Grünwald-type approximations and boundary conditions for one-sided fractional derivative operators

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Abstract

The focus of this thesis is two-fold. The first part investigates higher order numerical schemes for one-dimensional fractional-in-space partial differential equations in $L_1(\mathbb{R})$. The approximations for the (space) fractional derivative operators are constructed using a shifted Grünwald-Letnikov fractional difference formula. Rigorous error and stability analysis of the *Grünwald-type* numerical schemes for space-time discretisations of the associated Cauchy problem are carried out using (Fourier) multiplier theory and semigroup theory. The use of a transference principle facilitates the generalisation of the results from the L_1-setting to any function space where the translation (semi) group is strongly continuous. Furthermore, the results extend to the case when the fractional derivative operator is replaced by the fractional power of a (semi) group generator on an arbitrary Banach space. The second part is dedicated to the study of certain fractional-in-space partial differential equations associated with (truncated) Riemann-Liouville and first degree Caputo fractional derivative operators on $\Omega := [(0,1)]$. The boundary conditions encoded in the domains of the fractional derivative operators dictate the inclusion or exclusion of the end points of Ω. Elaborate technical constructions and detailed error analysis are carried out to show convergence of *Grünwald-type* approximations to fractional derivative operators on $X = C_0(\Omega)$ and $L_1[0,1]$. The well-posedness of the associated Cauchy problem on X is established using the approximation theory of semigroups. The culmination of the thesis is the result which shows convergence in the Skorohod topology of the well understood stochastic processes associated with Grünwald-type approximations to the processes governed by the corresponding fractional-in-space partial differential equations.

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Introduction

The fundamentals of fractional calculus and their applications have been treated by several authors, see for example [43, 46, 51, 82, 84, 86, 90, 95] and the references therein. Even though fractional derivatives have existed as long as their integer order counterparts only in recent decades have fractional derivative models become exciting new tools in the study of practical problems in displines as diverse as physics [13, 19, 20, 26, 71, 72, 73, 80, 81, 96, 110], finance [69, 89, 93], biology [5, 6] and hydrology [3, 15, 16, 17, 97, 98, 99]. This observation, that fractional derivative models are becoming increasingly popular among the wider scientific community, is the main motivation to study numerical schemes for fractional partial differential equations.

From a purely mathematical perspective, fractional partial differential equations can be thought of as generalizations of the corresponding classical partial differential equations. On occasions however, fractional partial differential equations do arise naturally as better theoretical models for practical problems in diverse scientific disciplines. For instance, in geophysical sciences [99], the authors employ the Lévy-Gnedenko generalised central limit theorem [41] and fractional conservation of mass arguments [98, 107] to derive fractional-in-space as well as fractional-in-time advection-dispersion equations. According to the authors, these fractional advection-dispersion equations provide better models for the motion of an ensemble of particles on Earth's surface as measured by the concentration (or mass) in space and time.

In general, particle transport phenomena may involve random states of motion as well as rest. Therefore, *jump length* and *waiting time* between motion of a particle can be viewed as random variables [15, 98, 99]. It is well known that if the probability density function that describes the jump length decays at least as fast as an exponential distribution, then jump length distribution has finite mean and variance. Further, assuming that the waiting time distribution has finite mean, the concentration (or mass) in this case may be adequately described by the classical advection-dispersion equation (Fokker Planck equation). The associated stochastic process, the so-called

Brownian motion with drift, is governed by the classical advection-dispersion equation whose fundamental solution is the Gaussian density [41]. The (normal) scaling of dispersion (Fickian or Boltzmann scaling), described by the standard deviation, is proportional to $t^{\frac{1}{2}}$.

The term anomalous diffusion is given to diffusion phenomena that cannot be adequately described by the classical advection-dispersion equations. One such scenario is when jump length follows an infinite-variance distribution. In addition to infinite-variance jump length distribution, assuming a finite-mean waiting time distribution, the authors in [99] show that the associated stochastic process (Lévy motion) is Markovian and is governed by fractional-in-space advection-dispersion equation whose fundamental solution is a Lévy α-stable density. The diffusion phenomena is referred to as *super-diffusion* because of the faster than normal $t^{\frac{1}{2}}$ scaling, as the scaling of dispersion in this case is proportional to $t^{\frac{1}{\alpha}}$ where $1 < \alpha < 2$ is the order of the fractional-in-space derivative used in the model. Fractional-in-space advection-dispersion equations arise as natural models when the velocity variations are heavy tailed. On the other hand, assuming infinite-mean waiting time distribution, the associated stochastic process (Lévy motion subordinated to an inverse Lévy process) is shown to be non-Markovian and is governed by a fractional-in-time advection-dispersion equation whose fundamental solution is a subordinated Lévy α-stable density. The diffusion phenomena inherits the name *sub-diffusion* in this case since the scaling of dispersion is proportional to $t^{\frac{\gamma}{2}}$ where $0 < \gamma < 1$ is the order of the fractional-in-time derivative used in the model. A comprehensive review of random walk and other theoretical models for anomalous sub-diffusion and super-diffusion as well as evidence of the occurence of anomalous dynamics in various fields such as biology, geophysics, physics and finance can be found in [80, 81].

The connection of fractional calculus with probability theory that we have briefly outlined above is interesting in its own right. Evidently, this provides an insight into the stochastic processes governed by fractional partial differential equations. More importantly, this link also provides new tools from probability theory that can be used in the search for numerical solutions for fractional partial differential equations. For instance, Feller investigated the semigroups generated by a certain pseudo differential operator and identified the underlying stochastic processes [40]. As it turns out, these processes are governed by a certain diffusion equation obtained by replacing the second-order space derivative by the pseudo differential operator in the classical diffusion equation [44, 45, 94]. The fundamental solutions of this diffusion equation generate all the Lévy stable densities with index $\alpha \in (0, 2]$. In [44, 45], this diffusion equation is revisited using a random walk model that employs the Grünwald-Letnikov difference scheme where the authors refer to the equation as Lévy-Feller diffusion equation and the processes governed by them as Lévy-Feller processes. Theoretical and numerical methods as well as connections with stochastic processes for various types of fractional partial differential equations have been investigated by several authors, see for example [3, 7, 9, 28, 70, 71, 72, 74, 79, 97] and the references therein. In [76], the authors provide an in-depth treatment on the connection of fractional calculus to stochastic processes from a probabilistic perspective.

Inevitably, two crucial issues have to be addressed in the study of fractional partial differential equations. Firstly, the existence and uniqueness of solutions; that is, whether the associated Cauchy problem is well-posed in the function space framework chosen for study. Secondly, the consistency and stability of numerical schemes used to solve the fractional partial differential equations. The latter is particularly important since the inclusion of an external forcing function and/or the imposition of boundary conditions, especially in practical applications, could make the task of finding analytical solutions elusive [109].

The well-posedness of the Cauchy problem associated with certain fractional-in-space partial differential equations on bounded domains in the L_2-setting has been investigated by many authors. By constructing appropriate function spaces and demonstrating equivalence to fractional Sobolev spaces, variational solutions to the steady state fractional advection-dispersion equations on bounded domains $\Omega \subset \mathbb{R}^n$ were investigated in [38, 39]. These authors employ the Lax-Milgram Lemma to show the existence and uniqueness of solutions in $L_2(\Omega)$. The authors in [34] study a very general class of non-local diffusion problems on bounded domains of $\mathbb{R}^d$ using non-local vector calculus and non-local, non-linear conservation laws developed in [33, 35]. According to the authors, certain fractional derivative models for anomalous diffusion are special cases of their non-local diffusion model. In particular, the authors claim that the fractional Laplacian and the symmetric version (with $\alpha = 2s$) of the more general (asymmetric) fractional derivative operator of [70] are special cases of their non-local operator. Moreover, they show the well-posedness of steady-state volume-constrained diffusion problems in $L_2(\Omega)$, $\Omega \subset \mathbb{R}^n$. To our knowledge, the issue of the well-posedness of the Cauchy problem associated with fractional-in-space partial differential equations on bounded domains in function spaces other than $L_2(\Omega)$ has not been completely resolved.

Several works have addressed the need for numerical methods to solve different types of fractional partial differential equations. For instance, higher order linear multi-step methods to solve Abel-Volterra integral equations, of which fractional differential equations form a sub-class [31], were made popular by [63] in the 1980s using convolution quadratures and fast fourier transforms [49]. Since then, linear multi-step methods have been used by many authors, see for example [61, 64, 65, 66], to numerically solve fractional integral equations and fractional partial differential equations. A review and some applications of these methods can be found in [67]. Algorithms as well as the difficulties encountered while implementing such numerical schemes were discussed in [31, 32]. Matrix methods for approximating fractional integrals and derivatives have been investigated in [87, 88]. In [109], a fractional weighted average finite difference method along with von Neumann stability analysis of the numerical schemes was carried out. An implicit numerical scheme for (time) fractional diffusion equation based on finite difference approximations was developed in [59].

In [62], the authors investigate computationally efficient numerical methods for fractional-in-space diffusion equation with insulated ends obtained by replacing the second order space derivative in the classical diffusion equation by a Caputo fractional derivative of order $1 < \alpha < 2$. They use an explicit finite difference method and the method of lines to obtain numerical solutions. Stability and convergence of the explicit finite difference numerical method along with its scaling restriction were discussed. A similar method was also used in [100] combined with Grünwald-Letnikov difference scheme for space discretisation to solve a fractional Fokker-Planck equation. In [75, 77, 78, 101, 102], the authors develop a fractional Crank-Nicolson scheme using a shifted Grünwald formula to solve fractional-in-space partial differential equations. In very recent works [103, 112], a third order numerical method using a weighted and shifted Grünwald difference scheme for (space) fractional diffusion equations in one and two dimensions was developed. The authors carry out the analysis of numerical stability and convergence with respect to discrete L_2-norm.

Space fractional derivative operators are non-local. Thus, they can be used to characterise influences from a distance, for example super-diffusion phenomena [99]. In this thesis, we are particularly interested in fractional-in-space partial differential equations which can be used to model such non-local behaviour in space. Our numerical approximations for the (space) fractional derivative operators are also constructed using a shifted Grünwald formula [44, 45, 77, 112] and so throughout this thesis we refer to them as *Grünwald-type approximations*.

In our study of fractional-in-space partial differential equations, we make an attempt to address the following fundamental issues that we believe are lacking in the literature.

- Construction of (higher order) numerical approximations for fractional derivative operators in $L_1(\mathbb{R})$ and in other function spaces.
- Stability and smoothing properties of (higher order) numerical approximations that yield optimal convergence rate with minimal regularity of initial data for space-time discretisations of the abstract Cauchy problem associated with fractional derivative operators.
- Truncation of fractional derivative operators on a bounded interval $\Omega \subset \mathbb{R}$ for combinations of boundary conditions such as Dirichlet, Neumann etc. that yield well-defined operators with desirable properties.
- The question of well-posedness of the abstract Cauchy problems associated with fractional derivative operators whose domains encode various boundary conditions in function spaces other than $L_2(\Omega)$, in particular, $L_1(\Omega)$ and $C_0(\Omega)$.
- Construction of approximations for (truncated) fractional derivative operators whose associated stochastic processes can be easily identified and understood.
- Convergence of the stochastic processes associated with the approximation operators to the corresponding stochastic processes associated with the truncated fractional derivative operators.

Thesis outline:

In the first part of this thesis, in Chapters 1 and 2, we explore convergence with error estimates for higher order Grünwald-type approximations of semigroups generated first by a fractional derivative operator on $L_1(\mathbb{R})$ and then, using a transference principle, by fractional powers of group or semigroup generators on arbitrary Banach spaces. The main motivation for the investigation of higher order schemes are the works of Meerschaert, Scheffler and Tadjeran [75, 77, 78, 101, 102]. In these articles, the authors explored consistency and stability of numerical schemes for fractional-in-space partial differential equations using a Grünwald formula with non-negative integer shift to approximate the fractional derivative operator. In particular, in [102], they showed consistency if the order of the spatial derivative is less or equal to 2. They obtained specific error term expansion for $f \in C^{4+n}(\mathbb{R})$, where n is the number of error terms, as well as proved stability of their fractional Crank-Nicolson scheme, using Gershgorin's Theorem to determine the spectrum of the Grünwald matrix. Richardson extrapolation was then employed to obtain second order convergence in space.

This consistency result was extended for a Grünwald formula with any shift $p \in \mathbb{R}$ in [4, Proposition 4.9] where the authors showed that for all

$$f \in X_\alpha(\mathbb{R}) := \{f \in L_1(\mathbb{R}) : \exists\, g \in L_1(\mathbb{R}) \text{ with } \hat{g}(k) = (-ik)^\alpha \hat{f}(k), k \in \mathbb{R}\},$$

the first order Grünwald scheme

$$A^\alpha_{h,p} f(x) = \frac{1}{\Gamma(-\alpha)} \frac{1}{h^\alpha} \sum_{m=0}^{\infty} \frac{\Gamma(m-\alpha)}{\Gamma(m+1)} f(x-(m-p)h) \tag{1}$$

converges in $L_1(\mathbb{R})$ to the fractional derivative operator $f^{(\alpha)}$ as $h \to 0+$. Here $\hat{g}(k) = \int_{-\infty}^{\infty} e^{ikx} g(x)\, dx$ denotes the Fourier transform of $g \in L_1(\mathbb{R})$ and for $f \in X_\alpha(\mathbb{R})$, $f^{(\alpha)} = g$ iff $(-ik)^\alpha \hat{f}(k) = \hat{g}(k)$ for all $k \in \mathbb{R}$. In Section 1.5 we improve this result further and develop higher order Grünwald-type approximations $\tilde{A}^\alpha_h$. In Corollary 1.6.3 we give a consistency error estimate of the form

$$\left\| \tilde{A}^\alpha_h f - f^{(\alpha)} \right\|_{L_1(\mathbb{R})} \leq C h^n \|f^{(\alpha+n)}\|_{L_1(\mathbb{R})} \tag{2}$$

for an n-th order scheme.

Using a Carlson-type inequality for periodic multipliers developed in Section 1.3.2 (Theorem 1.3.4) we investigate the stability and smoothing of Grünwald-type approximation schemes $\tilde{A}^\alpha_h$. The main tool is Theorem 2.1.1 which gives a sufficient condition for multipliers associated with difference schemes approximating the fractional derivative operator to lead to stable schemes with desirable smoothing. In particular, we show in Proposition 2.1.2, that stability for a numerical scheme using (1) to solve the Cauchy problem associated with fractional derivative operator $f^{(\alpha)}$ where $2q-1 < \alpha < 2q+1, q \in \mathbb{N}$ can only be achieved for a unique shift p in the Grünwald formula. That is, it is necessary that $p = q$ for $(-1)^{q+1} A^\alpha_{h,p}$ to generate bounded semigroups on $L_1(\mathbb{R})$ where the bound is uniform in h. Furthermore, in Theorem 2.1.6, we prove stability and smoothing of a second order scheme.

Developing the theory in L_1 allows in Section 2.2 the transference of the theory to fractional powers A^α of the generator $-A$ of a strongly continuous (semi-)group G on a Banach space $(X, \|\cdot\|)$, noting that $f(x-(m-p)h)$ in (1) will read as $G((m-p)h)f$ [4]. The abstract Grünwald approximations with the optimal shifts generate analytic semigroups, uniformly in h, as shown in Theorem 2.2.1. This is the main property needed in Corollary 2.2.3 to show that the error between $S_\alpha(t)f = e^{t(-1)^{q+1}A^\alpha} f$ and a fully discrete approximation u_n obtained via a Runge-Kutta method with stage order s, order $r \geq s+1$, and an $N+1$ order Grünwald approximation is bounded by

$$\|S_\alpha(t)f - u_n\| \leq C\left(n^{-r}\|f\| + h^{N+1}\left|\log\frac{t}{h^\alpha}\right| \|A^{N+1}f\|\right),\ h > 0,\ t = n\tau.$$

In error estimates, the smoothing of the numerical scheme is used in an essential way to reduce the regularity requirements on the initial data. Further, this yields error estimates of the numerical approximation schemes applied to Cauchy problem associated with fractional derivative operators in spaces where the translation semigroup is strongly continuous, such as $L_p(\mathbb{R})$, $1 \leq p < \infty$, $BUC(\mathbb{R})$, $C_0(\mathbb{R})$, etc. Using the abstract setting we can also conclude that the consistency error estimate (2) holds in those spaces, with the L_1 norm replaced by the appropriate norm. Section 2.3 marks the conclusion of the first part of this thesis with results of some numerical experiments, including a third order scheme, that highlight the efficiency of the higher order schemes as well as the sharpness of the error estimates depending on the smoothness of the initial data. The results from the first part of this thesis have been accepted for publication in Transactions of the American Mathematical Society and available online [8].

Let us turn our attention to the second part of this thesis. The Fokker-Planck equation of a Lévy stable process on $\mathbb{R}$ is a fractional-in-space partial differential equation. The (space) fractional derivative operator is non-local with infinite reach. In the second part of this thesis, in Chapters 3 and 4, we investigate (truncated) Riemann-Liouville and first degree Caputo fractional derivative operators of order $1 < \alpha < 2$ on a bounded interval, $\Omega := [(0, 1)]$. The interval Ω may or may not contain its end point(s) depending on the boundary conditions encoded by the domain of the (truncated) fractional derivative operator under consideration. We show convergence in the Skorohod topology of easily identifiable finite state (sub)-Markov processes to a (sub)-Markov process governed by the Fokker-Planck equation on Ω associated with the (truncated) fractional derivative operators. Observe that the fractional derivative operators that we consider below on function spaces defined on the interval Ω are *one-sided*. The approach employed in [34] applies only to the symmetric fractional derivative operators as mentioned earlier and therefore do not extend to one-sided fractional derivative operators. However, the boundary conditions that we consider can be interpreted as special cases of the volume constraints employed in [34] and related works (one-dimensional mass constraints).

The stage is set in Section 3.1 where we discuss the general theoretical framework. Here we exploit the fact that the convergence, uniformly for $t \in [0, t_0]$, of Feller semigroups on $C_0(\Omega)$ implies convergence of the corresponding processes in the Skorohod topology. To do this, we turn a finite state (sub)-Markov processes into a Feller process by creating parallel copies of the finite state processes whose transition matrices interpolate continuously. The main idea behind the construction of these (continuous) interpolation matrices is the division of the interval $[0, 1]$ into $n + 1$ grids of equal length h so that the (Feller) process can jump between grids only in multiples of h. The transition operators on

$$X = C_0(\Omega) \text{ or } L_1[0, 1].$$

are constructed using these interpolation matrices. These transition operators are then employed in Chapter 4 to construct the Grünwald-type approximations for the fractional derivative operators on X. In Section 4.3, we show that the Grünwald (transition) approximation operators are the generators of the backward or forward semigroups associated with the extended finite state (sub)-Markov processes and thus identify the processes associated with the Grünwald approximation operators.

The central objects of study, the one-sided fractional derivative operators, are introduced in Section 3.3.2. The one-sided fractional derivative operators are denoted in general by $(A, \mathrm{BC}) : \mathcal{D}((A, \mathrm{BC})) \subset X \to X$ whose domains $\mathcal{D}((A, \mathrm{BC}))$ encode a particular combination of boundary conditions denoted by BC. The boundary conditions that we consider are Dirichlet, Neumann and Neumann*, where the latter appears naturally in the adjoint formulation of the fractional derivative operators in $L_1[0, 1]$ with a right Neumann boundary condition. We consider functions of the form

$$f = I^\alpha g + ap_\alpha + bp_{\alpha-1} + cp_{\alpha-2} + dp_0, \ g \in X \tag{3}$$

as candidates for the domain of the fractional derivative operators, where $a, b, c, d \in \mathbb{R}$ are determined by the boundary conditions and $p_\beta = \frac{x^\beta}{\Gamma(\beta+1)}$. The crucial point to note here is the structure of the domains of the fractional derivative operators. That is, the domains are defined as the range of the corresponding fractional integral operators I^α, supplemented by a linear combination of some particular power functions with constant weights that encode the regularity as well as the boundary conditions BC satisfied by the functions in the domain.

In Sections 3.4 and 4.1, well-posedness of the associated one-dimensional fractional-in-space partial differential equations is established using the approximation theory of semigroups [2, 37, 52, 85]. That is, we show that the fractional derivative operators (A, BC) generate strongly continuous contraction semigroups on X. To do this, in Section 3.4, we show that (A, BC) are densely defined, closed operators and that $\mathrm{rg}(\lambda I - A)$ are dense in X for some $\lambda > 0$. To make use of the Lumer-Phillips Theorem we further require that the operators (A, BC) are dissipative which is established using the convergence property of the Grünwald-type approximations, Proposition 4.3.2.

The Grünwald-type approximation operators G^h are constructed in Chapter 4 using the general theory for numerical schemes developed in Sections 3.1 and 3.2. For the numerical scheme, the boundary conditions BC are encoded into the generic $n \times n$ shifted Grünwald matrix where $h = \frac{1}{n+1}$, $n \in \mathbb{N}$ given by

$$G^h_{n\times n} = \frac{1}{h^\alpha}\begin{pmatrix} b^l_1 & \mathcal{G}^\alpha_0 & 0 & \cdots & 0 \\ \vdots & \mathcal{G}^\alpha_1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ b^l_{n-1} & \mathcal{G}^\alpha_{n-2} & \cdots & \mathcal{G}^\alpha_1 & \mathcal{G}^\alpha_0 \\ b_n & b^r_{n-1} & \cdots & \cdots & b^r_1 \end{pmatrix}, \tag{4}$$

using the boundary weights b^l_i, b^r_i and b_n. The $n \times n$ shifted Grünwald matrices $G^h_{n\times n}$ play the role of the transition rate matrices of the underlying finite state sub-Markov processes.

In Section 4.2, we first discuss the adjoint formulation of the abstract Cauchy problem on X associated with the fractional derivative operators. In doing so, we list the corresponding fractional derivative operators on X that are approximated by the Grünwald transition operators constructed using these boundary weights. Following that we conjecture the physical interpretation of the stochastic processes that would give rise to these different boundary conditions BC and discuss our reasons behind the choice of the boundary weights b^l_i, b^r_i, and b_n that appear in the generic Grünwald matrix (4.1) in the $L_1[0,1]$ case. In Section 4.4 we provide some examples of numerical solutions to the Cauchy problem associated with the fractional derivative operators (A, BC) on $L_1[0,1]$ and the initial value $u_0 \in L_1[0,1]$.

In Section 4.3, we prove the key result, Proposition 4.3.2, that the Grünwald transition operators converge to the respective fractional derivative operators on X. That is, for each $f \in \mathcal{D}(A, \mathrm{BC})$ we show that there exist sequences $f_h \in X$ such that $f_h \to f$ and $G^h f_h \to Af$ in X for each of the fractional derivative operators (A, BC). This as it turns out involves detailed error analysis employing elaborate constructions of appoximations for the power functions p_β that appear in (3) above. This result is essential firstly to show that the fractional derivative operators (A, BC) are dissipative. Using Proposition 4.3.2, we also conclude that the semigroups generated by the operators (A, BC) are the strong (and uniform for t in compact intervals) limit of the semigroups generated by the Grünwald transition operators using the Trotter-Kato Theorem. As a consequence, the underlying Feller processes associated with Grünwald approximations converge in the Skorohod topology to the Feller processes governed by the corresponding fractional-in-space partial differential equations. This identifies the processes governed by the fractional-in-space partial differential equations with boundary conditions BC as limits of processes whose boundary behaviour is perfectly understood.

Chapter 1

Grünwald-type approximations of fractional derivative operators on the real line

In this chapter we study Grünwald-type approximations for the fractional derivative operators on $\mathbb{R}$. We explore convergence and conduct a detailed error analysis using Fourier multiplier theory. Following that, combining Grünwald formulae with different shifts and step sizes, higher order Grünwald-type approximations are constructed for the fractional derivative operators on $\mathbb{R}$. We show convergence of the higher order approximations to the fractional derivative operators with optimal convergence rate under minimal regularity assumptions.

1.1 Fractional derivative operators on $\mathbb{R}$

The error analysis of higher order Grünwald-type numerical approximations of fractional derivative operators on $\mathbb{R}$ is carried out using multiplier theory. To facilitate this, we define the fractional derivatives of $L_1(\mathbb{R})$-functions in the Fourier or Laplace space depending on the support of the function under consideration. To define the fractional derivative operator of order $\alpha \in \mathbb{R}^+$ using Fourier transform if $f \in L_1(\mathbb{R})$ and Laplace transform if $f \in L_1(\mathbb{R}^+)$, let us begin with the following two spaces.

Definition 1.1.1. Let $\alpha \in \mathbb{R}^+$ and $z^\alpha := |z|^\alpha e^{i\alpha \arg z}$ be as in Remark B.3.4. Then, we define the following two spaces:

1.
$$X_\alpha(\mathbb{R}) := \{f \in L_1(\mathbb{R}) : \exists\, g \in L_1(\mathbb{R}) \text{ with } \hat{g}(k) = (-ik)^\alpha \hat{f}(k),\ k \in \mathbb{R}\},$$

where $\hat{f}(k)$ and $\hat{g}(k)$ denote the Fourier transforms of f and g, respectively, given by (B.3).

2.
$$X_\alpha(\mathbb{R}^+) := \{f \in L_1(\mathbb{R}^+) : \exists\, g \in L_1(\mathbb{R}^+) \text{ with } \hat{g}(z) = (-z)^\alpha \hat{f}(z), \ \ \operatorname{Re} z \leq 0\},$$

where $\hat{f}(z)$ and $\hat{g}(z)$ denote the Laplace transforms of f and g, respectively, given by (B.6).

Here is the formal definition of the fractional derivative operator on $\mathbb{R}$ that we use in this thesis.

Definition 1.1.2. For $f \in X_\alpha(\mathbb{R})$, if $g \in L_1(\mathbb{R})$ and

$$(-ik)^\alpha \hat{f}(k) = \hat{g}(k), \ k \in \mathbb{R},$$

then we define

$$f^{(\alpha)} := g.$$

Along similar lines for $f \in X_\alpha(\mathbb{R}^+)$, we define $f^{(\alpha)} := g$, if $g \in L_1(\mathbb{R}^+)$ and $(-z)^\alpha \hat{f}(z) = \hat{g}(z)$ for $\operatorname{Re} z \leq 0$.

To keep the notation simple, we denote the norms in both these spaces by $\|f\|_\alpha$, that is, we set

$$\begin{aligned} \|f\|_\alpha &:= \left\|f^{(\alpha)}\right\|_{L_1(\mathbb{R})}, \text{ for } f \in X_\alpha(\mathbb{R}) \text{ and} \\ \|f\|_\alpha &:= \left\|f^{(\alpha)}\right\|_{L_1(\mathbb{R}^+)}, \text{ for } f \in X_\alpha(\mathbb{R}^+). \end{aligned} \tag{1.1}$$

To connect the above definition of fractional derivatives on $\mathbb{R}$ with the standard definitions of fractional derivatives found in the literature, we list the definitions of the Riemann-Liouville fractional integrals and derivatives.

Definition 1.1.3. Let $\alpha > 0$ and $f \in L_1(\mathbb{R})$, then the so called (left-sided) *Riemann-Liouville fractional integral* (if $a = 0$) or *Liouville fractional integral* (if $a = -\infty$) of order α, is defined by

$${}_aI_x^\alpha f(x) := \int_a^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds, \ x > a$$

where the lower limit of the integral $a \in \mathbb{R}$ is fixed or $a = -\infty$, while the upper limit $x \in \mathbb{R}$ is variable, see [86, p. 65] and [95, p. 33 and 94].

This definition of fractional integral is well-defined for any piecewise continous function $f \in L_1(a,b)$, however, for our purposes we take $L_1(\mathbb{R})$ as the domain of definition.

Definition 1.1.4. For $\alpha > 0$, let $n = \lceil \alpha \rceil$ denote the least integer greater than α, then the so called *Riemann-Liouville fractional derivative*, $D^\alpha : \mathcal{D}(D^\alpha) \to L_1(\mathbb{R})$ is defined by

$$D^\alpha f(x) := D^n \left({}_{-\infty}I_x^{n-\alpha} f(x) \right)$$

with domain

$$\mathcal{D}(D^\alpha) := \left\{ f \in L_1(\mathbb{R}) : {}_{-\infty}I_x^{n-\alpha} f \in W^{n,1}(\mathbb{R}) \right\},$$

where D^n denotes the integer order derivative of order n on $\mathbb{R}$ with respect to variable upper limit of the fractional integral and the Sobolev space $W^{n,1}(\mathbb{R})$ is given in Definition B.1.1.

Remark 1.1.5. Let $f \in \mathcal{D}(D^\alpha)$, then $\widehat{D^\alpha f}(k) = (-ik)^\alpha \hat{f}(k)$ [95, p. 137-139]. Thus, $f^{(\alpha)} = D^\alpha f$ a.e.; that is, the function $f^{(\alpha)}$ defined uniquely in Definition 1.1.2 is equal to the *Riemann-Liouville fractional derivative* $D^\alpha f$ given by (1.1.4) almost everywhere by the uniqueness of the Fourier transform of $L_1(\mathbb{R})$-functions.

1.2 Grünwald-type approximations

In the Grünwald-Letnikov approach to fractional calculus, the fractional derivative of arbitrary order $\alpha > 0$ is defined as the limit of the corresponding fractional difference quotient [86, 95],

$${}_aD_x^\alpha f(x) = \lim_{\substack{h\to 0\\ nh=x-a}} h^{-\alpha} \left(\sum_{m=0}^{n} \mathcal{G}_m^\alpha f(x-mh) \right), \tag{1.2}$$

where a and x are the lower and upper terminals, respectively and $\mathcal{G}_m^\alpha = (-1)^m \binom{\alpha}{m}$ are given by (A.1). Podlubny [86, p.63] demonstrates the equivalence of the Riemann-Liouville and the Grünwald-Letnikov definitions of the fractional derivative under the assumption that f is $m+1$-times differentiable where $\alpha < m+1$. Thus, in numerical schemes, it is natural to use the Grünwald-Letnikov formula (1.2) with a fixed step size h to approximate the fractional derivative operator. Here is the formal definition of the shifted Grünwald formula that we use to approximate the fractional derivative operator on $\mathbb{R}$, (also see [44], [77], [105]).

Definition 1.2.1. Let $f \in L_1(\mathbb{R})$ and $h > 0$, then the p-shifted *Grünwald formula* is given by

$$A^{\alpha}_{h,p} f(x) := \frac{1}{h^{\alpha}} \sum_{m=0}^{\infty} \mathcal{G}^{\alpha}_{m} f(x - (m-p)h), \tag{1.3}$$

where the shift $p \in \mathbb{R}$ and the properties of the Grünwald coefficients, $\mathcal{G}^{\alpha}_{m} = (-1)^m \binom{\alpha}{m}$ can be found in Appendix A.

Observe the shift p used in the argument of the function f compared to (1.2) above. Meerschaert et al. in [77] used this Grünwald formula, a modified version of (1.2), with a non-negative integer shift p, to numerically approximate the Riemann-Liouville fractional derivative. The authors also proved stability of their numerical scheme for space-fractional advection dispersion equation with $1 < \alpha < 2$, using the Grünwald formula for space discretisation with shift $p = 1$ and the implicit Euler method for time discretisation. In the same article in Remark 2.5, and in [102] in Remark 3.2, the authors mention that the Grünwald formula with no shift or any shift p yields first order consistency of numerical schemes for fractional-in-space partial differential equations. However, they attribute the best performance of the numerical schemes to the optimal shift p obtained by minimising $\left|p - \frac{\alpha}{2}\right|$. In [44], the authors refer to the optimal shift as a "clever" shift using which yields a consistent approximation for the Riemann-Liouville fractional derivative for sufficiently smooth functions. We show in Chapter 2, that more is true of these remarks, that in fact the numerical schemes using the Grünwald formula with integer shift are stable if and only if the shift p is optimal.

Remark 1.2.2. We make a clarification of the convention that we adopt at this juncture. When we apply the shifted Grünwald formula (1.3) to a function $f \in L_1(\mathbb{R}^+)$, we always assume implicitly that f is extended to $L_1(\mathbb{R})$ by setting $f(x) = 0$ for $x < 0$. To keep matters simple, we will also refer to this extended function as f. In the case when the shift $p > 0$ we regard $A^{\alpha}_{h,p}$ as an operator on $L_1(\mathbb{R})$. However, in the case when $p \leq 0$, with this convention, one can verify that the support of $A^{\alpha}_{h,p} f$ is contained in $\mathbb{R}^+$ and hence $A^{\alpha}_{h,p}$ can be regarded as an operator on $L_1(\mathbb{R}^+)$. Indeed, for $p \leq 0$ if $x < 0$, then for all $m \in \mathbb{N}$, $x - (m-p)h \leq x + ph \leq x < 0$ which, in view of our convention, implies that $f(x - (m-p)h) = 0$ and hence $A^{\alpha}_{h,p} f(x) = 0$.

The following result of Tadjeran et al. [102], that the numerical schemes that employ the shifted Grünwald formula yields second order consistency is our main motivation to explore higher order schemes. Let the Sobolev space $W^{1,3+n}(\mathbb{R})$ be given by Definition B.1.1 and $1 < \alpha < 2$. Then, for $f \in W^{1,3+n}(\mathbb{R})$, the authors showed that the error term expansion for the numerical approximation of the Riemann-Liouville fractional derivative of order α by the (non-negative integer) shifted Grünwald formula is given by

$$A^{\alpha}_{h,p}f(x) - D^{\alpha}f(x) = \sum_{l=1}^{n-1}\left(a_l D^{\alpha+l}f(x)\right)h^l + O(h^n),$$

where the constants a_l are independent of f, h and x. In the same paper, the authors proved stability of their fractional Crank-Nicolson scheme by determining the spectrum of the Grünwald matrix associated with the shifted Grünwald formula using Gershgorin's Theorem. Moreover, their fractional Crank-Nicolson scheme was shown to be consistent with a second order in time and first order in space local truncation errors. Furthermore, they obtained a second order local truncation error in space employing the Richardson extrapolation method.

This result that the Grünwald approximation for the fractional derivative is consistent was further refined by Baeumer et. al. in [4, Proposition 4.9] to all $f \in X_{\alpha}(\mathbb{R})$ with any shift $p \in \mathbb{R}$ where X_{α} is given in Definition 1.1.1, that is, $A^{\alpha}_{h,p}f \to D^{\alpha}f$ in $L_1(\mathbb{R})$ as $h \to 0+$. A proof for the unshifted case, $p = 0$ with $\alpha > 0$ and $f \in X_{\alpha}(\mathbb{R}^+)$ can be found in [106, Theorem 13]. We generalise these results in Theorem 1.6.2 under minimal regularity assumptions. That is, for $f \in X_{\alpha+\beta}(\mathbb{R})$ the convergence rate of the first order Grünwald-type approximation for the fractional derivative operator in $L_1(\mathbb{R})$ can be further fine-tuned to the order h^{β}, $0 < \beta \leq 1$ as $h \to 0+$. If $p \leq 0$, then the same convergence rate holds in $L_1(\mathbb{R}^+)$ for $f \in X_{\alpha+\beta}(\mathbb{R}^+)$. Following that, in Section 1.5, we construct higher-order Grünwald-type approximations for the fractional derivative operator on $\mathbb{R}$, by combining Grünwald formulae with different shifts p and accuracy h such that the lower order error terms cancel out. We then conduct a detailed error analysis using Fourier multipliers and conclude this chapter with the main result, Corollary 1.6.3 on the consistency of the higher order schemes.

To begin with, we show that the shifted Grünwald formula given in Definition 1.2.1, maps $L_1(\mathbb{R})$ into $L_1(\mathbb{R})$ and derive an explicit formula for its Fourier transform.

Lemma 1.2.3. *Let $f \in L_1(\mathbb{R})$, $\alpha \in \mathbb{R}^+$, $p \in \mathbb{R}$, $h > 0$ be fixed, and the shifted Grünwald formula $A^{\alpha}_{h,p}$ be given by* (1.3)*, then*

$$A^{\alpha}_{h,p}f \in L_1(\mathbb{R}).$$

Moreover, its Fourier transform is given by

$$\widehat{(A^{\alpha}_{h,p}f)}(k) = \omega_{\alpha,p}(-ikh)\widehat{f^{(\alpha)}}(k),$$

where $\omega_{\alpha,p}(z) = \left(\frac{1-e^{-z}}{z}\right)^{\alpha} e^{zp}$.

Proof. Take note that the series $\sum_{m=0}^{\infty} \mathcal{G}_m^\alpha$ is absolutely convergent, see (A.10). Thus, the iterated integral

$$\sum_{m=0}^{\infty} \int_{\mathbb{R}} |\mathcal{G}_m^\alpha f(x-(m-p)h)| \,\mathrm{d}x \le \sum_{m=0}^{\infty} |\mathcal{G}_m^\alpha| \, \|f\|_{L_1(\mathbb{R})} < \infty$$

and the use of Fubini's theorem [92, p. 141], is justified. Hence,

$$\begin{aligned}
\left\|A_{h,p}^\alpha f\right\|_{L_1(\mathbb{R})} &= \int_{\mathbb{R}} \left| \frac{1}{h^\alpha} \sum_{m=0}^{\infty} \mathcal{G}_m^\alpha f(x-(m-p)h) \right| \mathrm{d}x \\
&\le \frac{1}{h^\alpha} \sum_{m=0}^{\infty} |\mathcal{G}_m^\alpha| \int_{\mathbb{R}} |f(x-(m-p)h)| \,\mathrm{d}x \\
&\le \frac{1}{h^\alpha} \sum_{m=0}^{\infty} |\mathcal{G}_m^\alpha| \, \|f\|_{L_1(\mathbb{R})} < \infty.
\end{aligned}$$

Therefore, the Fourier transform of the shifted Grünwald formula exists and using Fubini's theorem once again, we have

$$\begin{aligned}
(\widehat{A_{h,p}^\alpha f})(k) &= h^{-\alpha} \sum_{m=0}^{\infty} \mathcal{G}_m^\alpha \int_{\mathbb{R}} e^{ikx} f(x-(m-p)h) \,\mathrm{d}x \\
&= h^{-\alpha} \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} e^{ik(m-p)h} \hat{f}(k)
\end{aligned}$$

The Fourier transform of the Grünwald formula can be written in the following product form using the Binomial series (A.4),

$$\begin{aligned}
(\widehat{A_{h,p}^\alpha f})(k) &= h^{-\alpha} \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} e^{ik(m-p)h} \hat{f}(k) \\
&= h^{-\alpha} e^{-ikhp} (1-e^{ikh})^\alpha \hat{f}(k) \\
&= h^{-\alpha} (-ikh)^\alpha \left(\frac{1-e^{ikh}}{-ikh} \right)^\alpha e^{-ikhp} \hat{f}(k) \\
&= \omega_{\alpha,p}(-ikh)(-ik)^\alpha \hat{f}(k) \\
&= \omega_{\alpha,p}(-ikh) \widehat{f^{(\alpha)}}(k),
\end{aligned} \tag{1.4}$$

where we have used Definition 1.1.2 in the last line and introduce the special function $\omega_{\alpha,p}(z) = \left(\frac{1-e^{-z}}{z}\right)^\alpha e^{zp}$. □

Remark 1.2.4. Let $\alpha \in \mathbb{R}^+$ such that $2q-1 < \alpha < 2q+1$ where $q \in \mathbb{N}$ and let

$$\psi(z) = (-1)^{q+1} h^{-\alpha} e^{-hpz} (1-e^{hz})^\alpha. \tag{1.5}$$

Note that the second line of (1.4) above shows that for any $f \in L_1(\mathbb{R})$ the Grünwald formula (1.3) can be expressed in the multiplier notation of Appendix C.3 as

$$A^{\alpha}_{h,p} = T_{(-1)^{q+1}\psi_{\alpha,h,p}}$$

where the *Grünwald multiplier* is given by

$$\psi_{\alpha,h,p}(k) := \psi(ik) = (-1)^{q+1}h^{-\alpha}e^{-ikhp}(1-e^{ikh})^{\alpha} = (-1)^{q+1}\omega_{\alpha,p}(-ikh)(-ik)^{\alpha}. \quad (1.6)$$

This fact that the Grünwald formula can be viewed as a multiplier operator naturally leads us in Section 1.3 to study inequalities that estimate multiplier norms.

Remark 1.2.5. The Grünwald multiplier in Remark 1.2.4 above, involves the function $\omega_{\alpha,p}$. In the error analysis of the Grünwald schemes, as we will see later, the function $\omega_{\alpha,p} : \mathbb{C} \to \mathbb{C}$, where $\alpha \in \mathbb{R}^+$ and $p \in \mathbb{R}$, plays a very important role. Hence, for easy reference we give it a special name, *omega function*,

$$\omega_{\alpha,p}(z) := \left(\frac{1-e^{-z}}{z}\right)^{\alpha} e^{zp}$$

and study this function in detail in Section 1.4.

1.3 Bound for multiplier norms

In this section we study *Carlson-type* inequalities that bound L_1 (Fourier) multiplier norms. We refer to Appendix C.3 for the definition and results relating to L_1-multipliers. These inequalities are not only crucial in our error analysis, but also important in their own right. Firstly, they prove to be particulary useful in scenarios where only the multipliers are known explicitly while the corresponding measures even when they exist may not be known. Secondly, these Carlson-type inequalities help bound multiplier operator norms by solely exploiting the properties of the multipliers and as a consequence help to show that, in fact, the multipliers under consideration are L_1-multipliers. Thirdly, these inequalities help estimate the L_1-norms of functions, defined as the inverse Fourier transforms of functions in L_r for $1 < r \leq 2$ through the L_r-norms of the Fourier transform and its derivative. Lastly, as we will see in the applications in Chapter 2, it turns out that it is essential to consider L_r-spaces for $r \neq 2$.

In some situations the Carlson-type inequality, given in Proposition 1.3.1 below, is not directly applicable. One such scenario is when the decay of the multiplier at infinity is insufficient for it to be in L_r but its derivative has more than necessary decay to be in L_r. In this situation, we employ a partition of unity, and derive a similar result, thereby enhancing the reach of the Carlson-type inequality. We do this in Section 1.3.1, where we study inequalities for multipliers which along with their generalised first derivatives belong to $L_r(\mathbb{R})$, for $1 < r \le 2$. Another scenario where the Carlson-type inequality cannot be applied is for periodic multipliers. In Section 1.3.2, we study similar inequalities for periodic multipliers.

1.3.1 Carlson-type inequality

The first inequality that we consider is a special case of a more general *Carlson-type* inequality, see [60, Theorem 5.10, p.107]. We give a simple proof here to keep our discussions self-contained. The case $r = 2$ is usually referred to in the literature as Carlson-Beurling Inequality, and can be found in [1, p.429] and [18, 24, 36].

Let $W^{r,1}(\mathbb{R})$, $r \ge 1$, denote the Sobolev space of $L_r(\mathbb{R})$-functions given in Definition B.1.1 and $\mathcal{F}(L_1)$ denote the ring of Fourier transforms of L_1-functions as in Remark B.3.2. The following result yields a sharp bound for the L_1-multiplier operator norm.

Proposition 1.3.1 (Carlson-type inequality). *If $\psi \in W^{r,1}(\mathbb{R})$, $1 < r \le 2$, then there exists $\xi \in L_1$ such that $\hat{\xi} = \psi$; that is, $\psi \in \mathcal{F}(L_1)$. Moreover, there exists a constant $C(r) > 0$, independent of ξ and ψ, such that*

$$\|\xi\|_{L_1} \le C(r) \, \|\psi\|_{L_r}^{\frac{1}{s}} \, \|\psi'\|_{L_r}^{\frac{1}{r}} ,$$

where $\frac{1}{r} + \frac{1}{s} = 1$.

Proof. Let $\psi \in W^{r,1}(\mathbb{R})$, $1 < r \le 2$ and set $\psi_-(k) := \psi(-k)$, $k \in \mathbb{R}$. Then, define the function $\xi := \frac{1}{2\pi}\widehat{\psi_-} \in L_s(\mathbb{R})$, where $\frac{1}{r} + \frac{1}{s} = 1$, see Remark B.2.2. First, note that $\|\xi\|_{L_1} = \left\|\frac{1}{2\pi}\widehat{\psi_-}\right\|_{L_1} = \frac{1}{2\pi}\left\|\hat{\psi}\right\|_{L_1}$ and $\hat{\psi}'(x) = (-ix)\hat{\psi}(x)$.

Moreover, recall the Hausdorff-Young-Titchmarsh inequality given by (B.1)

$$\left\|\hat{\psi}\right\|_{L_s} \le (2\pi)^{\frac{1}{s}} \, \|\psi\|_{L_r} , \psi \in L_r, \ 1 \le r \le 2, \ \frac{1}{r} + \frac{1}{s} = 1.$$

If $\psi \equiv \mathbf{0}$ there is nothing to prove. So let us assume that $\psi \not\equiv \mathbf{0}$, then using Hölder's inequality in the second line, (B.1) in the third and setting $v = \frac{\|\psi'\|_{L_r}}{(r-1)^{1/r}\|\psi\|_{L_r}}$ we have

$$\begin{aligned}
\|\xi\|_{L_1} = \frac{1}{2\pi}\left\|\hat{\psi}\right\|_{L_1} &= \frac{1}{2\pi}\left(\int_{|x|\le v}\left|\hat{\psi}(x)\right| dx + \int_{|x|>v}\left|\frac{1}{x}(x\hat{\psi}(x))\right| dx\right) \\
&\le \frac{2^{\frac{1}{r}}}{2\pi}\left(v^{\frac{1}{r}}\left\|\hat{\psi}\right\|_{L_s} + v^{\frac{-1}{s}}\left\|(\cdot)\hat{\psi}(\cdot)\right\|_{L_s}(r-1)^{-1/r}\right)
\end{aligned}$$

$$\leq 2^{\frac{1}{r}} (2\pi)^{\frac{1}{s}-1} \left(v^{\frac{1}{r}} \|\psi\|_{L_r} + v^{\frac{-1}{s}} \|\psi'\|_{L_r} (r-1)^{-1/r} \right)$$
$$= \frac{2}{\pi^{\frac{1}{r}} (r-1)^{\frac{1}{r^2}}} \|\psi\|_{L_r}^{\frac{1}{s}} \|\psi'\|_{L_r}^{\frac{1}{r}}.$$

Therfore, $\xi \in L_1$ and so $\hat{\xi}$ exists. Thus, $\psi(k) = \frac{1}{2\pi}\widehat{\hat{\psi}_-}(k) = \hat{\xi}(k)$ for almost all $k \in \mathbb{R}$ by the inversion formula for the Fourier transform. Since, ψ and $\hat{\xi}$ are continuous, this holds for all $k \in \mathbb{R}$. □

Remark 1.3.2. In fact, the preceding proof shows that ψ is an L_1-multiplier and the Carlson-type inquality can be rewritten in multiplier notation of Appendix C.3 as

$$\|T_\psi\|_{\mathcal{B}(L_1)} = \|\xi\|_{L_1} \leq C(r) \|\psi\|_{L_r}^{\frac{1}{s}} \|\psi'\|_{L_r}^{\frac{1}{r}},$$

where $C(r)$ is independent of ψ.

In some cases, the multiplier has insufficient decay to be in L_r while its generalised derivative might have more than necessary decay to be in L_r. In such scenarios, partition of unity turns out to be an excellent tool to bound the multiplier norm, see [21, 22, 50]. The following result is a version of Carlson-type inequality employing a partition of unity.

Corollary 1.3.3. *Let $\psi : \mathbb{R} \to \mathbb{C}$ and θ_j be such that $\sum_{j\in\mathbb{Z}} \theta_j(x) = 1$ for all $x \in \mathbb{R}$. If $\theta_j\psi \in W^{r,1}(\mathbb{R})$, $1 < r \leq 2$, for all j and $\sum_j \|\theta_j\psi\|_{L_r}^{\frac{1}{s}} \|(\theta_j\psi)'\|_{L_r}^{\frac{1}{r}} < \infty$, where $\frac{1}{r} + \frac{1}{s} = 1$, then $\psi \in \mathcal{F}(L_1)$; that is, there exists $\xi \in L_1(\mathbb{R})$ and a constant $C(r)$ independent of ξ and ψ, such that $\hat{\xi} = \psi$ and*

$$\|\xi\|_{L_1} \leq C(r) \sum_j \|\theta_j\psi\|_{L_r}^{\frac{1}{s}} \|(\theta_j\psi)'\|_{L_r}^{\frac{1}{r}}.$$

Proof. By design, $\psi(x) = \sum_j \theta_j(x)\psi(x)$ for almost all x. Let $\xi_j \in L_1(\mathbb{R})$ be such that $\hat{\xi}_j = \theta_j\psi$ by Proposition 1.3.1. By assumption, the partial sums

$$\left\| \sum_{j=-n}^{n} \xi_j \right\|_{L_1} \leq \sum_{j=-n}^{n} \|\xi_j\|_{L_1} \leq C(r) \sum_j \|\theta_j\psi\|_{L_r}^{\frac{1}{s}} \|(\theta_j\psi)'\|_{L_r}^{\frac{1}{r}} < \infty.$$

Thus, the series $\sum_j \xi_j$ converges to some $\xi \in L_1$ and

$$\|\xi\|_{L_1} \leq C(r) \sum_j \|\theta_j\psi\|_{L_r}^{\frac{1}{s}} \|(\theta_j\psi)'\|_{L_r}^{\frac{1}{r}}.$$

Hence, the use of Fubini's theorem is justified and

$$\hat{\xi} = \sum_j \hat{\xi}_j = \psi.$$

□

1.3.2 Carlson-type inequality for periodic multipliers

For periodic multipliers a Carlson-type inequality is not directly applicable as these are not Fourier transforms of L_1-functions. In [22] a suitable smooth cut-off function η with compact support was used where $\eta = 1$ in a neighborhood of $[-\pi, \pi]$, to estimate the multiplier norm of a periodic multiplier ψ by the non-periodic one $\eta\psi$. For the multiplier norm of $\eta\psi$ the above Carlson-type inequality can be then used. However, we prove a result similar to Proposition 1.3.1 for periodic multipliers which along with their derivatives have local L_r bounds. This makes the introduction of a cut-off function superfluous and hence simplifies the technicalities in later estimates.

Let $W^{r,1}_{per}[-\pi, \pi]$ below denote the Sobolev space of 2π-periodic functions as in Definition B.1.1.

Theorem 1.3.4. *Let $\psi \in W^{r,1}_{per}[-\pi, \pi]$, $1 < r \le 2$, then ψ is an L_1-multiplier and there is $C > 0$, independent of ψ, such that*

$$\|T_\psi\|_{\mathcal{B}(L_1)} \le |a_0| + C(r)\, \|\psi\|^{\frac{1}{s}}_{L_r[-\pi,\pi]}\, \|\psi'\|^{\frac{1}{r}}_{L_r[-\pi,\pi]},$$

where $\frac{1}{r} + \frac{1}{s} = 1$ and $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi(x)dx$, denotes the 0^{th} Fourier coefficient of ψ.

Proof. Since $\psi \in L_r[-\pi, \pi]$, it follows using Hölder's inequality that $\psi \in L_1[-\pi, \pi]$ and so we can define the k^{th} Fourier coefficient of ψ,

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx}\psi(x)dx, \; k \in \mathbb{Z}.$$

First, note that $|a_0| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |\psi(x)|\, dx < \infty$ and using integration by parts and the fact that ψ is absolutely continuous, we have that ika_k are the Fourier coefficients of ψ' . Next, recall the Hausdorff-Young inequality for Fourier series, see [23, p. 177],

$$\left(\sum_{k=-\infty}^{\infty} |a_k|^s\right)^{\frac{1}{s}} \le (2\pi)^{-\frac{1}{r}}\, \|\psi\|_{L_r[-\pi,\pi]}\,, \; 1 < r \le 2, \; \frac{1}{r} + \frac{1}{s} = 1$$

and Bellman's inequality, see [14] and [60, p. 25],

$$\left(\sum_{k=1}^{\infty} b_k\right)^{\alpha\beta+\alpha-\beta} \le C(\alpha,\beta)\sum_{k=1}^{\infty} b_k^{\alpha}\left(\sum_{k=1}^{\infty} k^{\beta} b_k^{\beta}\right)^{\alpha-1}, \; \alpha, \beta > 1, \; b_k \ge 0, \; k \in \mathbb{N}.$$

On setting, $\alpha = \beta = s$ and $b_k = |a_k|$, we have

$$\sum_{k=1}^{\infty} |a_k| \le C(r)\left(\sum_{k=1}^{\infty} |a_k|^s\right)^{\frac{1}{s^2}} \left(\sum_{k=1}^{\infty} |(ika_k)|^s\right)^{\frac{1}{s}\left(\frac{s-1}{s}\right)} \le C(r)\, \|\psi\|^{\frac{1}{s}}_{L_r[-\pi,\pi]}\, \|\psi'\|^{\frac{1}{r}}_{L_r[-\pi,\pi]}.$$

Moreover, the same inequality holds for $\sum_{k=-\infty}^{-1} |a_k|$. Thus,

$$\sum_{k=-\infty}^{\infty} |a_k| \leq |a_0| + C(r) \|\psi\|_{L_r[-\pi,\pi]}^{\frac{1}{s}} \|\psi'\|_{L_r[-\pi,\pi]}^{\frac{1}{r}} < \infty. \tag{1.7}$$

This implies that ψ is the point-wise limit of its Fourier series [23, p. 166], that is;

$$\psi(x) = \sum_{m=-\infty}^{\infty} a_k e^{ikx}, \text{ for all } x \in \mathbb{R}.$$

Furthermore, let $\mu := \sum_{k=-\infty}^{\infty} a_k \delta_k$, where δ_k is the Dirac measure at $k \in \mathbb{Z}$ given by (C.3.1). Then, the series converges in the total variation norm; that is, $\|\mu\|_{\mathrm{TV}} = \sum_{k=-\infty}^{\infty} |a_k| < \infty$ and $\mu \in \mathcal{M}_{\mathcal{B}}(\mathbb{R})$, see Appendix C.3. Moreover, the use of Fubini's theorem is justified and taking Fourier transforms term by term we have $\hat{\mu} = \psi$. Hence, ψ is an L_1-multiplier and

$$\|T_\psi\|_{\mathcal{B}(L_1)} = \|\mu\|_{\mathrm{TV}} = \sum_{k=-\infty}^{\infty} |a_k| = |a_0| + C(r) \|\psi\|_{L_r[-\pi,\pi]}^{\frac{1}{s}} \|\psi'\|_{L_r[-\pi,\pi]}^{\frac{1}{r}}.$$

□

Remark 1.3.5. Firstly, note that the term $|a_0|$ cannot be removed from the above estimate in general. To see this, consider $\psi \equiv 1$. Secondly, the fact that if $\psi' \in L_r[-\pi,\pi]$, $r > 1$, then the Fourier series of ψ is absolutely summable, was first proved in [104]. A multivariate version of (1.7) with a different proof to the one above can be found in [54].

1.4 Grünwald (periodic) multiplier operators

The Grünwald formula (1.3) can be viewed as a multiplier operator as mentioned in Remark 1.2.4. The Grünwald multiplier is given by

$$\psi_{\alpha,h,p}(k) = (-1)^{q+1} \omega_{\alpha,p}(-ikh)(-ik)^\alpha.$$

In the error analysis of the consistency of the Grünwald schemes, for instance, in Theorem 1.6.2 below, the function $\omega_{\alpha,p} : \mathbb{C} \to \mathbb{C}$, where $\alpha \in \mathbb{R}^+$ and $p \in \mathbb{R}$, plays a very important role. In this section, as mentioned in Remark 1.2.5, we study this *omega function*

$$\omega_{\alpha,p}(z) := \left(\frac{1 - e^{-z}}{z} \right)^\alpha e^{zp}. \tag{1.8}$$

We also need the particular case of the omega function when $\alpha = 1$ and $p = 0$, which we simply denote by

$$\omega(z) := \omega_{1,0}(z) = \frac{1 - e^{-z}}{z}. \tag{1.9}$$

The unshifted version of the omega function $\omega_{\alpha,0}(z)$, appears in the works of several authors, including Westphal in [106] and [51, Chapter III], Butzer et. al. in [51, Chapter I] and [90, p. 116], Samko et. al. in [95, Section 20] and Lanford and Robinson in [58]. To keep our discussion self-contained, we list and prove those properties of the omega function with arbitrary shift p that are relevant for our purposes.

To this end, let us begin by showing that the omega function is analytic in some neighbourhood of the origin. See Remark B.3.4 for the convention adopted in the definition of the function z^μ, where $z \in \mathbb{C}$ and fixed $\mu \in \mathbb{C}$. Let us rewrite the omega function as

$$\omega_{\alpha,p}(z) = (\omega(z))^\alpha e^{zp}. \tag{1.10}$$

As we take the negative real axis as the branch cut for the α-th power of $\omega(z)$ and since e^{zp} is analytic, the omega function, $\omega_{\alpha,p}(z)$ is analytic, except where $\omega(z)$ takes values on the negative real axis and at the origin. Note that the principal branch here is chosen such that

$$\lim_{z\to 0} \omega_{\alpha,p}(z) = 1, \tag{1.11}$$

since for $z \in \mathbb{R}$,

$$\lim_{z\to 0} \omega(z) = \lim_{z\to 0} \left(\frac{1 - e^{-z}}{z} \right) = \lim_{z\to 0} e^{-z} = 1$$

by L'Hôpital's rule and $\lim_{z\to 0} e^{zp} = 1$. Thus, $\omega_{\alpha,p}(z)$ is analytic in some neighbourhood of the origin as the singularity at $z = 0$ is removable, see [29, p. 103]. Hence, we can write $\omega_{\alpha,p}(z)$ as a power series, that is, there exists $R > 0, a^\alpha_{p,n} \in \mathbb{R}$ such that

$$\omega_{\alpha,p}(z) = \sum_{n=0}^{\infty} a^\alpha_{p,n} z^n \text{ for all } |z| < 2R. \tag{1.12}$$

Remark 1.4.1. For convenience in calculations, if $R > 1$ we set $R = 1$ and thus, in what follows $0 < R \leq 1$.

Before we study the properties of the omega function in detail, let us briefly look at the properties of the simplest case with $\alpha = 1$ and $p = 0$. Firstly, note the recurrence relation satisfied by the omega functions,

$$\omega_{\alpha,p}(z)\omega(z) = \omega_{\alpha+1,p}(z). \tag{1.13}$$

Differentiating (1.9) with respect to z we have

$$\omega'(z) = \frac{ze^{-z} - (1 - e^{-z})}{z^2} = \frac{e^{-z} - \omega(z)}{z}.$$

Since, $\omega(z)$ is analytic, we also have that $\omega'(z)$ is analytic in the same neighbourhood of the origin given in (1.12). Using the exponential series, we have

$$\begin{aligned}\omega(z) &= 1 - \frac{z}{2!} + \frac{z^2}{3!} \cdots \text{ and}\\ \omega'(z) &= -\frac{1}{2} + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} z^m}{(m+1)! + m!}.\end{aligned} \tag{1.14}$$

As a consequence, there exists a constant $C > 0$ such that for $|z| < R$, $|\omega(z)| \le C$ and $|\omega'(z)| \le C$. If $\mathrm{Re}(z) \ge 0$ and $|z| \ge R$, note that $|e^{-z}| = e^{-\mathrm{Re}(z)} \le 1$ and that for convenience in calculations we assume that $0 < R \le 1$, see comments following (1.12) above. Hence, we have the following bounds,

$$|\omega(z)| \le \begin{cases} C, & \text{for } |z| < R, \\ \frac{1+|e^{-z}|}{|z|} \le \frac{2}{|z|}, & \text{for } \mathrm{Re}(z) \ge 0 \text{ and } |z| \ge R, \end{cases} \tag{1.15}$$

and

$$|\omega'(z)| \le \begin{cases} C, & \text{for } |z| < R, \\ \left|\frac{e^{-z}}{z}\right| + \left|\frac{\omega(z)}{z}\right| \le \frac{1}{|z|} + \frac{2}{|z|^2} \le \frac{c}{|z|}, & \text{for } \mathrm{Re}(z) \ge 0 \text{ and } |z| \ge R. \end{cases} \tag{1.16}$$

The observations made so far on $\omega(z)$ yield the Taylor expansion and the bound for the omega function, more importantly, the bound for its derivative on the imaginary axis, all of which are used repeatedly in the error analysis.

Lemma 1.4.2. *Let $\alpha > 0$, $p \in \mathbb{R}$ and $\omega_{\alpha,p}$ be given by* (1.8). *Then, we have the following:*

1. *The omega function has the following Taylor expansion,*

$$\omega_{\alpha,p}(z) = 1 + (p - \frac{\alpha}{2})z + \frac{1}{24}(3\alpha^2 + \alpha + 12p^2 - 12\alpha p)z^2 + O(z^3)$$

   where $|z| < 2R$.

2. *There exists a constant $C > 0$ (chosen to be the maximum of the constants in the three scenarios below) such that*

$$|\omega_{\alpha,p}(z) - 1| \le \begin{cases} C|z| & \text{for } |z| < R, \\ C & \text{for } z \in i\mathbb{R}, \\ C & \text{for } \mathrm{Re} z \ge 0 \ \&\ p \le 0, \end{cases} \tag{1.17}$$

   where $2R$ is the radius of convergence as described in (1.12).

3. Let either $\alpha \geq 1$ and $k \in \mathbb{R}$ or $0 < \alpha < 1$ and $k \in [-\pi, \pi]$, then

$$\left| \frac{d}{dk}(\omega_{\alpha,p}(-ik)) \right| \leq C. \tag{1.18}$$

Proof. 1. Expanding $\omega_{\alpha,p}(z)$ given by (1.8), using the binomial series, exponential series and (1.14), we obtain

$$\begin{aligned} \omega_{\alpha,p}(z) &= \left(\frac{1-e^{-z}}{z}\right)^{\alpha} e^{zp} \\ &= \left(1 - \frac{z}{2!} + \frac{z^2}{3!} \cdots\right)^{\alpha} \left(1 + zp + \frac{(zp)^2}{2!} \cdots\right) \\ &= 1 + (p - \frac{\alpha}{2})z + \frac{1}{24}(3\alpha^2 + \alpha + 12p^2 - 12\alpha p)z^2 + O(z^3) \end{aligned} \tag{1.19}$$

where $|z| < 2R$.

2. The first inequality is clear from the fact that the power series expansion given by (1.19) is absolutely convergent for $|z| < 2R$ and so uniformly convergent for $|z| < R$. Indeed,

$$|\omega_{\alpha,p}(z) - 1| \leq \left|(p - \frac{\alpha}{2})z\right| + O(|z|^2) \leq C\,|z|\,.$$

We only need to show that the second and third inequality hold for $|z| \geq R$ as they hold for $|z| < R$ as a consequence of the first inequality.

So, let $|z| \geq R$ and either $z = ik$ for $k \in \mathbb{R}$ or $\mathrm{Re}(z) \geq 0$ & $p \leq 0$ and note that for $z = ik,\ k \in \mathbb{R},\ \left|e^{-ikp}\right| = 1$ and for $\mathrm{Re}(z) \geq 0$ & $p \leq 0$, $|e^{zp}| = e^{p\,\mathrm{Re}(z)} \leq 1$. Moreover, since it is the Laplace transform of an $L_1(\mathbb{R}^+)$-function, see Lemma 1.4.6 below, note that $\omega_{\alpha,p}(-z)$ is continuous for $\mathrm{Re}(z) \geq 0$. Thus, taking the modulus on both sides of (1.10) and using (1.15), we have

$$|\omega_{\alpha,p}(z)| = |\omega(z)|^{\alpha}\,|e^{zp}| \leq \left\{ \begin{array}{cl} C, & \text{for } |z| < R, \\ \frac{c}{|z|^{\alpha}} \leq C, & \text{for } |z| \geq R. \end{array} \right. \tag{1.20}$$

The proof of the first statement is complete on using the triangle inequality for $|\omega_{\alpha,p}(z) - 1|$.

3. Differentiating (1.10) with respect to z, we have

$$\omega'_{\alpha,p}(z) = \alpha\,(\omega(z))^{\alpha-1}\,\omega'(z)e^{zp} + p\,(\omega(z))^{\alpha}\,e^{zp} \tag{1.21}$$

and note that

$$\left|\frac{d}{dk}\,(\omega_{\alpha,p}(-ik))\right| = \left|(-i)\,\omega'_{\alpha,p}(-ik)\right| = \left|\omega'_{\alpha,p}(-ik)\right|$$

$$\leq \left(\alpha \left|\omega(-ik)\right|^{\alpha-1} \left|\omega'(-ik)\right| + |p| \left|\omega(-ik)\right|^{\alpha}\right) \left|e^{-ikp}\right|. \quad (1.22)$$

First, let $\alpha \geq 1$, then using (1.15) and (1.16), since $\alpha - 1 \geq 0$ and $\left|e^{-ikp}\right| = 1$, we have

$$\left|\frac{d}{dk}\left(\omega_{\alpha,p}(-ik)\right)\right| \leq \begin{cases} C, & \text{for } |k| < R, \\ \frac{c}{|k|^{\alpha}} \leq C, & \text{for } |k| \geq R. \end{cases} \quad (1.23)$$

Now, let $0 < \alpha < 1$ and $|z| < R$, since $\omega'_{\alpha,p}(z)$ is analytic we have

$$\left|\omega'_{\alpha,p}(z)\right| = \left|(p - \frac{\alpha}{2})\right| + O(|z|)$$

which implies that $\left|\omega'_{\alpha,p}(-ik)\right| \leq C$ for $|k| < R$ as the power series is absolutely convergent. To complete the proof, let $R \leq |k| \leq \pi$, then it is clear that $|\omega(-ik)|^{\alpha}$ is bounded in view of (1.15) and so the second term in (1.22) is bounded. Rewrite the first term as

$$\alpha \left|\omega(-ik)\right|^{\alpha-1} \left|\omega'(-ik)\right| = \alpha \left|\omega(-ik)\right|^{\alpha} \left|\omega'(-ik)\right| \left|\frac{1}{\omega(-ik)}\right|$$

and note that for $R \leq |k| \leq \pi$,

$$\left|\frac{1}{\omega(-ik)}\right| = \left|\frac{-ik}{1 - e^{ik}}\right| \leq \frac{\pi}{2\sin(R/2)}.$$

This completes the proof of the second statement in view of (1.15) and (1.16).

□

We now study the most important property of the omega function, namely, that the omega function is the Fourier transform of an L_1-function and identify this function explicitly. To this end, let us begin with the definition of power functions with support in $\mathbb{R}^+$. We use ϕ_β to denote the power function in this section, instead of p_β adopted later in Definition 3.6, in order to emphasise the fact that we are working on $\mathbb{R}$ instead of $[0, 1]$.

Definition 1.4.3. For $\alpha > 0$, we define the power function $\phi_{\alpha-1}$, with $\text{supp}(\phi_{\alpha-1}) \subset \mathbb{R}^+$, by

$$\phi_{\alpha-1}(x) := H(x)\frac{x^{\alpha-1}}{\Gamma(\alpha)}, \quad (1.24)$$

where the unit step function H is given by

$$H(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0. \end{cases} \quad (1.25)$$

We show that $\omega_{\alpha,p}(-ik)$ is the Fourier transform of the p-shifted fractional difference quotient of the power function with $h = 1$, given by

$$\Phi_p^\alpha(x) := \sum_{m=0}^{\infty} \mathcal{G}_m^\alpha \phi_{\alpha-1}(x - (m - p)), \tag{1.26}$$

where $\phi_{\alpha-1}$ is given by (1.24) and $\mathcal{G}_m^\alpha$ by (A.1). Take note that $\operatorname{supp}(\Phi_p^\alpha) \subset [-p, \infty) \cup \mathbb{R}^+$ and that the sum on the right has only finite number of terms for a fixed x, since by definition $\phi_{\alpha-1}(x - (m - p)) = 0$ for all $m \geq x + p$.

By Proposition B.3.5, we have the Fourier transform pair, $\hat{\phi}_{\alpha-1}(k) = (-ik)^{-\alpha}$. Clearly, for $\alpha > 0$, $\phi_{\alpha-1} \notin L_1(\mathbb{R})$ and so $\hat{\phi}_{\alpha-1}(k) = (-ik)^{-\alpha} \notin \mathcal{F}(L_1)$, where the ring of Fourier transforms of L_1-functions $\mathcal{F}(L_1)$, is given in Remark B.3.2. Nevertheless, we provide a proof similar to that of the proof of the Carlson-type inequality in Corollary 1.3.3 to show that for $\alpha > 0$,

$$\sum_{\substack{j \in \mathbb{Z} \\ j \neq 0}} \theta_j(k)(-ik)^{-\alpha} \in \mathcal{F}(L_1),$$

where θ_j for $j \in \mathbb{Z}$ are the functions in a partition of unity given in Definition 1.4.4 below.

According to the authors mentioned at the start of Section 1.4, the most difficult property to prove is the fact that $\Phi_0^\alpha \in L_1(\mathbb{R})$. Westphal in [106], uses a result on the asymptotic behaviour of power functions from Ingham's summability theory to show that Φ_0^α is integrable at infinity and in Lemma 2 of the same article, the author shows that

$$\omega_{\alpha,0}(z) = \left(\frac{1 - e^{-z}}{z}\right)^\alpha = \int_0^\infty e^{-zt} \Phi_0^\alpha(t)\, \mathrm{d}t, \quad \operatorname{Re} z \geq 0,$$

where $\Phi_0^\alpha \in L_1(\mathbb{R}^+)$ and $\int_0^\infty \Phi_0^\alpha(t)\, \mathrm{d}t = 1$. Take note that in Westphal's definition of the Laplace transform, e^{-zt} is used as the kernel and therefore the region of convergence is the right half plane. In [95], a smooth step function was used to split $\omega_{\alpha,0}$ into a sum of two functions, and using the fact that if $f \in L_1(\mathbb{R})$ and $f' \in L_2(\mathbb{R})$, then f is the Fourier transform of an $L_1(\mathbb{R})$ function, the authors show that $\omega_{\alpha,0}$ is the Fourier transform of an L_1 function. In [58], the authors prove this property using Fourier transform techniques and distribution theory. To keep our discussion self-contained, in Lemma 1.4.6, we prove a similar result using Carlson-type inequality given in Corollary 1.3.3; that is, we show that $\omega_{\alpha,p}(-ik)$ is the Fourier transform of $\Phi_p^\alpha \in L_1(\mathbb{R})$. In the proof of Lemma 1.4.6 we require a particular partition of unity which we define below.

Definition 1.4.4. Let R be as in (1.12). Then, we first define θ_0 such that $\mathrm{supp}(\theta_0) \subset [-R, R]$,

$$\theta_0(k) = \begin{cases} 1, & \text{for } |k| < R/2, \\ 2(R-|k|)/R, & \text{for } R/2 \le |k| < R, \\ 0 & \text{else.} \end{cases}$$

Next, define

$$\theta_1(k) = \begin{cases} 2(k-R/2)/R, & \text{for } R/2 \le k < R, \\ 1, & \text{for } R \le k < 1, \\ 2-k, & \text{for } 1 \le k < 2, \\ 0, & \text{else,} \end{cases}$$

so that $\mathrm{supp}(\theta_1) \subset [R/2, 2]$. For $j \ge 2$, let

$$\theta_j(k) = \begin{cases} (k-2^{j-2})/2^{j-2} & \text{for } 2^{j-2} < k < 2^{j-1}, \\ (2^j-k)/2^{j-1} & \text{for } 2^{j-1} \le k < 2^j, \\ 0 & \text{else,} \end{cases}$$

such that $\mathrm{supp}(\theta_j) \subset [2^{j-2}, 2^j]$. For $j < 0$, define $\theta_j(k) = \theta_{-j}(-k)$.

The following result plays a crucial role in the proof of Lemma 1.4.6.

Proposition 1.4.5. *Let $\alpha > 0$, then*

$$\sum_{\substack{j \in \mathbb{Z} \\ j \neq 0}} \theta_j(k)(-ik)^{-\alpha} \in \mathcal{F}(L_1),$$

where θ_j are given by Definition 1.4.4.

Proof. For convenience in calculations, for $j \neq 0$, let $\theta_{j,\alpha}(k) = \theta_j(k)(-ik)^{-\alpha}$ and write

$$\theta(k) = \sum_{\substack{j \in \mathbb{Z} \\ j \neq 0}} \theta_j(k)(-ik)^{-\alpha} = \sum_{\substack{j \in \mathbb{Z} \\ j \neq 0}} \theta_{j,\alpha}(k),$$

where θ_j are given in Definition 1.4.4. First note that $\mathrm{supp}(\theta_{\pm 1,\alpha}) \subset [\frac{R}{2}, 2]$, for $j \ge 2$, $\mathrm{supp}(\theta_{j,\alpha}) \subset [2^{j-2}, 2^j]$ and for $j \le -2$, $\mathrm{supp}(\theta_{j,\alpha}) \subset [-2^{-j}, -2^{-j-2}]$. Thus, the length of the support of $\theta_{j,\alpha}$ for $j \neq 0$ is less than $2^{|j|}$. Moreover, for $k \in [\frac{R}{2}, 2]$, $|k|^{-\alpha} \le \left(\frac{2}{R}\right)^{-\alpha} < C$ while for $k \in [2^{j-2}, 2^j]$ or $k \in [-2^{-j}, -2^{-j-2}]$, $|k|^{-\alpha} \le 2^{-\alpha(|j|-2)}$.

Hence, for each $j \neq 0$,

$$\|\theta_{j,\alpha}\|_{L_2}^{\frac{1}{2}} \le C(2^{-2(|j|-2)\alpha} 2^{|j|})^{1/4} = C 2^{\beta} 2^{|j|(1-2\alpha)/4} \le C 2^{|j|(1-2\alpha)/4}.$$

For $|j| \geq 2$, $\left|\theta_j'(k)\right| \leq 2^{-|j|+2}$, thus

$$\begin{aligned}
|\theta_{j,\alpha}'(k)| &\leq C\left(\alpha\,|\theta_j(k)|\,|k|^{-\alpha-1} + \left|\theta_j'(k)\right|\,|k|^{-\alpha}\right) \\
&\leq C2^{(|j|-2)(-\alpha-1)} + C2^{-|j|+2}2^{(|j|-2)(-\alpha)} \\
&\leq C2^{3+2\alpha}2^{-(\alpha+1)|j|} \\
&\leq C2^{-(\alpha+1)|j|}.
\end{aligned}$$

Moreover, for $|j| = 1$, $\left|\theta_{\pm 1}'(k)\right| \leq \frac{2}{R}$ and so $|\theta_{1,\alpha}'(k)| \leq C$. Hence, for each $j \neq 0$,

$$\|\theta_{j,\alpha}'\|_{L_2}^{\frac{1}{2}} \leq C(2^{-2(\alpha+1)|j|}2^{|j|})^{1/4} = C2^{-|j|(2\alpha+1)/4}.$$

Putting these together, there exists C independent of j such that,

$$\|\theta_{j,\alpha}\|_{L_2}^{\frac{1}{2}}\|\theta_{j,\alpha}'\|_{L_2}^{\frac{1}{2}} \leq C2^{|j|(1-2\alpha)/4}2^{-|j|(2\alpha+1)/4} \leq C2^{-\alpha|j|}.$$

Thus, $\theta_{j,\alpha} \in W^{2,1}(\mathbb{R})$, and using Proposition 1.3.1, there exists $\xi_{j,\alpha} \in L_1(\mathbb{R})$ such that $\theta_{j,\alpha} = \hat{\xi}_{j,\alpha}$ for each $j \neq 0$. Define

$$\xi_\alpha = \sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}} \xi_{j,\alpha},$$

then $\xi_\alpha \in L_1(\mathbb{R})$, since

$$\sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}} \|\theta_{j,\alpha}\|_{L_2}^{\frac{1}{2}}\|\theta_{j,\alpha}'\|_{L_2}^{\frac{1}{2}} \leq C\sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}} 2^{-\alpha|j|} < \infty.$$

Thus, the use of Fubini's theorem is justified and

$$\hat{\xi}_\alpha = \sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}} \hat{\xi}_{j,\alpha} = \sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}} \theta_{j,\alpha}.$$

□

Here is the lemma that lists the important properties of the omega function.

Lemma 1.4.6. *Let $\alpha \in \mathbb{R}^+$, $p \in \mathbb{R}$ and the functions $\omega_{\alpha,p}$ and Φ_p^α be given by* (1.8) *and* (1.26)*, respectively. Then the following hold:*

1. *$\widehat{\Phi_p^\alpha}(k) = \omega_{\alpha,p}(-ik)$, for $k \in \mathbb{R}$, where $\widehat{\Phi_p^\alpha}(k)$ denotes the Fourier transform of Φ_p^α. Moreover, $\omega_{\alpha,p} \in \mathcal{F}(L_1)$, where $\mathcal{F}(L_1)$ denotes the ring of Fourier transforms as in Remark B.3.2, that is, $\Phi_p^\alpha \in L_1(\mathbb{R})$.*

2. $\int_{\mathbb{R}} \Phi_p^\alpha(x)dx = 1.$

3. $\mathcal{L}\left(\Phi_p^\alpha\right)(z) = \omega_{\alpha,p}(-z)$, *for* $\mathrm{Re}(z) \le 0$ *and* $p \ge 0$, *where* $\mathcal{L}\left(\Phi_p^\alpha\right)(z)$ *denotes the Laplace transform of* Φ_p^α *given by B.6.*

Proof. 1. First note that $\widehat{\phi_{\alpha-1}}(k) = (-ik)^{-\alpha}$ by Proposition B.3.5. Consider the series given by (1.26),

$$\Phi_p^\alpha(x) = \sum_{m=0}^{\infty} \mathcal{G}_m^\alpha \phi_{\alpha-1}(x-(m-p)).$$

Observe that there are only finite number of terms in the sum for a fixed $x \in \mathbb{R}$, since for $x \le m-p$, $\phi_{\alpha-1}(x-(m-p)) = 0$ and that $\mathrm{supp}(\Phi_p^\alpha) \subset [-p,\infty) \cup \mathbb{R}^+$. Note also that each term in the series corresponds to a regular tempered distribution and the series $\sum_{m=0}^{\infty} \mathcal{G}_m^\alpha$ is absolutely convergent by (A.10), thus the distribution $\tilde{\Phi}_p^\alpha \in \mathcal{S}'$ where $\mathcal{S}'$ denotes the space of tempered distributions, see Appendix B.3. Hence, using Theorem B.3.3 we take the Fourier transform of (1.26) term by term and use the Binomial series (A.4) to obtain

$$\begin{aligned}
\widehat{\Phi_p^\alpha}(k) &= \mathcal{F}\left(\sum_{m=0}^{\infty} \mathcal{G}_m^\alpha \phi_{\alpha-1}(x-(m-p))\right) \\
&= \sum_{m=0}^{\infty} \mathcal{G}_m^\alpha \mathcal{F}\left(\phi_{\alpha-1}(x-(m-p))\right) \\
&= \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} e^{ik(m-p)} (-ik)^{-\alpha} \\
&= \frac{(1-e^{ik})^\alpha e^{-ikp}}{(-ik)^\alpha} = \omega_{\alpha,p}(-ik).
\end{aligned}$$

We now show that the omega function is the Fourier transform of an $L_1(\mathbb{R})$-function, that is, $\omega_{\alpha,p} \in \mathcal{F}(L_1)$, where $\mathcal{F}(L_1)$ is given in Remark B.3.2. Rewrite the omega function using the partition of unity given in Definition 1.4.4,

$$\begin{aligned}
\omega_{\alpha,p}(-ik) &= \sum_{j\in\mathbb{Z}} \theta_j(k)\omega_{\alpha,p}(-ik) \\
&= \theta_0(k)\omega_{\alpha,p}(-ik) + \left(1-e^{ik}\right)^\alpha e^{-ikp} \sum_{\substack{j\in\mathbb{Z} \\ j\neq 0}} \theta_j(k)(-ik)^{-\alpha}. \qquad (1.27)
\end{aligned}$$

Then, the first term belongs to $W^{2,1}(\mathbb{R})$, since

$$\mathrm{supp}(\theta_0(k)\omega_{\alpha,p}(-ik)) \subset [-R,R]$$

and $\omega_{\alpha,p}$ is analytic there by (1.12). Thus, the first term, $\theta_0(k)\omega_{\alpha,p}(-ik) \in \mathcal{F}(L_1)$ by Proposition 1.3.1.

Consider the second term and note that $\sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}} \theta_j(k)(-ik)^{-\alpha} \in \mathcal{F}(L_1)$ by Proposition 1.4.5, that is, there exists $\xi_\alpha \in L_1(\mathbb{R})$ such that

$$\sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}} \theta_j(k)(-ik)^{-\alpha} = \hat{\xi}_\alpha(k).$$

Moreover, using the Binomial series we have that

$$(1-e^{ik})^\alpha = \sum_{m=0}^{\infty} \mathcal{G}_m^\alpha e^{ikm}$$

where $\mathcal{G}_m^\alpha$ is given by (A.1). Since, the series $\sum_{m=0}^{\infty} \mathcal{G}_m^\alpha$ is absolutely convergent by (A.10) and

$$\left\| \sum_{m=0}^{\infty} \mathcal{G}_m^\alpha \xi_\alpha(\cdot - (m-p)) \right\|_{L_1} \leq \sum_{m=0}^{\infty} |\mathcal{G}_m^\alpha| \, \|\xi_\alpha\|_{L_1} < \infty,$$

the function

$$\sum_{m=0}^{\infty} \mathcal{G}_m^\alpha \xi_\alpha(x-(m-p)) \in L_1(\mathbb{R}).$$

Hence, the use of Fubini's theorem is justified below, and the second term of 1.27 is the Fourier transform of an L_1-function, since

$$\begin{aligned}\left(1-e^{ik}\right)^\alpha e^{-ikp} \sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}} \theta_j(k)(-ik)^{-\alpha} &= \sum_{m=0}^{\infty} \mathcal{G}_m^\alpha e^{ik(m-p)} \sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}} \theta_j(k)(-ik)^{-\alpha} \\ &= \sum_{m=0}^{\infty} \mathcal{G}_m^\alpha \mathcal{F}\left(\xi_\alpha(x-(m-p))\right)(k) \\ &= \mathcal{F}\left(\sum_{m=0}^{\infty} \mathcal{G}_m^\alpha \xi_\alpha(x-(m-p)) \right)(k).\end{aligned}$$

2. Let $\{k_n\} \subset \mathbb{R}$ be a decreasing sequence such that $\lim_{n\to\infty} k_n = 0$ and set $f_n(x) = e^{ik_nx}\Phi_p^\alpha(x)$. Then, since $\Phi_p^\alpha \in L_1(\mathbb{R})$, $f_n \in L_1(\mathbb{R})$ and $f_n \to \Phi_p^\alpha$. Moreover, $|f_n| \leq \left|\Phi_p^\alpha\right|$ for all $n \in \mathbb{N}$. Thus,

$$\int_{\mathbb{R}} \lim_{n\to\infty} \left(e^{ik_nx}\Phi_p^\alpha(x)\right) \mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}} e^{ik_nx}\Phi_p^\alpha(x)\,\mathrm{d}x$$

by Lebesgue's dominated convergence theorem. That is, in view of (1.11) we have shown that

$$\int_{\mathbb{R}} \Phi_p^\alpha(x)\,\mathrm{d}x = \lim_{n\to\infty} \omega_{\alpha,p}(-ik_n) = 1.$$

3. For $\mathrm{Re}(z) < 0$ the iterated integral

$$\begin{aligned}
&\sum_{m=0}^{\infty}\int_0^{\infty}\left|e^{zt}\mathcal{G}_m^{\alpha}\phi_{\alpha-1}(t-(m-p))\right|\,\mathrm{d}t\\
&=\sum_{m=0}^{\infty}|\mathcal{G}_m^{\alpha}|\int_{m-p}^{\infty}e^{\mathrm{Re}(z)t}\phi_{\alpha-1}(t-(m-p))\,\mathrm{d}t\\
&=\sum_{m=0}^{\infty}|\mathcal{G}_m^{\alpha}|\,e^{\mathrm{Re}(z)(m-p)}\int_0^{\infty}e^{\mathrm{Re}(z)\tau}\phi_{\alpha-1}(\tau))\,\mathrm{d}\tau\\
&=\sum_{m=0}^{\infty}|\mathcal{G}_m^{\alpha}|\,e^{\mathrm{Re}(z)(m-p)}\frac{1}{(-\mathrm{Re}(z))^{\alpha}},
\end{aligned}$$

is convergent in view of (A.10) where we have used (B.10) in the last line. Thus, the use of Fubini's theorem is justified and interchanging summation with integration, the Laplace transform for $\mathrm{Re}(z) < 0$ is given by,

$$\begin{aligned}
\mathcal{L}\left(\Phi_p^{\alpha}\right)(z) &= \int_0^{\infty}e^{zt}\sum_{m=0}^{\infty}\mathcal{G}_m^{\alpha}\phi_{\alpha-1}(t-(m-p))\,\mathrm{d}t\\
&=\sum_{m=0}^{\infty}\mathcal{G}_m^{\alpha}\int_0^{\infty}e^{zt}\phi_{\alpha-1}(t-(m-p))\,\mathrm{d}t\\
&=\sum_{m=0}^{\infty}(-1)^m\binom{\alpha}{m}e^{(m-p)z}(-z)^{-\alpha}=\frac{(1-e^z)^{\alpha}e^{-zp}}{(-z)^{\alpha}}=\omega_{\alpha,p}(-z),
\end{aligned}$$

where we have used (B.10), (B.7) and the Binomial series (A.4). The proof is complete using the fact that the Laplace transform of an $L_1(\mathbb{R}^+)$-function is continuous for $\mathrm{Re}(z) \le 0$.

