KMS states of graph algebras with a generalised gauge dynamics

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a thesis submitted for the degree of

Doctor of Philosophy

at the University of Otago, Dunedin,
New Zealand.

December 5, 2015

Abstract

The goal of this thesis is to study the KMS states of graph algebras with a generalised gauge dynamics.

We start by studying the KMS states of the Toeplitz algebra and graph algebra of a finite directed graph, each with an a generalised gauge dynamics. We characterise the KMS states of the Toeplitz algebra and find an isomorphism between measures and KMS states at large inverse temperatures. When the graph is strongly connected we can describe all of the KMS states, and we get a unique KMS state at the critical inverse temperature. Viewing the graph algebra as a quotient of the Toeplitz algebra we describe the KMS states of the graph algebra.

Next we study the KMS states of graph algebras for row-finite infinite directed graphs with no sources and the gauge action. We characterise the KMS states of the Toeplitz algebra and discuss KMS states at large inverse temperatures. We then show that problems occur at the critical inverse temperature.

Lastly we study the KMS states of the Toeplitz algebras and graph algebras for higher-rank graphs with a generalised gauge dynamics, using the same method as we did for finite graphs. We finish by studying the preferred dynamics of the system, where we get our best results.

Acknowledgements

Firstly, and most importantly, I thank my supervisors, Professor Iain Raeburn and Professor Astrid an Huef, for their insight and guidance. I also thank them for their patience, and I am extremely grateful for the time they have spent answering my questions and checking my work. In addition I thank them and the remainder of the Operator Algebra research group, Dr Lisa Orloff Clark, Associate Professor John Clark, Dr Sooran Kang, Ilija, Danie, Yosafat and especially Zahra, for not only being a supportive group to work with, but also a friendly group to socialise with. It has been a privilege to work with you all.

I would also like to thank the whole Mathematics & Statistics department for their support, particularly the administrative and IT support staff, as well as Harish, Fabien and George for helping me to get started.

For their encouragement and support I give thanks to my family, especially my parents Jane and Daryl, and sister Alex. I also thank my friends, including the staff and students at Arana College over the last eight years, especially Jamie Gilbertson.

Finally I would like to acknowledge the financial support of the University of Otago Doctoral Scholarship and the Stuart Residence Halls Council.

For Dad

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Chapter 1

Introduction

Let B be a C^*-algebra and G a locally compact group. An action of G on B is a homomorphism $\alpha : G \to \operatorname{Aut} B$ such that $g \mapsto \alpha_g(b)$ is continuous on G for each $b \in B$. Then (B, α) defines a *C^*-dynamical system*, and α is the *dynamics* of the system. Applications of C^*-dynamical systems include the study of the time evolution in quantum and statistical mechanics.

In the study of quantum mechanical systems an equilibrium state of a C^*-dynamical system can be described by a Kubo-Martin-Schwinger state, or *KMS state*. The behaviour of the KMS states varies with a parameter β called the "inverse temperature". A C^*-algebra has a set of *analytic* elements, and for $\beta \in (0, \infty)$ a state ϕ is a KMS state at inverse temperature β if $\phi(ab) = \phi(b\alpha_{i\beta}(a))$ for all a, b in this set. KMS states are of interest in many areas, including topological graphs [1], groupoids [36, 29], and number theory [25, 27].

A *directed graph* $E = (E^0, E^1, r, s)$ consists of vertices E^0 and edges E^1, and relations which tell us how they are related, called the range and source maps, r and s respectively. We can form a *graph algebra* by associating projections P_v to the vertices $v \in E^0$ and partial isometries S_e to the edges $e \in E^1$ which satisfy certain relations. We are interested in two such algebras: the graph algebra $C^*(E)$ (see Section 2.2.2) and the Toeplitz algebra $\mathcal{T}C^*(E)$ (see Section 2.2). The graph algebra $C^*(E)$ has been studied by, for example; Kumjian, Pask, Raeburn and Renault [24] using a groupoid model, and Kumjian, Pask and Raeburn in [23]. The Toeplitz algebra $\mathcal{T}C^*(E)$ was introduced in [14]. Write B for either $C^*(E)$ or $\mathcal{T}C^*(E)$. Then there is a natural action γ of the circle $\mathbb{T}$ on B called the *gauge action*, and we can lift the gauge action to a natural dynamics (action of the real numbers) α of $\mathbb{R}$ on B by setting $\alpha_t := \gamma_{e^{it}}$.

Fix $n \geq 2$. Introduced in [5], the Cuntz algebra $\mathcal{O}_n$ is the C^*-algebra universal for a family of isometries $\{S_1, \dots, S_n\}$ such that $\sum_{i=1}^n S_i S_i^* = 1$. A consequence of this relation is that the S_i have mutually orthogonal ranges. For $\mu = \mu_1 \dots \mu_{|\mu|}$ we write $S_\mu := S_{\mu_1} \dots S_{\mu_{|\mu|}}$. Then for a universal family $\{s_1, \dots, s_n\}$ we have

$$\mathcal{O}_n = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu_i, \nu_i \in \{1, \dots, n\}\}.$$

We can view $\mathcal{O}_n$ as the graph algebra $C^*(E)$ of the graph E with one vertex and n edges. In [30] Olesen and Pedersen showed that the system $(\mathcal{O}_n, \alpha)$ has a unique KMS state which occurs at inverse temperature $\ln n$.

Let $\mathcal{G}$ be a finite set of n elements and A be an $n \times n$ matrix with entries in $\{0, 1\}$, no zero rows or columns, and which is irreducible but not a permutation matrix. Introduced in [6], the Cuntz-Krieger algebra $\mathcal{O}_A$ is the C^*-algebra universal for a family of partial isometries $\{S_i\}_{i \in \mathcal{G}}$ satisfying

(a) $\sum_{j \in \mathcal{G}} S_j S_j^* = 1$, and

(b) $S_i^* S_i = \sum_{j \in \mathcal{G}} A_{i,j} S_j S_j^*$, for all $i \in \mathcal{G}$.

For $\mu = \mu_1 \dots \mu_{|\mu|}$ we write $S_\mu := S_{\mu_1} \dots S_{\mu_{|\mu|}}$. Then for a universal family $\{s_i\}_{i \in \mathcal{G}}$ we have

$$\mathcal{O}_A = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu_i, \nu_i \in \mathcal{G}\}.$$

Again we can view $\mathcal{O}_A$ as the graph algebra $C^*(E_A)$ of the graph E_A with vertex set $\mathcal{G}$ and edges from j to i whenever $A_{i,j} = 1$ (see [32, Remark 2.8] for example). Because of the assumptions on the matrix A, the graphs E_A have special properties:

(a) because A is required to be irreducible and not a permutation matrix, E_A has the condition that "every cycle has an entry";

(b) because the entries of A are in $\{0, 1\}$, for vertices i, j in E_A there is at most one edge from j to i; and

(c) because A does not have any zero rows or columns, E_A must have no sources or sinks.

Write $\rho(A)$ for the spectral radius of the matrix A. In [9] Enomoto, Fujii and Watatani showed that the system $(\mathcal{O}_A, \alpha)$ has a unique KMS state which occurs at inverse temperature $\ln(\rho(A))$. This result was generalised to infinite matrices by Exel and Laca in [12], but it was not until 2011 that Kajiwara and Watatani [21] pointed out that the result of [9] was not known for graphs with sources. They showed that the presence of sources gave extra KMS states. Subsequently an Huef, Laca, Raeburn and Sims [17] extended this result to arbitrary finite graphs E, that is, graphs which potentially have sinks, sources, and multiple edges between vertices. The methods of [17] were quite different: Exel and Laca [12] and Laca and Neshveyev [26] pointed out that the Toeplitz algebra has a much richer KMS structure than its quotient, and they use this idea to study the KMS states of $\mathcal{T}C^*(E)$ and $C^*(E)$. Their results for KMS states at the critical inverse temperature were for graphs with an irreducible vertex matrix; in [19] they studied graphs with a reducible vertex matrix.

Higher-rank graphs (also known as k-graphs) Λ (see Section 5.1) and their graph algebras $C^*(\Lambda)$ (see Section 5.2.1) were introduced by Kumjian and Pask in [22] as combinatorial models for higher-rank Cuntz-Krieger algebras. They also have Toeplitz algebras $\mathcal{T}C^*(\Lambda)$ (see Section 5.2), introduced by Raeburn and Sims in [33]. Taking B as either $C^*(\Lambda)$ or $\mathcal{T}C^*(\Lambda)$, there is a natural gauge action γ of the torus $\mathbb{T}^k$ on B, which lifts to a dynamics α of $\mathbb{R}^k$ on B. In [18], an Huef, Laca, Raeburn and Sims extended the method of [17] to study KMS states of $(\mathcal{T}C^*(\Lambda), \alpha)$ and $(C^*(\Lambda), \alpha)$. In addition, the KMS states of 2-graphs with one vertex have been extensively studied, for example by Yang in [38] and [39].

In this thesis we are interested in dynamics which are more general than those lifted from the natural gauge actions of $\mathbb{T}$ on the algebras, which we call *generalised gauge dynamics*. Fix $n \geq 2$. Let $\{s_1, \ldots, s_n\}$ be a universal family generating $\mathcal{O}_n$. For a collection of real numbers $\{y_j\}_{j=1}^n$ such that $y_j > 0$ for all j, there is an action α^y of $\mathbb{R}$ on $\mathcal{O}_n$ such that

$$\alpha_t^y(s_j) = e^{ity_j} S_j \qquad\qquad \text{for all } t \in \mathbb{R}.$$

We can recover the gauge dynamics from α^y by taking $y_j = 1$ for all j. Then Evans [10] extended the result of [30] to $(\mathcal{O}_n, \alpha^y)$, showing there is a unique KMS state at inverse temperature β if and only if $1 = e^{-\beta y_1} + \cdots + e^{-\beta y_n}$. Similarly, Exel and Laca [12] extended the results of [9] to show there is a unique KMS state of $\mathcal{O}_A$ with a generalised gauge dynamics (in the finite case). The KMS states of $\mathcal{O}_A$ with generalised gauge actions have also been studied by Exel in [11].

These results have been extended to study KMS states of the graph algebra $C^*(E)$ with a generalised gauge dynamics. Let $E = (E^0, E^1, r, s)$ be a finite directed graph and $\{p_v, s_e\}$ be the universal family generating $C^*(E)$. For $\{r_e \in (0, 1)\}_{e \in E^1}$ we can define an action α^r of $\mathbb{R}$ on $C^*(E)$ such that

$$\alpha^r_t(s_e) = e^{-it\log(r_e)}s_e.$$

Then Ionescu and Kumjian [20] used a groupoid model for $C^*(E)$ to get results about the KMS states of $(C^*(E), \alpha^r)$. In [3] Carlsen and Larsen study the KMS states of relative graph algebras, a generalisation of $C^*(E)$ and $\mathcal{T}C^*(E)$, and a generalised gauge dynamics. In [4] Christensen and Thomsen extend the results of [17] to include generalised gauge actions, but they do not include results about the Toeplitz algebra $\mathcal{T}C^*(E)$. None of them obtained a detailed description of all of the KMS states like the one we present here.

The goal of this thesis is to study the KMS states of Toeplitz algebras $\mathcal{T}C^*(E)$ and graph algebras $C^*(E)$ with a generalised gauge dynamics. To do this we define a generalised gauge dynamics and then use the method of [17] to study the KMS states of graph algebras for a finite directed graph. We then study the KMS states of graph algebras for infinite graphs with the gauge action by examining what happens to the method of [17] when we remove the assumption that the graph is finite. Finally we use the method of [18] to study the KMS states of graph algebras for higher-rank graphs with a generalised gauge dynamics.

In Chapter 2 we review preliminary material required in our study of KMS states. We start by recalling the definition of a directed graph $E = (E^0, E^1, r, s)$, and then define the Toeplitz algebra $\mathcal{T}C^*(E)$ and graph algebra $C^*(E)$. Next we recall the definition of the usual gauge action γ of $\mathbb{T}$ on $\mathcal{T}C^*(E)$. We then choose a function $y : E^1 \to (0, \infty)$ and use it to find a generalised gauge dynamics α^y of $\mathbb{R}$ on $\mathcal{T}C^*(E)$. Finally we present the definition of KMS states and characterise the KMS states of $(\mathcal{T}C^*(E), \alpha^y)$.

In Chapter 3 we study the KMS states of C^*-algebras for finite directed graphs with a generalised gauge dynamics. First we study the KMS states of $(\mathcal{T}C^*(E), \alpha^y)$ at large inverse temperatures. We then discuss the KMS states of $(\mathcal{T}C^*(E), \alpha^y)$ at the critical inverse temperature when E is strongly connected. Finally we get a generalised gauge dynamics $\overline{\alpha}^y$ of $\mathbb{R}$ on $C^*(E)$ and study the KMS states of $(C^*(E), \overline{\alpha}^y)$.

In Chapter 4 we attempt to extend the results of [17] to row-finite infinite graphs with no sources. Taking α to be the dynamics associated to the gauge action γ, we start the chapter by characterising the KMS states of $(\mathcal{T}C^*(E), \alpha)$. To study the KMS states of $(\mathcal{T}C^*(E), \alpha)$ at large inverse temperatures we present some background from Banach spaces and then find an isomorphism between KMS functionals and a subset of $\ell^1(E^0)$. We then show when the KMS states of $(\mathcal{T}C^*(E), \alpha)$ factor through $C^*(E)$. Finally we give an example from [3] to show that we cannot guarantee existence of the KMS states of $(C^*(E), \overline{\alpha})$ for infinite graphs.

In Chapter 5 we extend the results from Chapter 3 to higher-rank graphs, using the method of [18]. We first introduce the required background material for a higher-rank graph Λ and its C^*-algebras $\mathcal{T}C^*(\Lambda)$ and $C^*(\Lambda)$. We then choose a functor $y : \Lambda \to [0, \infty)$ and use it to define a generalised gauge dynamics α^y of $\mathbb{R}$ on $\mathcal{T}C^*(\Lambda)$. Next we present results about the characterisation of the KMS states of $(\mathcal{T}C^*(\Lambda), \alpha^y)$. We characterise the KMS states of $(\mathcal{T}C^*(\Lambda), \alpha^y)$ at large inverse temperatures, by describing an isomorphism between measures and the KMS states of $(\mathcal{T}C^*(\Lambda), \alpha^y)$ at large inverse temperatures. We use these results to describe KMS states of $(\mathcal{T}C^*(\Lambda), \alpha^y)$ at the critical inverse temperature. We then describe the dynamics $\overline{\alpha}^y$ and get results about the KMS states of $(C^*(\Lambda), \overline{\alpha}^y)$. Finally we present results about the KMS states of $(\mathcal{T}C^*(\Lambda), \alpha^y)$ and $(C^*(\Lambda), \overline{\alpha}^y)$ for a preferred dynamics, and give an example to illustrate these results.

We finish with an appendix containing useful results about the spectral radius of nonnegative matrices (including consequences of the Perron-Frobenius theorem) and the enumeration and convergence of sums.

Chapter 2

Preliminaries

In this chapter we present definitions and results used in later chapters. We first define a directed graph and its related notation. We then define $\mathcal{T}C^*(E)$ and discuss its relationship with the graph algebra $C^*(E)$. Next we present the definition of the gauge action γ, as well as a generalised gauge dynamics α^y. Finally we present the definition of KMS states and characterise the KMS states of $(\mathcal{T}C^*(E), \alpha^y)$.

2.1 Directed Graphs

In this section we introduce the concept of a directed graph. We use the conventions of [32] for directed graphs and, as in [17], borrow some notation from the higher-rank graph literature.

A *directed graph* $E = (E^0, E^1, r, s)$ consists of two sets E^0, E^1 and functions $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 are called *edges*. For each edge we call $s(e)$ the *source* of e and $r(e)$ the *range* of e. An edge $e \in E^1$ can therefore be thought of as travelling from $s(e)$ to $r(e)$. For a vertex $v \in E^0$ we define

$$vE^1 := \{e \in E^1 : r(e) = v\}.$$

A graph is *row-finite* if vE^1 is a finite set for every $v \in E^0$, and in this thesis we will only consider row-finite graphs. If $vE^1 = \emptyset$ then v is called a *source*. For vertices $v, w \in E^0$ we also define

$$vE^1w := \{e \in E^1 : r(e) = v, s(e) = w\}.$$

We can then define the *vertex matrix* A of a graph to be the $E^0 \times E^0$ matrix with entries

$$A_{v,w} = |vE^1w|.$$

A *path of length* n is a sequence $\mu = \mu_1\mu_2 \dots \mu_n$ of edges $\mu_i \in E^1$ such that $s(\mu_i) = r(\mu_{i+1})$ for $1 \leq i \leq n-1$. We extend r and s to paths by defining $r(\mu) := r(\mu_1)$ and $s(\mu) := s(\mu_n)$. We denote by E^n the set of paths of length n, and for $\mu \in E^n$ define $|\mu| := n$, the number of edges in the path μ. We define $E^* := \bigcup_{n\geq 0} E^n$. Then, for example, we define

$$vE^* := \{\mu \in E^* : r(e) = v\}.$$

A directed graph is *strongly connected* if for every pair of vertices $v, w \in E^0$ there exists a path $\mu \in E^*$ such that $s(\mu) = w$ and $r(\mu) = v$, that is, for all $v, w \in E^0$, $vE^*w \neq \emptyset$.

Corollary A.12 tells us that the way we enumerate sums of nonnegative numbers doesn't matter. For example, taking $f : E^0 \to [0, \infty)$ we can write $\sum_{v\in E^0} f(x)$ without ambiguity, and we exploit this fact throughout.

2.2 The Toeplitz algebra $\mathcal{T}C^*(E)$

In this section we introduce Toeplitz-Cuntz-Krieger families and their algebras, using the definition of [14].

Let E be a row-finite directed graph. A *Toeplitz-Cuntz-Krieger E-family* $\{P_v, S_e\}$ consists of mutually orthogonal projections $\{P_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$ such that

(TCK1) $S_e^*S_e = P_{s(e)}$ for all $e \in E^1$, and

(TCK2) $P_v \geq \sum_{e\in vE^1} S_eS_e^*$, for all $v \in E^0$.

Note here that because E is assumed to be row-finite the sum in (TCK2) is finite. It is proved in, for example, [14, Proposition 1.3] that there is a C^*-algebra $\mathcal{T}C^*(E)$ generated by a Toeplitz-Cuntz-Krieger E-family $\{p_v, s_e\}$ which is universal in the sense that given any Toeplitz-Cuntz-Krieger E-family $\{Q_v, T_e\}$ in a C^*-algebra B, there is a homomorphism $\pi_{Q,T} : \mathcal{T}C^*(E) \to B$ such that $\pi_{Q,T}(s_e) = T_e$ for all $e \in E^1$ and $\pi_{Q,T}(p_v) = Q_v$ for all $v \in E^0$. We call $\mathcal{T}C^*(E)$ the *Toeplitz algebra* of E. We use the convention that a family denoted with lowercase letters (for example, $\{p_v, s_e\}$) has the universal property, whereas a family denoted with uppercase letters (for example, $\{Q_v, T_e\}$) can be any family.

We extend the partial isometries for edges to partial isometries for paths by defining

$$S_\mu := S_{\mu_1} \dots S_{\mu_{|\mu|}} \qquad\qquad \text{for } \mu \in E^*.$$

If $\mu \in E^*$ is a path then repeated applications of (TCK1) give

$$\begin{aligned} S_\mu^* S_\mu &= (S_{\mu_1} S_{\mu_2} \dots S_{\mu_n})^* S_{\mu_1} S_{\mu_2} \dots S_{\mu_n} \\ &= S_{\mu_n}^* \dots S_{\mu_2}^* (S_{\mu_1}^* S_{\mu_1}) S_{\mu_2} \dots S_{\mu_n} \\ &= S_{\mu_n}^* \dots S_{\mu_2}^* (P_{s(\mu_1)}) S_{\mu_2} \dots S_{\mu_n} \\ &= S_{\mu_n}^* \dots S_{\mu_2}^* (P_{r(\mu_2)}) S_{\mu_2} \dots S_{\mu_n} \\ &= S_{\mu_n}^* \dots (S_{\mu_2}^* S_{\mu_2}) \dots S_{\mu_n} \\ &\vdots \\ &= P_{s(\mu_n)} \\ &= P_{s(\mu)}. \end{aligned} \tag{2.1}$$

Lemma 2.1. *Let $\{P_v, S_e\}$ be a Toeplitz-Cuntz-Krieger E-family. Then*

$$S_e^* S_f = \delta_{e,f} P_{s(e)} \qquad\qquad \textit{for all } e, f \in E^1.$$

Proof. [17, Corollary 1.2] tells us that the projections $\{S_e S_e^* : e \in E^1\}$ are mutually orthogonal, and therefore

$$S_e^* S_f = S_e^* S_e S_e^* S_f S_f^* S_f = \begin{cases} S_e^* S_e & \text{if } e = f \\ S_e^* 0 S_f & \text{otherwise.} \end{cases}$$

So by (TCK1) $S_e^* S_f = \delta_{e,f} P_{s(e)}$. □

Corollary 2.2. *Let $\{P_v, S_e\}$ be a Toeplitz-Cuntz-Krieger E-family. Then, for $\mu, \nu, \sigma, \tau \in E^*$, we have*

$$(S_\mu S_\nu^*)(S_\sigma S_\tau^*) = \begin{cases} S_{\mu\sigma'} S_\tau^* & \textit{if } \sigma = \nu\sigma' \\ S_\mu S_{\tau\nu'}^* & \textit{if } \nu = \sigma\nu' \\ 0 & \textit{otherwise.} \end{cases} \tag{2.2}$$

We call this the product formula.

Proof. We prove this for a Toeplitz-Cuntz-Krieger E-family by adapting the proof of [32, Corollary 1.15], which was for a Cuntz-Krieger E-family.

We consider two cases, that $|\nu| \leq |\sigma|$ and that $|\sigma| < |\nu|$. First suppose that $n := |\nu| \leq |\sigma|$, and factor $\sigma = \alpha\sigma'$ with $|\alpha| = n$. Then

$$(S_\mu S_\nu^*)(S_\sigma S_\tau^*) = S_\mu S_\nu^*(S_\alpha S_{\sigma'})S_\tau^* = S_\mu(S_\nu^* S_\alpha)S_{\sigma'}S_\tau^*.$$

If $\nu = \alpha$, then Equation (2.1) implies that

$$(S_\mu S_\nu^*)(S_\sigma S_\tau^*) = S_\mu P_{s(\nu)} S_{\sigma'} S_\tau^* = S_\mu P_{r(\sigma')} S_{\sigma'} S_\tau^* = S_{\mu\sigma'} S_\tau^*.$$

If $\nu \neq \alpha$, let i be the smallest integer such that $\nu_i \neq \alpha_i$. Then applying Equation (2.1) gives

$$\begin{aligned}
S_\nu^* S_\alpha &= (S_{\nu_1} S_{\nu_2} \dots S_{\nu_n})^* S_{\alpha_1} S_{\alpha_2} \dots S_{\alpha_n} \\
&= S_{\nu_n}^* \dots S_{\nu_i}^* (S_{\nu_{i-1}}^* \dots S_{\nu_1}^* S_{\alpha_1} \dots S_{\alpha_{i-1}}) S_{\alpha_i} \dots S_{\alpha_n} \\
&= S_{\nu_n}^* \dots S_{\nu_i}^* P_{s(\nu_{i-1})} S_{\alpha_i} \dots S_{\alpha_n} \\
&= S_{\nu_n}^* \dots S_{\nu_i}^* P_{r(\nu_i)} S_{\alpha_i} \dots S_{\alpha_n} \\
&= S_{\nu_n}^* \dots S_{\nu_i}^* S_{\alpha_i} \dots S_{\alpha_n}.
\end{aligned}$$

Lemma 2.1 implies $S_{\nu_i}^* S_{\alpha_i} = 0$, so $S_\nu^* S_\alpha = 0$. Therefore

$$(S_\mu S_\nu^*)(S_\sigma S_\tau^*) = S_\mu(S_\nu^* S_\alpha)S_{\sigma'}S_\tau^* = 0.$$

Next suppose that $|\nu| > |\sigma|$. Then we can factor $\nu = \beta\nu'$ and run a similar argument to get

$$(S_\mu S_\nu^*)(S_\sigma S_\tau^*) = S_\mu P_{s(\sigma)} S_{\nu'}^* S_\tau^* = S_\mu P_{r(\nu')} S_{\nu'}^* S_\tau^* = S_\mu S_{\tau\nu'}^*,$$

if $\beta = \sigma$ and 0 otherwise. $\square$

The product formula implies that

$$\mathcal{T}C^*(E) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^*, s(\mu) = s(\nu)\}.$$

2.2.1 The finite path representation

We now introduce the *finite path representation*, which is used in later proofs to show properties of our universal family $\{p_v, s_e\}$.

Proposition 2.3. *Let E be a row-finite directed graph. Write h_μ for the point mass at $\mu \in E^*$, and let $\{Q_v : v \in E^0\}$ and $\{T_e : e \in E^1\}$ be the partial isometries on $\ell^2(E^*)$ such that*

$$Q_v h_\mu = \begin{cases} h_\mu & \textit{if } v = r(\mu) \\ 0 & \textit{otherwise} \end{cases}$$

and

$$T_e h_\mu = \begin{cases} h_{e\mu} & \textit{if } s(e) = r(\mu) \\ 0 & \textit{otherwise.} \end{cases}$$

Then $\{Q, T\}$ is a Toeplitz-Cuntz-Krieger E-family in $B(\ell^2(E^*))$. We call the representation $\pi_{Q,T} : \mathcal{T}C^*(E) \to B(\ell^2(E^*))$ such that $\pi_{Q,T}(p_v) = Q_v$ and $\pi_{Q,T}(s_e) = T_e$ the finite path representation.

Lemma 2.4. *Let h_μ be the point mass at $\mu \in E^*$. If $\{Q, T\}$ is the Toeplitz-Cuntz-Krieger E-family from Proposition 2.3 then*

$$T_e^* h_\nu = \begin{cases} h_{\nu'} & \textit{if } \nu = e\nu' \\ 0 & \textit{otherwise.} \end{cases}$$

Proof.

$$\begin{aligned} T_e^* h_\nu &= T_e^* T_{\nu_1} h_{\nu_2 \dots \nu_{|\nu|}} \\ &= \begin{cases} Q_{s(\nu_1)} h_{\nu_2 \dots \nu_{|\nu|}} & \text{if } \nu_1 = e \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} h_{\nu'} & \text{if } \nu = e\nu' \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$ □

Corollary 2.5. *Let h_μ be the point mass at $\mu \in E^*$. If $\{Q, T\}$ is the Toeplitz-Cuntz-Krieger E-family from Proposition 2.3 then $Q_v \neq \sum_{e \in vE^1} T_e T_e^*$.*

Proof. Applying Lemma 2.4,

$$\begin{aligned} \Big(Q_v - \sum_{e \in vE^1} T_e T_e^*\Big) h_v &= Q_v h_v - \sum_{e \in vE^1} T_e T_e^* h_v \\ &= h_v - \sum_{e \in vE^1} T_e 0 \\ &= h_v. \end{aligned}$$

In particular, we have $Q_v - \sum_{e \in vE^1} T_e T_e^* \neq 0$, so $Q_v \neq \sum_{e \in vE^1} T_e T_e^*$. □

2.2.2 The graph algebra $C^*(E)$

If a Toeplitz-Cuntz-Krieger E-family $\{P_v, S_e\}$ also satisfies

$$\text{(CK2)} \qquad P_v = \sum_{e \in F} S_e S_e^* \qquad \text{for all } v \in E^0,$$

then $\{P_v, S_e\}$ is a *Cuntz-Krieger E-family.* As with the Toeplitz algebra there is a C^*-algebra $C^*(E)$ generated by a Cuntz-Krieger E-family $\{p_v, s_e\}$ which is universal in the sense that given any Cuntz-Krieger E-family $\{Q_v, T_e\}$ in a C^*-algebra B there is a homomorphism $\pi_{Q,T} : C^*(E) \to B$ such that $\pi_{Q,T}(s_e) = T_e$ for all $e \in E^1$ and $\pi_{Q,T}(p_v) = Q_v$ for all $v \in E^0$ ([32, Proposition 1.21]). We call $C^*(E)$ the graph algebra of E.

The following result tells us how $\mathcal{T}C^*(E)$ and $C^*(E)$ are related.

Lemma 2.6. *Let $\{p_v, s_e\}$ be the universal Toeplitz-Cuntz-Krieger E-family which generates $\mathcal{T}C^*(E)$. Let J be the ideal generated by $\{p_v - \sum_{e \in vE^1} s_e s_e^* : v \in E^0\}$, and $q : \mathcal{T}C^*(E) \to \mathcal{T}C^*(E)/J$ be the quotient map. Write $\overline{s}_e = q(s_e)$ and $\overline{p}_v = q(p_v)$. Then $(\mathcal{T}C^*(E)/J, \{\overline{p}_v, \overline{s}_e\})$ is universal for Cuntz-Krieger E-families, that is,*

(a) $\{\overline{p}_v, \overline{s}_e\}$ is a Cuntz-Krieger E-family which generates $\mathcal{T}C^*(E)/J$; and

(b) if $\{P_v, S_e\}$ is a Cuntz-Krieger E-family in a C^*-algebra B then there exists a homomorphism $\overline{\pi}_{P,S} : \mathcal{T}C^*(E)/J \to B$ such that $\overline{\pi}_{P,S}(\overline{s}_e) = S_e$ and $\overline{\pi}_{P,S}(\overline{p}_v) = P_v$.

Proof. To prove part (a) we need to show that $\{\overline{p}_v : v \in E^0\}$ is a mutually orthogonal set, $\overline{s}_e^* \overline{s}_e = \overline{p}_{s(e)}$ for all $e \in E^1$, $\overline{p}_v = \sum_{e \in vE^1} \overline{s}_e \overline{s}_e^*$ whenever v is not a source, and that $\{\overline{p}_v, \overline{s}_e\}$ generates $\mathcal{T}C^*(E)/J$. To show that $\{\overline{p}_v : v \in E^0\}$ is a mutually orthogonal set, fix $v, w \in E^0$. Then, because q is a homomorphism,

$$\overline{p}_v \overline{p}_w = q(p_v)q(p_w) = q(p_v p_w) = q(0) = 0.$$

To show that $\overline{s}_e^* \overline{s}_e = \overline{p}_{s(e)}$ for all $e \in E^1$, fix $e \in E^1$. Then, because q is a homomorphism,

$$\overline{s}_e^* \overline{s}_e = q(s_e^*)q(s_e) = q(s_e^* s_e) = q(p_{s(e)}) = \overline{p}_{s(e)}.$$

Fix $v \in E^0$ such that v is not a source. We aim to show that $\overline{p}_v = \sum_{e \in vE^1} \overline{s}_e \overline{s}_e^*$. Then

$$0 = q\Big(p_v - \sum_{e \in vE^1} s_e s_e^*\Big) = q(p_v) - q\Big(\sum_{e \in vE^1} s_e s_e^*\Big) = \overline{p}_v - \sum_{e \in vE^1} q(s_e s_e^*).$$

This implies that

$$\overline{p}_v = \sum_{e\in vE^1} q(s_e s_e^*) = \sum_{e\in vE^1} q(s_e)q(s_e^*) = \sum_{e\in vE^1} \overline{s}_f \overline{s}_f^*.$$

Finally, since $\mathcal{T}C^*(E)$ is generated by $\{p_v, s_e\}$, $\{q(p_v), q(s_e)\} = \{\overline{p}_v, \overline{s}_e\}$ generates $\mathcal{T}C^*(E)/J$.

To prove part (b) we want to get a homomorphism on $\mathcal{T}C^*(E)$ and then prove it factors through a homomorphism of $\mathcal{T}C^*(E)/J$. We then need to prove this homomorphism has the required properties.

Because $\{P_v, S_e\}$ is a Cuntz-Krieger E-family in B, it is a Toeplitz-Cuntz-Krieger E-family in B, and the universal property of $(\mathcal{T}C^*(E), \{p_v, s_e\})$ gives a homomorphism $\pi_{P,S} : \mathcal{T}C^*(E) \to B$ such that $\pi_{P,S}(s_e) = S_e$, $\pi_{P,S}(p_v) = P_v$. Because $\pi_{P,S}$ is a homomorphism, $\ker \pi_{P,S}$ is a closed ideal. Since $\{P_v, S_e\}$ is a Cuntz-Krieger E-family, we have

$$\pi_{P,S}\Big(p_v - \sum_{e\in vE^1} s_e s_e^*\Big) = P_v - \sum_{e\in vE^1} S_e S_e^* = 0,$$

so $p_v - \sum_{e\in vE^1} s_e s_e^* \in \ker \pi_{P,S}$. Since J is the closed ideal generated by $\{p_v - \sum_{e\in vE^1} s_e s_e^*\}$ it is the smallest closed ideal containing $\{p_v - \sum_{e\in vE^1} s_e s_e^*\}$. So $J \subseteq \ker \pi_{P,S}$. Therefore there exists a homomorphism $\overline{\pi}_{P,S} : \mathcal{T}C^*(E)/J \to B$ such that $\pi_{P,S} = \overline{\pi}_{P,S} \circ q$.

Now that we have our homomorphism we need to check it has the required properties. Fix $\overline{s}_e \in \mathcal{T}C^*(E)/J$. Then

$$\overline{\pi}_{P,S}(\overline{s}_e) = \overline{\pi}_{P,S}(q(s_e)) = \pi_{P,S}(s_e) = S_e.$$

Fix $\overline{p}_v \in \mathcal{T}C^*(E)/J$. Then

$$\overline{\pi}_{P,S}(\overline{p}_v) = \overline{\pi}_{P,S}(q(p_v)) = \pi_{P,S}(p_v) = P_v.$$

Thus $\overline{\pi}_{P,S}$ has the required properties. □

Remark 2.7. Since $(\mathcal{T}C^*(E)/J, \{\overline{p}_v, \overline{s}_e\})$ has the universal property which determines the Cuntz-Krieger algebra, $(\mathcal{T}C^*(E)/J, \{\overline{p}_v, \overline{s}_e\})$ is canonically isomorphic to the Cuntz-Krieger algebra [32, Corollary 1.22]. From now on we use this isomorphism to identify $C^*(E)$ with this quotient, and $\ker q$ with the ideal J.

2.3 The gauge action γ

The Toeplitz algebra $\mathcal{T}C^*(E)$ carries a gauge action γ of $\mathbb{T}$, which satisfies

$$\gamma_z(s_\mu s_\nu^*) = z^{|\mu|-|\nu|} s_\mu s_\nu^*.$$

The gauge action has an associated dynamics $\alpha : \mathbb{R} \to \operatorname{Aut} \mathcal{T}C^*(E)$ defined by $\alpha_t := \gamma_{e^{it}}$. Since the quotient map q of $\mathcal{T}C^*(E)$ onto $C^*(E)$ is gauge-invariant, there is also an a corresponding action $\overline{\alpha}$ of $\mathbb{R}$ on $C^*(E)$.

2.4 A generalised gauge dynamics α^y

In [17] an Huef, Laca, Raeburn and Sims study KMS states of $\mathcal{T}C^*(E)$ and $C^*(E)$ with the gauge action γ. In this section we define an action α^y of R on $\mathcal{T}C^*(E)$ which is more general than the gauge action. We begin with a function $y : E^1 \to (0, \infty)$, and extend it to a function $y : E^* \to (0, \infty)$ by

$$y(\mu) = \sum_{j=1}^{|\mu|} y(\mu_j).$$

To define our generalised action using this function we use the method of [32, Proposition 2.1], noting here we are working in $\mathcal{T}C^*(E)$ rather than $C^*(E)$, and $\mathbb{R}$ rather than $\mathbb{T}$. We cannot define our action on $\mathbb{T}$ and use this to get an action on $\mathbb{R}$, as is done with the gauge action, because

$$\gamma_z^y(s_e) = z^{y(e)} s_e \qquad \text{for every } e \in E^1$$

would not be well-defined, since $z^{\frac{1}{2}}$ is ambiguous, for example.

Proposition 2.8. *Let E be a row-finite directed graph. Choose a weight function $y : E^1 \to (0, \infty)$. Then there is an action $\alpha^y : t \mapsto \alpha_t^y$ of $\mathbb{R}$ on $\mathcal{T}C^*(E)$ such that*

$$\alpha_t^y(s_e) = e^{ity(e)} s_e \qquad \text{for every } e \in E^1 \tag{2.3}$$

and

$$\alpha_t^y(p_v) = p_v \qquad \text{for every } v \in E^0. \tag{2.4}$$

To prove this we use the following lemma, stated here for convenience.

Lemma 2.9 ([14, Corollary 4.2])**.** *Let E be a directed graph. Suppose that $\{P_v, S_e\}$ and $\{Q_v, T_e\}$ are Toeplitz-Cuntz-Krieger E-families such that each P_v and Q_v is nonzero, and such that*

$$P_v \neq \sum_{e \in E^1 v} S_e S_e^* \quad \text{and} \quad Q_v \neq \sum_{e \in E^1 v} T_e T_e^*$$

for every vertex v which emits at most finitely many edges. Then there is an isomorphism π of $C^*(P_v, S_e)$ onto $C^*(Q_v, T_e)$ such that $\pi(P_v) = Q_v$ for all $v \in E^0$ and $\pi(S_e) = T_e$ for all $e \in E^1$.

