map $f : R/D \longrightarrow R/D$ by $f(r + D) = ra + D$. It can be easily checked that f is a well-defined homomorphism. Now, consider the s.e.s.

$$0 \longrightarrow R/D \xrightarrow{f} R/D \xrightarrow{\pi} R/(Ra + D) \longrightarrow 0 \quad (*),$$

where $\pi(r + D) = r + (Ra + D)$. Tensoring $(*)$ with M gives the long exact sequence for Tor with the following segment:

$$\mathrm{Tor}_n(R/D, M) \longrightarrow \mathrm{Tor}_n(R/(Ra + D), M) \longrightarrow \mathrm{Tor}_{n-1}(R/D, M) .$$

If $\mathrm{w.dim}_R(R/D, M) < n - 1$, then $\mathrm{Tor}_n(R/D, M) = 0 = \mathrm{Tor}_{n-1}(R/D, M)$ and so the segment gives $\mathrm{Tor}_n(R/(Ra + D), M) = 0$, contradicting the first part of the proof. Thus we have $\mathrm{w.dim}(R/D) \geq n - 1$. Moreover, $\mathrm{Tor}_n(R/D, X) = 0$ for any R-module X since: (i) if $\mathrm{w.dim}(X) = n$ this follows from Lemma 4.3, while

(ii) if $\mathrm{w.dim}(X) < n$ then $\mathrm{Tor}_n(Y, X) = 0$ for any R-module Y.

This, together with the above, we get $\mathrm{w.dim}_R(R/D) = n - 1$ as desired. □

Definition 4.20. Let M be an R-module and $a \in R$. Then a is called a **zero-divisor on** M if $am = 0$ for some $m \in M$, $m \neq 0$.

Lemma 4.21. *Let $a \in \mathfrak{m} \setminus D$ be a zero-divisor on $\mathrm{Tor}_{i-1}(R/D, R/D)$ for some $i \geq 1$. Then $\mathrm{Tor}_{i-1}(R/(Ra + D), R/D) \neq 0$. Furthermore, if $\mathrm{w.gl.dim}(R) = n < \infty$, then $\mathrm{Tor}_{n-1}(R/(Ra + D), R/D) \neq 0$ if and only if a is a zero-divisor on $\mathrm{Tor}_{n-2}(R/D, R/D)$.*

Proof. We have an s.e.s.

$$0 \longrightarrow R/D \xrightarrow{\mu_a} R/D \xrightarrow{\pi} R/(Ra + D) \longrightarrow 0\,,$$

where μ_a is multiplication by a and $\pi(r + D) = r + Ra + D$ for all $r \in R$. Tensoring this with R/D gives a long exact sequence for Tor with the following segment for each $i \geq 1$.

$$\mathrm{Tor}_i(R/(Ra + D), R/D) \longrightarrow \mathrm{Tor}_{i-1}(R/D, R/D) \xrightarrow{\mu_a^*} \mathrm{Tor}_{i-1}(R/D, R/D). \quad (*)$$

If $\mathrm{Tor}_i(R/(Ra+D), R/D) = 0$, then the induced multiplication μ_a^* will be a monomorphism. Then for any nonzero $x \in \mathrm{Tor}_{i-1}(R/D, R/D)$ we would have $\mu_a^*(x) \neq 0$, i.e., $ax \neq 0$ and so a is not a zero-divisor on $\mathrm{Tor}_{i-1}(R/D, R/D)$. This contradiction proves the first part of the lemma.

To prove the second, we take $i = n - 1$. For the required converse we show that if $\mathrm{Tor}_{n-2}(R/D, R/D) \neq 0$ and $a \in \mathfrak{m} \setminus D$ then $\mathrm{Tor}_{n-1}(R/(Ra + D), R/D) \neq 0$. By Lemma 4.19, $\mathrm{w.dim}_R(R/D) = n - 1$. Therefore, by Lemma 4.3, we have $\mathrm{Tor}_{n-1}(R/D, R/D) = 0$. Then the following exact segment

$$\begin{tikzcd}
\mathrm{Tor}_{n-1}(R/D, R/D) \arrow[r] & \mathrm{Tor}_{n-1}(R/(Ra + D), R/D) \arrow[d, "f"] & \\
& \mathrm{Tor}_{n-2}(R/D, R/D) \arrow[r, "\mu_a^*"] & \mathrm{Tor}_{n-2}(R/D, R/D)
\end{tikzcd}$$

implies that $\mathrm{Tor}_{n-1}(R/(Ra+D), R/D) \simeq \ker(\mu_a^*)$. Hence if $\mathrm{Tor}_{n-1}(R/(Ra+D), R/D) \neq 0$

then $\ker(\mu_a^*) \neq 0$ and so there is a nonzero element $m \in \mathrm{Tor}_{n-2}(R/D, R/D)$ such that $ma = 0$. Thus a is a zero-divisor on $\mathrm{Tor}_{n-2}(R/D, R/D)$. $\square$

Lemma 4.22. *Let* $\mathrm{w.gl.dim}(R) = n$. *If n is some finite integer then there exists a prime ideal P such that the following statements hold:*

(i) *$D_P \neq R_P P$ and there exists a nonzero element $w \in \mathrm{Tor}_{n-2}^{R_P}(R_P/D_P, R_P/D_P)$ such that for every $c \in R_P P$ we have $c^q w = 0$ for some $q \in \mathbb{N}$.*

(ii) *For every $c \in R_P P \setminus D_P$, $\mathrm{Tor}_{n-1}^{R_P}(R_P/(R_P c + D_P), R_P/D_P) \neq 0$.*

(iii) *Every element of $R_P P$ is a zero-divisor.*

(iv) $\mathrm{w.gl.dim}(R_P) = n$.