□

Remark 1.4.7. Note that for $\alpha \ge 1$ one does not need to use a partition of unity. For $\alpha \ge 1$ we include here a simpler proof. Let $\alpha = 1$, then $\omega_{1,p}(-ik) = \left(\frac{1-e^{ik}}{-ik}\right)e^{-ikp}$ and using (B.7) in view of Proposition B.3.5,

$$\omega_{1,p}(-ik) = \frac{e^{-ikp}}{-ik} - \frac{e^{ik}e^{-ikp}}{-ik} = \mathcal{F}\left(H(x+p) - H(x+p-1)\right), \tag{1.28}$$

where the unit step function H is given by (1.25). Clearly, $H(x+p) - H(x+p-1) \in L_1(\mathbb{R})$. Let $\alpha > 1$, then using the bounds given in the proof of Lemma 1.4.2, namely, the bound in (1.20) for $|\omega_{\alpha,p}(-ik)|$ and the bound in (1.23) for $\left|\frac{d(\omega_{\alpha,p}(-ik))}{dk}\right|$, we have that

$$\|\omega_{\alpha,p}(-i\cdot)\|_{L_2}^2 \le \int_{|k|<R} C\,\mathrm{d}k + \int_{|k|\ge R}\frac{c}{|k|^{2\alpha}}\mathrm{d}k < \infty$$

and the same is true for its derivative,

$$\left\| \frac{d\left(\omega_{\alpha,p}(-i\cdot)\right)}{dk}) \right\|_{L_2}^2 \leq \int_{|k|<R} C\,\mathrm{d}k + \int_{|k|\geq R} \frac{c}{|k|^{2\alpha}}\mathrm{d}k < \infty.$$

Hence, using Proposition 1.3.1 with $r = 2$, $\omega_{\alpha,p} \in W^{2,1}(\mathbb{R})$ and we have the required result.

1.5 Construction of higher order Grünwald-type approximations

In this section we construct higher order Grünwald-type approximations for the fractional derivative operator. To start with, let us rewrite the Fourier transform of the Grünwald formula given by (1.4) as the sum of the Fourier transforms of the fractional derivative operator and the error term.

$$\begin{aligned}\widehat{(A_h^\alpha f)}(k) &= (-ik)^\alpha \omega_{\alpha,p}(-ikh)\hat{f}(k) \\ &= (-ik)^\alpha \hat{f}(k) + (-ik)^\alpha(\omega_{\alpha,p}(-ikh) - 1)\hat{f}(k) \\ &= \widehat{f^{(\alpha)}}(k) + (-ik)^\alpha(\omega_{\alpha,p}(-ikh) - 1)\hat{f}(k), \end{aligned} \tag{1.29}$$

where $\omega_{\alpha,p}(z)$ is given by (1.8). Thus, the Fourier transform of the error in the approximation of the fractional derivative operator by the Grünwald formula is given by

$$\begin{aligned}\hat{\zeta}_{h,p}(k) &= \widehat{(A_h^\alpha f)}(k) - \widehat{f^{(\alpha)}}(k) \\ &= (-ik)^\alpha(\omega_{\alpha,p}(-ikh) - 1)\hat{f}(k) \\ &= (\omega_{\alpha,p}(-ikh) - 1)\widehat{f^{(\alpha)}}(k). \end{aligned} \tag{1.30}$$

Remark 1.5.1. Recall the Taylor expansion of $\omega_{\alpha,p}$ given by (1.12), that is, there exists $R > 0, a_{p,n}^\alpha \in \mathbb{R}$ such that

$$\omega_{\alpha,p}(z) = \sum_{n=0}^{\infty} a_{p,n}^\alpha z^n \text{ for all } |z| < 2R.$$

Since $a_{p,0}^\alpha = 1$, $\omega_{\alpha,p}(z) - 1 = \sum_{n=1}^\infty a_{p,n}^\alpha z^n$ and so we refer to the shifted Grünwald formula as *first order Grünwald-type approximation* which we justify in Theorem 1.6.2. Further, recall Remark 1.2.4 where we expressed the Grünwald formula (1.3) in the multiplier notation of Appendix C.3 as

$$A_{h,p}^\alpha = T_{(-1)^{q+1}\psi_{\alpha,h,p}}$$

where $p, k \in \mathbb{R}$, $\alpha \in \mathbb{R}^+$ such that $2q-1 < \alpha < 2q+1$ for $q \in \mathbb{N}$, and

$$\psi_{\alpha,h,p}(k) = (-1)^{q+1} h^{-\alpha} e^{-ikhp}(1-e^{ikh})^{\alpha}.$$

To construct higher order approximations we combine Grünwald formulae with different weights b_j, shifts p_j and accuracy $c_j h$ in such a way that the lower order error terms cancel out. To this end, let us consider the linear combination of the error terms $\hat{\zeta}_h$ in the Fourier space. Let $N \geq 0$ and for $0 \leq j \leq N$, let $b_j, p_j \in \mathbb{R}$ with

$$\sum_{j=0}^{N} b_j = 1 \tag{1.31}$$

and $c_j > 0$, then using (1.30) we have

$$\sum_{j=0}^{N} b_j \hat{\zeta}_{c_j h, p_j}(k) = \sum_{j=0}^{N} b_j (\omega_{\alpha,p_j}(-ikc_j h) - 1) \widehat{f^{(\alpha)}}(k) = \left(\sum_{j=0}^{N} b_j \omega_{\alpha,p_j}(-ikc_j h) - 1 \right) \widehat{f^{(\alpha)}}(k).$$

To be precise, for each $1 \leq n \leq N$ we require that the Taylor coefficients $a^{\alpha}_{p_j,n}$ of $\omega_{\alpha,p_j}(z)$ satisfy

$$\sum_{j=0}^{N} b_j a^{\alpha}_{p_j,n} = 0. \tag{1.32}$$

Since, $a^{\alpha}_{p_j,0} = 1$ for each j, we have that $\sum_{j=0}^{N} b_j a^{\alpha}_{p_j,0} = \sum_{j=0}^{N} b_j = 1$. Therefore, there exist d^{α}_j such that

$$\sum_{j=0}^{N} b_j \omega_{\alpha,p_j}(c_j z) - 1 = \sum_{j=N+1}^{\infty} d^{\alpha}_j z^j$$

for $|z| < 2R$. With the above preparation, we are ready to define the higher order Grünwald-type approximation for the fractional derivative operator.

Definition 1.5.2. Let $\alpha \in \mathbb{R}^+$ such that $2q-1 < \alpha < 2q+1$ for $q \in \mathbb{N}$ and let $f \in X_{\alpha+N+\beta}(\mathbb{R})$, where $N \geq 0$, $0 < \beta \leq 1$. Moreover, for $0 \leq j \leq N$, let $b_j, p_j \in \mathbb{R}$ with $\sum_{j=0}^{N} b_j = 1$ and $c_j > 0$. Then, we define an $N+1$ (higher) order Grünwald-type approximation for the fractional derivative operator of order α by

$$\tilde{\mathcal{A}}^{\alpha}_h f := \sum_{j=0}^{N} b_j A^{\alpha}_{c_j h, p_j} f = T_{(-1)^{q+1} \sum_{j=0}^{N} b_j \psi_{\alpha,c_j h,p_j}} f. \tag{1.33}$$

The terminology used in this definition; that is, order $N+1$, will be justified in Corollary 1.6.3.

Let us rewrite the Fourier transform of the error term to facilitate the errror analysis in the next section. The Fourier transform of the error term of the first order Grünwald approximation given by (1.30) can be written as

$$\begin{aligned}
\hat{\zeta}_h(k) &= (-ik)^{\alpha}(\omega_{\alpha,p}(-ikh)-1)\hat{f}(k) \\
&= h^{\beta}\frac{(\omega_{\alpha,p}(-ikh)-1)}{(-ikh)^{\beta}}(-ik)^{\alpha+\beta}\hat{f}(k) \\
&= h^{\beta}\Psi^{\alpha}_{\beta,0,p}(kh)\widehat{f^{(\alpha+\beta)}}(k). \qquad (1.34)
\end{aligned}$$

where

$$\Psi^{\alpha}_{\beta,N,p}(k) = \frac{\omega_{\alpha,p}(-ik)-\sum_{n=0}^{N} a^{\alpha}_{p,n}(-ik)^n}{(-ik)^{N+\beta}} \qquad (1.35)$$

and $\omega_{\alpha,p}(z)$ is given by (1.8), a^{α}_j by (1.12) and $0<\beta\le 1$, $N\in\mathbb{N}$.

Let us obtain the Fourier transform of the error term of the higher order Grünwald approximation. Recall (1.31); that is, $\sum_{j=0}^{N} b_j = 1$ and for $1\le n\le N$ by (1.32), we have that $\sum_{j=0}^{N} b_j a^{\alpha}_{p_j,n} = 0$. So, in the calculations below, we are justified in replacing $\sum_{j=0}^{N} b_j$ in the second line by

$$\sum_{j=0}^{N} b_j \sum_{n=0}^{N} a^{\alpha}_{p_j,n}(-ikc_jh)^n.$$

We repeat the same argument that led to (1.34) and obtain

$$\begin{aligned}
\hat{\zeta}_h(k) &= \left(\widehat{\tilde{\mathcal{A}}^{\alpha}_h f - f^{(\alpha)}}\right)(k) = \sum_{j=0}^{N} b_j\left(\omega_{\alpha,p_j}(-ikc_jh)-1\right)(-ik)^{\alpha}\hat{f}(k) \\
&= \sum_{j=0}^{N} b_j\left(\omega_{\alpha,p_j}(-ikc_jh)-\sum_{n=0}^{N} a^{\alpha}_{p_j,n}(-ikc_jh)^n\right)(-ik)^{\alpha}\hat{f}(k) \\
&= h^{N+\beta}\sum_{j=0}^{N} b_j c_j^{N+\beta}\frac{\omega_{\alpha,p_j}(-ikc_jh)-\sum_{n=0}^{N} a^{\alpha}_{p_j,n}(-ikc_jh)^n}{(-ikc_jh)^{N+\beta}}(-ik)^{\alpha+N+\beta}\hat{f}(k) \\
&= h^{N+\beta}\sum_{j=0}^{N} b_j c_j^{N+\beta}\Psi^{\alpha}_{\beta,N,p_j}(kc_jh)\widehat{f^{(\alpha+N+\beta)}}(k), \qquad (1.36)
\end{aligned}$$

where $a^{\alpha}_{p_j,n}$ are the Taylor coefficients of ω_{α,p_j} and $\Psi^{\alpha}_{\beta,N,p_j}$ are given by (1.35).

1.6 Consistency of higher order Grünwald-type approximations

In this section we show that the higher order Grünwald-type approximations converge to fractional derivative operator in $L_1 := L_1(\mathbb{R})$. But first, we need the following important technical lemma which shows that the function $\Psi^{\alpha}_{\beta,N,p}$ given by (1.35) is the Fourier transform of an L_1-function. This, as can be seen from the Fourier transforms of the error terms given by (1.34) and (1.36), turns out to be crucial in obtaining a bound for the L_1-norm of the error term in the higher order Grünwald-type approximations for the fractional derivative operator.

Lemma 1.6.1. *Let*

$$\Psi^{\alpha}_{\beta,N,p}(k) = \frac{\omega_{\alpha,p}(-ik) - \sum_{n=0}^{N} a^{\alpha}_{p,n}(-ik)^n}{(-ik)^{N+\beta}},$$

where $\omega_{\alpha,p}(z)$ is given by (1.8), $a^{\alpha}_{p,n}$ by (1.12)*, $N \in \mathbb{N}_0$ and $0 < \beta \leq 1$. Then,*

$$\Psi^{\alpha}_{\beta,N,p} \in \mathcal{F}(L_1);$$

that is, $\Psi^{\alpha}_{\beta,N,p}$ is the Fourier transform of some $\xi^{\alpha}_{\beta,N,p} \in L_1(\mathbb{R})$ with $\operatorname{supp}(\xi^{\alpha}_{\beta,N,p}) \subset [-p,\infty) \cup \mathbb{R}^+$.

Proof. First note that, since $\omega_{\alpha,p}(z)$ is analytic for $|z| < 2R$ (see Section 1.4), and

$$\lim_{z\to 0}\left(z\left(\frac{\omega_{\alpha,p}(z) - \sum_{n=0}^{N} a^{\alpha}_{p,n} z^n}{z^{N+\beta}}\right)\right) = \lim_{z\to 0}\left(\sum_{n=N+1}^{\infty} a^{\alpha}_{p,n} z^{n+1-N-\beta}\right) = 0,$$

the singularity at the origin is removable. Hence, $\Psi^{\alpha}_{\beta,N,p}(k)$ is analytic in $[-2R, 2R]$. Moreover, note that each term of $\Psi^{\alpha}_{\beta,N,p}(k)$ is the Fourier transform of a tempered distribution satisfying the support condition in view of the proof of Part 1 of Lemma 1.4.6 and Proposition B.3.5. Thus, all that remains to show is that $\Psi^{\alpha}_{\beta,N,p}$ is the Fourier transform of an L_1 function. We divide the proof into four parts.

1. Let $N, p = 0$ and $\beta < 1$. This was proved in [106, Lemma 5] using results from summability theory. Note that $a^{\alpha}_{0,0} = 1$ and consider,

$$\Psi^{\alpha}_{\beta,0,0}(k) = \frac{\omega_{\alpha,0}(-ik) - 1}{(-ik)^{\beta}}.$$

Using the partition of unity given by Definition 1.4.4, we rewrite $\Psi^{\alpha}_{\beta,0,0}(k)$ as

$$\begin{aligned}
\Psi^{\alpha}_{\beta,0,0}(k) &= \sum_{j\in\mathbb{Z}} \theta_j(k)\Psi^{\alpha}_{\beta,0,0}(k) \\
&= \theta_0(k)\Psi^{\alpha}_{\beta,0,0}(k) + \sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}} \theta_j(k)\frac{\omega_{\alpha,0}(-ik) - 1}{(-ik)^{\beta}} \\
&= \theta_0(k)\Psi^{\alpha}_{\beta,0,0}(k) - \sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}} \theta_j(k)(-ik)^{-\beta} + \omega_{\alpha,0}(-ik)\sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}} \theta_j(k)(-ik)^{-\beta}.
\end{aligned}$$

Note that $\text{supp}(\theta_0(k)\Psi^\alpha_{\beta,0,0}(k)) \subset [-R, R]$ and $\Psi^\alpha_{\beta,0,0}(k)$ is analytic there. Thus, the first term $\theta_0(k)\Psi^\alpha_{\beta,0,0}(k) \in W^{2,1}(\mathbb{R})$ and by Proposition 1.3.1, the Fourier transform of an L_1-function. By Proposition 1.4.5, the second term is the Fourier transform of an L_1-function. The third term is the product of Fourier transforms of L_1-functions in view of Lemma 1.4.6 and thus, the Fourier transform of convolution of L_1-functions, and hence the Fourier transform of an L_1-function. Therefore, $\Psi^\alpha_{\beta,0,0} \in \mathcal{F}(L_1)$.

2. Let $N = 0$, $p \neq 0$ and $\beta < 1$. Note that $a^\alpha_{p,0} = 1$. Then, writing $\omega_{\alpha,p}(-ik) = \omega_{\alpha,0}(-ik)e^{-ikp}$ we split $\Psi^\alpha_{\beta,0,p}(k)$ as follows,
$$\Psi^\alpha_{\beta,0,p}(k) = \frac{\omega_{\alpha,p}(-ik) - 1}{(-ik)^\beta} = \frac{\omega_{\alpha,0}(-ik)e^{-ikp} - 1}{(-ik)^\beta} = e^{-ikp}\Psi_{\beta,0,0}(k) + \frac{e^{-ikp} - 1}{(-ik)^\beta}.$$
The first term is the Fourier transform of an L_1-function as a consequence of the first case in view of (B.7). To see that the second term is the Fourier transform of an L_1-function, let us rewrite using the partition of unity given in Definition 1.4.4,
$$\frac{e^{-ikp} - 1}{(-ik)^\beta} = \theta_0(k)\frac{e^{-ikp} - 1}{(-ik)^\beta} + e^{-ikp}\sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}}\theta_j(k)(-ik)^{-\beta} - \sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}}\theta_j(k)(-ik)^{-\beta}.$$
Note that $\text{supp}(\theta_0(k)\frac{e^{-ikp}-1}{(-ik)^\beta}) \subset [-R, R]$ and $\frac{e^{-ikp}-1}{(-ik)^\beta}$ is analytic there and so $\theta_0(k)\frac{e^{-ikp}-1}{(-ik)^\beta} \in W^{2,1}(\mathbb{R})$, thus, by Proposition 1.3.1, the Fourier transform of an L_1-function. The other two terms are Fourier terms of L_1-functions in view of Proposition 1.4.5 and (B.7). Hence, $\Psi^\alpha_{\beta,0,p} \in \mathcal{F}(L_1)$.

3. Let $\alpha \geq 1$ and consider the remaining two possibilities, either $N = 0$ and $\beta = 1$ or $N \geq 1$ and $0 < \beta \leq 1$. Note that Carlson-type inequality given in Proposition 1.3.1 cannot be applied directly since $\Psi^\alpha_{\beta,N,p} \notin L_2(\mathbb{R})$ for $\beta < 1/2$. Therefore, we use Corollary 1.3.3, choosing a particular partition of unity $(\theta_j)_{j\in\mathbb{Z}}$ given by Definition 1.4.4. Rewriting,
$$\Psi^\alpha_{\beta,N,p}(k) = \sum_{j\in\mathbb{Z}}\theta_j(k)\Psi^\alpha_{\beta,N,p}(k),$$
we show that $\Psi^\alpha_{\beta,N,p} \in \mathcal{F}(L_1)$.

First, for $j = 0$, as $\Psi^\alpha_{\beta,N,p}$ is analytic in $(-2R, 2R)$ we have that $\theta_0\Psi^\alpha_{\beta,N,p} \in W^{2,1}(\mathbb{R})$. For $j \neq 0$, we split the remaining series into two terms,
$$\sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}}\theta_j(k)\Psi^\alpha_{\beta,N,p} = \sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}}\theta_j(k)\frac{\omega_{\alpha,p}(-ik) - \sum_{n=0}^{N-1} a^\alpha_{p,n}(-ik)^n}{(-ik)^{N+\beta}} - a^\alpha_{p,N}\sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}}\theta_j(k)(-ik)^{-\beta}$$

Now, note that by Proposition 1.4.5 the second term is the Fourier transform of an L_1-function. Consider the first term and for convenience in calculations let us write

$$T_j(k) = \theta_j(k)\frac{\omega_{\alpha,p}(-ik) - \sum_{n=0}^{N-1} a^{\alpha}_{p,n}(-ik)^n}{(-ik)^{N+\beta}},\ j \neq 0.$$

Then, for $j \geq 2$, $\mathrm{supp}(\theta_j) \subset [2^{j-2}, 2^j]$ and for $j \leq 2$, $\mathrm{supp}(\theta_j) \subset [-2^j, -2^{j-2}]$ so that the length of the support of θ_j is less that $2^{|j|}$. Also, $\left|\theta_j'(k)\right| \leq 2^{-|j|+2}$ for each $j \neq 0$, except when $j = \pm 1$ in which case $\left|\theta_{\pm 1}'(k)\right| \leq \frac{2}{R}$. Moreover, in view of Lemma 1.4.2, note that both $\omega_{\alpha,p}$ and $\omega_{\alpha,p}'$ are bounded and so the numerator of $T_j(k)$ and its derivative are bounded. Also, recall that either $N = 0$ and $\beta = 1$ or $N \geq 1$, so in either case, the exponent in the denominator of the $T_j(k)$ is at least one. Hence, there exists C independent of j such that

$$|T_j(k)| \leq |\theta_j(k)| \left[\frac{|\omega_{\alpha,p}(-ik)|}{|k|^{N+\beta}} + \sum_{n=0}^{N-1} \frac{\left|a^{\alpha}_{p,n}\right|}{|k|^{N+\beta-n}}\right] \leq \frac{C}{|k|} \leq C2^{2-|j|}$$

and

$$\begin{aligned} |T_j'(k)| \leq \left|\theta_j'(k)\right| &\left[\frac{|\omega_{\alpha,p}(-ik)|}{|k|^{N+\beta}} + \sum_{n=0}^{N-1} \frac{\left|a^{\alpha}_{p,n}\right|}{|k|^{N+\beta-n}}\right] \\ &+ |\theta_j(k)| \left[\frac{\left|\omega_{\alpha,p}'(-ik)\right|}{|k|^{N+\beta}} + \frac{(N+\beta)\left|\omega_{\alpha,p}(-ik)\right|}{|k|^{N+\beta+1}}\right] \\ &+ |\theta_j(k)| \sum_{n=0}^{N-1} \frac{(N+\beta-n)\left|a^{\alpha}_{p,n}\right|}{|k|^{N+\beta-n+1}} \\ \leq \frac{C}{|k|}&\left(\left|\theta_j'(k)\right| + 1\right) \leq C2^{2-|j|}. \end{aligned}$$

The length of the support of T_j is less than $2^{|j|}$. Therefore,

$$\|T_j\|_{L_2}^{\frac{1}{2}} \leq C(2^{4-2|j|}2^{|j|})^{1/4} \leq C2^{-|j|/4}$$

and the same holds for $\|T_j'\|_{L_2}^{\frac{1}{2}}$. Hence,

$$\sum_{\substack{j\in\mathbb{Z} \\ j\neq 0}} \|T_j\|_{L_2}^{\frac{1}{2}} \|(T_j)'\|_{L_2}^{\frac{1}{2}} < \infty.$$

The proof of this case is complete on applying Corollary 1.3.3.

4. Let $0 < \alpha < 1$ and consider the two possibilities once again, that is, either $N = 0$ and $\beta = 1$ or $N \geq 1$ and $0 < \beta \leq 1$. The Carlson-type inequality employing a partition of unity cannot be applied for this case, since for $\alpha < 1$, $\omega'_{\alpha,p}$ is not bounded and if $\alpha \leq 1/2$, $\omega'_{\alpha,p}$ is not even locally in L_2. Instead, we use induction on N. First, let $N = 0$, $\beta = 1$. Now, using (1.13) we have

$$\begin{aligned}\omega_{\alpha,p}(-ik)\Psi^1_{1,0,0}(k) &= \omega_{\alpha,p}(-ik)\frac{\omega(-ik)-1}{-ik} \\ &= \frac{\omega_{\alpha+1,p}(-ik)-1}{-ik} - \frac{\omega_{\alpha,p}(-ik)-1}{-ik} \\ &= \Psi^{\alpha+1}_{1,0,p}(k) - \Psi^{\alpha}_{1,0,p}(k).\end{aligned}$$

Hence, $\Psi^{\alpha}_{1,0,p} \in \mathcal{F}(L_1)$, since the convolution of L_1-functions is an L_1-function for the left hand side of the equation above we have that $\omega_{\alpha,p}\Psi^1_{1,0,0} \in \mathcal{F}(L_1)$ and by case 3 above we have that $\Psi^{\alpha+1}_{1,0,p} \in \mathcal{F}(L_1)$.

To complete the proof, note that we already established that the assertion holds for $N = 0$ and $\beta < 1$ in case 1, so let us assume it holds for some N; that is, assume that $\Psi^{\alpha}_{\beta,N,p}$ is the Fourier transform of an L_1 function satisfying the support condition. Then, using the fact that the convolution of L_1 functions is an L_1 function and the fact that we established the assertion for $\alpha = 1$, we obtain that the product $\Psi^{\alpha}_{\beta,N,p}\Psi^1_{1,0,0}$ is also the Fourier transform of an L_1 function satisfying the support condition as the support of the inverse of the second factor of the product is contained in $\mathbb{R}^+$. Indeed, using Proposition B.3.5 we have

$$\begin{aligned}\Psi^1_{1,0,0}(k) = \frac{\omega_{1,0}(-ik)-1}{-ik} &= (-ik)^{-2} - e^{ik}(-ik)^{-2} - (-ik)^{-1} \\ &= \mathcal{F}\left(H(x)\phi_1(x) - H(x-1)\phi_1(x-1) - H(x)\right).\end{aligned}$$

Furthermore, using (1.13) we have

$$\begin{aligned}\Psi^{\alpha}_{\beta,N,p}(k)\Psi^1_{1,0,0}(k) &= \left(\frac{\omega_{\alpha,p}(-ik) - \sum_{n=0}^{N} a^{\alpha}_{p,n}(-ik)^n}{(-ik)^{N+\beta}}\right)\left(\frac{\omega(-ik)-1}{-ik}\right) \\ &= \frac{\omega_{\alpha+1,p}(-ik) - \omega(-ik)\sum_{n=0}^{N} a^{\alpha}_{p,n}(-ik)^n}{(-ik)^{N+1+\beta}} \\ &\quad - \frac{\omega_{\alpha,p}(-ik) - \sum_{n=0}^{N} a^{\alpha}_{p,n}(-ik)^n}{(-ik)^{N+1+\beta}} \\ &= \frac{\omega_{\alpha+1,p}(-ik) - \sum_{n=0}^{N+1} a^{\alpha+1}_{p,n}(-ik)^n}{(-ik)^{N+1+\beta}} \\ &\quad - \sum_{n=0}^{N} a^{\alpha}_{p,n}\frac{\omega(-ik) - \sum_{m=0}^{N+1-n} a^1_{0,m}(-ik)^m}{(-ik)^{N+1-n+\beta}}\end{aligned}$$

$$- \frac{\omega_{\alpha,p}(-ik) - \sum_{n=0}^{N+1} d_n(-ik)^n}{(-ik)^{N+1+\beta}}$$
$$= \Psi^{\alpha+1}_{\beta,N+1,p}(k) - \sum_{n=0}^{N} a^{\alpha}_{p,n} \Psi^{1}_{\beta,N+1-n,0} - \varPsi(k),$$

where $\omega_{\alpha,p}(z)$ and $\omega(z)$ are given by (1.8) and 1.9, respectively, $a^{\alpha+1}_{p,n}$ are the Taylor coefficients of $\omega_{\alpha+1,p}$, $a^{1}_{0,m}$ are the Taylor coefficients of ω and d_n are such that the equality holds. By case 3 above, the first two terms are Fourier transforms of L_1 functions. This implies that $\varPsi \in \mathcal{F}(L_1)$. Thus, $\varPsi$ at $k = 0$ has to be bounded and therefore $d_j = a^{\alpha}_{p,n}$ and $\varPsi = \Psi^{\alpha}_{\beta,N+1,p}$.

□

We are ready to prove the main result of this section. The following theorem and its corollary, under the assumption that $f \in L_1(\mathbb{R})$ has fractional derivative of order $\alpha + \beta$ or $\alpha + N + \beta$, respectively, not only show that the Grünwald-type approximations converge to the fractional derivative operator in the L_1-setting, but also give the convergence rate in terms of the regularity parameter β.

Theorem 1.6.2. *Let $0 \leq \beta \leq 1$, $X_\alpha(\mathbb{R})$ and $X_\alpha(\mathbb{R}^+)$ be given by Definition 1.1.1 and $\|f\|_\alpha$ be as in (1.1).*

1. *If $f \in X_{\alpha+\beta}(\mathbb{R})$, then there exists a constant $C > 0$ such that*

$$\left\|A^{\alpha}_{h,p} f - f^{(\alpha)}\right\|_{L_1(\mathbb{R})} \leq C h^{\beta} \|f\|_{\alpha+\beta}$$

   as $h \to 0+$.

2. *If $p \leq 0$ and $f \in X_{\alpha+\beta}(\mathbb{R}^+)$, then there exists a constant $C > 0$ such that*

$$\left\|A^{\alpha}_{h,p} f - f^{(\alpha)}\right\|_{L_1(\mathbb{R}^+)} \leq C h^{\beta} \|f\|_{\alpha+\beta}$$

   as $h \to 0+$.

3. *In particular when $\beta = 0$, that is, if $f \in X_\alpha$, then*

$$\left\|A^{\alpha}_{h,p} f - f^{(\alpha)}\right\|_{L_1(\mathbb{R})} \to 0$$

   as $h \to 0+$.

Proof. 1. To prove the first statement, recall that we can rewrite the Fourier transform of the error term, $\hat{\zeta}_h(k)$ as a constant multiple of the product of Fourier transforms given by (1.34),

$$\hat{\zeta}_h(k) = h^\beta \Psi^\alpha_{\beta,0,p}(kh)\widehat{f^{(\alpha+\beta)}}(k).$$

Now by assumption $f^{(\alpha+\beta)} \in L_1(\mathbb{R})$, see Definitions 1.1.1 and 1.1.2 for details. By Lemma 1.6.1, $\Psi^\alpha_{\beta,0,p}(k) = \frac{\omega_{\alpha,p}(-ik)-1}{(-ik)^\beta}$ is the Fourier transform of an L_1 function $\xi^\alpha_{\beta,0,p}$ satisfying the support condition, $\mathrm{supp}(\xi^\alpha_{\beta,N,p}) \subset [-p,\infty) \cup \mathbb{R}^+$. This implies that $\hat{\zeta}_h$ is the product of Fourier transforms of L_1-functions and so the Fourier transform of the convolution of L_1-functions and therefore the Fourier transform of an L_1-function, ζ_h. In fact, this argument of product of Fourier transforms holds true for all $f \in L_1(\mathbb{R})$ and thus $\Psi^\alpha_{\beta,0,p}(k)$ is an L_1-multiplier. Hence, using the notation of Appendix C.3 we have

$$\zeta_h = h^\beta T_{\Psi^\alpha_{\beta,0,p}(h\cdot)} f^{(\alpha+\beta)}.$$

Note that using (C.4) followed by (C.3) we have

$$\|T_{\Psi^\alpha_{\beta,0,p}(h\cdot)}\|_{\mathcal{B}(L_1(\mathbb{R}))} = \|T_{\Psi^\alpha_{\beta,0,p}(\cdot)}\|_{\mathcal{B}(L_1(\mathbb{R}))} = \left\|\xi^\alpha_{\beta,0,p}\right\|_{L_1(\mathbb{R})}.$$

Moreover, Lemma 1.6.1 implies that $\left\|\xi^\alpha_{\beta,0,p}\right\|_{L_1(\mathbb{R})} \le C$ for some constant C and hence we obtain the required norm estimate

$$\begin{aligned}
\|\zeta_h\|_{L_1(\mathbb{R})} &= \left\|h^\beta T_{\Psi^\alpha_{\beta,0,p}(h\cdot)} f^{(\alpha+\beta)}\right\|_{L_1(\mathbb{R})} \\
&\le h^\beta \|T_{\Psi^\alpha_{\beta,0,p}(h\cdot)}\|_{\mathcal{B}(L_1(\mathbb{R}))} \left\|f^{(\alpha+\beta)}\right\|_{L_1(\mathbb{R})} \\
&= h^\beta \left\|\xi^\alpha_{\beta,0,p}\right\|_{L_1(\mathbb{R})} \left\|f^{(\alpha+\beta)}\right\|_{L_1(\mathbb{R})} \\
&\le Ch^\beta \|f\|_{\alpha+\beta}.
\end{aligned}$$

2. The second statement is proved using the same calculations in view of Remark 1.2.2 where we clarified the convention for extending the functions from $L_1(\mathbb{R}^+)$ to $L_1(\mathbb{R})$ and justified that for $p \le 0$, the Grünwald formula $A^\alpha_{h,p}$ can be considered as an operator on $L_1(\mathbb{R}^+)$. Thus, extending $f \in L_1(\mathbb{R}^+)$ to the left by zero and noting that if $p \le 0$, then $\mathrm{supp}(\zeta_h) \subset \mathbb{R}^+$ the result follows by using Lemma 1.6.1 once again as in this case $\mathrm{supp}(\xi^\alpha_{\beta,N,p}) \subset \mathbb{R}^+$. A proof of this result using functional calculus techniques for the unshifted case, that is, when $p = 0$ and $f \in X_{\alpha+\beta}(\mathbb{R}^+)$ can be found in [106, Theorem 13].

3. As mentioned earlier, the case $\beta = 0$ was proved in [4, Proposition 4.9] and is included in the statement of the theorem here for the sake of completeness.

$\square$

Corollary 1.6.3. *Let $0 < \beta \leq 1$, $N \in \mathbb{N}$ and $\tilde{A}_h^\alpha$ be an $N+1$ order Grünwald approximation. Then there exists $C > 0$ such that $f \in X_{\alpha+N+\beta}(\mathbb{R})$ implies that*

$$\left\| \tilde{A}_h^\alpha f - f^{(\alpha)} \right\|_{L_1(\mathbb{R})} \leq Ch^{N+\beta} \|f\|_{\alpha+N+\beta}.$$

as $h \to 0^+$. If $p_j \leq 0$ for all $0 \leq j \leq N$ and $f \in X_{\alpha+N+\beta}(\mathbb{R}_+)$ then

$$\left\| \tilde{A}_h^\alpha f - f^{(\alpha)} \right\|_{L_1(\mathbb{R}_+)} \leq Ch^{N+\beta} \|f\|_{\alpha+N+\beta}.$$

Proof. In view of (1.36), the Fourier transform of the error term for the higher order Grünwald-type approximation is given by

$$\hat{\zeta}_h(k) = h^{N+\beta} \sum_{j=0}^{N} b_j c_j^{N+\beta} \Psi_{\beta,N,p_j}^\alpha (kc_j h) \widehat{f^{(\alpha+N+\beta)}}(k).$$

By Lemma 1.6.1,

$$\Psi(k) = \sum_{j=0}^{N} b_j c_j^{N+\beta} \Psi_{\beta,N,p_j}^\alpha (kc_j)$$

is the finite sum of Fourier transforms of L_1 functions. Thus, in the multiplier notation of Appendix C.3 we have that

$$\zeta_h = h^{N+\beta} T_{\Psi(h\cdot)} f^{(\alpha+N+\beta)}$$

Hence, as a consequence of the previous theorem and using the same arguments there in, $\hat{\zeta}_h$ is the Fourier transform of an L_1 function with

$$\|\zeta_h\|_{L_1(\mathbb{R})} \leq Ch^{N+\beta} \|f\|_{\alpha+N+\beta}.$$

The second statement follows along the same lines taking into account the support condition satisfied by ζ_h. $\square$

Chapter 2

Semigroups generated by Grünwald-type approximations

In this chapter we investigate the stability and smoothing of numerical schemes for fractional-in-space partial differential equations that employ Grünwald-type approximations for space discretisation. First, we discuss a general technical result which gives a sufficient condition for multipliers associated with difference schemes approximating fractional derivative operators to lead to stable schemes with desirable smoothing. Following that we study the first order Grünwald scheme and give an example of a second order Grünwald scheme for fractional-in-space partial differential equations. In Section 2.2 we generalise the theory to fractional powers of generators of strongly continuous (semi) groups in an abstract function space setting using the so-called *transference principle* [4]. We conclude this chapter with results of some numerical experiments.

2.1 Semigroups generated by periodic multipliers approximating fractional derivative operators

We begin this section with an important result that gives a sufficient condition for multipliers associated with the numerical scheme approximating the fractional derivative to lead to stable scheme with desirable smoothing. In error estimates, the smoothing of the numerical scheme will be used in an essential way to reduce the regularity requirements on the initial data, and obtain optimal convergence rates when considering space-time discretizations of abstract Cauchy problems with fractional derivatives or, more generally, fractional powers of operators in Section 2.2. As mentioned in Section 1.3, observe below in Theorem 2.1.1 that the spaces $L_r(\mathbb{R})$ where $r \neq 2$ are essential when dealing with $\alpha \leq \frac{1}{2}$.

Let $AC[-\pi, \pi]$ denote the space of periodic functions that are absolutely continuous and $W^{r,1}_{per}[-\pi, \pi]$ denote the Sobolev space of $L_r[-\pi, \pi]$-functions as in Definition B.1.1. We refer to Definition C.1.1 for the definitions of the space of bounded linear operators on L_1, $\mathcal{B}(L_1)$, and the operator norm $\|\cdot\|_{\mathcal{B}(L_1)}$.

Theorem 2.1.1. *Let $\alpha \in \mathbb{R}^+$ and $\psi \in AC[-\pi, \pi]$ such that the following conditions are satisfied:*

(i) $|\psi(k)| \leq C\,|k|^\alpha$ *for some* $C > 0$,

(ii) $|\psi'(k)| \leq C'\,|k|^{\alpha-1}$ *for some* $C' > 0$,

(iii) $\mathrm{Re}(\psi(k)) \leq -c\,|k|^\alpha$ *for some* $c > 0$.

Then, $\psi \in W^{r,1}_{per}[-\pi, \pi]$ *where* $r = 2$ *if* $\alpha > \frac{1}{2}$ *and* $r < \frac{1}{1-\alpha}$ *if* $\alpha \leq \frac{1}{2}$. *Moreover,*

(a) $\|T_{e^{t\psi}}\|_{\mathcal{B}(L_1)} \leq K$ *for* $t \geq 0$,

(b) $\left\|T_{\psi e^{t\psi}}\right\|_{\mathcal{B}(L_1)} \leq \frac{M}{t}$ *for* $t > 0$,

where K *and* M *depend on* c, C *and* C' *above.*

Proof. Let r, s denote the Hölder conjugates where

$$1 < r \leq 2 \quad \text{and} \quad \frac{1}{r} + \frac{1}{s} = 1. \tag{2.1}$$

In what follows, if $\alpha \leq \frac{1}{2}$ set $r < \frac{1}{1-\alpha}$, otherwise, set $r = 2$, and keep in mind that $r(\alpha - 1) > -1$. As a consequence of the assumptions,

$$\psi \in W^{r,1}_{per}[-\pi, \pi] \text{ and } e^{t\psi} \in W^{r,1}_{per}[-\pi, \pi] \text{ for } t \geq 0. \tag{2.2}$$

Thus, by Theorem 1.3.4, both ψ and $e^{t\psi}$ are L_1-multipliers. Using the formal properties of Fourier transforms one can verify that the operator associated with the periodic multiplier $e^{t\psi}$ coincides with the semigroup generated by the operator T_ψ; that is, $(T_{e^{t\psi}})_{t\geq 0} = (e^{tT_\psi})_{t\geq 0}$ and by Theorem 1.3.4,

$$\|T_{e^{t\psi}}\|_{\mathcal{B}(L_1)} \leq |a_0| + C\left\|e^{t\psi}\right\|^{\frac{1}{s}}_{L_r}\left\|(e^{t\psi})'\right\|^{\frac{1}{r}}_{L_r}. \tag{2.3}$$

Firstly,

$$|a_0| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{t\psi(k)}dk\right| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{t\,\mathrm{Re}(\psi(k))}\,\mathrm{d}k \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ct|k|^\alpha}\,\mathrm{d}k \leq 1,$$

where we have used Assumption (iii). Next, using Assumption (iii) together with the substitution $\tau = t^{\frac{1}{\alpha}}|k|$,

$$\begin{aligned}\left\|e^{t\psi}\right\|_{L_r[-\pi,\pi]}^{\frac{1}{s}} = &\left(\int_{-\pi}^{\pi}\left|e^{t\psi(k)}\right|^r dk\right)^{\frac{1}{rs}} \leq C\left(\int_{-\pi}^{\pi} e^{rt\,\mathrm{Re}(\psi(k))}\,\mathrm{d}k\right)^{\frac{1}{rs}} \\ \leq &C\left(\int_{-\pi}^{\pi} e^{-rct|k|^\alpha}\,\mathrm{d}k\right)^{\frac{1}{rs}} \leq C\left(\frac{1}{t^{\frac{1}{\alpha}}}\int_{\mathbb{R}} e^{-rc|\tau|^\alpha}\,\mathrm{d}\tau\right)^{\frac{1}{rs}} \leq Ct^{\frac{-1}{\alpha rs}}. \qquad (2.4)\end{aligned}$$

Making use of Assumptions (ii) and (iii), we have

$$\left|\frac{d}{dk}\left(e^{t\psi(k)}\right)\right|^r = \left|t\frac{d\psi(k)}{dk}e^{t\psi(k)}\right|^r \leq C\left(t^r\left|k\right|^{r(\alpha-1)}\right)e^{-rct|k|^\alpha},$$

Since $r(\alpha-1) > -1$, an application of (2.1) and the substitution $\tau = t^{\frac{1}{\alpha}}|k|$, yields

$$\left\|\left(e^{t\psi}\right)'\right\|_{L_r[-\pi,\pi]}^{\frac{1}{r}} \leq C\left(t^{\frac{r-1}{\alpha}}\int_{\mathbb{R}}\tau^{r(\alpha-1)}e^{-rc\tau^\alpha}d\tau\right)^{\frac{1}{r^2}} \leq Ct^{\frac{1}{\alpha rs}}. \qquad (2.5)$$

Thus, the proof of (a) is complete in view of (2.3), (2.4) and (2.5).

In view of (2.2), $\psi e^{t\psi} \in W^{r,1}_{per}[-\pi,\pi]$. Thus, using Theorem 1.3.4,

$$\left\|T_{\psi e^{t\psi}}\right\|_{\mathcal{B}(L_1)} \leq |a_0| + C\left\|\psi e^{t\psi}\right\|_{L_r[-\pi,\pi]}^{\frac{1}{s}}\left\|\left(\psi e^{t\psi}\right)'\right\|_{L_r[-\pi,\pi]}^{\frac{1}{r}}. \qquad (2.6)$$

Note that

$$|a_0| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|\psi(k)|\,e^{t\,\mathrm{Re}(\psi(k))}\,\mathrm{d}k \leq C\int_{-\pi}^{\pi}|k|^\alpha\,e^{-ct|k|^\alpha}\,\mathrm{d}k \leq C/t,$$

where we have used Assumptions (i) and (iii). The use of the substitution $\tau = t^{\frac{1}{\alpha}}|k|$ and the Assumptions (i) and (iii), yield

$$\left\|\psi e^{t\psi}\right\|_{L_r[-\pi,\pi]}^{\frac{1}{s}} \leq C\left(\int_{\mathbb{R}}|k|^{r\alpha}e^{-rct|k|^\alpha}\,\mathrm{d}k\right)^{\frac{1}{rs}} \leq Ct^{-\frac{1}{s}-\frac{1}{\alpha rs}}. \qquad (2.7)$$

We also have by virtue of (2.5) and the three assumptions,

$$\begin{aligned}\left|\frac{d}{dk}(\psi e^{t\psi})(k)\right|^r &= \left|\psi(k)\frac{d}{dk}\left(e^{t\psi(k)}\right) + \frac{d\psi(k)}{dk}e^{t\psi(k)}\right|^r \\ &\leq 2^{r-1}\left(t^r\left|\psi(k)\right|^r + 1\right)\left|(\psi(k))'\right|^r\left|e^{t\psi(k)}\right|^r \\ &\leq C(|k|^{r(\alpha-1)} + t^r|k|^{r(2\alpha-1)})e^{-rct|k|^\alpha}.\end{aligned}$$

Thus, using (2.1) and the substitution $\tau = t^{\frac{1}{\alpha}}|k|$, and noting that

$$r(2\alpha-1) > r(\alpha-1) > -1,$$

$$\begin{aligned}\left\|\frac{d}{dk}(\psi e^{t\psi})\right\|_{L_r[-\pi,\pi]}^{\frac{1}{r}} \leq &Ct^{-\frac{1}{r}+\frac{1}{\alpha rs}}\left(\int_{\mathbb{R}}(|\tau|^{r(\alpha-1)} + |\tau|^{r(2\alpha-1)})e^{-rc\tau^\alpha}\,\mathrm{d}\tau\right)^{\frac{1}{r^2}} \\ \leq &Ct^{-\frac{1}{r}+\frac{1}{\alpha rs}}, \qquad (2.8)\end{aligned}$$

and the proof of (b) is complete in view of (2.1), (2.6), (2.7) and (2.8). □

2.1.1 First-order Grünwald-type approximation

Let us consider the multiplier associated with the shifted Grünwald formula given in Remarks 1.2.4 and 1.5.1,

$$\psi_{\alpha,h,p}(k) = (-1)^{q+1}h^{-\alpha}e^{-ikhp}(1-e^{ikh})^{\alpha} = (-1)^{q+1}\omega_{\alpha,p}(-ikh)(-ik)^{\alpha}, \tag{2.9}$$

where $p, k \in \mathbb{R}$, $h, \alpha \in \mathbb{R}^{+}$ such that $2q-1 < \alpha < 2q+1$, $q \in \mathbb{N}$, and $\omega_{\alpha,p}(z)$ is given by (1.8). Note that,

$$\psi_{\alpha,h,p}(k) = h^{-\alpha}\psi_{\alpha,1,p}(kh). \tag{2.10}$$

We first show that the range of the multiplier associated with the shifted Grünwald formula is completely contained in a half-plane if and only if the (integer) shift is optimal.

Proposition 2.1.2. *Let $\psi_{\alpha,h,p}$ be given by* (2.9) *with shift $p \in \mathbb{Z}$, $k \in \mathbb{R}$, $h, \alpha \in \mathbb{R}^{+}$ such that $2q-1 < \alpha < 2q+1$, $q \in \mathbb{N}$. Then*

(a) $\psi_{\alpha,h,p}$ satisfies Assumptions (i) and (ii) of Theorem 2.1.1 with C, C' independent of h.

(b) $\mathrm{Re}(\psi_{\alpha,h,p})$ *does not change sign if and only if $\left|p-\frac{\alpha}{2}\right| < \frac{1}{2}$ if and only if $\psi_{\alpha,h,p}$ satisfies the Assumption (iii) of Theorem 2.1.1 with c independent of h.*

Proof. First, let us recall (1.17) and (1.18); that is, for $k \in [-\pi, \pi]$,

$$|\omega_{\alpha,p}(-ikh)| \le C \text{ and } \left|\omega'_{\alpha,p}(-ikh)\right| \le C.$$

Thus, in view of (2.9), for $k \in [-\pi, \pi]$,

$$|\psi_{\alpha,h,p}(k)| \le C\,|k|^{\alpha} \tag{2.11}$$

and

$$\left|\frac{d\psi_{\alpha,h,p}(k)}{dk}\right| = h^{-\alpha}\left|-\alpha ih(-ikh)^{\alpha-1}\omega_{\alpha,p}(-ikh) + (-ikh)^{\alpha}\omega'_{\alpha,p}(-ikh)\right| \le C'|k|^{\alpha-1}, \tag{2.12}$$

for some $C, C' > 0$ independent of h. This completes the proof of Statement (a).

For the proof of Statement (b), it is sufficient to consider $h = 1$ in view of (2.10) and so let $\psi := \psi_{\alpha,1,p}$. Moreover, since ψ is 2π-periodic and $\psi(k) = \overline{\psi(-k)}$, it is also sufficient to consider $k \in [0, \pi]$. Rewrite $\psi(k)$ as follows,

$$\psi(k) = (-1)^{q+1}e^{-ipk}\left(e^{\frac{ik}{2}}(e^{\frac{-ik}{2}} - e^{\frac{ik}{2}})\right)^{\alpha} = (-1)^{q+1}e^{i(\frac{\alpha}{2}-p)k}\left(-2i\sin\left(\frac{k}{2}\right)\right)^{\alpha}.$$

Using the fact that for $x \geq 0$, $(-ix)^\alpha = x^\alpha e^{-i\alpha\frac{\pi}{2}}$, see Remark B.3.4, we obtain

$$\psi(k) = (-1)^{q+1} 2^\alpha \sin^\alpha\left(\frac{k}{2}\right) e^{i\left((\frac{\alpha}{2}-p)k-\frac{\alpha\pi}{2}\right)}; \quad 0 \leq k \leq \pi.$$

Therefore,

$$\begin{aligned}\mathrm{Re}\,(\psi(k)) =&(-1)^{q+1} 2^\alpha \sin^\alpha\left(\frac{k}{2}\right)\cos\left((\frac{\alpha}{2}-p)k-\frac{\alpha\pi}{2}\right)\\ =&(-1)^{q+1-p} 2^\alpha \sin^\alpha\left(\frac{k}{2}\right)\cos\left((\frac{\alpha}{2}-p)(k-\pi)\right),\end{aligned} \tag{2.13}$$

where we have used the fact that for $p \in \mathbb{N}$, $\cos(\theta) = (-1)^p \cos(\theta \pm p\pi)$. Clearly, as $0 \leq k \leq \pi$, in view of (2.13), $\mathrm{Re}\,(\psi(k))$ changes sign if and only if $\left|\frac{\alpha}{2}-p\right| > \frac{1}{2}$. Note that by assumption $\alpha \neq 2q-1$ for $q \in \mathbb{N}$ and so $|\frac{\alpha}{2}-p| \neq \frac{1}{2}$. Furthermore, Assumption (iii) of Theorem 2.1.1 implies that there is no sign change.

To complete the proof, all that remains to be shown is that $\left|p-\frac{\alpha}{2}\right| < \frac{1}{2}$ implies that $\psi_{\alpha,h,p}$ satisfies Assumption (iii) of Theorem 2.1.1. To this end, first note the fact that c is independent of h follows from (2.10). Further, note that if $2q-1 < \alpha < 2q+1$, then $\left|p-\frac{\alpha}{2}\right| < \frac{1}{2}$ implies that $p = q$. For $0 \leq x \leq \pi$, $\sin(x/2) \geq x/\pi$ and $\cos((\frac{\alpha}{2}-p)(x-\pi)) \geq \cos(-(\frac{\alpha}{2}-p)\pi)$. Thus,

$$\begin{aligned}\mathrm{Re}\,(\psi(k)) =& -2^\alpha \sin^\alpha\left(\frac{k}{2}\right)\cos\left((\frac{\alpha}{2}-p)(k-\pi)\right)\\ \leq& -2^\alpha\left(\frac{k}{\pi}\right)^\alpha \cos\left((\frac{\alpha}{2}-p)\pi\right) = -k^\alpha 2^\alpha \cos\left((\frac{\alpha}{2}-p)\pi\right)/\pi^\alpha.\end{aligned} \tag{2.14}$$

This completes the proof of Statement (b). □

Remark 2.1.3. Let us note in passing that $\left(p-\frac{\alpha}{2}\right)$ is the coefficient of z in the Taylor expansion of the omega function given in Lemma 1.4.2.

Next, we show that with the optimal shift; that is, $p = q$, the operators T_{ψ_h} generate strongly continuous semigroups on $L_1(\mathbb{R})$, and in the case when $p = 0$; that is, $0 < \alpha < 1$, on $L_1(\mathbb{R}^+)$, that are bounded uniformly in h. Recall, in particular for an L_1-multiplier, that the range of ψ_h is always contained in the spectrum of the associated operator T_{ψ_h}, [1, Lemma 8.1.1]. This, in view of Proposition 2.1.2 and Definition C.2.6, implies that, if the (integer) shift is not optimal, then the operator T_ψ cannot be sectorial. Hence, in view of Theorem C.2.8, the semigroups $(T_{e^{t\psi}})_{t\geq 0}$ generated by T_{ψ_h} will not be uniformly bounded. On the other hand, in those cases when the shift is optimal, we in fact show that the semigroups $(T_{e^{t\psi}})_{t\geq 0}$ generated by T_{ψ_h} are uniformly analytic in h; that is, there exists $M > 0$ such that the uniform estimate $\|T_{\psi_h e^{t\psi_h}}\|_{\mathcal{B}(L_1)} \leq Mt^{-1}$ holds for $t, h > 0$. As it turns out, this fact will have significance when proving error estimates for Grünwald-type numerical schemes for abstract Cauchy problems with fractional derivatives or, more generally, fractional powers of operators in Section 2.2.

Theorem 2.1.4. *Let $\alpha \in \mathbb{R}^+$, $2p-1 < \alpha < 2p+1$, $p \in \mathbb{N}$, and consider the Grünwald multiplier,*

$$\psi_h(k) = (-1)^{p+1} h^{-\alpha} e^{-ikhp} \left(1 - e^{ikh}\right)^\alpha .$$

Then the following hold:

(a) $\{T_{e^{t\psi_h}}\}_{t\geq 0}$ are strongly continuous semigroups on $L_1(\mathbb{R})$ that are bounded uniformly in $h > 0$ and $t \geq 0$. In particular, if $1 < \alpha < 2$, then $\{T_{e^{t\psi_h}}\}_{t\geq 0}$ is a positive contraction semigroup on $L_1(\mathbb{R})$ and for $0 < \alpha < 1$, on $L_1(\mathbb{R}^+)$.

(b) The semigroups $\{T_{e^{t\psi_h}}\}_{t\geq 0}$ are uniformly analytic in $h > 0$; that is, there exists $M > 0$ such that the uniform estimate $\|T_{\psi_h e^{t\psi_h}}\|_{\mathcal{B}(L_1)} \leq Mt^{-1}$ holds for $t, h > 0$.

Proof. Proof of (a): To begin, note that $\psi_h(k) = \psi_{\alpha,h,p}(k)$ with $q = p$ given by (2.9), also consult Remark 1.2.4. We only have to show that $\|T_{e^{t\psi_h}}\|_{\mathcal{B}(L_1)} \leq K$, for all $t \geq 0$ and $h > 0$, for some $K \geq 1$ and strong continuity follows by Theorem C.3.6. Furthermore, it is enough to consider $h = 1$ in view of (C.4) and (2.10), so let $\psi := \psi_1$. Consider, either $0 < \alpha < 1$ or $1 < \alpha < 2$ so that the optimal shift $p = 0$ or 1, respectively. Taking Remark 1.2.4 into account, we have

$$\begin{aligned}(T_\psi f)(x) &= (-1)^{p+1}(\sum_{m=0}^{\infty}(-1)^m \binom{\alpha}{m} f(x-(m-p))) \\ &= -\binom{\alpha}{p} f(x) + (-1)^{p+1} \sum_{m=0, m\neq p}^{\infty} (-1)^m \binom{\alpha}{m} f(x-(m-p)) \\ &= (-\binom{\alpha}{p} I\, f)(x) + (T_{\tilde{\psi}} f)(x).\end{aligned}$$

Since $(-1)^{p+1}(-1)^m\binom{\alpha}{m} \geq 0$, for $m \neq p$, it follows that $T_{\tilde{\psi}}$ is a positive operator on $L_1(\mathbb{R})$ (or, $L_1(\mathbb{R}_+)$ for $0 < \alpha < 1$ by recalling Remark 1.2.2 and so is $e^{tT_{\tilde{\psi}}} = T_{e^{t\tilde{\psi}}}$. Therefore, noting the fact that $\sum_{m=0}^{\infty}(-1)^m\binom{\alpha}{m} = 0$,

$$T_{e^{t\psi}} = e^{tT_\psi} = e^{t(-\binom{\alpha}{p} I + T_{\tilde{\psi}})} = e^{-\binom{\alpha}{p} t} e^{tT_{\tilde{\psi}}} \geq 0,$$

and

$$\|T_{e^{t\psi}}\|_{L_1} \leq e^{-\binom{\alpha}{p}t} e^{t\|T_{\tilde{\psi}}\|_{L_1}} = e^{-\binom{\alpha}{p}t} e^{(-1)^{p+1}\sum_{m=0,m\neq p}^{\infty}(-1)^m\binom{\alpha}{m}t} = 1.$$

Let now $\alpha > 2$, then by Proposition 2.1.2, ψ_h satisfies the hypothesis of Theorem 2.1.1 and the proof of (a) is complete.

Proof of (b): Let $\alpha > 0$, then the statement follows from Theorem 2.1.1 in view of Proposition 2.1.2 and Theorem C.2.8, since $\mathrm{rg}(T(t)) \subset \mathcal{D}(T_\psi) = L_1(\mathbb{R})$ for all $t > 0$. $\square$

2.1.2 Examples of second order stable Grünwald-type approximations

Let $\alpha \in \mathbb{R}^+,\ 2q-1 < \alpha < 2q+1,$ and $q \in \mathbb{N}$. Consider the mixture of multipliers associated with Grünwald formulae yielding a second order approximation,

$$\phi_h(k) := a\psi_{\alpha,h,p_1}(k) + (1-a)\psi_{\alpha,2h,p_2}(k),$$

where $\psi_{\alpha,h,p}(k) = (-1)^{q+1}h^{-\alpha}e^{-ipkh}(1-e^{ikh})^\alpha$, the multiplier associated with the p-shifted Grünwald formula given by (2.9). Moreover, in view of Definition 1.5.2 we also write

$$\tilde{\mathcal{A}}_h^\alpha = (-1)^{q+1}T_{\phi_h}.$$

It is worth noting here that there are many combinations of $\alpha, a, p_1,$ and p_2 that would yield a second order approximation; however, only some are stable. A simple calculation, in view of (1.31) and (1.32), verifies the fact that if $0 < \alpha < 1$, then $a = 2, p_1 = p_2 = 0$ give a second order approximation. Similarly, for $1 < \alpha < 2$, $a = 2 - \frac{2}{\alpha},\ p_1 = 1,\ p_2 = \frac{1}{2}$ give a second order approximation. Therefore, we focus on the stability of these second order schemes.

Proposition 2.1.5. *Let $\phi_h(k)$ be as above, where $a = 2,\ p_1 = p_2 = 0$, if $0 < \alpha < 1$ and $a = 2 - \frac{2}{\alpha},\ p_1 = 1,\ p_2 = \frac{1}{2}$, if $1 < \alpha < 2$. Then ϕ_h satisfies the assumptions of Theorem 2.1.1 with constants c, C and C' independent of h.*

Proof. Assumptions (i) and (ii) of Theorem 2.1.1 are clearly satisfied in view of (2.11) and (2.12) as they hold for any $p \in \mathbb{R}$. Let us show that Assumption (iii) of Theorem 2.1.1 is satisfied as well.

Note that

$$\phi_h(k) = h^{-\alpha}\phi_1(hk).$$

Thus, it is sufficient to consider the case $h = 1$, so let $\phi := \phi_1$. That is,

$$\phi(k) = \phi_1(k) = (-1)^{q+1}\left(ae^{-ip_1k}(1-e^{ik})^\alpha + (1-a)2^{-\alpha}e^{-i2p_2k}(1-e^{i2k})^\alpha\right)$$

and note that ϕ is 2π-periodic. By symmetry, we only need to consider the real part for $0 \le k \le \pi$, and so using (2.13) and the double angle formula we have

$$\mathrm{Re}(\phi(k)) = (-1)^{q+1} 2^\alpha \sin^\alpha\left(\frac{k}{2}\right)\left(a\cos A + (1-a)\cos^\alpha\left(\frac{k}{2}\right)\cos B\right),$$

where $A = (\frac{\alpha}{2} - p_1)k - \frac{\alpha\pi}{2}$ and $B = (\alpha - 2p_2)k - \frac{\alpha\pi}{2}$.

We consider for $0 \le k \le \pi$, the function

$$F(k) = (-1)^{q+1}\left(a\cos A + (1-a)\cos B\right)$$

and show that $F(k) \le F(\pi) < 0$. Then, since $a > 0$ and p_1 is the optimal shift, if $\mathrm{Re}\left((1-a)\psi_h^{p_2}(k)\right) < 0$, by (2.14),

$$\mathrm{Re}(\phi(k)) \le -ak^\alpha 2^\alpha \cos\left((\frac{\alpha}{2} - p)\pi\right)/\pi^\alpha.$$

If $\mathrm{Re}\left((1-a)\psi_h^{p_2}(k)\right) > 0$, we will have the estimate

$$\mathrm{Re}(\phi(k)) \le F(\pi)2^\alpha \sin^\alpha\left(\frac{k}{2}\right) \le k^\alpha F(\pi)2^\alpha/\pi^\alpha.$$

An easy check shows that $F(\pi) < 0$. It remains to show that $F'(k) > 0$. Now, as in both cases $2p_1 = p_2$ and $(1-a)(\alpha - 2p_2) = -a\left(\frac{\alpha}{2} - p_1\right)$,

$$\begin{aligned} F'(k) =& (-1)^{q+1}\left(-a\left(\frac{\alpha}{2} - p_1\right)\sin(A) - (1-a)\left(\alpha - 2p_2\right)\sin(B)\right) \\ =& C(\alpha)\left(\sin(A) - \sin(B)\right) = 2C(\alpha)\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \end{aligned}$$

where $C(\alpha) = -a(\frac{\alpha}{2} - p_1)$, $\frac{A+B}{2} = (\frac{3\alpha}{4} - p_1)k - \frac{\alpha\pi}{2}$, and $\frac{A-B}{2} = \frac{-\alpha k}{4}$. Checking the range of the arguments, the cosine factor is negative if $\alpha > 1$ and positive if $\alpha < 1$, the sine factor is always negative, and hence $F'(k) > 0$. ☐

As a consequence, the bounds for the next result obtained from Theorem 2.1.1 are independent of h, and thus we have the following result on the stability and smoothing of the second order schemes.

Theorem 2.1.6. *Let $\phi_h(k)$ and a be as in Proposition 2.1.5. If $0 < \alpha < 1$ and $a = 2$, or if $1 < \alpha < 2$ and $a = 2 - \frac{2}{\alpha}$, then $\{T_{e^{t\phi_h}}\}_{t \ge 0}$ are semigroups on $L_1(\mathbb{R}^+)$ or $L_1(\mathbb{R})$, respectively, that are uniformly bounded in h and t, strongly continuous, and uniformly analytic in h.*

Proof. The statement follows from Theorem 2.1.1 in view of Proposition 2.1.5 and Theorem C.2.8. ☐

2.2 Application to fractional powers of operators

Let X be a Banach space and $-A$ be the generator of a strongly continuous group of bounded linear operators $\{G(t)\}_{t\in\mathbb{R}}$ on X with $\|G(t)\|_{\mathcal{B}(X)} \le M$ for all $t \in \mathbb{R}$ for some $M \ge 1$. If μ is a bounded Borel measure on $\mathbb{R}$ and if we set $\psi(z) := \hat{\mu}(z) = \int_{-\infty}^{\infty} e^{zs}\, d\mu(s)$, $(z = ik)$, we may define the bounded linear operator

$$\psi(-A)x := \int_{\mathbb{R}} G(s)x \,\mathrm{d}\mu(s), \; x \in X. \tag{2.15}$$

It is well known that the map $\psi \to \psi(-A)$ is an algebra homomorphism and is called the Hille-Phillips functional calculus, see for example, [55]. That is, if $\psi = \hat{\mu}$ and $\phi = \hat{\nu}$, for some bounded Borel measures μ and ν, then $(\phi + \psi)(-A) = \phi(-A) + \psi(-A)$, $(\phi \cdot \psi)(-A) = \phi(-A)\psi(-A)$ and $(c\phi)(-A) = c\phi(-A)$, $c \in \mathbb{C}$. A simple transference principle shows, see e.g. [4, Theorem 3.1], that,

$$\|\psi(-A)\|_{\mathcal{B}(X)} \le M \|T_{k\mapsto\psi(ik)}\|_{\mathcal{B}(L_1(\mathbb{R}))}. \tag{2.16}$$

Note that if $\operatorname{supp}\mu \subset \mathbb{R}^+$, then we may take $-A$ to be the generator of a strongly continuous semigroup and the properties of the Hille-Phillips functional calculus (2.15) and the transference principle (2.16) still holds.

Let $2p - 1 < \alpha < 2p + 1$, $\alpha \in \mathbb{R}^+$, $p \in \mathbb{N}$ and $\{\mu_t\}_{t\ge 0}$ be the family of Borel measures on $\mathbb{R}$ such that $\hat{\mu_t}(z) = e^{t(-1)^{p+1}(-z)^{\alpha}}$, $(z = ik)$. Then the operator family given by

$$S_\alpha(t)x := \int_{\mathbb{R}} G(s)x \,\mathrm{d}\mu_t(s), \; x \in X, t \ge 0,$$

is a uniformly bounded (analytic) semigroup of bounded linear operators on X, see [4, Theorems 4.1 and 4.6] for the group case. In case $0 < \alpha < 1$, we have $\operatorname{supp}\mu_t \subset \mathbb{R}^+$ and hence $-A$ is allowed to be a semigroup generator and G to be a strongly continuous semigroup and the analyticity of S_α holds, see [11] and [108]. The fractional power A^α of A is then defined to be the generator of S_α multiplied by $(-1)^{p+1}$. We note that the fractional power of A may be defined via an unbounded functional calculus for group generators (or, semigroup generators), formally given by $f_\alpha(-A)$, where $f_\alpha(z) = (-z)^\alpha$. This coincides with the definition given here for groups and, in case $0 < \alpha < 1$, for semigroups, see [4] and [11] for more details. Thus, for the additional case of $-A$ being a semigroup generator and $\alpha > 1$, we just set $A^\alpha = f_\alpha(-A)$ as in [11].

The following theorem shows the rate of convergence for the Grünwald formula approximating fractional powers of operators in this general setting. The extension to cases beyond the shift group is useful in several applications, for instance, applications in hydrology that employ the flow group [4, 10]. For the sake of notational simplicity we only give the first order version; the higher order version follows exactly along the same lines and is discussed in Corollary 2.2.3.

Theorem 2.2.1. *Let X be a Banach space and $p \in \mathbb{N}$. Assume that $-A$ is the generator of a strongly continuous group, in case $p = 0$, semigroup, of uniformly bounded linear operators $\{G(t)\}_{t\in\mathbb{R}}$ on X. Define*

$$\Phi^p_{\alpha,h} x = h^{-\alpha} \sum_{m=0}^{\infty} \mathcal{G}^\alpha_m G((m-p)h)x = (-1)^{q+1}\psi_{\alpha,h,p}(-A)x, \; x \in X,$$

where $\mathcal{G}^\alpha_m = (-1)^m\binom{\alpha}{m}$, $\psi_{\alpha,h,p}(z)$ is given by (1.5) and $\alpha \in \mathbb{R}^+$ such that $2q-1 < \alpha < 2q+1, \; q \in \mathbb{N}$.

Then, as $h > 0$, we have

$$\|\Phi^p_{\alpha,h} x - A^\alpha x\| \leq Ch\|A^{\alpha+1}x\|, \; x \in \mathcal{D}(A^{\alpha+1}). \tag{2.17}$$

Furthermore, if $p = q$, then $(-1)^{p+1}\Phi^p_{\alpha,h}$ generate $\{S^p_{\alpha,h}(t)\}_{t\geq 0}$, strongly continuous semigroups of linear operators on X that are uniformly bounded in $h > 0$ and $t \geq 0$ and uniformly analytic in $h > 0$; that is, there is $M > 0$ such that $\|S^p_{\alpha,h}(t)\|_{\mathcal{B}(X)} \leq M$ and $\|\Phi^p_{\alpha,h} S^p_{\alpha,h}(t)\|_{\mathcal{B}(X)} \leq Mt^{-1}$ for all $t, h > 0$.

Proof. If $\hat{g}$ is defined by $\hat{g}(z) = \frac{\omega_{\alpha,p}(-z)-1}{-z}$, $\operatorname{Re} z \leq 0$, then by Lemma 1.6.1 with $N = 0$ and $\beta = 1$ and (C.4) we have that

$$\|T_{k\mapsto h\hat{g}(ikh)}\|_{\mathcal{B}(L_1(\mathbb{R}))} \leq Ch.$$

In case $p = 0$, we have that $\operatorname{supp}(g) \subset \mathbb{R}^+$ and hence,

$$\|T_{k\mapsto h\hat{g}(ikh)}\|_{\mathcal{B}(L_1(\mathbb{R}^+))} \leq Ch.$$

Therefore, by the transference estimate (2.16),

$$\|h\hat{g}(-hA)\|_{\mathcal{B}(X)} \leq Ch$$

for some $C > 0$. Thus, if $x \in \mathcal{D}(A^{\alpha+1})$, then

$$\|h\hat{g}(-hA)A^{\alpha+1}x\| \leq Ch\|A^{\alpha+1}x\|.$$

Using the unbounded functional calculus developed in [4] (in case $p = 0$ see [11]), we have, for $x \in \mathcal{D}(A^{\alpha+1})$,

$$h\hat{g}(-hA)A^{\alpha+1}x = \left[h\hat{g}(hz)(-z)^{\alpha+1}\Big|_{z=-A}\right] x$$

$$= \left[\left(h^{-\alpha} \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} e^{(m-p)h(z)} - (-z)^\alpha \right) \Big|_{z=-A} \right] x$$
$$= \Phi^p_{\alpha,h} x - A^\alpha x$$

and the proof of (2.17) is complete.

The strong continuity of $S^p_{\alpha,h}$ follow from Theorem 2.1.4 and [4, Theorem 4.1], where the latter theorem establishes the transference of strong continuity, from $L_1(\mathbb{R})$ to a general Banach space X (see, [11, Theorem 5.1] for the same result in the unilateral case). Finally, the operator norm estimates follow from the L_1-norm estimates in Theorem 2.1.4 and the transference estimate (2.16), noting that by the functional calculus of [4] and [11] it follows that $(\psi_{\alpha,h,p} e^{t\psi_{\alpha,h,p}})(-A) = (-1)^{p+1} \Phi^p_{\alpha,h} S^p_{\alpha,h}(t)$ for $t > 0$ where $\psi_{\alpha,h,p}$ is given by (1.5) and $p = q$. □

The stability and consistency estimates of Theorem 2.2.1 allow us to obtain unconditionally convergent numerical schemes for the associated Cauchy problem in the abstract setting together with error estimates. To demonstrate this, we use the optimally shifted first order Grünwald scheme as "spatial" approximation together with a first order scheme for time stepping, the Backward (Implicit) Euler scheme, to match the spatial order. Let $2p-1 < \alpha < 2p+1,\ \alpha \in \mathbb{R}^+,\ p \in \mathbb{N}$, X be a Banach space and $-A$ be the generator of a uniformly bounded strongly continuous group (semigroup if $p = 0$) of operators on X and set $A_\alpha := (-1)^{p+1} A^\alpha$. Consider the abstract Cauchy problem

$$\dot{u}(t) = A_\alpha u(t);\ u(0) = x,$$

with solution operator $\{S_\alpha(t)\}_{t \geq 0}$, where, as we already mentioned, S_α is a uniformly bounded analytic semigroup as shown in [4, Theorem 4.6] and in [108] for $0 < \alpha < 1$; that is, when $-A$ is a semigroup generator. For its numerical approximation set

$$\frac{u_{n+1} - u_n}{\tau} = (-1)^{p+1} \Phi^p_{\alpha,h} u_{n+1};\ u_0 = x, n = 0, 1, 2, ...;$$

that is, with $A_{\alpha,h} := (-1)^{p+1} \Phi^p_{\alpha,h}$,

$$u_n = (I - \tau A_{\alpha,h})^{-n} x, n = 1, 2, ...$$

We have the following smooth data error estimate.

Theorem 2.2.2. *Let* $2p-1 < \alpha < 2p+1,\ \alpha \in \mathbb{R}^+,\ p \in \mathbb{N},\ n \in \mathbb{N}$, *and* $0 < \varepsilon \leq 1$. *Let* X *be a Banach space and* $-A$ *be the generator of a uniformly bounded strongly continuous group (semigroup if* $p = 0$*) of operators on* X *and set* $t = n\tau$. *If* $x \in \mathcal{D}(A^{1+\varepsilon})$, *then*

$$\|S_\alpha(t)x - u_n\| \leq C(n^{-1}\|x\| + h\frac{\alpha t^{\frac{\varepsilon}{\alpha}}}{\varepsilon}\|A^{1+\varepsilon}x\|),\ n = 1, 2, ...; t > 0, \tag{2.18}$$

and, if $x \in \mathcal{D}(A)$, then

$$\|S_\alpha(t)x - u_n\| \le C(n^{-1}\|x\| + (1+\alpha)h\left|\log\frac{t}{h^\alpha}\right| \|Ax\|), \;\; n = 1,2,...; t > 0. \tag{2.19}$$

Proof. To show (2.18), we split the error as

$$S_\alpha(t)x - u_n = S_\alpha(t)x - S^p_{\alpha,h}(t)x + S^p_{\alpha,h}(t)x - u_n := e_1 + e_2.$$

It was shown in Theorem 2.2.1 that $S^p_{\alpha,h}$ are bounded analytic semigroups on X, uniformly in h. Therefore, $\|e_2\| \le Cn^{-1}\|x\|,\; n \in \mathbb{N}$, as shown in [30], with C independent of h and t. To bound e_1 we use the fact that all operators appearing commute being functions of A, to write

$$e_1 = S_\alpha(t)x - S^p_{\alpha,h}(t)x = \int_0^t (A_\alpha - A_{\alpha,h})S_\alpha(r)S^p_{\alpha,h}(t-r)x\,\mathrm{d}r \tag{2.20}$$

Note that the analyticity of the semigroup S_α implies that there is a constant M such that for $0 \le \varepsilon \le 1$, the estimate $\|A_\alpha^{1-\varepsilon}S_\alpha(t)\| \le Mt^{\varepsilon-1}$ holds for all $t > 0$. Then, by Theorem 2.2.1,

$$\begin{aligned}\|e_1\| &\le Ch\int_0^t \|A^{\alpha+1}S_\alpha(r)S^p_{\alpha,h}(t-r)x\|\,\mathrm{d}r \\ &= Ch\int_0^t \|A_\alpha^{1-\frac{\varepsilon}{\alpha}}S_\alpha(r)S^p_{\alpha,h}(t-r)A^{1+\varepsilon}x\|\,\mathrm{d}r \le Ch\frac{\alpha t^{\frac{\varepsilon}{\alpha}}}{\varepsilon}\|A^{1+\varepsilon}x\|,\end{aligned}$$

which completes the proof of (2.18).

To show (2.19), write e_1 in (2.20) as

$$e_1 = \int_0^{h^\alpha}(A_\alpha - A_{\alpha,h})S_\alpha(r)S^p_{\alpha,h}(t-r)x\,\mathrm{d}r + \int_{h^\alpha}^t (A_\alpha - A_{\alpha,h})S_\alpha(r)S^p_{\alpha,h}(t-r)x\,\mathrm{d}r$$

It is already known that $A_{\alpha,h}x \to A_\alpha x$ as $h \to 0+$ for all $x \in \mathcal{D}(A_\alpha)$, see [4, Proposition 4.9] and [106], and hence we have stability $\|A_{\alpha,h}x - A_\alpha x\| \le C\|A_\alpha x\|$ for all $x \in \mathcal{D}(A_\alpha)$. Therefore,

$$\begin{aligned}\|e_1\| \le& C\int_0^{h^\alpha}\|A_\alpha^{1-\frac{1}{\alpha}}S_\alpha(r)Ax\|\,\mathrm{d}r + Ch\left|\int_{h^\alpha}^t \|A_\alpha S_\alpha(r)Ax\|\,\mathrm{d}r\right| \\ \le& C(h\alpha + h\left|\log\frac{t}{h^\alpha}\right|)\|Ax\|.\end{aligned}$$

□

Note that the condition $x \in \mathcal{D}(A^{1+\varepsilon})$ in (2.18) might be hard to check for $\epsilon \neq 1$, depending on A and the Banach space X. However, one can always use $\varepsilon = 1$ as $\mathcal{D}(A^2)$ is usually quite explicit. We also obtain convergence and error estimates of stable higher order schemes (such as the second order Grünwald formulae introduced in Section 2.1.2).

Corollary 2.2.3. *Let $\alpha \in \mathbb{R}^+$ with $2q-1 < \alpha < 2q+1$, $q \in \mathbb{N}$ and let*

$$\Psi_{\alpha,h} := (-1)^{q+1} \sum_{j=0}^{N} b_j \Phi^{p_j}_{\alpha,c_j h} = \sum_{j=0}^{N} b_j \psi_{\alpha,c_j h,p_j}(-A)$$

be an $N+1$-order Grünwald approximation, where $\psi_{\alpha,h,p}(z)$ is given by (1.5) *and b_j, c_j, p_j are as defined in* (1.33). *Assume the multiplier $\sum_{j=0}^{N} b_j \psi_{\alpha,c_j h,p_j}(k)$, where $\psi_{\alpha,h,p}(k)$ is given by* (1.6)*, satisfies (i)-(iii) of Theorem 1.6.2 with constants independent of h. If one solves the Cauchy problem*

$$\dot{u}(t) = \Psi_{\alpha,h} u(t);\ u(0) = x,$$

with a strongly A-stable Runge-Kutta method with stage order s and order $r \geq s+1$, then, denoting the discrete solution by u_n at time level $t = n\tau$,

$$\|S_\alpha(t)x - u_n\| \leq C\left(n^{-r}\|x\| + h^{N+1}\left|\log\frac{t}{h^\alpha}\right| \|A^{N+1}x\|\right),\ h > 0,\ t = n\tau,$$

for all $x \in \mathcal{D}(A^{N+1})$.

Proof. It is straight forward to see that Theorem 2.2.1 holds for $\Psi_{\alpha,h}$; i.e.,

$$\|\Psi_{\alpha,h}x - A^\alpha x\| \leq Ch^{N+1}\|A^{\alpha+N+1}x\|,\ x \in \mathcal{D}(A^{\alpha+N+1}),$$

$\|e^{t\Psi_{\alpha,h}}\|_{\mathcal{B}(X)} \leq M$ and $\|\Psi_{\alpha,h}e^{t\Psi_{\alpha,h}}\|_{\mathcal{B}(X)} \leq M/t$. Following the proof of Theorem 2.2.2 we obtain the "spatial" error estimate

$$\|S_\alpha(t)x - e^{t\Psi_{\alpha,h}}x\| \leq Ch^{N+1}\left|\log\frac{t}{h^\alpha}\right| \|A^{N+1}x\|. \tag{2.21}$$

Since the analyticity of the semigroups $e^{t\Psi_{\alpha,h}}$ is uniform in h (the constant M does not depend on h), the statement follows from [68, Theorem 3.2] (see also [91]). □

Remark 2.2.4. The error estimates (2.18) and (2.19) are almost optimal in terms of the regularity of the data. We conjecture that one could remove the slight growth in t from (2.18) or the logarithmic factor in (2.19) by considering the L_1-case again and using the theory of Fourier multipliers on Besov spaces. Then use the transference principle to derive the abstract result. We do not pursue this issue here any further.

Remark 2.2.5. The convergence rate given in Corollary 2.2.3 can be extended using the stability estimate

$$\|S_\alpha(t)x - e^{t\Psi_{\alpha,h}}x\| \leq C\|x\|$$

and (2.21) to certain real interpolation spaces as in [57, Corollary 4.4]. We note that while the spaces $\mathcal{D}(A^s)$ endowed with the graph norm are, in general, not interpolation spaces, they are embedded within appropriate interpolation spaces (see, for example, [47, Corollary 6.6.3]) and therefore we obtain

$$\|S_\alpha(t)x - u_n\| \le C\left(n^{-r}\|x\| + h^s\left|\log\frac{t}{h^\alpha}\right|^{\frac{s}{N+1}}(\|A^s x\| + \|x\|)\right),\ h > 0,\ t = n\tau,$$

for all $x \in \mathcal{D}(A^s),\ s \in [0, N+1]$. Also note that we indeed have convergence of u_n to $S_\alpha(t)x$ for all $x \in X$ as $\tau \to 0$ and $h \to 0$ by Lax's Equivalence Theorem as a consequence of stability and consistency. The order of convergence, however, might be very low depending on x.

Remark 2.2.6. In recent work [27], Chen and Deng studied fourth order accurate schemes for certain fractional diffusion equations. Their approach is very different to ours, which we briefly compare and contrast below. Firstly, the authors develop a fourth order approximation different to our higher order Grünwald-type approximation for the (space) fractional derivative using Lubich's fractional linear multistep methods [64]. The authors refer to it as a weighted and shifted Lubich difference (WSLD) operator as a shift similar to ours is employed, see Definitions 1.2.1 and 1.5.2. In Theorem 2.4, the authors obtain point error estimates for their fourth order approximation for the fractional derivative under the assumption that the function along with the fractional derivative of order $\alpha + 4$ *as well as their Fourier transforms* belong to $L_1(\mathbb{R})$. In comparison, the regularity assumptions that we require to obtain higher order approximations are very minimal; that is, we only require that the function along with the fractional derivative of order $\alpha + 4$ belong to $L_1(\mathbb{R})$. Then, using Fourier multiplier theory and powerful Carlson-type inequalities, we obtain error estimates (see Corollary 1.6.3) in the L_1-setting which are sharp in view of the minimal regularity assumptions. Moreover, these error estimates from the L_1-setting can be used to obtain point estimates similar to the ones obtained by these authors, with less stringent regularity assumptions compared to theirs, using the transference principle (see Corollary 2.2.3). Lastly, the authors exploit the Toeplitz matrix structure of their scheme and employ the Grenander-Szergö Theorem to demonstrate stability. On the other hand, we use semigroup theory to show stability and smoothing of our numerical scheme. For future work, it might be worth considering their approach for showing stability of the numerical scheme, as we found our proof of stability for an explicit numerical example very technical and tedious, even in the case of a third order scheme.

2.3 Numerical results

In this section we give the results of two numerical experiments. The first is to explore the effect of the regularity of the initial distribution on the rate of convergence as needed for Corollary 2.2.3, the other is to see how well a second and third order scheme fare in the numerical experiment done by Tadjeran et. al [102].

Example 2.3.1. We consider $X = L_1[0,1]$ and $A = (d/dx)$ with

$$D(A) = \{f : f' \in L_1, f(0) = 0\}$$

and $\alpha = 0.8$. We approximate the solution to the Cauchy problems

$$u'(t) = -A^\alpha u(t); u(0) = f_i, \quad i = 1, 2, 3$$

at $t = 1$ with

$$f_1(x) = x^{-0.3}, \qquad f_2(x) = x^{0.7}, \qquad f_3(x) = x^{1.7},$$

with first and second order Grünwald schemes (as in Proposition 2.1.5) as well as via a convolution of f_i with an α-stable density approximated using Zolotarev's integral representation [83], which gives the exact solution but both the convolution and the density are computed numerically on a very fine grid. Note that $-A^\alpha$ denotes the Riemann-Liouville fractional derivative on the interval $[0, 1]$ which we study in detail in Chapters 3 and 4. Further, note that $f_1 \notin D(A)$, $f_2 \in D(A)$ but $f_2 \notin D(A^2)$ and $f_3 \in D(A^2)$. However, $f_1 \in D(A^{\beta_1})$ for $\beta_1 < 0.7$ and $f_2 \in D(A^{\beta_2})$ for $\beta_2 < 1.7$. By Remark 2.2.5 we expect about 0.7-order convergence for both schemes in case of $u_0 = f_1$, and first order convergence for the first order scheme for the other initial conditions. We expect about 1.7-order convergence for the second order scheme in case of $u_0 = f_2$ and second order convergence in case of $u_0 = f_3$. For the temporal discretization we use MATLAB's ode45, a fourth-order Runge-Kutta method with a forced high degree of accuracy in order to investigate the pure spatial discretization error. We see in Figure 2.1 that we obtain the expected convergence in all cases.

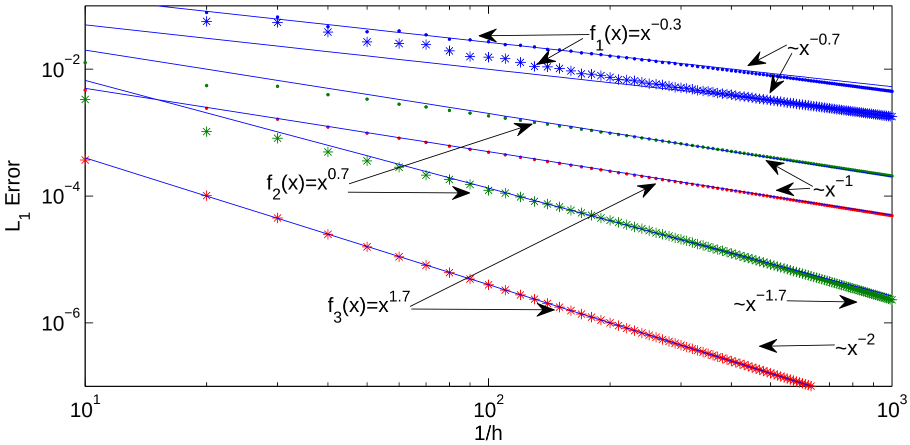

Figure 2.1: L_1-error for different initial conditions f_i and a first $(\cdot)$ and second order $(*)$ scheme. Note the less than first order convergence for a "bad" initial condition; i.e. one that is not in the domain of A. Also note the less than second order convergence for a second order scheme but first order convergence for the first order scheme for an initial condition that is in the domain of A but not in the domain of A^2.

Example 2.3.2. Even though our theoretical framework is not directly applicable, because the fractional differential operator appearing in (2.22) is defined on a finite domain with boundary conditions and has a multiplicative perturbation and hence it is not a fractional power of an auxiliary operator, we apply the second and third order approximations to the problem investigated by Tadjeran et al. [102], namely approximating the solution to

$$\frac{\partial u(x,t)}{\partial t} = \frac{\Gamma(2.2)}{3!} x^{2.8} \frac{\partial^{1.8} u(x,t)}{\partial x^{1.8}} - (1+x)e^{-t}x^3; u(x,0) = x^3 \tag{2.22}$$

on the interval $[0, 1]$ with boundary conditions $u(0, t) = 0, u(1, t) = e^{-t}$. The exact solution is given by $e^{-t}x^3$, which can be verified directly.