Proof of Proposition 2.8. Fix $t \in \mathbb{R}$. We first aim to apply Lemma 2.9 to get an isomorphism α_t^y satisfying Equation (2.3) and Equation (2.4). We know $\{p_v, s_e\}$ is a Toeplitz-Cuntz-Krieger E-family which generates $\mathcal{T}C^*(E)$, so

$$\{p_v, e^{ity(e)}s_e\}$$

is also a Toeplitz-Cuntz-Krieger E-family which generates $\mathcal{T}C^*(E)$. To see that $p_v - \sum_{e\in vE^1} s_e s_e^* \neq 0$, fix $v \in E^0$. Let $\pi_{Q,T}$ be the finite path representation from Proposition 2.3, then Corollary 2.5 tells us that

$$\pi_{Q,T}(p_v - \sum_{e\in vE^1} s_e s_e^*) = Q_v - \sum_{e\in vE^1} T_e T_e^* \neq 0.$$

It follows that $p_v - \sum_{e\in vE^1} s_e s_e^* \neq 0$. Because

$$e^{ity(e)}s_e(e^{ity(e)}s_e)^* = e^{ity(e)}e^{-ity(e)}s_e s_e^* = s_e s_e^*,$$

we have

$$p_v - \sum_{e\in vE^1} e^{ity(e)}s_e(e^{ity(e)}s_e)^* = p_v - \sum_{e\in vE^1} s_e s_e^* \neq 0.$$

We have therefore satisfied the assumptions of Lemma 2.9 and there exists an isomorphism $\alpha_t^y : \mathcal{T}C^*(E) \to \mathcal{T}C^*(E)$ such that $\alpha_t^y(s_e) = e^{ity(e)}s_e$ and $\alpha_t^y(p_v) = p_v$.

We now show that $\alpha^y : t \mapsto \alpha_t^y$ is a homomorphism of $\mathbb{R}$ into $\operatorname{Aut} \mathcal{T}C^*(E)$. For $t, x \in \mathbb{R}$, the automorphisms $\alpha_t^y \circ \alpha_x^y$ and α_{t+x}^y agree on the generators $\{p_v, s_e\}$: looking at s_e we have

$$\begin{aligned}
(\alpha_t^y \circ \alpha_x^y)(s_e) &= \alpha_t^y(\alpha_x^y(s_e)) \\
&= \alpha_t^y(e^{ixy(e)}s_e) \\
&= e^{ity(e)}e^{ixy(e)}s_e \\
&= e^{i(t+x)y(e)}s_e \\
&= \alpha_{t+x}^y(s_e).
\end{aligned}$$

Similarly, looking at p_v we have

$$(\alpha_t^y \circ \alpha_x^y)(p_v) = \alpha_t^y(\alpha_x^y(p_v)) = \alpha_t^y(p_v) = p_v = \alpha_{t+x}^y(p_v).$$

Since they agree on generators they agree on all of $\mathcal{T}C^*(E)$, and α^y is a homomorphism of $\mathbb{R}$ into $\operatorname{Aut} \mathcal{T}C^*(E)$.

Finally, we need to show that α^y is continuous. Fix $t \in \mathbb{R}$, $a \in \mathcal{T}C^*(E)$ and $\epsilon > 0$. Choose a finite linear combination $c = \sum \lambda_{\mu,\nu} s_\mu s_\nu^*$, such that $\|a - c\| < \frac{\epsilon}{3}$. For $\mu \in E^*$ we have

$$\begin{aligned}\alpha_t^y(s_\mu) &= \alpha_t^y(s_{\mu_1} s_{\mu_2} \dots s_{\mu_{|\mu|}}) \\ &= e^{ity(\mu_1)} s_{\mu_1} e^{ity(\mu_2)} s_{\mu_2} \dots e^{ity(\mu_{|\mu|})} s_{\mu_{|\mu|}} \\ &= e^{it(y(\mu_1)+y(\mu_2)+\cdots+y(\mu_{|\mu|}))} s_{\mu_1} s_{\mu_2} \dots s_{\mu_{|\mu|}} \\ &= e^{ity(\mu)} s_\mu.\end{aligned}$$

Since scalar multiplication is continuous, so is

$$\begin{aligned}x \mapsto \alpha_x^y(c) &= \alpha_x^y\big(\sum \lambda_{\mu,\nu} s_\mu s_\nu^*\big) \\ &= \sum \lambda_{\mu,\nu} \alpha_x^y(s_\mu s_\nu^*) \\ &= \sum \lambda_{\mu,\nu} \alpha_x^y(s_\mu) \alpha_w^y(s_\nu^*) \\ &= \sum \lambda_{\mu,\nu} e^{ix(y(\mu)-y(\nu))} s_\mu s_\nu^*.\end{aligned}$$

So there exists $\delta > 0$ such that

$$|x - t| < \delta \Rightarrow \|\alpha_x^y(c) - \alpha_t^y(c)\| < \frac{\epsilon}{3}.$$

Since automorphisms of C^*-algebras preserve the norm, we have $\|\alpha_t^y(a-c)\| < \frac{\epsilon}{3}$. Thus for $|x - t| < \delta$ we have

$$\|\alpha_x^y(a) - \alpha_t^y(c)\| \leq \|\alpha_x^y(a - c)\| + \|\alpha_x^y(c) - \alpha_t^y(c)\| + \|\alpha_t^y(a - c)\| < 3(\frac{\epsilon}{3}) = \epsilon.$$

Thus α^y is continuous, as required. □

Remark 2.10. If we take y to be the function such that $y(e) = 1$ for all $e \in E^1$, then $y(\mu) = |\mu|$ for all $\mu \in E^*$. Then α^y is the dynamics lifted from the gauge action, as studied in [17], for example.

2.5 KMS states

In this section we introduce the concept of KMS states, using the definitions from [17, Section 1] and [27, Section 7].

Suppose that α is an action of $\mathbb{R}$ on a C^*-algebra B. An element b in B is *analytic* for the action α if the function $t \mapsto \alpha_t(b)$ is the restriction to $\mathbb{R}$ of an entire function on $\mathbb{C}$. Let B^{a} be the set of analytic elements of B. For $\beta \in (0, \infty)$, a state ϕ of B is a *KMS state at inverse temperature* β *for* α, or a *KMS*$_\beta$ *state for* (B, α), if it satisfies the following *KMS condition*:

$$\phi(ab) = \phi\Big(b\alpha_{i\beta}(a)\Big) \qquad \text{for all } a, b \in B^{\mathrm{a}}. \tag{2.5}$$

In fact, [31, Proposition 8.12.3] tells us that it suffices to check Equation (2.5) for a set of analytic elements which spans a dense subset of B.

In the case where B is $\mathcal{T}C^*(E)$, for every $\mu, \nu \in E^*$, the function $t \mapsto \alpha_t^y(s_\mu s_\nu^*) = e^{it(y(\mu)-y(\nu))}s_\mu s_\nu^*$ is the restriction of the entire function

$$z \mapsto e^{iz(y(\mu)-y(\nu))}s_\mu s_\nu^*.$$

The elements $s_\mu s_\nu^*$ are therefore analytic. Since they span a dense subspace of $\mathcal{T}C^*(E)$, it follows from [31, Proposition 8.12.3] that a state ϕ of $(\mathcal{T}C^*(E), \alpha^y)$ is a KMS$_\beta$ state for $\beta \in (0, \infty)$ if and only if

$$\phi((s_\mu s_\nu^*)(s_\sigma s_\tau^*)) = \phi((s_\sigma s_\tau^*)\alpha_{i\beta}^y(s_\mu s_\nu^*))$$

for all $\mu, \nu, \sigma, \tau \in E^*$.

2.6 Characterising KMS states of $(\mathcal{T}C^*(E), \alpha^y)$

In [17, Proposition 2.1] the KMS states of $(\mathcal{T}C^*(E), \alpha)$ are characterised. In this section we use a similar method to characterise the KMS states of $(\mathcal{T}C^*(E), \alpha^y)$.

Proposition 2.11. *Let E be a row-finite directed graph. Choose a weight function $y : E^1 \to (0, \infty)$. Let $\alpha^y : \mathbb{R} \to \operatorname{Aut} \mathcal{T}C^*(E)$ be the action from Proposition 2.8. Let $\beta \in (0, \infty)$.*

(a) Let $\delta_{\mu,\nu}$ be the Kronecker delta function. If a linear functional ϕ satisfies

$$\phi(s_\mu s_\nu^*) = \delta_{\mu,\nu} e^{-\beta y(\mu)} \phi(p_{s(\mu)}) \qquad \textit{for all } \mu, \nu \in E^*. \tag{2.6}$$

then ϕ also satisfies the KMS condition.

(b) A state ϕ is a KMS$_\beta$ state of $(\mathcal{T}C^*(E), \alpha^y)$ if and only if it satisfies Equation (2.6)*.*

Proof. (a) Suppose that a linear functional ϕ satisfies Equation (2.6). To see that ϕ satisfies the KMS condition (Equation (2.5)), consider a pair of spanning elements $s_\mu s_\nu^*$

and $s_\sigma s_\tau^*$ in $\mathcal{T}C^*(E)$. Computations using the product formula (Equation (2.2)) give

$$\phi(s_\mu s_\nu^* s_\sigma s_\tau^*) = \begin{cases} \phi(s_{\mu\sigma'} s_\tau^*) & \text{if } \sigma = \nu\sigma' \\ \phi(s_\mu s_{\tau\nu'}^*) & \text{if } \nu = \sigma\nu' \\ 0 & \text{otherwise} \end{cases}$$

$$(2.7) \qquad = \begin{cases} e^{-\beta y(\tau)} \phi(p_{s(\tau)}) & \text{if } \sigma = \nu\sigma' \text{ and } \tau = \mu\sigma' \\ e^{-\beta y(\mu)} \phi(p_{s(\mu)}) & \text{if } \nu = \sigma\nu' \text{ and } \mu = \tau\nu' \\ 0 & \text{otherwise.} \end{cases}$$

Similarly,

$$\phi(s_\sigma s_\tau^* s_\mu s_\nu^*) = \begin{cases} \phi(s_{\sigma\mu'} s_\nu^*) & \text{if } \mu = \tau\mu' \\ \phi(s_\sigma s_{\nu\tau'}^*) & \text{if } \tau = \mu\tau' \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} e^{-\beta y(\nu)} \phi(p_{s(\nu)}) & \text{if } \mu = \tau\mu' \text{ and } \nu = \sigma\mu' \\ e^{-\beta y(\sigma)} \phi(p_{s(\sigma)}) & \text{if } \tau = \mu\tau' \text{ and } \sigma = \nu\tau' \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$(2.8) \qquad \phi(s_\sigma s_\tau^* \alpha_{i\beta}(s_\mu s_\nu^*)) = \begin{cases} e^{-\beta(y(\mu)-y(\nu))} e^{-\beta y(\nu)} \phi(p_{s(\nu)}) & \text{if } \mu = \tau\mu' \text{ and } \nu = \sigma\mu' \\ e^{-\beta(y(\mu)-y(\nu))} e^{-\beta y(\sigma)} \phi(p_{s(\sigma)}) & \text{if } \tau = \mu\tau' \text{ and } \sigma = \nu\tau' \\ 0 & \text{otherwise.} \end{cases}$$

If $\mu = \tau\mu'$ and $\nu = \sigma\mu'$, then $s(\mu) = s(\nu)$ and

$$\begin{aligned} \phi(s_\mu s_\nu^* s_\sigma s_\tau^*) &= e^{-\beta y(\mu)} \phi(p_{s(\mu)}) \\ &= e^{-\beta y(\mu)} \phi(p_{s(\nu)}) \\ &= e^{-\beta(y(\mu)-y(\nu))} e^{-\beta y(\nu)} \phi(p_{s(\nu)}) \\ &= \phi(s_\sigma s_\tau^* \alpha_{i\beta}^y (s_\mu s_\nu^*)). \end{aligned}$$

So far our computations have been very similar to those in [17]. For the next step we observe that y is additive, in the sense that if $\mu \in E^*$ such that $\mu = \mu'\mu''$ then $y(\mu) = y(\mu') + y(\mu'')$.

If $\tau = \mu\tau'$ and $\sigma = \nu\tau'$, then $s(\tau) = s(\sigma)$. Also, $y(\tau) = y(\mu) + y(\tau')$ and $y(\sigma) = y(\nu) + y(\tau')$, which implies that $y(\tau) = y(\mu) - y(\nu) + y(\sigma)$. So

$$\begin{aligned}\phi(s_\mu s_\nu^* s_\sigma s_\tau^*) &= e^{-\beta y(\tau)}\phi(p_{s(\tau)})\\ &= e^{-\beta y(\tau)}\phi(p_{s(\sigma)})\\ &= e^{-\beta(y(\mu)-y(\nu)+y(\sigma))}\phi(p_{s(\sigma)})\\ &= e^{-\beta(y(\mu)-y(\nu))}e^{-\beta y(\sigma)}\phi(p_{s(\sigma)})\\ &= \phi(s_\sigma s_\tau^* \alpha^y_{i\beta}(s_\mu s_\nu^*)).\end{aligned}$$

Otherwise at least one of $\mu = \tau\mu'$ and $\nu = \sigma\mu'$ fails and at least one of $\tau = \mu\tau'$ and $\sigma = \nu\tau'$ fails, in which case Equation (2.7) tells us that $\phi(s_\mu s_\nu^* s_\sigma s_\tau^*) = 0$ and Equation (2.8) tells us that $\phi(s_\sigma s_\tau^* \alpha_{i\beta}(s_\mu s_\nu^*)) = 0$.

Thus ϕ satisfies the KMS condition.

(b) In the forward direction suppose that ϕ is a KMS$_\beta$ state of $(\mathcal{T}C^*(E), \alpha^y)$. We want to show that ϕ satisfies Equation (2.6). Fix $\mu, \nu \in E^*$. The KMS condition gives

$$\phi(s_\mu s_\nu^*) = \phi(s_\nu^* \alpha^y_{i\beta}(s_\mu)) = \phi(s_\nu^* e^{-\beta y(\mu)} s_\mu) = e^{-\beta y(\mu)}\phi(s_\nu^* s_\mu). \tag{2.9}$$

We consider the cases $|\mu| = |\nu|$ and $|\mu| \neq |\nu|$ separately. For $|\mu| = |\nu|$, the product formula implies that $s_\nu^* s_\mu = \delta_{\nu,\mu} p_{s(\mu)}$, and Equation (2.9) gives

$$\phi(s_\mu s_\nu^*) = e^{-\beta y(\mu)}\phi(s_\nu^* s_\mu) = \delta_{\mu,\nu} e^{-\beta y(\mu)}\phi(p_{s(\mu)}).$$

Next, suppose that $|\mu| \neq |\nu|$. If μ doesn't extend ν and ν doesn't extend μ then the product formula tells us that $s_\nu^* s_\mu = 0$, and Equation (2.9) gives

$$\phi(s_\mu s_\nu^*) = e^{-\beta y(\mu)}\phi(s_\nu^* s_\mu) = 0.$$

Otherwise one of μ, ν extends the other, and since $y(e) > 0$ for all $e \in E^1$, $y(\mu) \neq y(\nu)$. Applying the KMS condition again to Equation (2.9) gives

$$\begin{aligned}\phi(s_\mu s_\nu^*) &= e^{-\beta y(\mu)}\phi(s_\nu^* s_\mu)\\ &= e^{-\beta y(\mu)}\phi(s_\mu \alpha^y_{i\beta}(s_\nu^*))\\ &= e^{-\beta y(\mu)}\phi(s_\mu e^{\beta y(\nu)} s_\nu^*)\\ &= e^{-\beta(y(\mu)-y(\nu))}\phi(s_\mu s_\nu^*),\end{aligned}$$

and since $e^{-\beta(y(\mu)-y(\nu))} \neq 1$, we have $\phi(s_\mu s_\nu^*) = 0$. Therefore ϕ satisfies Equation (2.6).

Conversely, suppose that ϕ satisfies Equation (2.6). Then part (a) tells us that ϕ satisfies the KMS condition. In addition ϕ is a state, so it is a KMS$_\beta$ state. $\square$

Chapter 3

KMS states of C^*-algebras for finite graphs with a generalised gauge dynamics

In [20, Section 4.3] Ionescu and Kumjian use a groupoid model for $C^*(E)$ to discuss KMS states on $C^*(E)$ with a generalised gauge dynamics. In [17] an Huef, Laca, Raeburn and Sims use direct arguments to describe the KMS states of both $\mathcal{T}C^*(E)$ and $C^*(E)$ with the gauge dynamics. The goal of this chapter is to extend the method of [17] to study the KMS states of both $\mathcal{T}C^*(E)$ and $C^*(E)$ with a generalised gauge dynamics.

First study the KMS states of $(\mathcal{T}C^*(E), \alpha^y)$ at large inverse temperatures. We then discuss the KMS states of $(\mathcal{T}C^*(E), \alpha^y)$ at the critical inverse temperature when E is strongly connected. Finally we get a generalised gauge dynamics $\overline{\alpha}^y$ of $\mathbb{R}$ on $C^*(E)$ and study the KMS states of $(C^*(E), \overline{\alpha}^y)$.

In this chapter we consider finite directed graphs, that is, directed graphs for which E^0 and E^1 are finite sets. Since the set E^1 is finite, vE^1 is a finite set for all $v \in E^0$, that is, E is row-finite. Sums over E^0 or E^1 (or their subsets) are finite sums, so we don't have to worry about convergence, and we can easily switch sums using the algebra of limits. Sums over E^* are usually infinite sums but we can use Tonelli's theorem and the dominated convergence theorem to ensure convergence.

3.1 KMS states of $(\mathcal{T}C^*(E), \alpha^y)$ and the subinvariance relation

In this section we use the function y and fix $\theta \in [0, \infty)$ to define a matrix $B(y, \theta)$ that will be central to our proofs in the remainder of the chapter. We then describe a subinvariance relation like that used in [17, Proposition 2.1(c)].

Definition 3.1. For $y : E^1 \to (0, \infty)$ and $\theta \in [0, \infty)$, let $B(y, \theta) = (B(y, \theta)_{v,w})$ be the $E^0 \times E^0$ matrix with entries

$$B(y, \theta)_{v,w} = \sum_{e \in vE^1 w} e^{-\theta y(e)},$$

where we take $B(y, \theta)_{v,w} = 0$ if $vE^1 w = \emptyset$.

Proposition 3.2. *Let E be a finite directed graph. Choose a weight function $y : E^1 \to (0, \infty)$. For $\theta \in [0, \infty)$ let $B(y, \theta)$ be the matrix from Definition 3.1. Let $\alpha^y : \mathbb{R} \to \operatorname{Aut} \mathcal{T}C^*(E)$ be the action from Proposition 2.8. Let $\beta \in (0, \infty)$. Suppose that ϕ is a KMS$_\beta$ state of $(\mathcal{T}C^*(E), \alpha^y)$, and define $m^\phi = (m^\phi_v)$ by $m^\phi_v = \phi(p_v)$. Then the vector $m^\phi \in [0, \infty)^{E^0}$ satisfies the subinvariance relation $B(y, \beta)m^\phi \leq m^\phi$ and $\|m^\phi\|_1 = 1$.*

Proof. First, each m^ϕ_v is non-negative because ϕ is a state, and therefore a positive functional. Next we show that m^ϕ satisfies the subinvariance relation $B(y, \beta)m^\phi \leq m^\phi$. Fix $v \in E^0$. If v is not a source then (TCK2) implies that $\phi(p_v) \geq \sum_{e \in vE^1} \phi(s_e s_e^*)$ and

$$\begin{aligned}
\sum_{e \in vE^1} \phi(s_e s_e^*) &= \sum_{e \in vE^1} \phi(s_e^* \alpha_{i\beta}(s_e)) \\
&= \sum_{e \in vE^1} e^{-\beta y(e)} \phi(s_e^* s_e) \\
&= \sum_{e \in vE^1} e^{-\beta y(e)} \phi(p_{s(e)}) \\
&= \sum_{e \in vE^1} e^{-\beta y(e)} m^\phi_{s(e)} \\
&= \sum_{w \in E^0} \sum_{e \in vE^1 w} e^{-\beta y(e)} m^\phi_w \\
&= \sum_{w \in E^0} B(y, \beta)_{v,w} m^\phi_w \\
&= (B(y, \beta)m^\phi)_v. \qquad (3.1)
\end{aligned}$$

Hence $(B(y,\beta)m^{\phi})_v \leq \phi(p_v) = m_v^{\phi}$. Otherwise v is a source. Then $B(y,\beta)_{v,w} = 0$ for all $w \in E^0$, and

$$(B(y,\beta)m^{\phi})_v = \sum_{w \in E^0} B(y,\beta)_{v,w} m_w^{\phi} = 0 \leq m_v^{\phi}.$$

Therefore $(B(y,\beta)m^{\phi})_v \leq m_v^{\phi}$ for all v, so m^{ϕ} satisfies $B(y,\beta)m^{\phi} \leq m^{\phi}$.

Finally we show that $\|m^{\phi}\|_1 = 1$. Note that $\sum_{v \in E^0} p_v$ is the identity of $\mathcal{T}C^*(E)$. Since ϕ is a state,

$$\|m^{\phi}\|_1 = \sum_{v \in E^0} m_v^{\phi} = \sum_{v \in E^0} \phi(p_v) = \phi\Big(\sum_{v \in E^0} p_v\Big) = \phi(1) = 1. \qquad \square$$

3.2 KMS states of $(\mathcal{T}C^*(E), \alpha^y)$ at large inverse temperatures

In this thesis, for $A \in M_n(\mathbb{C})$, we denote the spectral radius of A by $\rho(A)$. The purpose of this section is to study the KMS$_\beta$ states of $(\mathcal{T}C^*(E), \alpha^y)$ for which $1 > \rho(B(y,\beta))$. By Corollary A.14, the condition $1 > \rho(B(y,\beta))$ implies that $(I - B(y,\beta))$ is an invertible matrix, which is crucial in the following proofs. The following lemma tells us that if this condition holds for one β, it holds for all larger β', so we say such results are for "large inverse temperatures".

Lemma 3.3. *Let E be a finite directed graph. Choose a weight function $y : E^1 \to (0,\infty)$. For $\theta \in [0,\infty)$ let $B(y,\theta)$ be the matrix from Definition 3.1. If $\beta \in (0,\infty)$ satisfies $1 > \rho(B(y,\beta))$ and $\beta' \geq \beta$, then $1 > \rho(B(y,\beta'))$.*

Proof. Fix $\beta' \in (0,\infty)$ such that $\beta' \geq \beta$. Then $B(y,\beta')_{v,w} \leq B(y,\beta)_{v,w}$, and hence $0 \leq B(y,\beta') \leq B(y,\beta)$ in the sense of Section A.1. Then applying Corollary A.4 implies that $\rho(B(y,\beta')) \leq \rho(B(y,\beta))$. Thus $\rho(B(y,\beta')) < 1$. $\square$

Theorem 3.4. *Let E be a finite directed graph. Choose a weight function $y : E^1 \to (0,\infty)$. For $\theta \in [0,\infty)$ let $B(y,\theta)$ be the matrix from Definition 3.1. Assume that $\beta \in (0,\infty)$ satisfies $1 > \rho(B(y,\beta))$. Let $\alpha^y : \mathbb{R} \to \operatorname{Aut} \mathcal{T}C^*(E)$ be the action from Proposition 2.8.*

(a) For $w \in E^0$, the series $\sum_{\mu \in E^* w} e^{-\beta y(\mu)}$ converges with sum $x_w \geq 1$. Set $x := (x_w) \in [1,\infty)^{E^0}$, and consider $\epsilon \in [0,\infty)^{E^0}$. Define $m := (I - B(y,\beta))^{-1}\epsilon$. Then $m \in [0,\infty)^{E^0}$, and $\|m\|_1 = 1$ if and only if $\epsilon \cdot x = 1$.

(b) Suppose that $\epsilon \in [0,\infty)^{E^0}$ satisfies $\epsilon \cdot x = 1$, and set $m := (I - B(y,\beta))^{-1}\epsilon$. Then there is a KMS$_\beta$ state ϕ_ϵ of $(\mathcal{T}C^*(E), \alpha^y)$ satisfying

$$\phi_\epsilon(s_\mu s_\nu^*) = \delta_{\mu,\nu} e^{-\beta y(\mu)} m_{s(\mu)}. \tag{3.2}$$

(c) The set

$$\Sigma_\beta := \{\epsilon \in [0,\infty)^{E^0} : \epsilon \cdot x = 1\}$$

is a compact convex subset of $\mathbb{R}^{E^0}$ and $F : \epsilon \mapsto \phi_\epsilon$ is an affine homeomorphism of Σ_β onto the simplex of KMS$_\beta$ states of $(\mathcal{T}C^*(E), \alpha^y)$. For a KMS$_\beta$ state ϕ of $(\mathcal{T}C^*(E), \alpha^y)$ let $m^\phi = (m_v^\phi)$ be the vector with entries $m_v^\phi := \phi(p_v)$. Then the inverse of this isomorphism takes ϕ to $(I - B(y,\beta))m^\phi$.

Proof. (a) First we want to show that the series $\sum_{\mu \in E^* w} e^{-\beta y(\mu)}$ either converges or is finite. We start by showing that $\sum_{\mu \in E^* w} e^{-\beta y(\mu)}$ converges. Let $w \in E^0$ and fix $n \in \mathbb{N}$. Then

$$\sum_{\mu \in E^n w} e^{-\beta y(\mu)} = \sum_{v \in E^0} \sum_{\mu \in v E^n w} e^{-\beta y(\mu)} = \sum_{v \in E^0} B(y,\beta)^n_{v,w}. \tag{3.3}$$

Since $1 > \rho(B(y,\beta))$, Corollary A.14 tells us that the series $\sum_{n=0}^\infty B(y,\beta)^n$ converges in operator norm with sum $(I - B(y,\beta))^{-1}$. This implies that for every fixed $v \in E^0$ the series $\sum_{n=0}^\infty B(y,\beta)^n_{v,w}$ converges. Then, by the algebra of limits,

$$\begin{aligned}
\sum_{v \in E^0} \sum_{n=0}^\infty B(y,\beta)^n_{v,w} &= \sum_{v \in E^0} \Big(\lim_{N \to \infty} \sum_{n=0}^N B(y,\beta)^n_{v,w} \Big) \\
&= \lim_{N \to \infty} \sum_{v \in E^0} \sum_{n=0}^N B(y,\beta)^n_{v,w} \\
&= \lim_{N \to \infty} \sum_{n=0}^N \sum_{v \in E^0} B(y,\beta)^n_{v,w} \\
&= \sum_{n=0}^\infty \sum_{v \in E^0} B(y,\beta)^n_{v,w}.
\end{aligned}$$

Now Equation (3.3) tells us that

$$\sum_{v \in E^0} \sum_{n=0}^\infty B(y,\beta)^n_{v,w} = \sum_{n=0}^\infty \sum_{\mu \in E^n w} e^{-\beta y(\mu)},$$

so $\sum_{n=0}^{\infty}\sum_{\mu\in E^n w} e^{-\beta y(\mu)}$ converges, and

$$\sum_{\mu\in E^* w} e^{-\beta y(\mu)} = \sum_{n=0}^{\infty}\sum_{\mu\in E^n w} e^{-\beta y(\mu)}.$$

The sum is at least 1 because all the terms are non-negative and when $n = 0$, $B(y,\beta)^n_{v,v} = 1$.

Next, let $x := (x_w) \in [1,\infty)^{E^0}$ and fix $\epsilon \in [0,\infty)^{E^0}$. To see that $m \geq 0$, fix $v \in E^0$. We want to show that $m_v \geq 0$. We have

$$m_v = \Big((I - B(y,\beta))^{-1}\epsilon\Big)_v = \Big(\sum_{n=0}^{\infty} B(y,\beta)^n \epsilon\Big)_v.$$

Every element of $B(y,\beta)^n$ and ϵ is non-negative, so m_v must be non-negative.

Finally we want to show that $\|m\|_1 = 1$ if and only if $\epsilon \cdot x = 1$. Computing, we get

$$\begin{aligned}
\|m\|_1 &= \sum_{v\in E^0} m_v \\
&= \sum_{v\in E^0} ((I - B(y,\beta))^{-1}\epsilon)_v \\
&= \sum_{v\in E^0} \Big(\Big(\sum_{n=0}^{\infty} B(y,\beta)^n\Big)\epsilon\Big)_v \\
&= \sum_{v\in E^0}\sum_{n=0}^{\infty}\sum_{w\in E^0} B(y,\beta)^n_{v,w}\epsilon_w \\
&= \sum_{w\in E^0} \epsilon_w\Big(\sum_{\mu\in E^* w} e^{-\beta y(\mu)}\Big) \\
&= \sum_{w\in E^0} \epsilon_w x_w \\
&= \epsilon \cdot x.
\end{aligned}$$

Thus $\|m\|_1 = 1$ if and only if $\epsilon \cdot x = 1$.

(b) First we need to find a state of $(\mathcal{T}C^*(E), \alpha^y)$. We build a state using the finite path representation $\pi_{Q,T}$ defined in Proposition 2.3. For $\mu \in E^*$ we set

$$\Delta_\mu := e^{-\beta y(\mu)}\epsilon_{s(\mu)}, \tag{3.4}$$

and note that $\Delta_\mu \geq 0$. Taking h_μ as the point mass for $\mu \in E^*$, we aim to define ϕ_ϵ by

$$\phi_\epsilon(a) = \sum_{\mu\in E^*} \Delta_\mu(\pi_{Q,T}(a)h_\mu | h_\mu) \tag{3.5}$$

for $a \in \mathcal{T}C^*(E)$. To show that Equation (3.5) defines a state, we need to show it is a positive linear functional and that $\phi_\epsilon(1) = 1$.

We first claim that $\sum_{\mu \in E^*} \Delta_\mu = 1$. We start by showing that $\sum_{\mu \in vE^*} e^{-\beta y(\mu)} \epsilon_{s(\mu)}$ converges. Fix $v \in E^0$ and $n \in \mathbb{N}$. Then

$$\begin{aligned}
\sum_{\mu \in vE^n} e^{-\beta y(\mu)} \epsilon_{s(\mu)} &= \sum_{w \in E^0} \sum_{\mu \in vE^n w} e^{-\beta y(\mu)} \epsilon_w \\
&= \sum_{w \in E^0} B(y,\beta)^n_{v,w} \epsilon_w \\
&= (B(y,\beta)^n \epsilon)_v. \qquad (3.6)
\end{aligned}$$

We know that $\sum_{n=0}^{\infty} (B(y,\beta)^n \epsilon)_v$ converges with sum $((I - B(y,\beta))^{-1}\epsilon)_v$. Since, by Equation (3.6),

$$\sum_{n=0}^{\infty} (B(y,\beta)^n \epsilon)_v = \sum_{n=0}^{\infty} \sum_{\mu \in vE^n} e^{-\beta y(\mu)} \epsilon_{s(\mu)},$$

the latter converges absolutely. Using Equation (3.4), we can write

$$\begin{aligned}
m_v &:= ((I - B(y,\beta))^{-1}\epsilon)_v \qquad (3.7) \\
&= \sum_{n=0}^{\infty} \sum_{\mu \in vE^n} e^{-\beta y(\mu)} \epsilon_{s(\mu)} \\
&= \sum_{\mu \in vE^*} e^{-\beta y(\mu)} \epsilon_{s(\mu)} \\
&= \sum_{\mu \in vE^*} \Delta_\mu.
\end{aligned}$$

So $\sum_{\mu \in vE^*} \Delta_\mu$ converges as well. Finally,

$$\sum_{\mu \in E^*} \Delta_\mu = \sum_{v \in E^0} \Big(\sum_{\mu \in vE^*} \Delta_\mu \Big) = \sum_{v \in E^0} m_v = \|m\|_1 = 1.$$

Thus $\sum_{\mu \in E^*} \Delta_\mu = 1$, as claimed.

We now use that $\sum_{\mu \in E^*} \Delta_\mu = 1$ to prove that Equation (3.5) converges for all $a \in \mathcal{T}C^*(E)$. Fix $a \in \mathcal{T}C^*(E)$. Then applying the Cauchy-Schwarz inequality,

$$\begin{aligned}
0 &\leq |\Delta_\mu (\pi_{Q,T}(a) h_\mu | h_\mu)| \\
&= |\Delta_\mu| |(\pi_{Q,T}(a) h_\mu | h_\mu)| \\
&\leq \Delta_\mu \|\pi_{Q,T}(a) h_\mu\| \|h_\mu\| \\
&\leq \Delta_\mu \|\pi_{Q,T}(a)\| \|h_\mu\|^2 \\
&\leq \Delta_\mu \|a\| \cdot 1.
\end{aligned}$$

Since $\sum_{\mu\in E^*}\Delta_\mu$ converges, $\sum_{\mu\in E^*}\Delta_\mu\|a\|$ converges and the comparison test tells us that $\sum_{\mu\in E^*}|\Delta_\mu(\pi_{Q,T}(a)h_\mu|h_\mu)|$ converges. Thus $\sum_{\mu\in E^*}\Delta_\mu(\pi_{Q,T}(a)h_\mu|h_\mu)$ converges absolutely.

Now we need to show that ϕ_ϵ defines a positive functional on $\mathcal{T}C^*(E)$. Linearity of ϕ_ϵ follows from linearity of the inner product in the first variable. Next we show that $\phi_\epsilon(a^*a)\geq 0$. Since

$$\phi_\epsilon(a^*a)=\sum_{\mu\in E^*}\Delta_\mu(\pi_{Q,T}(a^*a)h_\mu|h_\mu)=\sum_{\mu\in E^*}\Delta_\mu(\pi_{Q,T}(a)h_\mu|\pi_{Q,T}(a)h_\mu)$$

is a sum of non-negative terms, $\phi_\epsilon(a^*a)\geq 0$. Therefore ϕ_ϵ is a positive functional.

Finally we need to show that $\phi_\epsilon(1)=1$. Applying Equation (3.7) gives

$$\begin{aligned}\phi_e(1)&=\sum_{\mu\in E^*}\Delta_\mu(\pi_{Q,T}(1)h_\mu|h_\mu)\\&=\sum_{\mu\in E^*}\Delta_\mu(1h_\mu|h_\mu)\\&=\sum_{\mu\in E^*}\Delta_\mu\|h_\mu\|^2\\&=\sum_{\mu\in E^*}\Delta_\mu\\&=1.\end{aligned}$$

Now ϕ_ϵ is a positive linear functional and $\phi_\epsilon(1)=1$, so it is a state.

Next we prove that ϕ_ϵ satisfies Equation (3.2). Fix $\lambda\in E^*$. Then

$$(\pi_{Q,T}(s_\mu s_\nu^*)h_\mu|h_\mu)=(T_\nu^*h_\lambda|T_\mu^*h_\lambda)=\begin{cases}1 & \text{if } \lambda=\mu\lambda'=\nu\lambda'\\ 0 & \text{otherwise.}\end{cases}$$

Since $\mu\lambda'=\nu\lambda'$ forces $\mu=\nu$, we have $\phi_\epsilon(s_\mu s_\nu^*)=0$ if $\mu\neq\nu$. So suppose that $\mu=\nu$. Then

$$\begin{aligned}\phi_\epsilon(s_\mu s_\mu^*)&=\sum_{\mu\in E^*}\Delta_\lambda(T_\mu^*h_\lambda|T_\mu^*h_\lambda)\\&=\sum_{\lambda=\mu\lambda'}e^{-\beta y(\mu\lambda')}\epsilon_{s(\lambda')}\\&=\sum_{\lambda=\mu\lambda'}e^{-\beta(y(\mu)+y(\lambda'))}\epsilon_{s(\lambda')}\\&=\sum_{\lambda=\mu\lambda'}e^{-\beta y(\mu)}e^{-\beta y(\lambda')}\epsilon_{s(\lambda')}\\&=e^{-\beta y(\mu)}\sum_{\lambda'\in s(\mu)E^*}\Delta_{\lambda'}.\end{aligned}$$

Now Equation (3.7) gives $\phi_\epsilon(s_\mu s_\mu^*) = e^{-\beta y(\mu)} m_{s(\mu)}$. Thus ϕ_ϵ satisfies Equation (3.2).

Since $\phi(p_v) = m_v$, $\phi_\epsilon(s_\mu s_\nu^*) = \delta_{\mu,\nu} e^{-\beta y(\mu)} \phi(p_{s(\mu)})$, and Proposition 2.11(b) implies that ϕ_ϵ is a KMS$_\beta$ state.

(c) We first show that Σ_β is a compact convex subset of $\mathbb{R}^{E^0}$, and then that F is a homeomorphism.

We show that Σ_β is compact by showing it is closed and bounded. To see that Σ_β is closed in $\mathbb{R}^{E^0}$, take $\{\epsilon_n\} \subset \Sigma_\beta$ and $\epsilon \in [0,\infty)^{E^0}$ such that $\epsilon_n \to \epsilon$. The dot product is continuous from $\mathbb{R}^{E^0} \times \mathbb{R}^{E^0} \to \mathbb{R}$, so $\epsilon_n \to \epsilon$ implies that $\epsilon_n \cdot x \to \epsilon \cdot x$. But $\epsilon_n \in \Sigma_\beta$ for all n, so $\epsilon_n \cdot x = 1$ for all n. Thus $\epsilon \cdot x = 1$, that is, $\epsilon \in \Sigma_\beta$. Thus Σ_β contains all of its limit points, and is therefore closed. To see that Σ_β is bounded, we show it is contained in a ball of finite radius. Take $\epsilon \in \Sigma_\beta$. This implies that $\epsilon \cdot x = 1$, that is, that $\sum_{v\in E^0} \epsilon_v x_v = 1$. Since $x_v \in [1,\infty)$ for all $v \in E^0$, $\sum_{v\in E^0} \epsilon_v \le 1$. Since $\epsilon_v \ge 0$ for all $v \in E^0$, we have $0 \le \epsilon_v \le 1$. Thus $\|\epsilon\|^2 = \sum_{v\in E^0} \epsilon_v^2 \le |E^0|$. This implies that $\Sigma_\beta \subseteq B[0, |E^0|] \subseteq \mathbb{R}^{E^0}$, so Σ_β is bounded. Since Σ_β is closed and bounded the Heine-Borel theorem tells us that it is compact.

Next we show that Σ_β is convex. Fix $k \in \mathbb{N}$. For $i \in \{1, \ldots, k\}$ take $c_i \in [0,1]$ such that $\sum_{i=0}^k c_i = 1$ and let $\{\epsilon_i\} \subset \Sigma_\beta$. Then

$$\begin{aligned}
\Big(\sum_{i=0}^k c_i\epsilon_i\Big) \cdot x &= \sum_{v\in E^0} \Big(\sum_{i=0}^k c_i\epsilon_i\Big)_v x_v \\
&= \sum_{v\in E^0} \sum_{i=0}^k c_i(\epsilon_i)_v x_v \\
&= \sum_{i=0}^k c_i \sum_{v\in E^0} (\epsilon_i)_v x_v \\
&= \sum_{i=0}^k c_i(\epsilon_i \cdot x) \\
&= \sum_{i=0}^k c_i \\
&= 1,
\end{aligned}$$

so $\epsilon = \sum_{i=0}^k c_i\epsilon_i \in \Sigma_\beta$. Thus Σ_β is convex. We now know that Σ_β is a compact convex subset of $\mathbb{R}^{E^0}$.