Proof. (i) By Lemma 4.19, $\mathrm{w.gl.dim}(R/D) = n - 1$. Taking $M = R/D$ and replacing n by $n-1$ in Lemma 4.5 gives an $a \in \mathfrak{m} \setminus D$ with $\mathrm{Tor}_{n-1}(R/(Ra + D), R/D) \neq 0$. By Lemma 4.21 we see that a is a zero-divisor on $\mathrm{Tor}_{n-2}(R/D, R/D)$, i.e., there is a nonzero $w \in \mathrm{Tor}_{n-2}(R/D, R/D)$ such that $aw = 0$. Let $P = \{b \in \mathfrak{m} : b^q w = 0 \text{ for some } q \geq 1\}$. Then by Lemma 3.29, P is a prime ideal of R. Note that $a \in P$ and, since D $(= \mathrm{Nil}(R))$ is a prime ideal, $\frac{a}{1} \notin D_P$ (since if $\frac{a}{1} = \frac{d}{s}$ for some $d \in D$, $s \in R \setminus P$ we obtain $ast = dt$ for some $t \in R \setminus P$, in particular $ast \in D$ but, in contradiction, $a, s, t \notin D$). However $\frac{a}{1} \in R_P P$ and so $D_P \neq R_P P$. Now note that by Lemma 2.46, we have $(\mathrm{Tor}_{n-2}^{R}(R/D, R/D))_P = \mathrm{Tor}_{n-2}^{R_P}(R_P/D_P, R_P/D_P)$. We claim that $\frac{w}{1} \in (\mathrm{Tor}_{n-2}^{R}(R/D, R/D))_P$ is nonzero. If not, then for some $b \in R \setminus P$ we have $bw = 0$ in $\mathrm{Tor}_{n-2}(R/D, R/D)$ and so $b \in P$, a contradiction. Again by definition of P, if $c \in R_P P$, say $c = \frac{p}{t}$, where $p \in P$ and $t \notin R \setminus P$, then we have $p^q w = 0$ for some $q \geq 1$. Hence $c^q \frac{w}{1} = \frac{0}{1}$ in $(\mathrm{Tor}_{n-2}^{R}(R/D, R/D))_P = \mathrm{Tor}_{n-2}^{R_P}(R_P/D_P, R_P/D_P)$.

(ii) By item (i), if $c \in R_P P \setminus D_P$ then c is a zero-divisor on $\mathrm{Tor}_{n-2}^{R_P}(R_P/D_P, R_P/D_P)$. Thus by Lemma 4.21, $\mathrm{Tor}_{n-1}^{R_P}(R_P/(R_P c + D_P), R_P/D_P) \neq 0$.

(iii) Assume to the contrary that there is a regular element $c \in R_P P$. Then $c \in$

$R_P P \setminus D_P$ (since otherwise $c^2 = 0$). By Proposition 3.9 and Theorem 3.18, $D_P \subseteq R_P c + \mathrm{ann}(c) = R_P c$ and so $R_P c = R_P c + D_P$. By (ii) we get $\mathrm{Tor}_{n-1}^{R_P}(R_P/R_P c, R_P/D_P) \neq 0$. Moreover, we have the s.e.s.

$$0 \longrightarrow R_P \xrightarrow{\mu_c} R_P \xrightarrow{\pi} R_P/R_P c \longrightarrow 0,$$

where μ_c is multiplication by c and π is the natural projection, giving a free resolution for the R_P-module $R_P/R_P c$. Thus $\mathrm{w.dim}_{R_P}(R_P/R_P c) \leq 1$ and so, in particular, $\mathrm{Tor}_k^{R_P}(R_P/R_P c, R_P/D_P) = 0$ for all $k \geq 2$. However, by Proposition 4.9, we have $n \geq 3$ and, from (ii), $\mathrm{Tor}_{n-1}^{R_P}(R_P/R_P c, R_P/D_P) \neq 0$ and so we have a contradiction. Thus every element of $R_P P$ is a zero-divisor.

(iv) It suffices to show that $\mathrm{w.gl.dim}(R_P) \geq n$ since, by Theorem 2.52, $\mathrm{w.gl.dim}(R_P) \leq n$. By (ii) we have $\mathrm{w.dim}_{R_P}(R_P/D_P) \geq n-1$. However, if $\mathrm{w.gl.dim}(R_P) = n-1$, Lemma 4.19 implies that $\mathrm{w.dim}_{R_P}(R_P/D_P) < n-1$, a contradiction. Thus $\mathrm{w.gl.dim}(R_P) = n$. □

4.7 The finishing touch.

If the Bazzoni–Glaz Conjecture is false then, relabeling the ring R_P of Lemma 4.22 as simply R, properties (i)–(iv) of the Lemma show that there exists a local Gaussian ring $(R, \mathfrak{m})$ with $\mathrm{w.gl.dim}(R) = n < \infty$ with $n > 1$ satisfying the following condition.

(**A**) $\mathrm{Nil}(R) = D \neq \mathfrak{m}$ and $\mathrm{Tor}_{n-1}(R/(Ra + D), R/D) \neq 0$ for every $a \in \mathfrak{m} \setminus D$.

Note also that $\mathrm{w.gl.dim}(R) = n \geq 3$ by Proposition 4.9. Then, tensoring the s.e.s.

$$0 \longrightarrow D \xrightarrow{i} R \xrightarrow{\pi} R/D \longrightarrow 0\ .$$

by R/D produces a long exact sequence for Tor with the following segment

$$\mathrm{Tor}_{n-2}(R, R/D) \longrightarrow \mathrm{Tor}_{n-2}(R/D, R/D) \longrightarrow \mathrm{Tor}_{n-3}(D, R/D) \longrightarrow \mathrm{Tor}_{n-3}(R, R/D),$$

where $\mathrm{Tor}_{n-2}(R, R/D) = 0 = \mathrm{Tor}_{n-3}(R, R/D)$ since R is flat. Thus $\mathrm{Tor}_{n-2}(R/D, R/D) \simeq$

$\mathrm{Tor}_{n-3}(D, R/D)$, so by Lemma 4.22 (i) we can also assume R meets the following condition.