A second order approximation of the fractional derivative is given by Proposition 2.1.5. In order to obtain a third order approximation we consider

$$\phi_h(k) = a\psi_h^1 + b\psi_{2h}^{\frac{1}{2}} + c\psi_h^0$$

with the coefficients a, b and c such that ϕ_h is a third order approximation; i.e.

$$a = \frac{7 - 8\alpha + 3\alpha^2}{3(\alpha - 1)}, b = \frac{-7 + 3\alpha}{3(\alpha - 1)}, c = 1 - a - b.$$

A quick plot of $\phi_h(k)$ for $k \in \mathbb{R}$ strengthens the conjecture that the spectrum is in a sector in the left half plane and hence we expect stability and smoothing. We use again a fourth order Runge-Kutta method to solve the systems to $t = 1$. Table 1 suggests that we indeed have second and third order convergence with respect to the spatial discretization parameter Δx.

Δx	Error 2^{nd}	Error rate	Error 3^{rd}	Error rate
1/10	6.825×10^{-5}	-	9.180×10^{-6}	-
1/15	3.048×10^{-5}	$2.24 \approx (15/10)^2$	1.933×10^{-6}	$4.75 > (15/10)^3$
1/20	1.708×10^{-5}	$1.78 \approx (20/15)^2$	7.825×10^{-7}	$2.47 \approx (20/15)^3$
1/25	1.088×10^{-5}	$1.57 \approx (25/20)^2$	3.922×10^{-7}	$2 \approx (25/20)^3$

Table 2.1: Maximum error behaviour for second and third order Grünwald approximations.

Chapter 3

Boundary conditions for fractional-in-space partial differential equations

The Fokker-Planck equation of a Lévy stable process on $\mathbb{R}$ is a fractional (in space) partial differential equation. The (spatial) fractional derivative operator is non-local with infinite reach. In Chapters 3 and 4 we investigate the truncated fractional derivative operators on a bounded interval Ω with various boundary conditions. Moreover, we identify the stochastic processes whose marginal densities are the solutions to the fractional partial differential equations. That is, we show convergence of easily identifiable (sub)-Markov processes (that are essentially finite state), to a (sub)-Markov process governed by a Fokker-Planck equation on a bounded interval where the spatial operator is a truncated fractional derivative with appropriate boundary conditions. This will be achieved using the Trotter-Kato theorem [37, p. 209], regarding convergence of Feller semi-groups on $C_0(\Omega)$ and strongly continuous positive contraction semigroups on $L_1[0, 1]$, and hence showing process convergence [53, p. 331, Theorem 17.25].

3.1 Extension of a finite state Markov process to a Feller process on $[0, 1]$

To set the stage, let the matrix $G_{n\times n}$ denote the generator of a finite state sub-Markov process $(X_t^n)_{t\geq 0} \in \{1, \dots, n\}$. Then, for any function $f : \{1, \dots, n\} \mapsto \mathbb{R}$,

$$\left(e^{tG_{n\times n}} \left(f(1), \dots, f(n)\right)^{\mathrm{T}}\right)_i = E[f(X_t^n | X_0^n = i)].$$

Identifying f with a vector $\mathbf{f}$ via $\mathbf{f}_i = f(i)$, $(S(t))_{t\geq 0}$ with

$$S(t) : \mathbf{f} \mapsto e^{tG_{n\times n}}\mathbf{f}$$

is a family of bounded operators on $\ell_\infty^n = (\mathbb{R}^n, \|.\|_\infty)$ called the *backward semigroup* of the process $(X_t^n)_{t\geq 0}$. For $\mathbf{f} = \mathbf{e}_j$, $(S(t)\mathbf{f})_i$ gives the probability that $X_t^n = j$ given that $X_0^n = i$. This is in contrast to the *forward semigroup* $(T(t))_{t\geq 0}$ with

$$T(t) : \mathbf{g} \mapsto \mathbf{g}^T e^{tG_{n\times n}} = e^{tG^*_{n\times n}}\mathbf{g}$$

acting on $\mathbf{g} \in \ell_1^n = (\mathbb{R}^n, \|.\|_1)$, where $T(t)\mathbf{g}$ is the probability distribution of $(X_t^n)_{t\geq 0}$ given that the initial probability distribution of X_0^n is $\mathbf{g}$. Here $G^*_{n\times n}$ is the adjoint of $G_{n\times n}$ and $\mathbf{g}^T$ the transpose of $\mathbf{g}$. In particular, if $\mathbf{g} = \mathbf{e}_j$, $(T(t)\mathbf{g})_i$ gives the probability that $X_t^n = i$ given that $X_0^n = j$.

Remark 3.1.1 (**Transition rate matrix**). We refer to $G_{n\times n}$ and $G^*_{n\times n}$ as *transition rate matrices* for the spaces ℓ_∞^n and ℓ_1^n, respectively. Their diagonal entries $g_{i,i} \leq 0$ denote the total rate at which particles leave state i and their entries $g_{i,j} \geq 0,\ i \neq j$ signify the rate at which particles move from state i to state j or from state j to state i, respectively. The processes are referred to as *sub-Markovian* if the diagonal entries dominate the row and column sums, respectively, and note that in this case the particles exit the domain completely and so the mass is not preserved.

Recall that these concepts are extendable to general Markov processes taking values in a locally compact separable metric space Ω with the backward semigroup $(S(t))_{t\geq 0}$ acting on bounded functions and the forward semigroup $(T(t))_{t\geq 0}$ acting on Borel measures on Ω. Let $C_0(\Omega)$ with sup-norm denote the closure of the space of continuous functions with compact support in Ω. In particular, if $(S(t))_{t\geq 0}$ is a positive, strongly continuous contraction semigroup that leaves $C_0(\Omega)$ invariant, then $(X_t)_{t\geq 0}$ is called a Feller process. On the other hand, for any positive, strongly continuous, contraction semigroup $(S(t))_{t\geq 0}$ on $C_0(\Omega)$ (called Feller semigroups) there exists a process $(X_t)_{t\geq 0}$ with $(S(t))_{t\geq 0}$ as its backwards semigroup [53, Chapter 17]. Furthermore, a family of Feller semigroups converge strongly, uniformly for $t \in [0, t_0]$, to a Feller semigroup if and only if their respective Feller processes converge in the Skorokhod topology, see [53, p. 331, Theorem 17.25].

3.2 Construction of the transition operators

We begin with the bounded domain where all the action takes place. In what follows, we use

$$\Omega = [(0,1)] \tag{3.1}$$

to represent the interval $[0,1]$, which may or may not contain its endpoints, depending on the left and right boundary conditions that we impose on the fractional derivative operator. For a Dirichlet boundary condition (or absorbing boundary condition) we remove the endpoint from that side, which implies that $C_0(\Omega)$ will be the set of continuous functions that are zero at that endpoint. Note that if there are no absorbing boundary conditions then $C_0(\Omega) = C[0,1]$ as the closed interval $[0,1]$ is compact.

In order to exploit the fact that convergence, uniformly for $t \in [0, t_0]$, of Feller semigroups on $C_0(\Omega)$ implies convergence of the processes we turn a (n)-state (sub)-Markov process $(X_t^n)_{t\geq 0} \in \{1,\ 2, \dots\ , n\}$ to a Feller process $(\tilde{X}_t^n)_{t\geq 0} \in \Omega$ by having parallel copies of the finite state processes whose transition matrices interpolate continuously. The main idea here is to divide the interval $[0,1]$ into $n+1$ grids of equal length h so that the (Feller) process can jump between grids only in multiples of h. The transition rates for the (Feller) process $(\tilde{X}_t^n)_{t\geq 0}$ in the interval $[(i-1)h, ih]$ jumping up or down by jh interpolate continuously between the transition rates of sub-Markov process $(X_t^n)_{t\geq 0}$ being in state $i-1$ going to state $(i-1+j)$ and the transition rates of sub-Markov process $(X_t^n)_{t\geq 0}$ being in state i going to state $(i+j)$.

Let us make the necessary preparations in order to facilitate the definition of the transition operator.

Remark 3.2.1. To be precise, let $n \in \mathbb{N}$, $h = \frac{1}{n+1}$ and divide the interval $[0,1]$ into $n+1$ intervals such that the first n intervals are half open (on the right) while the $(n+1)^{\text{th}}$ (last) interval is closed. We can then uniquely determine each $x \in [0,1]$ by writing $\frac{x}{h}$ as the sum of its integer and fractional parts,

$$\frac{x}{h} = \left\lfloor \frac{x}{h} \right\rfloor + \left\{ \frac{x}{h} \right\}.$$

For convenience in calculations, we write $\iota(x) - 1 = \left\lfloor \frac{x}{h} \right\rfloor$ and $\lambda(x) = \left\{ \frac{x}{h} \right\}$, so that

$$\frac{x}{h} = (\iota(x) - 1) + \lambda(x),$$

where the two grid co-ordinate functions are defined as follows.

Definition 3.2.2. For each $x \in [0,1]$, the (step) function that returns the index of the grid under consideration, $\iota : x \to \{1,\ 2, \dots ,n+1\}$, is given by

$$\iota(x) = \begin{cases} i+1, & \text{if } i \leq \frac{x}{h} < i+1 \text{ for } i \in \{0, 1,\ 2, \dots ,n\}, \\ n+1, & \text{if } \frac{x}{h} = n+1. \end{cases}$$

The (sawtooth) function that returns the interpolant value, $\lambda : x \to [0,1]$ is given by

$$\lambda(x) = \begin{cases} \frac{x}{h} - i, & \text{if } i \leq \frac{x}{h} < i+1 \text{ for } i \in \{0, 1,\ 2, \dots ,n\}, \\ 1, & \text{if } \frac{x}{h} = n+1. \end{cases}$$

Note that the value $\lambda(x) = 1$ is taken only once at the right boundary $x = (n+1)h = 1$, while every other value $0 \leq \lambda(x) < 1$ is taken $n+1$-times.

Let $C\left([0,1]\ ;\ \mathbb{R}^{n+1}\right)$ denote the space of vector-valued continuous functions $v : [0,1] \to \mathbb{R}^{n+1}$. The projection operator $P_{n+1} : C[0,1] \to C\left([0,1]\ ;\ \mathbb{R}^{n+1}\right)$ is defined by

$$\left(P_{n+1}f\right)_j(\lambda) = f((\lambda + j - 1)h),\ f \in C[0,1],$$

where $\lambda \in [0,1]$ and $j \in \{1,\ 2, \dots ,n+1\}$. The embedding (gluing) operator is defined on the range of the projection operator; that is, let

$$\mathcal{D}(E_{n+1}) := \mathrm{rg}(P_{n+1}) \subset C\left([0,1]\ ;\ \mathbb{R}^{n+1}\right)$$

and define $E_{n+1} : \mathcal{D}(E_{n+1}) \to C[0,1]$ by $\left(E_{n+1}v\right)(x) = v_{i(x)}(\lambda(x))$, where $x \in [0,1]$ and domain

$$\mathcal{D}(E_{n+1}) = \left\{v \in C\left([0,1]\ ;\ \mathbb{R}^{n+1}\right) : v_{j+1}(0) = v_j(1) \text{ for } j = 1, \dots ,n\right\}.$$

Observe that E_{n+1} is a bounded operator and $\mathcal{D}(E_{n+1})$ is a closed subspace of $C\left([0,1];\mathbb{R}^{n+1}\right)$, thus E_{n+1} is closed. Moreover,

$$\begin{aligned} E_{n+1}\left(P_{n+1}f\right) &= f,\ f \in C[0,1], \\ P_{n+1}\left(E_{n+1}v\right) &= v, v \in \mathcal{D}(E_{n+1}). \end{aligned} \tag{3.2}$$

Let $G_{n\times n}$ denote a given $n \times n$ transition matrix on l_∞^n. Then, we construct the corresponding $(n+1)\times(n+1)$ *interpolation matrix*,

$$G_{n+1}(\lambda) = \begin{pmatrix} g_{1,1} & D^l(\lambda)g_{1,2} & \cdots & D^l(\lambda)g_{1,n} & 0 \\ N^l(\lambda)g_{2,1} & & & & N^r(\lambda)g_{1,n} \\ \vdots & & & & \vdots \\ N^l(\lambda)g_{i,1} & & (1-\lambda)g_{i-1,j-1} + \lambda g_{i,j} & & N^r(\lambda)g_{i-1,n} \\ \vdots & & & & \vdots \\ N^l(\lambda)g_{n,1} & & & & N^r(\lambda)g_{n-1,n} \\ 0 & D^r(\lambda)g_{n,1} & \cdots & D^r(\lambda)g_{n,n-1} & g_{n,n} \end{pmatrix}, \tag{3.3}$$

where the parameter $\lambda \in [0,1]$, $g_{i,j}$ are the entries of $G_{n\times n}$, and $D^l,\ N^l,\ D^r,\ N^r$ are *continuous interpolating functions* of the parameter λ such that $G_{n+1}(\lambda)$ is also a rate matrix for each $\lambda \in [0,1]$.

Remark 3.2.3. The interpolating functions are chosen in the following manner depending on the boundary conditions at hand. If the left boundary condition is Dirichlet, then we set $N^l = \mathbf{1}$ and take D^l to be a continuous function of the parameter λ that interpolates from 0 to 1. If the left boundary condition is not Dirichlet, then we set $D^l = \mathbf{1}$ and take N^l to be a continuous function that interpolates from 0 to 1. Similarly, if the right boundary condition is Dirichlet, then we set $N^r = \mathbf{1}$ and take D^r to be a continuous function that interpolates from 1 to 0. If the right boundary condition is not Dirichlet, then we set $D^r = \mathbf{1}$ and N^r to be a continuous function that intepolates from 1 to 0.

Lastly, observe that the interior entries of G_{n+1} are given by

$$[G_{n+1}(\lambda)]_{i,j} = (1-\lambda)g_{i-1,j-1} + \lambda g_{i,j} \text{ for } i,j \in \{2,3,\cdots,n\}.$$

Let us verify that $G_{n+1}(P_{n+1}f) \in \mathcal{D}(E_{n+1})$; that is,

$$\left(G_{n+1}(P_{n+1}f)\right)_{i+1}(0) = \left(G_{n+1}(P_{n+1}f)\right)_{i}(1),\ i = 1,\dots,n,$$

where for $k = 1,\dots,n+1$,

$$\left(G_{n+1}(P_{n+1}f)\right)_{k}(\lambda) = \sum_{j=1}^{n+1} [G_{n+1}(\lambda)]_{k,j} \left(P_{n+1}f\right)_j(\lambda).$$

Keep in mind that irrespective of the boundary conditions,

$$N^l(0)f(0) = 0,\ N^r(1)f(1) = 0, \text{ and } N^l(1),\ N^r(0),\ D^l(1),\ D^r(0) = 1.$$

First,

$$\left(G_{n+1}(P_{n+1}f)\right)_1(1) = g_{1,1}f(h) + \sum_{j=2}^{n} D^l(1)g_{1,j}f(jh) = \sum_{j=1}^{n} g_{1,j}f(jh).$$

For $k \in \{2,\cdots,n\}$,

$$\begin{aligned}\left(G_{n+1}(P_{n+1}f)\right)_k(1) &= N^l(1)g_{k,1}f(h) + \sum_{j=2}^{n} g_{k,j}f(jh) + N^r(1)g_{k-1,n}f(1)\\ &= \sum_{j=1}^{n} g_{k,j}f(jh)\end{aligned}$$

and

$$\begin{aligned}\left(G_{n+1}(P_{n+1}f)\right)_k(0) &= N^l(0)g_{k,1}f(0) + \sum_{j=2}^{n} g_{k-1,j-1}f((j-1)h) + N^r(0)g_{k-1,n}f(nh) \\ &= \sum_{j=1}^{n} g_{k-1,j}f(jh).\end{aligned}$$

Lastly,

$$\begin{aligned}\left(G_{n+1}(P_{n+1}f)\right)_{n+1}(0) &= \sum_{j=2}^{n} D^r(0)g_{n,j-1}f((j-1)h) + g_{n,n}f(nh) \\ &= \sum_{j=1}^{n} g_{n,j}f(jh).\end{aligned}$$

Hence, $G_{n+1}(P_{n+1}f) \in \mathcal{D}(E_{n+1})$.

With this preparation, we are in a position to define the transition operator.

Definition 3.2.4. The bounded transition operator $G : C[0,1] \to C[0,1]$ is given by

$$G : f \mapsto \left(E_{n+1}G_{n+1}P_{n+1}\right) f,$$

where

$$\left(E_{n+1}\left(G_{n+1}P_{n+1}f\right)\right)(x) = \left[G_{n+1}(\lambda(x))(P_{n+1}f)(\lambda(x))\right]_{\iota(x)}.$$

Remark 3.2.5. Let us make the following observations:

- The inclusion of interpolating functions D^l, D^r, N^l and N^r is necessary to ensure that G is a bounded operator on $C_0(\Omega)$. Firstly, they are necessary to ensure that $\lim_{x\to x_b} Gf(x) = 0$ for a Dirichlet boundary point $x_b \in [0,1] \setminus \Omega$. Secondly, they ensure the continuity of Gf at the grid points $x = ih$, $1 \le i \le n$.

  To see this, using Matrix 3.3, observe that

$$Gf(x) = \begin{cases} g_{1,1}f(\lambda(x)h) + \sum_{j=1}^{n-1} D^l(\lambda(x))g_{1,j+1}f((\lambda(x)+j)h), & \text{if } \iota(x) = 1, \\[2ex] N^l(\lambda(x))g_{\iota(x),1}f(\lambda(x)h) \\ \quad + \sum_{j=1}^{n-1}\left((1-\lambda(x))g_{\iota(x)-1,j} + \lambda(x)g_{\iota(x),j+1}\right) f((\lambda(x)+j)h) \\ \quad + N^r(\lambda(x))g_{\iota(x)-1,n}f((\lambda(x)+n)h)), & \text{if } 2 \le \iota(x) \le n, \\[2ex] \sum_{j=1}^{n-1} D^r(\lambda(x))g_{n,j}f((\lambda(x)+j)h) \\ \quad + g_{n,n}f((\lambda(x)+n)h)), & \text{if } \iota(x) = n+1. \end{cases}$$

For a left Dirichlet boundary point x_b^l, since

$$\iota(x) = 1,\ \lim_{x \downarrow x_b^l} \lambda(x) = 0,\ D^l(0) = 0 \text{ and } \lim_{x \downarrow x_b^l} f(x) \to f(0) = 0$$

we have that

$$\lim_{x \downarrow x_b^l} Gf(x) = 0.$$

Similarly, for a right Dirichlet boundary point x_b^r, since

$$\iota(x) = n+1,\ \lim_{x \uparrow x_b^r} \lambda(x) = 1,\ D^r(1) = 0 \text{ and } \lim_{x \uparrow x_b^r} f(x) \to f((n+1)h) = f(1) = 0$$

we have that

$$\lim_{x \uparrow x_b^r} Gf(x) = 0.$$

Next, consider the interior grid points $x = ih$, for $i = 1, \ldots n$. Approaching the grid point from the right, note that irrespective of the boundary conditions, in view of Definition 3.2.2, for $x \in [ih, (i+1)h)$ we have $\lim_{x \downarrow ih} \lambda(x) \to 0$ and $\iota(x) = i + 1$. Moreover, $N^r(0) = 1 = D^r(0)$ and if the left boundary condition is Dirichlet, then $f(0) = 0$, else $N^l(0) = 0$. Thus,

$$\lim_{x \downarrow ih} Gf(x) = \sum_{j=1}^{n} g_{i,j} f(jh).$$

On the other hand, approaching from the left, note that for $x \in [(i-1)h, ih)$, we have $\lim_{x \uparrow ih} \lambda(ih) \to 1$ and $\iota(x) = i$. Moreover, $D^l(1) = 1 = N^l(1)$ and if the right boundary condition is Dirichlet, then $f(1) = 0$, else $N^r(1) = 0$. Thus,

$$\lim_{x \uparrow ih} Gf(x) = \sum_{j=1}^{n} g_{i,j} f(jh).$$

Hence, Gf is continuous at each of the interior grid points, $x = ih$. Furthermore, the action of the transition operator coincides with the action of the rate matrix, that is,

$$Gf = G_{n \times n} \mathbf{f},$$

where $G_{n \times n}$ is the given transition matrix and $\mathbf{f} = (f(h), \ldots, f(nh))$.

- In view of (3.2) we have

$$S(t)f := e^{tG} f = \sum_{j=0}^{\infty} \frac{t^j}{j!} \left(E_{n+1} G_{n+1} P_{n+1}\right)^j f = E_{n+1} e^{tG_{n+1}} P_{n+1} f.$$

As E_{n+1} and P_{n+1} are positive contractions, $(S(t))_{t\geq 0}$ is a Feller semigroup if $(e^{tG_{n+1}})_{t\geq 0}$ is a positive contraction on $C\left([0,1];\mathbb{R}^{n+1}\right)$, which is the case if and only if $G_{n+1}(\lambda)$ generates a positive contraction semigroup on ℓ_∞^{n+1} for each $\lambda \in [0,1]$; that is, $G_{n+1}(\lambda)$ is a rate matrix whose row sums are non-positive.

- Note that the approximation operator usually used in numerical analysis that is obtained by linearly interpolating the vector $\tilde{G}\mathbf{f}$ is not the generator of a positive semigroup and so does not admit a straight forward stochastic interpretation.

In applications, one is usually interested in observing the evolution of the forward semigroup that acts on the space of bounded (complex) Borel measures, $\mathcal{M}_{\mathcal{B}}(\Omega)$. It is well known that $L_1[0,1]$ is isometrically isomorphic to a closed subspace of $\mathcal{M}_{\mathcal{B}}(\Omega)$. The forward semigroup denoted by $(T(t))_{t\geq 0}$ is the adjoint of $(S(t))_{t\geq 0}$ and the action of the generator G^* of $(T(t))_{t\geq 0}$ can be easily computed for $g \in L_1[0,1]$. To this end, we first extend the definitions of E_{n+1} and P_{n+1} to $L_1[0,1]$-functions.

Let $L_1\left([0,1];\mathbb{R}^{n+1}\right)$ denote the space of vector-valued *Bochner* integrable functions $v : [0,1] \to \mathbb{R}^{n+1}$, see [1, p. 13]. For $g \in L_1[0,1]$, define the projection operator $P_{n+1} : L_1[0,1] \to L_1\left([0,1];\mathbb{R}^{n+1}\right)$ by

$$\left(P_{n+1}g\right)_j(\lambda) = g((\lambda + j - 1)h),$$

where $\lambda \in [0,1]$ and $j \in \{1,\ 2, \dots\ , n+1\}$. Define the embedding (gluing) operator $E_{n+1} : L_1\left([0,1];\mathbb{R}^{n+1}\right) \to L_1[0,1]$ by $\left(E_{n+1}v\right)(x) = v_{i(x)}(\lambda(x))$. where $x \in [0,1]$. Note that

$$\begin{aligned}
&E_{n+1}\left(P_{n+1}g\right) = g,\ g \in L_1[0,1],\\
&P_{n+1}\left(E_{n+1}v\right) = v, v \in L_1\left([0,1];\mathbb{R}^{n+1}\right).
\end{aligned}$$

Let $g \in L_1[0,1]$ and to simplify notation in the calculation below let $P := P_{n+1}$ and $\lambda := \lambda(x)$. In view of Remark 3.2.1, for $(i-1)h \leq x < ih$ we have $x = (\lambda(x) + (i-1))\,h$. Thus, using the substitution $x = (\lambda + (i-1))\,h$ along with Definition 3.2.4 we have

$$\begin{aligned}
\int_0^1 (Gf)(x)g(x)\,\mathrm{d}x &= \sum_{i=1}^{n+1} \int_{(i-1)h}^{ih} \left(G_{n+1}(\lambda(x))(Pf)(\lambda(x))\right)_{\iota(x)} (Pg)_{\iota(x)}(\lambda(x))\,\mathrm{d}x\\
&= \sum_{i=1}^{n+1} \int_{(i-1)h}^{ih} \left(\sum_{j=1}^{n+1} \left[G_{n+1}(\lambda(x))\right]_{i,j} (Pf)_j(\lambda(x)) \right) (Pg)_i(\lambda(x))\,\mathrm{d}x\\
&= \sum_{i=1}^{n+1} \int_0^1 \left(\sum_{j=1}^{n+1} \left[G_{n+1}(\lambda)\right]_{i,j} (Pf)_j(\lambda) \right) (Pg)_i(\lambda)\,h\,\mathrm{d}\lambda
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n+1} \int_0^1 (Pf)_j(\lambda) \left(\sum_{i=1}^{n+1} \left[G_{n+1}^T(\lambda) \right]_{j,i} (Pg)_i(\lambda) \right) h \, \mathrm{d}\lambda \\
&= \sum_{j=1}^{n+1} \int_{(j-1)h}^{jh} (Pf)_j(\lambda(x)) \left(\sum_{i=1}^{n+1} \left[G_{n+1}^T(\lambda(x)) \right]_{j,i} (Pg)_i(\lambda(x)) \right) \mathrm{d}x \\
&= \int_0^1 f(x)(G^* g)(x) \, \mathrm{d}x.
\end{aligned}
$$

Therefore,

$$
G^* g = E_{n+1} G_{n+1}^T P_{n+1} g
$$

and thus, G^* leaves $L_1[0,1]$ invariant. Hence, we have shown the following proposition.

We refer to Appendix C for the definition of the part of an operator.

Proposition 3.2.6. *The part of the adjoint transition operator G^* in $L_1[0,1]$, denoted by $G^*\Big|_{L_1[0,1]} : L_1[0,1] \to L_1[0,1]$, is given by*

$$
G^*\Big|_{L_1[0,1]} : f \mapsto \left(E_{n+1} G_{n+1}^T P_{n+1} \right) f,
$$

where

$$
\left(E_{n+1} G_{n+1}^T P_{n+1} \right) f(x) = \left[G_{n+1}^T(\lambda(x))(P_{n+1} f)(\lambda(x)) \right]_{\iota(x)}.
$$

Proposition 3.2.7. *Let $G_{n \times n}$ be an $n \times n$ rate matrix with non-negative off-diagonal entries and non-positive row sums. For $\lambda \in [0,1]$, let the operator G be as in Definition 3.2.4 and assume that the interpolating functions D^l, D^r, N^l and N^r are such that $G_{n+1}(\lambda)$ is a rate matrix and G is a bounded operator on $C_0(\Omega)$. Then G generates a Feller semigroup on $C_0(\Omega)$ and $G^*\Big|_{L_1[0,1]}$ generates a strongly continuous positive contraction semigroup on $L_1[0,1]$.*

Proof. Let $(S(t))_{t \geq 0}$ denote the semigroup generated by G and $(T(t))_{t \geq 0}$ denote the dual semigroup generated by G^*. Firstly, note that $(S(t))_{t \geq 0}$ is a Feller semigroup on $C_0(\Omega)$ in view of Remark 3.2.5. Indeed, since G is a bounded operator, $(S(t))_{t \geq 0}$ is strongly continuous. The fact that $G_{n+1}(\lambda)$ is a transition rate matrix with non-positive row sums for each $\lambda \in [0,1]$ yields the fact $(S(t))_{t \geq 0}$ is a contraction semigroup. Lastly, positivity follows in view of the linear version of Kamke's theorem, see [2, p. 124]: that is, $e^{tG_{n+1}^*} \geq 0$ if and only if $g_{i,j}^* \geq 0$ for $i \neq j$.) The same argument yields the positivity of $(T(t))_{t \geq 0}$. Since $(S(t))_{t \geq 0}$ is a contraction semigroup and for all $t \geq 0$, $\|T(t)\| = \|S(t)\|$, we have that $(T(t))_{t \geq 0}$ is a contraction semigroup on $\mathcal{M}_{\mathcal{B}}(\Omega)$. Next, note that $L_1[0,1]$ is a closed subspace of $\mathcal{M}_{\mathcal{B}}(\Omega)$. Hence, as G^*, and thus $(T(t))_{t \geq 0}$ leaves $L_1[0,1]$ invariant, the part of G^* in $L_1[0,1]$ is the generator of $\left(T(t)\Big|_{L_1[0,1]} \right)_{t \geq 0}$, see [37, p. 43, 61]. $\square$

3.3 One-sided fractional derivative operators with different combinations of boundary conditions

In this section we define the one-sided fractional derivative operator, denoted in general by A, as a densely defined, closed, linear operator from its domain $\mathcal{D}(A) \subset X$ into X. The domain $\mathcal{D}(A)$ encodes a particular combination of boundary conditions, the interval $\Omega = [(0,1)]$ is given by (3.1) and

$$X = C_0(\Omega) \text{ or } L_1[0,1]. \tag{3.4}$$

We use $\|\cdot\|_X$ to denote the X-norm where

$$\|f\|_X := \begin{cases} \sup_{x\in\Omega} |f(x)|, & \text{if } X = C_0(\Omega) \\ \int_0^1 |f(x)| \,\mathrm{d}x, & \text{if } X = L_1[0,1] \end{cases} \tag{3.5}$$

We make the necessary preparations in Sections 3.4 and 4.1 and show that these fractional derivative operators are generators of positive contraction semigroups using the Lumer-Philips Theorem. We do this by approximating the fractional derivative operators by the (transition operators) generators of the backward or forward semigroups associated with the extended finite state (sub)-Markov processes.

3.3.1 Fractional integral operators and fractional derivatives

In preparation for the definition of the one-sided fractional derivative operators in Section 3.3.2, we first define the linear (*Riemann-Liouville*) fractional integral operators and study some of their properties that we use often in our discussions. Following that we give the explicit definition of fractional derivatives on X and discuss the motivation behind the choice of functions in the domains of the fractional derivative operators which encode various boundary conditions. We conclude this section with the properties of some special functions which play important roles in our study of the fractional derivative operators on X.

Remark 3.3.1. In what follows, when working in $C_0(\Omega)$, we use D^n to denote the classical integer-order derivative operator. On the other hand, when working in $L_1[0,1]$, $(D^n, W^{n,1}[0,1])$ for $n \in \mathbb{N}$ denotes the generalised integer-order derivative operator as in Definition B.1.1.

Let p_β denote the power function (monomial) given by

$$p_\beta(x) := \frac{x^\beta}{\Gamma(\beta+1)}, \text{ if } -1 < \beta < 0 \text{ for } x \in (0,1], \text{ or if } \beta \geq 0 \text{ for } x \in [0,1]. \tag{3.6}$$

Let p_0 denote the constant *one* function and $\mathbf{0}$ denote the *zero* function on $[0,1]$. We use p_β to denote the power function on $[0,1]$, instead of ϕ_β as given in Definition 1.4.3, to emphasise the fact that we are working on the interval $[0,1]$ and not on $\mathbb{R}$.

First note that, since $\|p_\beta\|_{L_1[0,1]} = \frac{1}{\Gamma(\beta+2)}$,

$$p_\beta \in L_1[0,1] \text{ and if } \beta \geq 0,\ p_\beta \in C[0,1]. \tag{3.7}$$

Moreover,

$$\begin{aligned} D^n p_\beta &= p_{\beta-n}, \text{ for } \beta > n-1, \\ Dp_0 &= \mathbf{0}. \end{aligned} \tag{3.8}$$

Let $\gamma > 0$ and $f \in L_1[0,1]$, then the fractional integral of order γ, see Definition 1.1.3, is given by

$$I^\gamma f(x) = \int_0^x p_{\gamma-1}(x-s)f(s)ds,\ x > 0. \tag{3.9}$$

If $\alpha,\ \beta > 0$ and $f \in L_1[0,1]$, then the fractional integrals have the following semigroup property, see [86, p. 67],

$$I^\alpha I^\beta f = I^{\alpha+\beta} f. \tag{3.10}$$

For $f \in AC[0,1]$, note that the derivative of the fractional integral is related to the fractional integral of the derivative by

$$D(I^\gamma f) = I^\gamma(Df) + f(0)p_{\gamma-1}. \tag{3.11}$$

To see this, let us use the substitution $u = x - s$ and write

$$I^\gamma f(x) = \int_0^x f(x-s)p_{\gamma-1}(s)ds.$$

Then,

$$\begin{aligned} D(I^\gamma f)(x) &= \lim_{h\to 0} \frac{\left[\int_0^{x+h} f(x+h-s)p_{\gamma-1}(s)ds - \int_0^x f(x-s)p_{\gamma-1}(s)ds\right]}{h} \\ &= \lim_{h\to 0}\left[\int_0^x \frac{f(x-s+h)-f(x-s)}{h} p_{\gamma-1}(s)ds\right] \\ &\qquad\qquad + \lim_{h\to 0}\left[\frac{1}{h}\int_x^{x+h} f(x-s+h)p_{\gamma-1}(s)ds\right] \\ &= I^\gamma(Df)(x) + f(0)p_{\gamma-1}(x), \end{aligned}$$

where we have used the Lebesgue local average in the second term.

Remark 3.3.2. For $\gamma > 0$, if $f \in L_1[0,1]$, then using (3.7) and Young's inequality (B.2), we have

$$\|I^\gamma f\|_{L_1[0,1]} \leq C\,\|p_{\gamma-1}\|_{L_1[0,1]}\,\|f\|_{L_1[0,1]} = C\frac{1}{\Gamma(\gamma+1)}\,\|f\|_{L_1[0,1]} < \infty.$$

Thus, $I^\gamma \in \mathcal{B}(L_1[0,1])$ for $\gamma > 0$ and hence, continuous on $L_1[0,1]$. Moreover, if $f \in C[0,1]$, then using (3.7) and Young's inequality (B.2) again

$$\|I^\gamma f\|_{L_\infty[0,1]} \leq \|p_{\gamma-1}\|_{L_1[0,1]}\,\|f\|_{L_\infty[0,1]} < \infty.$$

Moreover, since $I^\gamma f \in C[0,1]$, we have that $I^\gamma \in \mathcal{B}(C[0,1])$ for $\gamma > 0$ and hence, continuous on $C[0,1]$.

We end the discussion on fractional integral operators with the following crucial proposition.

Proposition 3.3.3. *Let $f \in L_1[0,1]$. Suppose that either $0 < \alpha < 1$ and f is bounded on $[0,\epsilon)$ for some $\epsilon > 0$, or $\alpha \geq 1$. Then*

$$I^\alpha f(0) := \lim_{x\downarrow 0} I^\alpha f(x) = 0.$$

Proof. First, let $0 < \alpha < 1$ and f be bounded on $[0,x]$, $x < \epsilon$. Then,

$$\begin{aligned}|I^\alpha f(x)| &\leq \int_0^x |p_{\alpha-1}(x-s)|\,|f(s)|\,ds \leq \sup_{s\in[0,x]}(|f(s)|)\int_0^x |p_{\alpha-1}(x-s)|\,ds \\ &= \sup_{s\in[0,x]}(|f(s)|)\frac{x^\alpha}{\Gamma(\alpha+1)} \to 0, \text{ as } x\downarrow 0.\end{aligned}$$

Next, let $\alpha = 1$, then $I^\alpha f(0) = If(0) = 0$ by the absolute continuity of the Lebesgue integral, see [56, p. 300]. Lastly, let $\alpha > 1$, then $p_{\alpha-1}$ is continuous, that is bounded on $[0,1]$. Thus,

$$\begin{aligned}|I^\alpha f(x)| &\leq \int_0^x |p_{\alpha-1}(x-s)|\,|f(s)|\,ds \\ &\leq \sup_{s\in[0,x]}(p_{\alpha-1}(x-s))\,\|f\|_{L_1[0,1]} \leq \frac{x^{\alpha-1}}{\Gamma(\alpha)}\,\|f\|_{L_1[0,1]} \to 0, \text{ as } x\downarrow 0.\end{aligned}$$

□

Let X be given by (3.4) and $1 < \alpha < 2$. Then, the first degree *Caputo* and the *Riemann-Liouville* fractional derivatives of order α on X are given by

$$D_c^\alpha f = DI^{2-\alpha}Df \text{ and } D^\alpha f = D^2 I^{2-\alpha} f, \tag{3.12}$$

respectively (see Definition 1.1.4). In what follows, if the context of the discussion applies to both these fractional derivatives, we simply use A to denote them. Similarly, fractional derivatives of order $\alpha - 1$ are given by

$$D_c^{\alpha-1} f = I^{2-\alpha} D f \text{ and } D^{\alpha-1} f = D I^{2-\alpha} f. \tag{3.13}$$

Before we can introduce the respective domains of the fractional derivative operators on X that encode various boundary conditions, we briefly discuss the motivation behind the choice of the functions in the domains. To define the domains of the fractional derivative operators on X (see Section 3.3.2), it is natural to employ the range of the corresponding fractional anti-derivative (integral) operators. That is, view the fractional derivative operators as the inverse of the corresponding fractional integral operators. However, we will see that it is not as straightforward as that. For instance, in the cases when the domains encode left and right Neumann boundary conditions, non-zero steady state is possible, thus the fractional derivative operators in these cases are not invertible.

It is easily verified that the fractional derivative A plays the role of left inverse of the fractional integral I^α. Indeed, for $f \in X$ and $1 < \alpha < 2$ in view of (3.10),

$$\begin{aligned} D^\alpha I^\alpha f &= D^2 I^{2-\alpha} I^\alpha f = D^2 I^2 f = f, \\ D_c^\alpha I^\alpha f &= D I^{2-\alpha} D I^\alpha f = D I^{2-\alpha} I^{\alpha-1} f = D I f = f. \end{aligned} \tag{3.14}$$

However, the fractional integral I^α is not the left inverse of A. In fact, let $W^{n,1}[0,1]$ denote the Sobolev space of $L_1[0,1]$ functions given in Definition B.1.1, then we have the following well known result, see for example [86, p. 70]. The next result is the motivation for using the range of the fractional integral operator along with a certain linear combination of power functions to define the domains of the fractional derivative operators.

Proposition 3.3.4. *Let $1 < \alpha < 2$. Then the following hold:*

1. *If $I^{2-\alpha} f \in W^{2,1}[0,1]$, then*

$$I^\alpha D^\alpha f = f - D^{\alpha-1} f(0) p_{\alpha-1} - I^{2-\alpha} f(0) p_{\alpha-2}.$$

2. *If $D_c^{\alpha-1} f \in W^{1,1}[0,1]$, then*

$$I^\alpha D_c^\alpha f = f - D_c^{\alpha-1} f(0) p_{\alpha-1} - f(0) p_0.$$

Proof. We make use of (3.10), (3.11) with $p_\alpha(0) = 0$, and (3.12) in what follows. Firstly, $I^{2-\alpha}f \in W^{2,1}[0,1]$, thus $I^{2-\alpha}f$ is continuously differentiable and $DI^{2-\alpha}f$ is absolutely continuous. Therefore, using the fact that if $u \in X$ is absolutely continuous then $IDu = u - u(0)p_0$, we have

$$\begin{aligned}
I^\alpha D^\alpha f &= DII^\alpha D^2 I^{2-\alpha} f = DI^\alpha ID\left(DI^{2-\alpha}f\right)\\
&= DI^\alpha\left(DI^{2-\alpha}f - DI^{2-\alpha}f(0)p_0\right) = D\left(I^{\alpha-1}ID\left(I^{2-\alpha}f\right) - D^{\alpha-1}f(0)p_\alpha\right)\\
&= D\left(I^{\alpha-1}\left(I^{2-\alpha}f - I^{2-\alpha}f(0)p_0\right) - D^{\alpha-1}f(0)p_\alpha\right)\\
&= D\left(If - D^{\alpha-1}f(0)p_\alpha - I^{2-\alpha}f(0)p_{\alpha-1}\right)\\
&= f - D^{\alpha-1}f(0)p_{\alpha-1} - I^{2-\alpha}f(0)p_{\alpha-2}.
\end{aligned}$$

Similarly, $D_c^{\alpha-1}f = I^{2-\alpha}Df$ is absolutely continuous and

$$\begin{aligned}
I^\alpha D_c^\alpha f &= DI^\alpha ID\left(I^{2-\alpha}Df\right)\\
&= DI^\alpha\left(I^{2-\alpha}Df - I^{2-\alpha}Df(0)p_0\right) = DI^2Df - I^{2-\alpha}Df(0)p_{\alpha-1}\\
&= IDf - D_c^{\alpha-1}f(0)p_{\alpha-1} = f - f(0)p_0 - D_c^{\alpha-1}f(0)p_{\alpha-1},
\end{aligned}$$

where we used (3.8). □

We require the following properties of the power functions in our discussion of the respective domains of the fractional derivative operators. The fractional integral of p_β of order $\nu > 0$ is given by, see [86, p. 72]

$$I^\nu p_\beta = p_{\beta+\nu}. \tag{3.15}$$

Let $\nu > 0$ and $\beta > \nu$, then

$$D^\nu p_\beta = D^n I^{n-\nu} p_\beta = D^n p_{\beta+n-\nu} = p_{\beta-\nu}, \tag{3.16}$$

where $n = \lceil \nu \rceil$. Note that, if $\beta > 0$, then in view of (3.11), $D^{\alpha-1}p_\beta = D_c^{\alpha-1}p_\beta$ since $p_\beta(0) = 0$ and thus $D^\alpha p_\beta = D_c^\alpha p_\beta$. Here is a summary of the fractional derivatives of the special power functions that are used in the definitions of the domains of the fractional derivative operators. In (3.17) below, when a result applies to both D_c^α and D^α we just denote the fractional derivative by A. Using (3.12), (3.15) and (3.16), for $1 < \alpha < 2$ we have

$$\begin{aligned}
Ap_\alpha &= p_0,\\
Ap_{\alpha-1} &= \mathbf{0},\\
D_c^\alpha p_0 &= \mathbf{0},\\
D^\alpha p_{\alpha-2} &= \mathbf{0}.
\end{aligned} \tag{3.17}$$

3.3.2 Fractional derivative operators on a bounded interval encoding various boundary conditions

As discussed in Section 3.3.1, we define the domain of the fractional derivative operators as the range of the fractional integral operator I^α, supplemented by a linear combination of some particular power functions with constant weights that encode the regularity as well as the boundary conditions satisfied by the functions in the domain.

Let $X = C_0(\Omega)$ or $L_1[0,1]$. We consider functions of the form

$$f = I^\alpha g + ap_\alpha + bp_{\alpha-1} + cp_{\alpha-2} + dp_0,\ g \in X \tag{3.18}$$

as candidates for the domain of the fractional derivative operator, where $a, b, c, d \in \mathbb{R}$ and $p_\beta = \frac{x^\beta}{\Gamma(\beta+1)}$ as given by (3.6).

Remark 3.3.5. Firstly, if $c \neq 0$ in (3.18), then $f \notin C_0(\Omega)$. Secondly, note that $D_c^\alpha p_{\alpha-2}$ and $D^\alpha p_0$ are not defined in $L_1[0,1]$. Therefore, for f given by (3.18), we set $c = 0$ in the case of $X = C_0(\Omega)$ and in the case of first degree Caputo fractional derivative operators on $L_1[0,1]$. We also set $d = 0$ for f given by (3.18) in the case of Riemann-Liouville fractional derivative operators. Similarly, various boundary conditions imply relations to be satisfied by the constants a, b, c, d.

In preparation for the definition (Definition 3.3.8) of fractional derivative operators on X, let us first define formally the boundary conditions that we investigate, namely, Dirichlet, Neumann and Neumann* boundary conditions where the latter appears naturally as the adjoint of the Neumann boundary condition for the Riemann-Liouville fractional derivative operator whose discussion we take up in Section 4.2.

Definition 3.3.6. Let A denote either the first degree Caputo fractional derivative operator D_c^α or the Riemann-Liouville fractional derivative operator D^α on X, where the corresponding fractional derivatives are given explicitly by (3.12). Let us write A in the form $A = DF$ for brevity, where F is either $D_c^{\alpha-1} = I^{2-\alpha}Df$ or $D^{\alpha-1} = DI^{2-\alpha}$.

1. **Dirichlet boundary conditions**: A function $f \in X$ satisfies the *left* Dirichlet boundary condition for the operator A if and only if f is continuous as $x \downarrow 0$ and $f(0) = 0$. Similarly, a function $f \in X$ satisifies the *right* Dirichlet boundary condition for the operator A if and only if f is continuous as $x \uparrow 1$ and $f(1) = 0$.

2. **Neumann boundary conditions**: A function $f \in X$ satisfies the *left* Neumann boundary condition for the operator A if and only if Ff is continuous as $x \downarrow$

0 and $Ff(0) = 0$. Similarly, a function $f \in X$ satisfies the *right* Neumann boundary condition for the operator A if and only if Ff is continuous as $x \uparrow 1$ and $Ff(1) = 0$. In particular, the zero flux condition $Ff(p) = 0$ at the boundary point $p = 0$ or 1, for the operator D_c^α is $I^{2-\alpha}Df(p) = 0$ and for the operator D^α is $DI^{2-\alpha}f(p) = 0$.

3. **Neumann* boundary condition**: If $X = C_0(\Omega)$, then a function $f \in C_0(\Omega)$ satisfies the *right* Neumann* boundary condition for the operator A if and only if Df is continuous as $x \uparrow 1$ and $Df(1) = 0$.

Remark 3.3.7. In general, we use the abbreviation BC to refer to some combination of left Dirichlet or Neumann boundary condition with right Dirichlet, Neumann or Neumann* boundary condition encoded by the domain of the fractional derivative operator A. The boundary conditions encoded by the domains are an essential part of the definition of the fractional derivative operators on X. To emphasise this we denote the fractional derivative operators on X by the pair (A, BC) instead of the conventional $(A, \mathcal{D}(A))$. For instance, if the domain of the fractional derivative operator D_c^α encodes a left Dirichlet and a right Neumann boundary condition, we write $(D_c^\alpha, \mathrm{DN})$. Also, when dealing with common properties of operators with a particular left or right boundary condition combined with any of the other possible boundary conditions, for example, a left Dirichlet boundary condition, we just write $(A, \mathrm{D}\bullet)$. However, for convenience when the boundary conditions at hand are obvious from the context we will just write A for the fractional derivative operator on X.

Definition 3.3.8. (A, BC) is called a *fractional derivative operator* on X, if the operator $A \in \{D_c^\alpha,\ D^\alpha\}$ and the respective domains are given as follows:

1. The domain of the first degree Caputo fractional derivative operator is given by

$$\mathcal{D}(D_c^\alpha, \mathrm{BC}) = \{f \in X : f = I^\alpha g + ap_\alpha + bp_{\alpha-1} + dp_0,\ g \in X\}, \tag{3.19}$$

where the constants $a, b, d \in \mathbb{R}$ satisfy the respective relations for BC listed in Table 3.1.

2. The domain of the Riemann-Liouville fractional derivative operator is given by

$$\mathcal{D}(D^\alpha, \mathrm{BC}) = \{f \in X : f = I^\alpha g + ap_\alpha + bp_{\alpha-1} + cp_{\alpha-2},\ g \in X\}, \tag{3.20}$$

where the constants $a, b, c \in \mathbb{R}$ satisfy the respective relations for BC listed in Table 3.1.

$X = C_0(\Omega), \mathcal{D}(A, \mathrm{BC}) = \{I^\alpha g + ap_\alpha + bp_{\alpha-1} + dp_0 : g \in C_0(\Omega)\},\ (0,1) \subset \Omega$		
Boundary condition		Constants in $\mathcal{D}(A, \mathrm{BC})$
Left Dirichlet	$f(0) = 0,\ \Omega \subset (0,1]$	$a = 0, d = 0$
Left Neumann	$D_c^{\alpha-1} f(0) = 0,\ [0,1) \subset \Omega$	$b = 0$
Right Dirichlet	$f(1) = 0,\ \Omega \subset [0,1)$	$a = 0,\ \frac{b}{\Gamma(\alpha)} + d = -I^\alpha g(1)$
Right Neumann	$D_c^{\alpha-1} f(1) = 0,\ (0,1] \subset \Omega$	$a + b = -Ig(1)$
Right Neumann*	$Df(1) = 0,\ (0,1] \subset \Omega$	$\frac{a+(\alpha-1)b}{\Gamma(\alpha)} = -I^{\alpha-1} g(1)$
$X = L_1[0,1], \mathcal{D}(A, \mathrm{BC}) = \{I^\alpha g + ap_\alpha + bp_{\alpha-1} + cp_{\alpha-2} + dp_0 : g \in L_1[0,1]\}$		
Boundary condition		Constants in $\mathcal{D}(A, \mathrm{BC})$
Left Dirichlet	$f(0) = 0$	$a = 0, c = 0, d = 0$
Left Neumann	$D_c^{\alpha-1} f(0) = 0$	$b = 0, c = 0$
Left Neumann	$D^{\alpha-1} f(0) = 0$	$b = 0, d = 0$
Right Dirichlet	$f(1) = 0$	$a = 0,\ \frac{b+(\alpha-1)c}{\Gamma(\alpha)} + d = -I^\alpha g(1)$
Right Neumann	$Ff(1) = 0$	$a + b = -Ig(1)$

Table 3.1: Relations satisfied by the constants a, b, c, d for BC.

Remark 3.3.9 (**Non-homogeneous fractional-in-space partial differential equations with non-zero boundary conditions**). To solve a non-homogeneous fractional-in-space partial differential equation for any set of given time-dependent boundary values it is sufficient to solve the corresponding homogeneous equation with zero boundary conditions. To see this, consider the Cauchy problem associated with the homogeneous fractional-in-space partial differential equation with zero boundary conditions BC,

$$\begin{aligned} u'(t) &= Au(t),\ t \geq 0, \\ u(0) &= u_0, \end{aligned} \tag{3.21}$$

where A denotes the fractional derivative operator on X whose domains encode BC. Let us assume (3.21) is well-posed and let $(T(t))_{t\geq 0}$ denote the solution operator on X (strongly continuous semigroup generated by A). Then, the mild solution to the Cauchy problem associated with the non-homogeneous fractional-in-space partial differential equation with zero boundary conditions [12],

$$u'(t) = Au(t) + f(t) \quad \text{for } t \in [0, \tau],\ f \in L_1([0,\tau], X)$$

$$u(0) = u_0,$$

is given by the variation of constants formula,

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)\mathrm{d}s.$$

Next, consider the following non-homogeneous fractional-in-space partial differential equations with non-zero boundary conditions,

$$\frac{\partial}{\partial t}w(t,x) = D^\alpha w(t,x) + f(t,x);\ w(0,x) = w_0(x), \tag{3.22}$$

where D^α denotes a fractional derivative operator on Ω and $f \in L_1([0,\tau],X)$. Let $g(t,x)$ be a function such that $D^\alpha g,\ \frac{\partial}{\partial t}g \in L_1([0,\tau],X)$ and satisfies the non-zero boundary conditions. Further, assume that u solves the non-homogeneous fractional-in-space partial differential equation with zero boundary conditions,

$$\frac{\partial}{\partial t}u(t,x) = Au(t,x) + D^\alpha g(t,x) - \frac{\partial}{\partial t}g(t,x) + f(t,x);\ u(0,x) = w_0(x) - g(0,x).$$

Then $w = u + g$ solves (3.22) and satisfies the non-zero boundary conditions.

In view of this, we only study well-posedness and numerical solutions for the abstract Cauchy problem (3.21) associated with fractional derivative operators (A, BC) on X whose domains encode combinations of (zero) boundary conditions.

Proposition 3.3.10. *The domains of the fractional derivative operators* (A, BC) *given in Definitions 3.3.8 are equivalent to the following:*

1. *The domain of the first degree Caputo fractional derivative operator,*

$$\mathcal{D}(D_c^\alpha, \mathrm{BC}) = \{f \in X : f = I^\alpha g + ap_\alpha + bp_{\alpha-1} + dp_0,\ g \in X \\ \text{such that } D_c^\alpha f \in X \text{ and } f \text{ satisfies BC}\}.$$

2. *The domain of the Riemann-Liouville fractional derivative operator,*

$$\mathcal{D}(D^\alpha, \mathrm{BC}) = \{f \in X : f = I^\alpha g + ap_\alpha + bp_{\alpha-1} + cp_{\alpha-2},\ g \in X \\ \text{such that } D^\alpha f \in X \text{ and } f \text{ satisfies BC}\}.$$

Proof. Let

$$f = I^\alpha g + ap_\alpha + bp_{\alpha-1} + cp_{\alpha-2} + dp_0,\ g \in X,$$

then a simple calculation reveals that if the constants $a,\ b,\ c,\ d$ satisfy the relations for BC given in the Table 3.1 then $D_c^\alpha f \in X$ and $D^\alpha f \in X$, respectively and f satisfies the respective boundary conditions of BC. Therefore, to demonstrate the equivalence of the definitions of the respective domains, we show that if f satisfies a particular left or right boundary condition and $Af \in X$, where $A \in \{D_c^\alpha,\ D^\alpha\}$, then the corresponding relations summarised in Table 3.1 are satisfied by the constants a, b, c, d.

To this end, let

$$f = I^\alpha g + ap_\alpha + bp_{\alpha-1} + cp_{\alpha-2} + dp_0,\ g \in X \text{ and } Af \in X.$$

Further, let F be either $D_c^{\alpha-1}$ or $D^{\alpha-1}$. Then, first observe that

$$Ff = Ig + ap_1 + bp_0. \tag{3.23}$$

Indeed, in view of (3.10) and (3.15),

$$\begin{aligned} D_c^{\alpha-1} f &= I^{2-\alpha} D \left(I^\alpha g + ap_\alpha + bp_{\alpha-1} + dp_0\right) \\ &= I^{2-\alpha} I^{\alpha-1} g + aI^{2-\alpha} p_{\alpha-1} + bI^{2-\alpha} p_{\alpha-2} \\ &= Ig + ap_1 + bp_0 \end{aligned}$$

and

$$\begin{aligned} D^{\alpha-1} f &= DI^{2-\alpha} \left(I^\alpha g + ap_\alpha + bp_{\alpha-1} + cp_{\alpha-2}\right) \\ &= DI^2 g + aDp_2 + bDp_1 + cDp_0 \\ &= Ig + ap_1 + bp_0. \end{aligned}$$

- *Left Dirichlet boundary condition*: Let f be continuous as $x \downarrow 0$ and $f(0) = 0$, then

  $$0 = I^\alpha g(0) + ap_\alpha(0) + bp_{\alpha-1}(0) + cp_{\alpha-2}(0) + dp_0(0).$$

  Note that $p_\alpha(0),\ p_{\alpha-1}(0) = 0$ and $I^\alpha g(0) = 0$ by Proposition 3.3.3. This implies that $c, d = 0$. Moreover, in the case when $X = C_0(\Omega)$, besides $g(0) = 0$ we also require that the image $Af \in C_0(\Omega)$; that is, $Af(0) = 0$. Thus, we have that $a = 0$, since using (3.23),

  $$0 = Af(0) = (DFf)(0) = \left(D\left(Ig + ap_1 + bp_0\right)\right)(0) = g(0) + a = a.$$

  On the other hand, in the case when $X = L_1[0,1]$, since $ap_\alpha = I^\alpha(ap_0)$, the constant a turns out to be redundant as the term ap_0 can be incorporated into $g \in L_1[0,1]$. Therefore, we set $a = 0$.

- *Left Neumann boundary condition*: Let $D_c^{\alpha-1}f(0) = 0$ or $D^{\alpha-1}f(0) = 0$, then in view of Remark 3.3.5, $c = 0$ or $d = 0$, respectively. Moreover, using (3.23) we have

$$0 = Ig(0) + ap_1(0) + bp_0(0) = b,$$

since $p_1(0) = 0$ and $Ig(0) = 0$ in view of Proposition 3.3.3.

- *Right Dirichlet boundary condition*: Let f be continuous as $x \uparrow 1$ and $f(1) = 0$, then

$$0 = I^\alpha g(1) + ap_\alpha(1) + bp_{\alpha-1}(1) + cp_{\alpha-2}(1) + dp_0(1).$$

Moreover, in the case when $X = C_0(\Omega)$ we require that $g, Af \in C_0(\Omega)$; that is, $g(1),\ Af(1) = 0$. Therefore, $a = 0$, since using (3.23),

$$0 = Af(1) = (DFf)(1) = (D(Ig + ap_1 + bp_0))(1) = g(1) + a = a.$$

On the other hand, in the case when $X = L_1[0,1]$, since $ap_\alpha = I^\alpha(ap_0)$, the constant a again turns out to be redundant as the term ap_0 can be incorporated into $g \in L_1[0,1]$. Therefore, we set $a = 0$. Hence,

$$\frac{b + (\alpha-1)c}{\Gamma(\alpha)} + d = -I^\alpha g(1).$$

Further, note that if $X = C_0(\Omega)$, since $c = 0$ in view of Remark 3.3.5, this reduces to

$$\frac{b}{\Gamma(\alpha)} + d = -I^\alpha g(1).$$

- *Right Neumann* boundary condition*: Note that $p_{\alpha-2} \notin C_0(\Omega)$ and we use this boundary condition only for $X = C_0(\Omega)$. Let Df be continuous as $x \uparrow 1$ and $Df(1) = 0$, then

$$D(I^\alpha g + ap_\alpha + bp_{\alpha-1} + dp_0) = I^{\alpha-1}g + ap_{\alpha-1} + bp_{\alpha-2}.$$

Thus,

$$0 = I^{\alpha-1}g(1) + ap_{\alpha-1}(1) + bp_{\alpha-2}(1)$$

and

$$\frac{a + (\alpha-1)b}{\Gamma(\alpha)} = -I^{\alpha-1}g(1).$$

- *Right Neumann boundary condition*: Let $D_c^{\alpha-1}f(1) = 0$ or $D^{\alpha-1}f(1) = 0$; that is, $Ff(1) = 0$, then using (3.23) we have

$$0 = Ig(1) + a + b.$$

This implies that if $Ff(1) = 0$, then

$$a + b = -Ig(1).$$

$\square$

3.4 Properties of the one-sided fractional derivative operators on a bounded interval

We prepare for the important task of showing that the closures of the fractional derivative operators (A, BC) generate contraction semigroups. To do this, we need to show the following

- (A, BC) are densely defined, dissipative operators.
- $\mathrm{rg}(\lambda I - A)$ are dense in X for some $\lambda > 0$.

Then, as a consequence of the Lumer-Phillips Theorem (Theorem C.2.9) we can conclude that their closures generate contraction semigroups on X. In fact, we show that the operators (A, BC) are closed. Hence, we actually conclude that the operators (A, BC) themselves generate contraction semigroups.

In the remainder of this section, we establish that (A, BC) are densely defined, closed operators, identify a core for each (A, BC) and show that $\mathrm{rg}(\lambda I - A)$ are dense in X. In Section 4.1, we take up the issue of dissipativity of the operators (A, BC).

We refer to Definition C.1.4 for the definition of invertible operators.

Proposition 3.4.1. *Let the operators* (A, BC) *be such that their domains* $\mathcal{D}(A, \mathrm{BC})$*, given by Definition 3.3.8, encode at least one Dirichlet boundary condition. Then the operators* (A, BC) *are invertible.*

Proof. Let the fractional derivative operator (A, BC) have either a left or right Dirichlet boundary condition. For each such (A, BC), we show that there is a bounded operator B on X such that $BAf = f$ for all $f \in \mathcal{D}(A, \mathrm{BC})$, and $Bg \in \mathcal{D}(A, \mathrm{BC})$ and $ABg = g$ for all $g \in X$.

First, making use of Table 3.1, note that $a = 0$ in Definition 3.3.8. Next, as long as one of the boundary conditions for the fractional derivative operator is Dirichlet, we show that B_c and B_r are the inverses of $(D_c^\alpha, \mathrm{BC})$ and (D^α, BC), respectively, where for $g \in X$

$$B_c g = I^\alpha g + b p_{\alpha-1} + d p_0,$$

$$B_r g = I^\alpha g + bp_{\alpha-1} + cp_{\alpha-2}, \tag{3.24}$$

and the relations satisfied by the constants b, c, d depending on $g \in X$ and BC are given in Table 3.1.

Clearly, since $a = 0$, $B_c g \in \mathcal{D}(D_c^\alpha, \mathrm{BC})$ and $B_r g \in \mathcal{D}(D^\alpha, \mathrm{BC})$, in view of Definition 3.3.8 and Table 3.1. Moreover, I^α is bounded on X in view of Remark 3.3.2 while the constants b, c, d depend continuously on g. Thus, in view of (3.7), B_c is a bounded operator on X and B_r is a bounded operator on $L_1[0,1]$. Note that,

$$\begin{aligned} D_c^\alpha B_c g &= D_c^\alpha \left(I^\alpha g + bp_{\alpha-1} + dp_0\right) = g, \\ D^\alpha B_r g &= D^\alpha \left(I^\alpha g + bp_{\alpha-1} + cp_{\alpha-2}\right) = g. \end{aligned} \tag{3.25}$$

which follows directly on applying (3.14) and (3.17).

For $f \in \mathcal{D}(A)$, making use of Proposition 3.3.4 we have

$$\begin{aligned} B_c D_c^\alpha f &= I^\alpha D_c^\alpha f + bp_{\alpha-1} + dp_0 \\ &= f + \left(b - D_c^{\alpha-1} f(0)\right) p_{\alpha-1} + (d - f(0))\, p_0 \end{aligned}$$

and

$$\begin{aligned} B_r D^\alpha f &= I^\alpha D^\alpha f + bp_{\alpha-1} + cp_{\alpha-2} \\ &= f + \left(b - D^{\alpha-1} f(0)\right) p_{\alpha-1} + \left(c - I^{2-\alpha} f(0)\right) p_{\alpha-2}. \end{aligned}$$

To complete the proof we show that $B_c D_c^\alpha f = f$ and $B_r D^\alpha f = f$. In what follows, we make use of Proposition 3.3.4 as required.

- *Left Dirichlet boundary condition*: In this case, since f is continuous as $x \downarrow 0$ and $f(0) = 0$, we have $I^{2-\alpha} f(0) = 0$ by Proposition 3.3.3. Moreover, $a, c, d = 0$. To complete the proof for the left Dirichlet boundary condition, we verify that in each of the scenarios below that either $b = D_c^{\alpha-1} f(0)$ or $b = D^{\alpha-1} f(0)$, as required.

  1. *Right Dirichlet boundary condition*: Let $f(1) = 0$, then

  $$\begin{aligned} b &= -\Gamma(\alpha) I^\alpha D_c^\alpha f(1) \\ &= -\Gamma(\alpha) \left(f(1) - D_c^{\alpha-1} f(0) p_{\alpha-1}(1) - f(0) p_0(1)\right) = D_c^{\alpha-1} f(0) \end{aligned}$$

  or

  $$\begin{aligned} b &= -\Gamma(\alpha) I^\alpha D^\alpha f(1) \\ &= -\Gamma(\alpha) \left(f(1) - D^{\alpha-1} f(0) p_{\alpha-1}(1) - I^{2-\alpha} f(0) p_{\alpha-2}(1)\right) = D^{\alpha-1} f(0). \end{aligned}$$

2. *Right Neumann* boundary condition*: Let $Df(1) = 0$, then

$$\begin{aligned} b &= -\Gamma(\alpha-1) I^{\alpha-1} D_c^\alpha f(1) \\ &= -\Gamma(\alpha-1) D\left(I^\alpha D_c^\alpha f\right)(1) = -\Gamma(\alpha-1)\left(Df(1) - D_c^{\alpha-1} f(0) p_{\alpha-2}(1)\right) \\ &= D_c^{\alpha-1} f(0). \end{aligned}$$

3. *Right Neumann boundary condition*: Let $Ff(1) = 0$, then

$$\begin{aligned} b &= -ID_c^\alpha f(1) = -ID\left(D_c^{\alpha-1} f\right)(1) \\ &= -D_c^{\alpha-1} f(1) + D_c^{\alpha-1} f(0) p_0(1) = D_c^{\alpha-1} f(0) \end{aligned}$$

or

$$\begin{aligned} b &= -ID^\alpha f(1) - ID\left(D^{\alpha-1} f\right)(1) \\ &= -D^{\alpha-1} f(1) + D^{\alpha-1} f(0) p_0(1) = D^{\alpha-1} f(0). \end{aligned}$$

- *Right Dirichlet boundary condition*: In this case, $f(1) = 0$ and $a = 0$. Note that we are only left to consider left Neumann boundary condition; that is, $D^{\alpha-1} f(0) = 0$ or $D_c^{\alpha-1} f(0) = 0$. We have that $b = 0$ and either

$$d = -I^\alpha D_c^\alpha f(1) = -f(1) + D_c^{\alpha-1} f(0) p_{\alpha-1}(1) + f(0) p_0(1) = f(0)$$

or

$$\begin{aligned} c &= -\Gamma(\alpha-1) I^\alpha D^\alpha f(1) \\ &= -\Gamma(\alpha-1)\left(f(1) - D^{\alpha-1} f(0) p_{\alpha-1}(1) - I^{2-\alpha} f(0) p_{\alpha-2}(1)\right) \\ &= I^{2-\alpha} f(0). \end{aligned}$$

Thus, we have shown that $B_c D_c^\alpha f = f$ and $B_r D^\alpha f = f$ for f in the respective domains. Hence, the operator (A, BC) are invertible if their domains encode at least one Dirichlet boundary condition. □

Remark 3.4.2. In the case of a left Neumann boundary condition combined with one of the following right Neumann boundary conditions, the operators (A, BC) are not invertible:

1. *Right Neumann* boundary condition*: In this case the constant d is undetermined.
2. *Right Neumann boundary condition*: The constant d is undetermined in the case of $(D_c^\alpha, \mathrm{BC})$ while the constant c is undetermined in the case of (D^α, BC).

Remark 3.4.3. In what follows, we use the phrase *polynomials belonging to* X and write $P \in X$ where $X = C_0(\Omega)$ or $L_1[0,1]$ and P are N^{th} degree polynomials of the form

$$P(x) = \sum_{m=0}^{N} k_m p_m(x), \ p_m(x) = \frac{x^m}{\Gamma(m+1)}, \ x \in [0,1].$$

Note that here we implicitly assume that the constants $k_m \in \mathbb{R}$ are such that $P \in X$. This is crucial especially when working in $C_0(\Omega)$. For instance, for a left Dirichlet boundary condition we require $P \in C_0(0,1)]$; that is, we require that $P(0) = 0$ and so $k_0 = 0$. On the other hand, for a left Neumann boundary condition, $P \in C[0,1)]$ and thus the constant k_0 need not be zero. For a right Dirichlet boundary condition we require $P \in C_0[(0,1)$, thus the constants k_m are such that $P(1) = 0$. However, this need not be the case for a right Neumann boundary condition. Lastly, note that we repeatedly make use of the well known fact that the polynomials belonging to X are dense in X, Stone-Weierstrass theorem [42, see p. 141 Corollary 4.50].

We refer to Definition C.1.2 for the definition of a core (subspace dense in the graph norm).

Theorem 3.4.4. *The fractional derivative operators* (A, BC) *given by Definition 3.3.8 with* $1 < \alpha < 2$ *are densely defined, closed operators on* X*, where* $X = C_0(\Omega)$ or $L_1[0,1]$ *given by* (3.4)*. Moreover,*

1. *The subspace*

$$\mathcal{C}(D_c^\alpha, \mathrm{BC}) = \{f_n : f_n = I^\alpha P_n + a_n p_\alpha + b_n p_{\alpha-1} + d_n p_0, \ n \in \mathbb{N}\} \tag{3.26}$$

is a core of the operator $(D_c^\alpha, \mathrm{BC})$*, where* $P_n = \sum_{m=0}^{N_n} k_m p_m \in X$ *and the constants* $a_n, \ b_n, \ d_n \in \mathbb{R}$ *are given in Table 3.1.*

2. *The subspace*

$$\mathcal{C}(D^\alpha, \mathrm{BC}) = \{f_n : f_n = I^\alpha P_n + a_n p_\alpha + b_n p_{\alpha-1} + c_n p_{\alpha-2}, \ n \in \mathbb{N}\} \tag{3.27}$$

is a core of the operator (D^α, BC)*, where* $P_n = \sum_{m=0}^{N_n} k_m p_m \in X$ *and the constants* $a_n, b_n, c_n \in \mathbb{R}$ *are given in Table 3.1.*

Proof. **Proof of** (A, BC) **are densely defined:**

Let (A, BC) be a fractional derivative operator on $X = C_0(\Omega)$ or $L_1[0,1]$. Then, given $\epsilon > 0$ and $\phi \in X$, we show that there exists $f \in \mathcal{D}(A, \mathrm{BC})$ such that

$$\|\phi - f\|_X < \epsilon.$$

To this end, let $\phi \in X$ and $\epsilon > 0$. Without loss of generality, if $X = L_1[0,1]$, we set $\phi(0) = \phi(1) = 0$. Let

$$\tilde{\phi} = \phi - (\phi(1) - \phi(0))\Gamma(\alpha+2)p_{\alpha+1} - \phi(0)p_0,$$

and note that $\tilde{\phi} \in C_0(0,1)$ or $L_1(0,1)$. As $C_c^\infty(0,1)$ is dense in $C_0(0,1)$ and $L_1(0,1)$, there exist $\theta \in C_c^\infty(0,1)$ such that

$$\left\|\tilde{\phi} - \theta\right\|_X = \|\phi - (\phi(1) - \phi(0))\Gamma(\alpha+2)p_{\alpha+1} - \phi(0)p_0 - \theta\|_X < \frac{\epsilon}{2}. \tag{3.28}$$

For such $\theta \in C_c^\infty(0,1)$, let

$$g_\theta = I^{2-\alpha}D^2\theta + (\phi(1) - \phi(0))\Gamma(\alpha+2)p_1$$

so that $I^\alpha g_\theta = \theta + (\phi(1) - \phi(0))\Gamma(\alpha+2)p_{\alpha+1}$. Next, for $0 < \delta := \delta(\epsilon) < 1$, define

$$g_\delta(x) = \begin{cases} 0, & \text{for } 0 \le x \le 1-\delta, \\ C(\delta)\Big(\delta p_1(x-1+\delta) - (\alpha+2)p_2(x-1+\delta)\Big), & \text{for } 1-\delta < x \le 1. \end{cases} \tag{3.29}$$

Let us set $g = g_\theta + g_\delta$ and construct the following function

$$f = \theta + (\phi(1) - \phi(0))\Gamma(\alpha+2)p_{\alpha+1} + I^\alpha g_\delta + \phi(0)p_0 = I^\alpha g + \phi(0)p_0.$$

Note that g_δ satisfies all the left boundary conditions. Therefore, we need to show that for each right boundary condition of BC there exists $\delta > 0$ and $C(\delta)$ such that $g \in X$ and the constants $a,\ b,\ c = 0$, and $d = \phi(0)$ if $X = C_0(\Omega)$ or $d = 0$ if $X = L_1[0,1]$.

- *Right Dirichlet boundary condition*: For f to belong to $\mathcal{D}(A, \bullet\mathrm{D})$ we need to find $C(\delta)$ such that $g \in X$ and $I^\alpha g(1) = -\phi(0)$; that is,

  $$0 = g(1) = g_\theta(1) + g_\delta(1).$$

  Thus, we set $g_\delta(1) = -g_\theta(1)$ and obtain

  $$C(\delta) = \frac{-g_\theta(1)}{(1 - \frac{\alpha+2}{2})}\delta^{-2}.$$

  Note that $I^\alpha g(1) = \phi(1) - \phi(0) = -\phi(0)$, since $\theta(1) = 0$ and $I^\alpha g_\delta(1) = 0$.

- *Right Neumann boundary condition*: For f to belong to $\mathcal{D}(A, \bullet\mathrm{N})$ we need to find $C(\delta)$ such that $g \in X$ and $Ig(1) = 0$; that is,

  $$0 = Ig(1) = Ig_\theta(1) + Ig_\delta(1).$$

  Thus, we set $Ig_\delta(1) = -Ig_\theta(1)$ and obtain

  $$C(\delta) = \frac{-Ig_\theta(1)}{(\frac{1}{2} - \frac{\alpha+2}{6})}\delta^{-3}.$$

- *Right Neumann* boundary condition*: For f to belong to $\mathcal{D}(A, \bullet \mathrm{N}^*)$ we need to find $C(\delta)$ such that $g \in X$ and $I^{\alpha-1}g(1) = 0$; that is,

$$0 = I^{\alpha-1}g(1) = I^{\alpha-1}g_\theta(1) + I^{\alpha-1}g_\delta(1).$$

Thus, we set $I^{\alpha-1}g_\delta(1) = -I^{\alpha-1}g_\theta(1)$ and obtain

$$C(\delta) = \frac{-I^{\alpha-1}g_\theta(1)}{\left(\frac{1}{\Gamma(\alpha+1)} - \frac{\alpha+2}{\Gamma(\alpha+2)}\right)}\delta^{-\alpha-1}.$$

To complete the proof, observe that for any BC, we have

$$|C(\delta)| < K\delta^{-3}.$$

Therefore,

$$\|I^\alpha g_\delta\|_{L_\infty[0,1]} = \sup_{x\in(1-\delta,1]} |I^\alpha g_\delta(x)| \leq |C(\delta)| \frac{2\delta^{\alpha+2}}{\Gamma(\alpha+2)} < C_1\delta^{\alpha-1}$$

and

$$\|I^\alpha g_\delta\|_{L_1[0,1]} \leq |C(\delta)| \frac{2\delta^{\alpha+3}}{\Gamma(\alpha+3)} < C_2\delta^\alpha.$$

Thus, for $\phi \in X$, taking $\delta < \left(\frac{\epsilon}{2C}\right)^{\frac{1}{\alpha-1}}$ where $C = \max\{C_1, C_2\}$ and in view of (3.28),

$$\|\phi - f\|_X \leq \|\phi - \theta - (\phi(1) - \phi(0))\Gamma(\alpha+2)p_{\alpha+1} - \phi(0)p_0\|_X + \|I^\alpha g_\delta\|_X < \epsilon.$$

Hence, (A, BC) are densely defined operators on X.

Proof of (A, BC) are closed:

First, note that by Proposition 3.4.1, if the domains encode at least one Dirichlet boundary condition, then the operators (A, BC) are invertible and hence closed, see Definition C.1.4. On the other hand, as mentioned in Remark 3.4.2, when we have a left Neumann boundary condition combined with a right Neumann or Neumann* boundary condition, (A, BC) are not invertible. For those cases, we show that if $\{f_n\} \subset \mathcal{D}(A, \mathrm{BC})$ such that $f_n \to f$ and $Af_n \to \phi$ in X, then $f \in \mathcal{D}(A, \mathrm{BC})$ and $Af = \phi$.

In what follows, NN represents a combination of left Neumann boundary condition combined with a right Neumann or Neumann* boundary condition. Consider the sequences $\{f_n^c\} \in \mathcal{D}(D_c^\alpha, \mathrm{NN})$ and $\{f_n\} \in \mathcal{D}(D^\alpha, \mathrm{NN})$ given by

$$f_n^c = I^\alpha g_n + a_n p_\alpha + b_n p_{\alpha-1} + d_n p_0,\ g_n \in X$$

and

$$f_n = I^\alpha g_n + a_n p_\alpha + b_n p_{\alpha-1} + c_n p_{\alpha-2},\ g_n \in X,$$

respectively. Firstly, using Table 3.1, since we have a left Neumann boundary condition, $b_n = 0$. Thus,

$$f_n^c = I^\alpha g_n + a_n p_\alpha + d_n p_0,$$

where

$$a_n = -Ig_n(1) \text{ or } a_n = -\Gamma(\alpha) I^{\alpha-1} g_n(1)$$

for a right Neumann or a right Neumann* boundary condition, respectively. Also,

$$f_n = I^\alpha g_n + a_n p_\alpha + c_n p_{\alpha-2},$$

where $a_n = -Ig_n(1)$ for a right Neumann boundary condition. In either scenario,

$$D_c^\alpha f_n^c,\ D^\alpha f_n = g_n + a_n p_0,$$

since $D_c^\alpha p_0 = \mathbf{0}$ and $D^\alpha p_{\alpha-2} = \mathbf{0}$ in view of (3.17).

Let $f_n^c \to f^c$, $f_n \to f$ and $D_c^\alpha f_n^c$, $D^\alpha f_n = g_n + a_n p_0 \to \phi$ in X. Then, in view of Remark 3.3.2, since I^α is continuous on X,

$$I^\alpha (g_n + a_n p_0) = I^\alpha g_n + a_n p_\alpha \to I^\alpha \phi.$$

Thus,

$$d_n p_0 = f_n^c - (I^\alpha g_n + a_n p_\alpha) \to f^c - I^\alpha \phi \in X$$

and

$$c_n p_{\alpha-2} = f_n - (I^\alpha g_n + a_n p_\alpha) \to f - I^\alpha \phi \in X.$$

Therefore, there exist d, c such that $d_n \to d$ and $c_n \to c$. Hence,

$$f^c = I^\alpha \phi + d p_0 \text{ and } f = I^\alpha \phi + c p_{\alpha-2}.$$

It remains to show that $f^c \in \mathcal{D}(D_c^\alpha, \mathrm{NN})$ and $f \in \mathcal{D}(D^\alpha, \mathrm{NN})$. To this end, using Table 3.1, note first that f^c and f satisfy the left Neumann boundary condition, since $b = 0$. To see that the right boundary conditions are satisfied note that:

1. For a right Neumann boundary condition, since I is continuous on X, we have

$$I\phi(1) = \lim_{n\to\infty} \left(I \left(g_n - Ig_n(1) p_0 \right)(1) \right) = \lim_{n\to\infty} \left(Ig_n(1) - Ig_n(1) p_1(1) \right) = 0.$$

2. For a right Neumann* boundary condition, since $I^{\alpha-1}$ is continuous on X, we have

$$\begin{aligned} I^{\alpha-1}\phi(1) &= \lim_{n\to\infty} \left(I^{\alpha-1} \left(g_n - \Gamma(\alpha) I^{\alpha-1} g_n(1) p_0 \right)(1) \right) \\ &= \lim_{n\to\infty} \left(I^{\alpha-1} g_n(1) - \Gamma(\alpha) I^{\alpha-1} g_n(1) p_{\alpha-1}(1) \right) = 0. \end{aligned}$$

Therefore, $f^c \in \mathcal{D}(D_c^\alpha, \mathrm{NN})$ and $f \in \mathcal{D}(D^\alpha, \mathrm{NN})$. Lastly, since $D_c^\alpha p_0 = \mathbf{0}$ and $D^\alpha p_{\alpha-2} = \mathbf{0}$ in view of (3.17), we have

$$D_c^\alpha f^c = D_c^\alpha (I^\alpha \phi + dp_0) = \phi$$

and

$$D^\alpha f = D^\alpha (I^\alpha + cp_{\alpha-2}) = \phi$$

Hence, the fractional derivative operators (A, BC) are closed in X.

Proof of $\mathcal{C}(A, \mathrm{BC})$ is a core of (A, BC):

Let $\mathcal{C}(A, \mathrm{BC})$ be given by (3.26) and (3.27) for $A = D_c^\alpha$ and $A = D^\alpha$, respectively. To show that $\mathcal{C}(A, \mathrm{BC})$ is a core of (A, BC), we show that for any $f \in \mathcal{D}(A, \mathrm{BC})$, there exists $f_n \subset \mathcal{C}(A, \mathrm{BC})$ such that $f_n \to f$ and $Af_n \to Af$.

To this end, let $f \in \mathcal{D}(A, \mathrm{BC})$ be given by

$$f = I^\alpha g + ap_\alpha + bp_{\alpha-1} + cp_{\alpha-2} + dp_0, \ g \in X.$$

Then, consider

$$f_n = I^\alpha P_n + a_n p_\alpha + b_n p_{\alpha-1} + c_n p_{\alpha-2} + d_n p_0,$$

where $P_n = \sum_{m=0}^{N_n} k_m p_m \in X$ and the constants a_n, b_n, c_n and d_n are given in Table 3.1. Note that $f_n \in \mathcal{C}(A, \mathrm{BC})$ and since the polynomials belonging to X are dense in X, for each $g \in X$, there exist polynomials $P_n \in X$ such that $P_n \to g$ in X.

Consider the following two cases:

1. *At least one Dirichlet boundary condition*: In these cases, using Table 3.1, observe that $a_n = 0$ while b_n, c_n and d_n are either zero or depend continuously on $I^\nu P_n$ for $\nu \in \{\alpha, \ 1, \ \alpha - 1\}$.

2. *No Dirichlet boundary condition*: In these cases, $b_n = 0$ since we have a left Neumann boundary condition while a_n depends continuously on $I^\nu P_n$ for $\nu \in \{1, \ \alpha - 1\}$.

   - *Right Neumann* boundary condition*: In this case set $d_n = d$.
   - *Right Neumann boundary condition*: For $A = D_c^\alpha$, set $d_n = d$ and for $A = D^\alpha$, set $c_n = c$.

In view of Remark 3.3.2, $I^\nu, \ \nu > 0$ is continuous on X, thus $I^\nu P_n \to I^\nu g$ in X for $\nu \in \{\alpha, \ 1, \ \alpha - 1\}$. Hence, $f_n \to f$ in X. Moreover, using (3.14) and (3.17) we have that

$$Af_n = P_n + a_n p_0 \text{ and } Af = g + ap_0.$$

Thus, $Af_n \to Af$ in X and hence, $\mathcal{C}(A, \mathrm{BC})$ is a core of (A, BC). $\square$

Let us turn to the task of showing that for each (A, BC), $\mathrm{rg}(\lambda I - A)$ is dense in X for some $\lambda > 0$ as part of the requirement of the Lumer-Phillips Theorem as mentioned at the start of Section 3.4. Since it is sufficient to show denseness of $\mathrm{rg}(\lambda I - A)$ for some $\lambda > 0$, let $\lambda = 1$. We show that $\mathrm{rg}(I - A)$ is dense in X using the fact that the polynomials belonging to X are dense in X. To be precise, we seek functions $\varphi \in X$ such that $(I - A)\varphi = P = \sum_{m=0}^{n} k_m p_m \in X$. In Theorem 3.4.5 we show that linear combinations of a certain variant of the two parameter functions of Mittag-Leffler type yield such candidates.

To this end, consider the two parameter function of Mittag-Leffler type, which we denote by $E^*_{\alpha,\beta}$ for $\alpha > 0$, defined by the series expansion,

$$E^*_{\alpha,\beta}(x) := \sum_{n=0}^{\infty} p_{n\alpha+\beta}(x), \text{ if } -1 < \beta < 0 \text{ for } x \in (0,1], \text{ or if } \beta \geq 0 \text{ for } x \in [0,1]. \quad (3.30)$$

Note the connection with the standard two parameter Mittag-Leffler function $E_{\alpha,\beta}$, $E^*_{\alpha,\beta}(x) = x^\beta E_{\alpha,\beta+1}(x^\alpha)$, see [48] for details. We choose this slightly different form of the two parameter Mittag-Leffler function in order to obtain the eigenfunctions as given in (3.39) below, which we require in the proof of Theorem 3.4.5. Using (3.7), we have

$$\left\|E^*_{\alpha,\beta}\right\|_{L_1[0,1]} \leq \sum_{n=0}^{\infty} \|p_{n\alpha+\beta}\|_{L_1[0,1]} \leq \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+\beta+2)} < \infty,$$

thus

$$E^*_{\alpha,\beta} \in L_1[0,1] \text{ and if } \beta \geq 0,\ E^*_{\alpha,\beta} \in C[0,1]. \quad (3.31)$$

Take note of the crucial recurrence relation

$$E^*_{\alpha,\beta} = p_\beta + E^*_{\alpha,\beta+\alpha}, \quad (3.32)$$

since

$$E^*_{\alpha,\beta}(x) = p_\beta(x) + \sum_{n=1}^{\infty} p_{n\alpha+\beta}(x) = p_\beta(x) + \sum_{k=0}^{\infty} p_{k\alpha+\beta+\alpha}(x) = p_\beta(x) + E^*_{\alpha,\beta+\alpha}(x).$$

Using (3.15) and the fact that I^γ is bounded on $L_1[0,1]$, see Remark 3.3.2, the Riemann-Liouville fractional integral of $E^*_{\alpha,\beta} \in L_1[0,1]$ of order $\nu > 0$ is given by

$$I^\nu\left(E^*_{\alpha,\beta}\right) = I^\nu\left(\sum_{n=0}^{\infty} p_{n\alpha+\beta}\right) = \sum_{n=0}^{\infty} p_{n\alpha+\beta+\nu} = E^*_{\alpha,\beta+\nu}. \quad (3.33)$$

Let $k \in \{1,2\}$, then in view of (3.8), for $\beta > k-1$,

$$D^k E^*_{\alpha,\beta} = D^k \sum_{n=0}^{\infty} p_{n\alpha+\beta} = \sum_{n=0}^{\infty} D^k p_{n\alpha+\beta} = \sum_{n=0}^{\infty} p_{n\alpha+\beta-k} = E^*_{\alpha,\beta-k}, \tag{3.34}$$

where to interchange summation with differentiation we have used the fact that the generalised integer-order derivative operator $\left(D^k, W^{k,1}[0,1]\right)$ given in Definition B.1.1 is closed in $L_1[0,1]$. Moreover, since $\beta > 0$ the same result holds true for $C[0,1]$, taking D^k to be the classical integer order derivative.