We now show that F is a homeomorphism by showing that it is surjective, injective and continuous.

To see that $F : \epsilon \mapsto \phi_\epsilon$ is surjective, let ϕ be a KMS$_\beta$ state. Proposition 3.2 tells us that $m^\phi := (\phi(p_v))$ satisfies $B(y,\beta)m^\phi \le m^\phi$ and $\|m^\phi\|_1 = 1$. Take $\epsilon :=$

$(I - B(y,\beta))m^{\phi}$. Then $m := (I - B(y,\beta))^{-1}\epsilon = m^{\phi}$. So $\|m\|_1 = 1$, and part (a) tells us that $\epsilon \cdot x = 1$. Then we can apply part (b), which tells us that there is a KMS$_\beta$ state ϕ_ϵ satisfying Equation (3.2), so that

$$\phi_\epsilon(s_\mu s_\nu^*) = \delta_{\mu,\nu} e^{-\beta y(\mu)} m_{s(\mu)} = \delta_{\mu,\nu} e^{-\beta y(\mu)} m^{\phi}_{s(\mu)} = \delta_{\mu,\nu} e^{-\beta y(\mu)} \phi(p_{s(\mu)}).$$

Then Equation (2.6) gives

$$\phi_\epsilon(s_\mu s_\nu^*) = \phi(s_\mu s_\nu^*).$$

Then linearity implies that $\phi_\epsilon(b) = \phi(b)$ for b in the dense $*$-subalgebra $\operatorname{span}\{s_\mu s_\nu^* : \mu, \nu \in E^*, s(\mu) = s(\nu)\}$. Therefore $\phi = \phi_\epsilon$, so F is surjective.

To see that F is injective, suppose that $F(\epsilon) = F(\epsilon')$, that is, $\phi_\epsilon = \phi_{\epsilon'}$. Let $m = (\phi_\epsilon(p_v))$ and $m' := (\phi_{\epsilon'}(p_v))$. Then $m = m'$. Now $\epsilon = (I - B(y,\beta))m = (I - B(y,\beta))m' = \epsilon'$. Thus F is injective.

To see that F is continuous, take $\{\epsilon_n\} \subset \Sigma_\beta$ and $\epsilon \in \Sigma_\beta$ such that $\epsilon_n \to \epsilon$. This implies that $((I - B(y,\beta))^{-1}\epsilon_n)_v \to ((I - B(y,\beta))^{-1}\epsilon)_v$ for all $v \in E^0$. Therefore, writing m_n for $(I - B(y,\beta))^{-1}\epsilon_n$, we have $(m_n)_v \to m_v$ for all $v \in E^0$. This in turn implies that

$$\phi_{\epsilon_n}(s_\mu s_\nu^*) = \delta_{\mu,\nu} e^{-\beta y(\mu)} (m_n)_{s(\mu)} \to \delta_{\mu,\nu} e^{-\beta y(\mu)} m_{s(\mu)} = \phi_\epsilon(s_\mu s_\nu^*),$$

for all $\mu, \nu \in E^*$. Then linearity implies that $\phi_{\epsilon_n}(b) \to \phi(b)$ for b in the dense $*$-subalgebra $\operatorname{span}\{s_\mu s_\nu^* : \mu, \nu \in E^*, s(\mu) = s(\nu)\}$. Therefore $\phi \to \phi_\epsilon$ in the weak* topology, so F is continuous.

Since F is a continuous bijection of a compact space Σ_β onto a Hausdorff space R^{E^0}, F is a homeomorphism.

Finally we show that F is affine. Fix $k \in \mathbb{N}$. Suppose that $\{\epsilon_i : 1 \le i \le k\} \subseteq \Sigma_\beta$ and $\{c_i \in [0,1] : 1 \le i \le k\}$ satisfy $\sum_{i=1}^k c_i = 1$. We need to check that $F(\sum_{i=0}^k c_i \epsilon_i) = \sum_{i=0}^k c_i F(\epsilon_i)$, that is, $\phi_{\sum_{i=0}^k c_i \epsilon_i}(a) = \sum_{i=0}^k c_i \phi_{\epsilon_i}(a)$ for all $a \in \mathcal{T}C^*(E)$. Fix $a \in \mathcal{T}C^*(\Lambda)$ and write $\epsilon = \sum_{i=1}^k c_i \epsilon_i$. Then

$$\begin{aligned}
\phi_\epsilon(a) &= \sum_{\mu \in E^0} \Delta_\mu(\pi_{Q,T}(a) h_\mu | h_\mu) \\
&= \sum_{\mu \in E^0} e^{-\beta y(\mu)} \Big(\sum_{i=0}^k c_i \epsilon_i\Big)_{s(\mu)} (\pi_{Q,T}(a) h_\mu | h_\mu) \\
&= \sum_{i=0}^k c_i \sum_{\mu \in E^0} e^{-\beta y(\mu)} (\epsilon_i)_{s(\mu)} (\pi_{Q,T}(a) h_\mu | h_\mu).
\end{aligned}$$

Let $\Delta_{\mu,i} := e^{-\beta y(\mu)}(\epsilon_i)_{s(\mu)}$. Then

$$\phi_\epsilon(a) = \sum_{i=0}^{k} c_i \sum_{\mu \in E^0} \Delta_{\mu,i}(\pi_{Q,T}(a)h_\mu | h_\mu) = \sum_{i=0}^{k} c_i \phi_{\epsilon_i}(a).$$

Thus F is affine. □

3.3 Finding the critical inverse temperature β_c

In this section we show that there exists a nonnegative number β_c such that

$$\rho(B(y, \beta_c)) = 1$$

and that this β_c is unique. In the next section we will use this β_c to show that there is a unique KMS$_{\beta_c}$ state of $(\mathcal{T}C^*(E), \alpha^y)$ when E is strongly connected. Since the behaviour of KMS$_{\beta_c}$ states change we call this the "critical inverse temperature".

To find β_c we apply the Perron-Frobenius theorem (for example, [7, Theorem 2.6]) to $B(y,\theta)$. This requires that $B(y,\theta)$ is irreducible, that is, that for every pair v, w of its index set there exists a positive integer m such that $(B(y,\theta)^m)_{v,w} > 0$. So, for a function $y : E^1 \to (0, \infty)$ and $\theta \in [0, \infty)$, we first show that $B(y,\theta)$ is irreducible when E is strongly connected (just as the vertex matrix A of E is irreducible if and only if E is strongly connected). We assume that E is strongly connected for the rest of the chapter.

Lemma 3.5. *Let E be a finite directed graph which is strongly connected. The matrix $B(y,\theta)$ from Definition 3.1 is irreducible.*

Proof. An $n \times n$ non-negative matrix T is irreducible if for every pair v, w of its index set, there exists a positive integer m such that $(T^m)_{v,w} > 0$. Since the exponential function takes positive values, $B(y,\theta)_{v,w} > 0$ when $vE^1w \neq \emptyset$. Fix $v, w \in E^0$. Then because E is strongly connected there exists a path from v to w. Let $m > 0$ be the length of such a path, and label the vertices in the path as $v = v_0, v_1, \ldots, v_n = w$. Then

$$B(y,\theta)^m_{v,w} \geq B(y,\theta)_{v_0,v_1} B(y,\theta)_{v_1,v_2} \ldots B(y,\theta)_{v_{n-1},v_n}.$$

Since each $B(y,\theta)_{v_i,v_{i+1}} > 0$, $B(y,\theta)^m_{v,w} > 0$, and $B(y,\theta)$ is irreducible. □

We now extract the following result from [8, Theorem 6.9.6] to show there exists a unique nonnegative number β_c such that $\rho(B(y, \beta_c)) = 1$.

Proposition 3.6. *Let E be a finite directed graph which is strongly connected and choose a weight function $y : E^1 \to (0, \infty)$. For $\theta \in [0, \infty)$ let $B(y, \theta)$ be the matrix from Definition 3.1. Then the function $\theta \mapsto \rho(B(y, \theta))$ is strictly decreasing and there exists a unique $\beta_c \in [0, \infty)$ such that $\rho(B(y, \beta_c)) = 1$.*

Proof. First we show that $\theta \mapsto \rho(B(y, \theta))$ is strictly decreasing. Fix $\theta, \theta_0 \in [0, \theta']$ such that $\theta > \theta_0$ and $v, w \in E^0$. If $vE^1w = \emptyset$ then $B(y, \theta)_{v,w} = B(y, \theta_0)_{v,w} = 0$. If $vE^1w \neq \emptyset$, since y takes positive values,

$$B(y, \theta)_{v,w} = \sum_{e \in vE^1w} e^{-\theta y(e)} < \sum_{e \in vE^1w} e^{-\theta_0 y(e)} = B(y, \theta_0)_{v,w}.$$

Take $\xi > 0$ to be the Perron-Frobenius eigenvector for $B(y, \theta_0)$, that is, the vector such that $B(y, \theta_0)\xi = \rho(B(y, \theta_0))\xi$. Since E is strongly connected every row has $w \in E^0$ such that $B(y, \theta)_{v,w} < B(y, \theta_0)_{v,w}$. In addition $\xi_w > 0$, so we have $B(y, \theta)\xi < B(y, \theta_0)\xi = \rho(B(y, \theta_0))\xi$. Thus Proposition A.5(d) implies that $\rho(B(y, \theta)) < \rho(B(y, \theta_0))$, that is, $\theta \mapsto \rho(B(y, \theta))$ is strictly decreasing.

Next we prove the existence of a non-negative solution to $\rho(B(y, \theta)) = 1$ by applying the intermediate value theorem.

We first claim the function $\theta \mapsto \rho(B(y, \theta))$ is continuous. Fix $\theta_0 \geq 0$. Lemma 3.5 tells us that $B(y, \theta_0)$ is irreducible, so Proposition A.7 tells there exists a vector $\xi > 0$ such that $B(y, \theta_0)\xi = \rho(B(y, \theta_0))\xi$. Define numbers $a := \min \xi_v$ and $b := \max \xi_v$; then a and b are both greater than zero because $\xi > 0$. Let $n = |E^0|$ and fix $\epsilon > 0$. The entries $B(y, \theta)_{v,w}$ of $B(y, \theta)$ are continuous functions of θ, so there exists $\delta > 0$ such that if $|\theta - \theta_0| < \delta$, then

$$|B(y, \theta)_{v,w} - B(y, \theta_0)_{v,w}| < \frac{a\epsilon}{nb} \tag{3.8}$$

for all $v, w \in E^0$. For $|\theta - \theta_0| < \delta$ and $v \in E^0$ we have

$$\begin{aligned}(B(y, \theta)\xi)_v &= \sum_{w \in E^0} B(y, \theta)_{v,w}\xi_w \\ &= \sum_{w \in E^0} B(y, \theta_0)_{v,w}\xi_w + \sum_{w \in E^0} (B(y, \theta)_{v,w} - B(y, \theta_0)_{v,w})\xi_w.\end{aligned}$$

Now by Equation (3.8),

$$\begin{aligned}(B(y, \theta)\xi)_v &< \sum_{w \in E^0} B(y, \theta_0)_{v,w}\xi_w + \sum_{w \in E^0} \frac{a\epsilon}{nb}\xi_w \\ &< \rho(B(y, \theta_0))\xi_v + n\frac{a\epsilon}{nb}b \\ &\leq (\rho(B(y, \theta_0)) + \epsilon)\xi_v \\ &= \Big((\rho(B(y, \theta_0)) + \epsilon)\xi\Big)_v.\end{aligned}$$

This implies that $B(y,\theta)\xi < (\rho(B(y,\theta_0)) + \epsilon)\xi$. Thus Proposition A.5(d) implies that $\rho(B(y,\theta)) < \rho(B(y,\theta_0)) + \epsilon$. We can do a similar calculation to show that $\rho(B(y,\theta)) > \rho(B(y,\theta_0)) - \epsilon$. For $|\theta - \theta_0| < \delta$ we have

$$\begin{aligned}(B(y,\theta)\xi)_v &= \sum_{w\in E^0} B(y,\theta)_{v,w}\xi_w \\ &= \sum_{w\in E^0} B(y,\theta_0)_{v,w}\xi_w - \sum_{w\in E^0} (B(y,\theta_0)_{v,w} - B(y,\theta)_{v,w})\xi_w.\end{aligned}$$

Now by Equation (3.8),

$$\begin{aligned}(B(y,\theta)\xi)_v &> \sum_{w\in E^0} B(y,\theta_0)_{v,w}\xi_w - \sum_{w\in E^0} \frac{a\epsilon}{nb}\xi_w \\ &> \rho(B(y,\theta_0))\xi_v - n\frac{a\epsilon}{nb}b \\ &\geq (\rho(B(y,\theta_0)) - \epsilon)\xi_v \\ &= \Big((\rho(B(y,\theta_0)) - \epsilon)\xi\Big)_v.\end{aligned}$$

This implies that $B(y,\theta)\xi > (\rho(B(y,\theta_0)) - \epsilon)\xi$. Thus Proposition A.5(c) implies that $\rho(B(y,\theta)) > \rho(B(y,\theta_0)) - \epsilon$. Therefore

$$|\theta - \theta_0| < \delta \Longrightarrow |\rho(B(y,\theta_0)) - \rho(B(y,\theta))| < \epsilon.$$

Thus $\theta \mapsto \rho(B(y,\theta))$ is continuous, as claimed.

Now that we know $\theta \mapsto \rho(B(y,\theta))$ is continuous we can apply the intermediate value theorem. First we need to find two endpoints. Take $\theta = 0$, then $B(y,0)$ is the vertex matrix A of E. Since the graph is strongly connected, each row of $B(y,0)$ has at least one entry greater than or equal to 1. In other words, for all $v \in E^0$ there exists $w \in E^0$ such that $B(y,0)_{v,w} \geq 1$. Let 1^n be the vector with all entries 1. Then $\sum_{w\in E^0} B(y,0)_{v,w} \geq 1$ for all $v \in E^0$, so $B(y,0)1^n \geq 1^n$. Thus Proposition A.5(a) implies that $\rho(B(y,0)) \geq 1$. Since θ is non-negative and y takes positive values, $B(y,\theta)_{v,w} \to 0$ as $\theta \to \infty$. Therefore, for large enough θ we have $B(y,\theta)_{v,w} \leq \frac{1}{2}$ for all $v, w \in E^0$. Take θ' to be such an θ. Then $B(y,\theta')1^n \leq (\frac{1}{2})1^n$. Then Proposition A.5(b) implies that $\rho(B(y,\theta')) \leq \frac{1}{2}$. Now, by applying the intermediate value theorem on the interval $[0,\theta']$, there is a solution $\beta_c \in (0,\theta')$ to the equation $\rho(B(y,\beta_c)) = 1$.

Finally, since the function $\theta \mapsto \rho(B(y,\theta))$ is strictly decreasing, β_c is unique. □

Remark 3.7. If we take y to be the function such that $y(e) = 1$ for all $e \in E^1$, then we get $B(y,\theta) = e^{-\theta}A$. Then $\rho(B(y,\theta)) = \rho(e^{-\theta}A) = e^{-\theta}\rho(A)$, so the uniqueness in Proposition 3.6 tells us that $\beta_c = \ln \rho(A)$. This is the critical inverse temperature of the KMS states for $\mathcal{T}C^*(E)$ and $C^*(E)$ with the gauge action, as studied in [17].

3.4 KMS states of $(\mathcal{T}C^*(E), \alpha^y)$ at the critical inverse temperature

In this section we show the existence and uniqueness of a KMS_{β_c} state for $(\mathcal{T}C^*(E), \alpha^y)$ when E is strongly connected. We call β_c the critical inverse temperature and also show that there are no KMS_β states of $(\mathcal{T}C^*(E), \alpha^y)$ when $\beta < \beta_c$.

Choose β_c to be the unique β_c such that $\rho(B(y, \beta_c)) = 1$, obtained by applying Proposition 3.6. We now show the existence of a KMS_{β_c} state, following the methods of [17, Proposition 4.1] and [17, Corollary 4.2].

Proposition 3.8. *Suppose that E is a finite directed graph which is strongly connected. Choose a weight function $y : E^1 \to (0, \infty)$. For $\theta \in [0, \infty)$ let $B(y, \theta)$ be the matrix from Definition 3.1. Let β_c be the unique β such that $\rho(B(y, \beta_c)) = 1$. Let $\alpha^y : \mathbb{R} \to \operatorname{Aut} \mathcal{T}C^*(E)$ be the action from Proposition 2.8. Then $(\mathcal{T}C^*(E), \alpha^y)$ has a KMS_{β_c} state.*

Proof. We first want to find m such that $\|m\|_1 = 1$ and $B(y, \beta_c)m \leq m$. Choose a decreasing sequence $\{\beta_n\} \subset (\beta_c, \infty)$ such that $\beta_n \to \beta_c$. Fix $n \in \mathbb{N}$. Take x as defined in Theorem 3.4(a). Choose $\epsilon_n \in [0, \infty)^{E^0}$ such that $\epsilon_n \cdot x = 1$. Then Theorem 3.4(a) implies that $m_n = (I - B(y, \beta_n))^{-1}\epsilon_n$ has $\|m_n\|_1 = 1$. Then

$$0 \leq \epsilon_n = (I - B(y, \beta_n))m_n = m_n - B(y, \beta_n)m_n,$$

so $B(y, \beta_n)m_n \leq m_n$. By passing to a subsequence, we may assume that $\{m_n\}$ converges pointwise to m, say. Then taking $n \to \infty$ tells us that $B(y, \beta_c)m \leq m$.

We now want to apply Theorem 3.4(b) to get a sequence of KMS_{β_n} states. Define $\epsilon'_n := (I - B(y, \beta_n))m$. We claim $\epsilon'_n \in [0, \infty)^{E^0}$. Since $\beta_n > \beta_c$ we have $0 \leq B(y, \beta_n)_{v,w} \leq B(y, \beta_c)_{v,w}$ for all $v, w \in E^0$ and therefore m satisfies $B(y, \beta_n)m \leq B(y, \beta_c)m \leq m$. Then

$$\epsilon'_n := (I - B(y, \beta_n))m = m - B(y, \beta_n)m \geq 0.$$

So $\epsilon'_n \in [0, \infty)^{E^0}$, as claimed. Thus we can apply Theorem 3.4(a) with $\beta = \beta_n$, which tells us that $\epsilon'_n \cdot x = 1$. We can then apply Theorem 3.4(b), which gives a KMS_{β_n} state ϕ_n satisfying

$$\phi_n(s_\mu s_\nu^*) = \delta_{\mu,\nu} e^{-\beta_n y(\mu)} m_{s(\mu)}. \tag{3.9}$$

Since the state space of $\mathcal{T}C^*(E)$ is weak* compact we may assume that by passing to a subsequence that the sequence $\{\phi_n\}$ converges to a state ϕ. Taking $n \to \infty$ in Equation (3.9) tells us that ϕ satisfies

$$\phi(s_\mu s_\nu^*) = \delta_{\mu,\nu} e^{-\beta_c y(\mu)} m_{s(\mu)}. \tag{3.10}$$

Proposition 2.11(b) then implies that ϕ is a KMS$_{\beta_c}$ state. □

We now follow the method of [17, Theorem 4.3] to show uniqueness of this KMS$_{\beta_c}$ state.

Theorem 3.9. *Suppose that E is a finite directed graph which is strongly connected. Choose a weight function $y : E^1 \to (0, \infty)$. For $\theta \in [0, \infty)$ let $B(y, \theta)$ be the matrix from Definition 3.1. Let $\alpha^y : \mathbb{R} \to \operatorname{Aut} \mathcal{T}C^*(E)$ be the action from Proposition 2.8. Let β_c be the unique number such that $\rho(B(y, \beta_c)) = 1$, which we know exists by Proposition 3.6. Let $\xi = (\xi_v)$ be the unimodular Perron-Frobenius eigenvector of the matrix $B(y, \beta_c)$.*

(a) The system $(\mathcal{T}C^*(E), \alpha^y)$ has a unique KMS$_{\beta_c}$ state ϕ. This state satisfies

$$\phi(s_\mu s_\nu^*) = \delta_{\mu,\nu} e^{-\beta_c y(\mu)} \xi_{s(\mu)}. \tag{3.11}$$

(b) If $\beta < \beta_c$, then $(\mathcal{T}C^*(E), \alpha^y)$ has no KMS$_\beta$ states.

Proof. (a) We proved existence of ϕ in Proposition 3.8. Proposition A.7 tells us that ξ is the unique vector such that $B(y, \beta_c)\xi = \rho(B(y, \beta_c))\xi$ and $\|\xi\|_1 = 1$. Then we can take ξ as m in Proposition 3.8, and Equation (3.10) tells us that ϕ satisfies Equation (3.11). To establish uniqueness, suppose that ψ is a KMS$_{\beta_c}$ state. Then Proposition 3.2 says that $m^\psi = (\psi(p_v))$ satisfies $B(y, \beta_c)m^\psi \le m^\psi$ and $\|m^\psi\|_1$. Since $\rho(B(y, \beta_c)) = 1$, applying Proposition A.8 gives $B(y, \beta_c)m^\psi = m^\psi$. Thus $m^\psi = \xi$. Finally, fix $\mu, \nu \in E^*$. Then Equation (3.11) gives

$$\phi(s_\mu s_\nu^*) = \delta_{\mu,\nu} e^{-\beta y(\mu)} \xi_{s(\mu)} = \delta_{\mu,\nu} e^{-\beta y(\mu)} m^\psi_{s(\mu)} = \delta_{\mu,\nu} e^{-\beta y(\mu)} \psi(p_{s(\mu)}).$$

Therefore Equation (2.6) tells us that

$$\phi(s_\mu s_\nu^*) = \psi(s_\mu s_\nu^*),$$

so $\phi = \psi$.

(b) Suppose that ϕ is a KMS$_\beta$ state of $(\mathcal{T}C^*(E), \alpha^y)$. Then Proposition 3.2 implies that $m^\phi := (\phi(p_v))$ satisfies $B(y, \beta)m^\phi \le m^\phi$. In other words, m^ϕ is subinvariant. Since $m^\phi \ge 0$ pointwise, Proposition A.9 implies that $1 \ge \rho(B(y, \beta))$. Therefore $\rho(B(y, \beta_c)) \ge \rho(B(y, \beta))$, so by Proposition 3.6, $\beta \ge \beta_c$. □

3.5 KMS states of $(C^*(E), \overline{\alpha}^y)$

In this section we define an action $\overline{\alpha}^y : \mathbb{R} \to \operatorname{Aut} C^*(E)$, and show when a KMS states of $(\mathcal{T}C^*(E), \alpha^y)$ factors through a KMS state of $(C^*(E), \overline{\alpha}^y)$. We then deduce uniqueness of the KMS state of $(C^*(E), \overline{\alpha}^y)$ from uniqueness of the KMS$_{\beta_c}$ state of $(\mathcal{T}C^*(E), \alpha^y)$.

Lemma 3.10. *The set*

$$P := \Big\{ p_v - \sum_{e \in vE^1} s_e s_e^* : v \in E^0,\ v \text{ is not a source} \Big\}$$

consists of elements which are fixed by α^y.

Proof. Fix $t \in \mathbb{R}$ and $v \in E^0$ such that v is not a source. Then

$$\begin{aligned} \alpha_t^y\Big(p_v - \sum_{e \in vE^1} s_e s_e^*\Big) &= \alpha_t^y(p_v) - \alpha_t^y\Big(\sum_{e \in vE^1} s_e s_e^*\Big) \\ &= p_v - \sum_{e \in vE^1} \alpha_t^y(s_e s_e^*) \\ &= p_v - \sum_{e \in vE^1} e^{it(y(e)-y(e))} s_e s_e^* \\ &= p_v - \sum_{e \in vE^1} e^0 s_e s_e^* \\ &= p_v - \sum_{e \in vE^1} s_e s_e^*. \end{aligned}$$

□

Remark 3.11. Recall that we are viewing $C^*(E)$ as the image of the quotient map $q : \mathcal{T}C^*(E) \to \mathcal{T}C^*(E)/J$ for the ideal J generated by P, and that the kernel of q is J (Remark 2.7). Lemma 3.10 tells us the set P consists of elements fixed by $\alpha_t^y : \ker q \to \ker q$, so it induces an automorphism $\overline{\alpha}_t^y$ of $C^*(E)$ such that

$$\overline{\alpha}_t^y(q(a)) = q(\overline{\alpha}_t^y(a))$$

for all $a \in C^*(E)$. We therefore have an action $\overline{\alpha}^y : \mathbb{R} \to \operatorname{Aut} C^*(E)$.

We now use the method of [17, Proposition 2.1(d)] to show when a KMS state of $(\mathcal{T}C^*(E), \alpha^y)$ factors through $C^*(E)$.

Proposition 3.12. *Let E be a finite directed graph. Let $\beta \in (0, \infty)$. Choose a weight function $y : E^1 \to (0, \infty)$. For $\theta \in [0, \infty)$ let $B(y, \theta)$ be the matrix from Definition 3.1. Let $\alpha^y : \mathbb{R} \to \operatorname{Aut} \mathcal{T}C^*(E)$ be the action from Proposition 2.8. Let ϕ be a KMS$_\beta$ state* *of* $(\mathcal{T}C^*(E), \alpha^y)$ *and* $m^\phi = (m^\phi_v)$ *be the vector with entries* $m^\phi_v = \phi(p_v)$*. Then* ϕ *factors through* $C^*(E)$ *if and only if*

$$(B(y,\beta)m^\phi)_v = m^\phi_v \qquad \textit{whenever } v \in E^0 \textit{ is not a source.} \tag{3.12}$$

Proof. In the forward direction assume that ϕ is a KMS_β state of $(\mathcal{T}C^*(E), \alpha^y)$ and that ϕ factors through $C^*(E)$; that is, there is a state ψ of $(C^*(E), \overline{\alpha}^y)$ such that ϕ is the composition $\psi \circ q$ with the quotient map q. Choose $v \in E^0$ and suppose that v is not a source. We want to show that $m^\phi_v = (B(y,\beta)m^\phi)_v$. Define $\overline{s}_e := q(s_e)$ for all $e \in E^1$ and $\overline{p}_v := q(p_v)$ for all $v \in E^0$. Lemma 2.6 then tells us that $\{\overline{p}_v, \overline{s}_e\}$ is a Cuntz-Krieger E-family which generates $C^*(E)$. Because $\overline{p}_v = \sum_{e\in vE^1} \overline{s}_e\overline{s}_e^*$,

$$\begin{aligned} m^\phi_v &= \phi(p_v) \\ &= \psi \circ q(p_v) \\ &= \psi(\overline{p}_v) \\ &= \psi\Big(\sum_{e\in vE^1} \overline{s}_e\overline{s}_e^*\Big) \\ &= \psi\Big(\sum_{e\in vE^1} q(s_e s_e^*)\Big) \\ &= \sum_{e\in vE^1} \psi \circ q(s_e s_e^*) \\ &= \sum_{e\in vE^1} \phi(s_e s_e^*). \end{aligned}$$

Thus Equation (3.1) implies that $m^\phi_v = (B(y,\beta)m^\phi)_v$.

For the reverse direction, we suppose that ϕ is a KMS_β state of $(\mathcal{T}C^*(E), \alpha^y)$ satisfying Equation (3.12). We want to apply [17, Lemma 2.2] to the set P defined in Lemma 3.10, noting $\ker q$ is the ideal generated by P (Remark 2.7).

First we need to show that we have met the conditions of [17, Lemma 2.2]. Choose $v \in E^0$ and suppose that v is not a source. Then $p_v \geq \sum_{e\in vE^1} s_e s_e^*$, so [32, Proposition A.1] implies that $p_v - \sum_{e\in vE^1} s_e s_e^*$ is a projection. So P is a set of projections. Lemma 3.10 tells us that P consists of elements fixed by α^y. Now we need a set $\mathcal{F}$ of analytic elements such that for all $a \in \mathcal{F}$ there exists an entire function f_a such that $\alpha^y_z(a) = f_a(z)a$. Here $\mathcal{F} = \{s_\mu s_\nu^* : \mu, \nu \in E^*, s(\mu) = s(\nu)\}$ and $\alpha^y_t(s_\mu s_\nu^*) = e^{it(y(\mu)-y(\nu))} s_\mu s_\nu^*$. So for $a = s_\mu s_\nu^* \in \mathcal{F}$, $f_a(z) = e^{iz(y(\mu)-y(\nu))}$ satisfies $\alpha^y_z(a) = f_a(z)a$.

Therefore $\mathcal{F}$ satisfies the conditions of [17, Lemma 2.2], and we can apply it. Fix a KMS state ϕ and projection $p \in P$. We want to show that $\phi(p) = 0$. Equation (3.1) implies that when $v \in E^0$ is not a source

$$\phi\Big(p_v - \sum_{e \in vE^1} s_e s_e^*\Big) = \phi(p_v) - \sum_{e \in vE^1} \phi(s_e s_e^*) = m_v^\phi - (B(y,\beta)m^\phi)_v = 0.$$

Thus [17, Lemma 2.2] tells us that ϕ factors through a state of $C^*(E)$. $\square$

We now apply Proposition 3.6 to show that there is a unique β for which a KMS$_\beta$ state of $(C^*(E), \overline{\alpha}^y)$ exists.

Proposition 3.13. *Let E be a finite directed graph which is strongly connected. For $\theta \in [0,\infty)$ let $B(y,\theta)$ be the matrix from Definition 3.1. Suppose that ϕ is a KMS$_\beta$ state of $(C^*(E), \overline{\alpha}^y)$. Then β satisfies*

$$\phi(p_v) = \sum_{w \in E^0} \sum_{e \in vE^1 w} e^{-\beta y(e)} \phi(p_w). \tag{3.13}$$

There is exactly one β which satisfies Equation (3.13)*, and it is the unique β such that*

$$\rho(B(y,\beta)) = 1.$$

Proof. Let $\beta \in (0,\infty)$. Suppose that ϕ is a KMS$_\beta$ state of $(C^*(E), \overline{\alpha}^y)$. Then, for $e \in E^1$, applying the KMS condition gives

$$\phi(s_e s_e^*) = \phi(s_e^* \alpha^y_{i\beta}(s_e)) = \phi(s_e^* e^{-\beta y(e)} s_e) = e^{-\beta y(e)} \phi(s_e^* s_e) = e^{-\beta y(e)} \phi(p_{s(e)}).$$

Fix $v \in E^0$. Then, by (CK2),

$$\phi(p_v) = \sum_{e \in vE^1} \phi(s_e s_e^*) = \sum_{e \in vE^1} e^{-\beta y(e)} \phi(p_{s(e)}) = \sum_{w \in E^0} \sum_{e \in vE^1 w} e^{-\beta y(e)} \phi(p_w).$$

Introduce the vector $m := (m_v) = (\phi(p_v))$. Then

$$m_v = \sum_{w \in E^0} B(y,\beta)_{v,w} m_w,$$

or alternatively $m = B(y,\beta)m$. Then Proposition A.9 implies $\rho(B(y,\beta)) = 1$. Now Proposition 3.6 tells us that β is unique. $\square$

Finally, we show uniqueness of the KMS state of $(C^*(E), \overline{\alpha}^y)$.

Theorem 3.14. *Suppose that E is a finite directed graph which is strongly connected. Choose a weight function $y : E^1 \to (0,\infty)$. For $\theta \in [0,\infty)$ let $B(y,\theta)$ be the matrix from Definition 3.1. Let $\alpha^y : \mathbb{R} \to \operatorname{Aut} \mathcal{T}C^*(E)$ be the action from Proposition 2.8. Let β_c be the real number such that $\rho(B(y,\beta_c)) = 1$. The state ϕ from Theorem 3.9 factors through a KMS$_{\beta_c}$ state $\overline{\phi}$ of $(C^*(E), \overline{\alpha}^y)$, and this is the only KMS state of $(C^*(E), \overline{\alpha}^y)$.*

Proof. Since ϕ is a KMS state, Proposition 3.2 tells us $m^\phi = (\phi(p_v))$ satisfies

$$B(y, \beta_c)m^\phi \leq m^\phi.$$

Then applying Proposition A.8 gives $B(y, \beta_c)m^\phi = m^\phi$. Since E is strongly connected it has no sources, so we can apply Proposition 3.12, which tells us that ϕ factors through a KMS_{β_c} state $\overline{\phi}$ of $(C^*(E), \overline{\alpha}^y)$.

To show uniqueness, suppose that ψ is a KMS_β state of $(C^*(E), \overline{\alpha}^y)$. Proposition 3.13 tells us that β satisfies $\rho(B(y, \beta)) = 1$. Proposition 3.12 implies that $B(y, \beta)m^{\psi\circ q} = m^{\psi\circ q}$. Now Theorem 3.9(a) implies that $\psi \circ q = \phi = \overline{\phi} \circ q$, so $\psi = \overline{\phi}$. $\square$

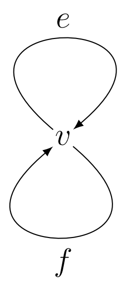
Remark 3.15. If we define y by $y(e) = 1$ for all $e \in E^1$, then $\beta_c = \ln \rho(A)$ (Remark 3.7). We therefore recover [17, Theorem 4.3] (which studies what happens at the critical inverse temperature for the gauge action) from Theorem 3.9 and Theorem 3.14.

Remark 3.16. Theorem 3.14 implies the main result of [20, Section 4.3]; that there exists a KMS_β state of $(C^*(E), \overline{\alpha}^y)$ if and only if β is the unique positive number for which Perron numbers exist.

3.6 Examples

We finish the chapter with two examples which show that interesting examples exist.

Example 3.17. Let E be a directed graph with one vertex v and two edges e and f, as follows



We want to find a nontrivial function y such that $(\mathcal{T}C^*(E), \alpha^y)$ has a unique KMS_{β_c} state. To do this we want to find y and β_c such that $\rho(B(y, \beta_c)) = 1$. By the definition of the spectral radius we want $e^{-\beta_c y(e)} + e^{-\beta_c y(f)} = 1$. Then

$$\frac{1}{e^{\beta_c y(e)}} + \frac{1}{e^{\beta_c y(f)}} = 1.$$

We can take $p, q \in (1, \infty)$ which are *conjugate indices*, that is

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then taking $y(e) = \frac{\ln p}{\beta_c}$ and $y(f) = \frac{\ln q}{\beta_c}$ satisfies our requirements. For example, we can take $p = 3$, $q = \frac{3}{2}$ and $\beta_c = 2$. Then Theorem 3.9 tells us there is a unique KMS_2 state for $(\mathcal{T}C^*(E), \alpha^y)$, and Theorem 3.14 then tells us this state factors through a KMS_2 state of $(C^*(E), \overline{\alpha}^y)$.

Remark 3.18. As we have remarked throughout the chapter (Remark 3.7 and Remark 3.15), by taking $y(e) = 1$ for all $e \in E^1$ we can use our results to study KMS states for the gauge action, as studied in [17]. The reason our results look a little different to [17] is that $B(1, \beta) = e^{-\beta}A$, so the $e^{-\beta}$ scaling term is contained within the matrix $B(y, \beta)$ rather than explicitly in the formulas.

Chapter 4

KMS states of C^*-algebras for infinite graphs with the gauge action

The goal of this chapter is to generalise [17] to study the KMS states of $(\mathcal{T}C^*(E), \alpha)$ and $(C^*(E), \overline{\alpha})$ for row-finite infinite graphs with no sources. By infinite we mean graphs $E = (E^0, E^1, r, s)$ for which E^0 and E^1 are potentially infinite sets, and recall that no sources means that $vE^1 \neq \emptyset$ for all $v \in E^0$.

Let α be the dynamics associated to the gauge action, as in Section 2.3. We start the chapter by characterising the KMS states of $(\mathcal{T}C^*(E), \alpha)$. To study the KMS states of $(\mathcal{T}C^*(E), \alpha)$ at large inverse temperatures we present some background from Banach spaces and then find an isomorphism between KMS functionals and a subset of $\ell^1(E^0)$. We then show when the KMS states of $(\mathcal{T}C^*(E), \alpha)$ factor through $C^*(E)$. Finally we give an example from [3] to show that we cannot guarantee existence of the KMS states of $(C^*(E), \overline{\alpha})$ for infinite graphs.

Remark 4.1. We do not attempt to extend the results of Chapter 3 to study the KMS states of $(\mathcal{T}C^*(E), \alpha^y)$ and $(C^*(E), \overline{\alpha}^y)$ row-finite infinite graphs with no sources. The reason for this is generalisations of the Perron-Frobenius theorem typically require a compactness hypothesis on the operator, and it would be difficult to show that the operator $\theta \mapsto B(y, \theta)$ from Chapter 3 is compact.

4.1 Characterising KMS states of $(\mathcal{T}C^*(E), \alpha)$

In this section we characterise the KMS states of $(\mathcal{T}C^*(E), \alpha)$.

When E has infinitely many vertices, $\mathcal{T}C^*(E)$ does not have an identity. Instead we use the following approximate identity.

Lemma 4.2. *Choose a listing $\{v_1, v_2, \dots\}$ of E^0. The sequence*

$$\{\sum_{n=1}^{N} p_{v_n}\}_N$$

is an approximate identity of $\mathcal{T}C^*(E)$.

Proof. This follows from [32, Lemma 2.10] (the argument in [32, Lemma 2.10] is for $C^*(E)$ but carries over for $\mathcal{T}C^*(E)$). □

Lemma 4.3. *Let ϕ be a state of $\mathcal{T}C^*(E)$ and for $v \in E^0$ define $m_v := \phi(p_v)$. Then $\sum_{v\in E^0} m_v = 1$. In particular, $m = (m_v) \in \ell^1(E^0)$.*

Proof. Choose a listing $\{v_1, v_2, \dots\}$ of E^0. Lemma 4.2 tells us that the sequence $\{\sum_{n=1}^{N} p_{v_n}\}_N$ is an approximate identity of $\mathcal{T}C^*(E)$. Then [35, Lemma A7(a)] implies that

$$\sum_{v\in E^0} m_v = \lim_{N\to\infty} \sum_{n=1}^{N} m_{v_n} = \lim_{N\to\infty} \sum_{n=1}^{N} \phi(p_{v_n}) = \lim_{N\to\infty} \phi\Big(\sum_{n=1}^{N} p_{v_n}\Big) = 1.$$

Thus $\|m\|_1 = 1$, so $m \in \ell^1(E^0)$. □

We can now characterise the KMS states of $(\mathcal{T}C^*(E), \alpha)$, using the method of [17, Proposition 2.1].