(**B**) There exists a nonzero element $w \in \mathrm{Tor}_{n-3}(D, R/D)$ such that for every $a \in \mathfrak{m}$, $a^q w = 0$ for some $q \geq 1$.

Our first result in this section is [DT, Lemma 6.5].

Lemma 4.23. *Suppose* $\mathrm{w.gl.dim}(R) = n < \infty$ *and* R *satisfies condition* (**A**)*. Then* R *also satisfies the following:*

(i) $\mathfrak{m}$ *is flat.*

(ii) *If* $I = \mathfrak{m}, D,$ *or* $\mathrm{ann}(c)$*, where* c *is any element in* $\mathfrak{m}$*, then* $I = I\mathfrak{m}$*. Furthermore, if* $x \in I$*, then there exists an element* $x' \in I$ *and* $a \in \mathfrak{m}$ *such that* $x = ax'$*.*

(iii) $\mathfrak{m}/D$ *is not finitely generated.*

Proof. (i) By Lemma 4.13 it suffices to show that R has Property 4.10. Suppose to the contrary that R does not admit Property 4.10. Then there exists $d \in D$ with $d \neq 0$ and $a \in \mathfrak{m} \setminus D$ such that $\mathrm{ann}(d) = Ra + D$. This gives $Rd \simeq R/\mathrm{ann}(d) = R/(Ra + D)$. By Lemma 4.19, we have $\mathrm{w.dim}(R/D) = n - 1$. Then, applying Corollary 2.57, there is a monomorphism $\mathrm{Tor}_{n-1}(Rd, R/D) \longrightarrow \mathrm{Tor}_{n-1}(R, R/D) = 0$. Consequently $\mathrm{Tor}_{n-1}(R/(Rd + D), R/D) = \mathrm{Tor}(Rd, R/D) = 0$ contradicting (**A**). Hence $\mathfrak{m}$ is flat.

(ii) Using part (i), we have the following flat resolution of $R/\mathfrak{m}$:

$$0 \longrightarrow \mathfrak{m} \longrightarrow R \longrightarrow R/\mathfrak{m} \longrightarrow 0,$$

so $\mathrm{w.dim}_R(R/\mathfrak{m}) = 1$. We show that $\mathrm{Tor}_1(R/I, R/\mathfrak{m}) = 0$ for each nominated ideal I.

First, if $I = \mathfrak{m}$, we obtain $\mathrm{Tor}_1(R/I, R/\mathfrak{m}) = 0$ by taking $M = R/I$ in Lemma 4.8. Next, if $I = D$ we have $\mathrm{Tor}_1(R/I, R/\mathfrak{m}) = 0$ using Lemma 4.3. Lastly, if $I = \mathrm{ann}(c)$, then $R/I \simeq Rc$ and so $\mathrm{Tor}_1(R/I, R/\mathfrak{m}) \simeq \mathrm{Tor}_1(Rc, R/\mathfrak{m})$. Moreover, by Corollary 2.57, the inclusion map $Rc \longrightarrow R$ induces a monomorphism $\mathrm{Tor}_1(Rc, R/\mathfrak{m}) \longrightarrow \mathrm{Tor}_1(R, R/\mathfrak{m})$. However, $\mathrm{Tor}_1(R, R/\mathfrak{m}) = 0$ since R is flat and so $0 = \mathrm{Tor}_1(Rc, R/\mathfrak{m}) \simeq \mathrm{Tor}_1(R/I, R/\mathfrak{m})$.

Since $\mathrm{Tor}_1(R/J, R/K) \simeq (J \cap K)/JK$ for any ideals J, K of R (see [Osb, Exercise 9, page 72]), since $\mathrm{Tor}_1(R/I, R/\mathfrak{m}) = 0$ for each I we get $(I \cap \mathfrak{m})/I\mathfrak{m} = 0$. Thus $I/I\mathfrak{m} = 0$ and so $I = I\mathfrak{m}$. Then, for any $x \in I$ we have $x \in I\mathfrak{m}$ and so there exist $x_1, \dots, x_s \in I$ and $a_1, \dots, a_s \in \mathfrak{m}$ such that $x = x_1a_1 + \cdots + x_sa_s$. Let J be the ideal generated by $a_1, \dots, a_s, x_1, \dots, x_s$. By Theorem 3.14 (b), $J/(J \cap \mathrm{ann}(J))$ is cyclic, say $J/(J \cap \mathrm{ann}(J)) = R(a + (J \cap \mathrm{ann}(J)))$ where $a \in J \subseteq \mathfrak{m}$. Then, for each $i = 1, \dots, s$, there is an $r_i \in R$ such that $a_i + (J \cap \mathrm{ann}(J)) = r_ia + (J \cap \mathrm{ann}(J))$. Thus $a_i - r_ia \in J \cap \mathrm{ann}(J)$ and so $a_i = r_ia + \lambda_i$ for some $\lambda_i \in J \cap \mathrm{ann}(J)$. Then $x = x_1a_1 + \cdots + x_sa_s = x_1(r_1a + \lambda_1) + \cdots + x_s(r_sa + \lambda_s) = (x_1r_1 + \cdots + x_sa_s)a$ since $x_i\lambda_i = 0$ for every i. Thus $x = x'a$, where $x' = x_1r_1 + \cdots + x_sa_s \in I$, as required.