Let $1 < \alpha < 2$ and $m \in \mathbb{N}_0$. We now list the derivatives and fractional derivatives of the two parameter Mittag-Leffler functions that we require in the proof of Theorem 3.4.5. The fractional derivatives of order $\alpha - 1$ are given by

$$\begin{aligned} D^{\alpha-1} E^*_{\alpha,\alpha+m} &= D I^{2-\alpha} E^*_{\alpha,\alpha+m} = E^*_{\alpha,m+1}, \\ D_c^{\alpha-1} E^*_{\alpha,\alpha+m} &= I^{2-\alpha} D E^*_{\alpha,\alpha+m} = I^{2-\alpha} E^*_{\alpha,\alpha+m-1} = E^*_{\alpha,m+1}, \end{aligned} \tag{3.35}$$

where we have used (3.33) and (3.34). Moreover, using (3.32) and (3.35), the fractional derivatives of order α are given by

$$\begin{aligned} D^{\alpha} E^*_{\alpha,\alpha+m} &= D D^{\alpha-1} E^*_{\alpha,\alpha+m} = E^*_{\alpha,m} = p_m + E^*_{\alpha,\alpha+m}, \\ D_c^{\alpha} E^*_{\alpha,\alpha+m} &= D I^{2-\alpha} D E^*_{\alpha,\alpha+m} = E^*_{\alpha,m} = p_m + E^*_{\alpha,\alpha+m}, \end{aligned} \tag{3.36}$$

We make use of the derivatives and fractional derivatives of these three functions, $E^*_{\alpha,0}$, $E^*_{\alpha,\alpha-1}$ and $E^*_{\alpha,\alpha-2}$ in the proof of the next theorem which are obtained using (3.17) (3.32), (3.33) and (3.34).

- First derivatives:

$$\begin{aligned} D E^*_{\alpha,0} &= D\left(p_0 + E^*_{\alpha,\alpha}\right) = E^*_{\alpha,\alpha-1}, \\ D E^*_{\alpha,\alpha-1} &= E^*_{\alpha,\alpha-2}. \end{aligned} \tag{3.37}$$

- Fractional derivatives of order $\alpha - 1$:

$$\begin{aligned} D_c^{\alpha-1} E^*_{\alpha,0} &= I^{2-\alpha} D E^*_{\alpha,0} = I^{2-\alpha} E^*_{\alpha,\alpha-1} = E^*_{\alpha,1}, \\ D_c^{\alpha-1} E^*_{\alpha,\alpha-1} &= I^{2-\alpha} D E^*_{\alpha,\alpha-1} = I^{2-\alpha} E^*_{\alpha,\alpha-2} = E^*_{\alpha,0}, \\ D^{\alpha-1} E^*_{\alpha,\alpha-1} &= D I^{2-\alpha} E^*_{\alpha,\alpha-1} = D E^*_{\alpha,1} = E^*_{\alpha,0}, \\ D^{\alpha-1} E^*_{\alpha,\alpha-2} &= D^{\alpha-1}\left(p_{\alpha-2} + E^*_{\alpha,2\alpha-2}\right) = D I^{2-\alpha} E^*_{\alpha,2\alpha-2} = E^*_{\alpha,\alpha-1}. \end{aligned} \tag{3.38}$$

$\varphi = -\sum_{m=0}^{N} k_m E^*_{\alpha,\alpha+m} + rE^*_{\alpha,\alpha-1} + sE^*_{\alpha,\alpha-2} + tE^*_{\alpha,0}$				
X	(A, BC)	r	s	t
X	(A, DD)	$\frac{\sum_{m=0}^{N} k_m E^*_{\alpha,\alpha+m}(1)}{E^*_{\alpha,\alpha-1}(1)}$	0	0
X	(A, DN)	$\frac{\sum_{m=0}^{N} k_m E^*_{\alpha,m+1}(1)}{E^*_{\alpha,0}(1)}$	0	0
$C_0(\Omega)$	$(D^\alpha_c, \mathrm{DN}^*)$	$\frac{\sum_{m=0}^{N} k_m E^*_{\alpha,\alpha+m-1}(1)}{E^*_{\alpha,\alpha-2}(1)}$	0	0
X	$(D^\alpha_c, \mathrm{ND})$	0	0	$\frac{\sum_{m=0}^{N} k_m E^*_{\alpha,\alpha+m}(1)}{E^*_{\alpha,0}(1)}$
X	$(D^\alpha_c, \mathrm{NN})$	0	0	$\frac{\sum_{m=0}^{N} k_m E^*_{\alpha,m+1}(1)}{E^*_{\alpha,\alpha-1}(1)}$
$C_0(\Omega)$	$(D^\alpha_c, \mathrm{NN}^*)$	0	0	$\frac{\sum_{m=0}^{N} k_m E^*_{\alpha,\alpha+m-1}(1)}{E^*_{\alpha,1}(1)}$
$L_1[0,1]$	(D^α, ND)	0	$\frac{\sum_{m=0}^{N} k_m E^*_{\alpha,\alpha+m}(1)}{E^*_{\alpha,\alpha-2}(1)}$	0
$L_1[0,1]$	(D^α, NN)	0	$\frac{\sum_{m=0}^{N} k_m E^*_{\alpha,m+1}(1)}{E^*_{\alpha,\alpha-1}(1)}$	0

Table 3.2: Constants r, s, t for φ given by (3.40).

- Fractional derivatives of order α:

$$\begin{aligned}
D^\alpha_c E^*_{\alpha,0} &= DD^{\alpha-1}_c E^*_{\alpha,0} = DE^*_{\alpha,1} = E^*_{\alpha,0},\\
D^\alpha_c E^*_{\alpha,\alpha-1} &= DD^{\alpha-1}_c E^*_{\alpha,\alpha-1} = DE^*_{\alpha,0} = E^*_{\alpha,\alpha-1},\\
D^\alpha E^*_{\alpha,\alpha-1} &= DD^{\alpha-1} E^*_{\alpha,\alpha-1} = DE^*_{\alpha,0} = E^*_{\alpha,\alpha-1},\\
D^\alpha E^*_{\alpha,\alpha-2} &= DD^{\alpha-1} E^*_{\alpha,\alpha-2} = E^*_{\alpha,\alpha-2}.
\end{aligned} \tag{3.39}$$

Theorem 3.4.5. *Let* (A, BC) *denote the fractional derivative operators on* X *as in Definition 3.3.8. Then,* $\mathrm{rg}(I - A)$ *are dense in* X *for each* (A, BC).

Proof. To show that $\mathrm{rg}(I-A)$ is dense in $X = C_0(\Omega)$ or $L_1[0,1]$ for each of the fractional derivative operators (A, BC), we show that the polynomials $P = \sum_{m=0}^{N} k_m p_m \in X$ are in $\mathrm{rg}(I - A)$.

To this end, let $P = \sum_{m=0}^{N} k_m p_m \in X$ as in Remark 3.4.3. Next, define

$$\varphi = -\sum_{m=0}^{N} k_m E^*_{\alpha,\alpha+m} + rE^*_{\alpha,\alpha-1} + sE^*_{\alpha,\alpha-2} + tE^*_{\alpha,0}, \tag{3.40}$$

where the constants r, s, t are given in Table 3.2 for each fractional derivative operator (A, BC).

We show that $\varphi \in X$, $\varphi \in \mathcal{D}(A, \mathrm{BC})$ and $(I - A)\varphi = P$. Firstly, note that if $X = C_0(\Omega)$, then $s = 0$ in (3.40). Recall that $k_m \in \mathbb{R}$ ensure that $P \in X$, then in view of (3.31) and using the constants r, s, t given in Table 3.2, it follows that $\varphi \in X$.

In view of Proposition 3.3.10, note that $\varphi \in \mathcal{D}(A, \mathrm{BC})$, if φ satisfies the respective boundary conditions BC for the operator (A, BC).

For each (A, BC), using (3.35), (3.36), (3.37), (3.38) and (3.39), it is easily verified that $\varphi \in X$ satisfies the respective BC. For instance, consider (A, DD). Then, using Table 3.2

$$\varphi_{\mathrm{DD}}(x) = -\sum_{m=0}^{N} k_m E^*_{\alpha,\alpha+m}(x) + \frac{\sum_{m=0}^{N} k_m E^*_{\alpha,\alpha+m}(1)}{E^*_{\alpha,\alpha-1}(1)} E^*_{\alpha,\alpha-1}(x).$$

Thus, $\varphi_{\mathrm{DD}}(0) = 0$ and $\varphi_{\mathrm{DD}}(1) = 0$. Moreover,

$$\begin{aligned}
A\varphi_{\mathrm{DD}}(x) &= -\sum_{m=0}^{N} k_m A E^*_{\alpha,\alpha+m}(x) + \frac{\sum_{m=0}^{N} k_m E^*_{\alpha,\alpha+m}(1)}{E^*_{\alpha,\alpha-1}(1)} A E^*_{\alpha,\alpha-1}(x) \\
&= -\sum_{m=0}^{N} k_m \left(p_m(x) + E^*_{\alpha,\alpha+m}(x)\right) + \frac{\sum_{m=0}^{N} k_m E^*_{\alpha,\alpha+m}(1)}{E^*_{\alpha,\alpha-1}(1)} E^*_{\alpha,\alpha-1}(x) \\
&= -P(x) + \varphi_{\mathrm{DD}}(x).
\end{aligned}$$

Since, $P \in X$, it also follows that $A\varphi_{\mathrm{DD}}(0) = 0$ and $A\varphi_{\mathrm{DD}}(1) = 0$. Hence, $\varphi_{\mathrm{DD}} \in \mathcal{D}(A, \mathrm{DD})$. Similar arguments work for the other boundary conditions BC.

To complete the proof, using (3.36) and (3.39), observe that

$$(I - A)\varphi = \sum_{m=0}^{N} k_m p_m = P.$$

Hence, $\mathrm{rg}(I - A)$ is dense in X since the polynomials belonging to X are dense in X. $\square$

Chapter 4

Grünwald-type approximations for fractional derivative operators on a bounded interval

In this chapter Grunwald-type (transition) approximation operators are constructed for the fractional derivative operators on X. The semigroups generated by the fractional derivative operators are shown to be the strong (and uniform for t in compact intervals) limit of the semigroups generated by the Grünwald transition operators. The underlying stochastic processes associated with the Grünwald approximation operators are identified. The highlight of the chapter is the result that shows the convergence in the Skorohod topology of the well-understood stochastic processes associated with the Grünwald (transition) approximation operators to the processes governed by the corresponding fractional-in-space partial differential equations.

4.1 Grünwald-type approximations for fractional derivative operators on $C_0(\Omega)$ and $L_1[0,1]$

In what follows, let $1 < \alpha < 2$ and $h = \frac{1}{n+1}$ for $n \in \mathbb{N}$. For the numerical scheme, the boundary conditions encoded by the domain of the fractional derivative operators

$X = C_0(\Omega), D(A,\mathrm{BC}) = \{I^\alpha g + ap_\alpha + bp_{\alpha-1} + dp_0 : g \in C_0(\Omega)\},\ (0,1) \subset \Omega$		
Boundary condition	Boundary weights for $G^h_{n\times n}$	$D(A,\mathrm{BC})$
$f(0) = 0$ $\Omega \subset (0,1]$	$b^l_i = \mathcal{G}^\alpha_i$ $b_n = b^r_n$ $D^l(\lambda) = \frac{\alpha\lambda}{\alpha\lambda+1-\lambda},\ N^l = \mathbf{1}$	$a = 0,\ d = 0$
$D^{\alpha-1}_c f(0) = 0$ $[0,1) \subset \Omega$	$b^l_i = -\mathcal{G}^{\alpha-1}_{i-1}$ $b_n = -\sum_{i=0}^{n-1} b^r_i$ $N^l(\lambda) = \lambda,\ D^l = \mathbf{1}$	$b = 0$
$f(1) = 0$ $\Omega \subset [0,1)$	$b^r_i = \mathcal{G}^\alpha_i$ $D^r(\lambda) = \frac{\alpha(1-\lambda)}{\alpha(1-\lambda)+\lambda},\ N^r = \mathbf{1}$	$a = 0,\ \frac{b}{\Gamma(\alpha)} + d = -I^\alpha g(1)$
$D^{\alpha-1}_c f(1) = 0$ $(0,1] \subset \Omega$	$b^r_i = -\mathcal{G}^{\alpha-1}_{i-1}$ $N^r(\lambda) = 1-\lambda,\ D^r = \mathbf{1}$	$a + b = -Ig(1)$
$Df(1) = 0$ $(0,1] \subset \Omega$	$b^r_0 = 0,\ b^r_1 = \mathcal{G}^\alpha_0 + \mathcal{G}^\alpha_1,$ $b^r_i = \mathcal{G}^\alpha_i$ $N^r(\lambda) = 1-\lambda,\ D^r = \mathbf{1}$	$\frac{a+(\alpha-1)b}{\Gamma(\alpha)} = -I^{\alpha-1}g(1)$

Table 4.1: Boundary conditions for $C_0(\Omega)$.

(A,BC) are built into the generic $n \times n$ shifted Grünwald matrix

$$G^h_{n\times n} = \frac{1}{h^\alpha}\begin{pmatrix} b^l_1 & \mathcal{G}^\alpha_0 & 0 & \cdots & 0 \\ \vdots & \mathcal{G}^\alpha_1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ b^l_{n-1} & \mathcal{G}^\alpha_{n-2} & \cdots & \mathcal{G}^\alpha_1 & \mathcal{G}^\alpha_0 \\ b_n & b^r_{n-1} & \cdots & \cdots & b^r_1 \end{pmatrix}, \tag{4.1}$$

using the boundary weights b^l_i, b^r_i and b_n which are listed in Tables 4.1 and 4.2 in terms of the Grünwald coefficients, for $C_0(\Omega)$ and $L_1[0,1]$, respectively. The $n \times n$ shifted Grünwald matrices $G^h_{n\times n}$ play the role of the transition rate matrices of the underlying finite state sub-Markov processes. The Grünwald transtion operators are constructed using the theory developed in Sections 3.1 and 3.2.

In Section 4.2, we first discuss the adjoint formulation of the abstract Cauchy problem on X associated with the fractional derivative operators (A,BC). In doing so, we list the corresponding fractional derivative operators on X that are approximated by the Grünwald transition operators. Following that we describe the physical reasons behind the choice of boundary weights b^l_i, b^r_i, and b_n that encode different boundary conditions BC and appear in the generic Grünwald matrix given by (4.1) in the $L_1[0,1]$

$X = L_1[0,1],\ D(A,\mathrm{BC}) = \{I^\alpha g + ap_\alpha + bp_{\alpha-1} + cp_{\alpha-2} + dp_0 : g \in L_1[0,1]\}$		
Boundary condition	Boundary weights for $G^h_{n\times n}$	$D(A,\mathrm{BC})$
$f(0) = 0$	$b^l_i = \mathcal{G}^\alpha_i$ $D^l(\lambda) = \frac{\alpha\lambda}{\alpha\lambda+1-\lambda},\ N^l = \mathbf{1}$	$a = 0,\ c = 0,\ d = 0$
$D^{\alpha-1}_c f(0) = 0$	$b^l_i = -\mathcal{G}^{\alpha-1}_{i-1}$ $N^l(\lambda) = \lambda,\ D^l = \mathbf{1}$	$b = 0,\ c = 0$
$D^{\alpha-1} f(0) = 0$	$b^l_0 = 0,\ b^l_1 = \mathcal{G}^\alpha_0 + \mathcal{G}^\alpha_1$ $b^l_i = \mathcal{G}^\alpha_i$ $N^l(\lambda) = \lambda,\ D^l = \mathbf{1}$	$b = 0,\ d = 0$
$f(1) = 0$	$b^r_i = \mathcal{G}^\alpha_i$ $b_n = b^l_n$ $D^r(\lambda) = \frac{\alpha(1-\lambda)}{\alpha(1-\lambda)+\lambda},\ N^r = \mathbf{1}$	$a = 0,\ \frac{b+(\alpha-1)c}{\Gamma(\alpha)} + d = -I^\alpha g(1)$
$Ff(1) = 0$	$b^r_i = -\mathcal{G}^{\alpha-1}_{i-1}$ $b_n = -\sum_{i=0}^{n-1} b^l_i$ $N^r(\lambda) = 1-\lambda,\ D^r = \mathbf{1}$	$a + b = -Ig(1)$

Table 4.2: Boundary conditions for $L_1[0,1]$.

case used in the construction of the Grünwald transition operators.

In Section 4.3, we first deal with the second part of the requirement of Lumer-Phillips Theorem as mentioned at the start of Section 3.4. That is, we show that the fractional derivative operators (A,BC) on X given by Definition 3.3.8 are dissipative. To do this, we prove a crucial result, Proposition 4.3.2, that the Grünwald transition operators constructed in Section 4.2 approximate the respective fractional derivative operators on X. Combining this with the fact that the Grünwald transition operators (G^h,BC) are dissipative, we show that (A,BC) are dissipative. Then, in view of Theorems 3.4.4 and 3.4.5, since (A,BC) are densely defined closed operators with dense $\mathrm{rg}(\lambda - A)$ in X, the operators (A,BC) generate strongly continuous contraction semigroups as a consequence of the Lumer-Phillips Theorem (C.2.9). Moreover, employing Proposition 4.3.2 we also show that the semigroups generated by the fractional derivative operators (A,BC) are the strong (and uniform for t in compact intervals) limit of the semigroups generated by the Grünwald transition operators using Trotter-Kato Theorem (C.2.10). As a consequence, the underlying Feller processes associated with (G^h,BC) converge in the Skorohod topology to the Feller process associated with the fractional derivative operators (A,BC) [53, p. 331, Theorem 17.25].

4.2 Adjoint formulation and boundary weights

In Section 3.3 we showed that each of the (left) one-sided fractional derivative operators (A, BC) are densely defined, closed linear operators and identified a core for each of them. As it turns out, because of a symmetry argument which we discuss below, it is sufficient to consider only the left-sided fractional derivative operators on both $C_0(\Omega)$ and $L_1[0,1]$.

Let us begin with the left-sided fractional derivatives $A^+ \in \{D_c^{\alpha,+}, D^{\alpha,+}\}$ of order $1 < \alpha < 2$ given by (3.12),

$$D_c^{\alpha,+} f = DI^{2-\alpha,+}Df \text{ and } D^{\alpha,+} f = D^2 I^{2-\alpha,+} f,$$

where

$$I^{\gamma,+} f(x) = \int_0^x p_{\gamma-1}(x-s) f(s) ds, \ \gamma > 0.$$

We use $+$ in the superscript to distinguish it from the right-sided fractional derivatives which we define below. Let $\gamma > 0$ and $f \in L_1[0,1]$, then the right-sided fractional integral of order γ is given by, see [86, p. 89]

$$I^{\gamma,-} f(x) = \int_x^1 p_{\gamma-1}(s-x) f(s) ds, \ x > 0.$$

Then, the right-sided *Riemann-Liouville* and the right-sided first degree *Caputo* fractional derivatives of order α are given by

$$D_c^{\alpha,-} f = (-D) I^{2-\alpha,-}(-D) f \text{ and } D^{\alpha,-} f = (-D)^2 I^{2-\alpha,-} f \tag{4.2}$$

respectively. In what follows, if the context of the discussion applies to both these fractional derivatives, we simply use A^- to denote them. Along similar lines, we also define

$$D_c^{\alpha-1,-} f = I^{2-\alpha,-}(-D) f \text{ and } D^{\alpha-1,-} f = (-D) I^{2-\alpha,-} f$$

We require the following properties in the proof of the next result. Firstly, for $\alpha, \ \beta > 0$ and $f \in L_1[0,1]$, the semigroup property,

$$I^{\alpha,\pm} I^{\beta,\pm} f = I^{\alpha+\beta,\pm} f. \tag{4.3}$$

Secondly, note that reversing the order of the double integral yields

$$\int_0^1 I^{\beta,+} f(x) g(x) \, \mathrm{d}x = \int_0^1 \Big(\int_0^x p_{\beta-1}(x-s) f(s) \, \mathrm{d}s \Big) g(x) \, \mathrm{d}x$$

$$= \int_0^1 \Big(\int_s^1 p_{\beta-1}(x-s) g(x)\, \mathrm{d}x \Big) f(s) \mathrm{d}s = \int_0^1 f(s) I^{\beta,-} g(s)\, \mathrm{d}s. \tag{4.4}$$

We refer to Appendix C for definitions of the adjoint operators and the part of operators. It is well known that, $L_1[0,1]$ is isometrically isomorphic to a closed subspace of the space of bounded (complex) Borel measures, $\mathcal{M}_{\mathcal{B}}(\Omega)$, namely, the subspace consisting of those measure which possess a density. Therefore, in what follows, we can explicitly identify the part of the adjoint of (A^+, BC) in $L_1[0,1]$.

Theorem 4.2.1. *Let (A^+, BC) be a left-sided fractional derivative operator on $C_0(\Omega)$ and (A^-, BC) be the corresponding right-sided fractional derivative operator on $L_1[0,1]$ whose domain encodes the same combination of boundary conditions* BC *as given in Table 4.3. Then, $(A^-, \mathrm{BC}) \subset (A^+, \mathrm{BC})^*|_{L_1[0,1]}$ and $(A^+, \mathrm{BC}) \subset (A^-, \mathrm{BC})^*|_{C_0(\Omega)}$, where $(A^+, \mathrm{BC})^*$ denotes the adjoint of (A^+, BC) on the space of bounded (complex) Borel measures, $\mathcal{M}_{\mathcal{B}}(\Omega)$ and $(A^-, \mathrm{BC})^*$ denotes the adjoint of (A^-, BC) on $L_\infty[0,1]$.*

Proof. We show that for all $\phi \in \mathcal{D}(A^+, \mathrm{BC})$ and for all $\psi \in \mathcal{D}(A^-, \mathrm{BC})$,

$$\int_0^1 A^+\phi(x)\psi(x)\, \mathrm{d}x = \int_0^1 \phi(x) A^-\psi(x)\, \mathrm{d}x.$$

First, let $\mathrm{BC} \in \{\mathrm{DD}, \mathrm{DN}, \mathrm{ND}, \mathrm{NN}\}$ and consider the pairs of operators $(D_c^{\alpha,+}, \mathrm{BC})$ on $C_0(\Omega)$ and $(D_c^{\alpha,-}, \mathrm{BC})$ on $L_1[0,1]$.

Let $\phi \in \mathcal{D}(D_c^{\alpha,+}, \mathrm{BC})$ and $\psi \in \mathcal{D}(D_c^{\alpha,-}, \mathrm{BC})$. Then, using integration by parts, we have

$$\begin{aligned} \int_0^1 D_c^{\alpha,+}\phi(x)\psi(x)\, \mathrm{d}x &= D_c^{\alpha-1,+}\phi(1)\psi(1) - D_c^{\alpha-1,+}\phi(0)\psi(0) \\ &\quad + \int_0^1 D_c^{\alpha-1,+}\phi(x)(-D)\psi(x)\, \mathrm{d}x. \end{aligned} \tag{4.5}$$

Similarly,

$$\begin{aligned} \int_0^1 \phi(x) D_c^{\alpha,-}\psi(x)\, \mathrm{d}x &= \phi(1) D_c^{\alpha-1,-}\psi(1) - \phi(0) D_c^{\alpha-1,-}\psi(0) \\ &\quad + \int_0^1 D\phi(x) D_c^{\alpha-1,-}\psi(x)\, \mathrm{d}x. \end{aligned} \tag{4.6}$$

Using (4.4) we have

$$\int_0^1 D_c^{\alpha-1,+}\phi(x)(-D)\psi(x)\, \mathrm{d}x = \int_0^1 I^{2-\alpha,+} D\phi(x)(-D)\psi(x)\, \mathrm{d}x$$

$$= \int_0^1 D\phi(x) I^{2-\alpha,-}(-D)\psi(x)\,\mathrm{d}x$$
$$= \int_0^1 D\phi(x) D_c^{\alpha-1,-}\psi(x)\,\mathrm{d}x.$$

Thus, the proof for the cases when $\mathrm{BC} \in \{\mathrm{DD}, \mathrm{DN}, \mathrm{ND}, \mathrm{NN}\}$ is complete on observing that the first two terms of (4.5) and (4.6) vanish upon using the respective boundary conditions from Tables 4.1 and 4.2.

Next, let us consider the pair $(D_c^{\alpha,+}, \mathrm{DN}^*)$ and $(D^{\alpha,-}, \mathrm{DN})$, and the pair $(D_c^{\alpha,+}, \mathrm{NN}^*)$ and $(D^{\alpha,-}, \mathrm{NN})$. In what follows, for convenience let us use the notation $\overline{f}(x) := f(1-x)$. Note using Table 4.2 that $\psi \in \mathcal{D}(D^{\alpha,-}, \bullet\mathrm{N})$ is given by

$$\psi(x) = I^{\alpha,-} g_1(x) + a_1 \overline{p}_\alpha(x) + c\overline{p}_{\alpha-2}(x),\ g_1 \in L_1[0,1],$$

since $b_1, d_1 = 0$. Next, using Table 4.1 note that $\phi \in \mathcal{D}(D_c^{\alpha}, \bullet\mathrm{N}^*)$ is given by

$$\phi(x) = I^{\alpha,+} g_0(x) + a_0 p_\alpha(x) + b p_{\alpha-1}(x) + d p_0(x),\ g_0 \in C_0(\Omega).$$

Observe that

$$D_c^{\alpha,+}\phi(x) = g_0(x) + a_0 p_0(x)$$

and

$$D^{\alpha,-}\psi(x) = g_1(x) + a_1 \overline{p}_0(x).$$

Then, using the facts

$$\int_0^1 p_\beta(x) f(x)\,\mathrm{d}x = I^{\beta+1,-} f(0)$$

and

$$\int_0^1 \overline{p}_\beta(x) f(x)\,\mathrm{d}x = I^{\beta+1,+} f(1),$$

along with (4.3), we have

$$\begin{aligned}
\int_0^1 D_c^{\alpha,+}\phi(x)\psi(x)\,\mathrm{d}x &= \int_0^1 \big(g_0(x) + a_0 p_0(x)\big)\left(I^{\alpha,-} g_1(x) + a_1\overline{p}_\alpha(x) + c\overline{p}_{\alpha-2}(x)\right)\mathrm{d}x \\
&= \int_0^1 g_0(x) I^{\alpha,-} g_1(x)\,\mathrm{d}x + a_1 I^{\alpha+1,+} g_0(1) + c I^{\alpha-1,+} g_0(1) \\
&\qquad + a_0 I^{1,-} I^{\alpha,-} g_1(0) + a_0 a_1 I^{\alpha+1,+} p_0(1) + a_0 c I^{\alpha-1,+} p_0(1) \\
&= \int_0^1 g_0(x) I^{\alpha,-} g_1(x)\,\mathrm{d}x \\
&\qquad + a_0 I^{\alpha+1,-} g_1(0) + a_1 I^{\alpha+1,+} g_0(1) + \frac{a_0 a_1}{\Gamma(\alpha+2)}
\end{aligned}$$

$$+ c\Big(I^{\alpha-1,+}g_0(1) + \frac{a_0}{\Gamma(\alpha)}\Big). \tag{4.7}$$

Similarly,

$$\begin{aligned}
&\int_0^1 \phi(x) D_c^{\alpha,-}\psi(x)\,\mathrm{d}x \\
&= \int_0^1 \Bigg(I^{\alpha,+}g_0(x) + a_0 p_\alpha(x) + b p_{\alpha-1}(x) + d p_0(x)\Bigg)\big(g_1(x) + a_1 \overline{p}_0(x)\big)\,\mathrm{d}x \\
&= \int_0^1 I^{\alpha,+}g_0(x)g_1(x)\,\mathrm{d}x + a_1 I^{1,+}I^{\alpha,+}g_0(1) + a_0 I^{\alpha+1,-}g_1(0) + a_0 a_1 I^{1,+}p_\alpha(1) \\
&\qquad + bI^{\alpha,-}g_1(0) + a_1 b I^{1,+}p_{\alpha-1}(1) + dI^{1,-}g_1(0) + a_1 d I^{1,+}p_0(1) \\
&= \int_0^1 I^{\alpha,+}g_0(x)g_1(x)\,\mathrm{d}x + a_0 I^{\alpha+1,-}g_1(0) + a_1 I^{\alpha+1,+}g_0(1) + \frac{a_0 a_1}{\Gamma(\alpha+2)} \\
&\qquad + b\Big(I^{\alpha,-}g_1(0) + \frac{a_1}{\Gamma(\alpha+1)}\Big) + d\Big(I^{1,-}g_1(0) + a_1\Big).
\end{aligned} \tag{4.8}$$

Firstly, observe that in view of (4.4), the first two lines of (4.7) and (4.8) are the same. To complete the proof, we now show that the other terms of (4.7) and (4.8) match as well if and only if the respective boundary conditions are satisfied.

For the operator $(D_c^{\alpha,+}, \mathrm{DN}^*)$, note that $a_0, d = 0$ and $b = -\Gamma(\alpha-1)I^{\alpha-1,+}g_0(1)$. For the operator $(D^{\alpha,-}, \mathrm{DN})$, note that $a_1 = 0$ and $c = -\Gamma(\alpha-1)I^{\alpha,-}g_1(0)$. Thus, the proof for the pair $(D_c^{\alpha,+}, \mathrm{DN}^*)$ and $(D^{\alpha,-}, \mathrm{DN})$ is complete. For the operator $(D_c^{\alpha,+}, \mathrm{NN}^*)$, note that $b = 0$ and $a_0 = -\Gamma(\alpha)I^{\alpha-1,+}g_0(1)$. For the operator $(D^{\alpha,-}, \mathrm{NN})$, note that $a_1 = -I^{1,-}g_1(0)$. Thus, the proof for the pair $(D_c^{\alpha,+}, \mathrm{NN}^*)$ and $(D^{\alpha,-}, \mathrm{NN})$ is complete. This also completes the proof of the theorem. □

Remark 4.2.2. In fact, we show in Corollary 4.3.5, that $(A^-, \mathrm{BC}) = (A^+, \mathrm{BC})^*|_{L_1[0,1]}$ and $(A^+, \mathrm{BC}) = (A^-, \mathrm{BC})^*|_{C_0(\Omega)}$. To do this, we require the dissipativity of the operators A^+ and A^- which is established using the convergence properties of the Grünwald approximations. However, it turns out that we only need to consider left-sided derivatives on both these spaces, which we justify below.

Let the (isomorphism) flip operator $\mathcal{J} : L_1[0,1] \to L_1[0,1]$ be given by $\mathcal{J}f(x) := f(1-x)$ and note that $\mathcal{J}^{-1} = \mathcal{J}$. Then,

$$D_c^{\alpha,-}f(x) = \mathcal{J}^{-1}D_c^{\alpha,+}\mathcal{J}f(x) \text{ and } D^{\alpha,-}f(x) = \mathcal{J}^{-1}D^{\alpha,+}\mathcal{J}f(x),$$

which follows on using the substitution $\tau = 1 - s$ in (4.2). In view of this relation and Corollary 4.3.5, we define the left-sided fractional derivative operators $(A^{\leftrightarrow}, \mathrm{BC})$ as the flipped versions of the part of the adjoint of (A^+, BC); that is, using the right-sided

$(A^+, \mathrm{BC}),\ C_0(\Omega)$	$(A^-, \mathrm{BC}),\ L_1[0,1]$	$(A^{\leftrightarrow}, \mathrm{BC}),\ L_1[0,1]$
$(D_c^{\alpha,+}, \mathrm{DN})$	$(D_c^{\alpha,-}, \mathrm{DN})$	$(D_c^{\alpha,+}, \mathrm{ND})$
$(D_c^{\alpha,+}, \mathrm{NN})$	$(D_c^{\alpha,-}, \mathrm{NN})$	$(D_c^{\alpha,+}, \mathrm{NN})$
$(D_c^{\alpha,+}, \mathrm{DD})$	$(D^{\alpha,-}, \mathrm{DD}) = (D_c^{\alpha,-}, \mathrm{DD})$	$(D_c^{\alpha,+}, \mathrm{DD})$
$(D_c^{\alpha,+}, \mathrm{ND})$	$(D^{\alpha,-}, \mathrm{ND}) = (D_c^{\alpha,-}, \mathrm{ND})$	$(D_c^{\alpha,+}, \mathrm{DN})$
$(D_c^{\alpha,+}, \mathrm{DN}^*)$	$(D^{\alpha,-}, \mathrm{DN})$	$(D^{\alpha,+}, \mathrm{ND})$
$(D_c^{\alpha,+}, \mathrm{NN}^*)$	$(D^{\alpha,-}, \mathrm{NN})$	$(D^{\alpha,+}, \mathrm{NN})$

Table 4.3: Corresponding fractional derivative operators on $C_0(\Omega)$ and $L_1[0,1]$.

fractional derivative operators (A^-, BC), we set

$$(A^{\leftrightarrow}, \mathrm{BC})f(x) = \mathcal{J}(A^-, \mathrm{BC})\mathcal{J}^{-1}f(x).$$

We use $\leftrightarrow$ instead of $+$ to emphasise the fact that upon reflection about $x = \frac{1}{2}$, the right-sided fractional derivative operators correspond to left-sided fractional derivative operators with the left and right boundary conditions of BC swapped. Since the semigroups generated by (A^-, BC) and $(A^{\leftrightarrow}, \mathrm{BC})$ are similar semigroups [37, p. 59], it is sufficient to only consider the corresponding left-sided fractional derivative operator $(A^{\leftrightarrow}, \mathrm{BC})$ on $L_1[0,1]$ as given in Table 4.3.

Consider the abstract Cauchy problem on $C_0(\Omega)$ associated to the left-sided fractional derivative operator (A^+, BC) and the initial value u_0,

$$\begin{aligned} u'(t) &= A^+ u(t) \quad \text{for } t \geq 0, \\ u(0) &= u_0. \end{aligned}$$

In view of Remark 4.2.2, we reformulate the abstract Cauchy problem in the adjoint scenario on $L_1[0,1]$ in terms of left-sided fractional derivative operators,

$$\begin{aligned} u'(t) &= A^{\leftrightarrow} u(t) \quad \text{for } t \geq 0, \\ u(0) &= u_0. \end{aligned}$$

We use the semigroups generated by the transition operators (G^h, BC) (see Definition 3.2.4) to approximate the semigroups generated by fractional derivative operators (A^+, BC) on $C_0(\Omega)$. The interpolation matrix $G^h_{n+1}(\lambda)$ used in the construction of (G^h, BC) is given by (4.9) below, where the parameter $\lambda \in [0,1]$, $\lambda' = 1 - \lambda$ and we have used the fact that the consecutive off-diagonal entries of the transition matrix

$G^h_{n\times n}$ given by 4.1 are equal except when the first column or the last row entry is involved, that is, $(G^h_{n\times n})_{i-1,j-1} = (G^h_{n\times n})_{i,j}$ if $j \neq 2$ and $i \neq n$. Note that, if the left boundary condition of BC is Dirichlet, then we set $N^l = \mathbf{1}$ and take $D^l(\lambda) = \frac{\alpha\lambda}{\alpha\lambda+\lambda'}$. On the other hand, if the left boundary condition is Neumann, then we set $D^l = \mathbf{1}$ and take $N^l(\lambda) = \lambda$. Similarly, if the right boundary condition of BC is Dirichlet, then we set $N^r = \mathbf{1}$ and take $D^r(\lambda) = \frac{\alpha\lambda'}{\alpha\lambda'+\lambda}$, else we set $D^l = \mathbf{1}$ and take $N^l(\lambda) = \lambda'$.

Similarly, to approximate the semigroups generated by the fractional derivative operators on $L_1[0,1]$, as discussed in Section 3.2, we could have used the adjoint transition operator G^{h*} given by Proposition 3.2.6 constructed using the interpolation matrix $G^{h*}_{n+1} = (G^h_{n+1})^T$. Further, recall that the domains of the fractional derivative operators given in Definition 3.3.8 involve the power functions p_β, $\beta \in \{\alpha, \alpha-1, \alpha-2, 0\}$. Therefore, in L_1 calculations we need to approximate $p_\beta(1-x)$. However, in view of Remark 4.2.2, using reflection about $x = \frac{1}{2}$, we obtain the flipped interpolation matrices $G^{h,\leftrightarrow}_{n+1}$ associated with the transition operators $(G^{h,\leftrightarrow}, \mathrm{BC})$ approximating the left-sided flipped versions $(A^{\leftrightarrow}, \mathrm{BC})$ of the right-sided fractional derivative operators (A^-, BC). As a consequence, the calculations in the $L_1[0,1]$ case (see Section 4.1) are simpler.

$$G^h_{n+1}(\lambda) =$$
$$\frac{1}{h^\alpha}\begin{pmatrix}
b^l_1 & D^l(\lambda)\mathcal{G}^\alpha_0 & 0 & \cdots & \cdots & \cdots & 0 \\
N^l(\lambda)b^l_2 & \lambda' b^l_1 + \lambda\mathcal{G}^\alpha_1 & \mathcal{G}^\alpha_0 & 0 & \cdots & \cdots & 0 \\
N^l(\lambda)b^l_3 & \lambda' b^l_2 + \lambda\mathcal{G}^\alpha_2 & \mathcal{G}^\alpha_1 & \mathcal{G}^\alpha_0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
N^l(\lambda)b^l_i & \lambda' b^l_{i-1} + \lambda\mathcal{G}^\alpha_{i-1} & \mathcal{G}^\alpha_{i-2} & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
N^l(\lambda)b_n & \lambda' b^l_{n-1} + \lambda b^r_{n-1} & \lambda'\mathcal{G}^\alpha_{n-2} + \lambda b^r_{n-2} & \cdots & \cdots & \lambda'\mathcal{G}^\alpha_1 + \lambda b^r_1 & N^r(\lambda)\mathcal{G}^\alpha_0 \\
0 & D^r(\lambda)b_n & D^r(\lambda)b^r_{n-1} & \cdots & \cdots & D^r(\lambda)b^r_2 & b^r_1
\end{pmatrix} \tag{4.9}$$

For $g \in L_1[0,1]$ recall Proposition 3.2.6,

$$(G^{h*}g)(x) = \left(E_{n+1}\left((G^h_{n+1})^T P_{n+1}g\right)\right)(x) = \left[(G^h_{n+1})^T(\lambda(x))(P_{n+1}g)(\lambda(x))\right]_{\iota(x)},$$

where for $x \in [0,1]$, λ and ι denote the fractional and integer parts of $\frac{x}{h}$ given by

Definition 3.2.2. Therefore, for $f \in L_1[0,1]$,

$$
\begin{aligned}
\left(G^{h,\leftrightarrow} f\right)(x) = \left(G^{h*}\overline{f}\right)(1-x) &= \left(\left(G^h_{n+1}\right)^T (\lambda(1-x)) \left(P_{n+1}\overline{f}\right)(\lambda(1-x))\right)_{\iota(1-x)} \\
&= \sum_{j=1}^{n+1} \left[G^h_{n+1}(\lambda(1-x))\right]_{j,\iota(1-x)} (P_{n+1}\overline{f})_j(\lambda(1-x)) \\
&= \sum_{j=1}^{n+1} \left[G^h_{n+1}(1-\lambda(x))\right]_{j,n+2-i} (P_{n+1}f)_{n+2-j}(\lambda(x)) \\
&= \sum_{j=1}^{n+1} \left[G^h_{n+1}(1-\lambda(x))\right]_{n+2-j,n+2-i} (P_{n+1}f)_j(\lambda(x)) \\
&= \sum_{j=1}^{n+1} \left[G^{h,\leftrightarrow}_{n+1}(\lambda(x))\right]_{i,j} (P_{n+1}f)_j(\lambda(x)),
\end{aligned}
$$

where setting $\lambda' = 1-\lambda$, we have $\left[G^{h,\leftrightarrow}_{n+1}(\lambda)\right]_{i,j} = \left[G^h_{n+1}(\lambda')\right]_{n+2-j,n+2-i}$. Hence, the transition operator $G^{h,\leftrightarrow}$ is given by

$$
(G^{h,\leftrightarrow} f)(x) = \left(E_{n+1}\left(G^{h,\leftrightarrow}_{n+1} P_{n+1} f\right)\right)(x) = \left[(G^{h,\leftrightarrow}_{n+1}(\lambda(x))(P_{n+1}f)(\lambda(x))\right]_{\iota(x)}, \quad (4.10)
$$

where the operators P_{n+1} and E_{n+1} are as discussed in Section 3.2.

Remark 4.2.3. In (4.12), the flipped adjoint interpolation matrix $G^{h,\leftrightarrow}_{n+1}(\lambda)$, is obtained by transposing the entries of $G^h_{n+1}(\lambda')$ about the (other) diagonal (the diagonal going from bottom left to top right). The interpolating functions D^l, D^r, N^l and N^r and the boundary weights b^l_i, b^r_i, b_n are given in Table 4.1 for G^h_{n+1} and in Table 4.2 for $G^{h,\leftrightarrow}_{n+1}$.

For the sake of consistency of notation, in $G^{h,\leftrightarrow}_{n+1}(\lambda)$, note that we have replaced λ' with λ in the arguments of D^l, D^r, N^l and N^r as well as relabelled the superscripts of these interpolating functions and the boundary weights. This is justified, since the interpolating function with index l is viewed as interpolating from zero to one and acting on the left boundary while the interpolating function with index r is viewed as interpolating from one to zero and acting on the right boundary. Moreover, in the Grünwald transition rate matrix the boundary weights with indices l and r are used to encode the left and right boundary conditions, respectively.

Therefore, for the L_1-case we follow the same convention and use b^l_i in the first column and b_i^r in the last row of

$$G_{n\times n}^{h,\leftrightarrow} = \frac{1}{h^\alpha}\begin{pmatrix} b_1^l & \mathcal{G}_0^\alpha & 0 & \cdots & 0 \\ \vdots & \mathcal{G}_1^\alpha & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ b_{n-1}^l & \mathcal{G}_{n-2}^\alpha & \cdots & \mathcal{G}_1^\alpha & \mathcal{G}_0^\alpha \\ b_n & b_{n-1}^r & \cdots & \cdots & b_1^r \end{pmatrix}. \tag{4.11}$$

However, note that as a result of the reflection, $x \mapsto 1-x$, the boundary weights of the Matrix $G_{n\times n}^{h,\leftrightarrow}$ now encode the mirrored boundary conditions of the corresponding boundary conditions (BC) encoded by the boundary weights of Matrix G_{n+1}^h. Lastly, compare matrices G_{n+1}^h and $G_{n+1}^{h,\leftrightarrow}$ and observe that the interpolating functions D and N have swapped roles.

$$G_{n+1}^h(\lambda') =$$

$$\frac{1}{h^\alpha}\begin{pmatrix} b_1^l & D^l(\lambda')\mathcal{G}_0^\alpha & 0 & \cdots & \cdots & \cdots & 0 \\ N^l(\lambda')b_2^l & \lambda b_1^l + \lambda'\mathcal{G}_1^\alpha & \mathcal{G}_0^\alpha & 0 & \cdots & \cdots & 0 \\ N^l(\lambda')b_3^l & \lambda b_2^l + \lambda'\mathcal{G}_2^\alpha & \mathcal{G}_1^\alpha & \mathcal{G}_0^\alpha & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ N^l(\lambda')b_i^l & \lambda b_{i-1}^l + \lambda'\mathcal{G}_{i-1}^\alpha & \mathcal{G}_{i-2}^\alpha & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ N^l(\lambda')b_n & \lambda b_{n-1}^l + \lambda' b_{n-1}^r & \lambda\mathcal{G}_{n-2}^\alpha + \lambda' b_{n-2}^r & \cdots & \cdots & \lambda\mathcal{G}_1^\alpha + \lambda' b_1^r & N^r(\lambda')\mathcal{G}_0^\alpha \\ 0 & D^r(\lambda')b_n & D^r(\lambda')b_{n-1}^r & \cdots & \cdots & D^r(\lambda')b_2^r & b_1^r \end{pmatrix}$$

$$x \to 1-x \qquad \Updownarrow \qquad l \leftrightarrow r$$

$$G_{n+1}^{h,\leftrightarrow}(\lambda) =$$

$$\frac{1}{h^\alpha}\begin{pmatrix} b_1^l & N^l(\lambda)\mathcal{G}_0^\alpha & 0 & \cdots & \cdots & \cdots & 0 \\ D^l(\lambda)b_2^l & \lambda' b_1^l + \lambda\mathcal{G}_1^\alpha & \mathcal{G}_0^\alpha & 0 & \cdots & \cdots & 0 \\ D^l(\lambda)b_3^l & \lambda' b_2^l + \lambda\mathcal{G}_2^\alpha & \mathcal{G}_1^\alpha & \mathcal{G}_0^\alpha & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ D^l(\lambda)b_i^l & \lambda' b_{i-1}^l + \lambda\mathcal{G}_{i-1}^\alpha & \mathcal{G}_{i-2}^\alpha & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ D^l(\lambda)b_n & \lambda' b_{n-1}^l + \lambda b_{n-1}^r & \lambda'\mathcal{G}_{n-2}^\alpha + \lambda b_{n-2}^r & \cdots & \cdots & \lambda'\mathcal{G}_1^\alpha + \lambda b_1^r & D^r(\lambda)\mathcal{G}_0^\alpha \\ 0 & N^r(\lambda)b_n & N^r(\lambda)b_{n-1}^r & \cdots & \cdots & N^r(\lambda)b_2^r & b_1^r \end{pmatrix} \tag{4.12}$$

Boundary weights for L_1-case: Let $g_{i,j}$ denote the entries of the transition rate matrix in the L_1-case given by (4.11). Then, recall from Section 3.1, that the entries $g_{i,j}$, for $i \neq j$ denote the rate at which particles jump from state j to i while for $i = j$ denote the total rate at which particles leave state j. Let us divide the interval $[0, 1]$ into n boxes (or grids) and visualise the state j as representing the j^{th} box. For an arbitrary entry $g_{i,j}$, if $i = j$, then $g_{j,j}$ corresponds to the rate at which particles jump out of the j^{th} box. On the other hand, if $i < j$, then $g_{i,j}$ corresponds to the rate at which particles jump backwards from j^{th} box and if $i > j$, then $g_{i,j}$ corresponds to the rate at which particles jump forwards from j^{th} box. Further, note that particles only jump one box backwards and the rest are forward jumps. Therefore, in the L_1-case we can physically interpret j^{th} column as keeping track of the rates of the particles jumping out of j^{th} box into other boxes.

Consider the infinite Grünwald matrix

$$\frac{1}{h^\alpha}\begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathcal{G}_1^\alpha & \mathcal{G}_0^\alpha & 0 & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathcal{G}_2^\alpha & \mathcal{G}_1^\alpha & \mathcal{G}_0^\alpha & 0 & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots \\ \ddots & \mathcal{G}_{n-1}^\alpha & \mathcal{G}_{n-2}^\alpha & \ddots & \ddots & \mathcal{G}_1^\alpha & \mathcal{G}_0^\alpha & \ddots \\ \ddots & \mathcal{G}_n^\alpha & \mathcal{G}_{n-1}^\alpha & \ddots & \ddots & \mathcal{G}_2^\alpha & \mathcal{G}_1^\alpha & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{4.13}$$

where the Grünwald coefficients $\mathcal{G}_m^\alpha = (-1)^m \binom{\alpha}{m}$ are given by (A.1). The left and right boundary conditions are dealt with independently by truncating the infinite matrix given by (4.13) to obtain the semi-infinite matrix

$$\frac{1}{h^\alpha}\begin{pmatrix} \boxed{b_1^l} & \boxed{\mathcal{G}_0^\alpha} & \boxed{0} & \cdots & \cdots & \cdots & \cdots & \cdots \\ \boxed{b_2^l} & \mathcal{G}_1^\alpha & \mathcal{G}_0^\alpha & 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & \ddots \\ \boxed{b_{n-1}^l} & \mathcal{G}_{n-2}^\alpha & \ddots & \ddots & \mathcal{G}_1^\alpha & \mathcal{G}_0^\alpha & \ddots & \ddots \\ \boxed{b_n} & \mathcal{G}_{n-1}^\alpha & \ddots & \ddots & \mathcal{G}_2^\alpha & \mathcal{G}_1^\alpha & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{4.14}$$

to deal with the left boundary and the semi-infinite matrix

$$\frac{1}{h^\alpha}\begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \ddots & \ddots & \ddots & \mathcal{G}_1^\alpha & \mathcal{G}_0^\alpha & 0 & \ddots & \ddots & \vdots \\ \ddots & \ddots & \ddots & \mathcal{G}_2^\alpha & \mathcal{G}_1^\alpha & \mathcal{G}_0^\alpha & 0 & \ddots & \vdots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \boxed{0} \\ \ddots & \ddots & \ddots & \mathcal{G}_{n-1}^\alpha & \mathcal{G}_{n-2}^\alpha & \ddots & \ddots & \mathcal{G}_1^\alpha & \boxed{\mathcal{G}_0^\alpha} \\ \cdots & \cdots & \cdots & \boxed{b_n} & \boxed{b_{n-1}^r} & \cdots & \cdots & \boxed{b_2^r} & \boxed{b_1^r} \end{pmatrix} \tag{4.15}$$

to deal with the right boundary.

Left boundary conditions: We use the first row and first column of the semi-infinite matrix given by (4.14) to encode the left boundary conditions. Note that the entries of the infinite Grünwald matrix in the columns above the first row entries of the semi-infinite matrix correspond to the rates of the particles that jump backward past the left boundary. Moreover, the entries of the infinite Grünwald matrix in the rows to the left of the first column entries of the semi-infinite matrix correspond to the rates of the particles that would jump forward past the left boundary; that is, the entries in the i^{th} row to the left of the first column entry corresponds to the rate of particles that would jump into i^{th} box.

- *Dirichlet*: In this case, we conjecture that we have an absorbing left boundary. The entries above the first row in the infinite matrix would therefore correspond to the rates of the particles that are lost by jumping backwards through the left boundary. However, since we expect the particles to be lost or killed after passing through the left boundary, no particles jump forward past the left boundary. Therefore, we set the entries to the left of the first column in the semi-infinite matrix (4.14) to zero and take $b_i^l = \mathcal{G}_i^\alpha$ for $i \geq 0$.

- *Neumann (Riemann-Liouville)*: In this case, we conjecture that we have no flux at the left boundary; that is, no particles jump backward or forward through the left boundary. Therefore, the entries above the first row in the infinite matrix which correspond to the rates of the particles that would have jumped backwards past the left boundary are set to zero and so we take $b_0^l = 0$. Moreover, as we expect no particles to jump forward past the left boundary, we set the entries to the left of the first column to zero and take $b_i^l = \mathcal{G}_i^\alpha$ for $i \geq 2$. Lastly, since we expect no flux at the left boundary, this implies that the mass of all the particles should be conserved. To ensure this, we need each column sum to be zero. Since all the column sums are zero except the first column, using (A.6), we take

$$b_1^l = -\sum_{k=2}^{\infty} \mathcal{G}_k^\alpha = \mathcal{G}_0^\alpha + \mathcal{G}_1^\alpha$$

to ensure the first column sum is zero.

- *Neumann (Caputo)*: In this case, firstly, we conjecture that we have no flux at the left boundary. Therefore, the entries above the first row in the infinite matrix which correspond to the rates of the particles that would have jumped backwards past the left boundary are set to zero; that is, we take $b_0^l = 0$. Moreover, we expect no particles to jump forward past the left boundary and so the entries to the left of the first column are set to zero. Secondly, note that the steady state solutions for the abstract Cauchy problem associated with the operator $(D_c^\alpha, \mathrm{BC})$ are constant functions, since $D_c^\alpha p_0 = \mathbf{0}$ in view of (3.17). Therefore, for the corresponding Grünwald transition matrix we conjecture that each row sum should be zero. Therefore, in view of (A.7), we take

$$b_i^l = -\sum_{k=0}^{i-1} \mathcal{G}_k^\alpha = -\mathcal{G}_{i-1}^{\alpha-1}.$$

In addition, using (A.6) note that $-\sum_{k=0}^{i-1} \mathcal{G}_k^\alpha = \mathcal{G}_i^\alpha + \sum_{k=i+1}^{\infty} \mathcal{G}_k^\alpha$ where the last term corresponds to the sum of the entries in the row of the infinite matrix to the left of the first column entry. Observe that the entries b_i^l for $i \in \{2, \cdots, n-1\}$ are larger than each of the other row entries; that is, the rate of particles jumping into i^{th} box from first box is significantly larger than the rate of particles jumping from the other boxes. This can be physically interpreted as the increased dispersion of the process at the left boundary as modelled by the first degree Caputo fractional derivative operator.

Right boundary conditions: We use the last row and last column of the semi-infinite matrix given by (4.15) to encode the right boundary condition. Note that the entries of the infinite Grünwald matrix in the columns below the last row entries correspond to the particles that jump forward past the right boundary. Moreover, the entries of the infinite Grünwald matrix in the rows to the right of the last column entries correspond to the rates of the particles that jump backward past the right boundary.

- *Dirichlet*: In this case, we conjecture that we have an absorbing right boundary. Therefore, the entries below the last row of the infinite matrix would correspond to the rates of the particles that are lost by jumping forward past the right boundary. However, since we expect that the particles are lost or killed after passing through the right boundary, no particles jump backward past the right boundary. Thus, we set the entries to the right of the last column to zero and take $b_i^r = \mathcal{G}_i^\alpha$ for $i \geq 0$. Moreover, note that the entry b_n is shared by both the last row and first column of $G_{n\times n}^h$, and so we retain $b_n = b_n^l$ depending on the left boundary condition.

- *Neumann*: We conjecture that we have no flux at the right boundary. Therefore, the entries below the last row that would correspond to the rate of particles that jump forward past the right boundary are set to zero. Similarly, the entries to the right of the last column that correspond to the rate of particles that jump backward past the right boundary are set to zero. Moreover, to conserve the mass of all the particles, we need to ensure that all the column sums are zero. Thus, we take $b_i^r = -\sum_{k=0}^{i-1} \mathcal{G}_k^\alpha = -\mathcal{G}_{i-1}^{\alpha-1}$ in view of (A.7) and similarly, $b_n = -\sum_{i=0}^{n-1} b_i^l$.

Thus, there are six possible combinations of boundary conditions BC that we encode in the Grünwald transition matrices in the L_1-case. In view of Remark 4.2.3 these correspond to the respective mirrored boundary conditions for the C_0-case. In Section 4.3 we show that the Grünwald transition operators (G^h, BC), constructed using these Grünwald transition matrices with the boundary weights as discussed above, approximate the respective fractional derivative operators (A, BC) on X as summarised in Table 4.3.

Remark 4.2.4. Note that irrespective of the left boundary condition of BC, $A = D_c^\alpha$ in the C_0-case. Indeed, for $f \in X$ with $f(0) = 0$, using (3.11); that is, $D(I^\gamma f) = I^\gamma(Df) + f(0)p_{\gamma-1}$ we have

$$D^\alpha f = D^2 I^{2-\alpha} f = D\left(I^{2-\alpha} Df - f(0)p_{1-\alpha}\right) = D_c^\alpha f. \tag{4.16}$$

Next note that the boundary weights b_i^r of the Grünwald transition matrices used to approximate $(D^\alpha, \bullet\mathrm{N})$ are the same as the boundary weights b_i^l corresponding to $(D_c^\alpha, \mathrm{N}\bullet)$. Lastly, note that the left Neumann boundary weights of the Grünwald transition matrices used in the approximation of the operators $(D^\alpha, \mathrm{N}\bullet)$ in $L_1[0,1]$ yield the right Neumann* boundary weights for the operators $(D_c^\alpha, \bullet\mathrm{N}^*)$ in $C_0(\Omega)$.

4.3 Semigroups and processes associated with Grünwald-type approximations and fractional derivative operators on X

Let us begin by recalling the Grünwald transition operator (G^h, BC) on $C_0(\Omega)$ given by Definition 3.2.4 and constructed using the interpolation matrix G^h_{n+1} given by (4.9) along with Table 4.1; that is, for $\phi \in C_0(\Omega)$,

$$(G^h\phi)(x) = \left(E_{n+1}\left(G^h_{n+1}P_{n+1}\phi\right)\right)(x) = \left[G^h_{n+1}(\lambda(x))(P_{n+1}\phi)(\lambda(x))\right]_{\iota(x)}. \qquad (4.17)$$

Similarly, the Grünwald transition operator $(G^{h,\leftrightarrow}, \mathrm{BC})$ on $L_1[0,1]$ is given by (4.10) and constructed using the interpolation matrix $G^{h,\leftrightarrow}_{n+1}$ given by (4.12) along with Table 4.2; that is, for $\phi \in L_1[0,1]$,

$$(G^{h,\leftrightarrow}\phi)(x) = \left(E_{n+1}\left(G^{h,\leftrightarrow}_{n+1}P_{n+1}\phi\right)\right)(x) = \left[(G^{h,\leftrightarrow}_{n+1}(\lambda(x))(P_{n+1}\phi)(\lambda(x))\right]_{\iota(x)}. \quad (4.18)$$

In Proposition 4.3.2, we show the following:

- For each of the fractional derivative operators (A^+, BC) on $C_0(\Omega)$ given in Table 4.3 and each $f \in \mathcal{C}(A^+, \mathrm{BC})$, there exists a sequence $\{f_h\} \subset C_0(\Omega)$ such that $f_h \to f$ and $G^h f_h \to A^+ f$.

- For each of the fractional derivative operators $(A^\leftrightarrow, \mathrm{BC})$ on $L_1[0,1]$ given in Table 4.3 and each $f \in \mathcal{C}(A^\leftrightarrow, \mathrm{BC})$, there exists a sequence $\{f_h\} \subset L_1[0,1]$ such that $f_h \to f$ and $G^{h,\leftrightarrow} f_h \to A^\leftrightarrow f$.

This is not trivial, since in general,

$$G^h\left(ap_\alpha + bp_{\alpha-1} + cp_{\alpha-2} + dp_0\right) \not\to A^+\left(ap_\alpha + bp_{\alpha-1} + cp_{\alpha-2} + dp_0\right).$$

For instance, $A^+p_{\alpha-1} = \mathbf{0}$ while $G^h p_{\alpha-1} \not\to \mathbf{0}$. In fact, showing that there exist such functions $f_h \in X$ turns out to be a tedious task involving elaborate constructions. Therefore, to keep the current discussion coherent, we delay the detailed proof of Proposion 4.3.2 to Section 4.5.

But first, we have the following lemma which identifies the stochastic processes associated with the Grünwald approximation operators.

Lemma 4.3.1. *Let the fractional derivative operators* (A^+, BC) *and* $(A^\leftrightarrow, \mathrm{BC})$ *be given by Definition 3.3.8 along with a core* $\mathcal{C}(A, \mathrm{BC})$ *as in Theorem 3.4.4, also see Remark 4.2.2. Further, let the corresponding Grünwald transition operators* (G^h, BC)

and $(G^{h,\leftrightarrow}, \mathrm{BC})$ *be given by* (4.17) *and* (4.18), *respectively. Then,* G^h *generate Feller semigroups on* $C_0(\Omega)$ *and* $G^{h,\leftrightarrow}$ *generate positive, strongly continuous contraction semigroups on* $L_1[0,1]$. *Furthermore, the operators* G^h *and* $G^{h,\leftrightarrow}$ *are dissipative.*

Proof. Firstly, let us verify using the boundary weights given in Tables 4.1 and 4.2 that the matrices

$$G^h_{n\times n} = \frac{1}{h^\alpha}\begin{pmatrix} b^l_1 & \mathcal{G}^\alpha_0 & 0 & \cdots & 0 \\ \vdots & \mathcal{G}^\alpha_1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ b^l_{n-1} & \mathcal{G}^\alpha_{n-2} & \cdots & \mathcal{G}^\alpha_1 & \mathcal{G}^\alpha_0 \\ b_n & b^r_{n-1} & \cdots & \cdots & b^r_1 \end{pmatrix}.$$

are indeed transition rate matrices by making use of the recursion formula (A.2), $\mathcal{G}^\alpha_{n+1} = \frac{n-\alpha}{n+1}\mathcal{G}^\alpha_n,\ \mathcal{G}^\alpha_0 = 1,\ n \in \mathbb{N}$. Note that all the entries of $G^h_{n\times n}$ except for the diagonal entries are non-negative, whereas the diagonal entries are all negative. Combining this fact with the binomial identity $\sum_{n=0}^\infty \mathcal{G}^\alpha_n = 0$ given by (A.5), we have that all row sums of $G^h_{n\times n}$ in the case when $X = C_0(\Omega)$ and all column sums of $G^h_{n\times n}$ in the case when $X = L_1[0,1]$ are non-positive. Thus, in view of Proposition 3.2.7, the operators G^h and $G^{h,\leftrightarrow}$ are bounded. Moreover, G^h generate Feller semigroups on $C_0(\Omega)$ and $G^{h,\leftrightarrow}$ generate positive, strongly continuous contraction semigroups on $L_1[0,1]$. Then, it also follows that the operators G^h and $G^{h,\leftrightarrow}$ are dissipative. □

We now state the crucial result of this section and give a brief sketch of the proof. The detailed proof is given in Section 4.5.

Proposition 4.3.2. *Let the fractional derivative operators* (A^+, BC) *and* $(A^\leftrightarrow, \mathrm{BC})$ *be given by Definition 3.3.8 along with a core as in Theorem 3.4.4, also see Remark 4.2.2. Further, let the corresponding Grünwald transition operators* (G^h, BC) *and* $(G^{h,\leftrightarrow}, \mathrm{BC})$ *be given by* (4.17) *and* (4.18), *respectively. Then, we have the following*

1. *For each* $f \in \mathcal{C}(A, \mathrm{BC})$ *there exists sequence* $\{f_h\} \subset C_0(\Omega)$ *such that* $f_h \to f$ *and* $G^h f_h \to A^+ f$ *in* $C_0(\Omega)$.

2. *For each* $f \in \mathcal{C}(A^\leftrightarrow, \mathrm{BC})$ *there exists sequence* $\{f_h\} \subset L_1[0,1]$ *such that* $f_h \to f$ *and* $G^{h,\leftrightarrow} f_h \to A^\leftrightarrow f$ *in* $L_1[0,1]$.

Proof. Let $1 < \alpha < 2$, $n \in \mathbb{N}$ and $h = \frac{1}{n+1}$. To simplify notation, let us write G^h for both G^h and $G^{h,\leftrightarrow}$ as well as A for both A^+ and $A^\leftrightarrow$. Consider a typical element $f \in \mathcal{C}(A, \mathrm{BC})$ given in Theorem 3.4.4 in its general form,

$$f = I^\alpha P + a p_\alpha + b p_{\alpha-1} + c p_{\alpha-2} + d p_0,$$

where the constants $a, b, c, d \in \mathbb{R}$ are given in Tables 4.1 and 4.2 for each BC, and polynomial $P = \sum_{m=0}^{N} k_m p_m \in X$ as in Remark 3.4.3. We construct functions $\{f_h\} \subset X$ and show that $f_h \to f$ and $G^h f_h \to Af$ in X for each of the fractional derivative operators (A, BC) on X given in Table 4.3. The functions f_h have the following general form

$$f_h = I^\alpha P + a_h \vartheta_h^\alpha + b_h \vartheta_h^{\alpha-1} + c_h \vartheta_h^{\alpha-2} + d\vartheta_h^0 + e_h, \; P \in X,$$

where $a_h, \; b_h, \; c_h \in \mathbb{R}$.

Outline of construction of $f_h \in X$: Firstly, the approximate power functions are constructed such that the following hold (see Section 4.5.1, Definition 4.5.1 and Lemma 4.5.3, and Section 4.5 for details):

1. The functions $\vartheta_h^\alpha, \vartheta_h^{\alpha-1}$ and ϑ_h^0 are constructed such that they converge in the X-norm to p_α, $p_{\alpha-1}$ and p_0, respectively.

2. The function $\vartheta_h^{\alpha-2}$ is constructed such that it converges to $p_{\alpha-2}$ in L_1-norm,

3. $G^h \vartheta_h^\beta \to Ap_\beta$ in the respective X-norm.

Secondly, the functions e_h are taken to be zero functions except in those cases when BC has a right Neumann boundary condition where we require the function e_h for $G^h f_h \to Af$, constructed such that $e_h \to \mathbf{0}$ in the X-norm. Thirdly, the real sequences $a_h, \; b_h, \; c_h$ are chosen such that they converge to $a, \; b, \; c$, respectively and $f_h \in X$. We specify $a_h, \; b_h, \; c_h$ in the detailed proof of Proposition 4.3.2 in Section 4.5.

Outline of proof of $f_h \to f$ and $G^h f_h \to Af$ in $C_0(\Omega)$:

For each of the fractional derivative operators (A, BC) on $C_0(\Omega)$ given in Table 4.3, and for each $f \in \mathcal{C}(A, \mathrm{BC})$ we show that there exists a sequence $\{f_h\} \subset C_0(\Omega)$ such that $f_h \to f$ and $G^h f_h \to Af$ in $C_0(\Omega)$. To this end, given $\epsilon > 0$ we show that there exists $\delta > 0$ such that for $h < \delta$,

$$\sup_{x\in\Omega} |f_h(x) - f(x)| < \epsilon \tag{4.19}$$

and

$$\sup_{x\in\Omega} \left|G^h f_h(x) - Af(x)\right| < \epsilon. \tag{4.20}$$

The proof of (4.19) follows from the construction outlined above. To show (4.20), we break the interval Ω into two parts, namely,

$$\Omega_1(h) := \Omega \cap [0, 1-2h) \text{ and } \Omega_2(h) := \Omega \cap [1-2h, 1]. \tag{4.21}$$

For $x \in \Omega_1(h)$ we show that

$$\sup_{x\in\Omega_1} \left|G^h f_h(x) - Af(x)\right| = O(h^\kappa),\ \kappa > 0.$$

As a consequence, there exists δ_1 such that for $h < \delta_1$,

$$\sup_{x\in\Omega_1(h)} \left|G^h f_h(x) - Af(x)\right| < \frac{\epsilon}{2}.$$

Similarly, for $x \in \Omega_2$ we show that

$$\sup_{x\in\Omega_2(h)} \left|G^h f_h(x) - Af(x)\right| = O(h^\kappa),\ \kappa > 0.$$

This implies that there exists δ_2 such that for $h < \delta_2$,

$$\sup_{x\in\Omega_2(h)} \left|G^h f_h(x) - Af(x)\right| < \frac{\epsilon}{2}.$$

Then, taking $\delta = \min\{\delta_1, \delta_2\}$, for $h < \delta$ we have $\Omega = \Omega_1(h) \cup \Omega_2(h)$ as well as uniform convergence on the interval Ω, (4.19) and (4.20).

Interval Ω_1:

1. We consider the common properties of the Grünwald approximations of operators with left Dirichlet boundary condition, $(A, \mathrm{D}\bullet)$ on Ω_1.

2. We consider the common properties of the Grünwald approximations of the operators with left Neumann boundary condition, $(A, \mathrm{N}\bullet)$ on Ω_1.

Interval Ω_2:

1. We first consider the common properties of the Grünwald approximations of the operators with right Dirichlet boundary condition, $(A, \bullet\mathrm{D})$ on Ω_2. Following that, we collate and complete the proof of Statement 1 of Proposition 4.3.2 for the operators (A, DD) and (A, ND) separately.

2. We first consider the common properties of the Grünwald approximations of the operators with right Neumann* boundary condition, $(A, \bullet\mathrm{N}^*)$ on Ω_2. Following that, we collate and complete the proof of Statement 1 of Proposition 4.3.2 for the operators (A, DN^*) and (A, NN^*) separately.

3. We first consider the common properties of the Grünwald approximations of the operators with right Neumann boundary condition, $(A, \bullet\mathrm{N})$ on Ω_2. Following that, we collate and complete the proof of Statement 1 of Proposition 4.3.2 for the operators (A, DN) and (A, NN) separately.

Outline of proof of $f_h \to f$ and $G^h f_h \to Af$ in $L_1[0,1]$:

For each of the fractional derivative operators (A, BC) on $L_1[0,1]$ given in Table 4.3, and for each $f \in \mathcal{C}(A, \mathrm{BC})$ we show that there exists a sequence $\{f_h\} \subset L_1[0,1]$ such that $f_h \to f$ and $G^h f_h \to Af$ in $L_1[0,1]$. To this end, given $\epsilon > 0$ we show that there exists $\delta > 0$ such that for $h < \delta$,

$$\|f_h - f\|_{L_1[0,1]} < \epsilon \qquad \text{and} \qquad \left\|G^h f_h - Af\right\|_{L_1[0,1]} < \epsilon.$$

1. We first consider the common properties of the Grünwald approximations of the operators with left Dirichlet boundary condition, $(A, \mathrm{D}\bullet)$. Following that, we collate and complete the proof of Statement 2 of Proposition 4.3.2 for the operators (A, DD) and (A, ND) separately.

2. We prove Statement 2 of Proposition 4.3.2 for the operator $(D_c^\alpha, \mathrm{ND})$.

3. We prove Statement 2 of Proposition 4.3.2 for the operator $(D_c^\alpha, \mathrm{NN})$.

4. We first consider the common properties of the Grünwald approximations of the operators with left Neumann boundary condition, $(D^\alpha, \mathrm{N}\bullet)$ on Ω_2. Following that, we collate and complete the proof of Statement 2 of Proposition 4.3.2 for the operators (D^α, ND) and (D^α, NN) separately.

$\square$

The following theorem and its corollaries are the most important results of the second part of this thesis. The following theorem establishes that the Grünwald-type approximations converge to the left-sided fractional derivative operators on X.

Theorem 4.3.3 (Trotter-Kato type approximation theorem). *Let the fractional derivative operators (A^+, BC) and $(A^\leftrightarrow, \mathrm{BC})$ be given by Definition 3.3.8 along with a core as in Theorem 3.4.4, also see Remark 4.2.2. Further, let the corresponding Grünwald transition operators (G^h, BC) and $(G^{h,\leftrightarrow}, \mathrm{BC})$ be given by (4.17) and (4.18), respectively. Then, we have the following:*

1. *The operators (G^h, BC) and (A, BC) generate Feller semigroups on $C_0(\Omega)$. The operators $(G^{h,\leftrightarrow}, \mathrm{BC})$ and $(A^\leftrightarrow, \mathrm{BC})$ generate positive, strongly continuous, contraction semigroups on $L_1[0,1]$.*

2. *The semigroups generated by (G^h, BC) converge strongly (and uniformly for $t \in [0, t_0]$) to the semigroup generated by (A, BC) on $C_0(\Omega)$. The semigroups generated by $(G^{h,\leftrightarrow}, \mathrm{BC})$ converge strongly (and uniformly for $t \in [0, t_0]$) to the semigroup generated by $(A^\leftrightarrow, \mathrm{BC})$ on $L_1[0,1]$.*

Proof. To simplify notation, let us write G^h for both G^h and $G^{h,\leftrightarrow}$ as well as A for both A^+ and $A^{\leftrightarrow}$. As a consequence of Proposition 4.3.2, for each $f \in \mathcal{C}(A, \mathrm{BC})$ there exists sequence $\{f_h\} \subset C_0(\Omega)$ such that $f_h \to f$ and $G^h f_h \to Af$ in $C_0(\Omega)$ and $\{f_h\} \subset L_1[0,1]$ such that $f_h \to f$ and $G^h f_h \to Af$ in $L_1[0,1]$. In view of Lemma 4.3.1, G^h are dissipative; that is,

$$\|(\lambda - G^h)f_h\| \geq \lambda \|f_h\|$$

for all $f_h \in X$ and all $\lambda > 0$. Thus, as $h \to 0$, in view of Proposition 4.3.2 we have

$$\|(\lambda - A)f\| \geq \lambda \|f\|$$

for all $f \in \mathcal{C}(A, \mathrm{BC})$ in the respective X-norms. As a consequence, this inequality holds for all $f \in \mathcal{D}(A, \mathrm{BC})$, and hence (A, BC) are dissipative. Furthermore, in view of Theorems 3.4.4 and 3.4.5, (A, BC) are densely defined closed operators with dense $\mathrm{rg}(\lambda - A)$ in X. Hence, the operators (A, BC) generate strongly continuous contraction semigroups as a consequence of the Lumer-Phillips Theorem (C.2.9). The proof of the first statement is complete in view of Lemma 4.3.1. The second statement follows using the Trotter-Kato theorem (C.2.10), in view of Proposition 4.3.2. □

We require the following lemma to prove the crucial Corollary 4.3.5 which explicitly identifies the part of the adjoint of the left-sided fractional derivative operator in $L_1[0,1]$ and the part of the adjoint of the right-sided fractional derivative operator in $C_0(\Omega)$.

Lemma 4.3.4. *Let the operators A, B be such that $A \subset B$, A is surjective and B is injective, then $A = B$.*

Proof. To see this, let $x \in \mathcal{D}(B)$. Then, since A is surjective and $A = B|_{\mathcal{D}(A)}$ there exists $w \in \mathcal{D}(A)$ such that $Bx = y = Aw = Bw$. Next, since B is injective, we have that $x = w$. Thus, $\mathcal{D}(B) \subset \mathcal{D}(A)$ and hence, $B \subset A$. □

As a consequence of Theorem 4.3.3, $(A^{\leftrightarrow}, \mathrm{BC})$ generate strongly continuous contraction semigroups on $L_1[0,1]$ and the semigroups generated by the corresponding right-sided fractional derivative operators (A^-, BC) (with mirrored boundary conditions) are similar semigroups in view of Remark 4.2.2, we have that (A^-, BC) also generate strongly continuous contraction semigroups on $L_1[0,1]$.

Corollary 4.3.5. *Let (A^+, BC) be a left-sided fractional derivative operator on $C_0(\Omega)$ and (A^-, BC) be the corresponding right-sided fractional derivative operator on $L_1[0,1]$ whose domain encodes the same combination of boundary conditions* BC *as given in Table 4.3. Then, $(A^-, \mathrm{BC}) = (A^+, \mathrm{BC})^*|_{L_1[0,1]}$ and $(A^+, \mathrm{BC}) = (A^-, \mathrm{BC})^*|_{C_0(\Omega)}$.*

Proof. Firstly, in view of Theorem 4.2.1 we have that

$$I - (A^-, \mathrm{BC}) \subset \big(I - (A^+, \mathrm{BC})^*\big)|_{L_1[0,1]} \text{ and } I - (A^+, \mathrm{BC}) \subset \big(I - (A^-, \mathrm{BC})^*\big)|_{C_0(\Omega)},$$

where I is the identity operator on the respective spaces. Secondly, in view of Theorem 4.3.3 and Remark 4.2.2, we have that (A^+, BC) and (A^-, BC) generate strongly continuous semigroups on $C_0(\Omega)$ and $L_1[0, 1]$, respectively. Therefore, in particular, $1 \in \rho((A^+, \mathrm{BC}))$ and $1 \in \rho((A^-, \mathrm{BC}))$. Thus, $I - (A^-, \mathrm{BC})$ is surjective and $I - (A^+, \mathrm{BC})^*$ is injective which implies that $\big(I - (A^+, \mathrm{BC})^*\big)|_{L_1[0,1]}$ is also injective. Then, in view of Lemma 4.3.4, we have that $I - (A^-, \mathrm{BC}) = \big(I - (A^+, \mathrm{BC})^*\big)|_{L_1[0,1]}$. Hence,

$$(A^-, \mathrm{BC}) = (A^+, \mathrm{BC})^*|_{L_1[0,1]}.$$

A similar argument holds for the pair, (A^+, BC) and $(A^-, \mathrm{BC})^*|_{C_0(\Omega)}$. □

The following result is the culmination of the second part of this thesis. This result identifies the stochastic processes, which are governed by fractional-in-space partial differential equations that employ fractional derivative operators (A, BC) on X with boundary conditions BC, as limits of processes whose boundary behaviour is perfectly understood. Let us briefly summarise the preparations made so far in Chapters 3 and 4 that yield this result in order to highlight its significance.

We began with transition rates of a finite state sub-Markov processes that signify the process jumping from state i to state j or from state j to state i for the spaces ℓ_∞^n and ℓ_1^n, respectively. The associated semigroups are the so-called backward semigroups $(e^{tG_{n\times n}})_{t\geq 0}$ in the case of ℓ_∞^n and the forward semigroups $(e^{tG^*_{n\times n}})_{t\geq 0}$ in the case of ℓ_1^n, where the matrices $G_{n\times n}$ and $G^*_{n\times n}$ are adjoint of each other. We extended the backward semigroups on ℓ_∞^n to $L_\infty(\Omega)$, restricted to its closed subspace $C_0(\Omega)$. Similarly, we extended the forward semigroups on ℓ_1^n to $\mathcal{M}_\mathcal{B}(\Omega)$, restricted to its closed subspace $L_1[0, 1]$. This general theory was then used in the construction Grünwald-type approximations for the fractional derivative operators on X.

We studied the properties of the (left-sided) one-sided fractional derivative operators (A^+, BC) on X. In Corollary 4.3.5, the part in $L_1[0, 1]$ of the adjoint of the left-sided fractional derivative operators on $C_0(\Omega)$ was explicitly identified as the right sided fractional derivative operator (A^-, BC) where the domains of both the operators encode the same boundary conditions BC. Moreover, in Remark 4.2.2, we justified that it is sufficient to consider only the semigroups generated by the left-sided fractional derivative operators on X, since the semigroups generated by the left-sided fractional derivative operators on $L_1[0,1]$ and the right-sided fractional derivative operators on $L_1[0,1]$ are similar semigroups.

In Chapter 3, we showed that the left-sided fractional derivative operators (A, BC) are densely defined closed operators on X and that $\mathrm{rg}(\lambda I - A)$ are dense in X. To conclude that the fractional derivative operators generate positive, strongly continuous contraction semigroups using the Lumer-Philips Theorem we established dissipativity of the left-sided fractional derivative operators on X using the Grünwald approximations. That is, in Proposition 4.3.2, we showed that the Grünwald approximations (G^h, BC) converge to the respective fractional derivative operators (A, BC). As a further consequence of Proposition 4.3.2, in Theorem 4.3.3, we established the convergence of the semigroups generated by the Grünwald approximations to those generated by the one-sided fractional derivative operators on X.

On the one hand, we showed that the Grünwald approximations generate positive, strongly continuous contraction semigroups on X, in particular, Feller semigroups $(S_h(t))_{t\geq 0}$ on $C_0(\Omega)$. On the other hand, we also showed that the fractional derivative operators generate positive, strongly continuous, contraction semigroups on X, in particular, Feller semigroups $(S(t))_{t\geq 0}$ on $C_0(\Omega)$. Therefore, there exist Feller processes $(X_t^h)_{t\geq 0}$ and $(X_t)_{t\geq 0}$ with $(S_h(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$, respectively, as their backward semigroups [53, Chapter 17]. Furthermore, in view of Theorem 4.3.3 the family of Feller semigroups $(S_h(t))_{t\geq 0}$ generated by Grünwald approximations converge strongly, uniformly for $t \in [0, t_0]$, to the Feller semigroups $(S(t))_{t\geq 0}$ generated by the fractional derivative operators on X. As a consequence, we have the following important result.

Corollary 4.3.6. *The stochastic processes $(X_t^h)_{t\geq 0}$ associated with the Grünwald approximations (G^h, BC) converge in the Skorokhod topology to the processes $(X_t)_{t\geq 0}$ associated with the fractional derivative operators (A, BC).*

Proof. The proof is complete in view of Theorem 4.3.3, Remark 4.2.2, Corollary 4.3.5 and [53, p. 331, Theorem 17.25]. □

This corollary marks the conclusion of the second part of this thesis. However, as mentioned at the start of Section 4.3, the detailed proof of Proposition 4.3.2 is provided in Section 4.5 and we make the necessary preparations for the same in Section 4.5.1. But first, we have the following section on the numerical solutions for the Cauchy problem associated with the fractional derivative operators on $L_1[0,1]$.

4.4 Examples of Grünwald schemes for Cauchy problems on $L_1[0,1]$

In this section we provide some examples of numerical solutions to the Cauchy problem associated with the fractional derivative operators (A, BC) on $L_1[0,1]$ and the initial value $u_0 \in L_1[0,1]$,

$$u'(t) = Au(t) \quad \text{for } t \geq 0, \quad u(0) = u_0, \tag{4.22}$$

that employs the Grünwald scheme. For the space discretisation of the Grünwald scheme, we use the Grünwald transition matrix,

$$G^h_{n\times n} = \frac{1}{h^\alpha}\begin{pmatrix} b^l_1 & \mathcal{G}^\alpha_0 & 0 & \cdots & 0 \\ \vdots & \mathcal{G}^\alpha_1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ b^l_{n-1} & \mathcal{G}^\alpha_{n-2} & \cdots & \mathcal{G}^\alpha_1 & \mathcal{G}^\alpha_0 \\ b_n & b^r_{n-1} & \cdots & \cdots & b^r_1 \end{pmatrix}, \tag{4.23}$$

where the boundary weights are given in Table 4.4.

Boundary condition	Boundary weights for $G^h_{n\times n}$
$f(0) = 0$	$b^l_i = \mathcal{G}^\alpha_i$
$D^{\alpha-1}_c f(0) = 0$	$b^l_i = -\mathcal{G}^{\alpha-1}_{i-1}$
$D^{\alpha-1} f(0) = 0$	$b^l_0 = 0,\ b^l_1 = \mathcal{G}^\alpha_0 + \mathcal{G}^\alpha_1,\ b^l_i = \mathcal{G}^\alpha_i$
$f(1) = 0$	$b^r_i = \mathcal{G}^\alpha_i,\ b_n = b^l_n$
$Ff(1) = 0$	$b^r_i = -\mathcal{G}^{\alpha-1}_{i-1},\ b_n = -\sum_{i=0}^{n-1} b^l_i$

Table 4.4: Boundary conditions and boundary weights for $L_1[0,1]$.

To illustrate the efficiency of the Grünwald scheme in handling different boundary conditions, we plot the numerical solutions to the Cauchy problem associated with the fractional derivative operators on $L_1[0,1]$ and the initial value

$$u_0(x) = \begin{cases} 25x - 7.5, & \text{for } 0.3 < x \leq 0.5, \\ -25x + 7.5, & \text{for } 0.5 < x < 0.7, \\ 0, & \text{otherwise.} \end{cases} \tag{4.24}$$

at various times below. For the numerical scheme, we take $\alpha = 1.5$, the time step $\Delta t = 0.01$ and the space grid size $h = 0.001$. The MATLAB codes used to obtain the numerical solutions are given in Appendix D.

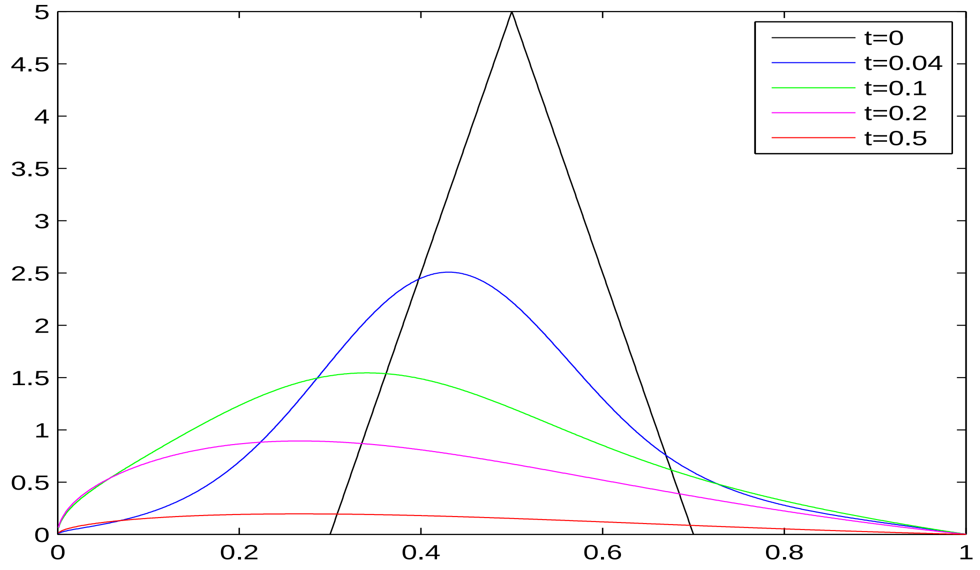

Figure 4.1: Time evolution of the numerical solution to the Cauchy problem associated with $(D_c^{1.5}, \mathrm{DD})$ and initial value u_0 given by (4.24).

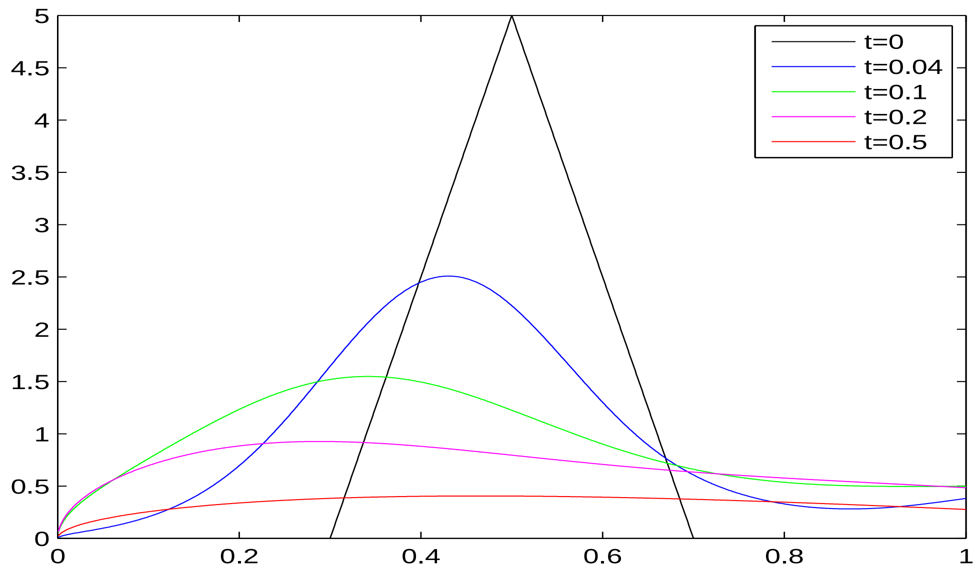

Figure 4.2: Time evolution of the numerical solution to the Cauchy problem associated with $(D_c^{1.5}, \mathrm{DN})$ and initial value u_0 given by (4.24). Take note of the build up at the right boundary.