Proposition 4.4. *Let E be a row-finite directed graph with no sources. Let $\beta \in (0, \infty)$. Let γ be the gauge action and define $\alpha_t := \gamma_{e^{it}}$.*

(a) A linear functional ϕ satisfies the KMS condition if and only if

$$\phi(s_\mu s_\nu^*) = \delta_{\mu,\nu} e^{-\beta|\mu|} \phi(p_{s(\mu)}) \qquad \textit{for all } \mu, \nu \in E^*. \tag{4.1}$$

(b) A state ϕ of $\mathcal{T}C^*(E)$ is a KMS$_\beta$ state of $(\mathcal{T}C^*(E), \alpha)$ if and only if ϕ satisfies Equation (4.1).

(c) Suppose that ϕ is a KMS$_\beta$ state of $(\mathcal{T}C^*(E), \alpha)$, and define $m^\phi = (m_v^\phi)$ by $m_v^\phi = \phi(p_v)$. Then m^ϕ satisfies the subinvariance relation $Am^\phi \le e^\beta m^\phi$. In addition, $\|m^\phi\|_1 = 1$ and

$$m^\phi \in \ell^1(E^0)^+ := \{(m_v) \in \ell^1(E^0) : m_v \ge 0\}.$$

Proof. (a) In the forward direction let ϕ be a linear functional satisfying the KMS condition (Equation (2.5)). We want to show that ϕ satisfies Equation (4.1). Fix $\mu, \nu \in E^*$. The KMS condition gives

$$\phi(s_\mu s_\nu^*) = \phi(s_\nu^* \alpha_{i\beta}(s_\mu)) = \phi(s_\nu^* e^{-\beta|\mu|} s_\mu) = e^{-\beta|\mu|}\phi(s_\nu^* s_\mu). \tag{4.2}$$

We consider the cases $|\mu| = |\nu|$ and $|\mu| \neq |\nu|$ separately. For $|\mu| = |\nu|$, the product formula (Equation (2.2)) implies that $s_\nu^* s_\mu = \delta_{\nu,\mu} p_{s(\mu)}$, and Equation (4.2) gives

$$\phi(s_\mu s_\nu^*) = e^{-\beta|\mu|}\phi(s_\nu^* s_\mu) = \delta_{\mu,\nu} e^{-\beta|\mu|}\phi(p_{s(\mu)}).$$

Next, suppose that $|\mu| \neq |\nu|$. If μ doesn't extend ν and ν doesn't extend μ then the product formula tells us that $s_\nu^* s_\mu = 0$, so Equation (4.2) gives

$$\phi(s_\mu s_\nu^*) = e^{-\beta|\mu|}\phi(s_\nu^* s_\mu) = 0.$$

Otherwise one of μ, ν extends the other, and $|\mu| \neq |\nu|$. Applying the KMS condition again to Equation (4.2) gives

$$\phi(s_\mu s_\nu^*) = e^{-\beta|\mu|}\phi(s_\nu^* s_\mu) = e^{-\beta|\mu|}\phi(s_\mu \alpha_{i\beta}(s_\nu^*)) = e^{-\beta|\mu|}\phi(s_\mu e^{\beta|\nu|} s_\nu^*) = e^{-\beta(|\mu|-|\nu|)}\phi(s_\mu s_\nu^*).$$

Since $e^{-\beta(|\mu|-|\nu|)} \neq 1$, it must be that $\phi(s_\mu s_\nu^*) = 0$, thus ϕ satisfies Equation (4.1).

Conversely, suppose that ϕ satisfies Equation (4.1). Applying Proposition 2.11(a) with $y(e) = 1$ for all $e \in E^0$ tells us that ϕ satisfies the KMS condition.

(b) In the forward direction let ϕ be a KMS state. Then ϕ is a linear functional satisfying the KMS condition, so part (a) tells us it satisfies Equation (4.1).

Conversely, suppose that ϕ is a state satisfying Equation (4.1). Then part (a) tells us ϕ satisfies the KMS condition, so it is a KMS state.

(c) Fix $v \in E^0$. Then, because the following sums are finite, applying Equation (4.1) gives

$$\begin{aligned}
\sum_{e \in vE^1} \phi(s_e s_e^*) &= \sum_{e \in vE^1} e^{-\beta}\phi(p_{s(e)}) \\
&= \sum_{e \in vE^1} e^{-\beta} m^\phi_{s(e)} \\
&= \sum_{w \in E^0} \sum_{e \in vE^1 w} e^{-\beta} m^\phi_w \\
&= e^{-\beta} \sum_{w \in E^0} |vE^1 w| m^\phi_w \\
&= e^{-\beta}(Am^\phi)_v \qquad (4.3)
\end{aligned}$$

Since E is row-finite and has no sources, vE^1 is finite and v is not a source, so we can use (TCK2) to get

$$(Am^\phi)_v = e^\beta \sum_{e \in vE^1} \phi(s_e s_e^*) \leq e^\beta \phi(p_v) = e^\beta m_v^\phi.$$

Next, Lemma 4.3 tells us that $\|m^\phi\|_1 = 1$ and $m^\phi \in \ell^1(E^0)$. Because ϕ is a positive linear functional, each $m_v^\phi \geq 0$, so $m^\phi \in \ell^1(E^0)^+$. □

4.2 An isomorphism between $\ell^1(E^0)$ and $c_0(E^0)^*$

We first give an explicit isomorphism between $\ell^1(E^0)$ and $c_0(E^0)^*$. This is well-known, but we include the proof as it is difficult to find a direct proof in the literature.

Theorem 4.5. *Suppose that E is a row-finite graph, and $m \in \ell^1(E^0)$. Then for each $f \in c_0(E^0)$ the series $\sum_{v \in E^0} m(v)f(v)$ converges absolutely, and $\Phi(m) : f \mapsto \sum_{v \in E^0} m(v)f(v)$ is a bounded linear functional with $\|\Phi(m)\| = \|m\|_1$. Then $\Phi : \ell^1(E^0) \to c_0(E^0)^*$ is an isometric isomorphism.*

Proof. Fix $f \in c_0(E^0)$. We first want to see that $\sum_{v \in E^0} |m(v)f(v)|$ converges. Since $|f(v)| \leq \|f\|_\infty$,

$$\sum_{v \in E^0} |m(v)f(v)| \leq \sum_{v \in E^0} |m(v)||f(v)| \leq \sum_{v \in E^0} |m(v)|\|f\|_\infty \leq \|m\|_1\|f\|_\infty.$$

Thus $\sum_{v \in E^0} |m(v)f(v)|$ converges with sum less than or equal to $\|m\|_1\|f\|_\infty$. Now since $\big|\Phi(m)(f)\big| \leq \|m\|_1\|f\|_\infty$, and

$$\|\Phi(m)\|_{c_0(E^0)^*} := \sup_{f \in c_0(E^0)} \frac{\big|\Phi(m)(f)\big|}{\|f\|_\infty},$$

we have that $\Phi(m)$ is a bounded linear functional with $\|\Phi(m)\|_{c_0(E^0)^*} \leq \|m\|_1$. To see that

$$\|\Phi(m)\|_{c_0(E^0)^*} \geq \|m\|_1,$$

fix $\epsilon > 0$. Choose a finite subset F of E^0 such that $\sum_{v \in F} |m(v)| > \|m\|_1 - \epsilon$. Define f_F by

$$f_F(v) := m(v)^{-1}|m(v)|$$

for $v \in \{v \in F : m(v) \neq 0\}$. Then $f_F \in c_c(E^0) \subseteq c_0(E^0)$. Now

$$\begin{aligned}
\|\Phi(m)\| &\geq |\Phi(m)(f_F)| \\
&\geq \sum_{v \in E^0} m(v) f_F(v) \\
&\geq \sum_{v \in F} m(v) m(v)^{-1} |m(v)| \\
&= \sum_{v \in F} |m(v)| \\
&> \|m\|_1 - \epsilon,
\end{aligned}$$

thus $\|\Phi(m)\|_{c_0(E^0)^*} \geq \|m\|_1$. Therefore $\|\Phi(m)\|_{c_0(E^0)^*} = \|m\|_1$.

To see that Φ is surjective let $y \in c_0(E^0)^*$. We want to show there exists $m \in \ell^1(E^0)$ such that $\Phi(m) = y$. Take, as a candidate, m defined by $m(v) = y(\chi_{\{v\}})$. We first want to see that $m \in \ell^1(E^0)$, which we do by showing $\|m\|_1 \leq \|y\|$. Let F_k be finite sets such that $F_k \subseteq F_{k+1}$ and $\bigcup_{k=1}^{\infty} F_k = E^0$ (these exist because E^0 is countable). For each $v \in E^0$ choose $\theta(v)$ such that $m(v) = |m(v)| e^{i\theta(v)}$. Define $g_k := \sum_{v \in F_k} e^{-i\theta(v)} \chi_v$. Then for all k, we have $\|g_k\|_\infty = 1$, and

$$\begin{aligned}
\|y\| &\geq |y(g_k)| \\
&\geq y\Big(\sum_{v \in F_k} e^{-i\theta(v)} \chi_{\{v\}}\Big) \\
&= \sum_{v \in F_k} e^{-i\theta(v)} y(\chi_{\{v\}}) \\
&= \sum_{v \in F_k} e^{-i\theta(v)} m(v) \\
&= \sum_{v \in F_k} |m(v)|.
\end{aligned}$$

Now the monotone convergence theorem tells us that the right hand side converges to $\|m\|_1$. Thus $m \in \ell^1(E^0)$. Next we show $\Phi(m)(f) = y(f)$ for all $f \in c_0(E^0)$. First, fix $g \in c_c(E^0)$. Then

$$\begin{aligned}
\Phi(m)(g) &= \sum_{v \in E^0} m(v) g(v) \\
&= \sum_{v \in E^0} y(\chi_{\{v\}}) g(v) \\
&= y\Big(\sum_{v \in E^0} \chi_{\{v\}} g(v)\Big) \\
&= y(g).
\end{aligned}$$

Finally, fix $f \in c_0(E^0)$ and choose $g \in c_c(E^0)$ such that $\|f - g\| < \frac{\epsilon}{2\|y\|_{\text{op}}}$. Then

$$\begin{aligned}
\|\Phi(m)(f) - y(f)\| &\leq \|\Phi(m)(f-g)\| + \|\Phi(m_n)(g) - y(g)\| + \|y(g-f)\| \\
&< \|\Phi(m)\| \frac{\epsilon}{2\|y\|_{\text{op}}} + 0 + \|y\|_{\text{op}} \frac{\epsilon}{2\|y\|_{\text{op}}} \\
&= \|m\|_1 \frac{\epsilon}{2\|y\|_{\text{op}}} + \frac{\epsilon}{2} \\
&\leq \|y\|_{\text{op}} \frac{\epsilon}{2\|y\|_{\text{op}}} + \frac{\epsilon}{2} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus $\Phi(m)(f) = y(f)$ for all $f \in c_0(E^0)$, therefore Φ is surjective.

To see that Φ is injective fix $m \in \ell^1(E^0)$ and suppose that $\Phi(m) = 0$. We want to show $m = 0$. Since $\Phi(m) = 0$, $\big[\Phi(m)\big](f) = 0$ for all $f \in c_0(E^0)$. In particular fix $w \in E^0$ and define $f_w := \chi_w$. Then

$$0 = \big[\Phi(m)\big](f_w) = \sum_{v \in E^0} m(v) f_w(v) = \sum_{v \in E^0} m(v) \chi_w(v) = m(w).$$

Because this is true for arbitrary $w \in E^0$, it is true for all $w \in E^0$, so $m = 0$. Thus Φ is injective.

Thus Φ is an isometric isomorphism. □

The following lemma tells us that pointwise convergence in $\ell^1(E^0)$ implies weak* convergence in $c_0(E^0)^*$, which we will use in our next proofs.

Lemma 4.6. *Suppose that $\{m_n\}$ is a norm-bounded sequence in $\ell^1(E^0)$. Then $\Phi(m_n) \to \Phi(m)$ in the weak* topology on $c_0(E^0)^*$ if and only if $m_n(v) \to m(v)$ for all $v \in E^0$.*

Proof. Suppose that $\Phi(m_n) \to \Phi(m)$. We want to show that $m_n(v) \to m(v)$ for all $v \in E^0$. Fix $v \in E^0$. Since $\Phi(m_n) \to \Phi(m)$,

$$\sum_{w \in E^0} m_n(w) f(w) \to \sum_{w \in E^0} m(w) f(w)$$

for all $f \in c_0(E^0)$. Let $f = \chi_v$. Then

$$\sum_{w \in E^0} m_n(w) \chi_v(w) \to \sum_{w \in E^0} m(w) \chi_v(w).$$

Thus $m_n(v) \to m(v)$.

Conversely, suppose that $m_n(v) \to m(v)$ for all $v \in E^0$. Fix $\epsilon > 0$ and $f \in c_0(E^0)$. We want to show that there exists $N \in \mathbb{N}$ such that if $n > N$ then $\|\Phi(m_n)(f) -$

$\Phi(m)(f)\| < \epsilon$. Let $M \in \mathbb{N}$ such that $\|m_n\|_1 < M$ for all $n \in \mathbb{N}$, and choose $g \in c_c(E^0)$ such that $\|f - g\| < \frac{\epsilon}{3M}$. Since $m_n \to m$ pointwise there exists N such that $n > N$ implies that

$$|m_n(v) - m(v)| < \frac{\epsilon}{3\|g\|_1}$$

for all $v \in \operatorname{supp}(g)$. Then, taking $n > N$,

$$\begin{aligned}
&\|\Phi(m_n)(f) - \Phi(m)(f)\| \\
&\qquad \leq \|\Phi(m_n)(f-g)\| + \|\Phi(m_n)(g) - \Phi(m)(g)\| + \|\Phi(m)(g-f)\| \\
&\qquad < \frac{\epsilon}{3} + \|\Phi(m_n)(g) - \Phi(m)(g)\| + \frac{\epsilon}{3} \\
&\qquad = \frac{2\epsilon}{3} + \Big| \sum_{v \in \operatorname{supp}(g)} m_n(v)g(v) - \sum_{v \in \operatorname{supp}(g)} m(v)g(v) \Big|.
\end{aligned}$$

Since $g \in c_c(E^0)$ implies that the sums are finite,

$$\begin{aligned}
\|\Phi(m_n)(f) - \Phi(m)(f)\| &\leq \frac{2\epsilon}{3} + \sum_{v \in \operatorname{supp}(g)} |m_n(v) - m(v)||g(v)| \\
&< \frac{2\epsilon}{3} + \sum_{v \in \operatorname{supp}(g)} \frac{\epsilon}{3\|g\|_1}|g(v)| \\
&= \frac{2\epsilon}{3} + \frac{\epsilon}{3\|g\|_1} \sum_{v \in \operatorname{supp}(g)} |g(v)| \\
&= \frac{2\epsilon}{3} + \frac{\epsilon}{3\|g\|_1}\|g\|_1 \\
&< \epsilon.
\end{aligned}$$

□

4.3 KMS functionals

A *KMS$_\beta$ functional* is a norm-decreasing positive linear functional which satisfies the KMS condition. Because the Toeplitz algebra $\mathcal{T}C^*(E)$ has no identity when E is infinite, the set of KMS states is not compact. This makes it difficult to find a homeomorphism between the simplex of KMS states and a subset of $\ell^1(E^0)^+$, as in [17, Theorem 3.1]. We therefore use the set of KMS functionals in our study of the KMS states of $(\mathcal{T}C^*(E), \alpha)$ at large inverse temperatures.

As in [28, pg. 272], if S is a subset of a locally convex space X we call the smallest convex set containing S the *convex hull* of S, denoted by $\operatorname{co}(S)$.

Lemma 4.7. *Fix $\beta \in (0, \infty)$. Let* $\mathrm{co}(\mathrm{KMS}_\beta \cup \{0\})$ *be the convex hull of the union of the KMS$_\beta$ states of $\mathcal{T}C^*(E)$ and the* 0 *functional. Then*

$$\begin{aligned}\mathrm{co}(\mathrm{KMS}_\beta \cup \{0\}) &= \{c\phi : \phi \textit{ is a KMS}_\beta \textit{ state}, c \in [0,1]\} \\ &= \{\textit{KMS}_\beta \textit{ functionals}\}\end{aligned}$$

is weak closed and compact.*

Proof. We first show that

$$\mathrm{co}(\mathrm{KMS}_\beta \cup \{0\}) = \{c\phi : \phi \text{ is a KMS}_\beta \text{ state}, c \in [0,1]\}. \tag{4.4}$$

The right hand side of Equation (4.4) is contained in $\mathrm{co}(\mathrm{KMS}_\beta \cup \{0\})$, so we only need to show $\mathrm{co}(\mathrm{KMS}_\beta \cup \{0\})$ is contained in the right hand side of Equation (4.4). The set $\mathrm{co}(\mathrm{KMS}_\beta \cup \{0\})$ is the intersection of all convex sets containing $\mathrm{KMS}_\beta \cup \{0\}$ and thus is the smallest convex subset of the KMS_β functionals containing $\mathrm{co}(\mathrm{KMS}_\beta \cup \{0\})$ ([28, pg. 272]). Therefore, to show it is contained in the right hand side of Equation (4.4) it suffices to show the right hand side contains the KMS_β states and 0 functional and is a convex set.

To see the right hand side contains the KMS_β states take $c = 1$, and to see it contains 0 take $c = 0$.

To see the right hand side is convex fix KMS_β states ϕ, ϕ' and $c, c' \in [0,1]$. Then $c\phi, c'\phi' \in \{c\phi : \phi \text{ is a KMS}_\beta \text{ state}, c \in [0,1]\}$. Fix $t \in [0,1]$, and take the convex combination $\Upsilon = tc\phi + (1-t)c'\phi'$. We want to show $\Upsilon \in \{c\phi : \phi \text{ is a KMS}_\beta \text{ state}, c \in [0,1]\}$. If $c\phi = c'\phi' = 0$ we have $\Upsilon = 0$, so we assume $c\phi \neq 0$. We claim that $\|\Upsilon\| \leq 1$ and $\frac{\Upsilon}{\|\Upsilon\|}$ is a KMS_{β_c} state. Since ϕ and ϕ' are states, and therefore of norm 1, by the triangle inequality gives

$$\|\Upsilon\| = \|tc\phi + (1-t)c'\phi'\| \leq t|c|\|\phi\| + (1-t)|c'|\|\phi'\| \leq t + 1 - t = 1.$$

Thus $\|\Upsilon\| \leq 1$. Next we show that $\frac{\Upsilon}{\|\Upsilon\|}$ is a KMS_{β_c} state. Since ϕ and ϕ' are KMS_β states, they are positive linear functionals, so by linearity $\frac{\Upsilon}{\|\Upsilon\|}$ is a positive linear functional. Also by linearity $\frac{\Upsilon}{\|\Upsilon\|}$ satisfies the KMS condition, so it only remains to show $\frac{\Upsilon}{\|\Upsilon\|}$ is a state, that is, $\|\frac{\Upsilon}{\|\Upsilon\|}\| = 1$. Since $c\phi \neq 0$ and ϕ is a state, there exists $a \in \mathcal{T}C^*(E)$ such that $c\phi(a^*a) > 0$. In addition, $c'\phi'(a^*a) \geq 0$, so $\Upsilon(a^*a) > 0$, and therefore $\Upsilon \neq 0$. Then $\|\frac{\Upsilon}{\|\Upsilon\|}\| = 1$, so $\frac{\Upsilon}{\|\Upsilon\|}$ is a state. Thus $\frac{\Upsilon}{\|\Upsilon\|}$ is a KMS_{β_c} state. Therefore we have proved our claim and $\Upsilon \in \{c\psi : \psi \text{ is a KMS}_\beta \text{ state}, c \in [0,1]\}$, so Equation (4.4) holds.

Next we show that

$$\{c\phi : \phi \text{ is a KMS}_\beta \text{ state}, c \in [0,1]\} = \{\text{KMS}_\beta \text{ functionals}\}. \tag{4.5}$$

Fix a KMS_β state ϕ and $c \in [0,1]$. Then it is clear $c\phi$ is a norm-decreasing positive linear functional satisfying the KMS condition, so the left hand side is contained in the right hand side. Conversely, fix a KMS_β functional ψ. We want to show $\psi \in \{c\phi : \phi \text{ is a } \mathrm{KMS}_\beta \text{ state}, c \in [0,1]\}$. If $\psi = 0$ take $c = 0$ and we are done, so assume $\psi \neq 0$. Then, noting

$$\psi = \|\psi\| \frac{\psi}{\|\psi\|},$$

since ψ is norm-decreasing and $\frac{\psi}{\|\psi\|}$ is of norm 1, and therefore a KMS_β state, $\psi \in \{c\phi : \phi \text{ is a } \mathrm{KMS}_\beta \text{ state}, c \in [0,1]\}$. Thus Equation (4.5) holds.

To see that $\mathrm{co}(\mathrm{KMS}_\beta \cup \{0\})$ is weak* closed suppose that $\{\phi_n\}$ is a sequence of KMS_β functionals such that $\phi_n \to \phi$ weak*. This means $\phi_n(a) \to \phi(a)$ for all $a \in \mathcal{T}C^*(E)$ so it follows that ϕ is a norm-decreasing positive linear functional. Since the ϕ_n are KMS_β functionals,

$$\phi_n(ab) = \phi_n(b\alpha_{i\beta}(a))$$

for all analytic $a, b \in \mathcal{T}C^*(E)$. The left hand side converges to $\phi(ab)$ and the right hand side to $\phi(b\alpha_{i\beta}(a))$, so ϕ satisfies the KMS condition. Thus $\mathrm{co}(\mathrm{KMS}_\beta \cup \{0\})$ is weak* closed.

Finally we show that $\mathrm{co}(\mathrm{KMS}_\beta \cup \{0\})$ is weak*-compact. The set of norm decreasing positive functionals is a convex weak*-compact set by [28, Theorem 5.1.8]. We have just shown $\mathrm{co}(\mathrm{KMS}_\beta \cup \{0\})$ is weak* closed and it is a subset of the norm decreasing positive functionals. Thus $\mathrm{co}(\mathrm{KMS}_\beta \cup \{0\})$ is weak*-compact. □

4.4 KMS states of $(\mathcal{T}C^*(E), \alpha)$ at large inverse temperatures

In this section we study the KMS states of $(\mathcal{T}C^*(E), \alpha)$ at large inverse temperatures by finding a homeomorphism between the KMS functionals and the following set.

Lemma 4.8. *Suppose that $x \in \ell^\infty(E^0)$. Then the set*

$$\Sigma_\beta := \Big\{\epsilon \in \ell^1(E^0)^+ : 0 \leq \sum_{v \in E^0} x_v \epsilon_v \leq 1\Big\}$$

is a weak-closed, convex, compact set in $\ell^1 = c_0^*$.*

Proof. First we show that Σ_β is closed in $\ell^1(E^0)$. Fix $\{\epsilon_n\} \in \Sigma_\beta$ and $\epsilon \in \ell^1(E^0)$ such that $\epsilon_n \to \epsilon$ weak*. We want to show that $\epsilon \in \Sigma_\beta$, that is, $0 \leq \sum_{v \in E^0} x_v \epsilon_v \leq 1$. Let F_k be finite sets such that $F_k \subseteq F_{k+1}$ and $\bigcup_{k=1}^{\infty} F_k = E^0$ (these exist because E^0 is countable). Then, since $\epsilon_n \to \epsilon$, $\sum_{v \in F_k} x_v(\epsilon_n)_v \to \sum_{v \in F_k} x_v \epsilon_v$. Since $\epsilon_n \in \Sigma_\beta$ for all n, $0 \leq \sum_{v \in F_k} x_v(\epsilon_n)_v \leq 1$, therefore $0 \leq \sum_{v \in F_k} x_v \epsilon_v \leq 1$. Then $0 \leq \sum_{v \in F_k} x_v \epsilon_v \leq \sum_{v \in F_{k+1}} x_v \epsilon_v \leq 1$ for all k, so we can apply the monotone convergence theorem to get $0 \leq \sum_{v \in E^0} x_v \epsilon_v \leq 1$. Therefore $\epsilon \in \Sigma_\beta$, so Σ_β is closed in $\ell^1(E^0)$.

Next we show that Σ_β is convex. Fix $y, z \in \Sigma_\beta$ and $c \in [0,1]$. We want to show that $cy + (1-c)z \in \Sigma_\beta$, that is $0 \leq \sum_{v \in E^0} x_v(cy + (1-c)z)_v \leq 1$. Since the sum is linear,

$$0 = c0 + (1-c)0 \leq c \sum_{v \in E^0} x_v y_v + (1-c) \sum_{v \in E^0} x_v z_v \leq c1 + (1-c)1 = 1.$$

Thus Σ_β is convex.

The Banach-Alaoglu theorem states that the closed unit ball of $\ell^1(E^0)$ is compact in the weak* topology, and Σ_β is weak*-closed inside the unit ball, so it is also compact. $\square$

To find a homeomorphism between $\mathrm{co}(\mathrm{KMS}_\beta \cup \{0\})$ and Σ_β we need that the vertex matrix $A = (|vE^1w|)_{v,w \in E^0}$ is bounded, so we add the following condition (which is stronger than locally finite).

Proposition 4.9. *Assume that $|s^{-1}(w)| \leq K$ for all $w \in E^0$. Then the vertex matrix A is a bounded linear operator on $\ell^1(E^0)$.*

Proof. Fix $\xi \in \ell^1(E^0)$. Then

$$\begin{aligned}
\|A\xi\|_1 &= \sum_{v \in E^0} |(A\xi)_v| \\
&= \sum_{v \in E^0} |\sum_{w \in E^0} A_{v,w}\xi_w| \\
&\leq \sum_{v \in E^0} \sum_{w \in E^0} |A_{v,w}\xi_w| \\
&= \sum_{v \in E^0} \sum_{w \in E^0} |A_{v,w}||\xi_w|.
\end{aligned}$$

Then, by Tonelli's theorem,

$$\begin{aligned}
\|A\xi\|_1 &\leq \sum_{w\in E^0}\Big(\sum_{v\in E^0} A_{v,w}\Big)|\xi_w| \\
&= \sum_{w\in E^0}\Big(\sum_{v\in E^0} |vE^1w|\Big)|\xi_w| \\
&= \sum_{w\in E^0} |s^{-1}(w)||\xi_w| \\
&\leq K \sum_{w\in E^0} |\xi_w| \\
&\leq K\|\xi\|_1.
\end{aligned}$$

Thus A is bounded with $\|A\|_{\mathrm{op}} \leq K$. □

We view A as a bounded operator on $\ell^1(E^0)$, so we write $\rho(A)$ for the spectral radius of $A \in B(\ell^1(E^0))$.

Theorem 4.10. *Let E be a row-finite directed graph with no sources such that $|s^{-1}(w)| \leq K$ for all $w \in E^0$. Take $\beta > \ln(\rho(A))$. Let γ be the gauge action and define $\alpha_t := \gamma_{e^{it}}$.*

(a) For $v \in E^0$, the sum $x_v = \sum_{\mu\in E^*v} e^{-\beta|\mu|}$ converges, and $x := (x_v) \in \ell^\infty(E^0)$. In addition, $x_v \geq 1$ for all $v \in E^0$. Let $\epsilon \in \ell^1(E^0)^+$ and define $m := (I - e^{-\beta}A)^{-1}\epsilon$. Then $m \in \ell^1(E^0)^+$ and $\|m\|_1 = \sum_{v\in E^0} x_v\epsilon_v$.

(b) Suppose that $\epsilon \in \ell^1(E^0)^+$ and set $m := (I - e^{-\beta}A)^{-1}\epsilon$. If $\sum_{v\in E^0} x_v\epsilon_v \leq 1$ there is a KMS$_\beta$ functional ϕ_ϵ of $(\mathcal{T}C^*(E), \alpha)$ satisfying

$$\phi_\epsilon(s_\mu s_\nu^*) = \delta_{\mu,\nu} e^{-\beta|\mu|} m_{s(\mu)}. \tag{4.6}$$

Moreover, if $\sum_{v\in E^0} x_v\epsilon_v = 1$, then ϕ_ϵ is a KMS$_\beta$ state.

(c) For a KMS functional ϕ define $m^\phi = (m^\phi_v)$ by $m^\phi_v = \phi(p_v)$. Then there is a linear homeomorphism $F : \mathrm{co}(\mathrm{KMS}_\beta \cup \{0\}) \to \Sigma_\beta$ defined by $F(\phi) := (I - e^{-\beta}A)m^\phi$ for all $\phi \in \mathrm{co}(\mathrm{KMS}_\beta \cup \{0\})$.

Proof of Theorem 4.10 (a). We first want to show that the series $\sum_{\mu\in E^*v} e^{-\beta|\mu|}$ converges. We start by showing that $\sum_{\mu\in E^*v} e^{-\beta|\mu|}$ converges. Fix $v \in E^0$. Then

$$\begin{aligned}\sum_{\mu\in E^*w} e^{-\beta|\mu|} &= \sum_{n=0}^{\infty}\sum_{\mu\in E^n w} e^{-\beta n}\\ &= \sum_{n=0}^{\infty}\sum_{v\in E^0} e^{-\beta n}|vE^n w|\\ &= \sum_{n=0}^{\infty}\sum_{v\in E^0} e^{-\beta n}A^n_{v,w}\end{aligned}$$

Then, applying Tonelli's theorem,

$$\sum_{\mu\in E^*w} e^{-\beta|\mu|} = \sum_{v\in E^0}\sum_{n=0}^{\infty} e^{-\beta n}A^n_{v,w}.$$

Proposition 4.9 implies that A is a bounded operator on $\ell^1(E^0)$. In addition, $\beta > \ln(\rho(A))$, so we can apply Corollary A.14, which tells us that the series $\sum_{n=0}^{\infty} e^{-\beta n}A^n$ converges in the operator norm to $(I - e^{-\beta}A)^{-1} \in B(\ell^1(E^0))$. This implies that for every fixed $v, w \in E^0$ the series $\sum_{n=0}^{\infty} e^{-\beta n}A^n_{v,w}$ converges to $(I - e^{-\beta}A)^{-1}_{v,w}$. Then, with $h_w \in \ell^1(E^0)$ the point mass at w,

$$\begin{aligned}x_w &= \sum_{\mu\in E^*w} e^{-\beta|\mu|}\\ &= \sum_{v\in E^0}\sum_{n=0}^{\infty} e^{-\beta n}A^n_{v,w}\\ &= \sum_{v\in E^0}(I - e^{-\beta}A)^{-1}_{v,w}\\ &= \sum_{v\in E^0}[(I - e^{-\beta}A)^{-1}h_w]_v.\end{aligned}$$

Since $(I - e^{-\beta}A)^{-1} \geq 0$,

$$x_w = \|(I - e^{-\beta}A)^{-1}h_w\|_1 \leq \|(I - e^{-\beta}A)^{-1}\|_{\text{op}}\|h_w\|_1 = \|(I - e^{-\beta}A)^{-1}\|_{\text{op}}. \tag{4.7}$$

Thus $x \in \ell^\infty(E^0)$ with $\|x\|_\infty \leq \|(I - e^{-\beta}A)^{-1}\|_{\text{op}}$. Each x_v is at least 1 because all the terms in the series Equation (4.7) are non-negative and when $n = 0$, $e^{-\beta n}A^n_{v,v} = 1$.

Next, fix $\epsilon \in \ell^1(E^0)^+$. Since $(I - e^{-\beta}A)^{-1} \in B(\ell^1(E^0))$, and $(I - e^{-\beta}A)^{-1} \geq 0$, $m := (I - e^{-\beta}A)^{-1}\epsilon \in \ell^1(E^0)^+$ too.

Finally,

$$\begin{aligned}
\|m\|_1 &= \sum_{v\in E^0} m_v \\
&= \sum_{v\in E^0} ((I-e^{-\beta}A)^{-1}\epsilon)_v \\
&= \sum_{v\in E^0} \Big(\Big(\sum_{n=0}^{\infty} e^{-\beta n}A^n\Big)\epsilon)_v \\
&= \sum_{v\in E^0}\sum_{n=0}^{\infty}\sum_{w\in E^0} e^{-\beta n}A^n_{v,w}\epsilon_w \\
&= \sum_{w\in E^0} \epsilon_w\Big(\sum_{\mu\in E^*w} e^{-\beta|\mu|}\Big) \\
&= \sum_{w\in E^0} \epsilon_w x_w. \qquad \square
\end{aligned}$$

To prove that ϕ_ϵ is a state in the next part of the theorem, we will need the following Lemma, a converse to [35, Lemma A7(a)].

Lemma 4.11. *Let ϕ be a norm-decreasing, positive linear functional, and $\{e_i\}$ an approximate identity in A such that $\phi(e_i) \to 1$. Then $\|\phi\| = 1$.*

Proof. The sequence $\{|\phi(e_i)|\}$ is bounded above by 1, and since ϕ is positive, it is increasing. Therefore $\{|\phi(e_i)|\}$ converges to its supremum, so

$$\sup_{\{e_i\}}\{|\phi(e_i)|\} = 1.$$

Then, since $\|e_i\| \geq 1$,

$$\|\phi\| = \sup_{\{\|a\|\leq 1\}}\{|\phi(a)|\} \geq \sup_{\{e_i\}}\{|\phi(e_i)|\} = 1.$$

In addition, ϕ is norm-decreasing, and in particular $\|\phi\| \leq 1$, so $\|\phi\| = 1$. $\square$

Proof of Theorem 4.10(b). Fix $\epsilon \in \ell^1(E^0)$ and take $m := (I - e^{-\beta}A)^{-1}\epsilon$. We first want to define ϕ_ϵ then show it is a positive linear functional on $(\mathcal{T}C^*(E), \alpha^y)$, and that it satisfies the KMS condition and Equation (4.6). We then assume that $\sum_{v\in E^0} x_v\epsilon_v = 1$ and show that ϕ_ϵ is a state.

We build a linear functional by representing $\mathcal{T}C^*(E)$ on $\ell^2(E^*)$. Let $\pi_{Q,T}$ be the finite path representation from Proposition 2.3. For $\mu \in E^*$ we set

$$\Delta_\mu := e^{-\beta|\mu|}\epsilon_{s(\mu)},$$

and note that $\Delta_\mu \geq 0$. We aim to define ϕ_e by

$$\phi_\epsilon(a) = \sum_{\mu\in E^*} \Delta_\mu(\pi_{Q,T}(a)h_\mu|h_\mu) \qquad \text{for } a \in \mathcal{T}C^*(E). \tag{4.8}$$

We first show that the sum in Equation (4.8) converges. To do this we claim that $\sum_{\mu\in E^*} \Delta_\mu = \|m\|_1$. We start by showing that $\sum_{\mu\in vE^*} e^{-\beta|\mu|}\epsilon_{s(\mu)}$ converges. Fix $v \in E^0$ and $n \in \mathbb{N}$. Then, since the graph is row-finite, that is, $|vE^n| < \infty$,

$$\begin{aligned}\sum_{\mu\in vE^n} e^{-\beta|\mu|}\epsilon_{s(\mu)} &= \sum_{w\in E^0}\sum_{\mu\in vE^n w} e^{-\beta|\mu|}\epsilon_w \\ &= \sum_{w\in E^0} e^{-\beta n}A^n_{v,w}\epsilon_w \\ &= (e^{-\beta n}A^n\epsilon)_v. \qquad (4.9)\end{aligned}$$

Now $\beta > \ln(\rho(A))$, so Corollary A.14 tells us that $\sum_{n=0}^\infty (e^{-\beta n}A^n\epsilon)_v$ converges with sum $((I - e^{-\beta}A)^{-1}\epsilon)_v = m_v$. Then, by Equation (4.9),

$$\sum_{n=0}^\infty (e^{-\beta n}A^n\epsilon)_v = \sum_{n=0}^\infty\sum_{\mu\in vE^n} e^{-\beta|\mu|}\epsilon_{s(\mu)},$$

and since E is row-finite the second sum is finite. Since the sum of non-negative entries over a countable set is independent of the listing of the set, we have

$$\sum_{n=0}^\infty\sum_{\mu\in vE^n} e^{-\beta|\mu|}\epsilon_{s(\mu)} = \sum_{\mu\in vE^*} e^{-\beta|\mu|}\epsilon_{s(\mu)} = \sum_{\mu\in vE^*}\Delta_\mu. \tag{4.10}$$

Therefore $\sum_{\mu\in vE^*}\Delta_\mu$ also converges with sum m_v. Then, by Tonelli's theorem,

$$\sum_{\mu\in E^*}\Delta_\mu = \sum_{v\in E^0}\Big(\sum_{\mu\in vE^*}\Delta_\mu\Big) = \sum_{v\in E^0} m_v = \|m\|_1. \tag{4.11}$$

We now use this to prove that Equation (4.8) converges for all $a \in \mathcal{T}C^*(E)$. Fix $a \in \mathcal{T}C^*(E)$, then, applying the Cauchy-Schwarz inequality,

$$\begin{aligned}0 &\leq |\Delta_\mu(\pi_{Q,T}(a)h_\mu|h_\mu)| \\ &= |\Delta_\mu||(\pi_{Q,T}(a)h_\mu|h_\mu)| \\ &\leq \Delta_\mu\|\pi_{Q,T}(a)h_\mu\|\|h_\mu\| \\ &\leq \Delta_\mu\|\pi_{Q,T}(a)\|\|h_\mu\|^2 \\ &\leq \Delta_\mu\|a\|\cdot 1. \qquad (4.12)\end{aligned}$$

Since $\sum_{\mu\in E^*}\Delta_\mu$ converges, $\sum_{\mu\in E^*}\Delta_\mu\|a\|$ converges and the comparison test tells us that $\sum_{\mu\in E^*}|\Delta_\mu(\pi_{Q,T}(a)h_\mu|h_\mu)|$ converges. Thus $\sum_{\mu\in E^*}\Delta_\mu(\pi_{Q,T}(a)h_\mu|h_\mu)$ converges absolutely for all $a\in\mathcal{T}C^*(E)$.