(iii) Suppose to the contrary that $\mathfrak{m}/D$ is finitely generated. Then there exist $m_1, \dots, m_t \in \mathfrak{m}$ such that $\mathfrak{m}/D = R(m_1 + D) + \cdots + R(m_t + D)$. Since R/D is a chain ring, the ideal $R(m_1 + D) + \cdots + R(m_t + D)$ in R/D is principal and so we may suppose that $\mathfrak{m}/D = Ra/D$, where $a \in \mathfrak{m}/D$. Hence $\mathfrak{m} = Ra + D$. Taking $I = \mathfrak{m}$ in (ii) shows that $\mathfrak{m} = \mathfrak{m}^2$ and so $Ra + D = (Ra + D)^2 = Ra^2 + Da + D^2 = Ra^2 + Da$. Thus $a = ra^2 + d$, for some $r \in R$ and $d \in D$. Hence $(1 - ra)a \in D$ and so, since $1 - ra$ is a unit because $a \in \mathfrak{m}$, we obtain $a \in D$. This contradiction shows that $\mathfrak{m}/D$ is not finitely generated. $\square$

We now present [DT, Lemma 6.6].

Lemma 4.24. *Suppose* $\mathrm{w.gl.dim}(R) = n < \infty$ *and* R *satisfies conditions* $(\mathbf{A})$ *and* $(\mathbf{B})$. *Then there is an element* $\bar{b} \in \mathfrak{m} \setminus D$ *such that*

for all $b \in \mathfrak{m}$ *for which* $\bar{b} \in Rb + D$, *there is an element* $w_b \in \mathrm{Tor}_{n-3}(D, R/D)$ *with* $D \subseteq \mathrm{ann}(w_b)$, $bw_b \neq 0$ *but* $b^q w_b = 0$ *for some* $q \in \mathbb{N}$.

Proof. If $n = 3$, then $\mathrm{Tor}_{n-3}(D, R/D) = \mathrm{Tor}_0(D, R/D) = D \otimes R/D = D/D^2 = D$ since $D^2 = 0$. Then $(\mathbf{B})$ gives a nonzero element $w \in D$ such that, for any $a \in \mathfrak{m}$, $a^q w = 0$ for some $q \geq 1$. Taking $I = D$ in Lemma 4.23, we see that there exists $\bar{b} \in \mathfrak{m}$ and $\bar{w} \in D$

such that $w = \bar{b}\bar{w}$. Take any $b \in \mathfrak{m}$ such that $\bar{b} \in Rb + D$, say $\bar{b} = rb + d$, where $r \in R$, $d \in D$. Let $w_b = r\bar{w} \in D = \mathrm{Tor}_{n-3}(D, R/D)$. Then $D \subseteq \mathrm{ann}(w_b)$ since $D^2 = 0$ and $bw_b = br\bar{w} = br\bar{w} + 0 = br\bar{w} + d\bar{w} = (br + d)\bar{w} = \bar{b}\bar{w} = w \neq 0$. Moreover, by (**B**), there exists $q \in \mathbb{N}$ such that $b^q w = 0$ and so $b^{q+1}w_b = b^q b w_b = b^q w = 0$. This shows that the result holds for $n = 3$.

Now suppose $n > 3$. Consider a free resolution of R/D

$$\cdots \longrightarrow R^{(X_{n-3})} \xrightarrow{\delta_{n-3}} R^{(X_{n-4})} \xrightarrow{\delta_{n-4}} \cdots \xrightarrow{\delta_3} R^{(X_2)} \xrightarrow{\delta_2} R^{(X_1)} \xrightarrow{\delta_1} R^{(X_0)} \xrightarrow{\delta_0} R/D \longrightarrow 0$$

where the X_i are index sets. Let K denote $\ker(\delta_{n-4})$ and $\sigma : K \longrightarrow R^{X_{(n-4)}}$ be the inclusion map. Next let $\bar{\sigma} : K \otimes D \longrightarrow D^{(X_{n-4})}$ $(= R^{(X_{n-4})} \otimes D)$ be the homomorphism induced by σ on tensoring the resolution by D. Then $\mathrm{Tor}_{n-3}(D, R/D) = \ker \bar{\sigma}$.

Then, by (**B**), there is a nonzero element $w \in \ker(\bar{\sigma})$ such that, for any $a \in \mathfrak{m}$, $a^q w = 0$ for some $q \geq 1$. Let $w = x_1 \otimes d_1 + x_2 \otimes d_2 + \cdots + x_s \otimes d_s$, where $x_i \in K$, $d_i \in D \setminus 0$ for each i. Taking $I = D$ in Lemma 4.23 (ii), we get $D = D\mathfrak{m}$ and so, for each $i = 1, \ldots, s$, there exist $a_i \in \mathfrak{m} \setminus D$, $d_i^* \in D$ such that $d_i = a_i d_i^*$. (Note that $a_i \notin D$ since otherwise $d_i = 0$.) Since R/D is a chain ring, the finitely generated ideal $\dfrac{Ra_1 + \cdots + Ra_s + D}{D}$ of R/D is principal and so there exists $a \in Ra_1 + \cdots + Ra_s$ such that $a_i \in Ra + D$ for every $i = 1, \ldots, s$, say $a_i = r_i a + \bar{d}_i$ for some $r_i \in R$, $\bar{d}_i \in D$. Then $d_i = a_i d_i^* = r_i a d_i^* + \bar{d}_i d_i^* = r_i a d_i^* = a d_i'$, where $d_i' = r_i d_i^* \in D$. Now let $w' = x_1 \otimes d_1' + \cdots + x_s \otimes d_s'$. Clearly $aw' = w$ and so $a\bar{\sigma}(w') = \bar{\sigma}(w) = 0$.