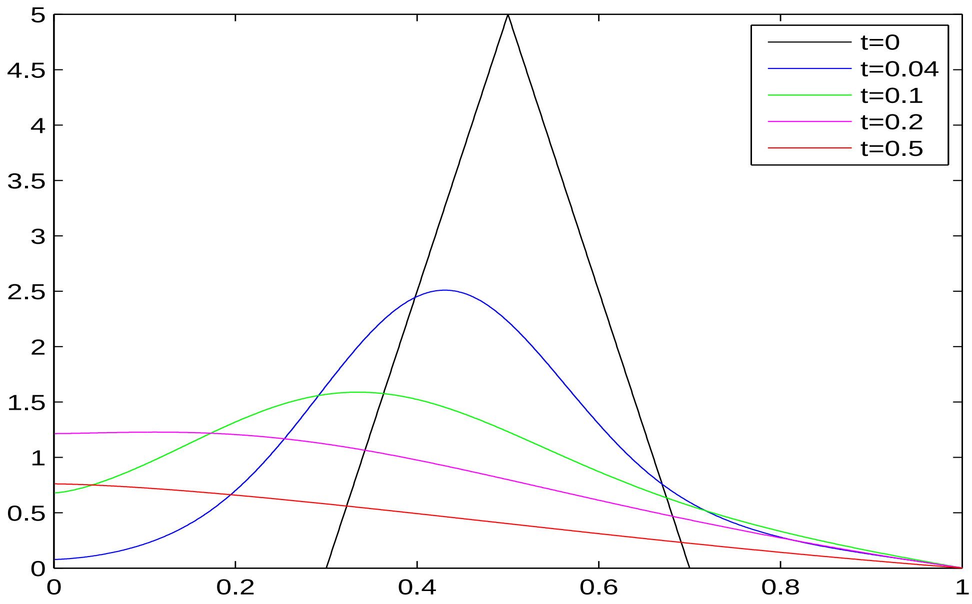

Figure 4.3: Time evolution of the numerical solution to the Cauchy problem associated with $(D_c^{1.5}, \mathrm{ND})$ and initial value u_0 given by (4.24).

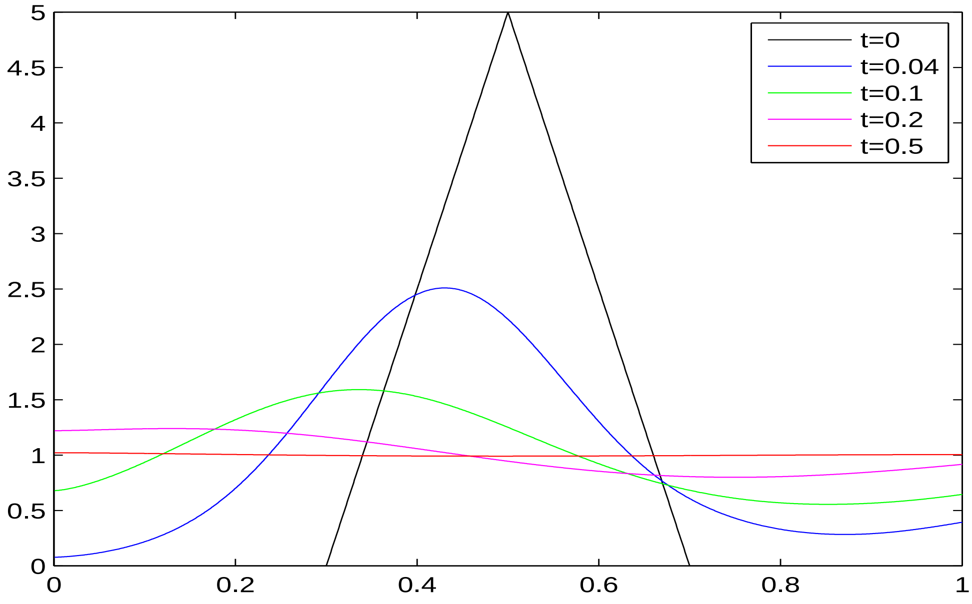

Figure 4.4: Time evolution of the numerical solution to the Cauchy problem associated with $(D_c^{1.5}, \mathrm{NN})$ and initial value u_0 given by (4.24).

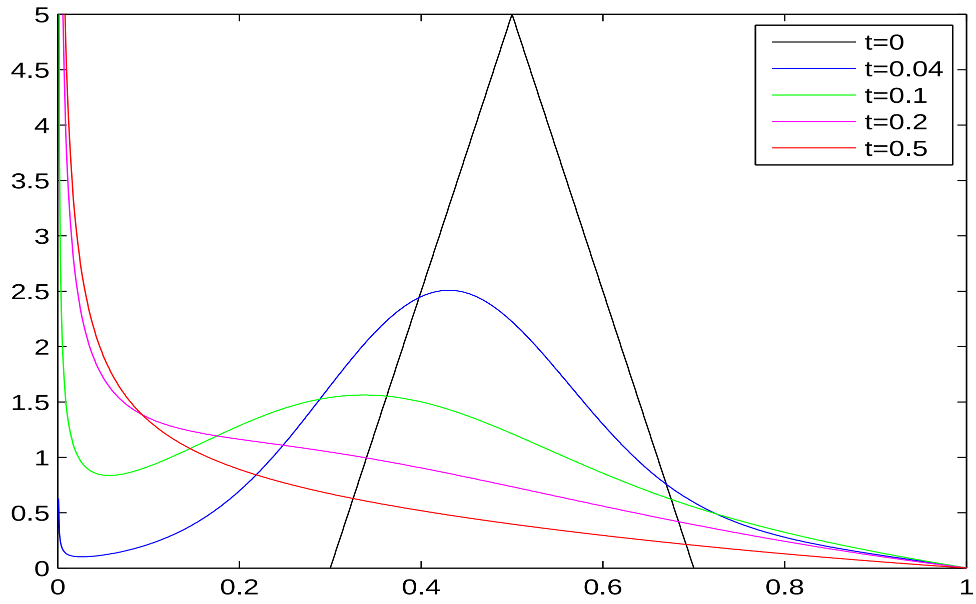

Figure 4.5: Time evolution of the numerical solution to the Cauchy problem associated with $(D^{1.5}, \mathrm{ND})$ and initial value u_0 given by (4.24).

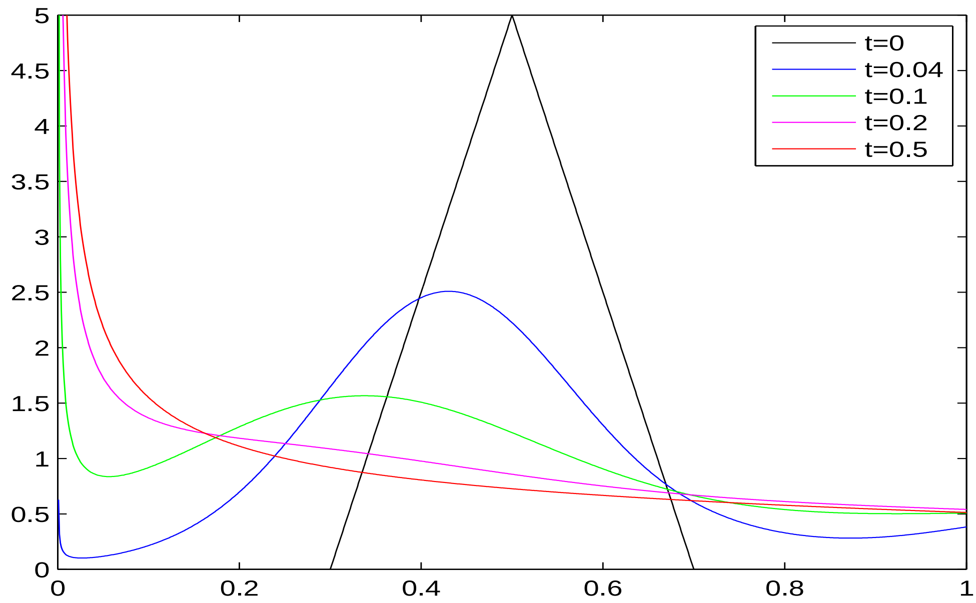

Figure 4.6: Time evolution of the numerical solution to the Cauchy problem associated with $(D^{1.5}, \mathrm{NN})$ and initial value u_0 given by (4.24).

4.5 Detailed proof of Proposition 4.3.2

The proof is divided into two main cases $X = C_0(\Omega)$ and $X = L_1[0,1]$ where the detailed proofs are given in Sections 4.5.2 and 4.5.3, respectively.

4.5.1 Construction of approximate power functions

In this section we make the necessary preparations for the proof of Proposition 4.3.2.

In what follows, let $1 < \alpha < 2$, $n \in \mathbb{N}$ and $h = \frac{1}{n+1}$. Moreover, for $x \in [0,1]$, let $\iota(x) - 1 = \left\lfloor \frac{x}{h} \right\rfloor$ and $\lambda(x) = \left\{ \frac{x}{h} \right\}$ denote the integer and fractional parts on $\frac{x}{h}$ as in Definition 3.2.2 and the Grünwald coefficients $\mathcal{G}_m^\alpha = (-1)^m \binom{\alpha}{m}$ be given by (A.1). Furthermore, to simplify notation, we write $\lambda := \lambda(x)$ and $\lambda' = 1 - \lambda$.

Definition 4.5.1. The approximate power functions for $C_0(\Omega)$ are defined as follows:

1.
$$\vartheta_h^\alpha(x) = h^\alpha \begin{cases} -\lambda', & \text{if } \iota(x) = 1, \\ \left(\lambda' \mathcal{G}_{\iota(x)-3}^{-\alpha-1} + \lambda \mathcal{G}_{\iota(x)-2}^{-\alpha-1}\right), & \text{if } \iota(x) \neq 1. \end{cases}$$

2.
$$\vartheta_h^{\alpha-1}(x) = h^{\alpha-1} \left(\lambda' \mathcal{G}_{\iota(x)-2}^{-\alpha} + \lambda \mathcal{G}_{\iota(x)-1}^{-\alpha}\right).$$

3.
$$\vartheta_h^0(x) = p_0(x).$$

The approximate power functions for $L_1[0,1]$ are defined as follows:

1.
$$\vartheta_h^\alpha(x) = h^\alpha \begin{cases} -\lambda', & \text{if } \iota(x) = 1, \\ \mathcal{G}_{\iota(x)-2}^{-\alpha-1}, & \text{if } \iota(x) \neq 1. \end{cases}$$

2.
$$\vartheta_h^{\alpha-1}(x) = h^{\alpha-1} \begin{cases} \frac{1}{\alpha}\left(\lambda' \mathcal{G}_0^{-\alpha} + \lambda \mathcal{G}_1^{-\alpha}\right), & \text{if } \iota(x) = 1, \\ \left(\lambda' \mathcal{G}_{\iota(x)-2}^{-\alpha} + \lambda \mathcal{G}_{\iota(x)-1}^{-\alpha}\right), & \text{if } \iota(x) \neq 1. \end{cases}$$

3.
$$\vartheta_h^0(x) = \begin{cases} \lambda, & \text{if } \iota(x) = 1, \\ p_0(x), & \text{if } \iota(x) \neq 1. \end{cases}$$

4.
$$\vartheta_h^{\alpha-2}(x) = h^{\alpha-2} \left((1-\theta(\lambda)) \mathcal{G}_{\iota(x)-2}^{-\alpha+1} + \theta(\lambda) \mathcal{G}_{\iota(x)-1}^{-\alpha+1}\right),$$

where $\theta(\lambda) = \frac{\lambda}{(\alpha-1)\lambda' + \lambda}$.

Remark 4.5.2 (**Canonical extension to** $[0,1+h]$). In the proof of Proposition 4.3.2, when dealing with the right Dirichlet and Neumann* boundary conditions, we require the values of the function f_h in the interval $(1,1+h]$. Therefore, the domain of f_h need to be extended to the interval $[0,1+h]$. To do this, we also extend the definitions of λ and ι given in Definition 3.2.2 to the interval $[0,1+h]$, but consider $x=1$ to belong to the interval $[1-h,1]$, that is, $\lambda(1)=1$ and $\iota(1)=n+1$ and similarly, $x=1+h$ to belong to the extended interval $(1,1+h]$, that is, $\lambda(1+h)=1$ and $\iota(1+h)=n+2$.

Let $\phi=\sum_{m=0}^{N}k_m p_{\alpha+m}$. Let us also assume that ϕ is canonically extended to the interval $[0,1+h]$ and $x\in[1-h,1]$. Then, the Taylor expansions around $x=1$ are given by

$$\begin{aligned}
&\phi(x-h)=\phi(1)-(2-\lambda)h\phi'(1)+O(h^2),\\
&\phi(x)=\phi(1)-(1-\lambda)h\phi'(1)+O(h^2),\\
&\phi(x+h)=\phi(1)+\lambda h\phi'(1)+O(h^2),
\end{aligned}\tag{4.25}$$

where observe that $\lambda:=\lambda(x)=\lambda(x-h)=\lambda(x+h)$ for $x\neq 1$.

We prove some of the properties of the interpolated functions that we use repeatedly.

Lemma 4.5.3. *Let the approximate power functions be as in Definition 4.5.1. Then* ϑ_h^{α}, $\vartheta_h^{\alpha-1}$ *and* ϑ_h^{0} *converge to* p_α, $p_{\alpha-1}$ *and* p_0, *respectively in the* X*-norm. Moreover,* $\vartheta_h^{\alpha-2}$ *converges to* $p_{\alpha-2}$ *in* $L_1[0,1]$*-norm.*

Proof. Firstly, observe that $\vartheta_h^0\to p_0$ in X-norm. In view of (A.9), on a fixed grid point $x=(k+1)h$, we have

$$\vartheta_h^{\alpha}((k+1)h)=h^{\alpha}\mathcal{G}_{k-1}^{-\alpha-1}=\frac{(h(k-1))^{\alpha}}{\Gamma(\alpha+1)}\left[1+O(k^{-1})\right]=p_\alpha((k-1)h)+O(h),$$

$$\vartheta_h^{\alpha-1}((k+1)h)=h^{\alpha-1}\mathcal{G}_{k}^{-\alpha}=\frac{(hk)^{\alpha-1}}{\Gamma(\alpha)}\left[1+O(k^{-1})\right]=p_{\alpha-1}(kh)+O(h)$$

and

$$\vartheta_h^{\alpha-2}((k+1)h)=h^{\alpha-2}\mathcal{G}_{k}^{-\alpha+1}=\frac{(hk)^{\alpha-2}}{\Gamma(\alpha-1)}\left[1+O(k^{-1})\right]=p_{\alpha-2}(kh)+O(h).$$

Then it follows that, as $h\to0$, $\vartheta_h^{\alpha}\to p_\alpha$, $\vartheta_h^{\alpha-1}\to p_{\alpha-1}$ in X-norm and $\vartheta_h^{\alpha-2}\to p_{\alpha-2}$ in $L_1[0,1]$-norm. □

Lemma 4.5.4. *Let* ϑ_h^{β} *for* $\beta\in\{\alpha,\alpha-1,\ \alpha-2\}$ *given in Definition 4.5.1 be canonically extended to the interval* $[0,1+h]$ *and set* $\lambda:=\lambda(x)=\lambda(x+h)$. *Then we have the following identities for* $x\in[1-h,1]$,

$$\frac{\vartheta_h^{\alpha-2}(x)}{\vartheta_h^{\alpha-2}(1)}-1=-(1-\theta(\lambda))\frac{\alpha-2}{n-2+\alpha},$$

$$\begin{aligned}
\frac{\vartheta_h^{\alpha-2}(x+h)}{\vartheta_h^{\alpha-2}(1)} - 1 &= \theta(\lambda)\frac{\alpha-2}{n+1}, \\
\frac{\vartheta_h^{\alpha-1}(x)}{\vartheta_h^{\alpha-1}(1)} - 1 &= -\lambda'\frac{\alpha-1}{n-1+\alpha}, \\
\frac{\vartheta_h^{\alpha-1}(x+h)}{\vartheta_h^{\alpha-1}(1)} - 1 &= \lambda\frac{\alpha-1}{n+1}, \\
\frac{\vartheta_h^{\alpha}(x)}{\vartheta_h^{\alpha}(1)} - 1 &= -\lambda'\frac{\alpha}{n-1+\alpha}, \\
\frac{\vartheta_h^{\alpha}(x+h)}{\vartheta_h^{\alpha}(1)} - 1 &= \lambda\frac{\alpha}{n}.
\end{aligned} \tag{4.26}$$

Moreover, for $x \in [1-2h, 1]$, we have

$$\begin{aligned}
\vartheta_h^{\alpha-1}(x) - \vartheta_h^{\alpha-1}(x+h) &= \vartheta_h^{\alpha-1}(1)\left(\frac{-(\alpha-1)}{k-1+\alpha} + O(h^2)\right) \\
\vartheta_h^{\alpha}(x) - \vartheta_h^{\alpha}(x+h) &= \vartheta_h^{\alpha}(1)\left(\frac{-\alpha}{k-1+\alpha} + O(h^2)\right),
\end{aligned} \tag{4.27}$$

where $k = \iota(x) - 1$.

Proof. Let $x \in [1-h, 1]$ and in the following calculations we make use of

$$\vartheta_h^{\alpha-2}(1) = h^{\alpha-2}\mathcal{G}_n^{-\alpha+1},\ \vartheta_h^{\alpha-1}(1) = h^{\alpha-1}\mathcal{G}_n^{-\alpha},\ \vartheta_h^{\alpha}(1) = h^{\alpha}\mathcal{G}_{n-1}^{-\alpha-1} \tag{4.28}$$

along with Definition (4.5.1) and (A.2) as required. Firstly, setting $\theta := \theta(\lambda)$ we have

$$\begin{aligned}
\frac{\vartheta_h^{\alpha-2}(x)}{\vartheta_h^{\alpha-2}(1)} &= \frac{h^{\alpha-2}\left((1-\theta)\mathcal{G}_{n-1}^{-\alpha+1} + \theta\mathcal{G}_n^{-\alpha+1}\right)}{h^{\alpha-2}\mathcal{G}_n^{-\alpha+1}} \\
&= (1-\theta)\frac{n}{n-2+\alpha} + \theta \\
&= 1 - (1-\theta)\frac{\alpha-2}{n-2+\alpha}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\vartheta_h^{\alpha-2}(x+h)}{\vartheta_h^{\alpha-2}(1)} &= \frac{h^{\alpha-2}\left((1-\theta)\mathcal{G}_n^{-\alpha+1} + \theta\mathcal{G}_{n+1}^{-\alpha+1}\right)}{h^{\alpha-2}\mathcal{G}_n^{-\alpha+1}} \\
&= (1-\theta) + \theta\frac{n+\alpha-1}{n+1} \\
&= 1 + \theta\frac{\alpha-2}{n+1}.
\end{aligned}$$

Secondly,

$$\frac{\vartheta_h^{\alpha-1}(x)}{\vartheta_h^{\alpha-1}(1)} = \frac{h^{\alpha-1}\left((1-\lambda)\mathcal{G}_{n-1}^{-\alpha} + \lambda\mathcal{G}_n^{-\alpha}\right)}{h^{\alpha-1}\mathcal{G}_n^{-\alpha}}$$

$$
\begin{aligned}
&= (1-\lambda)\frac{n}{n-1+\alpha} + \lambda \\
&= 1 - \lambda' \frac{\alpha - 1}{n-1+\alpha}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\vartheta_h^{\alpha-1}(x+h)}{\vartheta_h^{\alpha-1}(1)} &= \frac{h^{\alpha-1}\left((1-\lambda)\mathcal{G}_n^{-\alpha} + \lambda \mathcal{G}_{n+1}^{-\alpha}\right)}{h^{\alpha-1}\mathcal{G}_n^{-\alpha}} \\
&= (1-\lambda) + \lambda \frac{n+\alpha}{n+1} \\
&= 1 + \lambda \frac{\alpha-1}{n+1}.
\end{aligned}
$$

Thirdly,

$$
\begin{aligned}
\frac{\vartheta_h^{\alpha}(x)}{\vartheta_h^{\alpha}(1)} &= \frac{h^{\alpha}\left((1-\lambda)\mathcal{G}_{n-2}^{-\alpha-1} + \lambda \mathcal{G}_{n-1}^{-\alpha-1}\right)}{h^{\alpha}\mathcal{G}_{n-1}^{-\alpha-1}} \\
&= (1-\lambda)\frac{n-1}{n-1+\alpha} + \lambda \\
&= 1 - \lambda' \frac{\alpha}{n-1+\alpha}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\vartheta_h^{\alpha}(x+h)}{\vartheta_h^{\alpha}(1)} &= \frac{h^{\alpha}\left((1-\lambda)\mathcal{G}_{n-1}^{-\alpha-1} + \lambda \mathcal{G}_{n}^{-\alpha-1}\right)}{h^{\alpha}\mathcal{G}_{n-1}^{-\alpha-1}} \\
&= (1-\lambda) + \lambda \frac{n+\alpha}{n} \\
&= 1 + \lambda \frac{\alpha}{n}
\end{aligned}
$$

Next, for $x \in [1-h, 1]$ using the above identities

$$
\begin{aligned}
\vartheta_h^{\alpha-1}(x) - \vartheta_h^{\alpha-1}(x+h) &= \vartheta_h^{\alpha-1}(1)\left(\frac{-(\alpha-1)}{n-1+\alpha} + (\alpha-1)\lambda\left(\frac{1}{n-1+\alpha} - \frac{1}{n+1}\right)\right) \\
&= \vartheta_h^{\alpha-1}(1)\left(\frac{-(\alpha-1)}{n-1+\alpha} + O(h^2)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\vartheta_h^{\alpha}(x) - \vartheta_h^{\alpha}(x+h) &= \vartheta_h^{\alpha}(1)\left(\frac{-\alpha}{n-1+\alpha} + \alpha\lambda\left(\frac{1}{n-1+\alpha} - \frac{1}{n}\right)\right) \\
&= \vartheta_h^{\alpha}(1)\left(\frac{-\alpha}{n-1+\alpha} + O(h^2)\right).
\end{aligned}
$$

Note that if $x = 1-h$, since $\lambda = 0$, $O(h^2)$ terms above are absent.

Moreover, for $x \in [1-2h, 1-h)$, using (4.28) and $\mathcal{G}_k^\beta = \left(1 + \frac{\beta+1}{k-\beta}\right)\mathcal{G}_{k+1}^\beta$ given by (A.2), note that

$$
\begin{aligned}
\vartheta_h^\alpha(x) &= h^\alpha\left((1-\lambda)\mathcal{G}_{n-3}^{-\alpha-1} + \lambda\mathcal{G}_{n-2}^{-\alpha-1}\right) \\
&= h^\alpha\left((1-\lambda)\mathcal{G}_{n-2}^{-\alpha-1}\left(1 - \frac{\alpha}{n-2+\alpha}\right) + \lambda\mathcal{G}_{n-1}^{-\alpha-1}\left(1 - \frac{\alpha}{n-1+\alpha}\right)\right) \\
&= \vartheta_h^\alpha(x+h) + h^\alpha\left(-\mathcal{G}_{n-2}^{-\alpha-1}\frac{\alpha(1-\lambda)}{n-2+\alpha} - \mathcal{G}_{n-1}^{-\alpha-1}\frac{\alpha\lambda}{n-1+\alpha}\right) \\
&= \vartheta_h^\alpha(x+h) + h^\alpha\mathcal{G}_{n-1}^{-\alpha-1}\left(-\left(1 - \frac{\alpha}{n-1+\alpha}\right)\left(\frac{\alpha-\alpha\lambda}{n-2+\alpha}\right) - \frac{\alpha\lambda}{n-1+\alpha}\right) \\
&= \vartheta_h^\alpha(x+h) + \vartheta_h^\alpha(1)\left(\frac{-\alpha}{n-2+\alpha} + O(h^2)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\vartheta_h^{\alpha-1}(x) &= h^{\alpha-1}\left((1-\lambda)\mathcal{G}_{n-2}^{-\alpha} + \lambda\mathcal{G}_{n-1}^{-\alpha}\right) \\
&= h^{\alpha-1}\left((1-\lambda)\mathcal{G}_{n-1}^{-\alpha}\left(1 - \frac{\alpha-1}{n-2+\alpha}\right) + \lambda\mathcal{G}_{n}^{-\alpha}\left(1 - \frac{\alpha-1}{n-1+\alpha}\right)\right) \\
&= \vartheta_h^{\alpha-1}(x+h) + h^{\alpha-1}\left(-\mathcal{G}_{n-1}^{-\alpha}\frac{(\alpha-1)(1-\lambda)}{n-2+\alpha} - \mathcal{G}_{n}^{-\alpha}\frac{(\alpha-1)\lambda}{n-1+\alpha}\right) \\
&= \vartheta_h^{\alpha-1}(x+h) \\
&\qquad + h^{\alpha-1}\mathcal{G}_{n}^{-\alpha}\left(-\left(1 - \frac{\alpha-1}{n-1+\alpha}\right)\left(\frac{(\alpha-1)-(\alpha-1)\lambda}{n-2+\alpha}\right) - \frac{(\alpha-1)\lambda}{n-1+\alpha}\right) \\
&= \vartheta_h^{\alpha-1}(x+h) + \vartheta_h^{\alpha-1}(1)\left(\frac{-(\alpha-1)}{n-2+\alpha} + O(h^2)\right).
\end{aligned}
$$

□

Recall that for $\alpha > 0$, $f \in L_1(\mathbb{R})$ or $C_0(\mathbb{R})$ and $h > 0$, the shifted Grünwald formula (1.3) is given by

$$A_{h,p}^\alpha f(x) = \frac{1}{h^\alpha}\sum_{k=0}^{\infty}\mathcal{G}_k^\alpha f(x-(k-p)h),$$

where the Grünwald coefficients, $\mathcal{G}_m^\alpha = (-1)^m\binom{\alpha}{m}$ are given by (A.1).

We make use of this Grünwald formula repeatedly in the proofs of Propositions 4.5.7 and 4.3.2. For easy reference we rewrite them using the notation employed therein. Let $x \in [0, 1+h]$; that is, $x = (\lambda(x) + \iota(x) - 1)h$ as in Definition 3.2.2. Then, for $1 < \alpha < 2$ with shift $p = 1$ and for $0 < \alpha - 1 < 1$ with shift $p = 0$, we have

$$A_{h,1}^\alpha f(x) = \frac{1}{h^\alpha}\sum_{k=0}^{\iota(x)}\mathcal{G}_k^\alpha f((\lambda(x) + \iota(x) - 1 - (k-1))h),$$

$$A_{h,0}^{\alpha-1} f(x) = \frac{1}{h^{\alpha-1}} \sum_{k=0}^{\iota(x)-1} \mathcal{G}_k^{\alpha-1} f((\lambda(x) + \iota(x) - 1 - k)h) \tag{4.29}$$

We state the following corollary of Theorem 2.2.1 which we use repeatedly in the proof of Proposition 4.3.2.

Corollary 4.5.5 (Corollary of Theorem 2.2.1)**.** *Let $1 < \alpha < 2$, and let $\beta > \alpha$ in the case of $C_0(\Omega)$ and $\beta \geq \alpha$ in the case of $L_1[0,1]$. Further, let*

$$\tilde{p}_\beta(x) = \begin{cases} 0, & \text{if } x < 0, \\ p_\beta(x), & \text{if } 0 \leq x \leq 2, \\ \theta(x), & \text{if } x \geq 2, \end{cases}$$

where $\theta \in C_0^\infty(\mathbb{R})$ such that the extended function $\tilde{p}_\beta \in C_0(\mathbb{R})$ or *$L_1(\mathbb{R})$, respectively. Then, given $\epsilon > 0$ there exists $\delta > 0$ such that for all $h < \delta$,*

$$\left\| A_{h,p}^\alpha p_\beta - p_{\beta-\alpha} \right\|_{C_0(\Omega)} \leq \left\| A_{h,p}^\alpha \tilde{p}_\beta - D^\alpha \tilde{p}_\beta \right\|_{C_0(\mathbb{R})} < \epsilon, \ \beta > \alpha$$

and

$$\left\| A_{h,p}^\alpha p_\beta - p_{\beta-\alpha} \right\|_{L_1[0,1]} \leq \left\| A_{h,p}^\alpha \tilde{p}_\beta - D^\alpha \tilde{p}_\beta \right\|_{L_1(\mathbb{R})} < \epsilon, \ \beta \geq \alpha.$$

Remark 4.5.6. Let $\mathcal{P} = I^\alpha(P - P(0)p_0)$, $P = \sum_{m=0}^N k_m p_m \in C_0(\Omega)$. Note that, since $\mathcal{P}(0) = 0$, $D^\alpha \mathcal{P} = D_c^\alpha \mathcal{P}$. Therefore, assuming that $\mathcal{P}$ is canonically extended to $[0, 1+h]$, in view of Corollary 4.5.5 and Lemma 1.4.2, the error terms are given by

$$A_{h,p}^\alpha \mathcal{P}(x) = D_c^\alpha \mathcal{P}(x) + h\Big(p - \frac{\alpha}{2}\Big) D_c^{\alpha+1} \mathcal{P}(x) + O(h^2).$$

In view of Corollary 4.5.5 and Lemma 1.4.2, for $\mathcal{Q} = I^\alpha P$, $P = \sum_{m=0}^N k_m p_m \in L_1[0,1]$ we have the error terms,

$$A_{h,p}^\alpha \mathcal{Q}(x) = D^\alpha \mathcal{Q}(x) + h\Big(p - \frac{\alpha}{2}\Big) D^{\alpha+1} \mathcal{Q}(x) + O(h^2)$$

and the same holds with D^β replaced by D_c^β.

We conclude the preparation for the proof of Proposition 4.3.2 with the following result which contains only those properties of the approximate power functions that we require.

Proposition 4.5.7. *Let ϑ_h^β for $\beta \in \{\alpha, \alpha-2\}$and in the case when $X = C_0(\Omega)$, $\vartheta_h^{\alpha-1}$ be given by Definition 4.5.1 and canonically extended to the interval $[0, 1+h]$. Let $\iota(x) - 1 = \left\lfloor \frac{x}{h} \right\rfloor$ and $\lambda := \lambda(x) = \left\{ \frac{x}{h} \right\}$ denote the integer and fractional parts of $\frac{x}{h}$ as in Definition 3.2.2. Then the following hold:*

1.
$$A_{h,1}^{\alpha}\vartheta_h^{\alpha}(x) = 1 - \lambda'\mathcal{G}_{\iota(x)}^{\alpha},\ \iota(x) \geq 2.$$

2.
$$A_{h,0}^{\alpha-1}\vartheta_h^{\alpha}(x) = h\left(\iota(x) - 2 + \theta(\lambda) - \lambda'\mathcal{G}_{\iota(x)-1}^{\alpha-1}\right),\ \iota(x) \geq 2.$$

3.
$$A_{h,1}^{\alpha}\vartheta_h^{\alpha-1}(x) = 0,\ \iota(x) \geq 2.$$

4.
$$A_{h,0}^{\alpha-1}\vartheta_h^{\alpha-1}(x) = 1,\ \iota(x) \geq 2.$$

5.
$$A_{h,1}^{\alpha}\vartheta_h^{\alpha-2}(x) = 0,\ \iota(x) \geq 3.$$

6.
$$A_{h,0}^{\alpha-1}\vartheta_h^{\alpha-2}(x) = 0,\ \iota(x) \geq 3.$$

Proof. In what follows we make use of $\sum_{m=0}^{k}\mathcal{G}_m^q\mathcal{G}_{k-m}^Q = \mathcal{G}_k^{q+Q}$ given by (A.8) as required.

1. Let $\iota(x) \geq 2$, $\theta = \lambda$ or 1 and note using (4.29) that
$$A_{h,1}^{\alpha}\vartheta_h^{\alpha}(x) = h^{-\alpha}\sum_{k=0}^{\iota(x)}\mathcal{G}_k^{\alpha}\vartheta_h^{\alpha}\Big(x-(k-1)h\Big).$$
Then, since $\mathcal{G}_{-1}^{-\alpha-1} = 0$ by (A.3), we have
$$\begin{aligned}A_{h,1}^{\alpha}\vartheta_h^{\alpha}(x) &= \sum_{k=0}^{\iota(x)-1}\mathcal{G}_k^{\alpha}\left((1-\theta)\mathcal{G}_{\iota(x)-(k-1)-3}^{-\alpha-1} + \theta\mathcal{G}_{\iota(x)-(k-1)-2}^{-\alpha-1}\right) + (\lambda-1)\mathcal{G}_{\iota(x)}^{\alpha} \\ &= (1-\theta)\sum_{k=0}^{\iota(x)-2}\mathcal{G}_k^{\alpha}\mathcal{G}_{\iota(x)-(k-1)-3}^{-\alpha-1} + \theta\sum_{k=0}^{\iota(x)-1}\mathcal{G}_{\iota(x)-(k-1)-2}^{-\alpha-1} + (\lambda-1)\mathcal{G}_{\iota(x)}^{\alpha}.\end{aligned}$$
In view of (A.2) note that $\mathcal{G}_k^{-1} = 1$. Thus, the first term above reduces to
$$(1-\theta)\sum_{k=0}^{\iota(x)-2}\mathcal{G}_k^{\alpha}\mathcal{G}_{\iota(x)-2-k}^{-\alpha-1} = (1-\theta)\mathcal{G}_{\iota(x)-2}^{-1} = (1-\theta)$$
while the second term above reduces to
$$\theta\sum_{k=0}^{\iota(x)-1}\mathcal{G}_k^{\alpha}\mathcal{G}_{\iota(x)-1-k}^{-\alpha-1} = \theta\mathcal{G}_{\iota(x)-1}^{-1} = \theta.$$
Hence,
$$A_{h,1}^{\alpha}\vartheta_h^{\alpha}(x) = 1 - \lambda'\mathcal{G}_{\iota(x)}^{\alpha}.$$

2. Let $\iota(x) \geq 2$ and $\theta = \lambda$ or 1, then using (4.29) we have

$$A_{h,0}^{\alpha-1}\vartheta_h^\alpha(x) = h^{-(\alpha-1)} \sum_{k=0}^{\iota(x)-1} \mathcal{G}_k^{\alpha-1}\vartheta_h^\alpha\Big(x - kh\Big).$$

Note that $\mathcal{G}_k^{-2} = k+1$ by (A.2) and $\mathcal{G}_{-1}^{-\alpha-1} = 0$ by (A.3). Thus,

$$\begin{aligned}
A_{h,1}^{\alpha-1}\vartheta_h^\alpha(x) &= h \sum_{k=0}^{\iota(x)-2} \mathcal{G}_k^{\alpha-1}\left((1-\theta)\mathcal{G}_{\iota(x)-k-3}^{-\alpha-1} + \theta\mathcal{G}_{\iota(x)-k-2}^{-\alpha-1}\right) + h(\lambda-1)\mathcal{G}_{\iota(x)-1}^{\alpha-1} \\
&= h(1-\theta) \sum_{k=0}^{\iota(x)-3} \mathcal{G}_k^{\alpha-1}\mathcal{G}_{\iota(x)-3-k}^{-\alpha-1} + h\theta \sum_{k=0}^{\iota(x)-2} \mathcal{G}_k^{\alpha-1}\mathcal{G}_{\iota(x)-2-k}^{-\alpha-1} + h(\lambda-1)\mathcal{G}_{\iota(x)-1}^{\alpha-1} \\
&= h\Big((1-\theta)\mathcal{G}_{\iota(x)-3}^{-2} + \theta\mathcal{G}_{\iota(x)-2}^{-2} + (\lambda-1)\mathcal{G}_{\iota(x)-1}^{\alpha-1}\Big) \\
&= h\Big((1-\theta)(\iota(x)-2) + \theta(\iota(x)-1) + (\lambda-1)\mathcal{G}_{\iota(x)-1}^{\alpha-1}\Big) \\
&= h\left(\iota(x) - 2 + \theta - \lambda'\mathcal{G}_{\iota(x)-1}^{\alpha-1}\right).
\end{aligned}$$

3. Let $X = C_0(\Omega)$ and $\iota(x) \geq 2$, then in view of (4.29), using $\mathcal{G}_{-1}^{-\alpha} = 0$ we have

$$\begin{aligned}
A_{h,1}^{\alpha}\vartheta_h^{\alpha-1}(x) &= h^{-1}\left(\sum_{k=0}^{\iota(x)} \mathcal{G}_k^\alpha\Big((1-\lambda)\mathcal{G}_{\iota(x)-(k-1)-2}^{-\alpha} + \lambda\mathcal{G}_{\iota(x)-(k-1))-1}^{-\alpha}\Big)\right) \\
&= h^{-1}\Big((1-\lambda)\sum_{k=0}^{\iota(x)-1} \mathcal{G}_k^\alpha\mathcal{G}_{\iota(x)-1-k}^{-\alpha} + \lambda\sum_{k=0}^{\iota(x)} \mathcal{G}_k^\alpha\mathcal{G}_{\iota(x)-k}^{-\alpha}\Big) \\
&= h^{-1}\Big((1-\lambda)\mathcal{G}_{\iota(x)-1}^0 + \lambda\mathcal{G}_{\iota(x)}^0\Big) = 0,
\end{aligned}$$

since $\iota(x) \geq 2$ and $\mathcal{G}_k^0 = 0,\ k \geq 1$ by (A.2).

4. Let $X = C_0(\Omega)$ and $\iota(x) \geq 2$, then in view of (4.29), using $\mathcal{G}_{-1}^{-\alpha} = 0$ and $\mathcal{G}_k^{-1} = 1$ we have

$$\begin{aligned}
A_{h,0}^{\alpha-1}\vartheta_h^{\alpha-1}(x) &= h^{-(\alpha-1)} \sum_{k=0}^{\iota(x)-1} \mathcal{G}_k^{\alpha-1}\vartheta_h^{\alpha-1}\Big(x - kh\Big) \\
&= \sum_{k=0}^{\iota(x)-1} \mathcal{G}_k^{\alpha-1}\left((1-\lambda)\mathcal{G}_{\iota(x)-k-2}^{-\alpha} + \lambda\mathcal{G}_{\iota(x)-k-1}^{-\alpha}\right) \\
&= (1-\lambda) \sum_{k=0}^{\iota(x)-2} \mathcal{G}_k^{\alpha-1}\mathcal{G}_{\iota(x)-2-k}^{-\alpha} + \lambda \sum_{k=0}^{\iota(x)-1} \mathcal{G}_k^{\alpha-1}\mathcal{G}_{\iota(x)-1-k}^{-\alpha} \\
&= (1-\lambda)\mathcal{G}_{\iota(x)-2}^{-1} + \lambda\mathcal{G}_{\iota(x)-1}^{-1} = 1.
\end{aligned}$$

5. Let $X = L_1[0,1]$ and $\iota(x) \geq 3$, then in view of (4.29), using $\mathcal{G}_{-1}^{-\alpha+1} = 0$

$$\begin{aligned} A_{h,1}^{\alpha}\vartheta_h^{\alpha-2}(x) &= \frac{1}{h^2}\left(\sum_{k=0}^{\iota(x)} \mathcal{G}_k^{\alpha}\left((1-\theta)\mathcal{G}_{\iota(x)-1-k}^{-\alpha+1} + \theta\mathcal{G}_{\iota(x)-k}^{-\alpha+1}\right)\right) \\ &= \frac{1}{h^2}\left((1-\theta)\sum_{k=0}^{\iota(x)-1} \mathcal{G}_k^{\alpha}\mathcal{G}_{\iota(x)-1-k}^{-\alpha+1} + \theta\sum_{k=0}^{\iota(x)} \mathcal{G}_k^{\alpha}\mathcal{G}_{\iota(x)-k}^{-\alpha+1}\right) \\ &= \frac{1}{h^2}\left((1-\theta)\mathcal{G}_{\iota(x)-1}^{1} + \theta\mathcal{G}_{\iota(x)}^{1}\right) = 0, \end{aligned} \tag{4.30}$$

since $\mathcal{G}_k^1 = 0$ for $k \geq 2$.

6. Let $X = L_1[0,1]$ and $\iota(x) \geq 3$, then in view of (4.29), using $\mathcal{G}_{-1}^{-\alpha+1} = 0$

$$\begin{aligned} A_{h,1}^{\alpha-1}\vartheta_h^{\alpha-2}(x) &= \frac{1}{h}\left(\sum_{k=0}^{\iota(x)-1} \mathcal{G}_k^{\alpha-1}\left((1-\theta)\mathcal{G}_{\iota(x)-2-k}^{-\alpha+1} + \theta\mathcal{G}_{\iota(x)-1-k}^{-\alpha+1}\right)\right) \\ &= \frac{1}{h}\left((1-\theta)\sum_{k=0}^{\iota(x)-2} \mathcal{G}_k^{\alpha-1}\mathcal{G}_{\iota(x)-2-k}^{-\alpha+1} + \theta\sum_{k=0}^{\iota(x)-1} \mathcal{G}_k^{\alpha-1}\mathcal{G}_{\iota(x)-1-k}^{-\alpha+1}\right) \\ &= \frac{1}{h}\left((1-\theta)\mathcal{G}_{\iota(x)-2}^{0} + \theta\mathcal{G}_{\iota(x)-1}^{0}\right) = 0, \end{aligned} \tag{4.31}$$

since $\mathcal{G}_k^0 = 0$ for $k \geq 1$.

□

4.5.2 Proof of Proposition 4.3.2 for the case $X = C_0(\Omega)$

Proof. To simplify notation, we write $A := A^+$. For each of the fractional derivative operators (A, BC) on $C_0(\Omega)$ given in Table 4.5, and for each $f \in \mathcal{C}(A, \mathrm{BC})$ we show that there exists a sequence $\{f_h\} \subset C_0(\Omega)$ such that $f_h \to f$ and $G^h f_h \to Af$ in $C_0(\Omega)$. To this end, given $\epsilon > 0$ we show that there exists $\delta > 0$ such that for $h < \delta$,

$$\sup_{x\in\Omega} |f_h(x) - f(x)| < \epsilon \tag{4.32}$$

and

$$\sup_{x\in\Omega} \left|G^h f_h(x) - Af(x)\right| < \epsilon. \tag{4.33}$$

We show (4.32) when dealing each of the operators (A, BC) separately.

To show (4.33), we break the interval Ω into two parts, namely,

$$\Omega_1(h) := \Omega \cap [0, 1-2h) \text{ and } \Omega_2(h) := \Omega \cap [1-2h, 1]. \tag{4.34}$$

$(A^+,\mathrm{BC}),\ C_0(\Omega)$	$(A^-,\mathrm{BC}),\ L_1[0,1]$	$(A^\leftrightarrow,\mathrm{BC}),\ L_1[0,1]$
$(D_c^{\alpha,+},\mathrm{DN})$	$(D_c^{\alpha,-},\mathrm{DN})$	$(D_c^{\alpha,\leftrightarrow},\mathrm{ND})$
$(D_c^{\alpha,+},\mathrm{NN})$	$(D_c^{\alpha,-},\mathrm{NN})$	$(D_c^{\alpha,\leftrightarrow},\mathrm{NN})$
$(D_c^{\alpha,+},\mathrm{DD})$	$(D^{\alpha,-},\mathrm{DD})=(D_c^{\alpha,-},\mathrm{DD})$	$(D_c^{\alpha,\leftrightarrow},\mathrm{DD})$
$(D_c^{\alpha,+},\mathrm{ND})$	$(D^{\alpha,-},\mathrm{ND})=(D_c^{\alpha,-},\mathrm{ND})$	$(D_c^{\alpha,\leftrightarrow},\mathrm{DN})$
$(D_c^{\alpha,+},\mathrm{DN}^*)$	$(D^{\alpha,-},\mathrm{DN})$	$(D^{\alpha,\leftrightarrow},\mathrm{ND})$
$(D_c^{\alpha,+},\mathrm{NN}^*)$	$(D^{\alpha,-},\mathrm{NN})$	$(D^{\alpha,\leftrightarrow},\mathrm{NN})$

Table 4.5: Corresponding fractional derivative operators on $C_0(\Omega)$ and $L_1[0,1]$.

For $x\in\Omega_1(h)$ we show that

$$\sup_{x\in\Omega_1}\left|G^hf_h(x)-Af(x)\right|=O(h^\kappa),\ \kappa>0.$$

As a consequence, there exists δ_1 such that for $h<\delta_1$,

$$\sup_{x\in\Omega_1(h)}\left|G^hf_h(x)-Af(x)\right|<\frac{\epsilon}{2}. \tag{4.35}$$

Similarly, for $x\in\Omega_2$ we show that

$$\sup_{x\in\Omega_2(h)}\left|G^hf_h(x)-Af(x)\right|=O(h^\kappa),\ \kappa>0.$$

This implies that there exists δ_2 such that for $h<\delta_2$,

$$\sup_{x\in\Omega_2(h)}\left|G^hf_h(x)-Af(x)\right|<\frac{\epsilon}{2}. \tag{4.36}$$

Then, taking $\delta=\min\{\delta_1,\delta_2\}$, for $h<\delta$ we have $\Omega=\Omega_1(h)\cup\Omega_2(h)$ as well as uniform convergence on the interval Ω, (4.32) and (4.33).

Remark 4.5.8. With this line of argument in mind, in what follows, for $h<\delta$ we loosely use the phrases $f_h\to f$ *uniformly in* Ω and $G^hf_h\to Af$ *uniformly* on Ω_1 or Ω_2 to refer to (4.35) or (4.36), respectively.

Take note that in the proof we repeatedly make use of the following:

1. $f\in\mathcal{C}(A,\mathrm{BC})$ as in Theorem 3.4.4,

$$f=I^\alpha P+ap_\alpha+bp_{\alpha-1}+dp_0,$$

where the polynomial $P=\sum_{m=0}^N k_mp_m\in C_0(\Omega)$, see Remark 3.4.3. For convenience in calculations below, we rewrite $f\in\mathcal{C}(A,\mathrm{BC})$ as follows:

$$f=\mathcal{P}+(P(0)+a)p_\alpha+bp_{\alpha-1}+dp_0, \tag{4.37}$$

where $\mathcal{P} = I^\alpha(P - P(0)p_0)$. Then, using (3.17),

$$Af = A\mathcal{P} + (P(0) + a)p_0 = P + ap_0. \tag{4.38}$$

2. The interpolation matrix $G^h_{n+1}(\lambda)$ given by (4.9),

$$G^h_{n+1}(\lambda) =$$

$$\frac{1}{h^\alpha}\begin{pmatrix} b_1^l & D^l(\lambda)\mathcal{G}_0^\alpha & 0 & \cdots & \cdots & \cdots & 0 \\ N^l(\lambda)b_2^l & \lambda' b_1^l + \lambda\mathcal{G}_1^\alpha & \mathcal{G}_0^\alpha & 0 & \cdots & \cdots & 0 \\ N^l(\lambda)b_3^l & \lambda' b_2^l + \lambda\mathcal{G}_2^\alpha & \mathcal{G}_1^\alpha & \mathcal{G}_0^\alpha & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ N^l(\lambda)b_i^l & \lambda' b_{i-1}^l + \lambda\mathcal{G}_{i-1}^\alpha & \mathcal{G}_{i-2}^\alpha & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ N^l(\lambda)b_n & \lambda' b_{n-1}^l + \lambda b_{n-1}^r & \lambda'\mathcal{G}_{n-2}^\alpha + \lambda b_{n-2}^r & \cdots & \cdots & \lambda'\mathcal{G}_1^\alpha + \lambda b_1^r & N^r(\lambda)\mathcal{G}_0^\alpha \\ 0 & D^r(\lambda)b_n & D^r(\lambda)b_{n-1}^r & \cdots & \cdots & D^r(\lambda)b_2^r & b_1^r \end{pmatrix}.$$

3. Table 4.6 for the boundary weights $b_i^l,\ b_i^r,\ b_n$, the constants a, b, c, d and the interpolating functions $D^l,\ D^r,\ N^l,\ N^r$.

4. The Grünwald formula given by (4.29),

$$A^\alpha_{h,1}f(x) = \frac{1}{h^\alpha}\sum_{k=0}^{\iota(x)} \mathcal{G}_k^\alpha f((\lambda(x) + \iota(x) - 1 - (k-1))h)$$

where $x = (\lambda(x) + \iota(x) - 1)h$ as given by Definition (3.2.2) along with Corollary 4.5.5 and Remark 4.5.6.

5. The approximate power functions given by Definition 4.5.1 and as mentioned in Remark 4.5.2, we canonically extend the domains of $\mathcal{P}$, p_0 and ϑ_h^β for $\beta \in \{\alpha, \alpha - 1\}$ to the interval $[0, 1+h]$ when required.

Remark 4.5.9. For easy reference, let us recall the outline of the proof of Proposition 4.3.2 for the $C_0(\Omega)$ case as given in Section 4.3.

Outline of the structure of the proof:

- **Interval Ω_1:**

  1. We consider the common properties of the Grünwald approximations of operators with left Dirichlet boundary condition, $(A, \mathrm{D}\bullet)$ on Ω_1.

$X=C_0(\Omega), D(A,\mathrm{BC})=\{I^\alpha g+ap_\alpha+bp_{\alpha-1}+dp_0 : g\in C_0(\Omega)\},\ (0,1)\subset\Omega$		
Boundary condition	Boundary weights for $G^h_{n\times n}$	$D(A,\mathrm{BC})$
$f(0)=0$ $\Omega\subset(0,1]$	$b^l_i=\mathcal{G}^\alpha_i$ $b_n=b^r_n$ $D^l(\lambda)=\frac{\alpha\lambda}{\alpha\lambda+\lambda'},\ N^l=\mathbf{1}$	$a=0,\ d=0$
$D^{\alpha-1}_c f(0)=0$ $[0,1)\subset\Omega$	$b^l_i=-\mathcal{G}^{\alpha-1}_{i-1}$ $b_n=-\sum_{i=0}^{n-1}b^r_i$ $N^l(\lambda)=\lambda,\ D^l=\mathbf{1}$	$b=0$
$f(1)=0$ $\Omega\subset[0,1)$	$b^r_i=\mathcal{G}^\alpha_i$ $D^r(\lambda)=\frac{\alpha\lambda'}{\alpha\lambda'+\lambda},\ N^r=\mathbf{1}$	$a=0,\ \frac{b}{\Gamma(\alpha)}+d=-I^\alpha g(1)$
$D^{\alpha-1}_c f(1)=0$ $(0,1]\subset\Omega$	$b^r_i=-\mathcal{G}^{\alpha-1}_{i-1}$ $N^r(\lambda)=\lambda',\ D^r=\mathbf{1}$	$a+b=-Ig(1)$
$Df(1)=0$ $(0,1]\subset\Omega$	$b^r_0=0,\ b^r_1=\mathcal{G}^\alpha_0+\mathcal{G}^\alpha_1,$ $b^r_i=\mathcal{G}^\alpha_i$ $N^r(\lambda)=\lambda',\ D^r=\mathbf{1}$	$\frac{a+(\alpha-1)b}{\Gamma(\alpha)}=-I^{\alpha-1}g(1)$

Table 4.6: Boundary conditions for $C_0(\Omega)$.

2. We consider the common properties of the Grünwald approximations of the operators with left Neumann boundary condition, $(A,\mathrm{N}\bullet)$ on Ω_1.

- **Interval Ω_2:**

1. We first consider the common properties of the Grünwald approximations of the operators with right Dirichlet boundary condition, $(A,\bullet\mathrm{D})$ on Ω_2. Following that, we collate and complete the proof of Statement 1 of Proposition 4.3.2 for the operators (A,DD) and (A,ND) separately.

2. We first consider the common properties of the Grünwald approximations of the operators with right Neumann* boundary condition, $(A,\bullet\mathrm{N}^*)$ on Ω_2. Following that, we collate and complete the proof of Statement 1 of Proposition 4.3.2 for the operators (A,DN^*) and (A,NN^*) separately.

3. We first consider the common properties of the Grünwald approximations of the operators with right Neumann boundary condition, $(A,\bullet\mathrm{N})$ on Ω_2. Following that, we collate and complete the proof of Statement 1 of Proposition 4.3.2 for the operators (A,DN) and (A,NN) separately.

<u>**Interval Ω_1:**</u>

Let $\phi \in C_0(\Omega)$ be an arbitrary element, then note using Definition 3.2.4 that

$$G^h\phi(x) = \left(E_{n+1}\left(G^h_{n+1}P_{n+1}\phi\right)\right)(x) = \left[G^h_{n+1}(\lambda(x))(P_{n+1}\phi)(\lambda(x))\right]_{\iota(x)}.$$

For $x \in \Omega \cap [0,h)$; that is, for $x = \lambda h$ with $\iota(x) = 1$ this reduces to

$$G^h\phi(x) = \frac{1}{h^\alpha}\Big(b^l_1\phi(\lambda h) + D^l(\lambda)\mathcal{G}^\alpha_0\phi((\lambda+1)h)\Big), \tag{4.39}$$

while for $x \in [h, 1-2h)$, that is, for $x = (\lambda + \iota(x) - 1)h$ with $\iota(x) \in \{2, 3, \cdots, n-1\}$, we have

$$G^h\phi(x) = \frac{1}{h^\alpha}\left(N^l(\lambda)b^l_{\iota(x)}\phi(\lambda h) + \left(\lambda' b^l_{\iota(x)-1} + \lambda\mathcal{G}^\alpha_{\iota(x)-1}\right)\phi((\lambda+1)h)\right.$$
$$\left. + \sum_{k=0}^{\iota(x)-2}\mathcal{G}^\alpha_k\phi((\lambda+\iota(x)-1-(k-1))h)\right). \tag{4.40}$$

As mentioned in Remark 4.5.9, we first consider the common properties of the operators with a left Dirichlet boundary condition, $(A, \mathrm{D}\bullet)$ on the interval Ω_1. Following that we consider the common properties of the operators with a left Neumann boundary condition, $(A, \mathrm{N}\bullet)$ on the interval Ω_1.

Common properties for operators $(A, \mathrm{D}\bullet)$:

Note that $a,\ d = 0$ and $P \in C_0(\Omega)$ implies that $P(0) = 0$, see Remark 3.4.3. Thus, $f \in \mathcal{C}(A, \mathrm{D}\bullet)$ given by (4.37) reduces to

$$f = \mathcal{P} + bp_{\alpha-1}, \tag{4.41}$$

where $\mathcal{P} = I^\alpha P$. We take

$$f_h = \mathcal{P} + b_h\vartheta^{\alpha-1}_h + e_h. \tag{4.42}$$

Note that for $x \in \Omega_1$, irrespective of the right boundary condition (see (4.106) and (4.109) below), we always have that $e_h(x) = 0$ and $G^h e_h(x) = 0$. Also, since $\mathcal{P}(0) = 0$ and $\vartheta^{\alpha-1}_h(0) = 0$, we have

$$f_h(0) = 0. \tag{4.43}$$

Next, assuming $b_h \to b$, which we show when dealing with the operators (A, DD), (A, DN^*) and (A, DN) (see (4.61), (4.86) and (4.113) below), we show that

$$G^h f_h = G^h\left(\mathcal{P} + b_h\vartheta^{\alpha-1}_h\right) \to A\mathcal{P} = Af \tag{4.44}$$

uniformly on Ω_1 for the operators $(A, \mathrm{D}\bullet)$. Note that $N^l = \mathbf{1}$, $D^l(\lambda) = \frac{\alpha\lambda}{\alpha\lambda+\lambda'}$ and $b_i^l = \mathcal{G}_i^\alpha$. Hence, for an arbitrary $\phi \in X$, using (4.39) for $x \in \Omega \cap [0, h)$,

$$\begin{aligned} G^h\phi(x) &= \frac{1}{h^\alpha}\left(\mathcal{G}_1^\alpha\phi(\lambda h) + \frac{\alpha\lambda}{\alpha\lambda+\lambda'}\mathcal{G}_0^\alpha\phi((\lambda+1)h)\right) \\ &= \frac{1}{h^\alpha}\left(\frac{\alpha\lambda}{\alpha\lambda+\lambda'}\phi((\lambda+1)h) - \alpha\phi(\lambda h)\right). \end{aligned} \tag{4.45}$$

Using (4.40) for $x \in [h, 1-2h)$,

$$G^h\phi(x) = \frac{1}{h^\alpha}\sum_{k=0}^{\iota(x)}\mathcal{G}_k^\alpha\phi((\lambda+\iota(x)-1-(k-1))h) = A_{h,1}^\alpha\phi(x). \tag{4.46}$$

Let us deal with the first term, $\mathcal{P}$ of f_h. Observe that $\mathcal{P}(x) = O(x^{\alpha+1})$ as $x \downarrow 0$, since $P(0) = 0$. Moreover, since $P(0) = 0$, $A\mathcal{P}(x) = P(x) = O(x)$ as $x \downarrow 0$. Hence, setting $\phi = \mathcal{P}$ in (4.45) we have that

$$\sup_{x\in\Omega\cap[0,h)}\left|G^h\mathcal{P}(x) - A\mathcal{P}(x)\right| = O(h).$$

Next, setting $\phi = \mathcal{P}$ in (4.46), in view of Corollary 4.5.5, note that

$$\sup_{x\in[h,1-2h)}\left|G^h\mathcal{P}(x) - A\mathcal{P}(x)\right| = O(h).$$

Hence,

$$\sup_{x\in\Omega_1}\left|G^h\mathcal{P}(x) - A\mathcal{P}(x)\right| = O(h).$$

To complete the proof of (4.44) we show that $G^h\vartheta_h^{\alpha-1} \to Ap_{\alpha-1} = \mathbf{0}$. Firstly, for $x \in \Omega \cap [0, h)$, setting $\phi = \vartheta_h^{\alpha-1}$ in (4.45) and using $\mathcal{G}_{-1}^\alpha = 0$,

$$\begin{aligned} G^h\vartheta_h^{\alpha-1}(x) &= \frac{1}{h^\alpha}\Big(\frac{\alpha\lambda}{\alpha\lambda+\lambda'}\vartheta_h^{\alpha-1}((\lambda+1)h) - \alpha\vartheta_h^{\alpha-1}(\lambda h)\Big) \\ &= \frac{1}{h}\left(\frac{\alpha\lambda}{\alpha\lambda+\lambda'}\left(\lambda'\mathcal{G}_{\iota((\lambda+1)h)-2}^{-\alpha} + \lambda\mathcal{G}_{\iota((\lambda+1)h)-1}^{-\alpha}\right)\right. \\ &\qquad\qquad\left. - \alpha\Big(\lambda'\mathcal{G}_{\iota(\lambda h)-2}^{-\alpha} + \lambda\mathcal{G}_{\iota(\lambda h)-1}^{-\alpha}\Big)\right) \\ &= \frac{1}{h}\left(\frac{\alpha\lambda}{\alpha\lambda+\lambda'}\left(\lambda'\mathcal{G}_0^{-\alpha} + \lambda\mathcal{G}_1^{-\alpha}\right) - \alpha\Big(\lambda'\mathcal{G}_{-1}^{-\alpha} + \lambda\mathcal{G}_0^{-\alpha}\Big)\right) \\ &= \frac{1}{h}\left(\frac{\alpha\lambda}{\alpha\lambda+\lambda'}\left(\lambda'+\lambda\alpha\right) - \alpha\lambda\right) = 0. \end{aligned}$$

Next, for $x \in [h, 1-2h)$ setting $\phi = \vartheta_h^{\alpha-1}$ in (4.46) and using Proposition 4.5.7,

$$G^h\vartheta_h^{\alpha-1}(x) = A_{h,1}^\alpha\vartheta_h^{\alpha-1}(x) = 0.$$

Hence,

$$G^h f_h = G^h\left(\mathcal{P} + b_h \vartheta_h^{\alpha-1}\right) \to A\mathcal{P} = Af$$

uniformly in Ω_1. This completes the proof of (4.44) for the operators $(A, \mathrm{D}\bullet)$.

Common properties for operators $(A, \mathrm{N}\bullet)$**:**

First, note that $b = 0$ and so $f \in \mathcal{C}(A, \mathrm{N}\bullet)$ given by (4.37) reduces to

$$f = \mathcal{P} + (P(0) + a)p_\alpha + dp_0, \tag{4.47}$$

where $\mathcal{P} = I^\alpha(P - P(0)p_0),\ P \in C_0(\Omega)$. We take

$$f_h = \mathcal{P} + (P(0)k_h + a_h)\vartheta_h^\alpha + dp_0 + e_h. \tag{4.48}$$

Note that for $x \in \Omega_1$, irrespective of the right boundary condition (see (4.106) and (4.109) below), we always have that $e_h(x) = 0$ and $G^h e_h(x) = 0$. Assuming $(P(0)k_h + a_h) \to P(0) + a$, which we show when dealing with the operators (A, DD), (A, DN^*) and (A, DN) (see (4.70), (4.94) and (4.118) below), we show that

$$G^h f_h = G^h\left(\mathcal{P} + (P(0)k_h + a_h)\vartheta_h^\alpha + dp_0\right) \to A\mathcal{P} + (P(0) + a)p_0 = Af \tag{4.49}$$

uniformly on Ω_1 for the operators $(A, \mathrm{N}\bullet)$.

Further, note that $D^l = \mathbf{1}$, $N^l(\lambda) = \lambda$ and $b_i^l = -\mathcal{G}_{i-1}^{\alpha-1}$. Using (4.39) for $x \in [0, h)$,

$$\begin{aligned} G^h\phi(x) &= \frac{1}{h^\alpha}\left(-\mathcal{G}_0^{\alpha-1}\phi(\lambda h) + \mathcal{G}_0^\alpha \phi((\lambda+1)h)\right) \\ &= \frac{1}{h^\alpha}\left(\phi((\lambda+1)h) - \phi(\lambda h)\right). \end{aligned} \tag{4.50}$$

Using (4.40) for $x \in [h, 1-2h)$,

$$\begin{aligned} G^h\phi(x) = \frac{1}{h^\alpha}\Bigg(&- \lambda\mathcal{G}_{\iota(x)-1}^{\alpha-1}\phi(\lambda h) + \left(-\lambda'\mathcal{G}_{\iota(x)-2}^{\alpha-1} + \lambda\mathcal{G}_{\iota(x)-1}^\alpha\right)\phi((\lambda+1)h) \\ &+ \sum_{k=0}^{\iota(x)-2} \mathcal{G}_k^\alpha \phi((\lambda + \iota(x) - 1 - (k-1))h)\Bigg) \end{aligned} \tag{4.51}$$

Let us deal with the first term $\mathcal{P}$ of f_h. Note that $\mathcal{P}(x) = I^\alpha(P - P(0)p_0)(x) = O(x^{\alpha+1})$ and $A\mathcal{P}(x) = P(x) - P(0)p_0(x) = O(x)$ as $x \downarrow 0$. Thus, using (4.50) with $\phi = \mathcal{P}$ for $x \in [0, h)$ we have

$$\left|G^h\mathcal{P}(x) - A\mathcal{P}(x)\right| = O(h).$$

Next, setting $\phi = \mathcal{P}$ in (4.51), we have

$$G^h\mathcal{P}(x) = \frac{1}{h^\alpha}\Bigg(\Big(- \lambda\mathcal{G}_{\iota(x)-1}^{\alpha-1} - \mathcal{G}_{\iota(x)}^\alpha\Big)\mathcal{P}(\lambda h)$$

$$- \lambda' \left(\mathcal{G}^{\alpha-1}_{\iota(x)-2} + \mathcal{G}^{\alpha}_{\iota(x)-1} \right) \mathcal{P}((\lambda+1)h)$$

$$+ \sum_{k=0}^{\iota(x)} \mathcal{G}^{\alpha}_k \mathcal{P}((\lambda + \iota(x) - 1 - (k-1))h) \Bigg)$$

$$= \frac{1}{h^\alpha} \Bigg(\Big(- \mathcal{G}^{\alpha}_{\iota(x)} - \lambda \mathcal{G}^{\alpha-1}_{\iota(x)-1} \Big) \mathcal{P}(\lambda h)$$

$$- \lambda' \left(\mathcal{G}^{\alpha-1}_{\iota(x)-2} + \mathcal{G}^{\alpha}_{\iota(x)-1} \right) \mathcal{P}((\lambda+1)h) \Bigg) + A^{\alpha}_{h,1} \mathcal{P}(x). \qquad (4.52)$$

Firstly, $\sup_{x\in[h,1-2h)} \left| A^{\alpha}_{h,1}\mathcal{P} - A\mathcal{P} \right| = O(h)$ in view of Corollary 4.5.5. Secondly, since $\mathcal{P}(x) = O(x^{\alpha+1})$ as $x \downarrow 0$, the first two terms in (4.52),

$$\frac{1}{h^\alpha} \left| \Big(- \mathcal{G}^{\alpha}_{\iota(x)} - \lambda \mathcal{G}^{\alpha-1}_{\iota(x)-1} \Big) \mathcal{P}(\lambda h) - \lambda' \left(\mathcal{G}^{\alpha-1}_{\iota(x)-2} + \mathcal{G}^{\alpha}_{\iota(x)-1} \right) \mathcal{P}((\lambda+1)h) \Big) \right| = O(h)$$

Hence,

$$\sup_{x\in\Omega_1} \left| G^h\mathcal{P}(x) - A\mathcal{P}(x) \right| = O(h).$$

To complete the proof of (4.49), we show that for $x \in \Omega_1$,

$$G^h p_0(x) = 0 \text{ and } G^h \vartheta^{\alpha}_h(x) = 1.$$

First, let $\phi = p_0$ in (4.50), then $G^h p_0(x) = 0$ for $x \in [0, h)$. Next in view of (A.7) we have $\sum_{k=0}^{\iota(x)-2} \mathcal{G}^{\alpha}_k = \mathcal{G}^{\alpha-1}_{\iota(x)-2}$ and $\mathcal{G}^{\alpha-1}_{\iota(x)-1} = \mathcal{G}^{\alpha-1}_{\iota(x)-2} + \mathcal{G}^{\alpha}_{\iota(x)-1}$. Thus, setting $\phi = p_0$ in (4.51) for $x \in [h, 1-2h)$,

$$\begin{aligned}
G^h p_0(x) &= \frac{1}{h^\alpha} \left(- \lambda \mathcal{G}^{\alpha-1}_{\iota(x)-1} + \Big(- \lambda' \mathcal{G}^{\alpha-1}_{\iota(x)-2} + \lambda \mathcal{G}^{\alpha}_{\iota(x)-1} \Big) + \sum_{k=0}^{\iota(x)-2} \mathcal{G}^{\alpha}_k \right) \\
&= \frac{1}{h^\alpha} \Big(- \lambda \mathcal{G}^{\alpha-1}_{\iota(x)-1} - \lambda' \mathcal{G}^{\alpha-1}_{\iota(x)-2} + \lambda \mathcal{G}^{\alpha}_{\iota(x)-1} + \mathcal{G}^{\alpha-1}_{\iota(x)-2} \Big) \\
&= \frac{1}{h^\alpha} \Big(- \lambda \mathcal{G}^{\alpha-1}_{\iota(x)-1} + \lambda \mathcal{G}^{\alpha-1}_{\iota(x)-2} + \lambda \mathcal{G}^{\alpha}_{\iota(x)-1} \Big) = 0.
\end{aligned}$$

For $x \in [0, h)$, setting $\phi = \vartheta^{\alpha}_h$ in (4.50),

$$\begin{aligned}
G^h \vartheta^{\alpha}_h(x) &= \lambda' \mathcal{G}^{-\alpha-1}_{\iota((\lambda+1)h))-3} + \lambda \mathcal{G}^{-\alpha-1}_{\iota((\lambda+1)h))-2} - (\lambda - 1), \\
&= (1-\lambda) \mathcal{G}^{-\alpha-1}_{-1} + \lambda \mathcal{G}^{-\alpha-1}_{0} - (\lambda - 1) = 1,
\end{aligned}$$

since $\mathcal{G}^{-\alpha-1}_{-1} = 0$. Lastly, for $x \in [h, 1-2h)$, using (4.51), $\mathcal{G}^{-\alpha-1}_{-1} = 0$ and $\mathcal{G}^{\alpha-1}_{\iota(x)-1} = \mathcal{G}^{\alpha-1}_{\iota(x)-2} + \mathcal{G}^{\alpha}_{\iota(x)-1}$, we have

$$G^h \vartheta^{\alpha}_h(x) = \frac{1}{h^\alpha} \Bigg(- \lambda \mathcal{G}^{\alpha-1}_{\iota(x)-1} \vartheta^{\alpha}_h(\lambda h) + \left(-\lambda' \mathcal{G}^{\alpha-1}_{\iota(x)-2} + \lambda \mathcal{G}^{\alpha}_{\iota(x)-1} \right) \vartheta^{\alpha}_h((\lambda+1)h)$$

$$
\left. + \sum_{k=0}^{\iota(x)-2} \mathcal{G}_k^\alpha \vartheta_h^\alpha((\lambda + \iota(x) - 1 - (k-1))h) \right)
$$

$$
\begin{aligned}
&= -\lambda(\lambda-1)\mathcal{G}^{\alpha-1}_{\iota(x)-1} + \lambda(\lambda-1)\left(\mathcal{G}^{\alpha-1}_{\iota(x)-2} + \mathcal{G}^{\alpha}_{\iota(x)-1}\right) \\
&\qquad + (1-\lambda)\sum_{k=0}^{\iota(x)-2} \mathcal{G}_k^\alpha \mathcal{G}^{-\alpha-1}_{\iota(x)-(k-1)-3} + \lambda \sum_{k=0}^{\iota(x)-1} \mathcal{G}_k^\alpha \mathcal{G}^{-\alpha-1}_{\iota(x)-(k-1)-2} \\
&= (1-\lambda)\sum_{k=0}^{\iota(x)-2} \mathcal{G}_k^\alpha \mathcal{G}^{-\alpha-1}_{\iota(x)-2-k} + \lambda \sum_{k=0}^{\iota(x)-1} \mathcal{G}^{-\alpha-1}_{\iota(x)-1-k} \\
&= (1-\lambda)\mathcal{G}^{-1}_{\iota(x)-2} + \lambda \mathcal{G}^{-1}_{\iota(x)-1} = 1,
\end{aligned}
$$

since $\mathcal{G}_k^{-1} = 1$ in view of (A.2). Hence,

$$
G^h f_h = G^h\left(\mathcal{P} + (P(0)k_h + a_h)\vartheta_h^\alpha + dp_0\right) \to A\mathcal{P} + (P(0)+a)p_0 = Af
$$

uniformly in Ω_1. This completes the proof of (4.49) for the operators $(A, \mathrm{N}\bullet)$.

Interval Ω_2:

Let $\phi \in C_0(\Omega)$ be any arbitrary element, then using Definition 3.2.4 note that

$$
G^h\phi(x) = \left(E_{n+1}\left(G^h_{n+1}P_{n+1}\phi\right)\right)(x) = \left[G^h_{n+1}(\lambda(x))(P_{n+1}\phi)(\lambda(x))\right]_{\iota(x)}.
$$

For $x \in [1-2h, 1-h)$; that is, $x = (\lambda + n - 1)h$ with $\iota(x) = n$ we have

$$
\begin{aligned}
G^h\phi(x) = \frac{1}{h^\alpha}\Bigg(& N^l(\lambda)b_n\phi(\lambda h) + \left(\lambda' b^l_{n-1} + \lambda b^r_{n-1}\right)\phi((\lambda+1)h) \\
& + \sum_{k=1}^{n-2}\left(\lambda'\mathcal{G}_k^\alpha + \lambda b_k^r\right)\phi((\lambda + n - 1 - (k-1))h) \\
& + N^r(\lambda)\mathcal{G}_0^\alpha\phi((\lambda+n)h)\Bigg)
\end{aligned} \tag{4.53}
$$

and for $x \in \Omega \cap [1-h, 1]$; that is, $x = (\lambda + n)h$ with $\iota(x) = n+1$ we have

$$
\begin{aligned}
G^h\phi(x) = \frac{1}{h^\alpha}\Bigg(& D^r(\lambda)b_n\phi((\lambda+1)h) \\
& + \sum_{k=2}^{n-1} D^r(\lambda)b_k^r\phi((\lambda + n - (k-1))h) + b_1^r\phi((\lambda+n)h)\Bigg).
\end{aligned} \tag{4.54}
$$

As mentioned in Remark 4.5.9, we now consider the common properties of the operators with a right Dirichlet boundary condition, $(A, \bullet\mathrm{D})$. Following that we deal with the operators (A, DD) and (A, ND) separately.

Common properties for operators $(A, \bullet\mathrm{D})$:

Note that $a = 0$ and so $f \in \mathcal{C}(A, \bullet\mathrm{D})$ given by (4.37), reduces to

$$f = \mathcal{P} + P(0)p_\alpha + bp_{\alpha-1} + dp_0, \tag{4.55}$$

where $\mathcal{P} = I^\alpha(P - P(0)p_0)$. Moreover, the relation $\frac{b}{\Gamma(\alpha)} + d = -I^\alpha g(1)$ for $f \in \mathcal{C}(A, \bullet\mathrm{D})$ reads

$$\frac{b}{\Gamma(\alpha)} + d = -\mathcal{P}(1) - \frac{P(0)}{\Gamma(\alpha+1)}, \tag{4.56}$$

since $I^\alpha p_0(1) = p_\alpha(1) = \frac{1}{\Gamma(\alpha+1)}$.

Note that $N^r = \mathbf{1}$ and $b_i^r = \mathcal{G}_i^\alpha$. Thus, for arbitrary $\phi \in C_0(\Omega)$ and $x \in [1-2h, 1-h)$, (4.53) becomes

$$\begin{aligned} G^h\phi(x) &= \frac{1}{h^\alpha}\Bigg(N^l(\lambda)b_n\phi(\lambda h) + \Big(\lambda' b_{n-1}^l + \lambda\mathcal{G}_{n-1}^\alpha\Big)\phi((\lambda+1)h) \\ &\qquad + \sum_{k=1}^{n-2}\mathcal{G}_k^\alpha\phi((\lambda+n-k)h) + \mathcal{G}_0^\alpha\phi((\lambda+n)h)\Bigg) \\ &= \frac{1}{h^\alpha}\Bigg(N^l(\lambda)b_n\phi(\lambda h) + \Big(\lambda' b_{n-1}^l + \lambda\mathcal{G}_{n-1}^\alpha\Big)\phi((\lambda+1)h) \\ &\qquad + \sum_{k=0}^{n-2}\mathcal{G}_k^\alpha\phi((\lambda+n-1-(k-1))h)\Bigg). \end{aligned} \tag{4.57}$$

For $x \in \Omega \cap [1-h, 1]$, (4.54) becomes

$$\begin{aligned} G^h\phi(x) = \frac{1}{h^\alpha}\Bigg(& D^r(\lambda)b_n\phi((\lambda+1)h) + \sum_{k=2}^{n-1}D^r(\lambda)\mathcal{G}_k^\alpha\phi((\lambda+n-(k-1))h) \\ & + \mathcal{G}_1^\alpha\phi((\lambda+n)h)\Bigg). \end{aligned} \tag{4.58}$$

Proof of Statement 1 of Proposition 4.3.2 for the operator (A, DD):

In this case, we further have $d = 0$ and $P(0) = 0$. Thus, $f \in \mathcal{C}(A, \mathrm{DD})$ given by (4.55) reduces to

$$f = \mathcal{P} + bp_{\alpha-1}, \tag{4.59}$$

(compare $f \in \mathcal{C}(A, \mathrm{D}\bullet)$ given by (4.41)). We take

$$f_h = \mathcal{P} + b_h\vartheta_h^{\alpha-1}. \tag{4.60}$$

where

$$b_h = \frac{b}{\vartheta_h^{\alpha-1}(1)\Gamma(\alpha)}. \tag{4.61}$$

First, note that for a right Dirichlet boundary condition we do not require the function e_h. Further, note that this choice is the same as the sequences f_h constructed for $f \in \mathcal{C}(A, \mathrm{D}\bullet)$ given by (4.42) with $e_h = \mathbf{0}$. In view of (4.28) and (A.9), since $(n+1)h = 1$,

$$b_h = \frac{b}{\vartheta_h^{\alpha-1}(1)\Gamma(\alpha)} = \frac{b}{h^{\alpha-1}\mathcal{G}_n^{-\alpha}\Gamma(\alpha)} = \frac{b}{h^{\alpha-1}n^{\alpha-1}\left(1+O(n^{-1})\right)} = b + O(h). \tag{4.62}$$

Note in view of (4.56) that,

$$f_h(1) = \mathcal{P}(1) + b_h\vartheta_h^{\alpha-1}(1) = \mathcal{P}(1) + \frac{b}{\Gamma(\alpha)} = 0. \tag{4.63}$$

Proof of $f_h \to f$ for (A, DD):

Observe that $f_h \in C_0(\Omega)$, since $f_h(0) = 0 = f_h(1)$ in view of (4.43) and (4.63). Next, note that as $h \to 0$, $b_h \to b$ in view of (4.62), $\vartheta_h^{\alpha-1} \to p_{\alpha-1}$ in the sup-norm in view of Lemma 4.5.3. Thus,

$$f_h \to f \tag{4.64}$$

uniformly in Ω.

Proof of $G^h f_h \to Af$ for (A, DD):

Observe that, in view of (4.44), it only remains to show that

$$G^h f_h = G^h(\mathcal{P} + b_h\vartheta_h^{\alpha-1}) \to A\mathcal{P} = Af \tag{4.65}$$

uniformly on Ω_2.

Take note that $N^l = \mathbf{1}$, $D^r(\lambda) = \frac{\alpha\lambda'}{\alpha\lambda'+\lambda}$, $b_i^l = \mathcal{G}_i^\alpha$ and $b_n = b_n^r = \mathcal{G}_n^\alpha$. Therefore, for $x \in [1-2h, 1-h)$, (4.57) becomes

$$G^h\phi(x) = \frac{1}{h^\alpha}\sum_{k=0}^{n}\mathcal{G}_k^\alpha\phi((\lambda + (n-1) - (k-1))h) = A_{h,1}^\alpha\phi(x). \tag{4.66}$$

Notice the similarity with $(A, \mathrm{D}\bullet)$ case above of (4.66) with (4.46) where $\iota(x) = n$. This implies that the argument of $G^h f_h \to Af$ uniformly on $[h, 1-2h)$ made above for $(A, \mathrm{D}\bullet)$ holds true for (A, DD) for the interval $[1-2h, 1-h)$. Therefore, we only need to deal with the interval $\Omega \cap [1-h, 1]$.

Setting $\phi = f_h$, (4.58) becomes

$$G^h f_h(x) = \frac{1}{h^\alpha}\Bigg(D^r(\lambda)\mathcal{G}_n^\alpha f_h((\lambda+1)h) + \sum_{k=2}^{n-1} D^r(\lambda)\mathcal{G}_k^\alpha f_h((\lambda + n - (k-1))h)$$
$$+ \mathcal{G}_1^\alpha f_h((\lambda+n)h)\Bigg)$$

$$= \frac{1}{h^\alpha}\Bigg(D^r(\lambda)\sum_{k=0}^{n+1}\mathcal{G}_k^\alpha f_h((\lambda+n-(k-1))h) - D^r(\lambda)\mathcal{G}_{n+1}^\alpha f_h(\lambda h)$$

$$+ \big(1-D^r(\lambda)\big)\mathcal{G}_1^\alpha f_h((\lambda+n)h) - D^r(\lambda)\mathcal{G}_0^\alpha f_h((\lambda+(n+1))h)\Bigg),$$

$$= -\frac{1}{h^\alpha}D^r(\lambda)\mathcal{G}_{n+1}^\alpha f_h(\lambda h) + R(\lambda,h) + D^r(\lambda)A_{h,1}^\alpha f_h(x), \tag{4.67}$$

where

$$R(\lambda,h) = \frac{1}{h^\alpha}\Big(\big(1-D^r(\lambda)\big)\mathcal{G}_1^\alpha f_h((\lambda+n)h) - D^r(\lambda)\mathcal{G}_0^\alpha f_h((\lambda+(n+1))h)\Big).$$

We consider the three terms of (4.67) one by one and show the following:

- Firstly, the first term of (4.67),

$$\left|\frac{1}{h^\alpha}D^r(\lambda)\mathcal{G}_{n+1}^\alpha f_h(\lambda h)\right| = O(h^\alpha).$$

- Secondly, the second term of (4.67),

$$|R(\lambda,h)| = O(h^{2-\alpha}).$$

- Thirdly,

$$\big|D^r(\lambda)A_{h,1}^\alpha f_h(x) - Af(x)\big| = O(h).$$

Note that the first term of (4.67),

$$\left|\frac{1}{h^\alpha}D^r(\lambda)\mathcal{G}_{n+1}^\alpha f_h(\lambda h)\right| = O(h^\alpha)$$

since $\left|\mathcal{G}_{n+1}^\alpha\right| = O(h^{\alpha+1})$, $|\mathcal{P}(x)| = O(x^{\alpha+1})$ as $x \downarrow 0$ and $\left|\vartheta_h^{\alpha-1}(\lambda h)\right| = O(h^{\alpha-1})$. Next we show that the second term of (4.67),

$$|R(\lambda,h)| = O(h^{2-\alpha}).$$

If $\lambda = 1$, then $R(1,h) = 0$, since $f_h((n+1)h) = f_h(1) = 0$ in view of (4.63). Hence, assuming $\lambda \in [0,1)$, we have

$$\begin{aligned}
|R(\lambda,h)| &= \left|\frac{1}{h^\alpha}\Big(\big(1-D^r(\lambda)\big)\mathcal{G}_1^\alpha f_h((\lambda+n)h) - D^r(\lambda)\mathcal{G}_0^\alpha f_h((\lambda+n+1)h)\Big)\right| \\
&= \left|\frac{1}{h^\alpha}\Big(\frac{-\alpha\lambda f_h((\lambda+n)h) - \alpha\lambda' f_h((\lambda+n+1)h)}{\alpha\lambda'+\lambda}\Big)\right| \\
&= \left|\frac{1}{h^\alpha}\Big(\frac{\alpha\lambda f_h(x) + \alpha\lambda' f_h(x+h)}{\alpha\lambda'+\lambda}\Big)\right|,
\end{aligned} \tag{4.68}$$

where $x = (\lambda + n)h$. Using the Taylor series for $\mathcal{P}$ given by (4.25) and (4.56) with d, $P(0) = 0$, we have

$$\begin{aligned} f_h(x) &= \mathcal{P}(x) + b_h \vartheta_h^{\alpha-1}(x) \\ &= \mathcal{P}(1) - (1-\lambda)h\mathcal{P}'(1) + O(h^2) + b_h \vartheta_h^{\alpha-1}(x) \\ &= -\frac{b}{\Gamma(\alpha)} + b_h \vartheta_h^{\alpha-1}(x) - (1-\lambda)h\mathcal{P}'(1) + O(h^2) \\ &= R_1(\lambda, h) - \lambda' h \mathcal{P}'(1) + O(h^2). \end{aligned}$$

Using (4.26), (4.70) and (4.61),

$$R_1(\lambda, h) = b_h \vartheta_h^{\alpha-1}(x) - \frac{b}{\Gamma(\alpha)} = \frac{b}{\Gamma(\alpha)} \left(\frac{\vartheta_h^{\alpha-1}(x)}{\vartheta_h^{\alpha-1}(1)} - 1 \right) = -\frac{b\lambda'(\alpha-1)}{\Gamma(\alpha)(n-1+\alpha)}.$$

Similarly,

$$\begin{aligned} f_h(x+h) &= \mathcal{P}(x+h) + b_h \vartheta_h^{\alpha-1}(x+h) \\ &= \mathcal{P}(1) + \lambda h P'(1) + O(h^2) + b_h \vartheta_h^{\alpha-1}(x+h) \\ &= -\frac{b}{\Gamma(\alpha)} + b_h \vartheta_h^{\alpha-1}(x+h) + \lambda h P'(1) + O(h^2) \\ &= R_2(\lambda, h) + \lambda h P'(1) + O(h^2), \end{aligned}$$

where

$$R_2(\lambda, h) = b_h \vartheta_h^{\alpha-1}(x+h) - \frac{b}{\Gamma(\alpha)} = \frac{b}{\Gamma(\alpha)} \left(\frac{\vartheta_h^{\alpha-1}(x+h)}{\vartheta_h^{\alpha-1}(1)} - 1 \right) = \frac{b\lambda(\alpha-1)}{\Gamma(\alpha)(n+1)}.$$

Then, observe that

$$\alpha\lambda R_1(\lambda, h) + \alpha\lambda' R_2(\lambda, h) = \frac{\alpha(\alpha-1)b\lambda\lambda'}{\Gamma(\alpha)} \Big(\frac{1}{n+1} - \frac{1}{n-1+\alpha} \Big) = O(h^2).$$

Thus,

$$|R(\lambda, h)| = \frac{1}{h^\alpha} \left| \frac{-\alpha\lambda\lambda' h\mathcal{P}'(1) + \alpha\lambda\lambda' hP'(1) + O(h^2)}{\alpha\lambda' + \lambda} \right| = O(h^{2-\alpha}).$$

Lastly, consider the third term of (4.67), $D^r(\lambda) A^\alpha_{h,1} f_h(x)$. Using Proposition 4.5.7 note that $A^\alpha_{h,1} \vartheta_h^{\alpha-1}(x) = 0$, for $x \in [1-h, 1)$ and so $A^\alpha_{h,1} f_h = A^\alpha_{h,1} \mathcal{P}$. Next, for $f \in \mathcal{C}(A, \mathrm{DD})$, $Af = A\mathcal{P} = P$ in view of (4.38) with $a = 0$. Moreover, $P \in C_0(\Omega)$ and so $Af(1) = P(1) = 0$ and $D^r(1) = 0$. Furthermore, Af and D^r are continuous as $x \uparrow 1$ and $\lambda \uparrow 1$, respectively. Therefore, for $x \in [1-h, 1]$,

$$|D^r(\lambda) A\mathcal{P}(x) - A\mathcal{P}(x)| = O(h).$$

In view of Corollary 4.5.5,

$$\left|A^{\alpha}_{h,1}\mathcal{P}(x) - A\mathcal{P}(x)\right| = O(h).$$

Therefore,

$$\left|D^{r}(\lambda)A^{\alpha}_{h,1}\mathcal{P}(x) - A\mathcal{P}(x)\right| = O(h)$$

for $x \in [1-h, 1)$ since $|D^{r}(\lambda)| \leq 1$. Thus,

$$\left|D^{r}(\lambda)A^{\alpha}_{h,1}f_h(x) - Af(x)\right| = O(h).$$

Hence,

$$G^{h}f_h = G^{h}(\mathcal{P} + b_h\vartheta_h^{\alpha-1}) \to A\mathcal{P} = Af$$

uniformly on Ω_2; that is, the proof of (4.65) is complete. This also completes the proof of Statement 1 of Proposition 4.3.2 for the operator (A, DD), in view of (4.44) and (4.64).