We can now show that ϕ_ϵ defines a positive functional on $\mathcal{T}C^*(E)$. To do this we need to show linearity, that is, that $\phi_\epsilon(wa+zb)=w\phi_\epsilon(a)+z\phi_\epsilon(b)$ for all $a,b\in\mathcal{T}C^*(E)$ and $w,z\in\mathbb{C}$, and that $\phi_\epsilon(a^*a)\geq 0$ for all $a\in\mathcal{T}C^*(E)$. Fix $a,b\in\mathcal{T}C^*(E)$ and $c\in\mathbb{C}$. First we show that $\phi_\epsilon(wa+zb)=w\phi_\epsilon(a)+z\phi_\epsilon(b)$. Since Equation (4.8) converges for all $a\in\mathcal{T}C^*(E)$, using the algebra of series,

$$\begin{aligned}
w\phi_\epsilon(a)+z\phi_\epsilon(b) &= w\sum_{\mu\in E^*}\Delta_\mu(\pi_{Q,T}(a)h_\mu|h_\mu)+z\sum_{\mu\in E^*}\Delta_\mu(\pi_{Q,T}(b)h_\mu|h_\mu)\\
&=\sum_{\mu\in E^*}\Delta_\mu\Big(w(\pi_{Q,T}(a)h_\mu|h_\mu)+z(\pi_{Q,T}(b)h_\mu|h_\mu)\Big)\\
&=\sum_{\mu\in E^*}\Delta_\mu((w\pi_{Q,T}(a)+z\pi_{Q,T}(b))h_\mu|h_\mu)\\
&=\sum_{\mu\in E^*}\Delta_\mu(\pi_{Q,T}(wa+zb)h_\mu|h_\mu)\\
&=\phi_\epsilon(wa+zb).
\end{aligned}$$

Next we show that $\phi_\epsilon(a^*a)\geq 0$. By definition of ϕ_ϵ,

$$\phi_\epsilon(a^*a)=\sum_{\mu\in E^*}\Delta_\mu(\pi_{Q,T}(a^*a)h_\mu|h_\mu).$$

Since $\Delta_\mu(\pi_{Q,T}(a^*a)h_\mu|h_\mu)$ is nonnegative for each μ, the sum is nonnegative, and $\phi_\epsilon(a^*a)\geq 0$. Thus ϕ_ϵ is a positive linear functional.

Next we prove that ϕ_ϵ is norm-decreasing. Fix $a\in\mathcal{T}C^*(E)$, then

$$|\phi_\epsilon(a)|=|\sum_{\mu\in E^*}\Delta_\mu(\phi_{Q,T}(a)h_\mu|h_\mu)|\leq\sum_{\mu\in E^*}|\Delta_\mu(\phi_{Q,T}(a)h_\mu|h_\mu)|.$$

So by Equation (4.12),

$$|\phi_\epsilon(a)|\leq\sum_{\mu\in E^*}\Delta_\mu\|a\|.$$

By Equation (4.11), $|\phi_\epsilon(a)|\leq\|m\|_1\|a\|$, and by part (a), $|\phi_\epsilon(a)|\leq\sum_{v\in E^0}x_v\epsilon_v\|a\|$. Since $\sum_{v\in E^0}x_v\epsilon_v\leq 1$, $|\phi_\epsilon(a)|\leq\|a\|$. Therefore ϕ_ϵ is norm-decreasing.

Next we prove that ϕ_ϵ satisfies Equation (4.6). Fix $\lambda\in E^*$. Then

$$(\pi_{Q,T}(s_\mu s_\nu^*)h_\lambda|h_\lambda)=(T_\nu^*h_\lambda|T_\mu^*h_\lambda)=\begin{cases}1 & \text{if } \lambda=\mu\lambda'=\nu\lambda'\\ 0 & \text{otherwise.}\end{cases}$$

Since $\mu\lambda' = \nu\lambda'$ forces $\mu = \nu$, we have $\phi_\epsilon(s_\mu s_\nu^*) = 0$ if $\mu \neq \nu$. So suppose that $\mu = \nu$. Then Equation (4.10) gives $\sum_{\mu \in vE^*} \Delta_\mu = m_v$, so

$$\begin{aligned}
\phi_\epsilon(s_\mu s_\mu^*) &= \sum_{\mu \in E^*} \Delta_\lambda (T_\mu^* h_\lambda | T_\mu^* h_\lambda) \\
&= \sum_{\lambda = \mu\lambda'} e^{-\beta|\mu\lambda'|} \epsilon_{s_{(\lambda')}} \\
&= \sum_{\lambda = \mu\lambda'} e^{-\beta(|\mu| + |\lambda'|)} \epsilon_{s_{(\lambda')}} \\
&= \sum_{\lambda = \mu\lambda'} e^{-\beta|\mu|} e^{-\beta|\lambda'|} \epsilon_{s_{(\lambda')}} \\
&= e^{-\beta|\mu|} \sum_{\lambda' \in s(\mu)E^*} \Delta_{\lambda'} \\
&= e^{-\beta|\mu|} m_{s(\mu)}.
\end{aligned}$$

Thus ϕ_ϵ satisfies Equation (4.6).

Next we show that ϕ_ϵ satisfies the KMS condition. Fixing $v \in E^0$ and calculating $\phi_\epsilon(p_v)$, by Tonelli's theorem we get

$$\begin{aligned}
\phi_\epsilon(p_v) &= \sum_{\mu \in E^*} \Delta_\mu (\pi_{Q,T}(p_v) h_\mu | h_\mu) \\
&= \sum_{\mu \in E^*} \Delta_\mu (Q_v h_\mu | h_\mu) \\
&= \sum_{w \in E^0} \sum_{\mu \in wE^*} \Delta_\mu (Q_v h_\mu | h_\mu).
\end{aligned}$$

Then, since $Q_v h_\mu = 0$ if $v \neq s(\mu)$,

$$\phi_\epsilon(p_v) = \sum_{\mu \in vE^*} \Delta_\mu (Q_v h_\mu | h_\mu) = \sum_{\mu \in vE^*} \Delta_\mu = m_v.$$

Therefore

$$\phi_\epsilon(s_\mu s_\nu^*) = \delta_{\mu,\nu} e^{-\beta|\mu|} m_{s(\mu)} = \delta_{\mu,\nu} e^{-\beta|\mu|} \phi_\epsilon(p_{s(\mu)}).$$

Now Proposition 4.4(b) tells us that ϕ_ϵ is a KMS$_\beta$ functional.

Finally, assume that $\sum_{v \in E^0} x_v \epsilon_v = 1$. Then we want to show that ϕ_ϵ is a state, that is, that $\|\phi_\epsilon\| = 1$. Because $\sum_{v \in E^0} x_v \epsilon_v = 1$ part (a) tells us that $\|m\|_1 = 1$. Choose any listing $\{v_n\}$ of E^0. Lemma 4.2 tells us that the sequence $\{\sum_{n=1}^N p_{v_n}\}_N$ is an approximate identity of $\mathcal{T}C^*(E)$. Since $Q_v h_\mu = \delta_{v,r(\mu)} h_\mu$,

$$\phi_e\Big(\sum_{n=1}^N p_{v_n}\Big) = \sum_{\mu \in E^*} \Delta_\mu \Big(\sum_{n=1}^N Q_{v_n} h_\mu | h_\mu\Big) = \sum_{\mu \in E^*} \sum_{n=1}^N \Delta_\mu (Q_{v_n} h_\mu | h_\mu) = \sum_{\mu \in E^*} \sum_{n=1}^N \Delta_\mu.$$

Then, by the algebra of series,

$$\begin{aligned}\phi_e\Big(\sum_{n=1}^{N} p_{v_n}\Big) &= \sum_{n=1}^{N}\sum_{\mu\in v_nE^*}\Delta_\mu \\ &\to \sum_{n=1}^{\infty}\sum_{\mu\in v_nE^*}\Delta_\mu \\ &= \sum_{\mu\in E^*}\Delta_\mu \\ &= \|m\|_1 \\ &= 1.\end{aligned}$$

Now Lemma 4.11 implies ϕ_ϵ is a state. □

Proof of Theorem 4.10 (c). We aim to find a homeomorphism F between $\mathrm{co}(\mathrm{KMS}_\beta \cup \{0\})$ and Σ_β, noting that the topologies on both spaces is the weak* topology.

First we show that the range of F is contained in $\ell^1(E^0)$. Fix $\phi \in \mathrm{co}(\mathrm{KMS}_\beta \cup \{0\})$. Since $\|m^\phi\|_1 \le 1$,

$$\|F(\phi)\|_1 = \|(I - e^{-\beta}A)m^\phi\|_1 \le \|I - e^{-\beta}A\|_{\mathrm{op}}\|m^\phi\|_1 \le \|I - e^{-\beta}A\|_{\mathrm{op}}.$$

Since $A \in B(\ell^1(E^0))$ implies that $I - e^{-\beta}A \in B(\ell^1(E^0))$, its operator norm is finite, so $F(\phi) \in \ell^1(E^0)$.

Next we show that F is linear. Let $c, d \in \mathbb{R}$ and $\phi, \psi \in \mathrm{co}(\mathrm{KMS}_\beta \cup \{0\})$. We need to show that $F(c\phi + d\psi) = cF(\phi) + dF(\psi)$. Fix $v \in E^0$. Then

$$\begin{aligned}F(c\phi + d\psi)_v &= \Big[(I - e^{-\beta}A)m^{c\phi+d\psi}\Big]_v \\ &= \sum_{w\in E^0}(I - e^{-\beta}A)_{v,w}m_w^{c\phi+d\psi} \\ &= \sum_{w\in E^0}(I - e^{-\beta}A)_{v,w}[c\phi + d\psi](p_w) \\ &= c\sum_{w\in E^0}(I - e^{-\beta}A)_{v,w}\phi(p_w) + d\sum_{w\in E^0}(I - e^{-\beta}A)_{v,w}\psi(p_w) \\ &= \Big[cF(\phi) + dF(\psi)\Big]_v.\end{aligned}$$

Thus F is linear.

Since $\mathrm{co}(\mathrm{KMS}_\beta \cup \{0\})$ is weak*-compact (by Lemma 4.7), to prove that F is a homeomorphism it is enough to show that it is surjective, injective and continuous.

First we show that F is continuous. F maps elements from $\mathrm{co}(\mathrm{KMS}_\beta \cup \{0\}) \subseteq \mathcal{T}C^*(E)^*$ to $\Sigma_\beta \subseteq \ell^1(E^0)$. We want to show that F is weak* to weak* continuous, and we use Φ to identify $\ell^1(E^0)$ as $c_0(E^0)^*$. Suppose that $\phi_n \to \phi$ weak* in $\mathcal{T}C^*(E)^*$, that is, for all $a \in \mathcal{T}C^*(E)$, $\phi_n(a) \to \phi(a)$ in $\mathbb{C}$. We want to show that $\Phi \circ F(\phi_n) \to \Phi \circ F(\phi)$ weak* in $c_0(E^0)^*$. By Lemma 4.6, it suffices to show that $F(\phi_n)_v \to F(\phi)_v$ for all $v \in E^0$. Computing:

$$\begin{aligned} F(\phi_n)_v &= \Big((I - e^{-\beta}A)m^{\phi_n}\Big)_v \\ &= \sum_{w \in E^0} (I - e^{-\beta}A)_{v,w} m_w^{\phi_n} \\ &= \sum_{w \in E^0} (I - e^{-\beta}A)_{v,w} \phi_n(p_w) \\ &= \phi_n(p_v) - \sum_{w \in E^0} e^{-\beta} A_{v,w} \phi_n(p_w). \end{aligned}$$

Now, since $\phi_n(a) \to \phi(a)$ for all $a \in \mathcal{T}C^*(E)$ and the sum above is finite (because E is row-finite),

$$\begin{aligned} F(\phi_n)_v = \phi_n(p_v) &- \sum_{w \in E^0} e^{-\beta} A_{v,w} \phi_n(p_w) \\ &\to \phi(p_v) - \sum_{w \in E^0} e^{-\beta} A_{v,w} \phi(p_w) = F(\phi)_v. \end{aligned}$$

Thus F is weak* continuous.

Next we show that F is injective. Let $\phi \in \operatorname{co}(\mathrm{KMS}_\beta \cup \{0\})$ such that $F(\phi) = 0$. We want to show that $\phi = 0$, and it suffices to show that $\phi(s_\mu s_\nu^*) = 0$ for all $\mu, \nu \in E^*$. We have $F(\phi) = (I - e^{-\beta}A)m^\phi = 0$. Now $\beta > \ln(\rho(A))$ implies that $I - e^{-\beta}A$ is invertible in $B(\ell^1(E^0))$, and in particular is injective, so $m^\phi = 0$. Then $\phi(p_v) = 0$ for all $v \in E^0$. By Proposition 4.4(a), this implies that $\phi(s_\mu s_\nu^*) = 0$ for all $\mu, \nu \in E^*$, so $\phi = 0$ and F is injective.

Finally we show that F is surjective. Fix $\epsilon \in \Sigma_\beta$, and then part (b) gives ϕ_ϵ and $m = (I - e^{-\beta}A)^{-1}\epsilon$. Fix $\mu \in E^*$. Comparing Equation (4.6) with Equation (4.1) tells us $m_{s(\mu)} = \phi_\epsilon(p_{s(\mu)})$, and $m^\phi_{s(\mu)} = \phi_\epsilon(p_{s(\mu)})$ by definition. Therefore $m = m^{\phi_\epsilon}$, so

$$F(\phi_\epsilon) = (I - e^{-\beta}A)m^{\phi_\epsilon} = (I - e^{-\beta}A)m = (I - e^{-\beta}A)(I - e^{-\beta}A)^{-1}\epsilon = \epsilon,$$

so F is surjective.

Then, since F is surjective, injective and continuous, it is a homeomorphism. □

4.5 KMS states of $(C^*(E), \overline{\alpha})$

Since the quotient map q of $\mathcal{T}C^*(E)$ onto $C^*(E)$ is gauge-invariant we get an action of $\mathbb{R}$ on $C^*(E)$, which we call $\overline{\alpha}$. In this section we show when a KMS state of $(\mathcal{T}C^*(E), \overline{\alpha})$

factors through $C^*(E)$.

Proposition 4.12. *Let E be a row-finite directed graph with no sources. Let $\beta \in (0, \infty)$ and A be the vertex matrix of E. Let γ be the gauge action and define $\alpha_t := \gamma_{e^{it}}$. A KMS$_\beta$ state ϕ of $(\mathcal{T}C^*(E), \alpha)$ factors through $C^*(E)$ if and only if $(Am^\phi)_v = e^\beta m_v^\phi$ for all $v \in E^0$.*

Proof. We follow the method of [17, Proposition 2.1(d)]. Fix $v \in E^0$. By [17, Lemma 2.2] it suffices to check that $\phi\big(p_v - \sum_{e\in vE^1} s_e s_e^*\big) = 0$ if and only if $(Am^\phi)_v = e^\beta m_v^\phi$. Applying Equation (4.3),

$$\begin{aligned}
\phi\big(p_v - \sum_{f\in vE^1} s_e s_e^*\big) &= \phi(p_v) - \phi\Big(\sum_{f\in vE^1} s_e s_e^*\Big) \\
&= \phi(p_v) - \sum_{f\in vE^1} \phi(s_e s_e^*) \\
&= \phi(p_v) - e^{-\beta}(Am^\phi)_v \\
&= m^\phi - e^{-\beta}(Am^\phi)_v.
\end{aligned}$$

Multiplying both sides by e^β we see that $\phi\big(p_v - \sum_{e\in vE^1} s_e s_e^*\big) = 0$ if and only if $(Am^\phi)_v = e^\beta m_v^\phi$. $\square$

4.6 An example

The following example from Carlsen and Larsen [3, Example 7.2] shows that the existence of KMS states of $(C^*(E), \alpha)$ for infinite graphs is not guaranteed, even when the graph is strongly connected. It is proved here using the same method as [3, Example 7.2], but applying our results instead of those in [3].

Example 4.13. Let E be the graph defined as follows

$$\begin{tikzcd}
\cdots \arrow[r, bend left, "e_{-1}"] & v_0 \arrow[l, bend left, "f_{-1}"] \arrow[r, bend left, "e_0"] & v_1 \arrow[l, bend left, "f_0"] \arrow[r, bend left, "e_1"] & v_2 \arrow[l, bend left, "f_1"] \arrow[r, bend left, "e_2"] & \cdots \arrow[l, bend left, "f_2"]
\end{tikzcd}$$

Fix $\beta \in (0, \infty)$ and suppose that there exists a KMS$_\beta$ state ϕ of $(C^*(E), \overline{\alpha})$. Then $\psi := \phi \circ q$ is a KMS$_\beta$ state of $(\mathcal{T}C^*(E), \alpha)$. With $m^\psi := (\psi(p_v))$, Proposition 4.12 tells us that $(Am^\psi)_v = e^\beta m_v^\psi$ for all $v \in E^0$. Fix $n \in \mathbb{Z}$. Then

$$m^\psi(v_n) = \frac{1}{2}(m^\psi(v_{n-1}) + m^\psi(v_{n+1})).$$

So either $m^\psi(v_{n-1}) \geq m^\psi(v_n)$ or $m^\psi(v_{n+1}) \geq m^\psi(v_n)$. Without loss of generality suppose that $m^\psi(v_{n+1}) \geq m^\psi(v_n)$. By induction on $k \geq 1$,

$$m^\psi(v_{n+k}) = km^\psi(v_{n+1}) - (k-1)m^\psi(v_n) \geq m^\psi(v_n).$$

However

$$\sum_{k=0}^{\infty} m^\psi(v_{n+k}) \leq \sum_{v \in E^0} m^\psi(v_n) = 1,$$

so it must be the case that $m^\psi(v_n) = 0$. Thus $m^\psi = 0$, which contradicts that $\|m^\psi\|_1 = 1$ (Lemma 4.3). Therefore $(C^*(E), \overline{\alpha})$ has no KMS states.

Chapter 5

KMS states of C^*-algebras for higher-rank graphs with a generalised gauge dynamics

The goal of this chapter is to generalise Chapter 3 to higher-rank graphs.

We first introduce the required background material on higher-rank graphs and their C^*-algebras $\mathcal{T}C^*(\Lambda)$ and $C^*(\Lambda)$. We then describe our generalised gauge dynamics α^y. Next we characterise of the KMS states of $(\mathcal{T}C^*(\Lambda), \alpha^y)$. We then get a subinvariance relation and use this to characterise the KMS states of $(\mathcal{T}C^*(\Lambda), \alpha^y)$ at large inverse temperatures. With this characterisation we describe the KMS states of $(\mathcal{T}C^*(\Lambda), \alpha^y)$ at large inverse temperatures. We then discuss the KMS states of $(\mathcal{T}C^*(\Lambda), \alpha^y)$ at the critical inverse temperature. Next we describe the dynamics $\overline{\alpha}^y$ and show when a KMS state of $(\mathcal{T}C^*(\Lambda), \alpha^y)$ factors through $C^*(\Lambda)$. Finally we discuss the KMS states of $(\mathcal{T}C^*(\Lambda), \alpha^y)$ and $(C^*(\Lambda), \overline{\alpha}^y)$ for a preferred dynamics, for which we get our best result.

5.1 Higher-rank graphs

In this section we present the definition of a higher-rank graph. We use the same notation as [32, Chapter 10], [34] and [16].

Let k be a positive integer. We view $\mathbb{N}^k$ as a category with one object and write $\{e_i\}$ for the usual basis. Fix $m, n \in \mathbb{N}^k$. By $m \leq n$ we mean $m_i \leq n_i$ for all $i \in \{1, \dots, k\}$. We write the pointwise maximum of m and n as $m \vee n$; similarly the pointwise minimum is denoted $m \wedge n$.

A *higher-rank graph* or *k-graph* is a countable category $\Lambda = (\Lambda^0, \Lambda, r, s)$ together with a degree functor $d : \Lambda \to \mathbb{N}^k$ which satisfies the *factorisation property*: if $d(\lambda) = m + n$ then there are unique $\mu, \nu \in \Lambda$ with $d(\mu) = m, d(\nu) = n$ such that $\lambda = \mu\nu$. We call the elements $\lambda \in \Lambda$ paths.

For $n \in \mathbb{N}^k$ we define

$$\Lambda^n := \{\lambda \in \Lambda : d(\lambda) = n\}.$$

We call the elements of Λ^n *paths of degree n*. For $v \in E^0$ we define

$$\Lambda^n v := \{\lambda \in \Lambda^n : r(\lambda) = v\}$$

and

$$v\Lambda^n := \{\lambda \in \Lambda^n : s(\lambda) = v\}.$$

A k-graph (Λ, d) is row-finite if $v\Lambda^n$ is finite for all $n \in \mathbb{N}^k$ and $v \in \Lambda^0$, and has *no sources* if $v\Lambda^n$ is nonempty for all $n \in \mathbb{N}^k$ and $v \in \Lambda^0$. In this chapter we only consider higher-rank graphs which have no sources and are finite in the sense that Λ^n is finite for all $n \in \mathbb{N}^k$ (this, in particular, implies that Λ is row-finite). Notice that we are not asserting that Λ is a finite set.

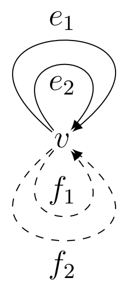
For $\mu, \nu \in \Lambda$, we say that λ is a *minimal common extension* of μ and ν if $d(\lambda) = d(\mu) \vee d(\nu)$ and $\lambda = \mu\mu' = \nu\nu'$ for some $\mu', \nu' \in \Lambda$. We write $\Lambda^{\min}(\mu, \nu)$ for the set of all minimal common extensions of μ and ν.

For $i \in \{1, \dots, k\}$ let A_i be the matrix with entries $(A_i)_{v,w} = |v\Lambda^{e_i}w|$. We call the A_i the *vertex matrices* of Λ. We say that Λ is *coordinatewise irreducible* if for all $i \in \{1, \dots, k\}$ the vertex matrix A_i is irreducible.

To visualise a k-graph Λ we can draw its *skeleton*, the directed graph $(\Lambda^0, \cup_{i=1}^k \Lambda^{e_i}, r, s)$ where an edge of degree i is drawn in colour c_i. The factorisation property then gives bijections between the c_ic_j-coloured paths of length 2 and the c_jc_i-coloured paths, and we think of each pair as a *commuting square* in the skeleton. Then [15, Remark 2.3] tells us that a complete collection of commuting squares determines the k-graph.

Example 5.1. Let Λ be the 2-graph defined by the skeleton (with edges of degree

$(1,0)$ in solid lines and edges of degree $(0,1)$ in dashed lines)



and the factorisation property

$$e_1 f_1 = f_1 e_1, \quad e_1 f_2 = f_1 e_2, \quad e_2 f_1 = f_2 e_1 \text{ and } e_2 f_2 = f_2 e_2.$$

This factorisation property gives the commuting squares:

$$\begin{tikzcd}
\bullet \arrow[d, dashed, "f_1"'] & \bullet \arrow[l, "e_1"'] \arrow[d, dashed, "f_1"] \\
\bullet & \bullet \arrow[l, "e_1"]
\end{tikzcd}
\quad
\begin{tikzcd}
\bullet \arrow[d, dashed, "f_1"'] & \bullet \arrow[l, "e_2"'] \arrow[d, dashed, "f_2"] \\
\bullet & \bullet \arrow[l, "e_1"]
\end{tikzcd}
\quad
\begin{tikzcd}
\bullet \arrow[d, dashed, "f_2"'] & \bullet \arrow[l, "e_1"'] \arrow[d, dashed, "f_1"] \\
\bullet & \bullet \arrow[l, "e_2"]
\end{tikzcd}
\quad
\begin{tikzcd}
\bullet \arrow[d, dashed, "f_2"'] & \bullet \arrow[l, "e_2"'] \arrow[d, dashed, "f_2"] \\
\bullet & \bullet \arrow[l, "e_2"]
\end{tikzcd}$$

Then we can write down all of the factorisations for a path $\lambda \in \Lambda$ with degree $(3,1)$.

$$\begin{tikzcd}
\bullet v \arrow[d, dashed, "f_1"] & \bullet v \arrow[l, "e_2"'] \arrow[d, dashed, "f_2"] & \bullet v \arrow[l, "e_2"'] \arrow[d, dashed, "f_2"] & \bullet v \arrow[l, "e_1"'] \arrow[d, dashed, "f_1"] \\
\bullet v & \bullet v \arrow[l, "e_1"] & \bullet v \arrow[l, "e_2"] & \bullet v \arrow[l, "e_2"]
\end{tikzcd}$$

Thus $\lambda = f_1 e_2 e_2 e_1 = e_1 f_2 e_2 e_2 = e_1 e_2 f_2 e_1 = e_1 e_2 e_2 f_1$.

5.2 The Toeplitz algebra $\mathcal{T}C^*(\Lambda)$

In this section we present the definition of Toeplitz-Cuntz-Krieger Λ-families and their algebras. We use the definition from [33, Section 7].

A *Toeplitz-Cuntz-Krieger Λ-family* $\{T\}$ consists of partial isometries $\{T_\lambda : \lambda \in \Lambda\}$ such that

(T1) $\{T_v : v \in \Lambda^0\}$ are mutually orthogonal projections;

(T2) $T_\lambda T_\mu = T_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$;

(T3) $T_\lambda^* T_\lambda = T_{s(\lambda)}$ for all λ;

(T4) for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, we have

$$T_v \geq \sum_{\lambda \in v\Lambda^n} T_\lambda T_\lambda^*;$$

(T5) for all $\mu, \nu \in \Lambda$, we have (interpreting any empty sums as 0)

$$T_\mu^* T_\nu = \sum_{(\eta,\zeta)\in\Lambda^{\min}(\mu,\nu)} T_\eta T_\zeta^*.$$

There is a C^*-algebra $\mathcal{T}C^*(\Lambda)$ generated by a Toeplitz-Cuntz-Krieger Λ-family $\{t_\lambda\}$ which is universal in the sense that: given any Toeplitz-Cuntz-Krieger Λ-family $\{T_\lambda\}$ in a C^*-algebra B there is a homomorphism $\pi_T : \mathcal{T}C^*(\Lambda) \to B$ such that $\pi_T(t_\lambda) = T_\lambda$ for all $\lambda \in \Lambda$ [33, Corollary 7.5]. We call $\mathcal{T}C^*(\Lambda)$ the *Toeplitz algebra* of Λ. In this thesis we only study $\mathcal{T}C^*(\Lambda)$ when Λ is finite as this implies $\mathcal{T}C^*(\Lambda)$ has an identity, that is $\sum_{v\in\Lambda^0} t_v = 1$. The literature usually defines $Q_v := T_v$ for $v \in \Lambda^0$ and calls $\{Q, T\}$ a Toeplitz-Cuntz-Krieger Λ-family. We avoid this definition so we don't cause confusion between a universal Toeplitz-Cuntz-Krieger Λ-family $\{q, t\}$ and the quotient map q used later in the chapter.

Lemma 5.2. *Suppose that (Λ, d) is a k-graph and B is a C^*-algebra generated by a Toeplitz-Cuntz-Krieger Λ-family $\{w\}$ such that, for every Toeplitz-Cuntz-Krieger Λ-family $\{T\}$ in a C^*-algebra C, there exists a homomorphism $\rho_T : B \to C$ satisfying $\rho_T(w_\lambda) = T_\lambda$. Then there exists an isomorphism $\pi_w : \mathcal{T}C^*(\Lambda) \to B$ such that $\pi_w(t_\lambda) = w_\lambda$.*

Proof. The universal property of $\mathcal{T}C^*(\Lambda)$ gives us a homomorphism $\pi_w : \mathcal{T}C^*(\Lambda) \to B$. The homomorphism π_w is onto because the range of π_w is a C^*-algebra containing $\{w_\lambda\}$, and hence is all of B. Since $\rho_T \circ \pi_w$ is the identity on $\{t_\lambda\}$ it is the identity on all of $\mathcal{T}C^*(\Lambda)$. Therefore $\pi_w(a) = 0$ implies that $a = \rho_T(\pi_w(a)) = 0$, and π_w is injective. □

The following proposition gives an example of a Toeplitz-Cuntz-Krieger Λ-family.

Proposition 5.3. *Let Λ be a finite k-graph. Write h_λ for the point mass at $\lambda \in \Lambda$, and let $\{T_\lambda : \lambda \in \Lambda\}$ be the operators on $\ell^2(\Lambda)$ such that*

$$T_\mu h_\lambda = \begin{cases} h_{\mu\lambda} & \text{if } s(\mu) = r(\lambda) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{T\}$ is a Toeplitz-Cuntz-Krieger Λ-family in $B(\ell^2(\Lambda))$. We call the representation $\pi_T : \mathcal{T}C^*(\Lambda) \to B(\ell^2(\Lambda))$ such that $\pi_T(t_\lambda) = T_\lambda$ the path representation, and π_T is faithful.

Proof. [33, Example 7.4] proves that $\{T\}$ is a Toeplitz-Cuntz-Krieger Λ-family and [33, Corollary 7.7] proves that π_T is faithful. $\square$

5.2.1 The graph algebra $C^*(\Lambda)$

A Toeplitz-Cuntz-Krieger Λ-family is a *Cuntz-Krieger* Λ-family if in addition we have

$$\text{(CK)} \qquad T_v = \sum_{\lambda \in v\Lambda^n} T_\lambda T_\lambda^* \qquad \text{for all } v \in \Lambda^0 \text{ and } n \in \mathbb{N}^k.$$

As with the Toeplitz algebra there is a C^*-algebra $C^*(\Lambda)$ generated by a universal Cuntz-Krieger Λ-family $\{t_\lambda\}$.

The following result tells us how $\mathcal{T}C^*(\Lambda)$ and $C^*(\Lambda)$ are related.

Lemma 5.4. *Let $\{t_\lambda\}$ be the universal Toeplitz-Cuntz-Krieger Λ-family which generates $\mathcal{T}C^*(\Lambda)$. Let J be the ideal generated by $\{t_v - \sum_{\lambda \in v\Lambda^{e_i}} t_\lambda t_\lambda^* : v \in \Lambda^0$ and $i \in \{1, \dots, k\}\}$, and $q : \mathcal{T}C^*(\Lambda) \to \mathcal{T}C^*(\Lambda)/J$ be the quotient map. Write $\bar{t}_\lambda = q(t_\lambda)$. Then $(\mathcal{T}C^*(\Lambda)/J, \{\bar{t}\})$ is universal for Cuntz-Krieger Λ-families, that is,*

(a) $\{\bar{t}\}$ is a Cuntz-Krieger Λ-family which generates $\mathcal{T}C^*(\Lambda)/J$; and

(b) if $\{T_\lambda\}$ is a Cuntz-Krieger Λ-family in a C^*-algebra B then there exists a homomorphism $\overline{\pi}_T : \mathcal{T}C^*(\Lambda)/J \to B$ such that $\overline{\pi}_T(\bar{t}_\lambda) = T_\lambda$.

Proof. To prove part (a) we need to show that $\{\bar{t}_\lambda : \lambda \in \Lambda\}$ is a Cuntz-Krieger Λ-family and that it generates $\mathcal{T}C^*(\Lambda)/J$. Since $\{t_\lambda : \lambda \in \Lambda\}$ is a Toeplitz-Cuntz-Krieger Λ-family and q is a homomorphism, $\{\bar{t}_\lambda : \lambda \in \Lambda\}$ is a set of partial isometries and (T1) - (T3) hold. To see that (CK) holds, fix $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. We aim to show that $\bar{t}_v = \sum_{\lambda \in v\Lambda^n} \bar{t}_\lambda \bar{t}_\lambda^*$. Then

$$0 = q\Big(t_v - \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^*\Big) = q(t_v) - q\Big(\sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^*\Big) = \bar{t}_v - \sum_{\lambda \in v\Lambda^n} q(t_\lambda t_\lambda^*).$$

This implies that

$$\bar{t}_v = \sum_{\lambda \in v\Lambda^n} q(t_\lambda t_\lambda^*) = \sum_{\lambda \in v\Lambda^n} q(t_\lambda) q(t_\lambda^*) = \sum_{\lambda \in v\Lambda^n} \bar{t}_\lambda \bar{t}_\lambda^*.$$

Thus (CK) holds. (CK) implies (T4) and (CK) combined with (T1)-(T3) implies (T5) ([22, Lemma 3.1]), so $\{\bar{t}_\lambda : \lambda \in \Lambda\}$ is a Cuntz-Krieger Λ-family. Since $\mathcal{T}C^*(\Lambda)$ is generated by $\{t_\lambda\}$, $\{q(t_\lambda)\} = \{\bar{t}_\lambda\}$ generates $\mathcal{T}C^*(\Lambda)/J$.

To prove part (b) we want to get a homomorphism on $\mathcal{T}C^*(\Lambda)$ and then prove it factors through a homomorphism of $\mathcal{T}C^*(\Lambda)/J$. We then need to prove this homomorphism has the required properties.

Because $\{T_\lambda\}$ is a Cuntz-Krieger Λ-family in B, it is a Toeplitz-Cuntz-Krieger Λ-family in B, and the universal property of $(\mathcal{T}C^*(\Lambda), \{t_\lambda\})$ gives a homomorphism $\pi_T : \mathcal{T}C^*(\Lambda) \to B$ such that $\pi_T(t_\lambda) = T_\lambda$. Because π_T is a homomorphism, $\ker \pi_T$ is a closed ideal. Since $\{T_\lambda\}$ is a Cuntz-Krieger Λ-family, we have

$$\pi_T\Big(t_v - \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^*\Big) = T_v - \sum_{\lambda \in v\Lambda^n} T_\lambda T_\lambda^* = 0,$$

so $t_v - \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^* \in \ker \pi_T$. Then, since J is the closed ideal generated by $\{t_v - \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^*\}$ it is the smallest closed ideal containing $\{t_v - \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^*\}$. So $J \subseteq \ker \pi_T$. Therefore there exists a homomorphism $\overline{\pi}_T : \mathcal{T}C^*(\Lambda)/J \to B$ such that $\pi_T = \overline{\pi}_T \circ q$.

Finally, we check it has the required property. Fix $\bar{t}_\lambda \in \mathcal{T}C^*(\Lambda)/J$. Then

$$\overline{\pi}_T(\bar{t}_\lambda) = \overline{\pi}_T(q(t_\lambda)) = \pi_T(t_\lambda) = T_\lambda. \qquad \square$$

Remark 5.5. Since $(\mathcal{T}C^*(\Lambda)/J, \{\bar{t}_\lambda\})$ has the universal property which determines the Cuntz-Krieger algebra, $(\mathcal{T}C^*(\Lambda)/J, \{\bar{t}_\lambda\})$ is canonically isomorphic to the Cuntz-Krieger algebra. From now on we use this isomorphism to identify $C^*(\Lambda)$ with this quotient and $\ker q$ with the ideal J.

5.3 A generalised action α^y

View $[0, \infty)$ as a category (with one object, morphisms $[0, \infty)$ and composition defined by addition). Then a *weight functor* $y : \Lambda \to [0, \infty)$ is a functor and by definition it satisfies the following relations:

(Y1) $y(\iota_v) = 0$ for all $v \in \Lambda^0$, and

(Y2) $y(\mu\nu) = y(\mu) + y(\nu)$ for all $\mu, \nu \in \Lambda$.

A *traversing path* $\lambda_1 \dots \lambda_{|d(\lambda)|} \in \Lambda$ for $\lambda \in \Lambda$ is a set of $\lambda_n \in \Lambda^{e_i}$ with $i \in \{1, \dots, k\}$ such that $\lambda = \lambda_1 \dots \lambda_{|d(\lambda)|}$. We first want to show that it suffices to define y on $\lambda_n \in \Lambda^{e_i}$ with $i \in \{1, \dots, k\}$. That is, if we define y on $\lambda_n \in \Lambda^{e_i}$ consistent with the factorisation property, then for all $\lambda \in \Lambda$, $y(\lambda)$ is the same no matter how we decompose λ to traversing paths.

Lemma 5.6. *Suppose that $y : \bigcup_i \Lambda^{e_i} \to [0,\infty)$ satisfies $y(f) + y(g) = y(g') + y(f')$ for all commuting squares $fg = g'f'$ in Λ. Suppose that $\mu_1 \dots \mu_N = \nu_1 \dots \nu_N$ are compositions of edges. Then $\sum_{n=1}^N y(\mu_n) = \sum_{n=1}^N y(\nu_n)$.*

Proof. For clarity we use the following notation in this proof. For $\lambda \in \Lambda$ and $n \leq d(\lambda)$, by the factorisation property we can write λ uniquely as $\lambda = \lambda_1\lambda_2$ with $d(\lambda_1) = n$ and $d(\lambda_2) = d(\lambda) - n$. We write $\lambda_1 = \lambda\big(0, n\big)$ and $\lambda_2 = \lambda\big(n, d(\lambda)\big)$.

We proceed by induction on N. When $N = 1$ the result is trivial. Assume that the result is true for $K \leq N$ and that $\mu_1 \dots \mu_{K+1} = \mu'_1 \dots \mu'_{K+1} = \mu$, say. If $d(\mu_{K+1}) = d(\mu'_{K+1}) = e_i$, then unique factorisation gives $\mu_{K+1} = \mu\big(d(\mu) - e_i, d(\mu)\big) = \mu'_{K+1}$. So suppose that $d(\mu_{K+1}) = e_i$ and $d(\mu'_{K+1}) = e_j$ with $i \neq j$. Factorise $\mu\big(0, d(\mu) - e_i\big) = \mu\big(0, d(\mu) - e_i - e_j\big)f$, $\mu\big(0, d(\mu) - e_j\big) = \mu\big(0, d(\mu) - e_i - e_j\big)e$ with $d(e) = e_i$ and $d(f) = e_j$. Then $\mu\big(0, d(\mu) - e_i\big) = \mu_1 \dots \mu_K$ and the inductive hypothesis implies that $y\big(\mu\big(0, d(\mu) - e_i - e_j\big)\big) + y(f) = \sum_{n=1}^K y(\mu_n)$. Similarly $y\big(\mu\big(0, d(\mu) - e_i - e_j\big)\big) + y(e) = \sum_{n=1}^K y(\mu'_n)$.