Next note that, since $\bar{\sigma}(w')$ is an element of $D^{(X_{n-4})}$, we may identify it with $(\lambda_1, \ldots, \lambda_t)$ where each λ_j is a nonzero element of D and $\{1, \ldots, t\} \subseteq X_{n-4}$ represents the (finite) support of $\bar{\sigma}(w')$ in $D^{(X_{n-4})}$. Since $a\bar{\sigma}(w') = 0$, we have $a\lambda_j = 0$ for each j. Now, taking $I = \mathrm{ann}(\lambda_j)$ in Lemma 4.23 (ii), we see that for each j there exists $c_j \in \mathrm{ann}(\lambda_j)$ such that $a \in c_j\mathfrak{m}$. Because R/D is a chain ring, the ideals $\dfrac{Rc_j + D}{D}$ form a chain in R/D and so for some $c \in \{c_1, \ldots, c_t\}$ we have $Rc + D \subseteq Rc_j + D$ for every j. Then for each j, $c = r_j c_j + d_j$ for some $r_j \in R$ and $d_j \in D$. Thus $c\lambda_j = r_j c_j \lambda_j + d_j \lambda_j = 0$ since $c_j \in \operatorname{ann}(\lambda_j)$ and $d_j \lambda_j \in D^2 = 0$. Now, since $a \in c\mathfrak{m}$, we have $a = c\bar{b}$ for some $\bar{b} \in \mathfrak{m} \setminus D$ (note that $\bar{b} \notin D$ since otherwise we get $a \in D$, a contradiction). Setting $\bar{w} = cw'$, we get $\bar{\sigma}(\bar{w}) = c\bar{\sigma}(w') = c(\lambda_1, \ldots, \lambda_t)$ since $c_j \in \operatorname{ann}(\lambda_j)$ for all j and so $\bar{w} \in \ker(\bar{\sigma})$. Also $\bar{b}\bar{w} = \bar{b}cw' = aw' = w \neq 0$ while, for any $d \in D$, we have since $d\bar{w} = dcw' = cd\sum_{i=1}^{s}(x_i \otimes d_i) = d\sum_{i=1}^{s}(cx_i \otimes dd_i) = 0$ since $dd_i \in D^2 = 0$.

Now choose $b \in \mathfrak{m}$ such that $\bar{b} \in Rb + D$. This gives $\bar{b} = rb + d$ for some $r \in R$, $d \in D$. Set $w_b = r\bar{w}$. Then, for any $d* \in D$, $d * w_b = rd * w = 0$ so $D \subseteq \operatorname{ann}(w_b)$. Moreover $bw_b = br\bar{w} = br\bar{w} + 0 = br\bar{w} + d\bar{w} = (br + d)\bar{w} = \bar{b}\bar{w} \neq 0$. By condition (**B**), for each $a \in \mathfrak{m}$, there exists $q \geq 1$ such that $b^q w = 0$. Then $b^{q+1}w_b = b^{q+1}r\bar{w} = b^{q+1}rcw' = b^q rbcw' = b^q(\bar{b} - d)cw' = b^q\bar{b}cw' - b^q dcw' = b^q aw' - b^q dcw' = b^q w - b^q c\sum_{i=1}^{s}(x_i \otimes dd_i) = 0 - b^q c\sum_{i=1}^{s}(x_i \otimes 0) = 0$, as desired. $\square$

Next we give [DT, Lemma 6.7] which gives further conditions a counterexample to the Bazzoni–Glaz Conjecture must have.

Lemma 4.25. *Let* $\text{w.gl.dim}(R) = n < \infty$. *If* R *satisfies conditions* (**A**) *and* (**B**) *then there is a* $P \in \operatorname{Spec}(R)$ *satisfying the following properties:*

(i) $D_P \neq PR_P$ *and each element of* PR_P *is a zero-divisor. Also, there is an element* $d \in D$ *such that* $d/1 \in D_P$ *is nonzero and, if* $c \in PR_P$, $(d/1)c^q = 0$ *for some* $q \in \mathbb{N}$.

(ii) *if* $c \in PR_P \setminus D_P$, *then* $\operatorname{Tor}_{n-1}^{R_P}(R_P/(R_P c + D_P), R_P/D_P) \neq 0$.

(iii) $\text{w.gl.dim}\, R_P = n$.

Proof. (i) Choose $\bar{b} \in \mathfrak{m} \setminus D$ as given by Lemma 4.24. Next choose $d \in \operatorname{ann}(\ \bar{b}\ )$, $d \neq 0$, and set $P = \{a \in \mathfrak{m} : a^q d = 0 \text{ for some } q \in \mathbb{N}\}$. Then $P \in \operatorname{Spec}(R)$ by Lemma 3.29 and $\bar{b} \in P$ since $\bar{b}d = 0$. We next show that $\bar{b}/1 \notin D_P$. To see this, suppose to the contrary that $(\bar{b}/1)^2 = 0/1$. Then there is an $s \in R \setminus P$ such that $s\bar{b}^2 = 0$. Recall that in this section we are assuming that $\mathrm{Nil}(R) = D$ and so D is a prime ideal. This gives either $s \in D$ or $\overline{b} \in D$. Since $D \subseteq P$, we can't have $s \in D$. On the other hand, by its definition, $\overline{b} \notin D$. Thus this contradiction gives $\overline{b}/1 \notin D_P$ and so $D_P \neq PR_P$. Note also that, by the definition of P, given any $c \in P$, there is a positive integer q such that $c^q d = 0$ and so $c^q(d/1) = 0$.