Proof of Statement 1 of Proposition 4.3.2 for the operator (A, ND):

In this case, we further have $b = 0$. Thus, $f \in \mathcal{C}(A, \mathrm{ND})$ given by (4.55) reduces to

$$f = \mathcal{P} + P(0)p_\alpha + dp_0,$$

(compare $f \in \mathcal{C}(A, \mathrm{N}\bullet)$ given by (4.47)). We take

$$f_h = \mathcal{P} + P(0)k_h\vartheta_h^{\alpha} + dp_0, \tag{4.69}$$

where

$$k_h = \frac{1}{\vartheta_h^{\alpha}(1)\Gamma(\alpha+1)}. \tag{4.70}$$

First, note that for a right Dirichlet boundary condition we do not require the function e_h. Further, note that this choice is the same as the sequences f_h constructed for $f \in \mathcal{C}(A, \mathrm{N}\bullet)$ given by (4.48) with $b = 0$ and $e_h = \mathbf{0}$.

In view of (4.28) and (A.9), since $(n+1)h = 1$,

$$k_h = \frac{1}{\vartheta_h^{\alpha}(1)\Gamma(\alpha+1)} = \frac{1}{h^{\alpha}\mathcal{G}_{n-1}^{-\alpha-1}\Gamma(\alpha+1)} = \frac{1}{h^{\alpha}(n-1)^{\alpha}\left(1 + O((n-1)^{-1})\right)} = 1 + O(h). \tag{4.71}$$

Proof of $f_h \to f$ for (A, ND):

Observe that as $h \to 0$, $k_h \to 1$ in view of (4.71) and $\vartheta_h^{\alpha} \to p_\alpha$ in the sup-norm in view of Lemma 4.5.3. Thus,

$$f_h \to f \tag{4.72}$$

uniformly in Ω.

Proof of $G^h f_h \to Af$ for (A, ND):

Observe that, in view of (4.44), it only remains to show that

$$G^h f_h = G^h(\mathcal{P} + P(0)k_h\vartheta_h^\alpha + dp_0) \to A\left(\mathcal{P} + P(0)p_\alpha\right) = Af. \tag{4.73}$$

uniformly on Ω_2.

Take note that $N^l(\lambda) = \lambda$, $b_i^l = -\mathcal{G}_{i-1}^{\alpha-1}$ and $b_n = -\sum_{i=0}^{n-1} b_i^r = -\sum_{i=0}^{n-1} \mathcal{G}_i^\alpha = -\mathcal{G}_{n-1}^{\alpha-1}$. Therefore, for $x \in [1-2h, 1-h)$, (4.57) becomes

$$\begin{aligned} G^h\phi(x) = -\frac{1}{h^\alpha}\Bigg(&\lambda\mathcal{G}_{n-1}^{\alpha-1}\phi(\lambda h) + \Big(-\lambda'\mathcal{G}_{n-2}^{\alpha-1} + \lambda\mathcal{G}_{n-1}^\alpha\Big)\phi((\lambda+1)h) \\ &+ \sum_{k=0}^{n-2}\mathcal{G}_k^\alpha\phi((\lambda+n-1-(k-1))h)\Bigg) \end{aligned} \tag{4.74}$$

Notice the similarity with $(A, \mathrm{N}\bullet)$ case above of (4.74) with (4.51) with $\iota(x) = n$. This implies that the argument of $G^h f_h \to Af$ uniformly on $[h, 1-2h)$ made above for $(A, \mathrm{N}\bullet)$ holds true for (A, ND) for the interval $[1-2h, 1-h)$. Therefore, we only need to deal with the interval $\Omega \cap [1-h, 1]$.

Setting $\phi = f_h$ and using $\mathcal{G}_{n-1}^{\alpha-1} + \mathcal{G}_n^\alpha = \mathcal{G}_n^{\alpha-1}$, (4.58) becomes

$$\begin{aligned} G^h f_h(x) &= \frac{1}{h^\alpha}\Bigg(-D^r(\lambda)\mathcal{G}_{n-1}^{\alpha-1}f_h((\lambda+1)h) + D^r(\lambda)\sum_{k=2}^{n-1}\mathcal{G}_k^\alpha f_h((\lambda+n-(k-1))h) \\ &\qquad + \mathcal{G}_1^\alpha f_h((\lambda+n)h)\Bigg), \\ &= \frac{1}{h^\alpha}\Bigg(-D^r(\lambda)\mathcal{G}_{n+1}^\alpha f_h(\lambda h) - D^r(\lambda)\mathcal{G}_n^{\alpha-1}f_h((\lambda+1)h) \\ &\qquad + D^r(\lambda)\sum_{k=0}^{n+1}\mathcal{G}_k^\alpha f_h((\lambda+n-(k-1))h) \\ &\qquad + \big(1-D^r(\lambda)\big)\mathcal{G}_1^\alpha f_h((\lambda+n)h) - D^r(\lambda)\mathcal{G}_0^\alpha f_h((\lambda+(n+1))h)\Bigg), \\ &= R(\lambda,h) + D^r(\lambda)\Big(A_{h,1}^\alpha f_h(x) - \frac{\mathcal{G}_{n+1}^\alpha f_h(\lambda h) + \mathcal{G}_n^{\alpha-1}f_h((\lambda+1)h)}{h^\alpha}\Big), \end{aligned} \tag{4.75}$$

where

$$R(\lambda,h) = \frac{1}{h^\alpha}\Big(\big(1-D^r(\lambda)\big)\mathcal{G}_1^\alpha f_h((\lambda+n)h) - D^r(\lambda)\mathcal{G}_0^\alpha f_h((\lambda+(n+1))h)\Big).$$

We consider the terms of (4.75) one by one and show the following:

- Firstly, the first term $|R(\lambda,h)| = O(h^{2-\alpha})$.
- Secondly,

$$\Big|D^r(\lambda)\Big(A^\alpha_{h,1}f_h(x) - \frac{\mathcal{G}^\alpha_{n+1}f_h(\lambda h) + \mathcal{G}^{\alpha-1}_n f_h((\lambda+1)h)}{h^\alpha}\Big) - Af(x)\Big| = O(h).$$

First, we show that the first term of (4.75), $|R(\lambda,h)| = O(h^{2-\alpha})$. That is,

$$\left|\frac{1}{h^\alpha}\Big(\big(1-D^r(\lambda)\big)\mathcal{G}^\alpha_1 f_h((\lambda+n)h) - D^r(\lambda)\mathcal{G}^\alpha_0 f_h((\lambda+(n+1))h)\Big)\right| = O(h^{2-\alpha}).$$

If $\lambda=1$, then $D^r(\lambda)=0$ and $f_h((n+1)h) = f_h(1) = 0$ in view of (4.63). Therefore, $R(1,h)=0$. Hence, assuming $\lambda\in[0,1)$ and using $D^r(\lambda)-1 = \frac{\lambda}{\alpha\lambda'+\lambda}$, we have

$$\begin{aligned}
|R(\lambda,h)| &= \left|\frac{1}{h^\alpha}\Big[\big(1-D^r(\lambda)\big)\mathcal{G}^\alpha_1 f_h((\lambda+n)h) - D^r(\lambda)\mathcal{G}^\alpha_0 f_h((\lambda+n+1)h)\Big]\right| \\
&= \left|\frac{1}{h^\alpha}\Big[\big(D^r(\lambda)-1\big)\alpha f_h((\lambda+n)h) - D^r(\lambda) f_h((\lambda+n+1)h)\Big]\right| \\
&= \left|\frac{1}{h^\alpha}\left(\frac{-\alpha\lambda f_h(x) - \alpha\lambda' f_h(x+h)}{\alpha\lambda'+\lambda}\right)\right|, \qquad (4.76)
\end{aligned}$$

where $x=(\lambda+n)h$. Using the Taylor series for $\mathcal{P}$ given by (4.25), (4.56) with $b=0$ and $k_h = \frac{1}{\vartheta^\alpha_h(1)\Gamma(\alpha+1)}$, we have

$$\begin{aligned}
f_h(x) &= \mathcal{P}(x) + P(0)k_h\vartheta^\alpha_h(x) + dp_0(x) \\
&= \mathcal{P}(1) - (1-\lambda)h\mathcal{P}'(1) + O(h^2) + P(0)k_h\vartheta^\alpha_h(x) + d \\
&= \frac{P(0)}{\Gamma(\alpha+1)}\left(\frac{\vartheta^\alpha_h(x)}{\vartheta^\alpha_h(1)} - 1\right) - (1-\lambda)h\mathcal{P}'(1) + O(h^2) \\
&= R_1(\lambda,h) - \lambda' h\mathcal{P}'(1) + O(h^2),
\end{aligned}$$

where

$$R_1(\lambda,h) = \frac{P(0)}{\Gamma(\alpha+1)}\left(\frac{\vartheta^\alpha_h(x)}{\vartheta^\alpha_h(1)} - 1\right).$$

Similarly,

$$\begin{aligned}
f_h(x+h) &= \mathcal{P}(x+h) + P(0)k_h\vartheta^\alpha_h(x+h) + dp_0(x+h) \\
&= \mathcal{P}(1) + \lambda hP'(1) + O(h^2) + P(0)k_h\vartheta^\alpha_h(x+h) + d \\
&= \frac{P(0)}{\Gamma(\alpha+1)}\left(\frac{\vartheta^\alpha_h(x+h)}{\vartheta^\alpha_h(1)} - 1\right) + \lambda hP'(1) + O(h^2) \\
&= R_2(\lambda,h) + \lambda hP'(1) + O(h^2),
\end{aligned}$$

where

$$R_2(\lambda,h)=\frac{P(0)}{\Gamma(\alpha+1)}\left(\frac{\vartheta_h^\alpha(x+h)}{\vartheta_h^\alpha(1)}-1\right).$$

Then using (4.26) observe that

$$-\alpha\lambda R_1(\lambda,h)-\alpha\lambda' R_2(\lambda,h)=\frac{P(0)\alpha^2\lambda\lambda'}{\Gamma(\alpha+1)}\left(\frac{1}{n-1+\alpha}-\frac{1}{n}\right)=O(h^2).$$

Thus, (4.76) yields

$$|R(\lambda,h)|=\frac{1}{h^\alpha}\left|\frac{\alpha\lambda\lambda' h\mathcal{P}'(1)-\alpha\lambda\lambda' hP'(1)+O(h^2)}{\alpha\lambda'+\lambda}\right|=O(h^{2-\alpha}).$$

Next, we split the second term of (4.75) by setting $g_h=f_h-dp_0=\mathcal{P}+P(0)k_h\vartheta_h^\alpha$,

$$\begin{aligned}
&D^r(\lambda)\Big(A^\alpha_{h,1}f_h(x)-\frac{\mathcal{G}^\alpha_{n+1}f_h(\lambda h)+\mathcal{G}^{\alpha-1}_n f_h((\lambda+1)h)}{h^\alpha}\Big)\\
&=-D^r(\lambda)\Big(\frac{\mathcal{G}^\alpha_{n+1}g_h(\lambda h)+\mathcal{G}^{\alpha-1}_n g_h((\lambda+1)h)}{h^\alpha}\Big)\\
&\qquad+dD^r(\lambda)\Big(A^\alpha_{h,1}p_0(x)-\frac{\mathcal{G}^\alpha_{n+1}p_0(\lambda h)+\mathcal{G}^{\alpha-1}_n p_0((\lambda+1)h)}{h^\alpha}\Big)\\
&\qquad\qquad+D^r(\lambda)A^\alpha_{h,1}g_h(x).
\end{aligned}$$

First note that $g_h(\lambda h)=O(h^\alpha)$ and $g_h((\lambda+1)h)=O(h^\alpha)$ since in view of Definition 4.5.1, $\vartheta_h^\alpha(x)=O(h^\alpha)$ for $x\in[0,2h]$ and $\mathcal{P}(x)=O(x^{\alpha+1})$ as $x\downarrow 0$. Therefore, since $|\mathcal{G}^{\alpha-1}_n|=O(h^\alpha)$, $\left|\mathcal{G}^\alpha_{n+1}\right|=O(h^{\alpha+1})$ and $|D^r(\lambda)|\le 1$,

$$\left|D^r(\lambda)\Big(\frac{\mathcal{G}^\alpha_{n+1}g_h(\lambda h)+\mathcal{G}^{\alpha-1}_n g_h((\lambda+1)h)}{h^\alpha}\Big)\right|=O(h^\alpha). \tag{4.77}$$

Next note that

$$dD^r(\lambda)\Big(A^\alpha_{h,1}p_0(x)-\frac{\mathcal{G}^\alpha_{n+1}p_0(\lambda h)+\mathcal{G}^{\alpha-1}_n p_0((\lambda+1)h)}{h^\alpha}\Big)=0, \tag{4.78}$$

since $A^\alpha_{h,1}p_0(x)=\frac{1}{h^\alpha}\sum_{k=0}^{n+1}\mathcal{G}^\alpha_k=\frac{1}{h^\alpha}\mathcal{G}^{\alpha-1}_{n+1}$ and $\mathcal{G}^\alpha_{n+1}+\mathcal{G}^{\alpha-1}_n=\mathcal{G}^{\alpha-1}_{n+1}$ in view of (A.7).

Using Corollary 4.5.5

$$\left|A^\alpha_{h,1}\mathcal{P}(x)-A\mathcal{P}(x)\right|=O(h).$$

In view of Proposition 4.5.7, note that for $x\in[1-h,1]$,

$$A^\alpha_{h,1}\vartheta^\alpha_h(x)=1-\lambda'\mathcal{G}^\alpha_{n+1}=1+O(h^{\alpha+1})$$

since $\left|\mathcal{G}^\alpha_{n+1}\right|=O(h^{\alpha+1})$. Moreover, since $k_h\to 1$ in view of (4.70) and $Ap_\alpha(x)=p_0(x)=1$, we have

$$\left|P(0)\left(k_hA^\alpha_{h,1}\vartheta^\alpha_h(x)-Ap_\alpha(x)\right)\right|=O(h^{\alpha+1}).$$

Thus,

$$\begin{aligned}
&\left|A^{\alpha}_{h,1}g_h(x) - Af(x)\right| \\
&= \left|A^{\alpha}_{h,1}\big(\mathcal{P}(x) + P(0)k_h\vartheta^{\alpha}_h(x)\big) - A\big(\mathcal{P}(x) + P(0)p_\alpha(x)\big)\right| = O(h). \qquad (4.79)
\end{aligned}$$

For $f \in \mathcal{C}(A, \mathrm{ND})$, since $P \in C_0(\Omega)$, note that $Af(1) = P(1) = 0$. Moreover, $D^r(\lambda) = \frac{\alpha\lambda'}{\alpha\lambda'+\lambda}$ and $D^r(1) = 0$. Furthermore, Af and D^r are continuous as $x \uparrow 1$ and $\lambda \uparrow 1$, respectively. Therefore, for $x \in [1-h, 1]$

$$|D^r(\lambda)Af(x) - Af(x)| = O(h).$$

Thus, for $x \in [1-h, 1)$, in view of (4.77), (4.78) and (4.79),

$$\left|D^r(\lambda)\Big(A^{\alpha}_{h,1}f_h(x) - \frac{\mathcal{G}^{\alpha}_{n+1}f_h(\lambda h) + \mathcal{G}^{\alpha-1}_{n}f_h((\lambda+1)h)}{h^\alpha}\Big) - Af(x)\right| = O(h).$$

Hence,

$$G^h f_h = G^h(\mathcal{P} + P(0)k_h\vartheta^{\alpha}_h + dp_0) \to A\left(\mathcal{P} + P(0)p_\alpha\right) = Af$$

uniformly on Ω_2; that is, the proof of (4.73) is complete. This also completes the proof of Statement 1 of Proposition 4.3.2 for the operator (A, ND), in view of (4.49) and (4.72).

As mentioned in Remark 4.5.9, we now consider the common properties of the operators with a right Neumann* boundary condition, $(A, \bullet\mathrm{N}^*)$. Following that we deal with the operators (A, DN^*) and (A, NN^*) separately.

Common properties for operators $(A, \bullet\mathrm{N}^*)$**:**

Note that $f \in \mathcal{C}(A, \bullet\mathrm{N}^*)$ is given by (4.37)

$$f = \mathcal{P} + (P(0) + a)p_\alpha + bp_{\alpha-1} + dp_0, \qquad (4.80)$$

where $\mathcal{P} = I^\alpha(P - P(0)p_0)$. The relation $\frac{a+(\alpha-1)b}{\Gamma(\alpha)} = -I^{\alpha-1}g(1)$, for $f \in \mathcal{C}(A, \bullet\mathrm{N}^*)$ reads

$$\frac{a + (\alpha-1)b}{\Gamma(\alpha)} = -I^{\alpha-1}P(1) = -\mathcal{P}'(1) - \frac{P(0)}{\Gamma(\alpha)}, \qquad (4.81)$$

since $\mathcal{P}' = I^{\alpha-1}(P - P(0)p_0)$ and $I^{\alpha-1}p_0(1) = p_{\alpha-1}(1) = \frac{1}{\Gamma(\alpha)}$.

Note that $b^r_0 = 0$, $b^r_1 = \mathcal{G}^\alpha_0 + \mathcal{G}^\alpha_1$, $b^r_i = \mathcal{G}^\alpha_i$, $D^r = \mathbf{1}$ and $N^r(\lambda) = \lambda'$. First, let $x \in [1-2h, 1-h)$ and setting $\phi = f_h$ in (4.53), we have

$$G^h f_h(x) = \frac{1}{h^\alpha}\Bigg(N^l(\lambda)b_n f_h(\lambda h) + \Big(\lambda' b^l_{n-1} + \lambda\mathcal{G}^\alpha_{n-1}\Big)f_h((\lambda+1)h)$$

$$
\begin{aligned}
&+\sum_{k=2}^{n-2}\Big(\lambda'\mathcal{G}_k^\alpha+\lambda\mathcal{G}_k^\alpha\Big)f_h((\lambda+n-k)h)\\
&\quad+\Big(\lambda'\mathcal{G}_1^\alpha+\lambda(\mathcal{G}_0^\alpha+\mathcal{G}_1^\alpha)\Big)f_h((\lambda+n-1)h)\\
&\qquad+\lambda'\mathcal{G}_0^\alpha f_h((\lambda+n)h)\Bigg)\\
=\frac{1}{h^\alpha}\Bigg(&N^l(\lambda)b_nf_h(\lambda h)+\Big(\lambda' b_{n-1}^l+\lambda\mathcal{G}_{n-1}^\alpha\Big)f_h((\lambda+1)h)\\
&\quad+\sum_{k=0}^{n-2}\mathcal{G}_k^\alpha f_h((\lambda+n-1-(k-1))h)\\
&\qquad+\lambda\Big(f_h((\lambda+n-1)h)-f_h((\lambda+n)h)\Big)\Bigg).
\end{aligned}
\tag{4.82}
$$

Next, let $x\in[1-h,1]$ then setting $\phi=f_h$ in (4.54), we have

$$
\begin{aligned}
G^hf_h(x)&=\frac{1}{h^\alpha}\Bigg(b_nf_h((\lambda+1)h)+\sum_{k=2}^{n-1}\mathcal{G}_k^\alpha f_h((\lambda+n-(k-1))h)\\
&\qquad+(\mathcal{G}_0^\alpha+\mathcal{G}_1^\alpha)f_h((\lambda+n)h)\Bigg)\\
&=\frac{1}{h^\alpha}\Bigg((b_n-\mathcal{G}_n^\alpha)f_h((\lambda+1)h)+\sum_{k=0}^{n}\mathcal{G}_k^\alpha f_h((\lambda+n-(k-1))h)\\
&\qquad+f_h((\lambda+n)h)-f_h((\lambda+n+1)h)\Bigg)
\end{aligned}
\tag{4.83}
$$

Proof of Statement 1 of Proposition 4.3.2 for the operator (A,DN^*)**:** In this case, we further have $a,\ d=0$ and $P(0)=0$. Thus, (4.80) reduces to

$$
f=\mathcal{P}+bp_{\alpha-1}. \tag{4.84}
$$

We take

$$
f_h=\mathcal{P}+b_h\vartheta_h^{\alpha-1}, \tag{4.85}
$$

where

$$
b_h=\frac{b(n-1+\alpha)h}{\vartheta_h^{\alpha-1}(1)\Gamma(\alpha)}, \tag{4.86}
$$

First, note that for a right Neumann* boundary condition we do not require the function e_h. Further, note that this choice is the same as the sequences f_h constructed for $f\in\mathcal{C}(A,\mathrm{D}\bullet)$ given by (4.42) with $e_h=\mathbf{0}$. Next note that since $(n+1)h=1$ and using

(4.28) and (A.9),

$$b_h = \frac{b(n-1+\alpha)h}{\vartheta_h^{\alpha-1}(1)\Gamma(\alpha)} = \frac{b(n-1+\alpha)h}{h^{\alpha-1}n^{\alpha-1}\left(1+O(n^{-1})\right)} = b + O(h). \tag{4.87}$$

Proof of $f_h \to f$ for (A, ND):

Observe that as $h \to 0$, $b_h \to b$ in view of (4.87) and $\vartheta_h^{\alpha-1} \to p_{\alpha-1}$ in the sup-norm in view of Lemma 4.5.3. Thus,

$$f_h \to f \tag{4.88}$$

uniformly in Ω.

Proof of $G^h f_h \to Af$ for (A, ND):

Observe that, in view of (4.44), it only remains to show that We show that

$$G^h f_h = G^h(\mathcal{P} + b_h \vartheta_h^{\alpha-1}) \to A\mathcal{P} = Af \tag{4.89}$$

uniformly on Ω_2.

Note that $N^l = \mathbf{1}$, $b_i^l = \mathcal{G}_i^\alpha$ and $b_n = b_n^r = \mathcal{G}_n^\alpha$. Therefore, for $x \in [1-2h, 1-h)$, (4.82) becomes

$$\begin{aligned}
G^h f_h(x) &= \frac{1}{h^\alpha}\Bigg(\sum_{k=0}^{n} \mathcal{G}_k^\alpha f_h((\lambda+n-1-(k-1))h) \\
&\qquad\qquad + \lambda\Big(f_h((\lambda+n-1)h) - f_h((\lambda+n)h)\Big)\Bigg) \\
&= A_{h,1}^\alpha f_h(x) + \frac{\lambda}{h^\alpha}\Big(f_h((\lambda+n-1)h) - f_h((\lambda+n)h)\Big),
\end{aligned} \tag{4.90}$$

Similarly, for $x \in [1-h, 1]$ (4.83) becomes

$$G^h f_h(x) = A_{h,1}^\alpha f_h(x) - \frac{1}{h^\alpha}\mathcal{G}_{n+1}^\alpha f_h(\lambda h) + \frac{1}{h^\alpha}\Big(f_h((\lambda+n)h) - f_h((\lambda+n+1)h)\Big). \tag{4.91}$$

Let us deal with the terms of (4.90) and (4.91) one by one. First, let us consider their first terms. Then, in view of Corollary 4.5.5, for $x \in [1-2h, 1]$ we have that

$$\left|A_{h,1}^\alpha \mathcal{P}(x) - A\mathcal{P}(x)\right| = O(h).$$

Moreover, in view of Proposition 4.5.7, $A_{h,1}^\alpha \vartheta_h^{\alpha-1} = 0$ for $x \in [1-2h, 1]$. Thus,

$$\left|A_{h,1}^\alpha f_h(x) - Af(x)\right| = O(h).$$

To complete the proof we show the following:

- The second term of (4.91),

$$\left| -\frac{1}{h^\alpha}\mathcal{G}^\alpha_{n+1} f_h(\lambda h) \right| = O(h^\alpha).$$

- For $x \in [1-2h, 1]$; that is, for $x = (\lambda + n - 1)h$ and $(\lambda + n)h$

$$\left| \frac{1}{h^\alpha}\Big(f_h(x) - f_h(x+h) \Big) \right| = O(h^{2-\alpha}).$$

First, note that $\mathcal{G}^\alpha_{n+1} = O(h^{\alpha+1})$ and $f_h((\lambda h) = O(h^{\alpha-1})$, since $\vartheta_h^{\alpha-1}(\lambda h) = O(h^{\alpha-1})$ in view of Definition 4.5.1 and $\mathcal{P}(x) = O(x^{\alpha+1})$ as $x \downarrow 0$. Thus, the first term of (4.91)

$$\left| -\frac{1}{h^\alpha}\mathcal{G}^\alpha_{n+1} f_h(\lambda h) \right| = O(h^\alpha).$$

For $x \in [1-2h, 1]$, using (4.27), we have

$$b_h\left(\vartheta_h^{\alpha-1}(x) - \vartheta_h^{\alpha-1}(x+h)\right) = \frac{-(\alpha-1)bh}{\Gamma(\alpha)} + O(h^2)$$

Moreover, using (4.25) and (4.81) with a, $P(0) = 0$,

$$\mathcal{P}(x) - \mathcal{P}(x+h) = -h\mathcal{P}'(1) + O(h^2) = \frac{(\alpha-1)bh}{\Gamma(\alpha)} + O(h^2).$$

Thus,

$$\left| \frac{1}{h^\alpha}\left(f_h(x) - f_h(x+h) \right) \right| = O(h^{2-\alpha}).$$

Hence,

$$G^h f_h = G^h(\mathcal{P} + P(0)k_h\vartheta_h^\alpha + dp_0) \to A\left(\mathcal{P} + P(0)p_\alpha\right) = Af$$

uniformly on Ω_2; that is, the proof of (4.89) is complete. This also completes the proof of Statement 1 of Proposition 4.3.2 for the operator (A, DN^*), in view of (4.44) and (4.88).

Proof of Statement 1 of Proposition 4.3.2 for the operator (A, NN^*):

In this case, we further have that $b = 0$. Thus, (4.80) reduces to

$$f = \mathcal{P} + (P(0) + a)p_\alpha + dp_0. \tag{4.92}$$

We take

$$f_h = \mathcal{P} + (P(0)k_h + a_h)\vartheta_h^\alpha + dp_0, \tag{4.93}$$

where

$$k_h = \frac{(n-1+\alpha)h}{\vartheta_h^\alpha(1)\Gamma(\alpha+1)}, \text{ and } a_h = \frac{a(n-1+\alpha)h}{\vartheta_h^\alpha(1)\Gamma(\alpha+1)} \tag{4.94}$$

First, note that for a right Neumann* boundary condition we do not require the function e_h. Further, note that this choice is the same as the sequences f_h constructed for $f \in \mathcal{C}(A, \mathrm{N}\bullet)$ given by (4.48) with $e_h = \mathbf{0}$. Next note that as $h \to 0$,

$$(P(0)k_h + a_h) \to (P(0) + a), \tag{4.95}$$

since $(n+1)h = 1$ and using (4.28) and (A.9), we have

$$\frac{(n-1+\alpha)h}{\vartheta_h^\alpha(1)\Gamma(\alpha+1)} = \frac{(n-1+\alpha)h}{h^\alpha(n-1)^\alpha\,(1+O((n-1)^{-1}))} = 1 + O(h).$$

Proof of $f_h \to f$ for (A, ND):

As $h \to 0$, $\vartheta_h^\alpha \to p_\alpha$ in the sup-norm in view of Lemma 4.5.3. Thus, in view of (4.95),

$$f_h \to f \tag{4.96}$$

uniformly in Ω.

Proof of $G^h f_h \to Af$ for (A, ND):

Observe that, in view of (4.49), it only remains to show that

$$G^h f_h = G^h\left(\mathcal{P} + (P(0)k_h + a_h)\vartheta_h^\alpha + dp_0\right) \to A\mathcal{P} + (P(0) + a)p_0 = Af \tag{4.97}$$

uniformly on Ω_2.

Note that $N^l(\lambda) = \lambda$, $b_i^l = -\mathcal{G}_{i-1}^{\alpha-1}$ and since $b_0^r = 0$, in view of (A.7),

$$b_n = -\sum_{i=0}^{n-1} b_i^r = -(\mathcal{G}_0^\alpha + \mathcal{G}_1^\alpha) - \sum_{i=2}^{n-1}\mathcal{G}_i^\alpha = -\sum_{i=0}^{n-1}\mathcal{G}_i^\alpha = -\mathcal{G}_{n-1}^{\alpha-1}.$$

Thus, using $\mathcal{G}_{n-2}^{\alpha-1} + \mathcal{G}_{n-1}^\alpha = \mathcal{G}_{n-1}^{\alpha-1}$, (4.82) becomes

$$\begin{aligned}
G^h f_h(x) = \frac{1}{h^\alpha}\Bigg(&-\lambda\mathcal{G}_{n-1}^{\alpha-1} f_h(\lambda h) + \Big(-\lambda'\mathcal{G}_{n-2}^{\alpha-1} + \lambda\mathcal{G}_{n-1}^\alpha\Big) f_h((\lambda+1)h) \\
&+ \sum_{k=0}^{n-2}\mathcal{G}_k^\alpha f_h((\lambda + n - 1 - (k-1))h) \\
&+ \lambda\Big(f_h((\lambda+n-1)h) - f_h((\lambda+n)h)\Big)\Bigg) \\
= \frac{1}{h^\alpha}\Bigg(&-\Big(\lambda\mathcal{G}_{n-1}^{\alpha-1} + \mathcal{G}_n^\alpha\Big) f_h(\lambda h) - \lambda'\Big(\mathcal{G}_{n-2}^{\alpha-1} + \mathcal{G}_{n-1}^\alpha\Big) f_h((\lambda+1)h)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=0}^{n}\mathcal{G}_k^{\alpha} f_h((\lambda+n-1-(k-1))h)\\
&\quad+\lambda\Big(f_h((\lambda+n-1)h)-f_h((\lambda+n)h)\Big)\Bigg)\\
&=A_{h,1}^{\alpha}f_h(x)-\frac{1}{h^{\alpha}}\Bigg(\Big(\lambda\mathcal{G}_{n-1}^{\alpha-1}+\mathcal{G}_n^{\alpha}\Big)f_h(\lambda h)+\lambda'\mathcal{G}_{n-1}^{\alpha-1}f_h((\lambda+1)h)\Bigg)\\
&\quad+\lambda\Bigg(\frac{f_h((\lambda+n-1)h)-f_h((\lambda+n)h)}{h^{\alpha}}\Bigg). \qquad (4.98)
\end{aligned}
$$

Next, note that $b_n-\mathcal{G}_n^{\alpha}=-\mathcal{G}_n^{\alpha-1}$ in view of (A.7). Thus, (4.83) becomes

$$
\begin{aligned}
G^h f_h(x)&=\frac{1}{h^{\alpha}}\Bigg(-\mathcal{G}_n^{\alpha-1}f_h((\lambda+1)h)+\sum_{k=0}^{n}\mathcal{G}_k^{\alpha}f_h((\lambda+n-(k-1))h)\\
&\quad+\Big(f_h((\lambda+n)h)-f_h((\lambda+n+1)h)\Big)\Bigg)\\
&=\frac{1}{h^{\alpha}}\Bigg(-\Big(\mathcal{G}_{n+1}^{\alpha}f_h(\lambda h)+\mathcal{G}_n^{\alpha-1}f_h((\lambda+1)h)\Big)\\
&\quad+\sum_{k=0}^{n+1}\mathcal{G}_k^{\alpha}f_h((\lambda+n-(k-1))h)\\
&\quad+\Big(f_h((\lambda+n)h)-f_h((\lambda+n+1)h)\Big)\Bigg)\\
&=A_{h,1}^{\alpha}f_h(x)-\frac{1}{h^{\alpha}}\Big(\mathcal{G}_{n+1}^{\alpha}f_h(\lambda h)+\mathcal{G}_n^{\alpha-1}f_h((\lambda+1)h)\Big)\\
&\quad+\frac{f_h((\lambda+n)h)-f_h((\lambda+n+1)h)}{h^{\alpha}}. \qquad (4.99)
\end{aligned}
$$

We deal with the terms of (4.98) and (4.99) one be one and show the following:

- For $x\in[1-2h,1-h)$,

$$
\left|A_{h,1}^{\alpha}f_h(x)-\frac{1}{h^{\alpha}}\Big((\lambda\mathcal{G}_{n-1}^{\alpha-1}+\mathcal{G}_n^{\alpha})f_h(\lambda h)+\lambda'\mathcal{G}_{n-1}^{\alpha-1}f_h((\lambda+1)h)\Big)-Af\right|=O(h). \quad (4.100)
$$

- For $x\in[1-h,1]$,

$$
\left|A_{h,1}^{\alpha}f_h(x)-\frac{1}{h^{\alpha}}\Big(\mathcal{G}_{n+1}^{\alpha}f_h(\lambda h)+\mathcal{G}_n^{\alpha-1}f_h((\lambda+1)h)\Big)-Af\right|=O(h). \qquad (4.101)
$$

- For $x\in[1-2h,1]$; that is, $x=(\lambda+n-1)h$ and $(\lambda+n)h$

$$
\left|\frac{f_h(x)-f_h(x+h)}{h^{\alpha}}\right|=O(h^{2-\alpha}). \qquad (4.102)
$$

Let $g_h = f_h - dp_0 = \mathcal{P} + (P(0)k_h + a_h)\vartheta_h^\alpha$. For $x \in [1-2h, 1]$, using Corollary 4.5.5

$$\left|A_{h,1}^\alpha \mathcal{P}(x) - A\mathcal{P}(x)\right| = O(h).$$

In view of Proposition 4.5.7, $A_{h,1}^\alpha \vartheta_h^\alpha(x) = 1 - \lambda' \mathcal{G}_{\iota(x)}^\alpha$, where $\iota(x) = n$ or $n+1$. Moreover, since $(P(0)k_h + a_h) \to P(0) + a$ in view of (4.95), $p_0(x) = 1$ and $\left|\mathcal{G}_{\iota(x)}^\alpha\right| = O(h^{\alpha+1})$, we have

$$\left|(P(0)k_h + a_h)A_{h,1}^\alpha \vartheta_h^\alpha(x) - (P(0)+a)p_0(x)\right| = O(h^{\alpha+1}).$$

Thus,

$$\begin{aligned}
&\left|A_{h,1}^\alpha g_h(x) - Af(x)\right| \\
&= \left|A_{h,1}^\alpha\big(\mathcal{P}(x) + (P(0)k_h + a_h)\vartheta_h^\alpha(x)\big) - \big(A\mathcal{P}(x) + (P(0)+a)p_0(x)\big)\right| = O(h).
\end{aligned} \tag{4.103}$$

Proof of (4.100): Observe that,

$$\begin{aligned}
&d\left|A_{h,1}^\alpha p_0(x) - \frac{1}{h^\alpha}\left(\left(\lambda \mathcal{G}_{n-1}^{\alpha-1} + \mathcal{G}_n^\alpha\right)p_0(\lambda h) + (1-\lambda)\mathcal{G}_{n-1}^{\alpha-1} p_0((\lambda+1)h)\right)\right| \\
&= \frac{d}{h^\alpha}\left|\sum_{k=0}^{n} \mathcal{G}_k^\alpha - \left(\lambda \mathcal{G}_{n-1}^{\alpha-1} + \mathcal{G}_n^\alpha + (1-\lambda)\mathcal{G}_{n-1}^{\alpha-1}\right)\right| \\
&= \frac{d}{h^\alpha}\left|\mathcal{G}_n^{\alpha-1} - (\mathcal{G}_n^\alpha + \mathcal{G}_{n-1}^{\alpha-1})\right| = 0.
\end{aligned}$$

Next, since $g_h(\lambda h) = O(h^\alpha)$, $g_h((\lambda+1)h) = O(h^\alpha)$, $|\mathcal{G}_n^\alpha| = O(h^{\alpha+1})$ and $\left|\mathcal{G}_{n-1}^{\alpha-1}\right| = O(h^\alpha)$ we have that

$$\left|\frac{1}{h^\alpha}\left(\left(\lambda \mathcal{G}_{n-1}^{\alpha-1} + \mathcal{G}_n^\alpha\right)g_h(\lambda h) + (1-\lambda)\mathcal{G}_{n-1}^{\alpha-1} g_h((\lambda+1)h)\right)\right| = O(h^\alpha).$$

Hence, in view of (4.103),

$$\begin{aligned}
&\left|A_{h,1}^\alpha f_h(x) - \frac{1}{h^\alpha}\left(\left(\lambda \mathcal{G}_{n-1}^{\alpha-1} + \mathcal{G}_n^\alpha\right)f_h(\lambda h) + (1-\lambda)\mathcal{G}_{n-1}^{\alpha-1} f_h((\lambda+1)h)\right) - Af(x)\right| \\
&= \left|A_{h,1}^\alpha g_h(x) - Af(x)\right| + O(h^\alpha) = O(h).
\end{aligned}$$

Proof of (4.101):

Observe that,

$$d\left|A_{h,1}^\alpha p_0(x) - \frac{1}{h^\alpha}\left(\mathcal{G}_{n+1}^\alpha p_0(\lambda h) + \mathcal{G}_n^{\alpha-1} p_0((\lambda+1)h)\right)\right|$$

$$= \frac{d}{h^\alpha}\left|\sum_{k=0}^{n+1}\mathcal{G}_k^\alpha - \left(\mathcal{G}_{n+1}^\alpha + \mathcal{G}_n^{\alpha-1}\right)\right| = \frac{d}{h^\alpha}\left|\mathcal{G}_{n+1}^{\alpha-1} - \mathcal{G}_{n+1}^{\alpha-1}\right| = 0.$$

Next, since $g_h(\lambda h) = O(h^\alpha)$, $\left|\mathcal{G}_{n+1}^\alpha\right| = O(h^{\alpha+1})$ and $|\mathcal{G}_n^{\alpha-1}| = O(h^\alpha)$ we have that

$$\left|\frac{1}{h^\alpha}\left(\mathcal{G}_{n+1}^\alpha g_h(\lambda h) + \mathcal{G}_n^{\alpha-1} g_h((\lambda+1)h)\right)\right| = O(h^\alpha).$$

Hence, in view of (4.103)

$$\left|A_{h,1}^\alpha f_h(x) - \frac{1}{h^\alpha}\left(\mathcal{G}_{n+1}^\alpha f_h(\lambda h) + \mathcal{G}_n^{\alpha-1} f_h((\lambda+1)h)\right) - Af(x)\right|$$
$$= \left|A_{h,1}^\alpha g_h(x) - Af(x)\right| + O(h^\alpha) = O(h).$$

Proof of (4.102): For $x \in [1-2h, 1]$, using (4.27), we have

$$(P(0)k_h + a_h)\left(\vartheta_h^\alpha(x) - \vartheta_h^\alpha(x+h)\right) = \frac{-(P(0)+a)h}{\Gamma(\alpha)} + O(h^2).$$

Moreover, using (4.25) and (4.81),

$$\mathcal{P}(x) - \mathcal{P}(x+h) = -h\mathcal{P}'(1) + O(h^2) = \frac{(P(0)+a)h}{\Gamma(\alpha)} + O(h^2).$$

Thus,

$$\left|\frac{f_h(x) - f_h(x+h)}{h^\alpha}\right| = O(h^{2-\alpha}).$$

Hence,

$$G^h f_h = G^h\left(\mathcal{P} + (P(0)k_h + a_h)\vartheta_h^\alpha + dp_0\right) \to A\mathcal{P} + (P(0)+a)p_0 = Af$$

uniformly on Ω_2; that is, the proof of (4.97) is complete. This also completes the proof of Statement 1 of Proposition 4.3.2 for the operator (A, NN^*), in view of (4.49) and (4.96).

As mentioned in Remark 4.5.9, we now consider the common properties of the operators with a right Neumann boundary condition, $(A, \bullet\mathrm{N})$. Following that we deal with the operators (A, DN) and (A, NN) separately.

Common properties for operators $(A, \bullet\mathrm{N})$**:**

Note that $f \in \mathcal{C}(A, \bullet\mathrm{N})$ is given by (4.37)

$$f = \mathcal{P} + (P(0)+a)p_\alpha + bp_{\alpha-1} + dp_0, \tag{4.104}$$

where $\mathcal{P} = I^\alpha(P - P(0)p_0)$. The relation $a + b = -Ig(1)$, for $f \in \mathcal{C}(A, \bullet\mathrm{N})$ reads

$$a + b = -IP(1) = -D_c^{\alpha-1}\mathcal{P}(1) - P(0), \tag{4.105}$$

since $D_c^{\alpha-1}\mathcal{P} = I(P - P(0)p_0)$ and $Ip_0(1) = p_1(1) = 1$.

When dealing with a right Neumann boundary condition it turns out that in order for $G^h f_h$ to converge to Af we need the term e_h in f_h which is constructed as follows:

$$e_h(x) = \begin{cases} 0, & \text{if } x \in \Omega \cap [0, 1-h), \\ -h^\alpha \lambda \left(A\mathcal{P}(1) + (P(0) + a)\right), & \text{if } x \in [1-h, 1]. \end{cases} \tag{4.106}$$

Then, $e_h \in C_0(\Omega)$ and as $h \to 0$, $e_h \to \mathbf{0}$.

Note that $N^r(\lambda) = \lambda'$, $D^r = \mathbf{1}$ and $b_i^r = -\mathcal{G}_{i-1}^{\alpha-1}$.

Let $\phi \in C_0(\Omega)$ be an arbitrary element, then using (4.53) for $x \in [1-2h, 1-h)$, we have

$$\begin{aligned} G^h\phi(x) = \frac{1}{h^\alpha}\Bigg(& N^l(\lambda) b_n \phi(\lambda h) + \left(\lambda' b_{n-1}^l - \lambda \mathcal{G}_{n-2}^{\alpha-1}\right)\phi((\lambda+1)h) \\ & + \sum_{k=1}^{n-2}\left(\lambda'\mathcal{G}_k^\alpha - \lambda\mathcal{G}_{k-1}^{\alpha-1}\right)\phi((\lambda + n - 1 - (k-1))h) \\ & + \lambda'\mathcal{G}_0^\alpha\phi((\lambda+n)h)\Bigg) \\ = \frac{1}{h^\alpha}\Bigg(& \left(N^l(\lambda) b_n - \lambda'\mathcal{G}_n^\alpha + \lambda\mathcal{G}_{n-1}^{\alpha-1}\right)\phi(\lambda h) \\ & + \lambda'\left(b_{n-1}^l + \mathcal{G}_{n-1}^\alpha\right)\phi((\lambda+1)h) \\ & + \lambda'\sum_{k=0}^{n}\mathcal{G}_k^\alpha\phi((\lambda + n - 1 - (k-1))h) \\ & - \lambda\sum_{k=0}^{n-1}\mathcal{G}_k^{\alpha-1}\phi((\lambda + n - 1 - k))h)\Bigg) \\ = \lambda' A_{h,1}^\alpha\phi(x) - \frac{\lambda}{h} & A_{h,0}^{\alpha-1}\phi(x) + \frac{1}{h^\alpha}\Bigg(\left(N^l(\lambda) b_n - \lambda'\mathcal{G}_n^\alpha + \lambda\mathcal{G}_{n-1}^{\alpha-1}\right)\phi(\lambda h) \\ & + \lambda'\left(b_{n-1}^l + \mathcal{G}_{n-1}^\alpha\right)\phi((\lambda+1)h)\Bigg). \end{aligned} \tag{4.107}$$

Using (4.54) for $x \in [1-h, 1]$, we have

$$G^h\phi(x) = \frac{1}{h^\alpha}\left(b_n\phi((\lambda+1)h) - \sum_{k=1}^{n-1}\mathcal{G}_{k-1}^{\alpha-1}\phi((\lambda + n - (k-1))h)\right)$$

$$= \frac{1}{h^\alpha}\left(\mathcal{G}_n^{\alpha-1}\phi(\lambda h) + \left(b_n + \mathcal{G}_{n-1}^{\alpha-1}\right)\phi((\lambda+1)h) - \sum_{k=0}^{n}\mathcal{G}_k^{\alpha-1}\phi((\lambda+n-k)h)\right)$$

$$= -\frac{1}{h}A_{h,0}^{\alpha-1}\phi(x) + \frac{1}{h^\alpha}\left(\mathcal{G}_n^{\alpha-1}\phi(\lambda h) + \left(b_n + \mathcal{G}_{n-1}^{\alpha-1}\right)\phi((\lambda+1)h)\right) \tag{4.108}$$

Moreover, it follows on using (4.9), (4.107) and (4.108) that

$$G^h e_h(x) = \begin{cases} 0, & \text{if } x \in \Omega \cap [0, 1-2h), \\ -\lambda\lambda'\left(A\mathcal{P}(1) + (P(0)+a)\right), & \text{if } x \in [1-2h, 1-h), \\ \lambda\left(A\mathcal{P}(1) + (P(0)+a)\right), & \text{if } x \in [1-h, 1]. \end{cases} \tag{4.109}$$

Next take note of the error terms for the Grünwald-type approximation of the fractional derivatives of $\mathcal{P}$ given in Remark 4.5.6. First, for $x \in [1-2h, 1-h)$,

$$\begin{aligned} A_{h,0}^{\alpha-1}\mathcal{P}(x) &= A_{h,\lambda-2}^{\alpha-1}\mathcal{P}(1) = D_c^{\alpha-1}\mathcal{P}(1) + h\Big((\lambda-2) - \frac{\alpha-1}{2}\Big)A\mathcal{P}(1) + O(h^2), \\ A_{h,1}^{\alpha}\mathcal{P}(x) &= A_{h,\lambda-2}^{\alpha}\mathcal{P}(1) = A\mathcal{P}(1) + O(h). \end{aligned} \tag{4.110}$$

Next, for $x \in [1-h, 1]$,

$$A_{h,0}^{\alpha-1}\mathcal{P}(x) = A_{h,\lambda-1}^{\alpha-1}\mathcal{P}(1) = D_c^{\alpha-1}\mathcal{P}(1) + h\Big((\lambda-1) - \frac{\alpha-1}{2}\Big)A\mathcal{P}(1) + O(h^2). \tag{4.111}$$

Moreover, since $\left|A_{h,1}^{\alpha}\mathcal{P}(x) - A\mathcal{P}(x)\right| = O(h)$ in view of Corollary 4.5.5, we have that

$$A\mathcal{P}(1) = A\mathcal{P}(x) + O(h),\ x \in [1-2h, 1]. \tag{4.112}$$

Proof of Statement 1 of Proposition 4.3.2 for the operator (A, DN):

In this case, we further have $a, d = 0$ and $P(0) = 0$. Thus, $f \in \mathcal{C}(A, \mathrm{DN})$ given by (4.104), reduces to

$$f = \mathcal{P} + bp_{\alpha-1}.$$

We take

$$f_h = \mathcal{P} + b_h\vartheta_h^{\alpha-1} + e_h$$

with

$$b_h = b + h\frac{\alpha-1}{2}A\mathcal{P}(1). \tag{4.113}$$

Note that this choice is the same as the sequences f_h constructed for $f \in \mathcal{C}(A, \mathrm{D}\bullet)$ given by (4.42).

Proof of $f_h \to f$ for (A, DN):

As $h \to 0$, $b_h \to b$, $e_h \to \mathbf{0}$ in the sup-norm and $\vartheta_h^{\alpha-1} \to p_{\alpha-1}$ in the sup-norm in view of Lemma 4.5.3. Thus,

$$f_h \to f \tag{4.114}$$

uniformly in Ω.

Proof of $G^h f_h \to Af$ *for* (A, DN):

Observe that, in view of (4.44), it only remains to show that

$$G^h f_h = G^h \left(\mathcal{P} + b_h \vartheta_h^{\alpha-1} + e_h\right) \to A\mathcal{P} = Af \tag{4.115}$$

uniformly on Ω_2.

Note that $N^l = \mathbf{1}$, $b_i^l = \mathcal{G}_i^\alpha$ and $b_n = -\mathcal{G}_{n-1}^{\alpha-1}$. Therefore, for $x \in [1-2h, 1-h)$, using $\mathcal{G}_n^\alpha + \mathcal{G}_{n-1}^{\alpha-1} = \mathcal{G}_n^{\alpha-1}$, (4.107) becomes

$$G^h \phi(x) = \lambda' A_{h,1}^\alpha \phi(x) - \frac{\lambda}{h} A_{h,0}^{\alpha-1} \phi(x) - \frac{\lambda'}{h^\alpha} \mathcal{G}_n^{\alpha-1} \phi(\lambda h). \tag{4.116}$$

For $x \in [1-h, 1]$, (4.108) becomes

$$G^h \phi(x) = -\frac{1}{h} A_{h,0}^{\alpha-1} \phi(x) + \frac{1}{h^\alpha} \mathcal{G}_n^{\alpha-1} \phi(\lambda h). \tag{4.117}$$

Let $x \in [1-2h, 1-h)$, then using (4.109) with $a = 0$, $P(0) = 0$ and (4.116), observe that

$$\begin{aligned} G^h f_h(x) &= G^h\Big((\mathcal{P}(x) + b_h \vartheta_h^{\alpha-1}(x)\Big) + G^h e_h(x) \\ &= (1-\lambda) A_{h,1}^\alpha \left(\mathcal{P}(x) + b_h \vartheta_h^{\alpha-1}(x)\right) - \frac{\lambda}{h} A_{h,0}^{\alpha-1} \left(\mathcal{P}(x) + b_h \vartheta_h^{\alpha-1}(x)\right) \\ &\qquad - (1-\lambda)\lambda A\mathcal{P}(1) - \frac{1}{h^\alpha}(1-\lambda)\mathcal{G}_n^{\alpha-1}\Big(\mathcal{P}(\lambda h) + b_h \vartheta_h^{\alpha-1}(\lambda h)\Big). \end{aligned}$$

Since, $\left|\mathcal{P}(\lambda h) + b_h \vartheta_h^{\alpha-1}(\lambda h)\right| = O(h^{\alpha-1})$ and $|\mathcal{G}_n^{\alpha-1}| = O(h^\alpha)$, the last term

$$\left|\frac{1}{h^\alpha}(1-\lambda)\mathcal{G}_n^{\alpha-1}\Big(\mathcal{P}(\lambda h) + b_h \vartheta_h^{\alpha-1}(\lambda h)\Big)\right| = O(h^{\alpha-1}).$$

In view of Proposition 4.5.7

$$A_{h,1}^\alpha \vartheta_h^{\alpha-1}(x) = 0 \text{ and } A_{h,0}^{\alpha-1} \vartheta_h^{\alpha-1}(x) = 1.$$

Thus, using (4.105), (4.110) and (4.112),

$$\begin{aligned} &\left|G^h f_h(x) - A\mathcal{P}(x)\right| \\ &= \Bigg|(1-\lambda)\left(A\mathcal{P}(1) + O(h)\right) \\ &\qquad - \frac{\lambda}{h}\left(D_c^{\alpha-1}\mathcal{P}(1) + h\Big((\lambda-2) - \frac{\alpha-1}{2}\Big)A\mathcal{P}(1) + O(h^2) + b_h\right) \end{aligned}$$

$$- (1-\lambda)\lambda A\mathcal{P}(1) - (A\mathcal{P}(1) + O(h)) \Bigg| + O(h^{\alpha-1})$$

$$= \Bigg| (1-\lambda)A\mathcal{P}(1)$$

$$+ \frac{\lambda b}{h} - \lambda(\lambda-2)A\mathcal{P}(1) + \frac{\lambda(\alpha-1)}{2}A\mathcal{P}(1) - \frac{\lambda b}{h} - \frac{\lambda(\alpha-1)}{2}A\mathcal{P}(1)$$

$$- (1-\lambda)\lambda A\mathcal{P}(1) - A\mathcal{P}(1) \Bigg| + O(h^{\alpha-1}) = O(h^{\alpha-1}).$$

Next, let $x \in [1-h, 1]$, then using (4.109) with $a = 0, P(0) = 0$ and (4.117), observe that

$$\begin{aligned} G^h f_h(x) &= G^h\Big(\mathcal{P}(x) + b_h \vartheta_h^{\alpha-1}(x)\Big) + G^h e_h(x) \\ &= -\frac{1}{h} A_{h,0}^{\alpha-1}\left(\mathcal{P}(x) + b_h \vartheta_h^{\alpha-1}(x)\right) + \lambda A\mathcal{P}(1) + \frac{1}{h^\alpha}\mathcal{G}_n^{\alpha-1}\big(\mathcal{P}(\lambda h) + b_h \vartheta_h^{\alpha-1}(\lambda h)\Big). \end{aligned}$$

Since $\left|\mathcal{P}(\lambda h) + b_h \vartheta_h^{\alpha-1}(\lambda h)\right| = O(h^{\alpha-1})$ and $|\mathcal{G}_n^{\alpha-1}| = O(h^\alpha)$, the last term

$$\left|\frac{1}{h^\alpha}\mathcal{G}_n^{\alpha-1}\Big(\mathcal{P}(\lambda h) + b_h \vartheta_h^{\alpha-1}(\lambda h)\Big)\right| = O(h^{\alpha-1}).$$

In view of Proposition 4.5.7

$$A_{h,0}^{\alpha-1}\vartheta_h^{\alpha-1}(x) = 1.$$

Thus, in view of (4.105), (4.111) and (4.112),

$$\begin{aligned} &\Big| G^h f_h(x) - A\mathcal{P}(x) \Big| \\ &= \Bigg| -\frac{1}{h}\left(D_c^{\alpha-1}\mathcal{P}(1) + h\Big((\lambda-1) - \frac{\alpha-1}{2}\Big)A\mathcal{P}(1) + O(h^2) + b_h\right) \\ &\qquad + \lambda A\mathcal{P}(1) - (A\mathcal{P}(1) + O(h)) \Bigg| + O(h^{\alpha-1}) \\ &= \Bigg| \frac{b}{h} - \Big((\lambda-1) - \frac{\alpha-1}{2}\Big)A\mathcal{P}(1) - \frac{b}{h} - \frac{\alpha-1}{2}A\mathcal{P}(1) \\ &\qquad + \lambda A\mathcal{P}(1) - A\mathcal{P}(1) \Bigg| + O(h^{\alpha-1}) = O(h^{\alpha-1}). \end{aligned}$$

Hence,

$$G^h f_h = G^h\left(\mathcal{P} + (P(0)k_h + a_h)\vartheta_h^\alpha + dp_0\right) \to A\mathcal{P} + (P(0) + a)p_0 = Af$$

uniformly on Ω_2; that is, the proof of (4.115) is complete. This also completes the proof of Statement 1 of Proposition 4.3.2 for the operator (A, DN), in view of (4.44) and (4.114).

Proof of Statement 1 of Proposition 4.3.2 for the operator (A, NN):

In this case, we further have $b = 0$. Thus, $f \in \mathcal{C}(A, \mathrm{NN})$ given by (4.104), reduces to

$$f = \mathcal{P} + (P(0) + a)p_\alpha + dp_0.$$

We take

$$f_h = \mathcal{P} + (P(0)k_h + a_h)\vartheta_h^\alpha + dp_0 + e_h$$

where

$$k_h = 1 + h \text{ and } a_h = a(1 + h) + h\frac{\alpha - 1}{2}A\mathcal{P}(1). \tag{4.118}$$

Proof of $f_h \to f$ for (A, NN):

Note that as $h \to 0$,

$$(P(0)k_h + a_h) = (P(0) + a)(1 + h) + h\frac{\alpha - 1}{2}A\mathcal{P}(1) \to (P(0) + a).$$

Moreover, as $h \to 0$, $e_h \to \mathbf{0}$ in the sup-norm and $\vartheta_h^\alpha \to p_\alpha$ in the sup-norm in view of Lemma 4.5.3. Thus,

$$f_h \to f \tag{4.119}$$

uniformly in Ω.

Proof of $G^h f_h \to Af$ for (A, NN):

Observe that, in view of (4.44), it only remains to show that

$$G^h f_h = G^h\left(\mathcal{P} + (P(0)k_h + a_h)\vartheta_h^\alpha + dp_0 + e_h\right) \to A\mathcal{P} + (P(0) + a)p_0 = Af \tag{4.120}$$

uniformly on Ω_2.

Note that $N^l(\lambda) = \lambda$, $b_i^l = -\mathcal{G}_{i-1}^{\alpha-1}$ and $b_n = \sum_{i=0}^{n-1}\mathcal{G}_{i-1}^{\alpha-1} = \sum_{i=0}^{n-2}\mathcal{G}_i^{\alpha-1}$. Therefore, for $x \in [1 - 2h, 1 - h)$, using $\mathcal{G}_{n-2}^{\alpha-1} + \mathcal{G}_{n-1}^{\alpha} = \mathcal{G}_{n-1}^{\alpha-1}$ (4.107) becomes

$$\begin{aligned} G^h\phi(x) = \lambda' A_{h,1}^\alpha\phi(x) - \frac{\lambda}{h}A_{h,0}^{\alpha-1}\phi(x) \\ + \frac{1}{h^\alpha}\Bigg(\Big(\lambda\sum_{i=0}^{n-1}\mathcal{G}_i^{\alpha-1} - \lambda'\mathcal{G}_n^\alpha\Big)\phi(\lambda h) \\ - \lambda'\mathcal{G}_{n-1}^{\alpha-1}\phi((\lambda + 1)h)\Bigg). \end{aligned} \tag{4.121}$$

For $x \in [1-h, 1]$, (4.108) becomes

$$G^h\phi(x) = -\frac{1}{h}A_{h,0}^{\alpha-1}\phi(x) + \frac{1}{h^\alpha}\left(\mathcal{G}_n^{\alpha-1}\phi(\lambda h) + \sum_{i=0}^{n-1}\mathcal{G}_i^{\alpha-1}\phi((\lambda+1)h)\right). \tag{4.122}$$

Let $x \in [1-2h, 1-h)$, then using (4.121) and (4.109), setting

$$g_h = \mathcal{P} + (P(0)k_h + a_h)\vartheta_h^\alpha$$

we have

$$\begin{aligned}
G^h f_h(x) &= G^h\left(\mathcal{P}(x) + (P(0)k_h + a_h)\vartheta_h^\alpha(x)\right) + dG^h p_0(x) + G^h e_h(x) \\
&= \lambda' A_{h,1}^\alpha g_h(x) - \frac{\lambda}{h}A_{h,0}^{\alpha-1}g_h(x) - \lambda\lambda'\Big(A\mathcal{P}(1) + (P(0)+a)\Big) \\
&+ d\Bigg(\lambda'\Big(A_{h,1}^\alpha p_0(x) - \frac{1}{h^\alpha}\mathcal{G}_n^\alpha p_0(\lambda h) - \frac{1}{h^\alpha}\mathcal{G}_{n-1}^{\alpha-1}p_0((\lambda+1)h)\Big) \\
&\qquad - \frac{\lambda}{h}\Big(A_{h,0}^{\alpha-1}p_0(x) - \frac{1}{h^{\alpha-1}}\sum_{i=0}^{n-1}\mathcal{G}_i^{\alpha-1}p_0(\lambda h)\Big)\Bigg) \\
&+ \frac{1}{h^\alpha}\left(\Big(\lambda\sum_{i=0}^{n-1}\mathcal{G}_i^{\alpha-1} - \lambda'\mathcal{G}_n^\alpha\Big)g_h(\lambda h) - \lambda'\mathcal{G}_{n-1}^{\alpha-1}g_h((\lambda+1)h)\right).
\end{aligned} \tag{4.123}$$

Since $|g_h(\lambda h)| = O(h^\alpha)$, $\left|\sum_{i=0}^{n-1}\mathcal{G}_i^{\alpha-1}\right| = O(h^{\alpha-1})$, $|\mathcal{G}_n^{\alpha-1}| = O(h^\alpha)$ and $|\mathcal{G}_n^\alpha| = O(h^{\alpha+1})$, the last line of (4.123)

$$\left|\frac{1}{h^\alpha}\left(\Big(\lambda\sum_{i=0}^{n-1}\mathcal{G}_i^{\alpha-1} - \lambda'\mathcal{G}_n^\alpha\Big)g_h(\lambda h) - \lambda'\mathcal{G}_{n-1}^{\alpha-1}g_h((\lambda+1)h)\right)\right| = O(h^{\alpha-1}).$$

Next in view of (A.7) observe that the second line of (4.123)

$$\begin{aligned}
&d\Bigg(\lambda'\Big(A_{h,1}^\alpha p_0(x) - \frac{1}{h^\alpha}\mathcal{G}_n^\alpha p_0(\lambda h) - \frac{1}{h^\alpha}\mathcal{G}_{n-1}^{\alpha-1}p_0((\lambda+1)h)\Big) \\
&\qquad - \frac{\lambda}{h}\Big(A_{h,0}^{\alpha-1}p_0(x) - \frac{1}{h^{\alpha-1}}\sum_{i=0}^{n-1}\mathcal{G}_i^{\alpha-1}p_0(\lambda h)\Big)\Bigg) \\
&= \frac{d}{h^\alpha}\left(\lambda'\Big(\sum_{i=0}^{n}\mathcal{G}_i^\alpha - \mathcal{G}_n^\alpha - \mathcal{G}_{n-1}^{\alpha-1}\Big) - \lambda\Big(\sum_{i=0}^{n-1}\mathcal{G}_i^{\alpha-1} - \sum_{i=0}^{n-1}\mathcal{G}_i^{\alpha-1}\Big)\right) = 0.
\end{aligned}$$

Consider the first line of (4.123) which we rewrite as

$$\begin{aligned}
&\lambda' A_{h,1}^\alpha g_h(x) - \frac{\lambda}{h}A_{h,0}^{\alpha-1}g_h(x) - \lambda\lambda'\Big(A\mathcal{P}(1) + (P(0)+a)\Big) \\
&= (1-\lambda)A_{h,1}^\alpha\mathcal{P}(x) - \frac{\lambda}{h}A_{h,0}^{\alpha-1}\mathcal{P}(x) - (1-\lambda)\lambda\Big(A\mathcal{P}(1) + (P(0)+a)\Big)
\end{aligned}$$

$$+ (P(0)k_h + a_h)\Big((1-\lambda)A^{\alpha}_{h,1}\vartheta^{\alpha}_h(x) - \frac{\lambda}{h}A^{\alpha-1}_{h,0}\vartheta^{\alpha}_h(x)\Big). \tag{4.124}$$

Using (4.105) and (4.110), the first line of (4.124) becomes

$$\begin{aligned}
&(1-\lambda)A^{\alpha}_{h,1}\mathcal{P}(x) - \frac{\lambda}{h}A^{\alpha-1}_{h,0}\mathcal{P}(x) - (1-\lambda)\lambda\Big(A\mathcal{P}(1) + (P(0)+a)\Big)\\
&= (1-\lambda)\left(A\mathcal{P}(1) + O(h)\right)\\
&\qquad - \frac{\lambda}{h}\left(D^{\alpha-1}_c\mathcal{P}(1) + h\Big((\lambda-2) - \frac{\alpha-1}{2}\Big)A\mathcal{P}(1) + O(h^2)\right)\\
&\qquad\qquad - (1-\lambda)\lambda\Big(A\mathcal{P}(1) + (P(0)+a)\Big)\\
&= A\mathcal{P}(1) + (P(0)+a)\\
&\qquad + (P(0)+a)\left(\frac{\lambda}{h} - 1 - \lambda + \lambda^2)\right) + \lambda\frac{\alpha-1}{2}A\mathcal{P}(1) + O(h)
\end{aligned}$$

In view of Proposition 4.5.7

$$\begin{aligned}
A^{\alpha}_{h,1}\vartheta^{\alpha}_h(x) &= 1 + (\lambda-1)\mathcal{G}^{\alpha}_n = 1 + O(h^{\alpha+1})\\
A^{\alpha-1}_{h,0}\vartheta^{\alpha}_h(x) &= h\left(n + \lambda - 2 + (\lambda-1)\mathcal{G}^{\alpha-1}_{n-1}\right) = 1 + (\lambda-3)h + O(h^{\alpha+1}).
\end{aligned}$$

Thus, using (4.118), the second line of (4.124) becomes

$$\begin{aligned}
&(P(0)k_h + a_h)\Big((1-\lambda)A^{\alpha}_{h,1}\vartheta^{\alpha}_h(x) - \frac{\lambda}{h}A^{\alpha-1}_{h,0}\vartheta^{\alpha}_h(x)\Big)\\
&= \left((P(0)+a)(1+h) + h\frac{\alpha-1}{2}A\mathcal{P}(1)\right)\Big(-\frac{\lambda}{h} + 1 + 2\lambda - \lambda^2 + O(h^{\alpha+1})\Big)\\
&= -(P(0)+a)\left(\frac{\lambda}{h} - 1 - \lambda + \lambda^2)\right) - \lambda\frac{\alpha-1}{2}A\mathcal{P}(1) + O(h).
\end{aligned}$$

Putting the two terms together and using (4.112), the first line of (4.123) (rewritten as (4.124)) becomes

$$\begin{aligned}
&(1-\lambda)A^{\alpha}_{h,1}\mathcal{P}(x) - \frac{\lambda}{h}A^{\alpha-1}_{h,0}\mathcal{P}(x) - (1-\lambda)\lambda\Big(A\mathcal{P}(1) + (P(0)+a)\\
&= A\mathcal{P}(1) + (P(0)+a) + O(h)\\
&= A\mathcal{P}(x) + (P(0)+a) + O(h)
\end{aligned}$$

and so

$$G^h f_h(x) = A\mathcal{P}(x) + (P(0)+a) + O(h^{\alpha-1}).$$

Hence, for $x \in [1-2h, 1-h)$

$$\left|G^h f_h(x) - Af(x)\right| = \left|G^h f_h(x) - A\mathcal{P}(x) + (P(0)+a)\right| = O(h^{\alpha-1}).$$

Proof of $G^h f_h \to Af$ on $[1-h,1]$: Using (4.122) and (4.109), on setting

$$g_h = \mathcal{P} + (P(0)k_h + a_h)\vartheta_h^\alpha$$

let us rewrite as

$$\begin{aligned}
G^h f_h(x) &= G^h\left(\mathcal{P}(x) + (P(0)k_h + a_h)\vartheta_h^\alpha(x)\right) + dG^h p_0(x) + G^h e_h(x) \\
&= -\frac{1}{h}A_{h,0}^{\alpha-1} g_h(x) + \lambda\Big(A\mathcal{P}(1) + (P(0)+a)\Big) \\
&\qquad\qquad + d\Bigg(-\frac{1}{h}A_{h,0}^{\alpha-1} p_0(x) \qquad\qquad (4.125)\\
&\qquad\qquad\quad + \frac{1}{h^\alpha}\Big(\mathcal{G}_n^{\alpha-1} p_0(\lambda h) + \sum_{i=0}^{n-1} \mathcal{G}_i^{\alpha-1} p_0((\lambda+1)h)\Big)\Bigg) \\
&\qquad\qquad + \frac{1}{h^\alpha}\Big(\mathcal{G}_n^{\alpha-1} g_h(\lambda h) + \sum_{i=0}^{n-1} \mathcal{G}_i^{\alpha-1} g_h((\lambda+1)h)\Big) \qquad\qquad (4.126)
\end{aligned}$$

Since $|g_h(\lambda h)| = O(h^\alpha)$, $\left|\sum_{i=0}^{n-1} \mathcal{G}_i^{\alpha-1}\right| = O(h^{\alpha-1})$ and $|\mathcal{G}_n^{\alpha-1}| = O(h^\alpha)$, the last term of (4.125),

$$\left|\frac{1}{h^\alpha}\Big(\mathcal{G}_n^{\alpha-1} g_h(\lambda h) + \sum_{i=0}^{n-1} \mathcal{G}_i^{\alpha-1} g_h((\lambda+1)h)\Big)\right| = O(h^{\alpha-1}).$$

Next, the second term of (4.125)

$$\begin{aligned}
&d\Bigg(-\frac{1}{h}A_{h,0}^{\alpha-1} p_0(x) + \frac{1}{h^\alpha}\Big(\mathcal{G}_n^{\alpha-1} p_0(\lambda h) + \sum_{i=0}^{n-1} \mathcal{G}_i^{\alpha-1} p_0((\lambda+1)h)\Big)\Bigg) \\
&= d\Bigg(-\frac{1}{h^\alpha}\Big(\sum_{i=0}^{n} \mathcal{G}_i^{\alpha-1} - \mathcal{G}_n^{\alpha-1} - \sum_{i=0}^{n-1} \mathcal{G}_i^{\alpha-1}\Big)\Bigg) = 0.
\end{aligned}$$

In view of Proposition 4.5.7

$$A_{h,0}^{\alpha-1}\vartheta_h^\alpha(x) = h\left(n + 1 + \lambda - 2 + (\lambda-1)\mathcal{G}_{n-1}^{\alpha-1}\right) = 1 + (\lambda-2)h + O(h^{\alpha+1}).$$

Consider the first term of (4.125), then using (4.105) and (4.111) we have

$$\begin{aligned}
&-\frac{1}{h}A_{h,0}^{\alpha-1} g_h(x) + \lambda\Big(A\mathcal{P}(1) + (P(0)+a)\Big) \\
&= -\frac{1}{h}A_{h,0}^{\alpha-1}\mathcal{P}(x) - (P(0)k_h + a_h)\frac{1}{h}A_{h,0}^{\alpha-1}\vartheta_h^\alpha(x) + \lambda\Big(A\mathcal{P}(1) + (P(0)+a)\Big) \\
&= -\frac{1}{h}\Big(D_c^{\alpha-1}\mathcal{P}(1) + h\Big((\lambda-1) - \frac{\alpha-1}{2}\Big)A\mathcal{P}(1) + O(h^2)\Big) \\
&\qquad - (P(0)k_h + a_h)\frac{1}{h}A_{h,0}^{\alpha-1}\vartheta_h^\alpha(x) + \lambda\Big(A\mathcal{P}(1) + (P(0)+a)\Big) \\
&= \frac{1}{h}(P(0)+a) - (\lambda-1)A\mathcal{P}(1) + \frac{\alpha-1}{2}A\mathcal{P}(1)
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{1}{h}\Big((P(0)+a)(1+h) + h\frac{\alpha-1}{2}A\mathcal{P}(1)\Big)\Big(1+(\lambda-2)h+O(h^{\alpha+1})\Big)\\
&\qquad\qquad + \lambda A\mathcal{P}(1) + \lambda(P(0)+a) + O(h)\\
&= \frac{1}{h}(P(0)+a) - (\lambda-1)A\mathcal{P}(1) + \frac{\alpha-1}{2}A\mathcal{P}(1)\\
&\quad - \frac{1}{h}(P(0)+a) - (P(0)+a) - \frac{\alpha-1}{2}A\mathcal{P}(1) - (\lambda-2)(P(0)+a)\\
&\qquad\qquad + \lambda A\mathcal{P}(1) + \lambda(P(0)+a) + O(h)\\
&= A\mathcal{P}(1) + (P(0)+a) + O(h).
\end{aligned}
$$

Therefore, using (4.112), since $A\mathcal{P}(1) = A\mathcal{P}(x) + O(h)$ we have

$$G^h f_h(x) = A\mathcal{P}(x) + (P(0)+a) + O(h^{\alpha-1}).$$

Thus, in view of (4.112) , for $x \in [1-h, 1]$

$$\left|G^h f_h(x) - Af(x)\right| = O(h^{\alpha-1}).$$

Hence,

$$G^h f_h = G^h\left(\mathcal{P} + (P(0)k_h + a_h)\vartheta_h^\alpha + dp_0\right) \to A\mathcal{P} + (P(0)+a)p_0 = Af$$

uniformly on Ω_2; that is, the proof of (4.97) is complete. This also completes the proof of Statement 1 of Proposition 4.3.2 for the operator (A, NN^*), in view of (4.49) and (4.96).

The proof of Statement 1 of Proposition 4.3.2 for all the fractional derivative operators on $C_0(\Omega)$ given in Table 4.5 is complete. □

4.5.3 Proof of Proposition 4.3.2 for the case $X = L_1[0,1]$

Proof. To simplify notation, we write $G^h := G^{h,\leftrightarrow}$ and $A := A^{\leftrightarrow}$. For each of the fractional derivative operators $(A^{\leftrightarrow}, \mathrm{BC}) := (A, \mathrm{BC})$ on $L_1[0,1]$ given in Table 4.7, and for each $f \in \mathcal{C}(A, \mathrm{BC})$ we show that there exists a sequence $\{f_h\} \subset L_1[0,1]$ such that $f_h \to f$ and $G^h f_h \to Af$ in $L_1[0,1]$. To this end, let $f \in \mathcal{C}(A, \mathrm{BC})$ as in Theorem 3.4.4,

$$f = \mathcal{Q} + ap_\alpha + bp_{\alpha-1} + cp_{\alpha-2} + dp_0, \tag{4.127}$$

where $\mathcal{Q} = I^\alpha P$ for some polynomial $P = \sum_{m=0}^N k_m p_m$. Then, in view of (3.17),

$$Af = A\mathcal{Q} + ap_0 = P + ap_0. \tag{4.128}$$

Keep in mind that we repeatedly make use of the following:

$(A^{+}, \mathrm{BC}),\ C_0(\Omega)$	$(A^{-}, \mathrm{BC}),\ L_1[0,1]$	$(A^{\leftrightarrow}, \mathrm{BC}),\ L_1[0,1]$
$(D_c^{\alpha,+}, \mathrm{DN})$	$(D_c^{\alpha,-}, \mathrm{DN})$	$(D_c^{\alpha,\leftrightarrow}, \mathrm{ND})$
$(D_c^{\alpha,+}, \mathrm{NN})$	$(D_c^{\alpha,-}, \mathrm{NN})$	$(D_c^{\alpha,\leftrightarrow}, \mathrm{NN})$
$(D_c^{\alpha,+}, \mathrm{DD})$	$(D^{\alpha,-}, \mathrm{DD}) = (D_c^{\alpha,-}, \mathrm{DD})$	$(D_c^{\alpha,\leftrightarrow}, \mathrm{DD})$
$(D_c^{\alpha,+}, \mathrm{ND})$	$(D^{\alpha,-}, \mathrm{ND}) = (D_c^{\alpha,-}, \mathrm{ND})$	$(D_c^{\alpha,\leftrightarrow}, \mathrm{DN})$
$(D_c^{\alpha,+}, \mathrm{DN}^*)$	$(D^{\alpha,-}, \mathrm{DN})$	$(D^{\alpha,\leftrightarrow}, \mathrm{ND})$
$(D_c^{\alpha,+}, \mathrm{NN}^*)$	$(D^{\alpha,-}, \mathrm{NN})$	$(D^{\alpha,\leftrightarrow}, \mathrm{NN})$

Table 4.7: Corresponding fractional derivative operators on $C_0(\Omega)$ and $L_1[0,1]$.

1. The Grünwald formula given by (4.29),
$$A_{h,1}^{\alpha} f(x) = \frac{1}{h^{\alpha}} \sum_{k=0}^{\iota(x)} \mathcal{G}_k^{\alpha} f((\lambda(x) + \iota(x) - 1 - (k-1))h)$$
where $x = (\lambda(x) + \iota(x) - 1)h$ as given by Definition (3.2.2).

2. The canonical extension of $\mathcal{Q}$ and for $x \in [nh, 1]$, the Taylor expansions of $\mathcal{Q}$ around $x = 1$ given by
$$\begin{aligned} \mathcal{Q}(x-h) &= \mathcal{Q}(1) - (2-\lambda)h\mathcal{Q}'(1) + O(h^2), \\ \mathcal{Q}(x) &= \mathcal{Q}(1) - (1-\lambda)h\mathcal{Q}'(1) + O(h^2), \\ \mathcal{Q}(x+h) &= \mathcal{Q}(1) + \lambda h\mathcal{Q}'(1) + O(h^2), \end{aligned} \tag{4.129}$$
where $\lambda := \lambda(x) = \lambda(x-h) = \lambda(x+h)$.

3. The interpolation matrix given by (4.12),

$$G_{n+1}^{h,\leftrightarrow}(\lambda) =$$
$$\frac{1}{h^{\alpha}} \begin{pmatrix} b_1^l & N^l(\lambda)\mathcal{G}_0^{\alpha} & 0 & \cdots & \cdots & \cdots & 0 \\ D^l(\lambda)b_2^l & \lambda' b_1^l + \lambda\mathcal{G}_1^{\alpha} & \mathcal{G}_0^{\alpha} & 0 & \cdots & \cdots & 0 \\ D^l(\lambda)b_3^l & \lambda' b_2^l + \lambda\mathcal{G}_2^{\alpha} & \mathcal{G}_1^{\alpha} & \mathcal{G}_0^{\alpha} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ D^l(\lambda)b_i^l & \lambda' b_{i-1}^l + \lambda\mathcal{G}_{i-1}^{\alpha} & \mathcal{G}_{i-2}^{\alpha} & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ D^l(\lambda)b_n & \lambda' b_{n-1}^l + \lambda b_{n-1}^r & \lambda'\mathcal{G}_{n-2}^{\alpha} + \lambda b_{n-2}^r & \cdots & \cdots & \lambda'\mathcal{G}_1^{\alpha} + \lambda b_1^r & D^r(\lambda)\mathcal{G}_0^{\alpha} \\ 0 & N^r(\lambda)b_n & N^r(\lambda)b_{n-1}^r & \cdots & \cdots & N^r(\lambda)b_2^r & b_1^r \end{pmatrix}$$

$X = L_1[0,1],\ D(A,\mathrm{BC}) = \{I^\alpha g + ap_\alpha + bp_{\alpha-1} + cp_{\alpha-2} + dp_0 : g \in L_1[0,1]\}$		
Boundary condition	Boundary weights for $G^h_{n\times n}$	$D(A,\mathrm{BC})$
$f(0) = 0$	$b^l_i = \mathcal{G}^\alpha_i$ $D^l(\lambda) = \frac{\alpha\lambda}{\alpha\lambda+\lambda'},\ N^l = \mathbf{1}$	$a = 0,\ c = 0,\ d = 0$
$D^{\alpha-1}_c f(0) = 0$	$b^l_i = -\mathcal{G}^{\alpha-1}_{i-1}$ $N^l(\lambda) = \lambda,\ D^l = \mathbf{1}$	$b = 0,\ c = 0$
$D^{\alpha-1} f(0) = 0$	$b^l_0 = 0,\ b^l_1 = \mathcal{G}^\alpha_0 + \mathcal{G}^\alpha_1$ $b^l_i = \mathcal{G}^\alpha_i$ $N^l(\lambda) = \lambda,\ D^l = \mathbf{1}$	$b = 0,\ d = 0$
$f(1) = 0$	$b^r_i = \mathcal{G}^\alpha_i$ $b_n = b^l_n$ $D^r(\lambda) = \frac{\alpha\lambda'}{\alpha\lambda'+\lambda},\ N^r = \mathbf{1}$	$a = 0,\ \frac{b+(\alpha-1)c}{\Gamma(\alpha)} + d = -I^\alpha g(1)$
$Ff(1) = 0$	$b^r_i = -\mathcal{G}^{\alpha-1}_{i-1}$ $b_n = -\sum_{i=0}^{n-1} b^l_i$ $N^r(\lambda) = \lambda',\ D^r = \mathbf{1}$	$a + b = -Ig(1)$

Table 4.8: Boundary conditions for $L_1[0,1]$.

4. The approximate power functions ϑ^β_h, $\beta \in \{\alpha, \alpha-1, \alpha-2, 0\}$ given by Definition 4.5.1 and their canonical extensions when required.

5. The error terms for the Grünwald-type approximation of the fractional derivative given in Remark 4.5.6,

$$
\begin{aligned}
&A^{\alpha-1}_{h,0}\mathcal{Q}(x) = A^{\alpha-1}_{h,\lambda-2}\mathcal{Q}(1) = D^{\alpha-1}\mathcal{Q}(1) + O(h),\ x \in [(n-1)h, nh) \\
&A^{\alpha-1}_{h,0}\mathcal{Q}(x) = A^{\alpha-1}_{h,\lambda-1}\mathcal{Q}(1) \\
&\qquad\qquad = D^{\alpha-1}\mathcal{Q}(1) + h\Big((\lambda-1) - \frac{\alpha-1}{2}\Big)D^\alpha\mathcal{Q}(1) + O(h^2),\ x \in [nh, 1], \\
&D^\alpha\mathcal{Q}(1) = A^\alpha_{h,1}\mathcal{Q}(x) + O(h),\ x \in [(n-1)h, 1]
\end{aligned} \tag{4.130}
$$

and the same hold with D^β replaced by D^β_c.

6. Table 4.8 for the boundary weights $b^l_i,\ b^r_i,\ b_n$, the constants a, b, c, d and the interpolating functions $D^l,\ D^r,\ N^l,\ N^r$.

Common properties for operators $(A, \mathrm{D}\bullet)$**:**

We first consider the common properties of the operators with a left Dirichlet boundary condition, $(A, \mathrm{D}\bullet)$. Following that we consider the right boundary conditions one by one and deal with the operators (A, DD) and (A, DN) separately.

In these cases, since $a, c, d = 0$, $f \in \mathcal{C}(A, \mathrm{D}\bullet)$ given by (4.127) reduces to

$$f = \mathcal{Q} + bp_{\alpha-1}. \tag{4.131}$$

Moreover, Af given by (4.128) reduces to

$$Af = A\mathcal{Q}. \tag{4.132}$$

Next, we take

$$f_h = \mathcal{Q} + b_h \vartheta_h^{\alpha-1} + e_h, \tag{4.133}$$

where the function $e_h = \mathbf{0}$ except when the right boundary condition is Neumann. Next, note that $D^l(\lambda) = \frac{\alpha\lambda}{\alpha\lambda+\lambda'}$, $N^l = \mathbf{1}$ and $b_i^l = \mathcal{G}_i^\alpha$. Next, observe that the approximate power function $\vartheta_h^{\alpha-1}$ in the $L_1[0,1]$-case is constructed such that the following three identities hold as they are crucial for both cases, (A, DD) and (A, DN).

For $x \in [0, h)$,

$$\begin{aligned}
&\frac{1}{h^\alpha}\Bigg(\mathcal{G}_1^\alpha \vartheta_h^{\alpha-1}(\lambda h) + \mathcal{G}_0^\alpha \vartheta_h^{\alpha-1}((\lambda+1)h)\Bigg) \\
&= \frac{1}{h}\Bigg(\mathcal{G}_1^\alpha\Big(\frac{\lambda'\mathcal{G}_0^{-\alpha} + \lambda\mathcal{G}_1^{-\alpha}}{\alpha}\Big) + \mathcal{G}_0^\alpha\big(\lambda'\mathcal{G}_0^{-\alpha} + \lambda\mathcal{G}_1^{-\alpha}\big)\Bigg) \\
&= -\frac{1}{h}\Bigg(\alpha\Big(\frac{\lambda'\mathcal{G}_0^{-\alpha} + \lambda\mathcal{G}_1^{-\alpha}}{\alpha}\Big) + \big(\lambda'\mathcal{G}_0^{-\alpha} + \lambda\mathcal{G}_1^{-\alpha}\big)\Bigg) = 0.
\end{aligned} \tag{4.134}$$

For $x \in [h, 1]$,

$$\begin{aligned}
&\frac{1}{h^\alpha}\Bigg(D^l(\lambda)\mathcal{G}_{\iota(x)}^\alpha \vartheta_h^{\alpha-1}(\lambda h) + \sum_{k=0}^{\iota(x)-1} \mathcal{G}_k^\alpha \vartheta_h^{\alpha-1}((\lambda + \iota(x) - 1 - (k-1))h)\Bigg) \\
&= \frac{1}{h}\Bigg(D^l(\lambda)\mathcal{G}_{\iota(x)}^\alpha\Big(\frac{\lambda'\mathcal{G}_0^{-\alpha} + \lambda\mathcal{G}_1^{-\alpha}}{\alpha}\Big) + \sum_{k=0}^{\iota(x)-1} \mathcal{G}_k^\alpha\big(\lambda'\mathcal{G}_{\iota(x)-1+k}^{-\alpha} + \lambda\mathcal{G}_{\iota(x)-k}^{-\alpha}\big)\Bigg) \\
&= \frac{1}{h}\Bigg(\lambda\mathcal{G}_{\iota(x)}^\alpha + \lambda' \sum_{k=0}^{\iota(x)-1} \mathcal{G}_k^\alpha \mathcal{G}_{\iota(x)-1+k}^{-\alpha} + \lambda \sum_{k=0}^{\iota(x)-1} \mathcal{G}_k^\alpha \mathcal{G}_{\iota(x)-k}^{-\alpha}\Bigg) \\
&= \frac{1}{h}\Bigg(\lambda' \sum_{k=0}^{\iota(x)-1} \mathcal{G}_k^\alpha \mathcal{G}_{\iota(x)-1+k}^{-\alpha} + \lambda \sum_{k=0}^{\iota(x)} \mathcal{G}_k^\alpha \mathcal{G}_{\iota(x)-k}^{-\alpha}\Bigg)
\end{aligned}$$

$$= \frac{1}{h}\left(\lambda'\mathcal{G}^0_{\iota(x)-1} + \lambda\mathcal{G}^0_{\iota(x)}\right) = 0, \tag{4.135}$$

where we have used (A.8) and the fact that $\mathcal{G}^0_k = 0$ for $k \geq 1$.