Now, since $f\mu_{K+1} = e\mu'_{K+1}$ is a commuting square in Λ, our hypothesis on y tells us $y(f) + y(\mu_{K+1}) = y(e) + y(\mu'_{K+1})$. Thus

$$\begin{aligned}
\sum_{n=1}^{K+1} y(\mu_n) &= \sum_{n=1}^{K} y(\mu_n) + y(\mu_{K+1}) \\
&= y\big(\mu\big(0, d(\mu) - e_i - e_j\big)\big) + y(f) + y(\mu_{K+1}) \\
&= y\big(\mu\big(0, d(\mu) - e_i - e_j\big)\big) + y(e) + y(\mu'_{K+1}) \\
&= \sum_{n=1}^{K} y(\mu'_n) + y(\mu'_{K+1}) \\
&= \sum_{n=1}^{K+1} y(\mu'_n).
\end{aligned}$$

So the inductive hypothesis holds for $K + 1$. □

Proposition 5.7. *Suppose that $y : \bigcup_i \Lambda^{e_i} \to [0,\infty)$ satisfies $y(f) + y(g) = y(g') + y(f')$ for all commuting squares $fg = g'f'$ in Λ. Define $y : \Lambda \to [0,\infty)$ by $y(\lambda) := \sum_{n=1}^{|d(\lambda)|} y(\lambda_n)$ for every traversing path $\lambda_1 \dots \lambda_{|d(\lambda)|} \in \Lambda$. Then y is a weight functor.*

Proof. Lemma 5.6 tells us that y is well-defined, so we just need to show that y is a functor. That y preserves identity morphisms is trivial. To see that y preserves composition of morphisms fix $\mu, \nu \in \Lambda$. Choose traversing paths $\mu_1 \dots \mu_{|d(\mu)|}$ for μ and $\nu_1 \dots \nu_{|d(\nu)|}$ for ν. Then

$$\lambda_1 \dots \lambda_{|d(\mu\nu)|} = \mu_1 \dots \mu_{|d(\mu)|}\nu_1 \dots \nu_{|d(\nu)|}$$

is a traversing path for $\mu\nu$. Since y is well-defined,

$$y(\mu\nu) = \sum_{i=1}^{|d(\mu\nu)|} y(\lambda_i) = \sum_{m=1}^{|d(\mu)|} y(\mu_m) + \sum_{n=1}^{|d(\nu)|} y(\nu_n) = y(\mu) + y(\nu). \qquad \square$$

We now use this weight functor to get a generalised gauge dynamics α^y of $\mathbb{R}$ on $\mathcal{T}C^*(\Lambda)$. As for directed graphs, we define our action directly on $\mathbb{R}$ rather than on $\mathbb{T}$.

Proposition 5.8. *Let (Λ, d) be a k-graph and choose a weight functor $y : \Lambda \to [0, \infty)$. Then there is an action $\alpha^y : r \mapsto \alpha^y_r$ of $\mathbb{R}$ on $\mathcal{T}C^*(\Lambda)$ such that*

$$\alpha^y_r(t_\lambda) = e^{iry(\lambda)}t_\lambda \textit{ for every } \lambda \in \Lambda.$$

Proof. Fix $r \in \mathbb{R}$. We first want to apply Lemma 5.2 to show that there exists an automorphism α^y_r of $\mathcal{T}C^*(\Lambda)$ such that $\alpha_r(t_\lambda) = e^{iry(\lambda)}t_\lambda$.

We claim $\{e^{iry(\lambda)}t_\lambda : \lambda \in \Lambda\}$ is a Toeplitz-Cuntz-Krieger Λ-family. Since $\{t_\lambda : \lambda \in \Lambda\}$ are partial isometries, so are $\{e^{iry(\lambda)}t_\lambda : \lambda \in \Lambda\}$. To see (T1) is true fix $v \in \Lambda^0$. By (Y1) $e^{iry(v)} = 1$ for $v \in \Lambda^0$, so

$$(e^{iry(v)}t_v)^2 = t_v^2 = t_v,$$

and

$$(e^{iry(v)}t_v)^* = t_v^* = t_v.$$

Thus $\{e^{iry(v)}t_v : v \in \Lambda^0\}$ are projections. In addition, they are mutually orthogonal: for $w \in \Lambda^0$, $w \neq v$ we have $(e^{iry(v)}t_v)(e^{iry(w)}t_w) = t_v t_w = 0$. For (T2) fix $\lambda, \mu \in \Lambda$ such that $s(\lambda) = r(\mu)$. Then, since $\lambda \mapsto e^{iry(\lambda)}$ is a functor,

$$(e^{iry(\lambda)}t_\lambda)(e^{iry(\mu)}t_\mu) = e^{iry(\lambda\mu)}t_\lambda t_\mu = e^{iry(\lambda\mu)}t_{\lambda\mu}.$$

For (T3) fix $\lambda \in \Lambda$. Then, since $e^{iry(\lambda)} \in \mathbb{T}$,

$$(e^{iry(\lambda)}t_\lambda)^*(e^{iry(\lambda)}t_\lambda) = t_\lambda^* e^{-iry(\lambda)}e^{iry(\lambda)}t_\lambda = t_\lambda^* t_\lambda = t_{s(\lambda)} = e^{iry(s(\lambda))}t_{s(\lambda)}.$$

For (T4) fix $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. Then, since $e^{iry(\lambda)} \in \mathbb{T}$,

$$\begin{aligned}\sum_{\lambda \in v\Lambda^n} (e^{iry(\lambda)} t_\lambda)(e^{iry(\lambda)} t_\lambda)^* &= \sum_{\lambda \in v\Lambda^n} e^{iry(\lambda)} t_\lambda t_\lambda^* e^{-iry(\lambda)} \\ &= \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^* \\ &\leq t_v \\ &= e^{iry(v)} t_v.\end{aligned}$$

Finally, for (T5) fix $\mu, \nu \in \Lambda^0$. Then, since $\lambda \mapsto e^{iry(\lambda)}$ is a functor and $e^{iry(\lambda)} \in \mathbb{T}$,

$$\begin{aligned}&\sum_{(\eta,\zeta) \in \Lambda^{\min}(\mu,\nu)} e^{iry(\eta)} t_\eta (e^{iry(\zeta)} t_\zeta)^* \\ &\qquad = e^{-iry(\mu)} e^{iry(\mu)} \Big(\sum_{(\eta,\zeta) \in \Lambda^{\min}(\mu,\nu)} e^{iry(\eta)} t_\eta t_\zeta^* e^{-iry(\zeta)} \Big) e^{-iry(\nu)} e^{iry(\nu)} \\ &\qquad = e^{-iry(\mu)} \Big(\sum_{(\eta,\zeta) \in \Lambda^{\min}(\mu,\nu)} e^{iry(\mu)} e^{iry(\eta)} t_\eta t_\zeta^* e^{-iry(\zeta)} e^{-iry(\nu)} \Big) e^{iry(\nu)} \\ &\qquad = e^{-iry(\mu)} \Big(\sum_{(\eta,\zeta) \in \Lambda^{\min}(\mu,\nu)} e^{iry(\mu\eta)} e^{-iry(\nu\zeta)} t_\eta t_\zeta^* \Big) e^{iry(\nu)}.\end{aligned}$$

Thus, since $(\eta, \zeta) \in \Lambda^{\min}(\mu, \nu)$ implies $\mu\eta = \nu\zeta$,

$$\begin{aligned}&\sum_{(\eta,\zeta) \in \Lambda^{\min}(\mu,\nu)} e^{iry(\eta)} t_\eta t_\zeta^* e^{-iry(\zeta)} \\ &\qquad = e^{-iry(\mu)} \Big(\sum_{(\eta,\zeta) \in \Lambda^{\min}(\mu,\nu)} e^{iry(\mu\eta)} e^{-iry(\mu\eta)} t_\eta t_\zeta^* \Big) e^{iry(\nu)} \\ &\qquad = e^{-iry(\mu)} \Big(\sum_{(\eta,\zeta) \in \Lambda^{\min}(\mu,\nu)} t_\eta t_\zeta^* \Big) e^{iry(\nu)} \\ &\qquad = e^{-iry(\mu)} t_\mu^* t_\nu e^{iry(\nu)}.\end{aligned}$$

Thus $\{e^{iry(\lambda)} t_\lambda : \lambda \in \Lambda\}$ is a Toeplitz-Cuntz-Krieger Λ-family, as claimed.

We next need to show that for every Toeplitz-Cuntz-Krieger Λ-family $\{T\}$ in a C^*-algebra B there exists a homomorphism $\rho_T : \mathcal{T}C^*(\Lambda) \to B$ satisfying $\rho_T(e^{iry(\lambda)} t_\lambda) = T_\lambda$. Fix a Toeplitz-Cuntz-Krieger Λ-family $\{T\}$. Then $\{e^{-iry(\lambda)} T_\lambda\}$ is also a Toeplitz-Cuntz-Krieger Λ-family. Then the universal property [33, Corollary 7.5] gives us a homomorphism $\pi_{e^{-iry(\lambda)} T_\lambda}$ such that $\pi_{e^{-iry(\lambda)} T_\lambda}(t_\lambda) = e^{-iry(\lambda)} T_\lambda$. Therefore,

$$\pi_{e^{-iry(\lambda)} T_\lambda}(e^{iry(\lambda)} t_\lambda) = e^{iry(\lambda)} (\pi_{e^{-iry(\lambda)} T_\lambda}(t_\lambda)) = e^{iry(\lambda)} (e^{-iry(\lambda)} T_\lambda) = T_\lambda.$$

Thus taking $\rho_T := \pi_{e^{-iry(\lambda)} T_\lambda}$ we have our required homomorphism, so Lemma 5.2 gives an isomorphism $\alpha_r^y : \mathcal{T}C^*(\Lambda) \to \mathcal{T}C^*(\Lambda)$ such that

$$\alpha(t_\lambda) = e^{iry(\lambda)} t_\lambda.$$

We now show that $r \mapsto \alpha^y_r$ is a homomorphism of $\mathbb{R}$ into $\operatorname{Aut} \mathcal{T}C^*(\Lambda)$. Fix $\lambda \in \Lambda$. For $r, x \in \mathbb{R}$, the isomorphisms $\alpha^y_r \circ \alpha^y_x$ and α^y_{r+x} agree on the generators t_λ since

$$\begin{aligned}
(\alpha^y_r \circ \alpha^y_x)(t_\lambda) &= \alpha^y_r(\alpha^y_x(t_\lambda)) \\
&= \alpha^y_r(e^{ixy(\lambda)} t_\lambda) \\
&= e^{iry(\lambda)} e^{ixy(\lambda)} t_\lambda \\
&= e^{i(r+x)y(\lambda)} t_\lambda \\
&= \alpha^y_{r+x}(t_\lambda).
\end{aligned}$$

Therefore they agree on all of $\mathcal{T}C^*(\Lambda)$, so α^y is a homomorphism of $\mathbb{R}$ into $\operatorname{Aut} \mathcal{T}C^*(\Lambda)$.

Finally, we need to show that α^y is continuous. Fix $r \in \mathbb{R}$, $a \in \mathcal{T}C^*(\Lambda)$ and $\epsilon > 0$. Choose a finite linear combination $c = \sum \eta_{\mu,\nu} t_\mu t_\nu^*$, such that $\|a - c\| < \frac{\epsilon}{3}$. Then

$$\begin{aligned}
x \mapsto \alpha^y_x(c) &= \alpha^y_x\Big(\sum \eta_{\mu,\nu} t_\mu t_\nu^*\Big) \\
&= \sum \eta_{\mu,\nu} \alpha^y_x(t_\mu t_\nu^*) \\
&= \sum \eta_{\mu,\nu} \alpha^y_x(t_\mu) \alpha^y_x(t_\nu^*) \\
&= \sum \eta_{\mu,\nu} e^{ixy(\mu)} t_\mu e^{ixy(\nu)} t_\nu^* \\
&= \sum \eta_{\mu,\nu} e^{ix(y(\mu)+y(\nu))} t_\mu t_\nu^*,
\end{aligned}$$

which is continuous because scalar multiplication is continuous. So there exists $\delta > 0$ such that

$$|x - r| < \delta \Rightarrow \|\alpha^y_x(c) - \alpha^y_r(c)\| < \frac{\epsilon}{3}.$$

Since automorphisms of C^*-algebras preserve the norm, we have $\|\alpha^y_r(a-c)\| < \frac{\epsilon}{3}$. Thus for $|x - r| < \delta$ we have

$$\|\alpha^y_x(a) - \alpha^y_r(a)\| \leq \|\alpha^y_x(a-c)\| + \|\alpha^y_x(c) - \alpha^y_r(c)\| + \|\alpha^y_r(a-c)\| < 3\Big(\frac{\epsilon}{3}\Big) = \epsilon.$$

Thus α^y is continuous, as required. □

5.4 Characterising KMS states of $(\mathcal{T}C^*(\Lambda), \alpha^y)$

In [18, Proposition 3.1] the KMS states of $(\mathcal{T}C^*(\Lambda), \alpha)$ are characterised. In this section we apply this method to characterise the KMS states of $(\mathcal{T}C^*(\Lambda), \alpha^y)$.

We first get a condition to describe the KMS states. For every $\mu, \nu \in \Lambda$, the function $r \mapsto \alpha^y_r(t_\mu t_\nu^*) = e^{ir(y(\mu)-y(\nu))} t_\mu t_\nu^*$ is the restriction of the entire function

$$z \mapsto e^{iz(y(\mu)-y(\nu))} t_\mu t_\nu^*.$$

The elements $t_\mu t_\nu^*$ are therefore analytic. Since they span a dense subspace of $\mathcal{T}C^*(\Lambda)$, it follows from [31, Proposition 8.12.3] that a state ϕ of $(\mathcal{T}C^*(\Lambda), \alpha^y)$ is a KMS$_\beta$ state for $\beta \in (0, \infty)$ if and only if

$$\phi((t_\mu t_\nu^*)(t_\sigma t_\tau^*)) = \phi((t_\sigma t_\tau^*)\alpha^y_{i\beta}(t_\mu t_\nu^*))$$

for all $\mu, \nu, \sigma, \tau \in \Lambda$.

Proposition 5.9. *Suppose that Λ is a finite k-graph with no sources. Let y be a weight functor. Let $\alpha^y : \mathbb{R} \to \mathcal{T}C^*(\Lambda)$ be the action given by Proposition 5.8. Suppose that $\beta \in (0, \infty)$ and ϕ is a state on $\mathcal{T}C^*(\Lambda)$.*

(a) If ϕ is a KMS$_\beta$ state of $(\mathcal{T}C^*(\Lambda), \alpha^y)$, then

$$\phi(t_\mu t_\nu^*) = \delta_{\mu,\nu} e^{-\beta y(\mu)} \phi(t_{s(\mu)}) \qquad \textit{for all } \mu, \nu \in \Lambda \textit{ with } d(\mu) = d(\nu). \tag{5.1}$$

(b) If

$$\phi(t_\mu t_\nu^*) = \delta_{\mu,\nu} e^{-\beta y(\mu)} \phi(t_{s(\mu)}) \qquad \textit{for all } \mu, \nu \in \Lambda, \tag{5.2}$$

then ϕ is a KMS$_\beta$ state of $(\mathcal{T}C^*(\Lambda), \alpha^y)$.

(c) If $y(\mu) = y(\nu)$ implies that $d(\mu) = d(\nu)$ for all $\mu, \nu \in \Lambda$, then ϕ is a KMS$_\beta$ state of $(\mathcal{T}C^*(\Lambda), \alpha^y)$ if and only if Equation (5.2) *holds.*

Proof. (a) Fix $\mu, \nu \in \Lambda$ such that $d(\mu) = d(\nu)$. Then $t_\nu^* t_\mu = \delta_{\mu,\nu} t_{s(\mu)}$. Then, since ϕ is a KMS$_\beta$ state, the KMS condition tells us that

$$\begin{aligned}
\phi(t_\mu t_\nu^*) &= \phi(t_\nu^* \alpha^y_{i\beta}(t_\mu)) \\
&= \phi(t_\nu^* e^{-\beta y(\mu)} t_\mu) \\
&= e^{-\beta y(\mu)} \phi(t_\nu^* t_\mu) \\
&= e^{-\beta y(\mu)} \phi(\delta_{\mu,\nu} t_{s(\mu)}) \\
&= \delta_{\mu,\nu} e^{-\beta y(\mu)} \phi(t_{s(\mu)}).
\end{aligned}$$

(b) Suppose that ϕ satisfies Equation (5.2), and consider two spanning elements $t_\mu t_\nu^*$ and $t_\sigma t_\tau^*$ with $s(\mu) = s(\nu)$ and $s(\sigma) = s(\tau)$ (if $s(\mu) \neq s(\nu)$, then $t_\mu t_\nu^* = t_\mu t_{s(\mu)} t_{s(\nu)} t_\nu^* = 0$). We want to verify the KMS condition, that is, show that

$$\phi(t_\mu t_\nu^* t_\sigma t_\tau^*) = \phi(t_\sigma t_\tau^* \alpha^y_{i\beta}(t_\mu t_\nu^*)) = e^{-\beta(y(\mu) - y(\nu))} \phi(t_\sigma t_\tau^* t_\mu t_\nu^*).$$

Equation (5.2) applied to (T5) implies that

$$\phi(t_\mu t_\nu^* t_\sigma t_\tau^*) = \sum_{(\alpha,\eta)\in\Lambda^{\min}(\nu,\sigma)} \phi(t_{\mu\alpha} t_{\tau\eta}^*)$$

$$= \sum_{(\alpha,\eta)\in\Lambda^{\min}(\nu,\sigma)} \delta_{\mu\alpha,\tau\eta} e^{-\beta y(\mu\alpha)} \phi(t_{s(\alpha)})$$

$$(5.3) \qquad = \sum_{(\alpha,\eta)\in\Lambda^{\min}(\nu,\sigma),\ \mu\alpha=\tau\eta} e^{-\beta y(\mu\alpha)} \phi(t_{s(\alpha)}).$$

Similarly,

$$(5.4) \qquad \phi(t_\sigma t_\tau^* t_\mu t_\nu^*) = \sum_{(\gamma,\zeta)\in\Lambda^{\min}(\tau,\mu),\ \sigma\gamma=\nu\zeta} e^{-\beta y(\sigma\gamma)} \phi(t_{s(\gamma)}).$$

We need to show that the indexing sets in the sums in Equation (5.3) and Equation (5.4) are closely related. To see this, suppose that $(\alpha,\eta) \in \Lambda^{\min}(\nu,\sigma)$ satisfies $\mu\alpha = \tau\eta$. Since $(\alpha,\eta) \in \Lambda^{\min}(\nu,\sigma)$, we have $d(\alpha)\wedge d(\eta) = 0$, because otherwise the extension is not minimal. [18, Lemma 3.2] tells us that since $d(\mu)+d(\alpha) = d(\tau)+d(\eta)$ and $d(\alpha)\wedge d(\eta) = 0$ we have $d(\mu)+d(\alpha) = d(\mu)\vee d(\tau)$; hence (η,α) belongs to $\Lambda^{\min}(\tau,\mu)$. The situation is symmetric, so we deduce that the map $(\alpha,\eta)\mapsto(\eta,\alpha)$ is a bijection of the index set in Equation (5.3) onto the index set in Equation (5.4).

Fix (α,η) in the index set in Equation (5.3). Since $s(\alpha) = s(\eta)$, $t_{s(\alpha)} = t_{s(\eta)}$. To verify the KMS condition, it therefore suffices for us to see that the summand of Equation (5.3) with index (α,η) and the summand of $e^{-\beta(y(\mu)-y(\nu))}\phi(t_\sigma t_\tau^* t_\mu t_\nu^*)$ with index (η,α) have the same coefficient. That is, we need to show that

$$e^{-\beta y(\mu\alpha)} = e^{-\beta(y(\mu)-y(\nu))} e^{-\beta y(\sigma\eta)}.$$

Since $\sigma\eta = \nu\alpha$ in the summand of $e^{-\beta(y(\mu)-y(\nu))}\phi(t_\sigma t_\tau^* t_\mu t_\nu^*)$,

$$\begin{aligned} e^{-\beta(y(\mu)-y(\nu))} e^{-\beta y(\sigma\eta)} &= e^{-\beta(y(\mu)-y(\nu))} e^{-\beta y(\nu\alpha)} \\ &= e^{-\beta(y(\mu)-y(\nu)+y(\nu\alpha))} \\ &= e^{-\beta(y(\mu)-y(\nu)+y(\nu)+y(\alpha))} \\ &= e^{-\beta(y(\mu)+y(\alpha))} \\ &= e^{-\beta y(\mu\alpha)}. \end{aligned}$$

Thus ϕ is a KMS$_\beta$ state.

(c) Let ϕ be a KMS$_\beta$ state. Take $\mu,\nu\in\Lambda$ with $s(\mu) = s(\nu)$. If $d(\mu) = d(\nu)$, then part (a) gives our result. Suppose that $d(\mu) \neq d(\nu)$. Applying the KMS condition twice gives

$$\begin{aligned}\phi(t_\mu t_\nu^*) &= \phi(t_\nu^* \alpha_{i\beta}(t_\mu)) \\ &= e^{-\beta y(\mu)}\phi(t_\nu^* t_\mu) \\ &= e^{-\beta y(\mu)}\phi(t_\mu \alpha_{i\beta}(t_\nu^*)) \\ &= e^{-\beta y(\mu)} e^{\beta y(\nu)} \phi(t_\mu t_\nu^*).\end{aligned}$$

Since $d(\mu) \neq d(\nu)$, we have $y(\mu) \neq y(\nu)$, hence $e^{\beta y(\mu)} \neq e^{\beta y(\nu)}$. Then $\phi(t_\mu t_\nu^*) = 0$. Thus ϕ satisfies Equation (5.2).

Conversely, part (b) shows that if Equation (5.2) holds then ϕ is a KMS state. $\square$

5.5 Generalised vertex matrices $B_i(y, \theta)$

In this section we present a generalised version of the vertex matrices A_i, which we will use in our analysis of the KMS states of $(\mathcal{T}C^*(\Lambda), \alpha^y)$ and $(C^*(\Lambda), \overline{\alpha}^y)$.

Definition 5.10. Suppose that Λ is a finite k-graph. Choose a weight functor $y : \Lambda \to [0, \infty)$ and $\theta \in [0, \infty)$. For $i \in \{1, \ldots, k\}$ let $B_i(y, \theta) = (B_i(y, \theta)_{v,w})$ be the $\Lambda^0 \times \Lambda^0$ matrix with entries

$$B_i(y, \theta)_{v,w} = \sum_{\lambda \in v\Lambda^{e_i} w} e^{-\theta y(\lambda)},$$

where we take $B_i(y, \theta)_{v,w} = 0$ if $v\Lambda^{e_i} w = \emptyset$.

We now present results about these matrices which will be used in our analysis of the KMS states of $(\mathcal{T}C^*(\Lambda), \alpha^y)$ and $(C^*(\Lambda), \overline{\alpha}^y)$.

Lemma 5.11. *Suppose that Λ is a finite k-graph. Choose a weight functor $y : \Lambda \to [0, \infty)$. For $\theta \in [0, \infty)$ and $i \in \{1, \ldots, k\}$ let $B_i(y, \theta)$ be the matrices from Definition 5.10. Then*

$$\{B_i(y, \theta) : i \in \{1, \ldots, k\}\}$$

pairwise commute.

Proof. Fix $i, j \in \{1, \ldots, k\}$. We want to show that $B_i(y, \theta)B_j(y, \theta) = B_j(y, \theta)B_i(y, \theta)$.

Fix $v, w \in \Lambda^0$. Then

$$\begin{aligned}(B_i(y,\theta)B_j(y,\theta))_{v,w} &= \sum_{u\in\Lambda^0} B_i(y,\theta)_{v,u}B_j(y,\theta)_{u,w} \\ &= \sum_{u\in\Lambda^0}\sum_{\lambda\in v\Lambda^{e_i}u}\sum_{\sigma\in u\Lambda^{e_j}w} e^{-\theta(y(\lambda)+y(\sigma))} \\ &= \sum_{\mu\in v\Lambda^{e_i+e_j}w} e^{-\theta y(\mu)}.\end{aligned}$$

Similarly,

$$(B_j(y,\theta)B_i(y,\theta))_{v,w} = \sum_{\nu\in v\Lambda^{e_j+e_i}w} e^{-\theta y(\nu)}.$$

The factorisation property tells us that $\Lambda^{e_i+e_j} = \Lambda^{e_j+e_i}$, thus elements of the set $\{B_i(y,\theta) : i \in \{1,\dots,k\}\}$ pairwise commute. □

Before the next definition we observe that since the matrices $B_i(y,\theta)$ commute we can unambiguously form this product.

Definition 5.12. Suppose that Λ is a finite k-graph. Choose a weight functor $y : \Lambda \to [0,\infty)$. For $\theta \in [0,\infty)$ and $i \in \{1,\dots,k\}$ let $B_i(y,\theta)$ be the matrices from Definition 5.10. Fix $n = (n_1,\dots,n_k) \in \mathbb{N}^k$ and define the matrix

$$B(y,\theta)^n := \prod_{i=1}^{k} B_i(y,\theta)^{n_i}.$$

It follows from this definition that

$$B(y,\theta)^n_{v,w} = \sum_{\lambda\in v\Lambda^n w} e^{-\theta y(\lambda)}.$$

Lemma 5.13. *Suppose that Λ is a coordinatewise irreducible finite k-graph. Choose a weight functor $y : \Lambda \to [0,\infty)$. For $\theta \in [0,\infty)$ and $i \in \{1,\dots,k\}$ let $B_i(y,\theta)$ be the matrices from Definition 5.10. Then the matrices $B_i(y,\theta)$ are irreducible.*

Proof. Fix $i \in \{1,\dots,k\}$. Since Λ is coordinatewise irreducible the vertex matrix A_i is irreducible. Therefore the coordinate graph $(\Lambda^0, \Lambda^{e_i}, r, s)$ is strongly connected. Thus applying Lemma 3.5 tells us $B_i(y,\theta)$ is irreducible. □

We can now prove a version of [18, Lemma 2.2], which we will use later in finding KMS_β states.

Proposition 5.14. *Suppose that $e\Lambda$ is a finite k-graph. Choose a weight functor $y : \Lambda \to [0, \infty)$. For $\theta \in [0, \infty)$ and $i \in \{1, \dots, k\}$ let $B_i(y, \theta)$ be the matrices from Definition 5.10. Suppose that $\beta \in (0, \infty)$ such that $\rho(B_i(y, \beta)) < 1$ for $i \in \{1, \dots, k\}$. Then the series $\sum_{n \in \mathbb{N}^k} B(y, \beta)^n$ converges in the operator norm to $\prod_{i=1}^{k}(I - B_i(y, \beta))^{-1}$.*

Proof. By Lemma 5.11 the matrices $B_i(y, \beta)$ commute, so the N^{th} partial sum is

$$\sum_{0 \le n \le N} B(y, \beta)^n = \sum_{0 \le n \le N} \prod_{i=1}^{k} B_i(y, \beta)^{n_i} = \prod_{i=1}^{k} \sum_{n_i=0}^{N_i} B_i(y, \beta)^{n_i}.$$

For each $i \in \{1, \dots, k\}$ we have $\rho(B_i(y, \beta)) < 1$, and hence by Corollary A.14,

$$\sum_{n_i=0}^{N_i} B_i(y, \beta)^{n_i}$$

converges to $(I - B_i(y, \beta))^{-1}$ in the operator norm as $N_i \to \infty$. Thus as $N \to \infty$ in $\mathbb{N}^k$, which means $N_i \to \infty$ for all $i \in \{1, \dots, k\}$, the product converges in the operator norm to $\prod_{i=1}^{k}(I - B_i(y, \beta))^{-1}$. $\square$

We can then generalise Proposition 3.6 to higher-rank graphs.

Theorem 5.15. *Suppose that Λ is a coordinatewise irreducible finite k-graph. Let $y : \Lambda \to [0, \infty)$ be a weight functor. For $\theta \in [0, \infty)$ and $i \in \{1, \dots, k\}$ let $B_i(y, \theta)$ be the matrices from Definition 5.10. Then, for each $i \in \{1, \dots, k\}$, the function $\theta \mapsto \rho(B_i(y, \theta))$ is strictly decreasing and there exists a unique $\beta_i \in [0, \infty)$ such that*

$$\rho(B_i(y, \beta_i)) = 1.$$

Proof. Fix $i \in \{1, \dots, k\}$ and apply Proposition 3.6. $\square$

5.6 KMS states of $(\mathcal{T}C^*(\Lambda), \alpha^y)$ and the subinvariance relation

In this section we describe a subinvariance relation like that used in [18, Section 4].

Proposition 5.16. *Suppose that Λ is a finite k-graph with no sources. Let $y : \Lambda \to [0, \infty)$ be a weight functor. For $\theta \in [0, \infty)$ and $i \in \{1, \dots, k\}$ let $B_i(y, \theta)$ be the matrices from Definition 5.10. Let $\alpha^y : \mathbb{R} \to \mathcal{T}C^*(\Lambda)$ be the action given by Proposition 5.8. Fix $\beta \in (0, \infty)$ and let ϕ be a KMS$_\beta$ state on $(\mathcal{T}C^*(\Lambda), \alpha^y)$. Define $m^\phi = (m^\phi_v)$ by $m^\phi_v = \phi(t_v)$. Then $m^\phi \in [0, \infty)^{\Lambda^0}$, $\|m^\phi\|_1 = 1$, and for every subset K of $\{1, \dots, k\}$ we have $\prod_{i \in K}(I - B_i(y, \beta))m^\phi \ge 0$ pointwise.*

Proof. First, because ϕ is positive, $m_v = \phi(t_v) \geq 0$ for all $v \in \Lambda^0$. Thus $m^\phi \in [0, \infty)^{\Lambda^0}$. Next, because $\mathcal{T}C^*(\Lambda)$ is finite, $\sum_{v \in \Lambda^0} t_v = 1$, so we have

$$\|m^\phi\|_1 = \sum_{v \in \Lambda^0} m_v = \sum_{v \in \Lambda^0} \phi(t_v) = \phi(1) = 1.$$

Fix $K \subseteq \{1, \ldots, k\}$, $J \subseteq K$ and $v \in \Lambda^0$. Write $e_J := \sum_{j \in J} e_j$ and

$$t_J = \sum_{\mu \in v\Lambda^{e_J}} t_\mu t_\mu^*,$$

with $t_i = t_{\{i\}}$. Then for $i \in K$, (T4) tells us that $t_v \geq t_i$, and since all the range projections commute, we have $\prod_{i \in K}(t_v - t_i) \geq 0$. Therefore $\phi(\prod_{i \in K}(t_v - t_i)) \geq 0$. Then, [18, Lemma 4.2] tells us

$$\prod_{i \in K}(t_v - t_i) = \sum_{J \subseteq K} (-1)^{|J|} t_J,$$

so

$$\begin{aligned}
0 &\leq \phi\Big(\sum_{J \subseteq K} (-1)^{|J|} t_J\Big) \\
&= \sum_{J \subseteq K} (-1)^{|J|} \phi(t_J) \\
&= \sum_{J \subseteq K} (-1)^{|J|} \Big(\sum_{\mu \in v\Lambda^{e_J}} \phi(t_\mu t_\mu^*) \Big).
\end{aligned}$$

Thus, by Equation (5.1),

$$\begin{aligned}
0 &\leq \sum_{J \subseteq K} (-1)^{|J|} \Big(\sum_{\mu \in v\Lambda^{e_J}} e^{-\beta y(\mu)} m_{s(\mu)} \Big) \\
&= \sum_{J \subseteq K} (-1)^{|J|} \Big(\sum_{w \in \Lambda^0} \sum_{\mu \in v\Lambda^{e_J} w} e^{-\beta y(\mu)} m_{s(\mu)} \Big) \\
&= \sum_{J \subseteq K} (-1)^{|J|} \Big(\sum_{w \in \Lambda^0} \Big(\sum_{\mu \in v\Lambda^{e_J} w} e^{-\beta y(\mu)} \Big) m_w \Big) \\
&= \sum_{J \subseteq K} (-1)^{|J|} \Big(\sum_{w \in \Lambda^0} B(y, \beta)^{e_J}_{v,w} m_w \Big) \\
&= \sum_{J \subseteq K} (-1)^{|J|} \Big(B(y, \beta)^{e_J} m^\phi \Big)_v \\
&= \sum_{J \subseteq K} (-1)^{|J|} \Big(\Big(\prod_{j \in J} B_j(y, \beta) \Big) m^\phi \Big)_v \\
&= \Big(\Big(\prod_{i \in K} (I - B_i(y, \beta)) \Big) m^\phi \Big)_v,
\end{aligned}$$

as required. $\square$

Corollary 5.17. *If there is a KMS$_\beta$ state of $(\mathcal{T}C^*(\Lambda), \alpha^y)$, then $0 \leq \rho(B_i(y, \beta)) \leq 1$ for $i \in \{1, \dots, k\}$.*

Proof. We follow the the method of [18, Corollary 4.3]. Define $m^\phi = (m^\phi_v)$ by $m^\phi_v = \phi(t_v)$. Applying Proposition 5.16(a) to the singleton sets $K = \{i\}$ shows that $(I - B_i(y, \beta))m^\phi \geq 0$ for $i \in \{1, \dots, k\}$. This says that for each $i \in \{1, \dots, k\}$, $m^\phi \geq B_i(y, \beta)m^\phi$ pointwise, therefore Proposition A.9 implies that $\rho(B_i(y, \beta)) \leq 1$. □

5.7 KMS states of $(\mathcal{T}C^*(\Lambda), \alpha^y)$ at large inverse temperatures

In [18, Section 5] the assumption of rational independence in [18, Theorem 5.1] is dropped to strengthen the characterisation of KMS$_\beta$ states in [18, Proposition 3.1] for large β. In this section we do the same, shedding the condition that $y(\mu) = y(\nu)$ implies that $d(\mu) = d(\nu)$ for all $\mu, \nu \in \Lambda$ used in Proposition 5.9(c). This gives a characterisation of KMS$_\beta$ states of $(\mathcal{T}C^*(\Lambda), \alpha^y)$ at large inverse temperatures.

Theorem 5.18. *Suppose that Λ is a finite k-graph with no sources. Let $y : \Lambda \to [0, \infty)$ be a weight functor. For $\theta \in [0, \infty)$ and $i \in \{1, \dots, k\}$ let $B_i(y, \theta)$ be the matrices from Definition 5.10. Let $\alpha^y : \mathbb{R} \to \mathcal{T}C^*(\Lambda)$ be the action given by Proposition 5.8. Suppose that $\rho(B_i(y, \beta)) < 1$ for $i \in \{1, \dots, k\}$. Then a state ϕ on $\mathcal{T}C^*(\Lambda)$ is a KMS$_\beta$ state for α^y if and only if*

$$\phi(t_\mu t_\nu^*) = \delta_{\mu,\nu} e^{-\beta y(\mu)} \phi(t_{s(\mu)}) \quad \textit{for all } \mu, \nu \in \Lambda. \tag{5.5}$$

To prove this we first need two lemmas.

Lemma 5.19. *Suppose that ϕ is a KMS$_\beta$ state of $(\mathcal{T}C^*(\Lambda), \alpha^y)$ for some $\beta \in (0, \infty)$, and that $\mu, \nu \in \Lambda$ satisfy $s(\mu) = s(\nu)$ and $y(\mu) = y(\nu)$. Then $\phi(t_\mu t_\mu^*) = \phi(t_\nu t_\nu^*)$ and $|\phi(t_\mu t_\nu^*)| \leq \phi(t_\mu t_\mu^*)$.*

Proof. The proof follows that of [18, Lemma 5.2]. By the KMS condition,

$$\phi(t_\mu t_\mu^*) = \phi(t_\mu t_{s(\mu)} t_\mu^*) = \phi(t_\mu t_\nu^* t_\nu t_\mu^*) = e^{-\beta(y(\mu) - y(\nu))} \phi(t_\nu t_\mu^* t_\mu t_\nu^*).$$

Therefore, since $y(\mu) = y(\nu)$,

$$\phi(t_\mu t_\mu^*) = \phi(t_\nu t_\mu^* t_\mu t_\nu^*) = \phi(t_\nu t_{s(\nu)} t_\nu^*) = \phi(t_\nu t_\nu^*).$$

Finally, the Cauchy-Schwarz inequality gives

$$|\phi(t_\mu t_\nu^*)|^2 \leq \phi(t_\mu t_\mu^*) \phi(t_\nu t_\nu^*) = \phi(t_\mu t_\mu^*)^2.$$ □

Lemma 5.20. *Suppose that ϕ is a KMS$_\beta$ state of $(\mathcal{T}C^*(\Lambda), \alpha^y)$ for some $\beta \in (0,\infty)$, and that $\mu, \nu \in \Lambda$ satisfy $s(\mu) = s(\nu)$.*

(a) If $\lambda \in \Lambda$ satisfies $\Lambda^{\min}(\mu\lambda, \nu\lambda) = \emptyset$ then $\phi(t_{\mu\lambda} t^*_{\nu\lambda}) = 0$.

(b) Let $n := (d(\mu) \vee d(\nu)) - d(\mu)$. For $j \in \mathbb{N}$ we have

$$\phi(t_\mu t^*_\nu) = \sum_{\lambda \in s(\mu)\Lambda^{jn}} \phi(t_{\mu\lambda} t^*_{\nu\lambda}). \tag{5.6}$$

Proof. The proof follows that of [18, Lemma 5.3].