(ii) By Lemma 4.21, applied to R_P instead of R, it suffices to show that every $c \in PR_P$ is a zero-divisor on $\mathrm{Tor}_{n-2}^{R_P}(R_P/D_P, R_P/D_P)$. To this end, consider the following s.e.s. of R_P-modules with i as the inclusion map and π as the natural epimorphism

$$0 \longrightarrow D_P \xrightarrow{i} R_P \xrightarrow{\pi} R_P/D_P \longrightarrow 0$$

Tensoring this sequence with R_P/D_P, we can then form the associated long Tor exact sequence, a part of which is given by

$$\begin{tikzcd}
0 = \mathrm{Tor}_{n-2}^{R_P}(R_P, R_P/D_P) \arrow[r] & \mathrm{Tor}_{n-2}^{R_P}(R_P/D_P, R_P/D_P) \arrow[d] & \\
& \mathrm{Tor}_{n-3}^{R_P}(D_P, R_P/D_P) \arrow[r] & \mathrm{Tor}_{n-3}^{R_P}(R_P, R_P/D_P) = 0.
\end{tikzcd}$$

From this we see that $\mathrm{Tor}_{n-2}^{R_P}(R_P/D_P, R_P/D_P) \simeq \mathrm{Tor}_{n-3}^{R_P}(D_P, R_P/D_P)$ and so it suffices to show that each c is a zero-divisor on $\mathrm{Tor}_{n-3}^{R_P}(D_P, R_P/D_P)$. Moreover, it is clear that we need only consider elements c of the form $b/1$ where $b \in P \setminus D$. Since R/D is a chain ring, given such a b and with $\overline{b}$ as above, the ideals $Rb + D$ and $R\overline{b} + D$ are comparable. This gives two cases to consider: either $\overline{b} \in Rb + D$ or $b \in R\overline{b} + D$.

Case 1: $\overline{b} \in Rb + D$. In this case, Lemma 4.24 tells us there is an element w_b in $\mathrm{Tor}_{n-3}(D, R/D)$ such that $D \subseteq \mathrm{ann}(w_b)$, $bw_b \neq 0$ but $b^q w_b = 0$ for some $q \in \mathbb{N}$. We now show that $w_b/1$ is a nonzero element of $(\mathrm{Tor}_{n-3}(D, R/D))_P \simeq \mathrm{Tor}_{n-3}^{R_P}(D_P, R_P/D_P)$. If not, then there is an $a \in R \setminus P$ such that $aw_b = 0$. Since $b \in P$ and R/D is a chain ring, we must have $Rb + D \subseteq Ra + D$ and this gives $b = ra + e$ say, where $r \in R$ and $e \in D$. Then

$bw_b = raw_b + ew_b = 0$, noting that $D \subseteq \mathrm{ann}(w_b)$. This is in contradiction to the defining properties of w_b and so $w_b/1$ is non-trivial in $\mathrm{Tor}_{n-3}^{R_P}(D_P, R_P/D_P)$ and, since $b^q(w_b/1) = 0$, b is a zero-divisor on $\mathrm{Tor}_{n-3}^{R_P}(D_P, R_P/D_P)$, as required.

Case 2: $b \in R\bar{b} + D$. Here we have $b = r\bar{b} + f$ where $r \in R$ and $f \in D$. As in case 1, with the roles of b and $\bar{b}$ reversed, there is a nonzero element $\overline{w} \in \mathrm{Tor}_{n-3}^{R_P}(D_P, R_P/D_P)$ such that $\bar{b}\overline{w} = 0$. Then, since $b\overline{w} = r\bar{b}\overline{w} + f\overline{w} = 0 + 0$, it follows that b is a zero-divisor on $\mathrm{Tor}_{n-3}^{R_P}(D_P, R_P/D_P)$, as required.

(iii). Since $\mathrm{w.gl.dim}(R) = \sup\{\mathrm{w.gl.dim}(R_P) : P \in \mathrm{Spec}(R)\}$ for any commutative ring R, it suffices to show that $\mathrm{w.gl.dim}(R_P) \geq n$. By (ii) we have $\mathrm{w.dim}_{R_P}(R_P/D_P) \geq n-1$ and so, by Lemma 4.19, $\mathrm{w.gl.dim}(R_P) \geq n$, as claimed. □

It follows from the properties of the ring $R_{\mathfrak{P}}$ of Lemma 4.25 that if the Bazzoni-Glaz conjecture is false then there exists a local Gaussian ring $(R, \mathfrak{m})$ such that $\mathrm{w.gl.dim}(R) = n < \infty$, all the elements of $\mathfrak{m}$ are zero-divisors, and the following two properties are satisfied:

(**A**)(As earlier,) $\mathrm{Nil}(R) = D \neq \mathfrak{m}$ and $\mathrm{Tor}_{n-1}(R/(Ra + D), R/D) \neq 0$ for every $a \in \mathfrak{m} \setminus D$.

(**C**) There is a nonzero element $d \in D$ such that, for any $c \in \mathfrak{m}$, we have $c^q d = 0$ for some $q \in \mathbb{N}$.

Lemma 4.26. *Let* $\mathrm{w.gl.dim}(R) = n < \infty$ *and suppose that* R *has properties* (**A**) *and* (**C**). *Then, for some* $a \in \mathfrak{m} \setminus D$, *we have* $\mathrm{Tor}_{n-1}(R/(a\mathfrak{m} + D), R/D) = 0$.

Proof. Choose an element $d \in D$ as given in property (**C**). By Lemma 4.23 (ii), taking $I = D$, there are elements $d^* \in D$ and $c \in \mathfrak{m}$ such that $d = cd^*$. Note that $c \in \mathfrak{m} \setminus D$ since $d \neq 0$. Using Lemma 4.23 (ii) again, this time taking $I = \mathfrak{m}$, there are elements $a, b \in \mathfrak{m}$ such that $c = ab$. Note that since $c \in \mathfrak{m} \setminus D$, we must have $a, b \in \mathfrak{m} \setminus D$ also.