For $x \in [(n-1)h, 1]$,

$$\begin{aligned}
&\frac{1}{h^\alpha}\left(D^l(\lambda)\mathcal{G}^{\alpha-1}_{\iota(x)-1}\vartheta^{\alpha-1}_h(\lambda h) + \sum_{k=0}^{\iota(x)-2}\mathcal{G}^{\alpha-1}_k\vartheta^{\alpha-1}_h((\lambda+\iota(x)-1-k)h)\right)\\
&= \frac{1}{h}\left(D^l(\lambda)\mathcal{G}^{\alpha-1}_{\iota(x)-1}\Big(\frac{\lambda'\mathcal{G}^{-\alpha}_0 + \lambda\mathcal{G}^{-\alpha}_1}{\alpha}\Big) + \sum_{k=0}^{\iota(x)-2}\mathcal{G}^{\alpha-1}_k\big(\lambda'\mathcal{G}^{-\alpha}_{\iota(x)-2+k} + \lambda\mathcal{G}^{-\alpha}_{\iota(x)-1-k}\big)\right)\\
&= \frac{1}{h}\left(\lambda\mathcal{G}^{\alpha-1}_{\iota(x)-1} + \lambda'\sum_{k=0}^{\iota(x)-2}\mathcal{G}^{\alpha-1}_k\mathcal{G}^{-\alpha}_{\iota(x)-2+k} + \lambda\sum_{k=0}^{\iota(x)-2}\mathcal{G}^{\alpha-1}_k\mathcal{G}^{-\alpha}_{\iota(x)-1-k}\right)\\
&= \frac{1}{h}\left(\lambda'\sum_{k=0}^{\iota(x)-2}\mathcal{G}^{\alpha-1}_k\mathcal{G}^{-\alpha}_{\iota(x)-2+k} + \lambda\sum_{k=0}^{\iota(x)-1}\mathcal{G}^{\alpha-1}_k\mathcal{G}^{-\alpha}_{\iota(x)-1-k}\right)\\
&= \frac{1}{h}\left(\lambda'\mathcal{G}^{-1}_{\iota(x)-1} + \lambda\mathcal{G}^{-1}_{\iota(x)}\right) = \frac{\lambda'+\lambda}{h} = \frac{1}{h},
\end{aligned} \tag{4.136}$$

where we have used (A.8) and the fact that $\mathcal{G}^{-1}_k = 1$ for $k \geq 0$.

Proof of Statement 2 **of Proposition 4.3.2 for the operator** (A, DD)**:**

In this case, we further have that $D^r(\lambda) = \frac{\alpha\lambda'}{\alpha\lambda'+\lambda}$, $N^r = \mathbf{1}$, $b^r_i = \mathcal{G}^\alpha_i$ and $b_n = \mathcal{G}^\alpha_n$. Moreover, the relation $\frac{b}{\Gamma(\alpha)} = -I^\alpha g(1)$ for $f \in \mathcal{C}(A, \mathrm{DD})$ reads

$$\frac{b}{\Gamma(\alpha)} = -\mathcal{Q}(1) \tag{4.137}$$

We do not require the function e_h and so we take $f_h = \mathcal{Q} + b_h\vartheta^{\alpha-1}_h$, where

$$b_h = \frac{b}{\Gamma(\alpha)\vartheta^{\alpha-1}_h(1)}. \tag{4.138}$$

This implies that $f_h \to f$ in $L_1[0,1]$, since $\left\|\vartheta^{\alpha-1}_h - p_{\alpha-1}\right\|_{L_1[0,1]} \to 0$ in view of Lemma 4.5.3. To show $\left\|G^h f_h - Af\right\|_{L_1[0,1]} \to 0$, we show the following:

1.

$$G^h b_h\vartheta^{\alpha-1}_h(x) = \begin{cases} 0, & x \in [0,(n-1)h),\\ -\frac{(D^r(\lambda)-1)\mathcal{Q}(1)}{h^\alpha} + O(h^{1-\alpha}), & x \in [(n-1)h, nh),\\ \frac{\mathcal{Q}(1)}{h^\alpha} + O(h^{1-\alpha}), & x \in [nh, 1].\end{cases} \tag{4.139}$$

2.

$$G^h\mathcal{Q}(x)$$

$$
= \begin{cases} A^\alpha_{h,1}\mathcal{Q}(x), & x\in[0,h),\\ A^\alpha_{h,1}\mathcal{Q}(x) + \frac{(D^l(\lambda)-1)\mathcal{G}^\alpha_{\iota(x)}\mathcal{Q}(\lambda h)}{h^\alpha}, & x\in[h,(n-1)h),\\ A^\alpha_{h,1}\mathcal{Q}(x) + \frac{(D^r(\lambda)-1)\mathcal{Q}(1)}{h^\alpha} & \\ \qquad\qquad + \frac{(D^l(\lambda)-1)\mathcal{G}^\alpha_n\mathcal{Q}(\lambda h)}{h^\alpha} + O(h^{1-\alpha}), & x\in[(n-1)h,nh),\\ A^\alpha_{h,1}\mathcal{Q}(x) - \frac{\mathcal{Q}(1)}{h^\alpha} - \frac{\mathcal{G}^\alpha_{n+1}\mathcal{Q}(\lambda h)}{h^\alpha} + O(h^{1-\alpha}), & x\in[nh,1]. \end{cases} \tag{4.140}
$$

Then, using (4.132), (4.139) and (4.140) note that

$$
\begin{aligned}
&\left\|G^h f_h - Af\right\|_{L_1[0,1]}\\
&= \int_0^1 \left|G^h\mathcal{Q}(x) + G^h b_h\vartheta_h^{\alpha-1}(x) - A\mathcal{Q}(x)\right| \mathrm{d}x\\
&\le \int_0^1 \left|A^\alpha_{h,1}\mathcal{Q}(x) - A\mathcal{Q}(x)\right| \mathrm{d}x + \sum_{i=2}^{n+1}\int_{(i-1)h}^{ih}\left|\frac{\mathcal{Q}(\lambda h)\mathcal{G}^\alpha_i}{h^\alpha}\right| \mathrm{d}x + O(h^{2-\alpha})\\
&\le \left\|A^\alpha_{h,1}\mathcal{Q} - A\mathcal{Q}\right\|_{L_1[0,1]} + O(h^{2-\alpha}),
\end{aligned}
$$

since $\mathcal{Q}(\lambda h) = O(h^\alpha)$ and

$$
h\sum_{i=0}^{\infty}|\mathcal{G}^\alpha_i| \to 0
$$

in view of (A.10). Using Corollary 4.5.5, as $h\to 0$, $\left\|A^\alpha_{h,1}\mathcal{Q} - A\mathcal{Q}\right\|_{L_1[0,1]} \to 0$, and hence $\left\|G^h f_h - Af\right\|_{L_1[0,1]} \to 0$ in the case of (A,DD).

Proof of (4.139): For $x\in[0,h)$, in view of (4.134)

$$
G^h b_h\vartheta^{\alpha-1}(x) = 0.
$$

For $x\in[h,(n-1)h)$, in view of (4.135),

$$
\begin{aligned}
G^h b_h\vartheta^{\alpha-1}(x) = \frac{b_h}{h}\Bigg(& D^l(\lambda)\mathcal{G}^\alpha_{\iota(x)}\Big(\frac{\lambda'\mathcal{G}^{-\alpha}_0 + \lambda\mathcal{G}^{-\alpha}_1)}{\alpha}\Big)\\
&+ \sum_{k=0}^{\iota(x)-1}\mathcal{G}^\alpha_k\big(\lambda'\mathcal{G}^{-\alpha}_{\iota(x)-1+k} + \lambda\mathcal{G}^{-\alpha}_{\iota(x)-k}\big)\Bigg) = 0.
\end{aligned}
$$

Next, let $x\in[(n-1)h,nh)$, then using (4.135) with $\iota(x)=n$ and (4.26) we have

$$
\begin{aligned}
G^h b_h\vartheta^{\alpha-1}(x) &= \frac{b_h}{h}\Bigg(D^l(\lambda)\mathcal{G}^\alpha_n\Big(\frac{\lambda'\mathcal{G}^{-\alpha}_0 + \lambda\mathcal{G}^{-\alpha}_1)}{\alpha}\Big) + \sum_{k=0}^{n-1}\mathcal{G}^\alpha_k\big(\lambda'\mathcal{G}^{-\alpha}_{n-1+k} + \lambda\mathcal{G}^{-\alpha}_{n-k}\big)\Bigg)\\
&\qquad + \frac{b_h(D^r(\lambda)-1)\vartheta_h^{\alpha-1}((\lambda+n)h)}{h^\alpha}\\
&= \frac{b(D^r(\lambda)-1)\vartheta_h^{\alpha-1}((\lambda+n)h)}{h^\alpha\Gamma(\alpha)\vartheta_h^{\alpha-1}(1)} = \frac{b(D^r(\lambda)-1)}{h^\alpha\Gamma(\alpha)}\Big(1 - \frac{\lambda'(\alpha-1)}{n-1+\alpha}\Big)
\end{aligned}
$$

$$= \frac{b(D^r(\lambda)-1)}{h^\alpha\Gamma(\alpha)} + O(h^{1-\alpha}) = -\frac{(D^r(\lambda)-1)\mathcal{Q}(1)}{h^\alpha} + O(h^{1-\alpha}),$$

where we have also used (4.137) in the last line. Lastly, let $x \in [nh, 1]$, then using $\mathcal{G}_{n+1}^\alpha = O(h^{\alpha+1})$, $\vartheta_h^{\alpha-1}(\lambda h) = O(h^{\alpha-1})$ for the first term below, followed by (4.135) with $\iota(x) = n+1$ and (4.26) we have

$$\begin{aligned}
G^h b_h \vartheta^{\alpha-1}(x) &= \frac{-b_h}{h^\alpha} D^l(\lambda)\mathcal{G}_{n+1}^\alpha \vartheta_h^{\alpha-1}(\lambda h) \\
&\quad + \frac{b_h}{h}\left(D^l(\lambda)\mathcal{G}_{n+1}^\alpha \Big(\frac{\lambda'\mathcal{G}_0^{-\alpha} + \lambda\mathcal{G}_1^{-\alpha}}{\alpha} \Big) + \sum_{k=0}^{n} \mathcal{G}_k^\alpha \big(\lambda'\mathcal{G}_{n+k}^{-\alpha} + \lambda\mathcal{G}_{n+1-k}^{-\alpha} \big) \right) \\
&\qquad - \frac{b_h \vartheta_h^{\alpha-1}((\lambda+n+1)h)}{h^\alpha} \\
&= -\frac{b\vartheta_h^{\alpha-1}((\lambda+n+1)h)}{h^\alpha\Gamma(\alpha)\vartheta_h^{\alpha-1}(1)} + O(h^\alpha) = -\frac{b}{h^\alpha\Gamma(\alpha)}\Big(1 + \frac{\lambda(\alpha-1)}{n+1}\Big) + O(h^\alpha) \\
&= -\frac{b}{h^\alpha\Gamma(\alpha)} + O(h^{1-\alpha}) = \frac{\mathcal{Q}(1)}{h^\alpha} + O(h^{1-\alpha}),
\end{aligned}$$

where we have also used (4.137) in the last line.

Proof of (4.140): Observe that

$$G^h\mathcal{Q}(x) = \begin{cases} A_{h,1}^\alpha \mathcal{Q}(x), & x \in [0, h), \\ A_{h,1}^\alpha \mathcal{Q}(x) + \frac{(D^l(\lambda)-1)\mathcal{G}_{\iota(x)}^\alpha \mathcal{Q}(\lambda h)}{h^\alpha}, & x \in [h, (n-1)h). \end{cases}$$

Therefore, let $x \in [(n-1)h, nh)$, then using the Taylor expansion (4.129) we have

$$\begin{aligned}
G^h\mathcal{Q}(x) &= \frac{1}{h^\alpha}\left((D^l(\lambda)-1)\mathcal{G}_n^\alpha \mathcal{Q}(\lambda h) + \sum_{k=0}^{n} \mathcal{G}_k^\alpha \mathcal{Q}\big((\lambda+n-1-(k-1))h\big) \right. \\
&\qquad \left. + (D^r(\lambda)-1)\mathcal{Q}((\lambda+n)h) \right) \\
&= \frac{1}{h^\alpha}\left((D^l(\lambda)-1)\mathcal{G}_n^\alpha \mathcal{Q}(\lambda h) + \sum_{k=0}^{n} \mathcal{G}_k^\alpha \mathcal{Q}\big((\lambda+n-1-(k-1))h\big) \right. \\
&\qquad \left. + (D^r(\lambda)-1)\big(\mathcal{Q}(1) + O(h)\big) \right) \\
&= A_{h,1}^\alpha \mathcal{Q}(x) + \frac{(D^r(\lambda)-1)\mathcal{Q}(1)}{h^\alpha} + \frac{(D^l(\lambda)-1)\mathcal{G}_n^\alpha \mathcal{Q}(\lambda h)}{h^\alpha} + O(h^{1-\alpha}).
\end{aligned}$$

Lastly, let $x \in [nh, 1]$, then using the Taylor expansion (4.129) we have

$$G^h\mathcal{Q}(x) = \frac{1}{h^\alpha}\left(-\mathcal{G}_{n+1}^\alpha \mathcal{Q}(\lambda h) + \sum_{k=0}^{n+1} \mathcal{G}_k^\alpha \mathcal{Q}\big((\lambda+n-(k-1))h\big) \right.$$

$$
\left. - \mathcal{Q}((\lambda+n+1)h) \right)
$$

$$
\begin{aligned}
&= \frac{-\mathcal{G}^{\alpha}_{n+1}\mathcal{Q}(\lambda h)}{h^{\alpha}} + A^{\alpha}_{h,1}\mathcal{Q}(x) - \frac{\mathcal{Q}(1)+O(h)}{h^{\alpha}} \\
&= A^{\alpha}_{h,1}\mathcal{Q}(x) - \frac{\mathcal{Q}(1)}{h^{\alpha}} - \frac{\mathcal{G}^{\alpha}_{n+1}\mathcal{Q}(\lambda h)}{h^{\alpha}} + O(h^{1-\alpha}).
\end{aligned}
$$

Proof of Statement 2 of Proposition 4.3.2 for the operator (A, DN)**:**

In this case, we further have that $N^r(\lambda) = \lambda'$, $D^r = \mathbf{1}$, $b^r_i = -\mathcal{G}^{\alpha-1}_{i-1}$ and $b_n = -\sum_{i=0}^{n-1}\mathcal{G}^{\alpha}_i = -\mathcal{G}^{\alpha-1}_{n-1}$. Moreover, the relation $b = -Ig(1)$ for $f \in \mathcal{C}(A, \mathrm{DN})$ reads

$$
b = -D^{\alpha-1}_c\mathcal{Q}(1). \tag{4.141}
$$

Let $f_h = \mathcal{Q} + b\vartheta^{\alpha-1}_h + e_h$. When dealing with a right Neumann boundary condition it turns out that we require the term e_h for $G^h f_h$ to converge which is constructed as follows:

$$
e_h(x) = \begin{cases} 0, & \text{if } x \in [0, nh), \\ -\lambda\big(\mathcal{Q}((\lambda+n)h) + b\vartheta^{\alpha-1}_h((\lambda+n)h)\big), & \text{if } x \in [nh, 1]. \end{cases} \tag{4.142}
$$

Then, it follows that as $h \to 0$, $\|e_h\|_{L_1[0,1]} \to 0$. This also implies that $f_h \to f$ in $L_1[0,1]$, since in view of Lemma 4.5.3, $\left\|\vartheta^{\alpha-1}_h - p_{\alpha-1}\right\|_{L_1[0,1]} \to 0$.

Moreover, observe that

$$
\begin{aligned}
&G^h e_h(x) \\
&= \begin{cases} 0, & \text{if } x \in [0, (n-1)h), \\ -\frac{\lambda}{h^{\alpha}}\big(\mathcal{Q}((\lambda+n)h) + b\vartheta^{\alpha-1}_h((\lambda+n)h)\big), & \text{if } x \in [(n-1)h, nh), \\ \frac{\lambda}{h^{\alpha}}\big(\mathcal{Q}((\lambda+n)h) + b\vartheta^{\alpha-1}_h((\lambda+n)h)\big), & \text{if } x \in [nh, 1]. \end{cases}
\end{aligned} \tag{4.143}
$$

To show that $\left\|G^h f_h - Af\right\|_{L_1[0,1]} \to 0$, we show the following:

1.

$$
\begin{aligned}
&G^h b\vartheta^{\alpha-1}_h(x) \\
&= \begin{cases} 0, & x \in [0, (n-1)h), \\ -\frac{\lambda b}{h} + \frac{\lambda b}{h^{\alpha}}\vartheta^{\alpha-1}_h((\lambda+n)h) + O(h^{\alpha-1}), & x \in [(n-1)h, nh), \\ -\frac{\lambda' b}{h} - \frac{\lambda b}{h^{\alpha}}\vartheta^{\alpha-1}_h((\lambda+n)h) + O(h^{\alpha-1}), & x \in [nh, 1]. \end{cases}
\end{aligned} \tag{4.144}
$$

2.

$$
G^h\mathcal{Q}(x)
$$

$$= \begin{cases} A^{\alpha}_{h,1}\mathcal{Q}(x), & x \in [0,h), \\ A^{\alpha}_{h,1}\mathcal{Q}(x) + \frac{(D^l(\lambda)-1)\mathcal{G}^{\alpha}_{\iota(x)}\mathcal{Q}(\lambda h)}{h^{\alpha}}, & x \in [h,(n-1)h), \\ A^{\alpha}_{h,1}\mathcal{Q}(x) + \frac{\left((\lambda - D^l(\lambda))\mathcal{G}^{\alpha-1}_{n-1} - \lambda'\mathcal{G}^{\alpha}_{n}\right)\mathcal{Q}(\lambda h)}{h^{\alpha}} & \\ \qquad + \frac{\lambda b}{h} + \frac{\lambda \mathcal{Q}((\lambda+n)h))}{h^{\alpha}} + C, & x \in [(n-1)h, nh), \\ A^{\alpha}_{h,1}\mathcal{Q}(x) + \frac{\lambda' \mathcal{G}^{\alpha-1}_{n}\mathcal{Q}(\lambda h)}{h^{\alpha}} & \\ \qquad + \frac{\lambda' b}{h} - \frac{\lambda \mathcal{Q}((\lambda+n)h)}{h^{\alpha}} + C, & x \in [nh, 1]. \end{cases} \tag{4.145}$$

Then, using (4.132), (4.143), (4.144) and (4.145) we have

$$\begin{aligned} &\left\|G^h f_h - Af\right\|_{L_1[0,1]} \\ &= \int_0^1 \left|G^h\mathcal{Q}(x) + G^h b_h \vartheta^{\alpha-1}_h(x) + G^h e_h(x) - A\mathcal{Q}(x)\right| \mathrm{d}x \\ &\le \int_0^1 \left|A^{\alpha}_{h,1}\mathcal{Q}(x) - A\mathcal{Q}(x)\right| \mathrm{d}x + \sum_{i=2}^{n-1}\int_{(i-1)h}^{ih} \left|\frac{\mathcal{Q}(\lambda h)\mathcal{G}^{\alpha}_i}{h^{\alpha}}\right| \mathrm{d}x \\ &\qquad\qquad + \int_{(n-1)h}^{nh} \left|\frac{\mathcal{Q}(\lambda h)\mathcal{G}^{\alpha-1}_n}{h^{\alpha}}\right| \mathrm{d}x + \int_{nh}^{1} \left|\frac{\mathcal{Q}(\lambda h)\mathcal{G}^{\alpha-1}_n}{h^{\alpha}}\right| \mathrm{d}x \\ &\le \left\|A^{\alpha}_{h,1}\mathcal{Q} - A\mathcal{Q}\right\|_{L_1[0,1]} + O(h), \end{aligned}$$

since $\mathcal{Q}(\lambda h) = O(h^{\alpha})$ and $\sum_{i=2}^{n+1} |\mathcal{G}^{\alpha}_i| < \infty$ in view of (A.10). Using Corollary 4.5.5, as $h \to 0$, $\left\|A^{\alpha}_{h,1}\mathcal{Q} - A\mathcal{Q}\right\|_{L_1[0,1]} \to 0$, and hence $\left\|G^h f_h - Af\right\|_{L_1[0,1]} \to 0$ in the case of (A, DN).

Proof of (4.144): For $x \in [0,(n-1)h)$, we showed in (4.134) and (4.135) that

$$G^h b_h \vartheta^{\alpha-1}(x) = 0.$$

Next, let $x \in [(n-1)h, nh)$, then rewriting $-\mathcal{G}^{\alpha-1}_{n-1} = -\lambda\mathcal{G}^{\alpha-1}_{n-1} - \lambda'\mathcal{G}^{\alpha-1}_{n} + \lambda'\mathcal{G}^{\alpha}_{n}$, and using (4.135) and (4.136) with $\iota(x) = n$, we have

$$\begin{aligned} G^h b \vartheta^{\alpha-1}(x) = \frac{b}{h^{\alpha}}\Bigg(&- D^l(\lambda)\mathcal{G}^{\alpha-1}_{n-1}\vartheta^{\alpha-1}_h(\lambda h) \\ &+ \sum_{k=1}^{n-1}(\lambda'\mathcal{G}^{\alpha}_k - \lambda\mathcal{G}^{\alpha-1}_{k-1})\vartheta^{\alpha-1}_h((\lambda + n - 1 - (k-1))h) \\ &+ \mathcal{G}^{\alpha}_0 \vartheta^{\alpha-1}_h((\lambda+n)h)\Bigg) \\ = \frac{b}{h^{\alpha}}\Bigg(&- \lambda' D^l(\lambda)\mathcal{G}^{\alpha-1}_n\vartheta^{\alpha-1}_h(\lambda h) \end{aligned}$$

$$+\lambda'\Big(D^l(\lambda)\mathcal{G}_n^\alpha\vartheta_h^{\alpha-1}(\lambda h)+\sum_{k=0}^{n-1}\mathcal{G}_k^\alpha\vartheta_h^{\alpha-1}((\lambda+n-1-(k-1))h)\Big)$$
$$-\lambda\Big(D^l(\lambda)\mathcal{G}_{n-1}^{\alpha-1}\vartheta_h^{\alpha-1}(\lambda h)+\sum_{k=0}^{n-2}\mathcal{G}_k^{\alpha-1}\vartheta_h^{\alpha-1}((\lambda+n-1-k)h)\Big)$$
$$\left.+\lambda\mathcal{G}_0^\alpha\vartheta_h^{\alpha-1}((\lambda+n)h)\right)$$
$$=-\frac{\lambda b}{h}+\frac{\lambda b}{h^\alpha}\vartheta_h^{\alpha-1}((\lambda+n)h)-\frac{b}{h^\alpha}\lambda'D^l(\lambda)\mathcal{G}_n^{\alpha-1}\vartheta_h^{\alpha-1}(\lambda h)$$
$$=-\frac{\lambda b}{h}+\frac{\lambda b}{h^\alpha}\vartheta_h^{\alpha-1}((\lambda+n)h)+O(h^{\alpha-1}),$$

since $\mathcal{G}_n^{\alpha-1}=O(h^\alpha)$ and $\vartheta_h^{\alpha-1}(\lambda h)=O(h^{\alpha-1})$. Lastly, let $x\in[nh,1]$, then using (4.136) we have

$$G^h b\vartheta^{\alpha-1}(x)=\frac{b}{h^\alpha}\left(-N^r(\lambda)\sum_{k=1}^{n-1}\mathcal{G}_k^{\alpha-1}\vartheta_h^{\alpha-1}((\lambda+n-k)h)\right.$$
$$\left.-\mathcal{G}_0^{\alpha-1}\vartheta_h^{\alpha-1}((\lambda+n)h)\right)$$
$$=\frac{b}{h^\alpha}\Big(\lambda'D^l(\lambda)\mathcal{G}_n^{\alpha-1}\vartheta_h^{\alpha-1}(\lambda h)-\lambda\mathcal{G}_0^{\alpha-1}\vartheta_h^{\alpha-1}((\lambda+n)h)\Big)$$
$$-\frac{\lambda'b}{h^\alpha}\left(D^l(\lambda)\mathcal{G}_n^{\alpha-1}\vartheta_h^{\alpha-1}(\lambda h)+\sum_{k=0}^{n-1}\mathcal{G}_k^{\alpha-1}\vartheta_h^{\alpha-1}((\lambda+n-k)h)\right)$$
$$=-\frac{\lambda'b}{h}-\frac{\lambda b}{h^\alpha}\vartheta_h^{\alpha-1}((\lambda+n)h)+\frac{\lambda'b}{h^\alpha}D^l(\lambda)\mathcal{G}_n^{\alpha-1}\vartheta_h^{\alpha-1}(\lambda h)$$
$$=-\frac{\lambda'b}{h}-\frac{\lambda b}{h^\alpha}\vartheta_h^{\alpha-1}((\lambda+n)h)+O(h^{\alpha-1}),$$

since $\mathcal{G}_n^{\alpha-1}=O(h^\alpha)$ and $\vartheta_h^{\alpha-1}(\lambda h)=O(h^{\alpha-1})$.

Proof of (4.145): Observe that

$$G^h\mathcal{Q}(x)=\begin{cases} A_{h,1}^\alpha\mathcal{Q}(x), & x\in[0,h),\\ A_{h,1}^\alpha\mathcal{Q}(x)+\frac{(D^l(\lambda)-1)\mathcal{G}_{\iota(x)}^\alpha\mathcal{Q}(\lambda h)}{h^\alpha}, & x\in[h,(n-1)h).\end{cases}$$

Let $x\in[(n-1)h,nh)$, then using (4.130) we have

$$G^h\mathcal{Q}(x)=\frac{1}{h^\alpha}\left(-D^l(\lambda)\mathcal{G}_{n-1}^{\alpha-1}\mathcal{Q}(\lambda h)+\sum_{k=1}^{n-1}(\lambda'\mathcal{G}_k^\alpha-\lambda\mathcal{G}_{k-1}^{\alpha-1})\mathcal{Q}((\lambda+n-1-(k-1))h)\right.$$
$$\left.+\mathcal{G}_0^\alpha\mathcal{Q}((\lambda+n)h)\right)$$
$$=\frac{1}{h^\alpha}\bigg(\Big(-D^l(\lambda)\mathcal{G}_{n-1}^{\alpha-1}-\lambda'\mathcal{G}_n^\alpha+\lambda\mathcal{G}_{n-1}^{\alpha-1}\Big)\mathcal{Q}(\lambda h)$$

$$
\begin{aligned}
&\quad + \lambda' \sum_{k=0}^{n} \mathcal{G}_k^{\alpha} \mathcal{Q}((\lambda+n-1-(k-1))h) \\
&\qquad - \lambda \sum_{k=0}^{n-1} \mathcal{G}_{k-1}^{\alpha-1}) \mathcal{Q}((\lambda+n-1-k)h) + \lambda \mathcal{G}_0^{\alpha} \mathcal{Q}((\lambda+n)h))\Bigg) \\
&= \frac{\Big((\lambda - D^l(\lambda))\mathcal{G}_{n-1}^{\alpha-1} - \lambda' \mathcal{G}_n^{\alpha}\Big)\mathcal{Q}(\lambda h)}{h^{\alpha}} + \frac{\lambda \mathcal{Q}((\lambda+n)h))}{h^{\alpha}} \\
&\qquad + \lambda' A_{h,1}^{\alpha} \mathcal{Q}(x) - \frac{\lambda}{h} A_{h,0}^{\alpha-1} \mathcal{Q}(x) \\
&= \frac{\Big((\lambda - D^l(\lambda))\mathcal{G}_{n-1}^{\alpha-1} - \lambda' \mathcal{G}_n^{\alpha}\Big)\mathcal{Q}(\lambda h)}{h^{\alpha}} + \frac{\lambda \mathcal{Q}((\lambda+n)h))}{h^{\alpha}} \\
&\qquad + A_{h,1}^{\alpha} \mathcal{Q}(x) + \frac{\lambda b}{h} + C.
\end{aligned}
$$

Lastly, let $x \in [nh, 1]$, then using (4.130) we have

$$
\begin{aligned}
G^h \mathcal{Q}(x) &= \frac{1}{h^{\alpha}} \Bigg(- N^r(\lambda) \sum_{k=1}^{n-1} \mathcal{G}_k^{\alpha-1} \mathcal{Q}((\lambda+n-k)h) \\
&\qquad\qquad - \mathcal{G}_0^{\alpha-1} \mathcal{Q}((\lambda+n)h) \Bigg) \\
&= \frac{1}{h^{\alpha}} \Big(\lambda' \mathcal{G}_n^{\alpha-1} \mathcal{Q}(\lambda h) - \lambda \mathcal{G}_0^{\alpha-1} \mathcal{Q}((\lambda+n)h) \Big) \\
&\qquad\qquad - \frac{\lambda'}{h^{\alpha}} \Bigg(\sum_{k=0}^{n} \mathcal{G}_k^{\alpha-1} \mathcal{Q}((\lambda+n-k)h) \Bigg) \\
&= \frac{1}{h^{\alpha}} \Big(\lambda' \mathcal{G}_n^{\alpha-1} \mathcal{Q}(\lambda h) - \lambda \mathcal{Q}((\lambda+n)h) \Big) - \frac{\lambda'}{h} A_{h,0}^{\alpha-1} \mathcal{Q}(x) \\
&= \frac{1}{h^{\alpha}} \Big(\lambda' \mathcal{G}_n^{\alpha-1} \mathcal{Q}(\lambda h) - \lambda \mathcal{Q}((\lambda+n)h) \Big) + \frac{\lambda' b}{h} + A_{h,1}^{\alpha} \mathcal{Q}(x) + C.
\end{aligned}
$$

Proof of Statement 2 of Proposition 4.3.2 for the operator $(D_c^{\alpha}, \mathrm{ND})$**:** In this case, since $a, b, c = 0$, $f \in \mathcal{C}(D_c^{\alpha}, \mathrm{ND})$ given by (4.127) reduces to

$$f = \mathcal{Q} + dp_0. \tag{4.146}$$

Moreover, Af given by (4.128) reduces to

$$D_c^{\alpha} f = D_c^{\alpha} \mathcal{Q}. \tag{4.147}$$

Next, we take

$$f_h = \mathcal{Q} + d\vartheta_h^0. \tag{4.148}$$

Then, $f_h \to f$ in $L_1[0,1]$, since in view of Lemma 4.5.3, $\vartheta_h^0 \to p_0$ in $L_1[0,1]$.

Note that $N^l(\lambda) = \lambda$, $D^l = \mathbf{1} = N^r$, $D^r(\lambda) = \frac{\alpha\lambda'}{\alpha\lambda'+\lambda}$, $b_i^l = -\mathcal{G}_{i-1}^{\alpha-1}$, $b_i^r = \mathcal{G}_i^\alpha$ and $b_n = -\mathcal{G}_{n-1}^{\alpha-1}$. Moreover, the relation $d = -I^\alpha g(1)$ for $f \in \mathcal{C}(D_c^\alpha, \mathrm{ND})$ reads

$$d = -\mathcal{Q}(1). \tag{4.149}$$

To show that $\left\|G^h f_h - D_c^\alpha f\right\|_{L_1[0,1]} \to 0$, we show the following:

1.
$$G^h d\vartheta_h^0(x) = \begin{cases} 0, & x \in [0,(n-1)h), \\ -\frac{D^r(\lambda)-1}{h^\alpha}\mathcal{Q}(1), & x \in [(n-1)h, nh), \\ \frac{1}{h^\alpha}\mathcal{Q}(1), & x \in [nh, 1]. \end{cases} \tag{4.150}$$

2.
$$\begin{aligned} &G^h\mathcal{Q}(x) \\ &= \begin{cases} A_{h,1}^\alpha \mathcal{Q}(x) \\ \qquad +\frac{1}{h^\alpha}\big(-\mathcal{G}_{\iota(x)}^{\alpha-1}\mathcal{Q}(\lambda h) - \lambda'\mathcal{G}_{\iota(x)-1}^{\alpha-1}\mathcal{Q}((\lambda+1)h)\big), & x \in [0,(n-1)h), \\ A_{h,1}^\alpha \mathcal{Q}(x) + \frac{1}{h^\alpha}\big(-\mathcal{G}_n^{\alpha-1}\mathcal{Q}(\lambda h) - \lambda'\mathcal{G}_{n-1}^{\alpha-1}\mathcal{Q}((\lambda+1)h)\big) \\ \qquad\qquad +\frac{D^r(\lambda)-1}{h^\alpha}\mathcal{Q}(1) + O(h^{1-\alpha}), & x \in [(n-1)h, nh), \\ A_{h,1}^\alpha \mathcal{Q}(x) + \frac{1}{h^\alpha}\big(-\mathcal{G}_{n+1}^{\alpha}\mathcal{Q}(\lambda h) - \mathcal{G}_n^{\alpha-1}\mathcal{Q}((\lambda+1)h)\big) \\ \qquad\qquad -\frac{1}{h^\alpha}\mathcal{Q}(1) + O(h^{1-\alpha}), & x \in [nh, 1]. \end{cases} \end{aligned} \tag{4.151}$$

Then, using (4.147), (4.150) and (4.151) we have

$$\begin{aligned} &\left\|G^h f_h - D_c^\alpha f\right\|_{L_1[0,1]} \\ &= \int_0^1 \left|G^h\mathcal{Q}(x) + G^h d\vartheta_h^0(x) - D_c^\alpha\mathcal{Q}(x)\right| \mathrm{d}x \\ &\le \int_0^1 \left|A_{h,1}^\alpha\mathcal{Q}(x) - D_c^\alpha\mathcal{Q}(x)\right| \mathrm{d}x + \sum_{i=1}^{n+1}\int_{(i-1)h}^{ih}\left|\frac{\mathcal{Q}(\lambda h)\mathcal{G}_i^{\alpha-1} + \mathcal{Q}((\lambda+1)h)\mathcal{G}_{i-1}^{\alpha-1}}{h^\alpha}\right| \mathrm{d}x \\ &\qquad\qquad\qquad\qquad\qquad\qquad + O(h^{2-\alpha}) \\ &\le \left\|A_{h,1}^\alpha\mathcal{Q} - D_c^\alpha\mathcal{Q}\right\|_{L_1[0,1]} + O(h^{2-\alpha}), \end{aligned}$$

since $\mathcal{Q}(\lambda h)$, $\mathcal{Q}((\lambda+1)h) = O(h^\alpha)$ and $\sum_{i=0}^{n+1}\left|\mathcal{G}_i^{\alpha-1}\right| < \infty$ in view of (A.10). Using Corollary 4.5.5, as $h \to 0$, $\left\|A_{h,1}^\alpha\mathcal{Q} - D_c^\alpha\mathcal{Q}\right\|_{L_1[0,1]} \to 0$. Hence,

$$\left\|G^h f_h - D_c^\alpha f\right\|_{L_1[0,1]} \to 0$$

in the case of $(D_c^\alpha, \mathrm{ND})$.

Proof of (4.150): Let $x \in [0,h)$, then

$$G^h d\vartheta_h^0(x) = \frac{d}{h^\alpha}\big(- \mathcal{G}_0^{\alpha-1}\lambda + \lambda\mathcal{G}_0^{\alpha}\big) = 0.$$

Let $x \in [h,(n-1)h)$, then using (A.7) and $\mathcal{G}_{\iota(x)-1}^{\alpha-1} = \mathcal{G}_{\iota(x)-2}^{\alpha-1} + \mathcal{G}_{\iota(x)-1}^{\alpha}$ we have

$$\begin{aligned}G^h d\vartheta_h^0(x) &= \frac{d}{h^\alpha}\Big(- \lambda\mathcal{G}_{\iota(x)-1}^{\alpha-1} - \lambda'\mathcal{G}_{\iota(x)-2}^{\alpha-1} + \lambda\mathcal{G}_{\iota(x)-1}^{\alpha} + \sum_{k=0}^{\iota(x)-2}\mathcal{G}_k^\alpha\Big)\\ &= \frac{d}{h^\alpha}\Big(- \lambda\mathcal{G}_{\iota(x)-1}^{\alpha-1} - \mathcal{G}_{\iota(x)-2}^{\alpha-1} + \lambda\mathcal{G}_{\iota(x)-2}^{\alpha-1} + \lambda\mathcal{G}_{\iota(x)-1}^{\alpha} + \mathcal{G}_{\iota(x)-2}^{\alpha-1}\Big) = 0.\end{aligned}$$

Next, using (A.7) and $\mathcal{G}_{n-1}^{\alpha-1} = \mathcal{G}_{n-2}^{\alpha-1} + \mathcal{G}_{n-1}^{\alpha}$, observe that for $x \in [(n-1)h, nh)$,

$$\begin{aligned}G^h d\vartheta_h^0(x) &= \frac{d}{h^\alpha}\Big(- \lambda\mathcal{G}_{n-1}^{\alpha-1} - \lambda'\mathcal{G}_{n-2}^{\alpha-1} + \lambda\mathcal{G}_{n-1}^{\alpha} + \sum_{k=1}^{n-2}\mathcal{G}_k^\alpha + D^r(\lambda)\mathcal{G}_0^\alpha\Big)\\ &= \frac{d}{h^\alpha}\Big(- \lambda\mathcal{G}_{n-1}^{\alpha-1} - \mathcal{G}_{n-2}^{\alpha-1} + \lambda\mathcal{G}_{n-2}^{\alpha-1} + \lambda\mathcal{G}_{n-1}^{\alpha} + \sum_{k=0}^{n-2}\mathcal{G}_k^\alpha + (D^r(\lambda)-1)\Big)\\ &= -\frac{\mathcal{Q}(1)}{h^\alpha}\big(D^r(\lambda)-1\big).\end{aligned}$$

and for $x \in [nh,1]$,

$$G^h d\vartheta_h^0(x) = \frac{d}{h^\alpha}\Big(- \mathcal{G}_{n-1}^{\alpha-1} + \sum_{k=1}^{n-1}\mathcal{G}_k^\alpha\Big) = \frac{\mathcal{Q}(1)}{h^\alpha}.$$

Proof of (4.151): Let $x \in [0,h)$, then

$$\begin{aligned}G^h\mathcal{Q}(x) &= \frac{-\mathcal{G}_0^{\alpha-1}\mathcal{Q}(\lambda h) + \lambda\mathcal{G}_0^\alpha\mathcal{Q}((\lambda+1)h)}{h^\alpha}\\ &= \frac{-\mathcal{G}_1^{\alpha-1}\mathcal{Q}(\lambda h) - \lambda'\mathcal{G}_0^{\alpha-1}\mathcal{Q}((\lambda+1)h)}{h^\alpha} + A_{h,1}^\alpha\mathcal{Q}(x)\end{aligned}$$

Let $x \in [h,(n-1)h)$, then using $\mathcal{G}_k^{\alpha-1} = \mathcal{G}_{k-1}^{\alpha-1} + \mathcal{G}_{k-1}^{\alpha}$

$$\begin{aligned}G^h\mathcal{Q}(x) &= \frac{1}{h^\alpha}\Bigg(- \mathcal{G}_{\iota(x)-1}^{\alpha-1}\mathcal{Q}(\lambda h) + \big(- \lambda'\mathcal{G}_{\iota(x)-2}^{\alpha-1} + \lambda\mathcal{G}_{\iota(x)-1}^{\alpha}\big)\mathcal{Q}((\lambda+1)h)\\ &\qquad + \sum_{k=0}^{\iota(x)-2}\mathcal{G}_k^\alpha\mathcal{Q}((\lambda+\iota(x)-1-(k-1))h)\Bigg)\\ &= \frac{1}{h^\alpha}\Bigg(- \mathcal{G}_{\iota(x)}^{\alpha-1}\mathcal{Q}(\lambda h) - \lambda'\mathcal{G}_{\iota(x)-1}^{\alpha-1}\mathcal{Q}((\lambda+1)h)\\ &\qquad + \sum_{k=0}^{\iota(x)}\mathcal{G}_k^\alpha\mathcal{Q}((\lambda+\iota(x)-1-(k-1))h)\Bigg)\end{aligned}$$

$$= \frac{1}{h^\alpha}\Big(- \mathcal{G}^{\alpha-1}_{\iota(x)}\mathcal{Q}(\lambda h) - \lambda'\mathcal{G}^{\alpha-1}_{\iota(x)-1}\mathcal{Q}((\lambda+1)h)\Big) + A^\alpha_{h,1}\mathcal{Q}(x).$$

Let $x \in [(n-1)h, nh)$, then using the Taylor expansion (4.129) we have

$$\begin{aligned}
G^h\mathcal{Q}(x) &= \frac{1}{h^\alpha}\Bigg(- \mathcal{G}^{\alpha-1}_{n-1}\mathcal{Q}(\lambda h) + \big(- \lambda'\mathcal{G}^{\alpha-1}_{n-2} + \lambda\mathcal{G}^{\alpha}_{n-1}\big)\mathcal{Q}((\lambda+1)h) \\
&\qquad + \sum_{k=1}^{n-2}\mathcal{G}^\alpha_k\mathcal{Q}((\lambda+n-1-(k-1))h) + D^r(\lambda)\mathcal{G}^\alpha_0\mathcal{Q}((\lambda+n)h)\Bigg) \\
&= \frac{1}{h^\alpha}\Bigg(- \mathcal{G}^{\alpha-1}_{n}\mathcal{Q}(\lambda h) - \lambda'\mathcal{G}^{\alpha-1}_{n-1}\mathcal{Q}((\lambda+1)h) + (D^r(\lambda)-1)\mathcal{G}^\alpha_0\mathcal{Q}((\lambda+n)h) \\
&\qquad + \sum_{k=0}^{n}\mathcal{G}^\alpha_k\mathcal{Q}((\lambda+n-1-(k-1))h)\Bigg) \\
&= \frac{1}{h^\alpha}\Big(- \mathcal{G}^{\alpha-1}_{\iota(x)}\mathcal{Q}(\lambda h) - \lambda'\mathcal{G}^{\alpha-1}_{\iota(x)-1}\mathcal{Q}((\lambda+1)h)\Big) \\
&\qquad + A^\alpha_{h,1}\mathcal{Q}(x) + \frac{(D^r(\lambda)-1)\mathcal{Q}((\lambda+n)h)}{h^\alpha} \\
&= \frac{1}{h^\alpha}\Big(- \mathcal{G}^{\alpha-1}_{\iota(x)}\mathcal{Q}(\lambda h) - \lambda'\mathcal{G}^{\alpha-1}_{\iota(x)-1}\mathcal{Q}((\lambda+1)h)\Big) \\
&\qquad + A^\alpha_{h,1}\mathcal{Q}(x) + \frac{(D^r(\lambda)-1)\mathcal{Q}(1)}{h^\alpha} + O(h^{1-\alpha}).
\end{aligned}$$

Let $x \in [nh, 1]$, then using the Taylor expansion (4.129) we have

$$\begin{aligned}
G^h\mathcal{Q}(x) &= \frac{1}{h^\alpha}\Bigg(- \mathcal{G}^{\alpha-1}_{n-1}\mathcal{Q}((\lambda+1)h) + \sum_{k=1}^{n-1}\mathcal{G}^\alpha_k\mathcal{Q}((\lambda+n-(k-1))h)\Bigg) \\
&= \frac{1}{h^\alpha}\Bigg(- \mathcal{G}^{\alpha}_{n+1}\mathcal{Q}(\lambda h) - \mathcal{G}^{\alpha-1}_{n}\mathcal{Q}((\lambda+1)h) - \mathcal{G}^\alpha_0\mathcal{Q}((\lambda+n+1)h) \\
&\qquad + \sum_{k=0}^{n+1}\mathcal{G}^\alpha_k\mathcal{Q}((\lambda+n-(k-1))h)\Bigg) \\
&= \frac{1}{h^\alpha}\Bigg(- \mathcal{G}^{\alpha}_{n+1}\mathcal{Q}(\lambda h) - \mathcal{G}^{\alpha-1}_{n}\mathcal{Q}((\lambda+1)h)\Bigg) + A^\alpha_{h,1}\mathcal{Q}(x) \\
&\qquad - \frac{1}{h^\alpha}\mathcal{Q}(1) + O(h^{1-\alpha}).
\end{aligned}$$

Proof of Statement 2 of Proposition 4.3.2 for the operator $(D^\alpha_c, \mathrm{NN})$:
In this case, since $b, c = 0$, $f \in \mathcal{C}(D^\alpha_c, \mathrm{NN})$ given by (4.127) reduces to

$$f = \mathcal{Q} + ap_\alpha + dp_0 \tag{4.152}$$

and Af given by (4.128) reads

$$D^\alpha_c f = D^\alpha_c\mathcal{Q} + ap_0. \tag{4.153}$$

Note that $N^l(\lambda) = \lambda$, $D^l = \mathbf{1} = D^r$, $N^r(\lambda) = 1-\lambda$, $b_i^l = -\mathcal{G}_{i-1}^{\alpha-1} = b_i^r$ and $b_n = -\sum_{i=0}^{n-1} b_i^l = \sum_{i=0}^{n-2} \mathcal{G}_i^{\alpha-1}$. Moreover, the relation $a = -Ig(1)$ for $f \in \mathcal{C}(D_c^\alpha, \mathrm{NN})$ reads

$$a = -D_c^{\alpha-1}\mathcal{Q}(1). \tag{4.154}$$

Next, we take

$$f_h = \mathcal{Q} + a\vartheta_h^\alpha + d\vartheta_h^0 + e_h. \tag{4.155}$$

When dealing with a right Neumann boundary condition we require the term e_h construced as follows:

$$e_h(x) = \begin{cases} 0, & \text{if } x \in [0, nh), \\ -\lambda\big(\mathcal{Q}((\lambda+n)h) + a\vartheta_h^\alpha((\lambda+n)h) + d\big), & \text{if } x \in [nh, 1]. \end{cases} \tag{4.156}$$

Then, as $h \to 0$, $\|e_h\|_{L_1[0,1]} \to 0$. This also implies that $f_h \to f$ in $L_1[0,1]$, since in view of Lemma 4.5.3, $\vartheta_h^\alpha \to p_\alpha$ and $\vartheta_h^0 \to p_0$ in $L_1[0,1]$.

Further, observe that

$$G^h e_h(x) = \begin{cases} 0, & \text{if } x \in [0, (n-1)h), \\ -\frac{\lambda}{h^\alpha}\big(\mathcal{Q}((\lambda+n)h) + a\vartheta_h^\alpha((\lambda+n)h) + d\big), & \text{if } x \in [(n-1)h, nh), \\ \frac{\lambda}{h^\alpha}\big(\mathcal{Q}((\lambda+n)h) + a\vartheta_h^\alpha((\lambda+n)h) + d\big), & \text{if } x \in [nh, 1]. \end{cases} \tag{4.157}$$

To show that $\left\|G^h f_h - D_c^\alpha f\right\|_{L_1[0,1]} \to 0$, we show the following:

1.

$$G^h d\vartheta_h^0(x) = \begin{cases} 0, & x \in [0, (n-1)h), \\ \frac{d\lambda}{h^\alpha}, & x \in [(n-1)h, nh), \\ -\frac{d\lambda}{h^\alpha}, & x \in [nh, 1]. \end{cases} \tag{4.158}$$

2.

$$\begin{aligned} &G^h a\vartheta_h^\alpha(x) \\ &= \begin{cases} a, & x \in [0, (n-1)h), \\ a + \frac{\lambda}{h} D_c^{\alpha-1}\mathcal{Q}(1) + \frac{\lambda a}{h^\alpha}\vartheta_h^\alpha((\lambda+n)h) + O(1), & x \in [(n-1)h, nh), \\ a + \frac{\lambda'}{h} D_c^{\alpha-1}\mathcal{Q}(1) - \frac{\lambda a}{h^\alpha}\vartheta_h^\alpha((\lambda+n)h) + O(1), & x \in [nh, 1]. \end{cases} \end{aligned} \tag{4.159}$$

3.

$$G^h\mathcal{Q}(x)$$

$$
= \begin{cases}
A^{\alpha}_{h,1}\mathcal{Q}(x) \\
\qquad +\frac{1}{h^{\alpha}}\big(-\mathcal{G}^{\alpha-1}_{\iota(x)}\mathcal{Q}(\lambda h) - \lambda'\mathcal{G}^{\alpha-1}_{\iota(x)-1}\mathcal{Q}((\lambda+1)h)\big), & x\in[0,(n-1)h), \\
A^{\alpha}_{h,1}\mathcal{Q}(x) - \frac{\lambda}{h}D^{\alpha-1}_{c}\mathcal{Q}(1) + \frac{\lambda}{h^{\alpha}}\mathcal{Q}((\lambda+n)h) + O(1) \\
\qquad +\frac{1}{h^{\alpha}}\Bigg(\Big(\sum_{i=0}^{n-2}\mathcal{G}^{\alpha-1}_{i} - \lambda'\mathcal{G}^{\alpha}_{n} + \lambda\mathcal{G}^{\alpha-1}_{n-1}\Big)\mathcal{Q}(\lambda h) \\
\qquad\qquad -\lambda'\mathcal{G}^{\alpha-1}_{n-1}\mathcal{Q}((\lambda+1)h)\Bigg), & x\in[(n-1)h,nh), \\
A^{\alpha}_{h,1}\mathcal{Q}(x) - \frac{\lambda'}{h}D^{\alpha-1}_{c}\mathcal{Q}(1) - \frac{\lambda}{h^{\alpha}}\mathcal{Q}((\lambda+n)h) + O(1) \\
\qquad +\frac{1}{h^{\alpha}}\Bigg(\lambda'\mathcal{G}^{\alpha-1}_{n}\mathcal{Q}(\lambda h) + \lambda'\sum_{i=0}^{n-1}\mathcal{G}^{\alpha-1}_{i}\mathcal{Q}((\lambda+1)h)\Bigg), & x\in[nh,1].
\end{cases}
\tag{4.160}
$$

Then, using (4.153), (4.158), (4.159) and (4.160) we have

$$
\begin{aligned}
&\left\|G^{h}f_{h} - D^{\alpha}_{c}f\right\|_{L_1[0,1]} \\
&= \int_0^1 \left|G^{h}\mathcal{Q}(x) + G^{h}a\vartheta^{\alpha}_{h}(x) + G^{h}d\vartheta^{0}_{h}(x) - \big(D^{\alpha}_{c}\mathcal{Q}(x) + ap_0(x)\big)\right|\mathrm{d}x \\
&\le \int_0^1 \left|A^{\alpha}_{h,1}\mathcal{Q}(x) - D^{\alpha}_{c}\mathcal{Q}(x)\right|\mathrm{d}x \\
&\qquad\qquad + \sum_{i=1}^{n-1}\int_{(i-1)h}^{ih}\left|\frac{\mathcal{Q}(\lambda h)\mathcal{G}^{\alpha-1}_{i} + \mathcal{Q}((\lambda+1)h)\mathcal{G}^{\alpha-1}_{i-1}}{h^{\alpha}}\right|\mathrm{d}x + O(h) \\
&\le \left\|A^{\alpha}_{h,1}\mathcal{Q} - D^{\alpha}_{c}\mathcal{Q}\right\|_{L_1[0,1]} + O(h),
\end{aligned}
$$

since $\mathcal{Q}(\lambda h),\ \mathcal{Q}((\lambda+1)h) = O(h^{\alpha})$ and $\sum_{i=0}^{n+1}\left|\mathcal{G}^{\alpha-1}_{i}\right| < \infty$ in view of (A.10). Using Corollary 4.5.5, as $h\to 0$, $\left\|A^{\alpha}_{h,1}\mathcal{Q} - D^{\alpha}_{c}\mathcal{Q}\right\|_{L_1[0,1]} \to 0$. Hence,

$$
\left\|G^{h}f_{h} - D^{\alpha}_{c}f\right\|_{L_1[0,1]} \to 0
$$

in the case of $(D^{\alpha}_{c}, \mathrm{NN})$.

Proof of (4.158): Let $x\in[0,h)$, then

$$
G^{h}d\vartheta^{0}_{h}(x) = \frac{d}{h^{\alpha}}\big(-\mathcal{G}^{\alpha-1}_{0}\lambda + \lambda\mathcal{G}^{\alpha}_{0}\big) = 0.
$$

Let $x\in[h,(n-1)h)$, then using (A.7) and $\mathcal{G}^{\alpha-1}_{\iota(x)-1} = \mathcal{G}^{\alpha-1}_{\iota(x)-2} + \mathcal{G}^{\alpha}_{\iota(x)-1}$ we have

$$
\begin{aligned}
G^{h}d\vartheta^{0}_{h}(x) &= \frac{d}{h^{\alpha}}\Big(-\lambda\mathcal{G}^{\alpha-1}_{\iota(x)-1} - \lambda'\mathcal{G}^{\alpha-1}_{\iota(x)-2} + \lambda\mathcal{G}^{\alpha}_{\iota(x)-1} + \sum_{k=0}^{\iota(x)-2}\mathcal{G}^{\alpha}_{k}\Big) \\
&= \frac{d}{h^{\alpha}}\Big(-\lambda\mathcal{G}^{\alpha-1}_{\iota(x)-1} - \mathcal{G}^{\alpha-1}_{\iota(x)-2} + \lambda\mathcal{G}^{\alpha-1}_{\iota(x)-2} + \lambda\mathcal{G}^{\alpha}_{\iota(x)-1} + \mathcal{G}^{\alpha-1}_{\iota(x)-2}\Big) = 0.
\end{aligned}
$$

Next, using (A.7) observe that for $x \in [(n-1)h, nh)$,

$$\begin{aligned}
G^h d\vartheta_h^0(x) &= \frac{d}{h^\alpha}\Big(\lambda\sum_{i=0}^{n-2}\mathcal{G}_i^{\alpha-1} - \lambda'\mathcal{G}_{n-2}^{\alpha-1} - \lambda\mathcal{G}_{n-2}^{\alpha-1} + \lambda'\sum_{k=1}^{n-2}\mathcal{G}_k^{\alpha} - \lambda\sum_{k=0}^{n-3}\mathcal{G}_k^{\alpha-1} + \mathcal{G}_0^{\alpha}\Big)\\
&= \frac{d}{h^\alpha}\Big(-\lambda'\mathcal{G}_{n-2}^{\alpha-1} + \lambda'\mathcal{G}_{n-2}^{\alpha-1} + (1-\lambda')\mathcal{G}_0^{\alpha}\Big) = \frac{d\lambda}{h^\alpha}
\end{aligned}$$

and for $x \in [nh, 1]$,

$$G^h d\vartheta_h^0(x) = \frac{d}{h^\alpha}\Big(\lambda'\sum_{i=0}^{n-2}\mathcal{G}_i^{\alpha-1} - \lambda'\sum_{k=1}^{n-2}\mathcal{G}_k^{\alpha-1} - \mathcal{G}_0^{\alpha-1}\Big) = -\frac{d\lambda}{h^\alpha}.$$

Proof of (4.159) : Let $x \in [0, h)$, then

$$G^h a\vartheta_h^\alpha(x) = a\big(-\mathcal{G}_0^{\alpha-1}\lambda' + \lambda\mathcal{G}_0^{\alpha}\big) = a.$$

Let $x \in [h, (n-1)h)$, then using (A.7), (A.8) and $\mathcal{G}_{\iota(x)-1}^{\alpha-1} = \mathcal{G}_{\iota(x)-2}^{\alpha-1} + \mathcal{G}_{\iota(x)-1}^{\alpha}$ we have

$$\begin{aligned}
G^h a\vartheta_h^\alpha(x) &= a\Big(\lambda'\mathcal{G}_{\iota(x)-1}^{\alpha-1} + (-\lambda'\mathcal{G}_{\iota(x)-2}^{\alpha-1} + \lambda\mathcal{G}_{\iota(x)-1}^{\alpha})\mathcal{G}_0^{-\alpha-1} + \sum_{k=0}^{\iota(x)-2}\mathcal{G}_k^{\alpha}\mathcal{G}_{\iota(x)-1-k}^{-\alpha-1}\Big)\\
&= a\Big(\lambda'\mathcal{G}_{\iota(x)-1}^{\alpha} - \lambda'\mathcal{G}_{\iota(x)-1}^{\alpha} + \sum_{k=0}^{\iota(x)-1}\mathcal{G}_k^{\alpha}\mathcal{G}_{\iota(x)-1-k}^{-\alpha-1}\Big)\\
&= a\mathcal{G}_{\iota(x)-1}^{-1} = a,
\end{aligned}$$

since $\mathcal{G}_{\iota(x)-1}^{-1} = 1$. Next, for $x \in [(n-1)h, nh)$, using (A.7), (A.8) and $\mathcal{G}_{n-1}^{\alpha-1} = \mathcal{G}_{n-2}^{\alpha-1} + \mathcal{G}_{n-1}^{\alpha}$ observe that

$$\begin{aligned}
G^h a\vartheta_h^\alpha(x) &= a\Big(-\lambda'\sum_{i=0}^{n-2}\mathcal{G}_i^{\alpha-1} + (-\lambda'\mathcal{G}_{n-2}^{\alpha-1} - \lambda\mathcal{G}_{n-2}^{\alpha-1})\mathcal{G}_0^{-\alpha-1}\\
&\qquad + \lambda'\sum_{k=1}^{n-2}\mathcal{G}_k^{\alpha}\mathcal{G}_{n-1-k}^{-\alpha-1} - \lambda\sum_{k=0}^{n-3}\mathcal{G}_k^{\alpha-1}\mathcal{G}_{n-2-k}^{-\alpha-1} + \mathcal{G}_0^{\alpha}\mathcal{G}_{n-1}^{-\alpha-1}\Big)\\
&= a\Big(-\lambda'\sum_{i=0}^{n-2}\mathcal{G}_i^{\alpha-1} - \lambda'(\mathcal{G}_{n-2}^{\alpha-1} + \mathcal{G}_{n-1}^{\alpha})\\
&\qquad + \lambda'\sum_{k=0}^{n-1}\mathcal{G}_k^{\alpha}\mathcal{G}_{n-1-k}^{-\alpha-1} - \lambda\sum_{k=0}^{n-2}\mathcal{G}_k^{\alpha-1}\mathcal{G}_{n-2-k}^{-\alpha-1} + \lambda\mathcal{G}_0^{\alpha}\mathcal{G}_{n-1}^{-\alpha-1}\Big)\\
&= a\Big(-\lambda'\sum_{i=0}^{n-1}\mathcal{G}_i^{\alpha-1} + \lambda'\mathcal{G}_{n-1}^{-1} - \lambda\mathcal{G}_{n-2}^{-2}\Big) + \frac{\lambda a}{h^\alpha}\vartheta_h^\alpha((\lambda+n)h)\\
&= a\Big(-\lambda'\sum_{i=0}^{n-1}\mathcal{G}_i^{\alpha-1} + \lambda' - \lambda(n-1)\Big) + \frac{\lambda a}{h^\alpha}\vartheta_h^\alpha((\lambda+n)h)
\end{aligned}$$

$$
\begin{aligned}
&= a - \frac{\lambda a}{h} + \frac{\lambda a}{h^\alpha}\vartheta_h^\alpha((\lambda+n)h) + O(1) \\
&= a + \frac{\lambda}{h} D_c^{\alpha-1}\mathcal{Q}(1) + \frac{\lambda a}{h^\alpha}\vartheta_h^\alpha((\lambda+n)h) + O(1),
\end{aligned}
$$

since $a = -D_c^{\alpha-1}\mathcal{Q}(1)$, $\mathcal{G}_{n-1}^{-1} = 1$, $\mathcal{G}_{n-2}^{-2} = n-1$ and $\sum_{i=0}^{n-1}\mathcal{G}_i^{\alpha-1} < \infty$ in view of (A.10).

Lastly, for $x \in [nh, 1]$, using (A.7), (A.8) and $\mathcal{G}_{n-1}^{\alpha-1} = \mathcal{G}_{n-2}^{\alpha-1} + \mathcal{G}_{n-1}^{\alpha}$ observe that

$$
\begin{aligned}
G^h a\vartheta_h^\alpha(x) &= a\Big(\lambda'\sum_{i=0}^{n-2}\mathcal{G}_i^{\alpha-1}\mathcal{G}_0^{-\alpha-1} - \lambda'\sum_{k=1}^{n-2}\mathcal{G}_k^{\alpha-1}\mathcal{G}_{n-1-k}^{-\alpha-1} - \mathcal{G}_0^{\alpha-1}\mathcal{G}_n^{-\alpha-1}\Big) \\
&= a\Big(\lambda'\sum_{i=0}^{n-1}\mathcal{G}_i^{\alpha-1}\mathcal{G}_0^{-\alpha-1} - \lambda'\sum_{k=0}^{n-1}\mathcal{G}_k^{\alpha-1}\mathcal{G}_{n-1-k}^{-\alpha-1} - \lambda\mathcal{G}_0^{\alpha-1}\mathcal{G}_n^{-\alpha-1}\Big) \\
&= a\lambda'\sum_{i=0}^{n-1}\mathcal{G}_i^{\alpha-1} - \lambda' a\mathcal{G}_{n-1}^{-2} - \frac{\lambda a}{h^\alpha}\vartheta_h^\alpha((\lambda+n)h) \\
&= a - \frac{\lambda' a}{h} - \frac{\lambda a}{h^\alpha}\vartheta_h^\alpha((\lambda+n)h) + O(1) \\
&= a + \frac{\lambda'}{h}D_c^{\alpha-1}\mathcal{Q}(1) - \frac{\lambda a}{h^\alpha}\vartheta_h^\alpha((\lambda+n)h) + O(1),
\end{aligned}
$$

since $a = -D_c^{\alpha-1}\mathcal{Q}(1)$, $\mathcal{G}_{n-1}^{-2} = n$ and $\sum_{i=0}^{n-1}\mathcal{G}_i^{\alpha-1} < \infty$ in view of (A.10).

Proof of (4.160): Let $x \in [0, h)$, then

$$
\begin{aligned}
G^h\mathcal{Q}(x) &= \frac{-\mathcal{G}_0^{\alpha-1}\mathcal{Q}(\lambda h) + \lambda\mathcal{G}_0^{\alpha}\mathcal{Q}((\lambda+1)h)}{h^\alpha} \\
&= \frac{-\mathcal{G}_1^{\alpha-1}\mathcal{Q}(\lambda h) - \lambda'\mathcal{G}_0^{\alpha-1}\mathcal{Q}((\lambda+1)h)}{h^\alpha} + A_{h,1}^\alpha\mathcal{Q}(x)
\end{aligned}
$$

Let $x \in [h, (n-1)h)$, then using $\mathcal{G}_k^{\alpha-1} = \mathcal{G}_{k-1}^{\alpha-1} + \mathcal{G}_{k-1}^{\alpha}$

$$
\begin{aligned}
G^h\mathcal{Q}(x) = \frac{1}{h^\alpha}\Bigg(&-\mathcal{G}_{\iota(x)-1}^{\alpha-1}\mathcal{Q}(\lambda h) + \big(-\lambda'\mathcal{G}_{\iota(x)-2}^{\alpha-1} + \lambda\mathcal{G}_{\iota(x)-1}^{\alpha}\big)\mathcal{Q}((\lambda+1)h) \\
&+ \sum_{k=0}^{\iota(x)-2}\mathcal{G}_k^{\alpha}\mathcal{Q}((\lambda+\iota(x)-1-(k-1))h)\Bigg) \\
= \frac{1}{h^\alpha}\Bigg(&-\mathcal{G}_{\iota(x)}^{\alpha-1}\mathcal{Q}(\lambda h) - \lambda'\mathcal{G}_{\iota(x)-1}^{\alpha-1}\mathcal{Q}((\lambda+1)h) \\
&+ \sum_{k=0}^{\iota(x)}\mathcal{G}_k^{\alpha}\mathcal{Q}((\lambda+\iota(x)-1-(k-1))h)\Bigg) \\
= \frac{1}{h^\alpha}\Big(&-\mathcal{G}_{\iota(x)}^{\alpha-1}\mathcal{Q}(\lambda h) - \lambda'\mathcal{G}_{\iota(x)-1}^{\alpha-1}\mathcal{Q}((\lambda+1)h)\Big) + A_{h,1}^\alpha\mathcal{Q}(x).
\end{aligned}
$$

Next, let $x \in [(n-1)h, nh)$, then using $\mathcal{G}_k^{\alpha-1} = \mathcal{G}_{k-1}^{\alpha-1} + \mathcal{G}_{k-1}^{\alpha}$ and the Taylor expansion (4.129) we have

$$
\begin{aligned}
G^h \mathcal{Q}(x) = \frac{1}{h^\alpha}\Bigg(& \sum_{i=0}^{n-2} \mathcal{G}_i^{\alpha-1}\mathcal{Q}(\lambda h) + \big(-\lambda' \mathcal{G}_{n-2}^{\alpha-1} - \lambda \mathcal{G}_{n-2}^{\alpha-1}\big)\mathcal{Q}((\lambda+1)h) \\
& + \lambda' \sum_{k=1}^{n-2} \mathcal{G}_k^{\alpha}\mathcal{Q}((\lambda+n-1-(k-1))h) \\
& - \lambda \sum_{k=0}^{n-3} \mathcal{G}_k^{\alpha-1}\mathcal{Q}((\lambda+n-1-k)h) \\
& + \mathcal{G}_0^{\alpha}\mathcal{Q}((\lambda+n)h)\Bigg) \\
= \frac{1}{h^\alpha}\Bigg(& \Big(\sum_{i=0}^{n-2} \mathcal{G}_i^{\alpha-1} - \lambda' \mathcal{G}_n^{\alpha} + \lambda \mathcal{G}_{n-1}^{\alpha-1}\Big)\mathcal{Q}(\lambda h) - \lambda' \mathcal{G}_{n-1}^{\alpha-1}\mathcal{Q}((\lambda+1)h) \\
& + \lambda' \sum_{k=0}^{n} \mathcal{G}_k^{\alpha}\mathcal{Q}((\lambda+n-1-(k-1))h) \\
& - \lambda \sum_{k=0}^{n-1} \mathcal{G}_k^{\alpha-1}\mathcal{Q}((\lambda+n-1-k)h) \\
& + \lambda \mathcal{G}_0^{\alpha}\mathcal{Q}((\lambda+n)h)\Bigg) \\
= \frac{1}{h^\alpha}\Bigg(& \Big(\sum_{i=0}^{n-2} \mathcal{G}_i^{\alpha-1} - \lambda' \mathcal{G}_n^{\alpha} + \lambda \mathcal{G}_{n-1}^{\alpha-1}\Big)\mathcal{Q}(\lambda h) - \lambda' \mathcal{G}_{n-1}^{\alpha-1}\mathcal{Q}((\lambda+1)h)\Bigg) \\
& + \lambda' A_{h,1}^{\alpha}\mathcal{Q}(x) - \frac{\lambda}{h} A_{h,0}^{\alpha-1}\mathcal{Q}(x) + \frac{\lambda}{h^\alpha}\mathcal{Q}((\lambda+n)h) \\
= \frac{1}{h^\alpha}\Bigg(& \Big(\sum_{i=0}^{n-2} \mathcal{G}_i^{\alpha-1} - \lambda' \mathcal{G}_n^{\alpha} + \lambda \mathcal{G}_{n-1}^{\alpha-1}\Big)\mathcal{Q}(\lambda h) - \lambda' \mathcal{G}_{n-1}^{\alpha-1}\mathcal{Q}((\lambda+1)h)\Bigg) \\
& + A_{h,1}^{\alpha}\mathcal{Q}(x) - \frac{\lambda}{h} D_c^{\alpha-1}\mathcal{Q}(1) + \frac{\lambda}{h^\alpha}\mathcal{Q}((\lambda+n)h) + O(1).
\end{aligned}
$$

Lastly, let $x \in [nh, 1]$, then using the Taylor expansion (4.129) we have

$$
\begin{aligned}
G^h \mathcal{Q}(x) = \frac{1}{h^\alpha}\Bigg(& \lambda' \sum_{i=0}^{n-2} \mathcal{G}_i^{\alpha-1}\mathcal{Q}((\lambda+1)h) - \lambda' \sum_{k=1}^{n-2} \mathcal{G}_k^{\alpha-1}\mathcal{Q}((\lambda+n-k)h) \\
& - \mathcal{G}_0^{\alpha-1}\mathcal{Q}((\lambda+n)h)\Bigg) \\
= \frac{1}{h^\alpha}\Bigg(& \lambda' \mathcal{G}_n^{\alpha-1}\mathcal{Q}(\lambda h) + \lambda' \sum_{i=0}^{n-1} \mathcal{G}_i^{\alpha-1}\mathcal{Q}((\lambda+1)h)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad - \lambda' \sum_{k=0}^{n} \mathcal{G}_k^{\alpha-1}\mathcal{Q}((\lambda+n-k)h) - \lambda\mathcal{G}_0^{\alpha-1}\mathcal{Q}((\lambda+n)h)\Bigg)\\
&= \frac{1}{h^\alpha}\left(\lambda'\mathcal{G}_n^{\alpha-1}\mathcal{Q}(\lambda h) + \lambda'\sum_{i=0}^{n-1}\mathcal{G}_i^{\alpha-1}\mathcal{Q}((\lambda+1)h)\right)\\
&\qquad - \frac{\lambda'}{h}A_{h,0}^{\alpha-1}\mathcal{Q}(x) - \frac{\lambda}{h^\alpha}\mathcal{Q}((\lambda+n)h)\\
&= \frac{1}{h^\alpha}\left(\lambda'\mathcal{G}_n^{\alpha-1}\mathcal{Q}(\lambda h) + \lambda'\sum_{i=0}^{n-1}\mathcal{G}_i^{\alpha-1}\mathcal{Q}((\lambda+1)h)\right)\\
&\qquad - \frac{\lambda'}{h}\Big(D_c^{\alpha-1}\mathcal{Q}(1) + h\big((\lambda-1) - \frac{\alpha-1}{2}\big)A^\alpha\mathcal{Q}(1) + O(h^2)\Big)\\
&\qquad - \frac{\lambda}{h^\alpha}\mathcal{Q}((\lambda+n)h)\\
&= \frac{1}{h^\alpha}\left(\lambda'\mathcal{G}_n^{\alpha-1}\mathcal{Q}(\lambda h) + \lambda'\sum_{i=0}^{n-1}\mathcal{G}_i^{\alpha-1}\mathcal{Q}((\lambda+1)h)\right)\\
&\qquad + A_{h,1}^{\alpha}\mathcal{Q}(x) - \frac{\lambda'}{h}D_c^{\alpha-1}\mathcal{Q}(1) - \frac{\lambda}{h^\alpha}\mathcal{Q}((\lambda+n)h) + O(1).
\end{aligned}
$$

Common properties for operators $(D^\alpha, \mathrm{N}\bullet)$**:**

We first consider the common properties of the operators $(D^\alpha, \mathrm{N}\bullet)$ and following that consider the right boundary conditions one by one and deal with the operators (D^α, ND) and (D^α, NN) separately.

In these cases, since $b, d = 0$, $f \in \mathcal{C}(D^\alpha, \mathrm{N}\bullet)$ given by (4.127) reduces to

$$f = \mathcal{Q} + ap_\alpha + cp_{\alpha-2}. \tag{4.161}$$

Moreover, Af given by (4.128) reduces to

$$D^\alpha f = D^\alpha \mathcal{Q} + ap_0. \tag{4.162}$$

Next, we take

$$f_h = \mathcal{Q} + a_h\vartheta_h^\alpha + c_h\vartheta_h^{\alpha-2} + e_h. \tag{4.163}$$

Note that $N^l(\lambda) = \lambda$, $D^l = \mathbf{1} = N^r$, $b_1^l = \mathcal{G}_0^\alpha + \mathcal{G}_1^\alpha$ and for $i \geq 2$, $b_i^l = \mathcal{G}_i^\alpha$.

Next recall $\vartheta_h^{\alpha-2}(x) = h^{\alpha-2}\Big((1-\theta)\mathcal{G}_{\iota(x)-2}^{-\alpha+1} + \theta\mathcal{G}_{\iota(x)-1}^{-\alpha+1}\Big)$ given in Definition 4.5.1 where $\theta := \theta(\lambda) = \frac{\lambda}{(\alpha-1)\lambda'+\lambda}$ and observe that the approximate power function $\vartheta_h^{\alpha-2}$ is constructed such that the following hold:

Let $x \in [0, h)$, then

$$G^h\vartheta_h^{\alpha-2}(x) = \frac{1}{h^2}\left((\mathcal{G}_0^\alpha + \mathcal{G}_1^\alpha)\theta\mathcal{G}_0^{-\alpha+1} + \lambda\mathcal{G}_0^\alpha\big((1-\theta)\mathcal{G}_0^{-\alpha+1} + \theta\mathcal{G}_1^{-\alpha+1}\big)\right)$$

$$= \frac{1}{h^2}\Bigg((1-\alpha)\theta + \lambda - \lambda\theta + (\alpha-1)\lambda\theta \Bigg)$$

$$= \frac{1}{h^2}\Bigg(-\theta\big((\alpha-1)\lambda' + \lambda\big) + \lambda \Bigg) = 0. \tag{4.164}$$

Let $x \in [h, 2h)$, then

$$\begin{aligned} G^h \vartheta_h^{\alpha-2}(x) &= \frac{1}{h^2}\Bigg(\mathcal{G}_2^\alpha \theta \mathcal{G}_0^{-\alpha+1} + \big(\lambda'(\mathcal{G}_0^\alpha + \mathcal{G}_1^\alpha) + \lambda \mathcal{G}_1^\alpha\big)\big((1-\theta)\mathcal{G}_0^{-\alpha+1} + \theta \mathcal{G}_1^{-\alpha+1}\big) \\ &\qquad + \mathcal{G}_0^\alpha\big((1-\theta)\mathcal{G}_1^{-\alpha+1} + \theta\mathcal{G}_2^{-\alpha+1}\big)\Bigg) \\ &= \frac{1}{h^2}\Bigg(\frac{\alpha(\alpha-1)}{2}\theta + (\lambda' - \alpha)\big((1-\theta) + (\alpha-1)\theta\big) \\ &\qquad + (1-\theta)(\alpha-1) + \frac{\alpha(\alpha-1)}{2}\theta \Bigg) \\ &= \frac{1}{h^2}\Bigg(\alpha(\alpha-1)\theta + \lambda'(1-\theta) - \alpha(1-\theta) \\ &\qquad + \lambda'(\alpha-1)\theta - \alpha(\alpha-1)\theta + (1-\theta)(\alpha-1) \Bigg) \\ &= \frac{1}{h^2}\Bigg(\theta\big((\alpha-1)\lambda' + \lambda\big) - \lambda \Bigg) = 0. \end{aligned} \tag{4.165}$$

Proof of Statement 2 of Proposition 4.3.2 for the operator (D^α, ND):

In this case, we further have that $a = 0$ and so $f \in \mathcal{C}(D^\alpha, \mathrm{ND})$ given by (4.161) reduces to

$$f = \mathcal{Q} + c p_{\alpha-2}. \tag{4.166}$$

Moreover, Af given by (4.162) reduces to

$$D^\alpha f = D^\alpha \mathcal{Q}. \tag{4.167}$$

Note that $N^r = 1$, $D^r(\lambda) = \frac{\alpha\lambda'}{\alpha\lambda'+\lambda}$, $b_i^r = \mathcal{G}_i^\alpha$ and $b_n = \mathcal{G}_n^\alpha$. Moreover, the relation $c = -\Gamma(\alpha-1) I^\alpha g(1)$ for $f \in \mathcal{C}(D^\alpha, \mathrm{ND})$ reads

$$c = -\Gamma(\alpha-1)\mathcal{Q}(1). \tag{4.168}$$

Next, setting $a_h = 0$ and $e_h = \mathbf{0}$ in (4.163) we have

$$f_h = \mathcal{Q} + c_h \vartheta_h^{\alpha-2}. \tag{4.169}$$

We also set

$$c_h = \frac{c}{\vartheta_h^{\alpha-2}(1)\Gamma(\alpha-1)}. \tag{4.170}$$

Then, $f_h \to f$ in $L_1[0,1]$, since in view of Lemma 4.5.3, $\vartheta_h^{\alpha-2} \to p_{\alpha-2}$ in $L_1[0,1]$.

To show that $\left\|G^h f_h - D^\alpha f\right\|_{L_1[0,1]} \to 0$, we show the following:

1.

$$G^h c_h \vartheta_h^{\alpha-2}(x) = \begin{cases} 0, & x \in [0,(n-1)h), \\ -\frac{D^r(\lambda)-1}{h^\alpha}\mathcal{Q}(1) + O(h^{1-\alpha}), & x \in [(n-1)h, nh), \\ \frac{1}{h^\alpha}\mathcal{Q}(1) + O(h^{1-\alpha}), & x \in [nh, 1]. \end{cases} \tag{4.171}$$

2.

$$\begin{aligned} &G^h \mathcal{Q}(x) \\ &= \begin{cases} A_{h,1}^\alpha \mathcal{Q}(x) + O(1), & x \in [0,2h), \\ A_{h,1}^\alpha \mathcal{Q}(x), & x \in [2h,(n-1)h) \\ A_{h,1}^\alpha \mathcal{Q}(x) + \frac{D^r(\lambda)-1}{h^\alpha}\mathcal{Q}(1) + O(h^{1-\alpha}), & x \in [(n-1)h, nh), \\ A_{h,1}^\alpha \mathcal{Q}(x) - \frac{1}{h^\alpha}\mathcal{Q}(1) + O(h^{1-\alpha}), & x \in [nh,1]. \end{cases} \end{aligned} \tag{4.172}$$

Then, using (4.167), (4.171) and (4.172) we have

$$\begin{aligned} &\left\|G^h f_h - D^\alpha f\right\|_{L_1[0,1]} \\ &= \int_0^1 \left|G^h \mathcal{Q}(x) + G^h c_h \vartheta_h^{\alpha-2}(x) - D^\alpha \mathcal{Q}(x)\right| \mathrm{d}x \\ &\le \int_0^1 \left|A_{h,1}^\alpha \mathcal{Q}(x) - D^\alpha \mathcal{Q}(x)\right| \mathrm{d}x + O(h^{2-\alpha}) \\ &= \left\|A_{h,1}^\alpha \mathcal{Q} - D^\alpha \mathcal{Q}\right\|_{L_1[0,1]} + O(h^{2-\alpha}). \end{aligned}$$

Using Corollary 4.5.5, as $h \to 0$, $\left\|A_{h,1}^\alpha \mathcal{Q} - D^\alpha \mathcal{Q}\right\|_{L_1[0,1]} \to 0$. Hence,

$$\left\|G^h f_h - D^\alpha f\right\|_{L_1[0,1]} \to 0$$

in the case of (D^α, ND).

Proof of (4.171): For $x \in [0,2h)$ it is clear that $G^h c_h \vartheta_h^{\alpha-2}(x) = 0$ in view of (4.164) and (4.165). Next, for $x \in [2h,(n-1)h)$, then $G^h \vartheta_h^{\alpha-2}(x) = A_{h,1}^\alpha \vartheta_h^{\alpha-1}(x) = 0$ in view of let $x \in [(n-1)h, nh)$, then using with $\iota(x) = n$, (4.170) and (4.26) we have

$$G^h c_h \vartheta_h^{\alpha-2}(x) = \frac{c_h}{h^2}\left(\mathcal{G}_n^\alpha \theta \mathcal{G}_0^{-\alpha+1} + \sum_{k=1}^{n-1} \mathcal{G}_k^\alpha \left((1-\theta)\mathcal{G}_{n-1-k}^{-\alpha+1} + \theta \mathcal{G}_{n-k}^{-\alpha+1}\right)\right)$$

$$
\begin{aligned}
&\qquad\qquad + \frac{c_h}{h^\alpha} D^r(\lambda)\mathcal{G}_0^\alpha \vartheta_h^{\alpha-2}((\lambda+n)h) \\
&= \frac{c_h}{h^2}\left(\mathcal{G}_n^\alpha \theta \mathcal{G}_0^{-\alpha+1} + \sum_{k=0}^{n-1} \mathcal{G}_k^\alpha \Big((1-\theta)\mathcal{G}_{n-1-k}^{-\alpha+1} + \theta \mathcal{G}_{n-k}^{-\alpha+1}\Big)\right) \\
&\qquad\qquad + \frac{c_h}{h^\alpha}\big(D^r(\lambda)-1\big)\mathcal{G}_0^\alpha \vartheta_h^{\alpha-2}((\lambda+n)h) \\
&= \frac{c_h}{h^\alpha}\big(D^r(\lambda)-1\big)\vartheta_h^{\alpha-2}((\lambda+n)h) \\
&= -\frac{(D^r(\lambda)-1)\mathcal{Q}(1)}{h^\alpha}\Big(\frac{\vartheta_h^{\alpha-2}((\lambda+n)h)}{\vartheta_h^{\alpha-2}(1)}\Big) \\
&= -\frac{D^r(\lambda)-1}{h^\alpha}\mathcal{Q}(1) + O(h^{1-\alpha}).
\end{aligned}
$$

Lastly, let $x \in [nh, 1]$, then using with $\iota(x) = n+1$ in the second line, (4.170) and (4.26) we have

$$
\begin{aligned}
G^h c_h \vartheta_h^{\alpha-2}(x) &= \frac{c_h}{h^2}\left(\sum_{k=1}^{n} \mathcal{G}_k^\alpha \Big((1-\theta)\mathcal{G}_{n-k}^{-\alpha+1} + \theta \mathcal{G}_{n+1-k}^{-\alpha+1}\Big)\right) \\
&= -\frac{c_h}{h^\alpha}\mathcal{G}_{n+1}^\alpha \vartheta_h^{\alpha-2}(\lambda h) \\
&\qquad + \frac{c_h}{h^2}\left(\mathcal{G}_{n+1}^\alpha \theta \mathcal{G}_0^{-\alpha+1} + \sum_{k=0}^{n} \mathcal{G}_k^\alpha \Big((1-\theta)\mathcal{G}_{n-1-k}^{-\alpha+1} + \theta \mathcal{G}_{n-k}^{-\alpha+1}\Big)\right) \\
&\qquad\qquad - \frac{c_h}{h^\alpha}\mathcal{G}_0^\alpha \vartheta_h^{\alpha-2}((\lambda+n+1)h) \\
&= -\frac{c_h}{h^\alpha}\mathcal{G}_0^\alpha \vartheta_h^{\alpha-2}((\lambda+n+1)h) + O(h^{\alpha-1}) \\
&= \frac{1}{h^\alpha}\mathcal{Q}(1)\frac{\vartheta_h^{\alpha-2}((\lambda+n+1)h)}{\vartheta_h^{\alpha-2}(1)} + O(h^{\alpha-1}) \\
&= \frac{\mathcal{Q}(1)}{h^\alpha} + O(h^{1-\alpha}),
\end{aligned}
$$

where we have also used the fact that $\left|\mathcal{G}_{n+1}^\alpha\right| = O(h^{\alpha+1})$ and $\left|\vartheta_h^{\alpha-2}(\lambda h)\right| = O(h^{\alpha-2})$ to simplify the first term in the second line.