(a) If $\Lambda^{\min}(\mu\lambda, \nu\lambda) = \emptyset$, then (T5) implies that $t^*_{\nu\lambda} t_{\mu\lambda} = 0$, and the KMS condition gives

$$\phi(t_{\mu\lambda} t^*_{\nu\lambda}) = e^{-\beta y(\mu\lambda)} \phi(t^*_{\nu\lambda} t_{\mu\lambda}) = 0.$$

(b) We proceed by induction on j. The inductive hypothesis is "Equation (5.6) holds for j". That the inductive hypothesis holds for $j = 0$ is trivial.

For the inductive step, suppose that Equation (5.6) holds for some $j \geq 0$. We want to show that Equation (5.6) holds for $j + 1$. We start by working with the summands on the right of Equation (5.6). Part (a) says $\phi(t_{\mu\lambda} t^*_{\nu\lambda}) = 0$ for $\lambda \in s(\mu)\Lambda^{jn}$ such that $\Lambda^{\min}(\mu\lambda, \nu\lambda) = \emptyset$, so suppose $\lambda \in s(\mu)\Lambda^{jn}$ such that $\Lambda^{\min}(\mu\lambda, \nu\lambda) \neq \emptyset$. The KMS condition implies that

$$\begin{aligned}
\phi(t_{\mu\lambda} t^*_{\nu\lambda}) &= \phi(t_{\nu\lambda} t^*_{\nu\lambda} t_{\mu\lambda} t^*_{\nu\lambda}) \\
&= \phi(t_{\mu\lambda} t^*_{\nu\lambda} \alpha^y_{i\beta}(t_{\nu\lambda} t^*_{\nu\lambda})) \\
&= e^{-\beta(y(\nu\lambda) - y(\nu\lambda))} \phi(t_{\mu\lambda} t^*_{\nu\lambda} t_{\nu\lambda} t^*_{\nu\lambda}) \\
&= \phi(t_{\nu\lambda} t^*_{\nu\lambda} t_{\mu\lambda} t^*_{\nu\lambda}).
\end{aligned}$$

Applying (T5) gives

$$\begin{aligned}
\phi(t_{\mu\lambda} t^*_{\nu\lambda}) &= \phi(t_{\nu\lambda} (t^*_{\nu\lambda} t_{\mu\lambda}) t_{\nu\lambda}) \\
&= \sum_{(\eta,\zeta) \in \Lambda^{\min}(\mu\lambda,\nu\lambda)} \phi(t_{\nu\lambda\eta} t^*_{\nu\lambda\zeta}) \\
&= \sum_{(\eta,\zeta) \in \Lambda^{\min}(\mu\lambda,\nu\lambda)} \phi(t_{\mu\lambda\eta} t^*_{\nu\lambda\zeta}).
\end{aligned}$$

Combining this with the induction hypothesis gives[1]

$$\phi(t_\mu t^*_\nu) = \sum_{\lambda \in s(\mu)\Lambda^{jn}} \phi(t_{\mu\lambda} t^*_{\nu\lambda}) = \sum_{\lambda \in s(\mu)\Lambda^{jn}} \sum_{(\eta,\zeta) \in \Lambda^{\min}(\mu\lambda,\nu\lambda)} \phi(t_{\mu\lambda\eta} t^*_{\nu\lambda\zeta}). \tag{5.7}$$

[1] In [18, (5.3)], $\phi(t_{\mu\lambda} t^*_{\nu\lambda})$ on the left hand side should be $\phi(t_\mu t^*_\nu)$. Then the final equation in the proof of [18, Lemma 5.3] should read $\phi(t_\mu t^*_\nu) = \sum_{\tau \in s(\mu)\Lambda^{(j+1)n}} \phi(t_{\mu\tau} t^*_{\nu\tau})$.

For $(\eta, \zeta) \in \Lambda^{\min}(\nu\lambda, \mu\lambda)$, we have

$$d(\mu\lambda) + d(\zeta) = d(\nu\lambda) \vee d(\mu\lambda) = \big(d(\mu) \vee d(\nu)\big) + d(\lambda),$$

which implies that $d(\zeta) = \big(d(\mu) \vee d(\nu)\big) - d(\mu) = n$. Thus $d(\lambda\zeta) = (j+1)n$. Now suppose that $\tau \in s(\mu)\Lambda^{(j+1)n}$ and $\phi(t_{\mu\tau}t_{\mu\tau}^*) \neq 0$. Then part (a) implies that there exists $(\gamma, \delta) \in \Lambda^{\min}(\mu\tau, \nu\tau)$, and then $\mu\tau\gamma = \nu\tau\delta$. But then with $\lambda := \tau(0, jn)$, the paths $\zeta := \tau(jn, (j+1)n)$ and $\eta := (\tau\delta)(jn, jn + (d(\mu) \vee d(\nu)) - d(\nu))$ give a pair (η, ζ) in $\Lambda^{\min}(\nu\lambda, \mu\lambda)$ such that $\tau = \lambda\zeta$. Thus Equation (5.7) gives

$$\phi(t_\mu t_\nu^*) = \sum_{\tau \in s(\mu)\Lambda^{(j+1)n}} \phi(t_{\mu\tau} t_{\nu\tau}^*),$$

and this is Equation (5.6) for $j+1$. □

Proof of Theorem 5.18. We follow the method of [18, Theorem 5.1]. First, suppose that ϕ is a KMS$_\beta$ state. Then we want to show that ϕ satisfies Equation (5.5). Fix $\mu, \nu \in \Lambda$. We then have two cases, $s(\mu) \neq s(\nu)$ and $s(\mu) = s(\nu)$.

If $s(\mu) \neq s(\nu)$, then $t_\mu t_\nu^* = t_\mu t_{s(\mu)} t_{s(\nu)} t_\nu^* = 0$, so the left hand side of Equation (5.5) is 0, and $\mu \neq \nu$, thus $\delta_{\mu,\nu} = 0$, so the right hand side of Equation (5.5) is 0.

Otherwise $s(\mu) = s(\nu)$, and we have a further two cases, $d(\mu) = d(\nu)$ or $d(\mu) \neq d(\nu)$. If $d(\mu) = d(\nu)$ then Proposition 5.9(a) tells us that Equation (5.5) holds.

Otherwise, $d(\mu) \neq d(\nu)$ and we have a final two cases, $y(\mu) \neq y(\nu)$ or $y(\mu) = y(\nu)$.

Suppose that $y(\mu) \neq y(\nu)$. Then $e^{\beta y(\mu)} \neq e^{\beta y(\nu)}$. Applying the KMS condition twice gives

$$\begin{aligned}\phi(t_\mu t_\nu^*) &= \phi(t_\nu^* \alpha_{i\beta}(t_\mu)) \\ &= e^{-\beta y(\mu)} \phi(t_\nu^* t_\mu) \\ &= e^{-\beta y(\mu)} \phi(t_\mu \alpha_{i\beta}(t_\nu^*)) \\ &= e^{-\beta y(\mu)} e^{\beta y(\nu)} \phi(t_\mu t_\nu^*),\end{aligned}$$

and since $e^{-\beta y(\mu)} e^{\beta y(\nu)} \neq 1$, this must equal zero. So Equation (5.5) holds.

So the final case is that $y(\mu) = y(\nu)$. Since $d(\mu) \neq d(\nu)$, at least one of $(d(\mu) \vee d(\nu)) - d(\mu)$ or $(d(\mu) \vee d(\nu)) - d(\nu)$ is nonzero. Since $\phi(t_\nu t_\mu^*) = 0$ if and only if $\phi(t_\mu t_\nu^*) = 0$, we may suppose that $n := (d(\mu) \vee d(\nu)) - d(\nu)$ is nonzero. Then for $j \in \mathbb{N}$, Lemma 5.20 gives

$$\phi(t_\mu t_\nu^*) = \sum_{\lambda \in s(\mu)\Lambda^{jn}} \phi(t_{\mu\lambda} t_{\nu\lambda}^*).$$

For each $\lambda \in s(\mu)\Lambda^{jn}$ we have $y(\mu\lambda) = y(\nu\lambda)$, and hence Lemma 5.19 implies that

$$\begin{aligned}
|\phi(t_\mu t_\nu^*)| &\leq \sum_{\lambda \in s(\mu)\Lambda^{jn}} |\phi(t_{\mu\lambda} t_{\nu\lambda}^*)| \\
&\leq \sum_{\lambda \in s(\mu)\Lambda^{jn}} \phi(t_{\mu\lambda} t_{\mu\lambda}^*) \\
&= \sum_{\lambda \in s(\mu)\Lambda^{jn}} e^{-\beta y(\mu\lambda)} \phi(t_{s(\lambda)}) \\
&= e^{-\beta y(\mu)} \sum_{w \in \Lambda^0} \sum_{\lambda \in s(\mu)\Lambda^{jn} w} e^{-\beta y(\lambda)} \phi(t_w).
\end{aligned}$$

Then by the definition of $B_i(y, \beta)$,

$$|\phi(t_\mu t_\nu^*)| \leq e^{-\beta y(\mu)} \sum_{w \in \Lambda^0} \Big(\prod_{i=1}^{k} B_i(y, \beta)^{jn_i} \Big)_{s(\mu), w} \phi(t_w). \tag{5.8}$$

For each $i \in \{1, \ldots, k\}$ such that $n_i > 0$, $B_i(y, \beta)^{jn_i}$ is the $(jn_i)^{\text{th}}$ term in the series $\sum_{m=0}^{\infty} B_i(y, \beta)^m$. Since $\rho(B_i(y, \beta)) < 1$, Corollary A.14 tells us that the series converges in the operator norm to $(I - B_i(y, \beta))^{-1}$. In particular, we have $B_i(y, \beta)^{jn_i} \to 0$ as $j \to \infty$. Since $n \neq 0$ there is at least one $i \in \{1, \ldots, k\}$ such that $n_i > 0$. Thus as $j \to \infty$, Equation (5.8) converges to 0, therefore $|\phi(t_\mu t_\nu^*)| = 0$.

Conversely, assume that Equation (5.5) holds for a state ϕ on $\mathcal{T}C^*(\Lambda)$. Then Proposition 5.9(b) tells us that ϕ is a KMS_β state. $\square$

We now use this characterisation to get an isomorphism between measures and the KMS_β states of $(\mathcal{T}C^*(\Lambda), \alpha^y)$ at large inverse temperatures, by which we mean $\beta \in (0, \infty)$ such that $1 > \rho(B_i(y, \beta))$ for all $i \in \{1, \ldots, k\}$. This crucially implies that $I - B_i(y, \beta)$ are invertible matrices for all $i \in \{1, \ldots, k\}$. The following lemma tells us that if this condition holds for one β, it holds for all larger β', hence the phrase "at large inverse temperatures" is a reasonable description.

Lemma 5.21. *Suppose that Λ is a finite k-graph with no sources. Let $y : \Lambda \to [0, \infty)$ be a weight functor. For $\theta \in [0, \infty)$ and $i \in \{1, \ldots, k\}$ let $B_i(y, \theta)$ be the matrices from Definition 5.10. If $\beta \in (0, \infty)$ satisfies $1 > \rho(B_i(y, \beta))$ for all $i \in \{1, \ldots, k\}$ and $\beta' \geq \beta$, then $1 > \rho(B(y, \beta'))$.*

Proof. Fix $i \in \{0, \ldots, k\}$. Fix $\beta' \in (0, \infty)$ such that $\beta' \geq \beta$. Then $B_i(y, \beta')_{v,w} \leq B_i(y, \beta)_{v,w}$, and hence $0 \leq B_i(y, \beta') \leq B_i(y, \beta)$ in the sense of Section A.1. Thus applying Corollary A.4 implies that $\rho(B_i(y, \beta')) \leq \rho(B_i(y, \beta))$. Thus $\rho(B_i(y, \beta')) < 1$. $\square$

Theorem 5.22. *Suppose that Λ is a finite k-graph with no sources. Let $y : \Lambda \to [0, \infty)$ be a weight functor. For $\theta \in [0, \infty)$ and $i \in \{1, \dots, k\}$ let $B_i(y, \theta)$ be the matrices from Definition 5.10. Let $\alpha^y : \mathbb{R} \to \mathcal{T}C^*(\Lambda)$ be the action given by Proposition 5.8. Assume that $\beta \in (0, \infty)$ satisfies $\rho(B_i(y, \beta)) < 1$ for all $i \in \{1, \dots, k\}$.*

(a) For $w \in \Lambda^0$, the series $\sum_{\mu \in \Lambda w} e^{-\beta y(\mu)}$ converges with sum $x_w \geq 1$. Set $x := (x_w) \in [1, \infty)^{\Lambda^0}$, and consider $\epsilon \in [0, \infty)^{\Lambda^0}$. Define $m := \prod_{i=1}^k (I - B_i(y, \beta))^{-1}\epsilon$. Then $m \in [0, \infty)^{\Lambda^0}$, $B_i(y, \beta)m \leq m$ for all $i \in \{1, \dots, k\}$; and $\|m\|_1 = 1$ if and only if $\epsilon \cdot x = 1$.

(b) Suppose that $\epsilon \in [0, \infty)^{\Lambda^0}$ satisfies $\epsilon \cdot x = 1$, and set $m := \prod_{i=1}^k (I - B_i(y, \beta))^{-1}\epsilon$. Then there is a KMS$_\beta$ state ϕ_ϵ of $(\mathcal{T}C^*(\Lambda), \alpha^y)$ satisfying

$$\phi_\epsilon(t_\mu t_\nu^*) = \delta_{\mu,\nu} e^{-\beta y(\mu)} m_{s(\mu)}. \tag{5.9}$$

(c) The set

$$\Sigma_\beta := \{\epsilon \in [0, \infty)^{\Lambda^0} : \epsilon \cdot x = 1\}$$

is a compact convex subset of $\mathbb{R}^{\Lambda^0}$ and $F : \epsilon \mapsto \phi_\epsilon$ is an affine homeomorphism of Σ_β onto the simplex of KMS$_\beta$ states of $(\mathcal{T}C^*(\Lambda), \alpha^y)$. For a KMS$_\beta$ state ϕ of $(\mathcal{T}C^*(\Lambda), \alpha^y)$ let $m^\phi = (m_v^\phi)$ be the vector with entries $m_v^\phi := \phi(t_v)$. Then the inverse of this isomorphism takes ϕ to $\prod_{i=1}^k (I - B_i(y, \beta))m^\phi$.

Proof. (a) We first show that the series $\sum_{\mu \in \Lambda w} e^{-\beta y(\mu)}$ converges. Let $w \in \Lambda^0$. Then

$$\sum_{\mu \in \Lambda w} e^{-\beta y(\mu)} = \sum_{n \in \mathbb{N}^k} \sum_{\Lambda^n w} e^{-\beta y(\mu)} = \sum_{n \in \mathbb{N}^k} \sum_{v \in \Lambda^0} B(y, \beta)^n_{v,w}. \tag{5.10}$$

Since $1 > \rho(B_i(y, \beta))$ for all $i \in \{1, \dots, k\}$ Proposition 5.14 tells us that

$$\sum_{n \in \mathbb{N}^k} B(y, \beta)^n$$

converges in operator norm with sum $\prod_{i=1}^k (I - B_i(y, \beta))^{-1}$.[2] This implies that for every fixed $v \in \Lambda^0$ the series $\sum_{n \in \mathbb{N}^k} B(y, \beta)^n_{v,w}$ converges, so Equation (5.10) converges. The sum is at least 1 because all the terms are non-negative and $B(y, \beta)^0_{v,v} = 1$.

[2] In the last paragraph of [18, pg. 278], $\sum_{n=0}^\infty e^{-\beta r \cdot n} A^n$ should be $\sum_{n \in \mathbb{N}^k} e^{-\beta r \cdot n} A^n$ and $\sum_{n=0}^\infty e^{-\beta r \cdot n} A^n(w, v)$ should be $\sum_{n \in \mathbb{N}^k} e^{-\beta r \cdot n} A^n(w, v)$.

For the next part, let $x := (x_v) \in [1,\infty)^{\Lambda^0}$ and fix $\epsilon \in [0,\infty)^{\Lambda^0}$. Because every element of $B(y,\beta)^n$ and ϵ is non-negative and,

$$m_v = \Big(\prod_{i=1}^{k}(I - B_i(y,\beta))^{-1}\epsilon\Big)_v = \Big(\sum_{n\in\mathbb{N}^k} B(y,\beta)^n\epsilon\Big)_v,$$

m_v is non-negative. Thus $m \in [0,\infty)^{\Lambda^0}$. Next, fix $j \in \{1,\dots,k\}$. Then, since $m \geq 0$ and $B_i(y,\beta) < 1$ for all $i \in \{1,\dots,k\}$,

$$(I - B_j(y,\beta))m = (I - B_j(y,\beta))\prod_{i=1}^{k}(I - B_i(y,\beta))^{-1}\epsilon = \prod_{i\neq j}(I - B_i(y,\beta))^{-1}\epsilon \geq 0,$$

so $B_j(y,\beta)m \leq m$. Finally,

$$\begin{aligned}
\|m\|_1 &= \sum_{v\in\Lambda^0} m_v \\
&= \sum_{v\in\Lambda^0}\Big(\prod_{i=1}^{k}(I - B_i(y,\beta))^{-1}\epsilon\Big)_v \\
&= \sum_{v\in\Lambda^0}\Big(\Big(\sum_{n\in\mathbb{N}^k} B(y,\beta)^n\Big)\epsilon\Big)_v \\
&= \sum_{v\in\Lambda^0}\sum_{n\in\mathbb{N}^k}\sum_{w\in\Lambda^0} B(y,\beta)^n_{v,w}\epsilon_w \\
&= \sum_{w\in\Lambda^0}\epsilon_w\Big(\sum_{\mu\in\Lambda w} e^{-\beta y(\mu)}\Big) \\
&= \sum_{w\in\Lambda^0}\epsilon_w x_w \\
&= \epsilon\cdot x.
\end{aligned}$$

Thus $\|m\|_1 = 1$ if and only if $\epsilon \cdot x = 1$.

(b) To find a KMS$_\beta$ state we first need to find a state of $(\mathcal{T}C^*(\Lambda), \alpha^y)$. We build a state by using the path representation π_T of $\mathcal{T}C^*(\Lambda)$ on $\ell^2(\Lambda)$ from Proposition 5.3. For $\lambda \in \Lambda$ we define

$$\Delta_\lambda := e^{-\beta y(\lambda)}\epsilon_{s(\lambda)},$$

and note that $\Delta_\lambda \geq 0$. We aim to define ϕ_ϵ by

$$\phi_\epsilon(a) = \sum_{\mu\in\Lambda}\Delta_\mu(\pi_T(a)h_\mu|h_\mu) \qquad \text{for } a \in \mathcal{T}C^*(\Lambda). \tag{5.11}$$

To see that Equation (5.11) defines a state we need to show that it is a positive linear functional and that $\phi_\epsilon(1) = 1$.

We first claim that $\phi_\epsilon(1) = \sum_{\mu\in\Lambda} \Delta_\mu = 1$. Fix $v \in \Lambda^0$. Then

$$\begin{aligned}
\sum_{\mu\in v\Lambda} \Delta_\mu &= \sum_{n\in\mathbb{N}^k} \sum_{\mu\in v\Lambda^n} e^{-\beta y(\mu)} \epsilon_{s(\mu)} \\
&= \sum_{n\in\mathbb{N}^k} \sum_{w\in\Lambda^0} \sum_{\mu\in v\Lambda^n w} e^{-\beta y(\mu)} \epsilon_w \\
&= \sum_{n\in\mathbb{N}^k} \sum_{w\in\Lambda^0} B(y,\beta)^n_{v,w} \epsilon_w \\
&= \sum_{n\in\mathbb{N}^k} (B(y,\beta)^n \epsilon)_v. \qquad (5.12)
\end{aligned}$$

Now, since $1 > \rho(B_i(y,\beta))$ for all $i \in \{1,\dots,k\}$ Proposition 5.14 implies that

$$\sum_{n\in\mathbb{N}^k} (B(y,\beta)^n \epsilon)_v$$

converges with sum

$$\sum_{\mu\in v\Lambda} \Delta_\mu = m_v = (\prod_{i=1}^{k} (I - B_i(y,\beta))^{-1} \epsilon)_v. \qquad (5.13)$$

Part (a) implies that $\|m\|_1 = 1$, so

$$\sum_{\mu\in\Lambda} \Delta_\mu = \sum_{v\in\Lambda^0} \sum_{\mu\in v\Lambda} \Delta_\mu = \sum_{v\in\Lambda^0} m_v = \|m\|_1 = 1. \qquad (5.14)$$

We now use that $\sum_{\mu\in\Lambda} \Delta_\mu = 1$ to prove that Equation (5.11) converges for all $a \in \mathcal{T}C^*(\Lambda)$. Fix $a \in \mathcal{T}C^*(\Lambda)$, then applying the Cauchy-Schwarz inequality

$$\begin{aligned}
0 &\le |\Delta_\mu(\pi_T(a)h_\mu|h_\mu)| \\
&= |\Delta_\mu||(\pi_T(a)h_\mu|h_\mu)| \\
&\le \Delta_\mu\|\pi_T(a)h_\mu\|\|h_\mu\| \\
&\le \Delta_\mu\|\pi_T(a)\|\|h_\mu\|^2 \\
&\le \Delta_\mu\|a\| \cdot 1.
\end{aligned}$$

Since $\sum_{\mu\in\Lambda} \Delta_\mu$ converges, $\sum_{\mu\in\Lambda} \Delta_\mu\|a\|$ converges and the comparison test tells us that $\sum_{\mu\in\Lambda} |\Delta_\mu(\pi_T(a)h_\mu|h_\mu)|$ converges with sum less than or equal to $\|a\|$, therefore $\sum_{\mu\in\Lambda} \Delta_\mu(\pi_T(a)h_\mu|h_\mu)$ converges absolutely for all $a \in \mathcal{T}C^*(\Lambda)$.

Since $\phi_\epsilon(a^*a) = \sum_{\mu\in\Lambda} \Delta_\mu(\pi(a)h_\mu|\pi(a)h_\mu) \ge 0$, ϕ_ϵ is a positive linear functional, and by Equation (5.14) $\phi_\epsilon(1) = 1$, so ϕ_ϵ is a state on $\mathcal{T}C^*(\Lambda)$.

We next verify that ϕ_ϵ satisfies Equation (5.9). Fix $\mu, \nu, \lambda \in \Lambda$. Then

$$(\pi_T(t_\mu t_\nu^*)h_\lambda|h_\lambda) = (T_\nu^* h_\lambda|T_\mu^* h_\lambda) = \begin{cases} 1 & \text{if } \lambda = \mu\lambda' = \nu\lambda' \\ 0 & \text{otherwise.} \end{cases}$$

By unique factorisation, $\mu\lambda' = \nu\lambda'$ implies that $\mu = \nu$, and hence $\phi_\epsilon(t_\mu t_\nu^*) = 0$ when $\mu \neq \nu$. So suppose that $\mu = \nu$. Since $\sum_{\mu \in v\Lambda} \Delta_\mu = m_v$, we have

$$\begin{aligned}
\phi_\epsilon(t_\mu t_\mu^*) &= \sum_{\mu \in \Lambda} \Delta_\lambda (T_\mu^* h_\lambda|T_\mu^* h_\lambda) \\
&= \sum_{\lambda = \mu\lambda'} e^{-\beta y(\mu\lambda')} \epsilon_{s(\lambda')} \\
&= \sum_{\lambda = \mu\lambda'} e^{-\beta(y(\mu)+y(\lambda'))} \epsilon_{s(\lambda')} \\
&= \sum_{\lambda = \mu\lambda'} e^{-\beta y(\mu)} e^{-\beta y(\lambda')} \epsilon_{s(\lambda')} \\
&= e^{-\beta y(\mu)} \sum_{\lambda' \in s(\mu)\Lambda} \Delta_{\lambda'}.
\end{aligned}$$

Then applying Equation (5.13), $\phi_\epsilon(t_\mu t_\mu^*) = e^{-\beta y(\mu)} m_{s(\mu)}$. Thus

$$\phi_\epsilon(t_\mu t_\nu^*) = \delta_{\mu,\nu} e^{-\beta y(\mu)} \phi(t_{s(\mu)}),$$

that is, ϕ_ϵ satisfies Equation (5.9). Equation (5.12) gives

$$\phi_\epsilon(t_{s(\mu)}) = \sum_{\lambda \in \Lambda} \Delta_\lambda (T_{s(\mu)} h_\lambda|h_\lambda) = \sum_{\lambda \in s(\mu)\Lambda} \Delta_\lambda = m_{s(\mu)}.$$

Thus Proposition 5.9 implies that ϕ_ϵ is a KMS$_\beta$ state.

(c) We first prove that Σ_β is a compact convex subset of $\mathbb{R}^\Lambda$, and then that F is a homeomorphism.

We show that Σ_β is compact by showing that it is closed and bounded. To see that Σ_β is closed in $\mathbb{R}^{\Lambda^0}$, take $\{\epsilon_n\} \subset \Sigma_\beta$ and $\epsilon \in [0, \infty)^{\Lambda^0}$ such that $\epsilon_n \to \epsilon$. The dot product is continuous from $\mathbb{R}^{\Lambda^0} \times \mathbb{R}^{\Lambda^0} \to \mathbb{R}$, so $\epsilon_n \to \epsilon$ implies that $\epsilon_n \cdot x \to \epsilon \cdot x$. But $\epsilon_n \in \Sigma_\beta$ for all n, so $\epsilon_n \cdot x = 1$ for all n. Thus $\epsilon \cdot x = 1$, and therefore $\epsilon \in \Sigma_\beta$. Thus Σ_β contains all of its limit points, and is therefore closed. To see that Σ_β is bounded, take $\epsilon \in \Sigma_\beta$. This implies that $\epsilon \cdot x = 1$, that is, that $\sum_{v \in \Lambda^0} \epsilon_v x_v = 1$. Since $x_v \in [1, \infty)$ for all $v \in \Lambda^0$, $\sum_{v \in \Lambda^0} \epsilon_v \leq 1$. Since $\epsilon_v \geq 0$ for all $v \in \Lambda^0$, we have $0 \leq \epsilon_v \leq 1$. Thus $\|\epsilon\|^2 = \sum_{v \in \Lambda^0} \epsilon_v^2 \leq |\Lambda^0|$. Thus Σ_β is bounded. Since Σ_β is closed and bounded, the Heine-Borel theorem tells us that it is compact.

Next we show that Σ_β is convex. Fix $k \in \mathbb{N}$. For $i \in \{1, \dots, k\}$ take $c_i \in [0,1]$ such that $\sum_{i=0}^k c_i = 1$ and let $\{\epsilon_i\} \subset \Sigma_\beta$. Then

$$\begin{aligned}
\Big(\sum_{i=0}^k c_i\epsilon_i\Big) \cdot x &= \sum_{v\in\Lambda^0} \Big(\sum_{i=0}^k c_i\epsilon_i\Big)_v x_v \\
&= \sum_{v\in\Lambda^0} \sum_{i=0}^k c_i(\epsilon_i)_v x_v \\
&= \sum_{i=0}^k c_i \sum_{v\in\Lambda^0} (\epsilon_i)_v x_v \\
&= \sum_{i=0}^k c_i(\epsilon_i \cdot x) \\
&= \sum_{i=0}^k c_i \\
&= 1,
\end{aligned}$$

so $\epsilon := \sum_{i=0}^k c_i\epsilon_i \in \Sigma_\beta$. Thus Σ_β is convex. Therefore Σ_β is a compact convex subset of $\mathbb{R}^{\Lambda^0}$.

We now show that F surjective, injective, and continuous.

To see that F is surjective, let ϕ be a KMS$_\beta$ state. Proposition 5.16(a) implies that $m^\phi = (\phi(t_v))$ satisfies

$$\epsilon := \prod_{i\in K}(I - B_i(y,\beta))m^\phi \geq 0,$$

and $\|m^\phi\|_1 = 1$ because ϕ is a state. Then $m := \prod_{i\in K}(I - B_i(y,\beta))^{-1}\epsilon = m^\phi$, so $\|m\|_1 = 1$, and part (a) tells us that $\epsilon \cdot x = 1$. Then we can apply part (b), which tells us that there is a KMS$_\beta$ state ϕ_ϵ satisfying Equation (5.9). Then, for $\mu, \nu \in \Lambda$,

$$\phi_\epsilon(t_\mu t_\nu^*) = \delta_{\mu,\nu}e^{-\beta y(\mu)}m^\phi_{s(\mu)} = \delta_{\mu,\nu}e^{-\beta y(\mu)}\phi(t_{s(\mu)}).$$

So by Equation (5.1) $\phi_\epsilon(t_\mu t_\nu^*) = \phi(t_\mu t_\nu^*)$. Then linearity implies that $\phi_\epsilon(b) = \phi(b)$ for $b \in \operatorname{span}\{t_\mu t_\nu^* : \mu, \nu \in \Lambda\}$. Thus $\phi = \phi_\epsilon$, so F is surjective.

To see show that F is injective, suppose $\epsilon, \epsilon' \in \Sigma_\beta$ such that $F(\epsilon) = F(\epsilon')$, that is, $\phi_\epsilon = \phi_{\epsilon'}$. Define $m := \prod_{i=1}^k(I - B_i(y,\beta))^{-1}\epsilon$ and $m' := \prod_{i=1}^k(I - B_i(y,\beta))^{-1}\epsilon'$. Since $\phi_\epsilon = \phi_{\epsilon'}$, by Equation (5.9) $m = m'$. Now,

$$\epsilon = \prod_{i\in K}(I - B_i(y,\beta))m = \prod_{i\in K}(I - B_i(y,\beta))m' = \epsilon',$$

and therefore F is injective.

To see that F is continuous, suppose that $\epsilon_n \to \epsilon$ in Σ_β. We want to show that $\phi_{\epsilon_n} \to \phi_\epsilon$. Since the states are all norm-bounded with norm 1 it suffices to show $\phi_{\epsilon_n}(b) \to \phi(b)$ for b in the dense $*$-subalgebra $\operatorname{span}\{t_\mu t_\nu^* : \mu, \nu \in \Lambda\}$. Since $\epsilon_n \to \epsilon$, we have that

$$(\prod_{i=1}^{k}(I - B_i(y,\beta))^{-1}\epsilon_n)_{s(\mu)} \to (\prod_{i=1}^{k}(I - B_i(y,\beta))^{-1}\epsilon)_{s(\mu)}$$

for all $\mu \in \Lambda$, therefore that $m(\epsilon_n)_{s(\mu)} \to m(\epsilon)_{s(\mu)}$ for all $\mu \in \Lambda$. This in turn implies that

$$\phi_{\epsilon_n}(t_\mu t_\nu^*) = \delta_{\mu,\nu} e^{-\beta y(\mu)} m(\epsilon_n)_{s(\mu)} \to \delta_{\mu,\nu} e^{-\beta y(\mu)} m(\epsilon)_{s(\mu)} = \phi_\epsilon(t_\mu t_\nu^*),$$

for all $\mu, \nu \in \Lambda$. Then linearity implies that $\phi_{\epsilon_n}(b) \to \phi(b)$ for $b \in \operatorname{span}\{t_\mu t_\nu^* : \mu, \nu \in \Lambda\}$. Therefore $\phi_{\epsilon_n} \to \phi$ in the weak* topology, so F is continuous.

Since F is a continuous bijection of Σ_β, a compact space, onto the simplex of KMS states of $(\mathcal{T}C^*(\Lambda), \alpha^y)$, a Hausdorff space, F^{-1} is continuous. Therefore F is a homeomorphism.

Finally we show that F is affine. Fix $k \in \mathbb{N}$. Suppose that $\{\epsilon_i : 1 \leq i \leq k\} \subseteq \Sigma_\beta$ and $\{c_i \in [0,1] : 1 \leq i \leq k\}$ satisfying $\sum_{i=1}^k c_i = 1$. We need to check that $F(\sum_{i=0}^k c_i\epsilon_i) = \sum_{i=0}^k c_i F(\epsilon_i)$, that is, $\phi_{\sum_{i=0}^k c_i\epsilon_i}(a) = \sum_{i=0}^k c_i\phi_{\epsilon_i}(a)$ for all $a \in \mathcal{T}C^*(\Lambda)$. Fix $a \in \mathcal{T}C^*(\Lambda)$. Then

$$\begin{aligned}
\phi_\epsilon(a) &= \sum_{\lambda\in\Lambda} \Delta_\lambda(\pi_T(a)h_\lambda|h_\lambda) \\
&= \sum_{\lambda\in\Lambda} e^{-\beta y(\lambda)}\Big(\sum_{i=0}^k c_i\epsilon_i\Big)_{s(\mu)}(\pi_T(a)h_\lambda|h_\lambda) \\
&= \sum_{i=0}^k c_i \sum_{\lambda\in\Lambda} e^{-\beta y(\lambda)}(\epsilon_i)_{s(\mu)}(\pi_T(a)h_\lambda|h_\lambda).
\end{aligned}$$

Let $\Delta_{\lambda,i} := e^{-\beta y(\lambda)}(\epsilon_i)_{s(\lambda)}$. Then

$$\phi_\epsilon(a) = \sum_{i=0}^k c_i \sum_{\lambda\in\Lambda} \Delta_{\lambda,i}(\pi_T(a)h_\lambda|h_\lambda) = \sum_{i=0}^k c_i\phi_{\epsilon_i}(a).$$

Thus F is affine. $\square$

5.8 Existence of KMS states of $(\mathcal{T}C^*(\Lambda), \alpha^y)$ at the critical inverse temperature

In this section we describe the KMS states of $(\mathcal{T}C^*(\Lambda), \alpha^y)$ at a critical inverse temperature β_c, where the behaviour of the KMS states changes.

Proposition 5.23. *Suppose that Λ is a coordinatewise irreducible finite k-graph. Let $y : \Lambda \to [0, \infty)$ be a weight functor. For $\theta \in [0, \infty)$ and $i \in \{1, \ldots, k\}$ let $B_i(y, \theta)$ be the matrices from Definition 5.10. Let $\alpha^y : \mathbb{R} \to \mathcal{T}C^*(\Lambda)$ be the action given by Proposition 5.8. For $i \in \{1, \ldots, k\}$ let β_i be the unique number satisfying $\rho(B_i(y, \beta_i)) = 1$ given by Theorem 5.15. Let $\beta_c = \max\{\beta_i : 1 \leq i \leq k\}$.*

(a) The system $(\mathcal{T}C^*(\Lambda), \alpha^y)$ has a KMS$_{\beta_c}$ state.

(b) If $\beta < \beta_c$, then $(\mathcal{T}C^*(\Lambda), \alpha^y)$ has no KMS$_\beta$ states.

Proof. (a) We first want to find m such that $B(y, \beta_c)m \leq m$. Choose a decreasing sequence $\{\beta_n\}$ such that $\beta_n \to \beta_c$. Fix $n \in \mathbb{N}$. Take x as defined in Theorem 5.22(a). Choose $\epsilon_n := [0, \infty)^{\Lambda^0}$ such that $\epsilon_n \cdot x = 1$. Define $m_n := \prod_{i=1}^k (I - B_i(y, \beta))^{-1} \epsilon_n$. Then Theorem 5.22(a) implies that $m_n \in [0, \infty)^{\Lambda^0}$ satisfies $B_i(y, \beta_n)m_n \leq m_n$ for all $i \in \{1, \ldots, k\}$ and $\|m_n\|_1 = 1$. By passing to a subsequence, we may assume that $\{m_n\}$ converges pointwise to $m \in [0, \infty)^{\Lambda^0}$ and $\|m\|_1 = 1$. Then taking $n \to \infty$ tells us that $B_i(y, \beta_c)m \leq m$ for all $i \in \{1, \ldots, k\}$.

We aim to define $\epsilon_n' := \prod_{i=1}^k (I - B_i(y, \beta_n))m$ and then to apply Theorem 5.22. To do this we first need to check that $\epsilon_n' \in [0, \infty)^{\Lambda^0}$. Fix $j \in \{1, \ldots, k\}$. Since $\beta_n > \beta_c$, we have $0 \leq B_j(y, \beta_n)_{v,w} \leq B_j(y, \beta_c)_{v,w}$ for all $v, w \in \Lambda^0$. Therefore m satisfies $B_j(y, \beta_n)m \leq B_j(y, \beta_c)m \leq m$. Now,

$$\epsilon_n' := \prod_{i=1}^k (I - B_i(y, \beta_n))m = \prod_{i=1}^k \Big(m - B_i(y, \beta_n)m\Big).$$

Since $B_i(y, \beta_n)m \leq m$ for all $i \in \{1, \ldots, k\}$, $\epsilon_n' \in [0, \infty)^{\Lambda^0}$, so the x from Theorem 5.22(a) with $\beta = \beta_n$ satisfies $\epsilon_n' \cdot x = 1$.

We can then apply Theorem 5.22(b), which gives a KMS$_{\beta_n}$ state ϕ_n satisfying

$$\phi_n(t_\mu t_\nu^*) = \delta_{\mu,\nu} e^{-\beta_n y(\mu)} m_{s(\mu)}$$

Since the state space of $\mathcal{T}C^*(\Lambda)$ is weak* compact we may assume that by passing to a subsequence that the sequence $\{\phi_n\}$ converges to a state ϕ. Proposition 5.9(a) (or [2, 5.2.3]) then implies that ϕ is a KMS$_{\beta_c}$ state.

(b) Suppose that ϕ is a KMS$_\beta$ state of $(\mathcal{T}C^*(\Lambda), \alpha^y)$. Then Corollary 5.17 tells us $\rho(B_i(y,\beta)) \leq 1$ for all $i \in \{1, \dots, k\}$. Since $\theta \mapsto \rho(B_i(y,\theta))$ is strictly decreasing (Theorem 5.15) and there exists $i \in \{1, \dots, k\}$ such that $\rho(B_i(y,\beta_c)) = 1$, $\beta \geq \beta_c$. $\square$

5.9 KMS states of $(C^*(\Lambda), \overline{\alpha}^y)$

In this section we get an action $\overline{\alpha}^y : \mathbb{R} \to \operatorname{Aut} C^*(\Lambda)$ and then show when a KMS state of $(\mathcal{T}C^*(\Lambda), \alpha^y)$ factors through a KMS state of $(C^*(\Lambda), \overline{\alpha}^y)$.