Now let $\mathfrak{X}$ denote the set of all ideals I_α of R such that $Ra + Rb + D \subseteq I_\alpha \subseteq \mathfrak{m}$ and I_α/D is finitely generated. Then $(R/(aI_\alpha + \mathfrak{m}); \phi_{\alpha,\beta})\}_{I\in\mathfrak{X}}$ is a direct system with $aI_\alpha + \mathfrak{m} < aI_\beta + \mathfrak{m}$ if $I_\alpha \subseteq I_\beta$ and, in this case $\phi_{\alpha,\beta} : R/(aI_\alpha + \mathfrak{m}) \to R/(aI_\beta + \mathfrak{m})$ is given by $\phi_{\alpha,\beta}(r + (aI_\alpha + \mathfrak{m})) = r + (aI_\beta + \mathfrak{m})$ for all $r \in R$. Then, taking $J_\alpha = aI_\alpha$ for each α and $A = D$ in Lemma 2.27, we get $\varinjlim_{I_\alpha\in\mathfrak{X}} R/(aI + D) = R/(a\mathfrak{m} + D)$. Then, by Lemma 2.46 (ii), we get that $\mathrm{Tor}_{n-1}(R/(a\mathfrak{m}+D), R/D) = \varinjlim_{I_\alpha\in\mathfrak{X}} \mathrm{Tor}_{n-1}(R/(aI+D), R/D)$. Thus, to show that $\mathrm{Tor}_{n-1}(R/(a\mathfrak{m}+D), R/D) = 0$, by the uniqueness requirement of direct limits (see Definition 2.22) it suffices to show that, given any $I_\alpha \in \mathfrak{X}$, the induced direct limit map

$$\mathrm{Tor}_{n-1}(R/(aI_\alpha + D), R/D) \to \varinjlim_{I_\alpha\in\mathfrak{X}} \mathrm{Tor}_{n-1}(R/(aI + D), R/D)$$

is trivial. To simplify notation, we replace the index α by 1.

Since I_1 is finitely generated modulo D and R/D is a chain ring, there is an element $b_1 \in \mathfrak{m} \setminus D$ for which $I_1 = Rb_1 + D$. By Lemma 4.23, $R/\mathfrak{m}$ is not finitely generated and so there is an ideal $I_2 \in \mathfrak{X}$ with $I_1 \subsetneq I_2$. As with I_1, there is an element $b_2 \in \mathfrak{m} \setminus D$ for which $I_2 = Rb_2 + D$. Since $I_1 \subsetneq I_2$, there is an $r \in R$ such that $b_1 = rb_2 + d_2$, for some $r \in R, d_2 \in D$. Note that $r \in \mathfrak{m}$ since otherwise $b_2 \in Rb_1 + D$, giving $I_1 = I_2$, a contradiction. Furthermore, $r \notin D$ since otherwise $b_1 = rb_2 + d_2 \in D$, again a contradiction. Also, by Theorem 3.18 we have $D \subseteq Rb_2 + \mathrm{ann}(b_2)$, so $d_2 = sb_2 + d_1$ for some $s \in R, d_1 \in \mathrm{ann}(b_2)$. It follows that $b_1 = rb_2 + d_2 = xb_2 + d_1$ where $x = r + s$. Since $b_2 \notin I_1$ we must have $x \in \mathfrak{m}$.

Next note that, from the definition of $\mathfrak{X}$, we have $a \in Rb_2 + D$. Also, if $t \in \mathrm{ann}(b_2)$ then, since $b_2 \in \mathfrak{m} \setminus D$, we have $t \in D$ by Lemma 3.21 and so $tD = 0$. From this it follows that $\mathrm{ann}(b_2) \subseteq \mathrm{ann}(a)$. Hence $ad_2 = 0$ and $ab_1 = axb_2 + ad_2 = axb_2$. In particular, this gives $\mathrm{ann}(ab_2) \subseteq \mathrm{ann}(ab_1)$. Now consider the following commutative diagram where the rows are exact, i is the inclusion map, and μ_{ab_2}, μ_x and μ_{ab_1} are the multiplications by

ab_2, x and ab_1 respectively.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{ann}(ab_2) \arrow[r] \arrow[d, "i"'] & R \arrow[r, "\mu_{ab_2}"] \arrow[d, "1_R"'] & Rab_2 \arrow[r] \arrow[d, "\mu_x"] & 0 \\
0 \arrow[r] & \operatorname{ann}(ab_1) \arrow[r] & R \arrow[r, "\mu_{ab_1}"'] & Rab_1 \arrow[r] & 0
\end{tikzcd}$$

We claim that the inclusion $i : \operatorname{ann}(ab_2) \to \operatorname{ann}(ab_1)$ is not onto. If otherwise, the first two downward maps in our diagram would be isomorphisms. Then two simple diagram chases show that μ_x is both one-to-one and onto, i.e., also an isomorphism. Now, by the definition of $\mathfrak{X}$, we have $b \in I_2$ and so $b = qb_2 + d_3$ for some $q \in R$ and $d_3 \in D$. Then, since $D^2 = 0$,

$$d = cd^* = abd^* = a(qb_2 + d_3)d^* = qd^*ab_2 \in Rab_2,$$

and so $x^i d \in Rab_2$ for all $i \geq 0$. By property (**C**) there is a positive integer k such that $x^k d = 0$ but $x^{k-1}d \neq 0$. This gives $\mu_x(x^{k-1}d) = 0$ and shows that μ_x is not an isomorphism, justifying the claim.