Proof of (4.172): Let $x \in [0, h)$, then

$$
\begin{aligned}
G^h \mathcal{Q}(x) &= \frac{1}{h^\alpha}\left((\mathcal{G}_0^\alpha + \mathcal{G}_1^\alpha)\mathcal{Q}(\lambda h) + \lambda \mathcal{G}_0^\alpha \mathcal{Q}((\lambda+1)h)\right) \\
&= \frac{1}{h^\alpha}\left(\mathcal{G}_1^\alpha \mathcal{Q}(\lambda h) + \mathcal{G}_0^\alpha \mathcal{Q}((\lambda+1)h)\right) \\
&\qquad + \frac{1}{h^\alpha}\left(\mathcal{G}_0^\alpha \mathcal{Q}(\lambda h) + (\lambda-1)\mathcal{G}_0^\alpha \mathcal{Q}((\lambda+1)h)\right) \\
&= A_{h,1}^\alpha \mathcal{Q}(x) + O(1),
\end{aligned}
$$

since $\mathcal{Q}(\lambda h),\ \mathcal{Q}((\lambda+1)h) = O(h^\alpha)$. Let $x \in [h, 2h)$, then

$$
\begin{aligned}
G^h\mathcal{Q}(x) &= \frac{1}{h^\alpha}\Bigg(\mathcal{G}_2^\alpha \mathcal{Q}(\lambda h) + \big(\lambda'(\mathcal{G}_0^\alpha + \mathcal{G}_1^\alpha) + \lambda\mathcal{G}_1^\alpha\big)\mathcal{Q}((\lambda+1)h) \\
&\qquad + \mathcal{G}_0^\alpha \mathcal{Q}((\lambda+2)h)\Bigg) \\
&= A_{h,1}^\alpha \mathcal{Q}(x) + \frac{1}{h^\alpha}\big(\lambda'(\mathcal{G}_0^\alpha + \mathcal{G}_1^\alpha) + (\lambda-1)\mathcal{G}_1^\alpha\big)\mathcal{Q}((\lambda+1)h) \\
&= A_{h,1}^\alpha \mathcal{Q}(x) + O(1),
\end{aligned}
$$

since $\mathcal{Q}((\lambda+1)h) = O(h^\alpha)$. For $x \in [2h, (n-1)h)$, it is clear that $G^h\mathcal{Q}(x) = A_{h,1}^\alpha \mathcal{Q}(x)$. Let $x \in [(n-1)h, nh)$, then using the Taylor expansion (4.129) we have

$$
\begin{aligned}
G^h\mathcal{Q}(x) &= \frac{1}{h^\alpha}\Bigg(\mathcal{G}_n^\alpha \mathcal{Q}(\lambda h) + \sum_{k=1}^{n-1} \mathcal{G}_k^\alpha \mathcal{Q}((\lambda + n - 1 - (k-1))h) \\
&\qquad + D^r(\lambda)\mathcal{G}_0^\alpha \mathcal{Q}((\lambda+n)h)\Bigg) \\
&= A_{h,1}^\alpha \mathcal{Q}(x) + \frac{D^r(\lambda) - 1}{h^\alpha}\mathcal{Q}((\lambda+n)h) \\
&= A_{h,1}^\alpha \mathcal{Q}(x) + \frac{D^r(\lambda) - 1}{h^\alpha}\mathcal{Q}(1) + O(h^{1-\alpha})
\end{aligned}
$$

Lastly, let $x \in [nh, 1]$, then using the Taylor expansion (4.129) we have

$$
\begin{aligned}
G^h\mathcal{Q}(x) &= \frac{1}{h^\alpha}\left(\sum_{k=1}^{n} \mathcal{G}_k^\alpha \mathcal{Q}((\lambda + n - (k-1)h)\right) \\
&= -\frac{1}{h^\alpha}\mathcal{G}_{n+1}^\alpha \mathcal{Q}(\lambda h) + A_{h,1}^\alpha \mathcal{Q}(x) - \frac{1}{h^\alpha}\mathcal{Q}((\lambda+n+1)h) \\
&= A_{h,1}^\alpha \mathcal{Q}(x) - \frac{1}{h^\alpha}\mathcal{Q}((\lambda+n+1)h) + O(h^{\alpha+1}) \\
&= A_{h,1}^\alpha \mathcal{Q}(x) - \frac{1}{h^\alpha}\mathcal{Q}(1) + O(h^{1-\alpha})
\end{aligned}
$$

since $\left|\mathcal{G}_{n+1}^\alpha\right| = O(h^{\alpha+1})$ and $|\mathcal{Q}(\lambda h)| = O(h^\alpha)$.

Proof of Statement 2 of Proposition 4.3.2 for the operator (D^α, NN)**:**

In this case, $f \in \mathcal{C}(D^\alpha, \mathrm{NN})$ is given by (4.161),

$$
f = \mathcal{Q} + ap_\alpha + cp_{\alpha-2}. \tag{4.173}
$$

Moreover, Af is given by (4.162),

$$
D^\alpha f = D^\alpha \mathcal{Q} + ap_0. \tag{4.174}
$$

Note that $N^r = \lambda'$, $D^r = \mathbf{1}$, $b_i^r = -\mathcal{G}_{i-1}^{\alpha-1}$ and $b_n = -\sum_{i=0}^{n-1} \mathcal{G}_i^{\alpha} = -\mathcal{G}_{n-1}^{\alpha-1}$. Moreover, the relation $a = -Ig(1)$ for $f \in \mathcal{C}(D^\alpha, \mathrm{NN})$ reads

$$a = -D^{\alpha-1}\mathcal{Q}(1). \tag{4.175}$$

Next, setting $a_h = a$ and $c_h = c$ in (4.163) we have

$$f_h = \mathcal{Q} + a\vartheta_h^\alpha(x) + c\vartheta_h^{\alpha-2} + e_h, \tag{4.176}$$

where since we are dealing with a right Neumann boundary condition we construct

$$e_h(x) \\ = \begin{cases} 0, & \text{if } x \in [0, nh), \\ -\lambda\big(\mathcal{Q}((\lambda+n)h) + a\vartheta_h^\alpha((\lambda+n)h) + c\vartheta_h^{\alpha-2}((\lambda+n)h)\big), & \text{if } x \in [nh, 1]. \end{cases} \tag{4.177}$$

Then, as $h \to 0$, $\|e_h\|_{L_1[0,1]} \to 0$. This also implies that $f_h \to f$ in $L_1[0,1]$, since in view of Lemma 4.5.3, $\vartheta_h^\alpha \to p_\alpha$ and $\vartheta_h^{\alpha-2} \to p_{\alpha-2}$ in $L_1[0,1]$.

Further, observe that

$$G^h e_h(x) = \begin{cases} 0, & \text{if } x \in [0, (n-1)h), \\ -\frac{\lambda}{h^\alpha}\big(\mathcal{Q}((\lambda+n)h) + a\vartheta_h^\alpha((\lambda+n)h) \\ \qquad +c\vartheta_h^{\alpha-2}((\lambda+n)h)\big), & \text{if } x \in [(n-1)h, nh), \\ \frac{\lambda}{h^\alpha}\big(\mathcal{Q}((\lambda+n)h) + a\vartheta_h^\alpha((\lambda+n)h) \\ \qquad +c\vartheta_h^{\alpha-2}((\lambda+n)h)\big), & \text{if } x \in [nh, 1]. \end{cases} \tag{4.178}$$

To show that $\left\|G^h f_h - D^\alpha f\right\|_{L_1[0,1]} \to 0$, we show the following:

1.

$$G^h c\vartheta_h^{\alpha-2}(x) = \begin{cases} 0, & x \in [0, (n-1)h), \\ \frac{\lambda c}{h^\alpha}\vartheta_h^{\alpha-2}((\lambda+n)h)) + O(h^{\alpha-2}), & x \in [(n-1)h, nh), \\ -\frac{\lambda c}{h^\alpha}\vartheta_h^{\alpha-2}((\lambda+n)h) + O(h^{\alpha-2}), & x \in [nh, 1]. \end{cases} \tag{4.179}$$

2.

$$G^h a\vartheta_h^\alpha(x) = \begin{cases} a + a\lambda'(\alpha-2), & x \in [0, h), \\ a + a\lambda'(1-\mathcal{G}_2^\alpha), & x \in [h, 2h), \\ a - a\lambda'\mathcal{G}_{\iota(x)}^\alpha, & x \in [2h, (n-1)h), \\ a + \frac{\lambda}{h}D^{\alpha-1}\mathcal{Q}(1) \\ \qquad +\frac{\lambda a}{h^\alpha}\vartheta_h^\alpha((\lambda+n)h)) + O(1), & x \in [(n-1)h, nh), \\ a + \frac{\lambda'}{h}D^{\alpha-1}\mathcal{Q}(1) \\ \qquad -\frac{\lambda a}{h^\alpha}\vartheta_h^\alpha((\lambda+n)h) + O(1), & x \in [nh, 1]. \end{cases} \tag{4.180}$$

3.

$$
\begin{aligned}
&G^h \mathcal{Q}(x) \\
&= \begin{cases} A^{\alpha}_{h,1}\mathcal{Q}(x) + O(1), & x \in [0, 2h) \\ A^{\alpha}_{h,1}\mathcal{Q}(x), & x \in [2h, (n-1)h) \\ A^{\alpha}_{h,1}\mathcal{Q}(x) - \frac{\lambda}{h} D^{\alpha-1}\mathcal{Q}(1) + \frac{\lambda}{h^\alpha}\mathcal{Q}((\lambda+n)h)) + O(1), & x \in [(n-1)h, nh), \\ A^{\alpha}_{h,1}\mathcal{Q}(x) - \frac{\lambda'}{h} D^{\alpha-1}\mathcal{Q}(1) - \frac{\lambda}{h^\alpha}\mathcal{Q}((\lambda+n)h) + O(1), & x \in [nh, 1]. \end{cases}
\end{aligned}
\tag{4.181}
$$

Then, using (4.174), (4.179) and (4.181) we have

$$
\begin{aligned}
&\left\| G^h f_h - D^\alpha f \right\|_{L_1[0,1]} \\
&= \int_0^1 \left| G^h \mathcal{Q}(x) + G^h a\vartheta^\alpha_h(x) + G^h c\vartheta^{\alpha-2}_h(x) + G^h e_h(x) - (D^\alpha \mathcal{Q}(x) + ap_0(x)) \right| \mathrm{d}x \\
&\le \int_0^1 \left| A^\alpha_{h,1}\mathcal{Q}(x) - D^\alpha \mathcal{Q}(x) \right| \mathrm{d}x + \sum_{i=3}^{n-1} \int_{(i-1)h}^{ih} \left| a\lambda' \mathcal{G}^\alpha_{i+1} \right| \mathrm{d}x + O(h^{\alpha-1}) \\
&= \left\| A^\alpha_{h,1}\mathcal{Q} - D^\alpha \mathcal{Q} \right\|_{L_1[0,1]} + O(h^{\alpha-1}),
\end{aligned}
$$

since $\sum_{i=3}^{n-1} \mathcal{G}^\alpha_{i+1} < \infty$ in view of (A.10). Using Corollary 4.5.5, as $h \to 0$,

$$
\left\| A^\alpha_{h,1}\mathcal{Q} - D^\alpha \mathcal{Q} \right\|_{L_1[0,1]} \to 0.
$$

Hence,

$$
\left\| G^h f_h - D^\alpha f \right\|_{L_1[0,1]} \to 0
$$

in the case of (D^α, NN).

Proof of (4.179): For $x \in [0, 2h)$ it is clear that $G^h c\vartheta^{\alpha-2}_h(x) = 0$ in view of (4.164) and (4.165). Next, for $x \in [2h, (n-1)h)$, in view of Proposition 4.5.7, $G^h c\vartheta^{\alpha-2}_h(x) = cA^\alpha_{h,1}\vartheta^{\alpha-2}_h(x) = 0$. Therefore, let $x \in [(n-1)h, nh)$, then in view of Proposition 4.5.7

$$
\begin{aligned}
G^h c\vartheta^{\alpha-2}_h(x) = \frac{c}{h^\alpha}\Bigg(&- \mathcal{G}^{\alpha-1}_{n-1}\vartheta^{\alpha-2}_h(\lambda h) \\
&+ \sum_{k=1}^{n-1} \left(\lambda' \mathcal{G}^\alpha_k - \lambda \mathcal{G}^{\alpha-1}_{k-1}\right)\vartheta^{\alpha-2}_h((\lambda + n - 1 - (k-1))h) \\
&+ \mathcal{G}^\alpha_0 \vartheta^{\alpha-2}_h((\lambda+n)h)\Bigg) \\
= \frac{c}{h^\alpha}\Bigg(&\big(- \mathcal{G}^{\alpha-1}_{n-1} + \lambda \mathcal{G}^{\alpha-1}_{n-1} - \lambda' \mathcal{G}^\alpha_n \big)\vartheta^{\alpha-2}_h(\lambda h)
\end{aligned}
$$

$$
\begin{aligned}
&+ \lambda' \sum_{k=0}^{n} \mathcal{G}_k^{\alpha} \vartheta_h^{\alpha-2}((\lambda + n - 1 - (k-1))h) \\
&- \lambda \sum_{k=0}^{n-1} \mathcal{G}_k^{\alpha-1} \vartheta_h^{\alpha-2}((\lambda + n - 1 - k)h) \\
&\qquad\qquad + \lambda \mathcal{G}_0^{\alpha} \vartheta_h^{\alpha-2}((\lambda + n)h)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{c}{h^\alpha} \mathcal{G}_n^{\alpha-1} \vartheta_h^{\alpha-2}(\lambda h) + \lambda' c A_{h,1}^{\alpha} \vartheta_h^{\alpha-2}(x) \\
&\qquad - \frac{\lambda c}{h} A_{h,0}^{\alpha-1} \vartheta_h^{\alpha-2}(x) + \frac{\lambda c}{h^\alpha} \vartheta_h^{\alpha-2}((\lambda + n)h)) \\
&= \frac{\lambda c}{h^\alpha} \vartheta_h^{\alpha-2}((\lambda + n)h)) + O(h^{\alpha-2}),
\end{aligned}
$$

since $\left|\vartheta_h^{\alpha-2}(\lambda h)\right| = O(h^{\alpha-2})$ and $\left|\mathcal{G}_{n-1}^{\alpha-1}\right| = O(h^\alpha)$. Lastly, let $x \in [nh, 1]$, then in view of Proposition 4.5.7

$$
\begin{aligned}
G^h c \vartheta_h^{\alpha-2}(x) &= \frac{c}{h^\alpha}\Bigg(- \lambda' \sum_{k=1}^{n-1} \mathcal{G}_k^{\alpha-1} \vartheta_h^{\alpha-2}((\lambda + n - k)h) - \mathcal{G}_0^{\alpha-1} \vartheta_h^{\alpha-2}((\lambda + n)h)\Bigg) \\
&= \frac{\lambda' c}{h^\alpha} \mathcal{G}_n^{\alpha-1} \vartheta_h^{\alpha-2}(\lambda h) - \frac{\lambda' c}{h^\alpha} \sum_{k=0}^{n} \mathcal{G}_k^{\alpha-1} \vartheta_h^{\alpha-2}((\lambda + n - k)h) \\
&\qquad\qquad - \frac{\lambda c}{h^\alpha} \mathcal{G}_0^{\alpha-1} \vartheta_h^{\alpha-2}((\lambda + n)h) \\
&= \frac{\lambda' c}{h^\alpha} \mathcal{G}_n^{\alpha-1} \vartheta_h^{\alpha-2}(\lambda h) - \frac{\lambda' c}{h} A_{h,0}^{\alpha-1} \vartheta_h^{\alpha-2}(x) - \frac{\lambda c}{h^\alpha} \vartheta_h^{\alpha-2}((\lambda + n)h) \\
&= -\frac{\lambda c}{h^\alpha} \vartheta_h^{\alpha-2}((\lambda + n)h) + O(h^{\alpha-2}),
\end{aligned}
$$

since $|\mathcal{G}_n^{\alpha-1}| = O(h^\alpha)$ and $\left|\vartheta_h^{\alpha-2}(\lambda h)\right| = O(h^{\alpha-2})$.

Proof of (4.180): Let $x \in [0, h)$, then

$$
\begin{aligned}
G^h a \vartheta_h^{\alpha}(x) &= \frac{a}{h^\alpha}\Bigg((\mathcal{G}_0^{\alpha} + \mathcal{G}_1^{\alpha}) \vartheta_h^{\alpha}(\lambda h) + \lambda \mathcal{G}_0^{\alpha} \vartheta_h^{\alpha}((\lambda + 1)h)\Bigg) \\
&= a\Big((1-\alpha)(\lambda - 1) + \lambda\Big) \\
&= a + a\lambda'(\alpha - 2).
\end{aligned}
$$

Let $x \in [h, 2h)$, then

$$
\begin{aligned}
G^h a \vartheta_h^{\alpha}(x) = \frac{a}{h^\alpha}\Bigg(&\mathcal{G}_2^{\alpha} \vartheta_h^{\alpha}(\lambda h) + \big(\lambda'(\mathcal{G}_0^{\alpha} + \mathcal{G}_1^{\alpha}) + \lambda \mathcal{G}_1^{\alpha}\big) \vartheta_h^{\alpha}((\lambda + 1)h) \\
&\qquad\qquad + \mathcal{G}_0^{\alpha} \vartheta_h^{\alpha}((\lambda + 2)h)\Bigg)
\end{aligned}
$$

$$= a + a\lambda'(1 - \mathcal{G}_2^\alpha).$$

For $x \in [2h, (n-1)h)$, in view of Proposition 4.5.7,

$$G^h a\vartheta_h^\alpha(x) = aA_{h,1}^\alpha \vartheta_h^\alpha(x) = a - a\lambda' \mathcal{G}_{\iota(x)}^\alpha.$$

Therefore, let $x \in [(n-1)h, nh)$, then in view of Proposition 4.5.7

$$\begin{aligned}
G^h a\vartheta_h^\alpha(x) &= \frac{a}{h^\alpha}\Bigg(- \mathcal{G}_{n-1}^{\alpha-1}\vartheta_h^\alpha(\lambda h) \\
&\qquad + \sum_{k=1}^{n-1} \Big(\lambda' \mathcal{G}_k^\alpha - \lambda \mathcal{G}_{k-1}^{\alpha-1}\Big)\vartheta_h^\alpha((\lambda + n - 1 - (k-1))h) \\
&\qquad + \mathcal{G}_0^\alpha \vartheta_h^\alpha((\lambda+n)h)\Bigg) \\
&= \frac{a}{h^\alpha}\Bigg(\big(- \mathcal{G}_{n-1}^{\alpha-1} + \lambda \mathcal{G}_{n-1}^{\alpha-1} - \lambda' \mathcal{G}_n^\alpha\big)\vartheta_h^\alpha(\lambda h) \\
&\qquad + \lambda' \sum_{k=0}^{n} \mathcal{G}_k^\alpha \vartheta_h^\alpha((\lambda + n - 1 - (k-1))h) \\
&\qquad - \lambda \sum_{k=0}^{n-1} \mathcal{G}_k^{\alpha-1}\vartheta_h^\alpha((\lambda + n - 1 - k)h) \\
&\qquad + \lambda \mathcal{G}_0^\alpha \vartheta_h^\alpha((\lambda+n)h)\Bigg) \\
&= \lambda' a \mathcal{G}_n^{\alpha-1} + \lambda' a A_{h,1}^\alpha \vartheta_h^\alpha(x) - \frac{\lambda a}{h} A_{h,0}^{\alpha-1}\vartheta_h^\alpha(x) + \frac{\lambda a}{h^\alpha}\vartheta_h^\alpha((\lambda+n)h)) \\
&= \lambda' a \mathcal{G}_n^{\alpha-1} + \lambda' a(1 - \lambda' \mathcal{G}_n^\alpha) \\
&\qquad - \frac{\lambda a}{h}\Big(h(n - 1 - \lambda' \mathcal{G}_{n-1}^{\alpha-1})\Big) + \frac{\lambda a}{h^\alpha}\vartheta_h^\alpha((\lambda+n)h)) \\
&= \lambda' a \mathcal{G}_n^{\alpha-1} + a - \lambda a - (\lambda')^2 a \mathcal{G}_n^\alpha \\
&\qquad - \frac{\lambda a}{h}\Big(1 + h(-2 - \lambda' \mathcal{G}_{n-1}^{\alpha-1})\Big) + \frac{\lambda a}{h^\alpha}\vartheta_h^\alpha((\lambda+n)h)) \\
&= a + \frac{\lambda}{h} D^{\alpha-1}\mathcal{Q}(1) + \frac{\lambda a}{h^\alpha}\vartheta_h^\alpha((\lambda+n)h)) + O(1),
\end{aligned}$$

where we replaced $a = -D^{\alpha-1}\mathcal{Q}(1)$ in view of (4.175).

Lastly, let $x \in [nh, 1]$, then in view of Proposition 4.5.7

$$\begin{aligned}
G^h a\vartheta_h^\alpha(x) &= \frac{a}{h^\alpha}\Bigg(- \lambda' \sum_{k=1}^{n-1} \mathcal{G}_k^{\alpha-1}\vartheta_h^\alpha((\lambda + n - k)h) - \mathcal{G}_0^{\alpha-1}\vartheta_h^\alpha((\lambda+n)h)\Bigg) \\
&= \frac{\lambda' a}{h^\alpha}\mathcal{G}_n^{\alpha-1}\vartheta_h^\alpha(\lambda h) - \frac{\lambda' a}{h^\alpha}\sum_{k=0}^{n} \mathcal{G}_k^{\alpha-1}\vartheta_h^\alpha((\lambda + n - k)h)
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad - \frac{\lambda a}{h^\alpha}\mathcal{G}_0^{\alpha-1}\vartheta_h^\alpha((\lambda+n)h) \\
&= \frac{\lambda' a}{h^\alpha}\mathcal{G}_n^{\alpha-1}\vartheta_h^\alpha(\lambda h) - \frac{\lambda' a}{h}A_{h,0}^{\alpha-1}\vartheta_h^\alpha(x) - \frac{\lambda a}{h^\alpha}\vartheta_h^\alpha((\lambda+n)h) \\
&= \frac{\lambda' a}{h^\alpha}\mathcal{G}_n^{\alpha-1}\vartheta_h^\alpha(\lambda h) - \frac{\lambda' a}{h}\Big(h(n-\lambda'\mathcal{G}_n^{\alpha-1})\Big) - \frac{\lambda a}{h^\alpha}\vartheta_h^\alpha((\lambda+n)h) \\
&= \frac{\lambda' a}{h^\alpha}\mathcal{G}_n^{\alpha-1}\vartheta_h^\alpha(\lambda h) - \frac{\lambda' a}{h}\Big(1-h-h\lambda'\mathcal{G}_n^{\alpha-1})\Big) - \frac{\lambda a}{h^\alpha}\vartheta_h^\alpha((\lambda+n)h) \\
&= a + \frac{\lambda'}{h}D^{\alpha-1}\mathcal{Q}(1) - \frac{\lambda a}{h^\alpha}\vartheta_h^\alpha((\lambda+n)h) + O(1),
\end{aligned}
$$

where we replaced $a = -D^{\alpha-1}\mathcal{Q}(1)$ in view of (4.175).

Proof of (4.181): Let $x \in [0, h)$, then

$$
\begin{aligned}
G^h\mathcal{Q}(x) &= \frac{1}{h^\alpha}\Bigg((\mathcal{G}_0^\alpha + \mathcal{G}_1^\alpha)\mathcal{Q}(\lambda h) + \lambda\mathcal{G}_0^\alpha\mathcal{Q}((\lambda+1)h)\Bigg) \\
&= \frac{1}{h^\alpha}\Bigg(\mathcal{G}_1^\alpha\mathcal{Q}(\lambda h) + \mathcal{G}_0^\alpha\mathcal{Q}((\lambda+1)h)\Bigg) \\
&\qquad\qquad + \frac{1}{h^\alpha}\Bigg(\mathcal{G}_0^\alpha\mathcal{Q}(\lambda h) + (\lambda-1)\mathcal{G}_0^\alpha\mathcal{Q}((\lambda+1)h)\Bigg) \\
&= A_{h,1}^\alpha\mathcal{Q}(x) + O(1),
\end{aligned}
$$

since $\mathcal{Q}(\lambda h),\ \mathcal{Q}((\lambda+1)h) = O(h^\alpha)$. Let $x \in [h, 2h)$, then

$$
\begin{aligned}
G^h\mathcal{Q}(x) &= \frac{1}{h^\alpha}\Bigg(\mathcal{G}_2^\alpha\mathcal{Q}(\lambda h) + \big(\lambda'(\mathcal{G}_0^\alpha + \mathcal{G}_1^\alpha) + \lambda\mathcal{G}_1^\alpha\big)\mathcal{Q}((\lambda+1)h) + \mathcal{G}_0^\alpha\mathcal{Q}((\lambda+2)h)\Bigg) \\
&= A_{h,1}^\alpha\mathcal{Q}(x) + \frac{1}{h^\alpha}\big(\lambda'\mathcal{G}_0^\alpha\big)\mathcal{Q}((\lambda+1)h) \\
&= A_{h,1}^\alpha\mathcal{Q}(x) + O(1),
\end{aligned}
$$

since $\mathcal{Q}((\lambda+1)h) = O(h^\alpha)$. For $x \in [2h, (n-1)h)$, it is clear that $G^h\mathcal{Q}(x) = A_{h,1}^\alpha\mathcal{Q}(x)$. Next, let $x \in [(n-1)h, nh)$, then using (4.130) we have

$$
\begin{aligned}
G^h\mathcal{Q}(x) &= \frac{1}{h^\alpha}\Bigg(-\mathcal{G}_{n-1}^{\alpha-1}\mathcal{Q}(\lambda h) + \sum_{k=1}^{n-1}\Big(\lambda'\mathcal{G}_k^\alpha - \lambda\mathcal{G}_{k-1}^{\alpha-1}\Big)\mathcal{Q}((\lambda+n-1-(k-1))h) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \mathcal{G}_0^\alpha\mathcal{Q}((\lambda+n)h)\Bigg) \\
&= \frac{1}{h^\alpha}\Bigg(\big(-\mathcal{G}_{n-1}^{\alpha-1} + \lambda\mathcal{G}_{n-1}^{\alpha-1} - \lambda'\mathcal{G}_n^\alpha\big)\mathcal{Q}(\lambda h) \\
&\qquad\qquad + \lambda'\sum_{k=0}^{n}\mathcal{G}_k^\alpha\mathcal{Q}((\lambda+n-1-(k-1))h)
\end{aligned}
$$

$$- \lambda \sum_{k=0}^{n-1} \mathcal{G}_k^{\alpha-1} \mathcal{Q}((\lambda + n - 1 - k)h)$$

$$\left. + \lambda \mathcal{G}_0^{\alpha} \mathcal{Q}((\lambda + n)h) \right)$$

$$\begin{aligned}
&= -\frac{1}{h^\alpha} \mathcal{G}_n^{\alpha-1} \mathcal{Q}(\lambda h) + \lambda' A_{h,1}^\alpha \mathcal{Q}(x) - \frac{\lambda}{h} A_{h,0}^{\alpha-1} \mathcal{Q}(x) + \frac{\lambda}{h^\alpha} \mathcal{Q}((\lambda + n)h)) \\
&= \lambda' A_{h,1}^\alpha \mathcal{Q}(x) - \frac{\lambda}{h} A_{h,0}^{\alpha-1} \mathcal{Q}(x) + \frac{\lambda}{h^\alpha} \mathcal{Q}((\lambda + n)h)) + O(h^\alpha) \\
&= A_{h,1}^\alpha \mathcal{Q}(x) - \frac{\lambda}{h} D^{\alpha-1} \mathcal{Q}(1) + \frac{\lambda}{h^\alpha} \mathcal{Q}((\lambda + n)h)) + O(1),
\end{aligned}$$

since $|\mathcal{Q}(\lambda h)| = O(h^\alpha)$ and $\left|\mathcal{G}_{n-1}^{\alpha-1}\right| = O(h^\alpha)$. Lastly, let $x \in [nh, 1]$, then using (4.130) we have

$$\begin{aligned}
G^h \mathcal{Q}(x) &= \frac{1}{h^\alpha} \left(- \lambda' \sum_{k=1}^{n-1} \mathcal{G}_k^{\alpha-1} \mathcal{Q}((\lambda + n - k)h) - \mathcal{G}_0^{\alpha-1} \mathcal{Q}((\lambda + n)h) \right) \\
&= \frac{\lambda'}{h^\alpha} \mathcal{G}_n^{\alpha-1} \mathcal{Q}(\lambda h) - \frac{\lambda'}{h^\alpha} \sum_{k=0}^{n} \mathcal{G}_k^{\alpha-1} \mathcal{Q}((\lambda + n - k)h) - \frac{\lambda}{h^\alpha} \mathcal{G}_0^{\alpha-1} \mathcal{Q}((\lambda + n)h) \\
&= -\frac{\lambda'}{h} A_{h,0}^{\alpha-1} \mathcal{Q}(x) - \frac{\lambda}{h^\alpha} \mathcal{Q}((\lambda + n)h) + O(h^\alpha) \\
&= -\frac{\lambda'}{h} \left(D^{\alpha-1} \mathcal{Q}(1) + h\left((\lambda - 1) - \frac{\alpha - 1}{2} \right) D^\alpha \mathcal{Q}(1) + O(h^2) \right) \\
&\qquad - \frac{\lambda}{h^\alpha} \mathcal{Q}((\lambda + n)h) + O(h^\alpha) \\
&= A_{h,1}^\alpha \mathcal{Q}(x) - \frac{\lambda'}{h} D^{\alpha-1} \mathcal{Q}(1) - \frac{\lambda}{h^\alpha} \mathcal{Q}((\lambda + n)h) + O(1),
\end{aligned}$$

since $|\mathcal{G}_n^{\alpha-1}| = O(h^\alpha)$ and $|\mathcal{Q}(\lambda h)| = O(h^\alpha)$.

The proof of Statement 2 of Proposition 4.3.2 for all the fractional derivative operators on $L_1[0, 1]$ given in Table 4.7 is complete.

This also completes the proof of Proposition 4.3.2. □

Appendix A

Properties of Grünwald coefficients

The *Grünwald coefficients*, which we denote by $\mathcal{G}_m^\alpha$, play a significant role in our study of numerical approximations of fractional derivative operators. We list here their relevant properties, see [84, p. 16–21].

The Grünwald coefficients expressed as a quotient of Gamma functions are related to the binomial coefficients by

$$\mathcal{G}_m^\alpha = \frac{\Gamma(m-\alpha)}{\Gamma(-\alpha)\Gamma(m+1)} = (-1)^m \binom{\alpha}{m} = \binom{m-\alpha-1}{m}, \tag{A.1}$$

where $\alpha \in \mathbb{R}$ and $m \in \mathbb{N}_0$. Note the recurrence relation

$$\mathcal{G}_{n+1}^\alpha = \frac{n-\alpha}{n+1}\mathcal{G}_n^\alpha,\ \mathcal{G}_0^\alpha = 1,\ n \in \mathbb{N} \tag{A.2}$$

and that

$$\mathcal{G}_m^\alpha = 0,\ \text{if } m \leq -1,\ m \in \mathbb{Z}. \tag{A.3}$$

Setting $x = 1$ in the Binomial series,

$$(1-x)^\alpha = \sum_{m=0}^{\infty} \mathcal{G}_m^\alpha x^m, \tag{A.4}$$

we have

$$\sum_{m=0}^{\infty} \mathcal{G}_m^\alpha = 0. \tag{A.5}$$

As a consequence, we have the following relation between the partial and tail sums of the Grünwald coefficients,

$$\sum_{m=0}^{k} \mathcal{G}_m^\alpha = -\sum_{m=k+1}^{\infty} \mathcal{G}_m^\alpha. \tag{A.6}$$

The partial sum of the Grünwald coefficients is also given by

$$\sum_{m=0}^{k} \mathcal{G}_m^\alpha = \mathcal{G}_k^{\alpha-1}. \tag{A.7}$$

The following summation formula for the product of binomial coefficients is well known which we rewrite in terms of Grünwald coefficients,

$$\sum_{m=0}^{k} \mathcal{G}_m^q \mathcal{G}_{k-m}^Q = (-1)^k \sum_{m=0}^{k} \binom{q}{m}\binom{Q}{k-m} = (-1)^k \binom{q+Q}{k} = \mathcal{G}_k^{q+Q}. \tag{A.8}$$

The quotient of Gamma functions has the asymptotic behaviour,

$$\frac{\Gamma(m-\alpha)}{\Gamma(m+1)} = m^{-1-\alpha}\left[1 + \frac{\alpha(\alpha+1)}{2m} + O(m^{-2})\right],\ m \to \infty,$$

where $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. As a consequence, for $\alpha \notin \mathbb{N}_0$, we have the asymptotics

$$\mathcal{G}_m^\alpha = \frac{m^{-1-\alpha}}{\Gamma(-\alpha)}\left[1 + O(m^{-1})\right] \tag{A.9}$$

and the absolute convergence of the series for $\alpha > -1$,

$$\sum_{m=0}^{\infty} |\mathcal{G}_m^\alpha| < \infty. \tag{A.10}$$

Appendix B

Reminder on function spaces, distributions and transforms

B.1 Function spaces

We say that a property holds *locally* on $\mathbb{R}$ if it holds for any finite interval. We collate the definitions of the various function spaces that we work with.

Definition B.1.1 (Function spaces).

1. Let $\Omega \subset \mathbb{R}$, then $L_r(\Omega)$ for $1 \le r \le \infty$, denotes the space of complex-valued Lebesgue measurable functions on Ω that satisfy $\|f\|_{L_r(\Omega)} < \infty$, where the $L_r(\Omega)$-norm is given by

$$\|f\|_{L_r(\Omega)} = \begin{cases} \left(\int_\Omega |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} & \text{if } 1 \le p < \infty, \\ \|f\|_\infty = \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| & \text{if } p = \infty. \end{cases}$$

   It is well known that $L_r(\Omega)$ with the usual pointwise addition and scalar multiplication can be made into a *Banach space* by identifying functions that are equal almost everywhere.

2. $AC[a,b]$ denotes the space of absolutely continuous functions on $[a,b]$; that is, for $u \in AC[a,b]$, there exists $v \in L_1[a,b]$ such that $u(x) = u(a) + \int_a^x v(t)dt$ for $x \in [a,b]$.

3. Let D^n denote the operator theoretic n^{th} power of the *generalised derivative operator* D, where $D : AC[0,1] \subset L_1[0,1] \to L_1[0,1]$ is given by $Df(x) = f'(x)$ for almost all $x \in [0,1]$.

4. $W^{r,1}(\mathbb{R})$ for $r \geq 1$, denotes the Sobolev space of $L_r(\mathbb{R})$-functions with generalised first derivative in $L_r(\mathbb{R})$; that is, $f \in W^{r,1}(\mathbb{R})$ if $f \in L_r(\mathbb{R})$, f is locally absolutely continuous, and $Df \in L_r(\mathbb{R})$.

5. $W^{r,1}_{per}[-\pi, \pi]$ for $r \geq 1$, denotes the Sobolev space of 2π-periodic functions g on $\mathbb{R}$ where both g and its generalized derivative Dg belong to $L_r[-\pi, \pi]$.

6. Let $W^{1,n}[0,1]$ for $n \in \mathbb{N}$, denote the Sobolev space of functions such that the function along with its generalised derivatives up to order n belong to $L_1[0,1]$,

$$W^{1,n}[0,1] = \left\{u \in L_1[0,1] : D^k u \in L_1[0,1],\ k = 1, 2, ...n\right\}.$$

In particular, for $n = 1$, $W^{1,1}[0,1] = AC[0,1]$ while for $n = 2$, u is continuously differentiable and $Du \in AC[0,1]$ so that $D^2 u \in L_1[0,1]$.

We list some of the well known inequalities.

1. The well known Hausdorff-Young-Titchmarsh inequality [23, p.211] is given by

$$\|\hat{u}\|_{L_s} \leq (2\pi)^{\frac{1}{s}} \|u\|_{L_r}, u \in L_r,\ 1 \leq r \leq 2,\ \frac{1}{r} + \frac{1}{s} = 1. \tag{B.1}$$

2. Let $p, r, s \geq 1$ be such that $\frac{1}{r} + \frac{1}{s} = \frac{1}{p} + 1$. Let $f \in L_r(\mathbb{R})$, $g \in L_s(\mathbb{R})$. Then, the Young's inequality for the convolution $f * g$ is given by

$$\|f * g\|_{L_p(\mathbb{R})} \leq C \|f\|_{L_r(\mathbb{R})} \|g\|_{L_s(\mathbb{R})}. \tag{B.2}$$

B.2 Fourier and Laplace Transforms

Definition B.2.1 (Fourier transform). If $f \in L_1(\mathbb{R})$, then we define the *Fourier transform* $\hat{f}$ of f by

$$\hat{f}(k) = \int_{\mathbb{R}} e^{ikx} f(x)\, \mathrm{d}x, \quad k \in \mathbb{R}. \tag{B.3}$$

Moreover, if $\hat{f} \in L_1(\mathbb{R})$, then the inverse Fourier transform is given by

$$\check{f}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ikx} \hat{f}(k)\, \mathrm{d}k, \quad x \in \mathbb{R}. \tag{B.4}$$

Remark B.2.2. It is well known that the Fourier transform of $f \in L_1(\mathbb{R})$ is bounded and belongs to $C_0(\mathbb{R})$. For $1 \leq r \leq 2$, to define the Fourier transform of $f \in L_r(\mathbb{R})$ (and similarly, the inverse Fourier transform of $\hat{f} \in L_r(\mathbb{R})$) where $L_r(\mathbb{R})$ is given by Definition B.1.1, it is first defined for $f \in L_r(\mathbb{R}) \cap L_1(\mathbb{R})$ as in Definition B.3 above.

Then, using the fact that $L_r(\mathbb{R}) \cap L_1(\mathbb{R})$ is dense in $L_r(\mathbb{R})$, the definition above is then extended to the whole of $L_r(\mathbb{R})$ using (B.1). Thus, the Fourier transform of $f \in L_r(\mathbb{R})$ is a function $\hat{f} \in L_s(\mathbb{R})$, where $\frac{1}{r} + \frac{1}{s} = 1$.

For $f \in W^{n,r}(\mathbb{R})$ where $1 \leq r \leq 2$, the Fourier transform of the integer order derivative is well known and is given by [23, page 195]

$$\widehat{D^n f}(k) = (-ik)^n \hat{f}(k). \tag{B.5}$$

We also make note of the following theorem on the uniqueness of Fourier transforms, see [92, p. 187].

Theorem B.2.3. *If $f \in L_1(\mathbb{R})$ and $\hat{f}(k) = 0$ for all $k \in \mathbb{R}$, then $f(x) = 0$ almost everywhere.*

For functions that are defined only on the half-line $\mathbb{R}^+$, the Laplace transform is defined as follows.

Definition B.2.4. Let $f \in L_1(\mathbb{R}^+)$ then we define the *Laplace transform* by

$$\hat{f}(z) = \int_{\mathbb{R}^+} e^{zt} f(t)\, \mathrm{d}t, \ \mathrm{Re}(z) \leq 0. \tag{B.6}$$

Moreover, it is well known that the Laplace transform of $L_1(\mathbb{R}^+)$-function is analytic for $\mathrm{Re}(z) < 0$ and continuous for $\mathrm{Re}(z) \leq 0$.

Remark B.2.5. Note that when there is no confusion, we use the same notation $\hat{f}$ and $\check{f}$ to denote both the Fourier and Laplace transforms and their inverses, respectively. However, if and when the need arises, we use $\mathcal{F}(f)$ and $\mathcal{L}(f)$ for Fourier and Laplace transforms, respectively.

Lastly, both the Fourier and Laplace transforms have the following translation property,

$$\begin{aligned} \mathcal{F}(f(x-a))(k) &= e^{iak}\mathcal{F}(f(x))(k), \ a \in \mathbb{R} \\ \mathcal{L}(f(x-a))(z) &= e^{az}\mathcal{L}(f(x))(z), \ a \in \mathbb{R}^+. \end{aligned} \tag{B.7}$$

B.3 Distributions

The contents of this section are adapted from [25, 111].

Let $\mathcal{S}$ denote the space of *Schwartz* functions of rapid descent on $\mathbb{R}$. Let $\mathcal{S}'$ denote the space of *tempered distributions* (continuous, linear functionals on $\mathcal{S}$). If f is a locally integrable function, then the so-called *regular distribution* $\tilde{f} \in \mathcal{S}'$ is given by

$$< \tilde{f}, \theta >= \int_{\mathbb{R}} f(t)\theta(t)\,\mathrm{d}t, \;\; \theta \in \mathcal{S}. \tag{B.8}$$

We now define the distributional Fourier and inverse Fourier transforms.

Definition B.3.1. If $\tilde{f} \in \mathcal{S}'$, then we define the *Fourier transform* $\hat{\tilde{f}} \in \mathcal{S}'$ by

$$\left\langle \hat{\tilde{f}}, \theta \right\rangle = \left\langle \tilde{f}, \hat{\theta} \right\rangle,$$

where $\theta \in \mathcal{S}$ and $\hat{\theta}$ is given by (B.3). In a similar fashion, for $\tilde{g} \in \mathcal{S}'$, the inverse Fourier transform $\check{\tilde{g}} \in \mathcal{S}'$ is defined by

$$\langle \check{\tilde{g}}, \psi \rangle = \langle \tilde{g}, \check{\psi} \rangle,$$

where $\psi \in \mathcal{S}$ and $\check{\psi}$ is given by (B.4).

Remark B.3.2. The regular distributions corresponding to the ordinary Fourier and inverse Fourier transforms are uniquely determined. We denote the ring of (distributional) Fourier transforms of L_1-functions by $\mathcal{F}(L_1)$. In general, $\tilde{f} \in \mathcal{F}(L_1)$ if there exists $g \in L_1(\mathbb{R})$ such that $\tilde{f} = \hat{\tilde{g}}$, where $\tilde{g}$ denotes the regular distribution corresponding to the function g.

For $\tilde{f}_n, \tilde{f} \in \mathcal{S}'$, we say that $\tilde{f}_n$*converges in* $\mathcal{S}'$ *to* $\tilde{f}$ if $\left\langle \tilde{f}_n, \theta \right\rangle \to \left\langle \tilde{f}, \theta \right\rangle$ for all $\theta \in \mathcal{S}$.

Theorem B.3.3. *Let the distributional Fourier and inverse Fourier transforms be as in Definition B.3.1. Then we have the following:*

1. *The Fourier transform and its inverse are continuous linear mappings of* $\mathcal{S}'$ *onto itself.*

2. *If the series* $\sum_{n=1}^{\infty} \tilde{f}_n$ *converges in* $\mathcal{S}'$ *to* $\tilde{f}$*, then*

$$\hat{\tilde{f}} = \mathcal{F}\left(\sum_{n=1}^{\infty} \tilde{f}_n\right) = \sum_{n=1}^{\infty} \mathcal{F}\left(\tilde{f}_n\right) \in \mathcal{S}',$$

that is, the distributional Fourier transformation can be applied to a series term by term.

Remark B.3.4. To define the function z^α where $z \in \mathbb{C}$ and $\alpha \in \mathbb{R}$ is fixed, following the standard practice in complex analysis [29], we take the negative real axis as the branch cut and define $z^\alpha := e^{\alpha \log z} = |z|^\alpha e^{i\alpha \arg z}$ where $-\pi < \arg(z) < \pi$ and $\log z$ denotes the principal branch of the logarithm, defined on the open connected set $\mathbb{C}/\{z \in \mathbb{R} : z \leq 0\}$.

The *Gamma function* has the following well-known integral representation,

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, \mathrm{d}t, \ \mathrm{Re}(\alpha) > 0. \tag{B.9}$$

This integral representation of the gamma function has the following general version, see [95, p. 137],

$$\int_0^\infty \phi_{\alpha-1}(t) e^{-zt}\, \mathrm{d}t = z^{-\alpha}, \ z \neq 0, \tag{B.10}$$

where $\phi_{\alpha-1}$ is given by (1.24), $\mathrm{Re}(\alpha) > 0$, if $\mathrm{Re}(z) > 0$ and $0 < \mathrm{Re}(\alpha) < 1$, if $\mathrm{Re}(z) = 0$.

We conclude with a result similar to B.10 for Fourier transforms, that we require in Section 1.4. We use $(-ik)^{-\alpha}$ for $\alpha > 0$ to denote both the function and the corresponding regular distribution in $\mathcal{S}'$.

Proposition B.3.5. *Let* $\phi_{\alpha-1}$ *be given by* (1.24)*, then the distributional Fourier transform of* $\phi_{\alpha-1}$ *is a regular tempered distribution,*

$$\hat{\tilde{\phi}}_{\alpha-1}(k) = (-ik)^{-\alpha}, \ \alpha \in \mathbb{R}^+,$$

Proof. We give a brief sketch of the proof. Let $\alpha\,, \epsilon \in \mathbb{R}^+$ and consider $\tilde{f}_\epsilon$, the regular distribution associated with the function $f_\epsilon(x) = e^{-\epsilon x}\phi_{\alpha-1}(x)$. Then, for each $\epsilon > 0$, $f_\epsilon \in L_1(\mathbb{R})$, $\tilde{f}_\epsilon \in \mathcal{S}'$ and $\tilde{f}_\epsilon$ converges to $\tilde{\phi}_{\alpha-1}$ in $\mathcal{S}'$ as $\epsilon \to 0$. Moreover, the distributional Fourier transform $\hat{\tilde{f}}_\epsilon \in \mathcal{S}'$ is given by the regular distribution, see [25, p. 139]

$$\hat{\tilde{f}}_\epsilon(k) = (\epsilon - ik)^{-\alpha} = \frac{e^{-i\alpha \tan^{-1}(\frac{k}{\epsilon})}}{(k^2 + \epsilon^2)^{\frac{\alpha}{2}}}.$$

As $\epsilon \to 0$, $\hat{\tilde{f}}_\epsilon$ converges to $(-ik)^{-\alpha}$ in $\mathcal{S}'$. $\square$

Appendix C

Reminder on operator theory, semigroups and multipliers

The contents of this appendix are adapted from the treatises, [1, 4, 37, 47]. In what follows, let $(X, \|\cdot\|_X)$ denote a Banach space.

C.1 Operator theory

Let $(A, \mathcal{D}(A))$ denote a linear operator $A : \mathcal{D}(A) \subset X \to X$ with domain $\mathcal{D}(A)$.

Definition C.1.1. Let $\mathcal{B}(X)$ denote the space of all bounded linear operators $A : X \to X$ with *operator norm* given by

$$\|A\|_{\mathcal{B}(X)} := \sup \{\|Af\|_X : \|f\|_X = 1\}, \ A \in \mathcal{B}(X).$$

Definition C.1.2 (Graph norm and core). The *graph norm* of $(A, \mathcal{D}(A))$ is given by

$$\|f\|_A := \|f\|_X + \|Af\|_X, \ f \in \mathcal{D}(A).$$

A subspace $\mathcal{C}(A)$ of $\mathcal{D}(A)$ is called a *core* for A if it is dense in $\mathcal{D}(A)$ in the graph norm.

Definition C.1.3 (Closed operator). Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(A)$ be an arbitrary sequence such that $\|f_n - f\|_X \to 0$ and $\|Af_n - g\|_X \to 0$. Then, $(A, \mathcal{D}(A))$ is said to be *closed*, if $f \in \mathcal{D}(A)$ and $Af = g$.

Definition C.1.4 (Invertible operator). An operator A on X is said to be *invertible* if there exists $B \in \mathcal{B}(X)$ such that $BAf = f$ for all $f \in \mathcal{D}(A)$, and $Bg \in \mathcal{D}(A)$ and $ABg = g$ for all $g \in X$. Moreover, A is invertible if and only if A is closed, the range $\mathrm{rg}(A) = X$ and the kernel $\mathrm{Ker}(A) = \{\mathbf{0}\}$, see [1, p. 462].

Definition C.1.5. For a closed linear operator $(A, \mathcal{D}(A))$, its resolvent set is given by

$$\rho(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is invertible}\}$$

and its spectrum by $\sigma(A) := \mathbb{C}/\rho(A)$. The operator $R(\lambda, A) := (\lambda I - A)^{-1} \in \mathcal{B}(X)$ is called the *resolvent* operator, where I denotes the identity operator on X.

Definition C.1.6 (Dissipative operator). A linear operator $(A, \mathcal{D}(A))$ is called *dissipative* if

$$\|(\lambda I - A)f\|_X \geq \lambda \|f\|_X ,$$

for all $\lambda > 0$ and $f \in \mathcal{D}(A)$.

If an operator A has a dense domain in X, we refer to A as a *densely defined* operator and we define its adjoint operator on the dual space X^* as follows.

Definition C.1.7 (Adjoint operator). For a densely defined operator $(A, \mathcal{D}(A))$ on X, the adjoint operator $(A^*, \mathcal{D}(A^*))$ on X^* is defined by

$$\begin{aligned}
&\mathcal{D}(A^*) := \{x^* \in X^* : \ \exists\, y^* \in X^* \text{ such that } \langle Ax, x^*\rangle = \langle x, y^*\rangle \ \forall x \in \mathcal{D}(A)\},\\
&A^*x^* = y^* \text{ for } x^* \in \mathcal{D}(A^*).
\end{aligned}$$

Let Y be a Banach space that is continuously embedded in X denoted by $Y \hookrightarrow X$.

Definition C.1.8 (Part of an operator). The *part* of A in Y is the operator $A|_Y$ defined by

$$\begin{aligned}
&A|_Y y := Ay\\
&\mathcal{D}(A|_Y) := \{y \in \mathcal{D}(A) \cap Y : Ay \in Y\}
\end{aligned}$$

Definition C.1.9 (Positive operator). Let $f : X \to \mathbb{R}$ where $X = C_0(\Omega)$ or $L_1(\Omega)$ as given in Definition B.1.1 and Ω is a locally compact topological space. Then f is called a *positive function* on $C_0(\Omega)$ $(L_1(\Omega))$ denoted by $f \geq 0$, if $f(x) \geq 0$ for all (almost all) $x \in \Omega$. A linear operator $T : X \to X$ is called a *positive operator*, if $Tf \geq 0$ whenever $f \geq 0$.

C.2 Semigroups

Definition C.2.1 (Semigroup). A family $(T(t))_{t\geq 0}$ of bounded linear operators on X, is called a (one-parameter) *semigroup* (on X), if

$$T(t+s) = T(t)T(s), \text{ for all } t, s \geq 0, \tag{C.1}$$

$$T(0) = I,$$

where I denotes the identity operator on X. Moreover, if (C.1) is satisfied for all $t, s \in \mathbb{R}$, then $(T(t))_{t\in\mathbb{R}}$ is called a (one-parameter) *group* on X.

Definition C.2.2 (Strongly continuous semigroup). A semigroup is called a *strongly continuous* semigroup or C_0-semigroup, if the functions $t \mapsto T(t)f$ are continuous from $[0,\infty)$ into X for all $f \in X$. Moreover, a group is called a strongly continuous group or C_0-group, if the functions $t \mapsto T(t)f$ are continuous from $\mathbb{R}$ into X for all $f \in X$.

Definition C.2.3 (Bounded semigroup). A strongly continuous semigroup (group) is called *bounded*, if for some $M \geq 1$, $\|T(t)\|_{\mathcal{B}(X)} \leq M$ for all $t \geq 0$ ($t \in \mathbb{R}$). A strongly continuous semigroup (group) is called *contractive*, if $\|T(t)\|_{\mathcal{B}(X)} \leq 1$ for all $t \geq 0$ ($t \in \mathbb{R}$).

Definition C.2.4 (Generator of a semigroup). The generator $A : \mathcal{D}(A) \subset X \to X$ of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X is the operator

$$Af := \lim_{h\downarrow 0} \frac{T(h)f - f}{h},$$

defined for every $f \in \mathcal{D}(A)$ where

$$\mathcal{D}(A) := \left\{ f \in X : \lim_{h\downarrow 0} \frac{T(h)f - f}{h} \text{ exists} \right\}$$

Definition C.2.5 (Positive semigroup). A semigroup $(T(t))_{t\geq 0}$ on X is called *positive* if $T(t) \geq 0$ for all $t \geq 0$.

Moreover, if the semigroup has generator A then the semigroup is positive if and only if $R(\lambda, A) \geq 0$ for sufficiently large real λ.

Let the sector Σ_δ be defined by

$$\Sigma_\delta := \{\lambda \in \mathbb{C} : |\arg \lambda| < \delta\} / \{0\}.$$

Definition C.2.6 (Sectorial operator). A closed linear operator A with dense domain $\mathcal{D}(A)$ in X is called *sectorial* (of angle θ), if there exists $0 < \theta \leq \frac{\pi}{2}$ such that the following are satisfied:

- The sector $\Sigma_{\frac{\pi}{2}+\theta}$ is contained in the resolvent set $\rho(A)$ given by Definition C.1.5.

- For each $\epsilon \in (0, \theta)$ there exists a constant $M_\epsilon \geq 1$ such that

$$\|R(\lambda, A)\|_{\mathcal{B}(X)} \leq \frac{M_\epsilon}{|\lambda|}, \text{ for all } \lambda \in \overline{\Sigma}_{\frac{\pi}{2}+\theta-\epsilon} / \{0\},$$

where the resolvent operator $R(\lambda, A)$ is given by Definition C.1.5.

Definition C.2.7 (Analytic semigroup). A family $(T(z))_{z \in \Sigma_\delta \cup \{0\}}$ of bounded linear operators is called an *analytic* semigroup (of angle $\delta \in (0, \frac{\pi}{2}]$) if

1. $T(0) = I$ and $T(z_1 + z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in \Sigma_\delta$.
2. The map $z \mapsto T(z)$ is analytic in Σ_δ.
3. $\lim_{z \to 0} T(z)f = f$ for all $f \in X$, $z \in \Sigma_{\delta'}$ and $0 < \delta' < \delta$.

Moreover, if $\|T(z)\|_{\mathcal{B}(X)}$ is bounded in the sector $\Sigma_{\delta'}$ for each $0 < \delta' < \delta$, then $(T(z))_{z \in \Sigma_\delta \cup \{0\}}$ is called a *bounded analytic* semigroup.

We state the following well-known important theorems without proof. For proofs we refer the reader to, for example [37].

Theorem C.2.8. *Let A be an operator on X with domain $\mathcal{D}(A)$. Then the following statements are equivalent:*

1. *A is sectorial.*
2. *A generates a bounded analytic semigroup $(T(z))_{z \in \Sigma_\delta \cup \{0\}}$ on X.*
3. *A generates a bounded strongly continuous semigroup $(T(t))_{t \geq 0}$ on X such that $\mathrm{rg}(T(t)) \subset \mathcal{D}(A)$ for all $t > 0$, and*

$$M := \sup_{t > 0} \|tAT(t)\|_{\mathcal{B}(X)} < \infty.$$

Theorem C.2.9 (Lumer-Phillips theorem). *For a densely defined, dissipative operator $(A, \mathcal{D}(A))$ on X the following statements are equivalent:*

1. *The closure $\overline{A}$ of A generates a contraction semigroup.*
2. *$\mathrm{rg}(\lambda - A)$ is dense in X for some (hence all) $\lambda > 0$.*

Theorem C.2.10 (Trotter-Kato approximation theorem). *Let $(T_n(t))_{t\geq 0}$, $n \in \mathbb{N}$ and $(T(t))_{t\geq 0}$ be strongly continuous semigroups on X with generators A_n and A, respectively. Further, assume that they satisfy the estimate*

$$\|T(t)\|_{\mathcal{B}(L_1)}, \ \|T_n(t)\|_{\mathcal{B}(L_1)} \leq Me^{\omega t}, \text{ for all } t \geq 0, \ n \in \mathbb{N},$$

for some constants $M \geq 1$, $\omega \in \mathbb{R}$. Let $\mathcal{C}(A)$ be a core for A and consider the following statements:

(a) $\mathcal{C}(A) \subset \mathcal{D}(A_n)$ for all $n \in \mathbb{N}$ and $A_n x \to Ax$ for all $x \in \mathcal{C}(A)$.

(b) For each $x \in \mathcal{C}(A)$, there exists $x_n \in \mathcal{D}(A_n)$ such that

$$x_n \to x \text{ and } A_n x_n \to Ax.$$

(c) $R(\lambda, A_n)x \to R(\lambda, A)x$ for all $x \in X$ and some, hence all $\lambda > \omega$.

(d) $T_n(t)x \to T(t)x$ for all $x \in X$, uniformly for $t \in [0, t_0]$.

Then we have the following implications

$$(a) \Rightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d),$$

while $(b) \not\Rightarrow (a)$.

Definition C.2.11. The initial value problem

$$\begin{aligned} u'(t) &= Au(t) \quad \text{for } t \geq 0, \\ u(0) &= x, \end{aligned} \tag{C.2}$$

is called the *abstract Cauchy problem* associated to $(A, \mathcal{D}(A))$ and the initial value x.

Definition C.2.12. Let $u : \mathbb{R}^+ \to X$. Then

- u is called a *classical* solution of (C.2) if u is continuously differentiable with respect to X, $u(t) \in \mathcal{D}(A)$ for all $t \geq 0$, and (C.2) holds.
- u is called a *mild* solution of (C.2), if u is continuous, $\int_0^t u(s)\,\mathrm{d}s \in \mathcal{D}(A)$ for all $t \geq 0$ and
$$u(t) = A\int_0^t u(s)\,\mathrm{d}s + x.$$

Theorem C.2.13. *If $(A, \mathcal{D}(A))$ is the generator of the strongly continuous semigroup $(T(t))_{t\geq 0}$, then:*

- *For every $x \in \mathcal{D}(A)$, the function*

$$u : t \mapsto u(t) := T(t)x$$

is the unique classical solution of (C.2).

- *For every $x \in X$, the function*

$$u : t \mapsto u(t) := T(t)x$$

is the unique mild solution of (C.2).

Definition C.2.14. The abstract Cauchy problem (C.2) is called *well-posed* if the following hold:

1. For every $x \in \mathcal{D}(A)$, there exists a unique classical solution $u(\cdot, x)$ for (C.2).

2. $\mathcal{D}(A)$ is dense in X.

3. For every sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(A)$ such that $\lim_{n\to\infty} x_n = 0$, one has

$$\lim_{n\to\infty} u(t, x_n) = 0$$

uniformly in compact intervals $[0, t_0]$.

Theorem C.2.15. *For a closed operator $A : \mathcal{D}(A) \subset X \to X$, the associated abstract Cauchy problem* (C.2) *is well-posed if and only if A generates a strongly continuous semigroup on X.*

C.3 Multipliers

In what follows, let $\mathcal{B}(\mathbb{R})$ denote the Borel σ-algebra and $\mathcal{M}_{\mathcal{B}}(\mathbb{R})$ the set of all bounded (complex) Borel measures on $\mathbb{R}$.

Definition C.3.1. The *Dirac* measure concentrated at $k \in \mathbb{R}$ is denoted by δ_k where for $E \in \mathcal{B}(\mathbb{R})$,

$$\delta_k(E) = \sum_{x\in E} f(x)$$

and

$$f(x) = \begin{cases} 1, & \text{if } x = k, \\ 0, & \text{otherwise.} \end{cases}$$

A collection of sets $\{E_m\}$, is called a *partition* of $E \in \mathcal{B}(\mathbb{R})$, if $E = \cup_{m=1}^{\infty} E_m$ and for $i \neq j$, $E_i \cap E_j = \emptyset$.

Definition C.3.2. The *total variation* of a (complex) Borel measure μ on $\mathbb{R}$ is a bounded, positive measure denoted by $|\mu| : \mathbb{R} \to [0, \infty)$, where the set function $|\mu|$ is defined on $\mathcal{B}(\mathbb{R})$ by

$$|\mu|(E) = \sup \sum_{m=1}^{\infty} |\mu(E_m)|,$$

the supremum being taken over all partitions $\{E_m\}$ of E. On setting the *total variation norm* to be

$$\|\mu\|_{\mathrm{TV}} = |\mu|(\mathbb{R}),$$

and defining addition and scalar multiplication in the usual manner, $(\mathcal{M}_{\mathcal{B}}(\mathbb{R}), \|\cdot\|_{\mathrm{TV}})$ is a normed vector space, see [92, Chapter 6].

We state the following well-known result without proof, see [92, Theorem 6.13].

Theorem C.3.3. *Let μ be a complex Borel measure on $\mathbb{R}$ with density $\rho \in L_1(\mathbb{R})$; that is, $\mu(\mathrm{d}x) = \rho(x)dx$. Then,*

$$\|\mu\|_{\mathrm{TV}} = \|\rho\|_{L_1}.$$

Definition C.3.4. Let ψ be a complex-valued measurable function, $\psi : \mathbb{R} \to \mathbb{C}$. Then ψ is called a L_1*-(Fourier) multiplier*, if for each $f \in L_1(\mathbb{R})$, there exists $\phi \in L_1(\mathbb{R})$ such that $\psi \hat{f} = \hat{\phi}$ where $\hat{f}$ and $\hat{\phi}$ are the Fourier transforms of f and ϕ, respectively as defined by (B.3).

Definition C.3.5. If ψ is an L_1-multiplier, then we define the operator associated with ψ, $T_\psi : L_1(\mathbb{R}) \to L_1(\mathbb{R})$, by $T_\psi f := \phi$ where $\phi = \mathcal{F}^{-1}(\psi \mathcal{F}(f))$.

We list here the properties that we use often.

- T_ψ is an everywhere defined closed operator, thus by the closed graph theorem, $T_\psi \in \mathcal{B}(L_1(\mathbb{R}))$, where $\mathcal{B}(L_1(\mathbb{R}))$ is given by Definition C.1.1.

- For an L_1-multiplier, the range of the function ψ is contained in the spectrum $\sigma(T_\psi)$ of the operator associated with the multiplier ψ, see Definition C.1.5 for the definition of spectrum.

- A necessary and sufficient condition for ψ to be an L_1-multiplier is that ψ is the Fourier-Stieltjes transform of a bounded (complex) Borel measure; that is,

$$\psi(k) = \hat{\mu}(k) = \int_{-\infty}^{\infty} e^{ikx} \, \mu(\mathrm{d}x)$$

for some $\mu \in \mathcal{M}_{\mathcal{B}}(\mathbb{R})$. Furthermore, $\|T_\psi\|_{\mathcal{B}(L_1)} = \|\mu\|_{\mathrm{TV}}$.

- If the measure μ has a density distribution ρ; that is, $\mu(\mathrm{d}x) = \rho(x)dx$ then

$$\|T_\psi\|_{\mathcal{B}(L_1)} = \|\rho\|_{L_1} = \left\|\check{\psi}\right\|_{L_1}. \tag{C.3}$$

- Finally, we have the invariance of the operator norm under translation and scaling. Let ψ be an L_1-multiplier, $a \in \mathbb{R}$ and $h > 0$. Define $\psi^a(k) := \psi(k+a)$ and $\psi_h(k) := \psi(hk)$. Then, ψ^a and ψ_h are multipliers and

$$\|T_\psi\|_{\mathcal{B}(L_1)} = \|T_{\psi_h}\|_{\mathcal{B}(L_1)} = \|T_{\psi^a}\|_{\mathcal{B}(L_1)}. \tag{C.4}$$

We conclude this appendix with the following result, see [1, Proposition 8.1.3].

Theorem C.3.6. *Let ψ be an L_1-multiplier, then the following statements are equivalent:*

1. *$e^{t\psi}$ is an L_1-multiplier and there exist constants $M, \omega \geq 0$ such that*

$$\|T_{e^{t\psi}}\|_{\mathcal{B}(L_1)} \leq Me^{\omega t},\ t \geq 0.$$

2. *T_ψ generates a strongly continuous semigroup on $L_1(\mathbb{R})$); that is, $(e^{tT_\psi})_{t\geq 0} = (T_{e^{t\psi}})_{t\geq 0}$ is a strongly continuous semigroup on $L_1(\mathbb{R})$.*

Appendix D

MATLAB codes for Grünwald schemes

```
%%% Timeevolutionplots
%% This script plots the numerical solution to Equation (4.22) in
%  Section 4.4 at different time points,
%  by Harish Sankaranarayanan Oct 2014 as part of the PHD Thesis

%% Define Variables
alpha=1.5;
n=1000; % # of x grid points
t=[.04,.1,.2,.5];
% snapshots plotted at 0 and given times (has to be at least size 2)
BC=1:6; % make a plot for each BC
% (D^alpha_c,DD) set A=1
% (D^alpha_c,DN) set A=2
% (D^alpha_c,ND) set A=3
% (D^alpha_c,NN) set A=4
% (D^alpha,ND)   set A=5
% (D^alpha,NN)   set A=6

%% initial value function

u0fun=@(x) (x>0.3&x<=0.5).*(x-0.3)*25+(x>0.5&x<0.7).*(0.7-x)*25;
```

```
%% initialise solution
h=1/(n+1);
x=(h:h:1-h)';
u0=u0fun(x);

%% loop for different figures according to different BC
for A=BC,
    figure(A)
    GMatrix=GrunwaldMatrixBC(A,alpha,n);
    options=odeset('Jacobian',GMatrix);
    % Use MATLAB ODE solver
    [~,sol]=ode15s(@(~,x) GMatrix*x,[0,t],u0,options);
    plot(x,sol);
    xlim([0,1]);ylim([0,5])
    legend('t=0','t=0.04','t=0.1','t=0.2','t=0.5')
end

function M = GrunwaldMatrixBC(A,alpha,n)
% This function computes the entries of the n x n Grunwald
% matrix given by Equation (4.23)
% for the fractional derivative operators
% on L_1 [0,1] given in Table 4.4 .
% alpha = 1.5
% n denotes the size of the matrix

% (D^alpha_c,DD) set A=1
% (D^alpha_c,DN) set A=2
% (D^alpha_c,ND) set A=3
% (D^alpha_c,NN) set A=4
% (D^alpha,ND)   set A=5
% (D^alpha,NN)   set A=6

%% build lower triangular n by n Grunwald matrix
M=zeros(n);

shift=1;
```

```
w=1;
% construct Grunwald coefficients given by Equation (A.1)
for k=1:n+shift;
    M=M+diag(w*ones(n-abs(k-1-shift),1),1-k+shift);
    w=w*(k-alpha-1)/k;
end
%% sum(A,2) adds the row entries while sum(A) adds the column entries
% M is the Transition matrix for L1 with default A=1 for DalphacDD
%% Boundary weights
switch A
    case 2, %DalphacDN
        M(end,:)=-sum(M(1:end-1,:)); % boundary weights b^r_i
        M(end,1)=M(end,1)-M(1,2); % overwrite b_n
    case 3, %DalphacND
        M(:,1)= -sum((M(:,2:end)),2); % boundary weights b^l_i
        M(end,1)= M(end,1)-M(end-1,end);% overwrite b_n
    case 4, %DalphacNN
        M(:,1)= -sum((M(:,2:end)),2); % boundary weights b^l_i
        M(end,:)=-sum(M(1:end-1,:)); % boundary weights b^r_i
    case 5, %DalphaND
        M(1,1)=M(1,1)+M(1,2); % change only b_l_1
    case 6, %DalphaNN
        M(1,1)=M(1,1)+M(1,2); % change only b_l_1
        M(end,:)=-sum(M(1:end-1,:)); % boundary weights b^r_i and b_n
end
%% scale Matrix with h^(-alpha)
M=(n+1)^alpha* M;
```

References

[1] W. Arendt, C. J. K Batty, M Hieber, and F Neubrander, *Vector-valued Laplace transforms and Cauchy problems*, Springer, 2000.

[2] W. Arendt, A. Grabosch, G. Greiner, U. Groh, H. P. Lotz, U. Moustakas, R. Nagel, F. Neubrnader, and U. Schlotterbeck, *One-parameter Semigroups of Positive Operators*, Springer-Verlag, 1986.

[3] B. Baeumer, D. A Benson, M. M. Meerschaert, and S. W. Wheatcraft, *Subordinated advection-dispersion equation for contaminant transport*, Water Resources Research **37** (2001), no. 6, 1543–1550.

[4] B. Baeumer, M. Haase, and M. Kovács, *Unbounded functional calculus for bounded groups with applications*, Journal of Evolution Equations **9** (2009), no. 1, 171–195.

[5] B. Baeumer, M. Kovács, and M. M. Meerschaert, *Fractional Reproduction-Dispersal Equations and Heavy Tail Dispersal Kernels*, Bulletin of Mathematical Biology **69** (2007), no. 7, 2281–2297.

[6] ———, *Numerical solutions for fractional reaction-diffusion equations*, Computers and Mathematics with Applications **55** (2008), no. 10, 2212–2226, Advanced Numerical Algorithms for Large-Scale Computations.

[7] B. Baeumer, M. Kovács, M. M. Meerschaert, R. L. Schilling, and P. Straka, *Reflected stable subordinators for fractional Cauchy problems*, Transactions of the American Mathematical Society (2014), 1–22, To appear.

[8] B. Baeumer, M. Kovács, and H. Sankaranarayanan, *Higher order Grünwald approximations of fractional derivatives and fractional powers of operators*, Transactions of the American Mathematical Society **Online** (September 4, 2014), 1–21, DOI: http://dx.doi.org/10.1090/S0002-9947-2014-05887-X.

[9] B. Baeumer, M. M. Meerschaert, and E. Nane, *Space–time duality for fractional diffusion*, Journal of Applied Probability **46** (2009), 1100–1115.

[10] B. Baeumer, Y. Zhang, and R. Schumer, *Incorporating the influence of subgrid heterogeneity in regional-scale contaminant transport models*, Groundwater (2013), 1–20, To appear.

[11] A. V. Balakrishnan, *Fractional powers of closed operators and the semigroups generated by them*, Pacific Journal of Mathematics **10** (1960), no. 2, 419–437.

[12] J. M. Ball, *Strongly continuous semigroups, weak solutions, and the variation of constants formula*, Proceedings of the American Mathematical Society **63** (1977), no. 2, 370–373.

[13] E. Barkai, R. Metzler, and J. Klafter, *From continuous time random walks to the fractional Fokker-Planck equation*, Physical Review E **61** (2000), no. 1, 132.

[14] R. Bellman, *An integral inequality*, Duke Mathematics Journal **10** (1943), 547–550.

[15] D. A. Benson, *The Fractional Advection–Dispersion Equation: Development and Application*, Ph.D. thesis, University of Nevada, 1998.

[16] D. A. Benson, C. Tadjeran, M. M. Meerschaert, I. Farnham, and G. Pohll, *Radial fractional-order dispersion through fractured rock*, Water Resources Research **40** (2004), no. 12, W12416.

[17] D. A. Benson, S. W. Wheatcraft, and M. M. Meerschaert, *Application of a fractional advection-dispersion equation*, Water Resources Research **36** (2000), no. 6, 1403–1412.

[18] A. Beurling, *Sur les intégrales de Fourier absolutement convergentes et leur application à une transformation fonctionelle*, C. R. Neuvième Congreès Math. Scandinaves 1938 **II.8; V.1, 4, VII.13** (1939), 345–366.

[19] A. Blumen, G. Zumofen, and J. Klafter, *Transport aspects in anomalous diffusion: Lévy walks*, Physical Review A **40** (1989), no. 7, 3964.

[20] J. P. Bouchaud and A. Georges, *Anomalous diffusion in disordered media - statistical mechanisms, models and physical applications*, Physics Reports-Review Section of Physics Letters **195** (1990), no. 4-5, 127–293.

[21] P. Brenner and V. Thomée, *On rational approximation of semigroups*, SIAM Journal of Numerical Analysis **16** (1979), no. 4, 683–694.

[22] P. Brenner, V. Thomée, and L. B. Wahlbin, *Besov Spaces and Applications to Difference Methods for Initial Value Problems*, Springer-Verlag, Berlin Heidelberg New York, 1975.

[23] P. L. Butzer and R. J. Nessel, *Fourier analysis and approximation*, Academic Press, New York and London, 1971.

[24] F. Carlson, *Une Inégalité*, Arkiv för Matematik **25B** (1935), 1–5.

[25] D. C. Champeney, *A handbook of Fourier theorems*, Cambridge University Press, 1987.

[26] A. S. Chaves, *A fractional diffusion equation to describe Lévy flights*, Physics Letters A **239** (1998), no. 1, 13–16.

[27] M. Chen and W. Deng, *Fourth Order Accurate Scheme for the Space Fractional Diffusion Equations*, SIAM Journal on Numerical Analysis **52** (2014), no. 3, 1418–1438.

[28] Z. Q. Chen, M. M. Meerschaert, and E. Nane, *Space–time fractional diffusion on bounded domains*, Journal of Mathematical Analysis and Applications **393** (2012), no. 2, 479–488.

[29] J. B. Conway, *Functions of One Complex Variable*, Springer-Verlag, 1973.

[30] M. Crouzeix, S. Larsson, S. Piskarev, and V. Thomée, *The stability of rational approximations of analytic semigroups*, BIT Numerical Mathematics **33** (1993), no. 1, 74–84.

[31] K. Diethelm, J. M. Ford, N. J. Ford, and M. Weilbeer, *Pitfalls in fast numerical solvers for fractional differential equations*, Journal of Computational and Applied Mathematics **186** (2006), no. 2, 482–503.

[32] K. Diethelm, N. J. Ford, A. D. Freed, and Y. Luchko, *Algorithms for the fractional calculus: a selection of numerical methods*, Computer Methods in Applied Mechanics and Engineering **194** (2005), no. 6, 743–773.

[33] Q. Du, M. Gunzburger, R. B. Lehoucq, and K. Zhou, *Analysis and approximation of nonlocal diffusion problems with volume constraints*, SIAM Review **54** (2012), no. 4, 667–696.

[34] Q. Du, M. Gunzburger, R. B. Lehoucq, and K. Zhou, *A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws*, Mathematical Models and Methods in Applied Sciences **23** (2013), no. 03, 493–540.

[35] Q. Du, J. R. Kamm, R. B. Lehoucq, and M. L. Parks, *A new approach for a nonlocal, nonlinear conservation law*, SIAM Journal on Applied Mathematics **72** (2012), no. 1, 464–487.

[36] R. E. Edwards, *On functions which are Fourier transforms*, Proceedings of the American Mathematical Society **5** (1954), 71–78.

[37] K. J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000.

[38] V. J Ervin and J. P. Roop, *Variational formulation for the stationary fractional advection dispersion equation*, Numerical Methods for Partial Differential Equations **22** (2006), no. 3, 558–576.

[39] V. J. Ervin and J. P. Roop, *Variational solution of fractional advection dispersion equations on bounded domains in* $\mathbb{R}^d$, Numerical Methods for Partial Differential Equations **23** (2007), no. 2, 256–281.

[40] W. Feller, *On a generalization of Marcel Riesz potentials and the semi-groups generated by them*, Meddelanden Lunds Universitets Matematiska Seminarium (Comm. Sém. Mathém. Université de Lund) (1952), 73–81.

[41] ———, *An Introduction to Probability Theory and Its Applications Volume 2*, John Wiley and Sons, Inc., New York, London and Sydney, 1966.

[42] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, John Wiley and Sons Inc., 1999.

[43] R. Gorenflo, *Fractional Calculus: Some Numerical Methods*, CISM Lecture Notes, FRACALMO pre-print, Springer Verlag, Wien and New York **378** (2001), 277–290.

[44] R. Gorenflo and F. Mainardi, *Random walk models for space-fractional diffusion processes*, Fractional Calculus and Applied Analysis **1** (1998), no. 2, 167–191.

[45] ———, *Of Lévy-Feller Diffusion by Random Walk*, Journal for Analysis and its Applications **18** (1999), no. 2, 1–16.

[46] ———, *Fractional Calculus: Integral and Differential Equations of Fractional order*, CISM Lecture Notes, FRACALMO pre-print, Springer Verlag, Wien and New York **378** (2000), 223–276.

[47] M. Haase, *The functional calculus for sectorial operators*, Birkhäuser Verlag, Basel-Boston-Berlin, 2006.

[48] H. J. Haubold, A. M. Mathai, and R. K. Saxena, *Mittag-Leffler functions and their applications*, Journal of Applied Mathematics **2011** (2011), 1–51.

[49] P. Henrici, *Fast Fourier Methods in Computational Complex Analysis*, SIAM Review **21** (1979), no. 4, 481–527.

[50] M. Hieber, *Integrated semigroups and differential operators on L^p spaces*, Mathematische Annalen **291** (1991), 1–16.

[51] R. Hilfer (ed.), *Applications of Fractional Calculus in Physics*, World Scientific Publishing Co. Pte. Ltd., 2000.

[52] E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, vol. 31, American Mathematical Society, 1957.

[53] O. Kallenberg, *Foundations of modern probability*, Springer, 1997.

[54] A. Kamaly, *Fritz Carlson's inequality and its application*, Mathematica Scandinavica **86** (2000), 100–108.

[55] S. Kantorovitz, *On the operational calculus for groups of operators*, Proceedings of the American Mathematical Society **26** (1970), 603–608.

[56] A. N. Kolmogorov and S. V. Fomin, *Introductory Real Analysis*, Prentice-Hall, Inc., 1970.

[57] M. Kovács, *On the convergence of rational approximations of semigroups on intermediate spaces*, Mathematics of Computation **76(257)** (2007), 273–286.

[58] O. E. Lanford and D. W. Robinson, *Fractional Powers of Generators of Equicontinuous Semigroups and Fractional Derivatives*, Journal of Australian Mathematical Society (Series A) **46** (1989), 473–504.

[59] T. A. M. Langlands and B. I. Henry, *The accuracy and stability of an implicit solution method for the fractional diffusion equation*, Journal of Computational Physics **205** (2005), 719–736.

[60] L. Larsson, L. Maligranda, J. Pečarić, and L. Persson, *Multiplicative inequalities of Carlson type and interpolation*, World Scientific Publishing Co. Pte. Ltd., 2006.

[61] R. Lin and F. Liu, *Fractional high order methods for the nonlinear fractional ordinary differential equation*, Nonlinear Analysis: Theory, Methods & Applications **66** (2007), no. 4, 856–869.

[62] F. Liu, V. Anh, and I. Turner, *Numerical solution of the space fractional Fokker–Planck equation*, Journal of Computational and Applied Mathematics **166** (2004), no. 1, 209–219.

[63] C. Lubich, *Fractional linear multistep methods for Abel-Volterra integral equations of the second kind*, Mathematics of computation **45** (1985), no. 172, 463–469.

[64] ______, *Discretized Fractional Calculus*, SIAM Journal of Mathematical Analysis **17** (1986), no. 3, 704–719.

[65] ______, *Convolution quadrature and discretized operational calculus. I*, Numerische Mathematik **52** (1988), no. 2, 129–145.

[66] ______, *Convolution quadrature and discretized operational calculus. II*, Numerische Mathematik **52** (1988), no. 4, 413–425.

[67] ______, *Convolution quadrature revisited*, BIT Numerical Mathematics **44** (2004), no. 3, 503–514.

[68] C. Lubich and A. Ostermann, *Runge-Kutta methods for parabolic equations and convolution quadrature*, Mathematics of Computation **60** (1993), no. 201, 105–131.

[69] F. Mainardi, M. Raberto, R. Gorenflo, and E. Scalas, *Fractional calculus and continuous-time finance II: the waiting-time distribution*, Physica A: Statistical Mechanics and its Applications **287** (2000), no. 3, 468–481.

[70] M. M. Meerschaert, D. A. Benson, and B. Baeumer, *Multidimensional advection and fractional dispersion*, Physical Review E **59** (1999), no. 5, 5026.

[71] ———, *Operator Lévy motion and multiscaling anomalous diffusion*, Physical Review E **63** (2001), no. 2, 021112.

[72] M. M. Meerschaert, D. A. Benson, H-P. Scheffler, and B. Baeumer, *Stochastic solution of space-time fractional diffusion equations*, Physical Review E **65** (2002), no. 4, 041103.

[73] M. M. Meerschaert, D. A. Benson, H-P. Scheffler, and P. Becker-Kern, *Governing equations and solutions of anomalous random walk limits*, Physical Review E **66** (2002), no. 6, 060102.

[74] M. M. Meerschaert, J. Mortensen, and S. W. Wheatcraft, *Fractional vector calculus for fractional advection–dispersion*, Physica A: Statistical Mechanics and its Applications **367** (2006), 181–190.

[75] M. M. Meerschaert, H-P. Scheffler, and C. Tadjeran, *Finite difference methods for two-dimensional fractional dispersion equation*, Journal of Computational Physics **211** (2006), no. 1, 249–261.

[76] M. M. Meerschaert and A. Sikorskii, *Stochastic Models for Fractional Calculus*, De Gruyter Studies in Mathematics, De Gruyter, 2011.

[77] M. M. Meerschaert and C. Tadjeran, *Finite difference approximations for fractional advection-dispersion flow equations*, Journal of Computational and Applied Mathematics **172** (2004), 65–77.

[78] ———, *Finite difference approximations for two-sided space-fractional partial differential equations*, Applied Numerical Mathematics **56** (2006), no. 1, 80–90.

[79] M. M. Meerschaert, Y. Zhang, and B. Baeumer, *Particle tracking for fractional diffusion with two time scales*, Computers & mathematics with applications **59** (2010), no. 3, 1078–1086.

[80] R. Metzler and J. Klafter, *The random walk's guide to anomalous diffusion: a fractional dynamics approach*, Physics reports **339** (2000), no. 1, 1–77.

[81] ______, *The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics*, Journal of Physics A: Mathematical and General **37** (2004), no. 31, R161.

[82] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons Inc., 1993.

[83] J.P. Nolan, *An algorithm for evaluating stable densities in Zolotarev's (M) parameterization*, Mathematical and Computer Modelling **29** (1999), no. 1012, 229 – 233.

[84] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, 1974.

[85] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.

[86] I. Podlubny, *Fractional Differential Equations*, Academic Press, 1999.

[87] ______, *Matrix approach to discrete fractional calculus*, Fractional Calculus and Applied Analysis **3** (2000), no. 4, 359–386.

[88] I. Podlubny, A. Chechkin, T. Skovranek, Y. Q. Chen, and B. M. V. Jara, *Matrix approach to discrete fractional calculus II: Partial fractional differential equations*, Journal of Computational Physics **228** (2009), no. 8, 3137–3153.

[89] M. Raberto, E. Scalas, and F. Mainardi, *Waiting-times and returns in high-frequency financial data: an empirical study*, Physica A: Statistical Mechanics and its Applications **314** (2002), no. 1, 749–755.

[90] B. Ross (ed.), *Fractional Calculus and Its Applications*, Springer-Verlag, 1975.

[91] M. N. Le Roux, *Semidiscretization in time for parabolic problems*, Mathematics of Computation **33** (1979), no. 147, 919–931.

[92] W. Rudin, *Real and Complex Analysis*, McGraw-Hill Book Company, 1966.

[93] L Sabatelli, S Keating, J Dudley, and P Richmond, *Waiting time distributions in financial markets*, The European Physical Journal B-Condensed Matter and Complex Systems **27** (2002), no. 2, 273–275.

[94] S. G. Samko, *Fractional integration and differentiation of variable order*, Analysis Mathematica **21** (1995), no. 3, 213–236.

[95] S. G. Samko, A.A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, 1993.

[96] E. Scalas, R. Gorenflo, and F. Mainardi, *Fractional calculus and continuous-time finance*, Physica A: Statistical Mechanics and its Applications **284** (2000), no. 1, 376–384.

[97] R. Schumer, D. A. Benson, M. M. Meerschaert, and B. Baeumer, *Multiscaling fractional advection-dispersion equations and their solutions*, Water Resources Research **39** (2003), no. 1, 1022.

[98] R. Schumer, D. A. Benson, M. M. Meerschaert, and S. W. Wheatcraft, *Eulerian derivation of the fractional advection-dispersion equation*, Jounal of Contaminant Hydrology **48** (2001), no. 1-2, 69–88.

[99] R. Schumer, M. M. Meerschaert, and B. Baeumer, *Fractional advection-dispersion equations for modeling transport at the Earth surface*, Journal of Geophysical Research **114** (2009), no. F4, F00A07.

[100] S. Shen and F. Liu, *Error analysis of an explicit finite difference approximation for the space fractional diffusion equation with insulated ends*, ANZIAM Journal **46** (2005), C871–C887.

[101] C. Tadjeran and M. M. Meerschaert, *A second-order accurate numerical method for the two-dimensional fractional diffusion equation*, Journal of Computational Physics **220** (2007), no. 2, 813 – 823.

[102] C. Tadjeran, M. M. Meerschaert, and H-P. Scheffler, *A second-order accurate numerical approximation for the fractional diffusion equation*, Journal of Computational Physics **213** (2006), no. 1, 205–213.

[103] W. Y. Tian, H. Zhou, and W. Deng, *A class of second order difference approximation for solving space fractional diffusion equations*, arXiv preprint arXiv:1201.5949 **0** (2012), 0.

[104] L. Tonelli, *Sulla convergencza assoluta delle serie di Fourier*, Rend. dei Lincei **2(6)** (1925), 142–149.

[105] V. K. Tuan and R. Gorenflo, *Extrapolation to the limit for numerical fractional differentiation*, ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik **75** (1995), no. 8, 646–648.

[106] U. Westphal, *An approach to fractional powers of operators via fractional differences*, Proceedings of the London Mathematical Society **3** (1974), no. 3, 557–576.

[107] S. W. Wheatcraft and M. M. Meerschaert, *Fractional conservation of mass*, Advances in Water Resources **31** (2008), no. 10, 1377–1381.

[108] K. Yosida, *Fractional powers of infinitesimal generators and the analyticity of the semi-groups generated by them*, Proceedings of the Japan Academy **36** (1960), no. 3, 86–89.

[109] S. B. Yuste, *Weighted average finite difference methods for fractional diffusion equations*, Journal of Computational Physics **216** (2006), no. 1, 264–274.

[110] G. M. Zaslavsky, *Chaos, fractional kinetics, and anomalous transport*, Physics Reports **371** (2002), no. 6, 461–580.

[111] A. H. Zemanian, *Distribution Theory and Transform Analysis, An Introduction to Generalised Functions with Applications*, Dover publications Inc., New York, 1965.

[112] H. Zhou, W. Y. Tian, and W. Deng, *Quasi-compact finite difference schemes for space fractional diffusion equations*, Journal of Scientific Computing **56** (2013), no. 1, 45–66.