Lemma 5.24. *The set*

$$P := \Big\{ t_v - \sum_{\lambda \in v\Lambda^{e_i}} t_\lambda t_\lambda^* : v \in \Lambda^0 \text{ and } i \in \{1, \dots, k\} \Big\}$$

consists of elements which are fixed by α^y.

Proof. Fix $r \in \mathbb{R}$, $v \in E^0$ and $i \in \{1, \dots, k\}$. Then

$$\begin{aligned}
\alpha_r^y\Big(t_v - \sum_{\lambda \in v\Lambda^{e_i}} t_\lambda t_\lambda^*\Big) &= \alpha_r^y(t_v) - \alpha_r^y\Big(\sum_{\lambda \in v\Lambda^{e_i}} t_\lambda t_\lambda^*\Big) \\
&= t_v - \sum_{\lambda \in v\Lambda^{e_i}} \alpha_r^y(t_\lambda t_\lambda^*) \\
&= t_v - \sum_{\lambda \in v\Lambda^{e_i}} e^{it(y(\lambda) - y(\lambda))} t_\lambda t_\lambda^* \\
&= t_v - \sum_{\lambda \in v\Lambda^{e_i}} e^0 t_\lambda t_\lambda^* \\
&= t_v - \sum_{\lambda \in v\Lambda^{e_i}} t_\lambda t_\lambda^*.
\end{aligned}$$

$\square$

Remark 5.25. Recall that we are viewing $C^*(\Lambda)$ as the image of the quotient map $q : \mathcal{T}C^*(\Lambda) \to \mathcal{T}C^*(\Lambda)/J$ for the ideal J generated by P, and q as the ideal J generated by P (Remark 5.5). Then $\alpha_r^y : \ker q \to \ker q$, and $\ker q$ consists of elements which are fixed by α_r^y, so it induces an automorphism $\overline{\alpha}^y$ of $C^*(\Lambda)$ such that

$$\overline{\alpha}_r^y(q(a)) = q(\overline{\alpha}_r^y(a))$$

for all $a \in C^*(\Lambda)$. We therefore have an action $\overline{\alpha}^y : \mathbb{R} \to \operatorname{Aut} C^*(\Lambda)$.

Proposition 5.26. *Suppose that Λ is a finite k-graph with no sources. Let y be a weight functor and $\alpha^y : \mathbb{R} \to \mathcal{T}C^*(\Lambda)$ be the action given by Proposition 5.8. Fix $\beta \in (0, \infty)$ and ϕ a KMS$_\beta$ state on $(\mathcal{T}C^*(\Lambda), \alpha^y)$. For $i \in \{1, \dots, k\}$ let $B_i(y,\theta)$ be the matrices from Definition 5.10. Define $m^\phi = (m_v^\phi)$ by $m_v^\phi = \phi(t_v)$. A KMS$_\beta$ state ϕ factors through $C^*(\Lambda)$ if and only if $B_i(y,\beta)m^\phi = m^\phi$ for every $i \in \{1, \dots, k\}$.*

Proof. As in the proof of [18, Lemma 4.1(b)] it suffices to check the Cuntz-Krieger relation (CK) on the generators e_i of $\mathbb{N}^k$. By Remark 5.5 we view $C^*(\Lambda)$ as the quotient of $\mathcal{T}C^*(\Lambda)$ by the ideal J generated by P from Lemma 5.24 . For each generating projection of J, Equation (5.2) tells us that

$$\begin{aligned}
\phi\Big(t_v - \sum_{\lambda\in v\Lambda^{e_i}} t_\lambda t_\lambda^*\Big) &= m_v^\phi - \sum_{\lambda\in v\Lambda^{e_i}} e^{-\beta y(\lambda)}\phi(t_{s(\lambda)}) \\
&= m_v^\phi - \sum_{w\in\Lambda^0}\sum_{\lambda\in v\Lambda^{e_i}w} e^{-\beta y(\lambda)}\phi(t_{s(\lambda)}) \\
&= m_v^\phi - \sum_{w\in\Lambda^0}\Big(\sum_{\lambda\in v\Lambda^{e_i}w} e^{-\beta y(\lambda)}\Big)\phi(t_w) \\
&= m_v^\phi - \sum_{w\in\Lambda^0} B_i(y,\beta)_{v,w}m_w \\
&= m_v^\phi - (B_i(y,\beta)m^\phi)_v. \qquad (5.15)
\end{aligned}$$

If ϕ factors through a state of $C^*(\Lambda)$, then the left-hand side of Equation (5.15) vanishes, and we have $m^\phi - B_i(y,\beta)m^\phi = 0$. Suppose on the other hand that $m^\phi = B_i(y,\beta)m^\phi$. Then Equation (5.15) implies that ϕ vanishes on the generators of J. Now, as in [18, Lemma 4.1(b)], each of these generating projections is fixed by the action α^y, and for each spanning element $a = t_\mu t_\nu^*$ of $\mathcal{T}C^*(\Lambda)$, the analytic function $f_a(z) := e^{iz(y(\mu)-y(\nu))}$ satisfies $\alpha_z^y(a) = f_a(z)a$. Thus [17, Lemma 2.2] implies that ϕ vanishes on the ideal J, and hence factors through a state of $C^*(\Lambda) = \mathcal{T}C^*(\Lambda)/J$. □

Corollary 5.27. *If there is a KMS$_\beta$ state of $(C^*(\Lambda),\overline{\alpha}^y)$, then $\rho(B_i(y,\beta)) = 1$ for all $i \in \{1,\dots,k\}$.*

Proof. We follow the method of [18, Corollary 4.4]. Let ϕ be a KMS$_\beta$ state of $(C^*(\Lambda),\overline{\alpha}^y)$. Proposition 5.26 implies that the vector $m^\phi := (\phi(t_v))$ satisfies $B_i(y,\beta)m^\phi = m^\phi$ for all $i \in \{1,\dots,k\}$. Since each $B_i(y,\beta)$ is irreducible (Lemma 5.13), Proposition A.9 implies that $\rho(B_i(y,\beta)) = 1$. □

5.10 The preferred dynamics

In this section we show that under certain conditions there is a *preferred dynamics* in which there is a unique KMS$_{\beta_c}$ state. Corollary 5.17 and Corollary 5.27 imply that there is a relationship between the dynamics α^y and the range of possible inverse temperatures β. In particular, Corollary 5.27 shows that the only possible inverse temperature for a KMS state on $(C^*(\Lambda),\overline{\alpha}^y)$ satisfies $\rho(B_i(y,\beta)) = 1$ for all $i \in \{1,\dots,k\}$. In other words the β_i such that $\rho(B_i(y,\beta_i)) = 1$ satisfy $\beta = \beta_i$ for all $i \in \{1,\dots,k\}$. This tells us that we get our best possible results about KMS states when the β_i coincide. We therefore normalise the dynamics α^y to ensure this is the case, and refer to the normalised dynamics as "the preferred dynamics". The preferred dynamics for $\mathcal{T}C^*(\Lambda)$ with the gauge action was studied in [18, Section 7], and the preferred dynamics for a 2-graph with one vertex was studied in [39].

For matrices $A_1,\dots,A_k \in M_n(\mathbb{R})$ write $\ln(\rho(A))$ for the vector

$$(\ln(\rho(A_1)),\dots,\ln(\rho(A_k))).$$

For a weight functor $y : \Lambda \to [0,\infty)$, define $\overline{y} : \Lambda \to [0,\infty)$ by

$$\overline{y}(\mu) = y(\mu) + \frac{1}{\beta_c}\ln(\rho(B(y,\beta_c))) \cdot d(\mu).$$

Then $\overline{y}$ is also a weight functor. We can find the preferred dynamics of the system $(\mathcal{T}C^*(\Lambda),\alpha^{\overline{y}})$ and we can describe all the KMS states for this preferred dynamics.

Remark 5.28. In [18, Section 7], the preferred dynamics for $(\mathcal{T}C^*(\Lambda),\alpha)$ is studied. Taking A_i as the vertex matrix with entries $A_{v,w} = |v\Lambda^{e_i}w|$, it occurs when $r := \ln(\rho(\Lambda)) = (\ln(\rho(A_1)),\dots,\ln(\rho(A_k)))$. If the $\ln(\rho(A_i))$ are rationally independent we can describe all of the KMS states for the preferred dynamics and the critical inverse temperature occurs at $\beta_c = 1$.

First we define the unimodular Perron-Frobenius eigenvector.

Lemma 5.29. *Suppose that Λ is a coordinatewise irreducible finite k-graph. Let $y : \Lambda \to [0,\infty)$ be a weight functor. For $\theta \in [0,\infty)$ and $i \in \{1,\dots,k\}$ let $B_i(y,\theta)$ be the matrices from Definition 5.10. Then the unimodular Perron-Frobenius eigenvectors of the $B_i(y,\theta)$ are equal for $i \in \{1,\dots,k\}$; we denote this vector ξ^Λ and call it the unimodular Perron-Frobenius eigenvector of Λ.*

Proof. The matrices $B_i(y,\theta)$ are irreducible (Lemma 5.13) and commute (Lemma 5.11), so we can apply [18, Lemma 2.1], which implies that their unimodular Perron-Frobenius eigenvectors are equal. □

Theorem 5.30. *Suppose that Λ is a coordinatewise irreducible finite k-graph. Let $y : \Lambda \to [0,\infty)$ be a weight functor. For $\theta \in [0,\infty)$ and $i \in \{1,\dots,k\}$ let $B_i(y,\theta)$ be the matrices from Definition 5.10. Let $\alpha^y : \mathbb{R} \to \mathcal{T}C^*(\Lambda)$ be the action given by Proposition 5.8. Choose $\beta_c \in (0,\infty)$, and define*

$$\overline{y}(\mu) := y(\mu) + \frac{1}{\beta_c}\ln(\rho(B(y,\beta_c))) \cdot d(\mu)$$

for all $\mu \in \Lambda$. Let ξ^Λ be the unimodular Perron-Frobenius eigenvector for Λ from Lemma 5.29. If $\overline{y}(\mu) = \overline{y}(\nu)$ implies that $d(\mu) = d(\nu)$ for all $\mu, \nu \in \Lambda$, then the system $(\mathcal{T}C^*(\Lambda), \alpha^{\overline{y}})$ has a unique KMS$_{\beta_c}$ state ϕ. The state satisfies

$$(5.16) \qquad \phi(t_\mu t_\nu^*) = \delta_{\mu,\nu} e^{-\beta_c \overline{y}(\mu)} \xi^\Lambda_{s(\mu)} \qquad \text{for all } \mu, \nu \in \Lambda,$$

and factors through a state $\overline{\phi}$ of the quotient $C^*(\Lambda)$. The state $\overline{\phi}$ is the only KMS state for $(C^*(\Lambda), \overline{\alpha}^{\overline{y}})$.

Proof. Fix $j \in \{1, \dots, k\}$. We first show that $\rho(B_j(\overline{y}, \beta_c)) = 1$. Now,

$$\begin{aligned}
B_j(\overline{y}, \beta_c)_{v,w} &= \sum_{\lambda \in v\Lambda^{e_j} w} e^{-\beta_c (y(\lambda) + \frac{1}{\beta_c} \ln(\rho(B(y,\beta_c))) \cdot d(\lambda))} \\
&= \sum_{\lambda \in v\Lambda^{e_j} w} e^{-\beta_c (y(\lambda) + \frac{1}{\beta_c} \ln(\rho(B_j(y,\beta_c))))} \\
&= \sum_{\lambda \in v\Lambda^{e_j} w} e^{-\beta_c y(\lambda)} e^{-\ln(\rho(B_j(y,\beta_c)))} \\
&= \sum_{\lambda \in v\Lambda^{e_j} w} \frac{e^{-\beta_c y(\lambda)}}{\rho(B_j(y,\beta_c)))} \\
&= \frac{1}{\rho(B_j(y,\beta_c)))} \sum_{\lambda \in v\Lambda^{e_j} w} e^{-\beta_c y(\lambda)} \\
&= \frac{1}{\rho(B_j(y,\beta_c)))} B_j(y,\beta_c)_{v,w}.
\end{aligned}$$

Therefore $B_j(\overline{y}, \beta_c) = \frac{1}{\rho(B_j(y,\beta_c)))} B_j(y, \beta_c)$, so $\rho(B_j(\overline{y}, \beta_c)) = 1$.

To show that a KMS$_{\beta_c}$ state exists, choose a decreasing sequence $\{\beta_n\}$ such that $\beta_n \to \beta_c$. Since $\beta_n > \beta_c$, applying Theorem 5.22(b) to $\epsilon := \prod_{i=1}^k (I - B(\overline{y}, \beta_n))\xi^\Lambda$ gives a KMS$_{\beta_n}$ state satisfying

$$(5.17) \qquad \phi_n(t_\mu t_\nu^*) = \delta_{\mu,\nu} e^{-\beta_n \overline{y}(\mu)} \xi^\Lambda_{s(\mu)}.$$

Since the state space of $\mathcal{T}C^*(\Lambda)$ is weak* compact we may assume that the sequence $\{\phi_n\}$ converges to a state ϕ. Letting $n \to \infty$ in Equation (5.17) shows that ϕ satisfies Equation (5.16). Thus, since $\overline{y}(\mu) = \overline{y}(\nu)$ implies that $d(\mu) = d(\mu)$, Proposition 5.9(c) implies that ϕ is a KMS$_{\beta_c}$ state.

To establish uniqueness, suppose that ψ is a KMS$_{\beta_c}$ state. Then Proposition 5.16(a) says that $m^\psi = (\psi(t_v))$ satisfies $B_j(\overline{y}, \beta_c) m^\psi \le m^\psi$. Then Proposition A.8 implies that $B_j(\overline{y}, \beta_c) m^\psi = m^\psi$. Now, since $B_i(\overline{y}, \beta_c) m^\psi = m^\psi$ for all $i \in \{1, \dots, k\}$ and ξ^Λ is the Perron-Frobenius eigenvector, $m^\psi = \xi^\Lambda$. Finally, fix $t_\mu, t_\nu^* \in \mathcal{T}C^*(\Lambda)$. Then, since

$\overline{y}(\mu) = \overline{y}(\nu)$ implies that $d(\mu) = d(\mu)$, Proposition 5.9(c) implies that

$$\begin{aligned}\psi(t_\mu t_\nu^*) &= \delta_{\mu,\nu} e^{-\beta_c \overline{y}(\mu)} \psi(t_{s(\mu)}) \\ &= \delta_{\mu,\nu} e^{-\beta_c \overline{y}(\mu)} m^\psi_{s(\mu)} \\ &= \delta_{\mu,\nu} e^{-\beta_c \overline{y}(\mu)} \xi^\Lambda_{s(\mu)}.\end{aligned}$$

So Equation (5.16) implies that $\psi(t_\mu t_\nu^*) = \phi(t_\mu t_\nu^*)$. Thus $\phi = \psi$.

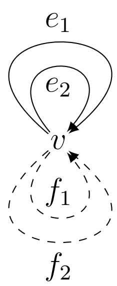
Since $B_i(\overline{y}, \beta_c) m^\phi = m^\phi$ for all $i \in \{1, \ldots, k\}$, Proposition 5.26 implies that ϕ factors through a state of $C^*(\Lambda)$.

To see that $\overline{\phi}$ is the only KMS state of $(C^*(\Lambda), \overline{\alpha}^{\overline{y}})$, suppose that ψ is a KMS$_\beta$ state of $(C^*(\Lambda), \overline{\alpha}^{\overline{y}})$. Then $\psi \circ q$ is a KMS state of $(\mathcal{T}C^*(\Lambda), \alpha^{\overline{y}})$, and Proposition 5.26 shows that $B_i(\overline{y}, \beta) m^{\psi \circ q} = m^{\psi \circ q}$ for all $i \in \{1, \ldots, k\}$. Then the Perron-Frobenius theorem implies that $\rho(B_i(\overline{y}, \beta)) = 1$ for all $i \in \{1, \ldots, k\}$. Thus uniqueness of the KMS$_{\beta_c}$ state of $(\mathcal{T}C^*(\Lambda), \alpha^{\overline{y}})$ implies that $\psi = \overline{\phi}$. $\square$

5.11 An example

We finish with an example which shows Theorem 5.30 gives examples not covered by [18].

Example 5.31. Let (Λ, d) be the 2-graph defined by the skeleton



and the factorisation property

$$e_1 f_1 = f_1 e_1, \quad e_1 f_2 = f_1 e_2, \quad e_2 f_1 = f_2 e_1 \text{ and } e_2 f_2 = f_2 e_2.$$

Fix $\beta_c \in (0, \infty)$. Define $y : \Lambda \to [0, \infty)$ on edges by

$$y(e_1) = 1 \qquad y(e_2) = \sqrt{2} \qquad y(f_1) = \sqrt{3} \qquad y(f_2) = \sqrt{2} + \sqrt{3} - 1.$$

Then Proposition 5.7 tells us y is a well-defined weight functor. From Definition 5.10 we get the 1×1 matrices

$$B_1(y, \beta_c) = \left[e^{-\beta_c} + e^{-\sqrt{2}\beta_c} \right] \qquad B_2(y, \beta_c) = \left[e^{-\sqrt{3}\beta_c} + e^{-\beta_c(\sqrt{2}+\sqrt{3}-1)} \right].$$

Then

$$\frac{1}{\beta_c}\ln(\rho(B_1(y,\beta_c))) = \frac{1}{\beta_c}\ln\left(e^{-\beta_c} + e^{-\sqrt{2}\beta_c}\right)$$

and

$$\frac{1}{\beta_c}\ln(\rho(B_2(y,\beta_c))) = \frac{1}{\beta_c}\ln\left(e^{-\sqrt{2}\beta_c} + e^{-\beta_c(\sqrt{2}+\sqrt{3}-1)}\right).$$

Then we define

$$\overline{y}(\mu) := y(\mu) + \begin{bmatrix} \frac{1}{\beta_c}\ln\left(e^{-\beta_c} + e^{-\sqrt{2}\beta_c}\right) \\ \frac{1}{\beta_c}\ln\left(e^{-\sqrt{3}\beta_c} + e^{-\beta_c(\sqrt{2}+\sqrt{3}-1)}\right) \end{bmatrix} \cdot d(\mu).$$

Now

$$\begin{aligned} B_1(\overline{y},\beta_c) &= \begin{bmatrix} e^{-\beta_c(1+\frac{1}{\beta_c}\ln(e^{-\beta_c}+e^{-\sqrt{2}\beta_c}))} + e^{-\beta_c(\sqrt{2}+\frac{1}{\beta_c}\ln(e^{-\beta_c}+e^{-\sqrt{2}\beta_c}))} \end{bmatrix} \\ &= \begin{bmatrix} \frac{e^{-\beta_c}+e^{-\sqrt{2}\beta_c}}{e^{\beta_c\frac{1}{\beta_c}\ln(e^{-\beta_c}+e^{-\sqrt{2}\beta_c})}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{e^{-\beta_c}+e^{-\sqrt{2}\beta_c}}{e^{-\beta_c}+e^{-\sqrt{2}\beta_c}} \end{bmatrix} \\ &= \begin{bmatrix} 1 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} &B_2(\overline{y},\beta_c) \\ &\quad= \begin{bmatrix} e^{-\beta_c(\sqrt{3}+\frac{1}{\beta_c}\ln(e^{-\sqrt{3}\beta_c}+e^{-\beta_c(\sqrt{2}+\sqrt{3}-1)}))} + e^{-\beta_c((\sqrt{2}+\sqrt{3}-1)+\frac{1}{\beta_c}\ln(e^{-\sqrt{3}\beta_c}+e^{-\beta_c(\sqrt{2}+\sqrt{3}-1)}))} \end{bmatrix} \\ &\quad= \begin{bmatrix} \frac{e^{-\sqrt{3}\beta_c}+e^{-\beta_c(\sqrt{2}+\sqrt{3}-1)}}{e^{\beta_c\frac{1}{\beta_c}\ln(e^{-\sqrt{3}\beta_c}+e^{-\beta_c(\sqrt{2}+\sqrt{3}-1)})}} \end{bmatrix} \\ &\quad= \begin{bmatrix} \frac{e^{-\sqrt{3}\beta_c}+e^{-\beta_c(\sqrt{2}+\sqrt{3}-1)}}{e^{-\sqrt{3}\beta_c}+e^{-\beta_c(\sqrt{2}+\sqrt{3}-1)}} \end{bmatrix} \\ &\quad= \begin{bmatrix} 1 \end{bmatrix}, \end{aligned}$$

so $\rho(B_j(\overline{y},\beta_c)) = 1$ for all $j \in \{1,\dots,k\}$.

Then, by Theorem 5.30, the system $(\mathcal{T}C^*(\Lambda), \alpha^{\overline{y}})$ has a unique KMS β_c state ϕ, and it satisfies

$$\phi(t_\mu t_\nu^*) = \delta_{\mu,\nu} e^{-\beta_c \overline{y}(\mu)} \xi^\Lambda_{s(\mu)}.$$

Since $y(e_1) \neq y(e_2)$, we can't write $y(\mu)$ as $r' \cdot d(\mu)$ for some $r' \in [0,\infty)^k$, so $\overline{y}(\mu)$ is not of the form $r \cdot d(\mu)$ for some $r \in [0,\infty)^k$ and this example is not just an example of the preferred dynamics from [18].

Appendix A

Appendix

A.1 The spectral radius of nonnegative matrices

For $A \in M_n(\mathbb{C})$, the *spectral radius* of A, denoted $\rho(A)$, is the maximal absolute value of all of the eigenvalues of A. For $A, B \in M_n(\mathbb{R})$ we write $A \leq B$ if $a_{ij} \leq b_{ij}$ for all $1 \leq i, j \leq n$. In this section we study nonnegative real matrices, that is, $A = (a_{ij}) \in M_n(\mathbb{R})$ such that $A \geq 0$. We show that the spectral radius of nonnegative real matrices are decreasing, that is, $A \leq B$ implies $\rho(A) \leq \rho(B)$.

Lemma A.1. *Let $A \in M_n(\mathbb{R})$. Then $\|A\|_{L(\mathbb{R}^n)} = \|A\|_{L(\mathbb{C}^n)}$.*

Proof. First we show that $\|A\|_{L(\mathbb{C}^n)} \leq \|A\|_{L(\mathbb{R}^n)}$. Fix $x, y \in \mathbb{R}^n$. Then

$$\|x + iy\|^2 = \sum_{j=1}^{n} |x_j + iy_j|^2.$$

Because x_j and y_j are real numbers,

$$\|x + iy\|^2 = \sum_{j=1}^{n} |x_j|^2 + |y_j|^2 = \sum_{j=1}^{n} x_j^2 + \sum_{j=1}^{n} y_j^2 = \|x\|^2 + \|y\|^2. \tag{A.1}$$

Let $z = x + iy$. Then

$$\begin{aligned}
\|Az\|^2 &= \|Ax + iAy\|^2 \\
&= \|Ax\|^2 + (Ax \mid iAy) + (iAy \mid Ax) + \|iAy\|^2 \\
&= \|Ax\|^2 + \|Ay\|^2 \\
&\leq (\|A\|_{L(\mathbb{R}^n)}\|x\|)^2 + (\|A\|_{L(\mathbb{R}^n)}\|y\|)^2 \\
&= \|A\|_{L(\mathbb{R}^n)}^2(\|x\|^2 + \|y\|^2),
\end{aligned}$$

and Equation (A.1) gives

$$\|Az\|^2 \le \|A\|_{L(\mathbb{R}^n)}^2 \|x+iy\|^2 = \|A\|_{L(\mathbb{R}^n)}^2 \|z\|^2.$$

Then because $\|A\|_{L(\mathbb{C}^n)}$ is the smallest $c \in \mathbb{R}$ such that $\|Az\| \le c\|z\|$, $\|A\|_{L(\mathbb{C}^n)} \le \|A\|_{L(\mathbb{R}^n)}$.

Conversely,

$$\begin{aligned} \|A\|_{L(\mathbb{R}^n)} &= \sup\{\|Ax\| : x \in \mathbb{R}^n : \|x\| = 1\} \\ &\le \sup\{\|Ax\| : x \in \mathbb{C}^n : \|x\| = 1\} \\ &= \|A\|_{L(\mathbb{C}^n)}. \end{aligned}$$

Thus $\|A\|_{L(\mathbb{R}^n)} = \|A\|_{L(\mathbb{C}^n)}$. □

Lemma A.2. *If $A \ge 0$, then $\|A\|_{L(\mathbb{R}^n)} = \sup\{\|Ax\| : x \in [0,\infty)^n, \|x\| = 1\}$.*

Proof. Fix $x \in \mathbb{R}^n$. Since $A \ge 0$,

$$\begin{aligned} \|Ax\|^2 &= \sum_{j=1}^n |(Ax)_j|^2 \\ &= \sum_{j=1}^n \Big| \sum_{k=1}^n a_{jk} x_k \Big|^2 \\ &\le \sum_{j=1}^n \Big| \sum_{k=1}^n a_{jk} |x_k| \Big|^2 \\ &= \sum_{j=1}^n |(A|x|)_j|^2 \\ &= \|A|x|\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} \|A\|_{L(\mathbb{R}^n)} &= \sup\{\|Ax\| : x \in \mathbb{R}^n, \|x\| = 1\} \\ &= \sup\{\|A|x|\| : x \in \mathbb{R}^n, \|x\| = 1\} \\ &= \sup\{\|Ax\| : x \in [0,\infty)^n, \|x\| = 1\}. \end{aligned}$$ □

Corollary A.3. *If $0 \le A \le B$, then $\|A\|_{L(\mathbb{R}^n)} \le \|B\|_{L(\mathbb{R}^n)}$.*

Proof. Fix $x \in [0,\infty)^n$. Then, since $A \le B$ entrywise,

$$\|Ax\|^2 = \sum_{j=1}^n (Ax)_j^2 \le \sum_{j=1}^n (Bx)_j^2 = \|Bx\|^2$$

Then, by Lemma A.2,

$$\begin{aligned}\|A\|_{L(\mathbb{R}^n)} &= \sup\{\|Ax\| : x \in [0,\infty)^n, \|x\| = 1\} \\ &\leq \sup\{\|Bx\| : x \in [0,\infty)^n, \|x\| = 1\} \\ &= \|B\|_{L(\mathbb{R}^n)}.\end{aligned}$$ □

Corollary A.4. *If* $0 \leq A \leq B$, *then* $\rho(A) \leq \rho(B)$.

Proof. Since $0 \leq A \leq B$ we have $0 \leq A^k \leq B^k$ for all $k \in \mathbb{N}$. Then applying Corollary A.3 we get $\|A^k\|_{L(\mathbb{R}^n)} \leq \|B^k\|_{L(\mathbb{R}^n)}$ for all $k \in \mathbb{N}$. Then Lemma A.1 tells us that $\|A^k\|_{L(\mathbb{C}^n)} \leq \|B^k\|_{L(\mathbb{C}^n)}$ for all $k \in \mathbb{N}$. Now $L(\mathbb{C}^n)$ is a Banach algebra over $\mathbb{C}$, so we can use the spectral radius formula to get

$$\rho(A) = \lim_{k\to\infty} \|A^k\|^{\frac{1}{k}} \leq \lim_{k\to\infty} \|B^k\|^{\frac{1}{k}} = \rho(B).$$ □

The following results from Perron-Frobenius theory are used throughout this thesis and stated here for convenience. We use the reference [7] as our main source for the theory, because it makes it clear when the individual hypotheses are used.

Proposition A.5 ([7, Proposition 2.2])**.** *Let* $A \in M_n(\mathbb{R})$ *be a nonnegative matrix,* $x \in \mathbb{R}^n$ *be a positive vector and* λ *be a nonnegative number.*

(a) If $\lambda x \leq Ax$, *then* $\lambda \leq \rho(A)$.

(b) If $Ax \leq \lambda x$, *then* $\rho(A) \leq \lambda$.

(c) If $\lambda x < Ax$, *then* $\lambda < \rho(A)$.

(d) If $Ax < \lambda x$, *then* $\rho(A) < \lambda$.

Corollary A.6 ([7, Corollary 2.3])**.** *Let* $A \in M_n(\mathbb{R})$ *be a nonnegative matrix and* $x \in \mathbb{R}^n$ *be a positive vector. If* $Ax = \lambda x$ *then* $\lambda = \rho(A)$.

A.1.1 The Perron-Frobenius theorem

We now present consequences of the Perron-Frobenius Theorem (for example, [7, Theorem 2.6]) used throughout this thesis.

A nonnegative matrix $T \in M_n(\mathbb{R})$ is *irreducible* if for every pair $1 \leq i, j \leq n$ there exists a positive integer m such that $(T^m)_{i,j} > 0$. For $x = (x_i) \in \mathbb{R}^n$ we write $x > 0$ if $x_i > 0$ for all $1 \leq i \leq n$.

Proposition A.7 ([7, Corollary 2.2])**.** *Let $A \in M_n(\mathbb{R})$ be nonnegative and irreducible. Then there exists a unique vector $x > 0$ such that $Ax = \rho(A)x$ and $\|x\|_1 = 1$.*

Proposition A.8. *Let $A \in M_n(\mathbb{R})$ be nonnegative and irreducible. If $x \geq 0$ such that $x \neq 0$ and either $Ax \geq \rho(A)x$ or $Ax \leq \rho(A)x$, then $Ax = \rho(A)x$.*

Proof. We apply the Perron-Frobenius theorem and then follow the method of [7, Proposition 2.4]. Since A is a nonnegative irreducible square matrix we can apply the Perron-Frobenius theorem to get a positive left eigenvector y. Then $y^T A = \rho(A)y^T$, which implies that

$$y^T(Ax - \rho(A)x) = \rho(A)y^T x - \rho(A)y^T x = 0.$$

Because $y > 0$ and either $Ax \geq \rho(A)x$ or $Ax \leq \rho(A)x$, this implies $Ax = \rho(A)x$. □

Proposition A.9. *Let $A \in M_n(\mathbb{R})$ be nonnegative and irreducible and λ be a nonnegative number. If $x \geq 0$ such that $x \neq 0$ and $Ax \geq \lambda x$, then $\rho(A) \leq \lambda$. Moreover, if $Ax = \lambda x$, then $\rho(A) = \lambda$.*

Proof. We use the method of [37, Theorem 1.6(a)] to show that $x > 0$ and then apply Proposition A.5(b). Fix i satisfying $1 \leq i \leq n$. Since $Ax \leq \lambda x$, we have $A^k x \leq \lambda^k x$ for every $k \in \mathbb{N}$. We write the elements of A^k as $a_{ij}^{(k)}$, then

$$\sum_{j=1}^{n} a_{ij}^{(k)} x_j \leq \lambda^k x_i.$$

Now, since A is irreducible, for all j there exists k_j such that $a_{ij}^{(k_j)} > 0$. Choose j such that $x_j > 0$ (at least one exists because $x \neq 0$), then $a_{ij}^{(k_j)} x_j > 0$, so it follows that $x_i > 0$. Then Proposition A.5(b) tells us that $\rho(A) \leq \lambda$. If in addition $Ax = \lambda x$, Proposition A.8 tells us $\rho(A) = \lambda$. □

A.2 Enumeration and convergence of sums

Throughout this thesis we will use sums of the form $\sum_{x \in X}$ in which the index set X is possibly infinite. In this section we define what we mean by such sums. Fix a set X. We use the σ-algebra of all subsets of X.

Lemma A.10. *Let m be a measure on X and $f : X \to [0, \infty)$ a measurable function. Then there is a measure ν on X such that*

$$\nu(F) = \int_F f\,dm. \tag{A.2}$$

Proof. We assume that Equation (A.2) and need to show that ν is a measure. To do this we need to show that $\nu(\emptyset) = 0$ and that ν is countably additive, that is, if $\{X_n\}_{n=1}^{\infty}$ is a sequence of disjoint sets, then $\nu(\bigcup_{n=1}^{\infty} X_n) = \sum_{n=1}^{\infty} \nu(X_n)$. That $\nu(\emptyset) = 0$ is trivial:

$$\nu(\emptyset) = \int_{\emptyset} f dm = 0.$$

To show the countable additivity of ν, write $Y = \bigcup_{n=1}^{\infty} X_n$, such that the X_n's are disjoint and measurable. We first look at the characteristic function $\chi_{\bigcup_{n=1}^{\infty} X_n}$. Fix $x \in X$. Since the X_n's are disjoint,

$$\chi_{\bigcup_{n=1}^{\infty} X_n}(x) = \begin{cases} 1 & \text{if } x \in \bigcup_{n=1}^{\infty} X_n \\ 0 & \text{otherwise} \end{cases} = \sum_{n=1}^{\infty} \chi_{X_n}(x)$$

Then, by the algebra of limits,

$$\begin{aligned} \nu(Y) &= \nu(\bigcup_{n=1}^{\infty} X_n) \\ &= \int_{\bigcup_{n=1}^{\infty} X_n} f dm \\ &= \int f \chi_{\bigcup_{n=1}^{\infty} X_n} dm \\ &= \int f\Big(\sum_{n=1}^{\infty} \chi_{X_n}\Big) dm \\ &= \int f\Big(\lim_{N\to\infty} \sum_{n=1}^{N} \chi_{X_n}\Big) dm \\ &= \int \lim_{N\to\infty} \sum_{n=1}^{N} \Big(f\chi_{X_n}\Big) dm. \end{aligned}$$

Now, define $f_N := \sum_{n=1}^{N} f\chi_{X_n}$. Since

$$f_{N+1} = \sum_{n=1}^{N+1} f\chi_{X_n} = f\chi_{X_{N+1}} + f_N \geq f_N,$$

we can apply the monotone convergence theorem [13, 2.14] with f_N to get that

$$\begin{aligned}\int \lim_{N\to\infty} \sum_{n=1}^{N} \Big(f\chi_{X_n}\Big) dm &= \lim_{N\to\infty} \int \sum_{n=1}^{N} \Big(f\chi_{X_n}\Big) dm \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \int \Big(f\chi_{X_n}\Big) dm \\ &= \sum_{n=1}^{\infty} \int f\chi_{X_n} dm \\ &= \sum_{n=1}^{\infty} \int_{X_n} f dm \\ &= \sum_{n=1}^{\infty} \nu(X_n).\end{aligned}$$

Thus $\nu(Y) = \sum_{n=1}^{\infty} \nu(X_n)$ and ν is a measure, as required. □

Theorem A.11. *Let X be a set and $f : X \to [0, \infty)$ a function. Let m be the counting measure on X, and define*

$$\sum_{x\in X} f(x) := \int_X f dm.$$

Let $\nu(F)$ be given by Lemma A.10. If $F \subseteq X$, then $\sum_{x\in F} f(x) = \nu(F)$. If $X = \bigsqcup_{n=1}^{\infty} F_n$ is a disjoint union, then

$$\sum_{x\in X} f(x) = \sum_{n=1}^{\infty} \Big(\sum_{x\in F_n} f(x) \Big).$$

Proof. First we show that $\sum_{x\in F} f(x) = \nu(F)$. Because $\nu(F) = \int_F f dm$ we have,

$$\sum_{x\in F} f(x) := \int_F f dm = \nu(F). \tag{A.3}$$

Next, Lemma A.10 tells us that ν is a measure. Then, by Equation (A.3),

$$\sum_{x\in X} f(x) = \nu(X) = \nu(\bigsqcup_{n=1}^{\infty} F_n) = \sum_{n=1}^{\infty} \nu(F_n) = \sum_{n=1}^{\infty} \sum_{x\in F_n} f(x).$$ □

Corollary A.12. *Suppose that X is a countable set, and $\{x_k : k \in \mathbb{N}\}$ is an enumeration of X. Then for every function $f : X \to [0, \infty)$ the series $\sum_{n=1}^{\infty} f(x_n)$ has sum $\sum_{x\in X} f(x)$.*

Proof. Apply Theorem A.11 with $F_n = \{x_n\}$. Then

$$\sum_{x\in X} f(x) = \sum_{n=1}^{\infty} \Big(\sum_{x\in\{x_n\}} f(x) \Big) = \sum_{n=1}^{\infty} f(x_n).$$ □

The following results are used throughout this thesis to show some of our infinite sums converge. Thanks to Iain Raeburn for the proofs, which we were unable to find in the literature.

Proposition A.13. *Let A be a Banach algebra with identity, and suppose that a is an element of A such that $\sum_{n=0}^{\infty} \|a^n\|$ converges. Then $1-a$ is invertible, with*

$$(1-a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

Proof. We have to show that the sequence $\{S_N\}$ of partial sums $S_N = \sum_{n=0}^{N} a^n$ converges in A to an inverse for $1-a$. If $M < N$, we have

$$\|S_N - S_M\| = \Big\| \sum_{n=M+1}^{N} a^n \Big\| \leq \sum_{n=M+1}^{N} \|a^n\| = \sum_{n=1}^{N} \|a^n\| - \sum_{n=1}^{M} \|a^n\|. \tag{A.4}$$

Since $\sum \|a^n\|$ converges, its partial sums form a Cauchy sequence, and Equation (A.4) implies that $\{S_N\}$ is a Cauchy sequence. Because A is complete, $\{S_N\}$ converges to an element $b := \sum_{n=0}^{\infty} a^n$ of A. From the continuity of multiplication, we have

$$(1-a)b = \lim_{N\to\infty} (1-a)\Big(\sum_{n=0}^{N} a^n\Big) = \lim_{N\to\infty} (1-a^{N+1}),$$

which is 1 because the summands in the convergent series $\sum \|a^n\|$ must go to 0. Similarly $b(1-a) = 1$, and $1-a$ is invertible with inverse b. $\square$

Corollary A.14. *Suppose that A is a Banach algebra with identity and $a \in A$. If $\lambda \in \mathbb{C}$ satisfies $|\lambda| > \rho(a)$, then the series $\sum_{n=0}^{\infty} \lambda^{-n} a^n$ converges in norm in A with sum $(1-\lambda^{-1}a)^{-1}$.*

Proof. The spectral radius formula implies that

$$\|\lambda^{-n}a^n\|^{1/n} = |\lambda^{-n}|^{1/n}\|a^n\|^{1/n} = |\lambda|^{-1}\|a^n\|^{1/n} \to |\lambda|^{-1}\rho(a) < 1.$$

Thus the nth root test implies that the series $\sum_{n=0}^{\infty} \|\lambda^{-n}a^n\|$ converges, and the result follows from Proposition A.13. $\square$

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