Now note that, since D is prime and $a, b_1 \in \mathfrak{m} \setminus D$, we also have $ab_1 \in \mathfrak{m} \setminus D$ and so, by Lemma 3.21, $\operatorname{ann}(ab_1) \subseteq D$. The claim proved above then establishes an element $\lambda \in D$ such that $\lambda \in \operatorname{ann}(ab_1)$ but $\lambda \notin \operatorname{ann}(ab_2)$. From this we get $Rab_1 + D \subseteq \operatorname{ann}(\lambda) \subsetneq Rab_2 + D$, which in turn gives $aI_1 + D \subseteq \operatorname{ann}(\lambda) \subsetneq aI_2 + D$. Then we have natural epimorphisms

$$R/(aI_1 + D) \to R/\operatorname{ann}(\lambda) \to R/(aI_2 + D). \quad (\dagger)$$

Note that $R/\operatorname{ann}(\lambda) \simeq R\lambda \subsetneq R$ and, by Lemma 4.19, $\operatorname{w.dim}(R/D) = n - 1$. Then, by Lemma 2.57, we have a monomorphism $\operatorname{Tor}_{n-1}(R\lambda, R/D) \to \operatorname{Tor}_{n-1}(R, R/D) = 0$ and so $\operatorname{Tor}_{n-1}(R/\operatorname{ann}(\lambda), R/D) = 0$. From this, the natural map $\operatorname{Tor}_{n-1}(R/(aI_1 + D), R/D) \to \operatorname{Tor}_{n-1}(R/(aI_2 + D), R/D)$ is trivial since, from ($\dagger$), it is the composition of the two trivial maps $\operatorname{Tor}_{n-1}(R/(aI_1+D), R/D) \to \operatorname{Tor}_{n-1}(R/(\operatorname{ann}(\lambda), R/D)$ and $\operatorname{Tor}_{n-1}(R/\operatorname{ann}(\lambda), R/D) \to \operatorname{Tor}_{n-1}(R/(aI_2+D), R/D)$. Consequently, the natural map $\operatorname{Tor}_{n-1}(R/(aI_1+D), R/D) \to \varinjlim_{I\in\mathfrak{X}} \operatorname{Tor}_{n-1}(R/(aI+D), R/D)$ is also trivial, completing the proof. $\square$

We now complete the proof of the Bazzoni-Glaz conjecture.

Theorem 4.27. *Let* $(R, \mathfrak{m})$ *be a non-reduced local Gaussian ring. If* $\operatorname{Nil}(R)^2 = 0$ *then* $\operatorname{w.gl.dim}(R) = \infty$.

Proof. By way of contradiction, assume that $\operatorname{w.gl.dim}(R) = n < \infty$. Then, by Proposition 4.9, $n \geq 3$. Moreover, by Lemma 4.22, we may assume that R satisfies properties (**A**) and (**C**). By Lemma 4.26, there is an element $a \in \mathfrak{m}\setminus D$ such that $\operatorname{Tor}_{n-1}(R/(a\mathfrak{m}+D), R/D) = 0$. We have the following s.e.s.

$$0 \longrightarrow (aR+D)/(a\mathfrak{m}+D) \xrightarrow{f} R/(a\mathfrak{m}+D) \xrightarrow{g} R/(aR+D) \longrightarrow 0$$

where the monomorphism f is given by $ar + d + (a\mathfrak{m} + D) \mapsto r + (a\mathfrak{m} + D)$ and the epimorphism g by $r + (a\mathfrak{m}+D) \mapsto r + (aR+D)$ for all $r \in R, d \in D$. Tensoring with R/D we get a long Tor exact sequence of which the following is a segment:

$$\begin{tikzcd} \operatorname{Tor}_{n-1}(R/(a\mathfrak{m}+D), R/D) \arrow[r] & \operatorname{Tor}_{n-1}(R/(aR+D), R/D) \arrow[d] \\ & \operatorname{Tor}_{n-2}((aR+D)/(a\mathfrak{m}+D), R/D). \end{tikzcd}$$

Now define $\phi : (aR+D)/(a\mathfrak{m}+D) \longrightarrow R/\mathfrak{m}$ by $\phi(ar+(a\mathfrak{m}+D)) = r+\mathfrak{m}$ for all $r \in R$. To see that ϕ is well-defined, suppose that $r, s \in R$ with $ar+(a\mathfrak{m}+D) = as+(a\mathfrak{m}+D)$. Then there is an $m \in \mathfrak{m}$ such that $a(r-s-m) = ar-as-am \in D$. Since $D = \operatorname{Nil}(R)$ is a prime ideal, this gives either $r-s-m \in D$ or $a \in D$. The latter contradicts the definition of a and so $r-s-m \in D$. Then $r-s \in \mathfrak{m}$ and so $\phi(ar+(a\mathfrak{m}+D)) = r+\mathfrak{m} = s+\mathfrak{m} = \phi(as+(a\mathfrak{m}+D))$, as required. From here it is straightforward to show that ϕ is an isomorphism.

Next, by Lemma 4.23 (i), $\mathfrak{m}$ is flat and so $\mathrm{w.dim}(R/\mathfrak{m}) \leq 1$. Then, using Corollary 4.4 and recalling that $n \geq 3$, the isomorphism above gives $\mathrm{Tor}_{n-2}((aR+D)/(a\mathfrak{m}+D), R/D) = 0$. Since, as noted earlier, we also have $\mathrm{Tor}_{n-1}(R/(a\mathfrak{m}+D), R/D) = 0$, the exact segment gives $\mathrm{Tor}_{n-1}(R/(aR+D), R/D) = 0$. This contradicts property (**A**) so we are finished. $\square$

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