C^*-algebras generated by semigroups of partial isometries

Ilija Tolich

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Abstract

This thesis examines the C^*-algebras associated to semigroups of partial isometries. There are many interesting examples of C^*-algebras generated by families of partial isometries, for example the C^*-algebras associated to directed graphs and the C^*-algebras associated to inverse semigroups.

In 1992 Nica introduced a class of partially ordered groups called quasi-lattice ordered groups, and studied the C^*-algebras generated by semigroups of isometries satisfying a covariance condition. We have adapted Nica's construction for semigroups of partial isometries associated to what we call doubly quasi-lattice ordered groups. For each doubly quasi-lattice ordered group we construct two algebras: a concretely defined reduced algebra, and a universal algebra generated by a covariant family of partial isometries. We examine when representations of the universal algebra are faithful, and this gives rise to a notion of amenability for doubly quasi-lattice ordered groups.

We prove several recognition theorems for amenability. In particular, we prove that the universal and reduced algebras are isomorphic if and only if the doubly quasi-lattice ordered group is amenable. Further, we prove that if there is an order preserving homomorphism from a doubly quasi-lattice ordered group to an amenable group, then the quasi-lattice ordered group is amenable and the associated universal algebra is nuclear.

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Contents

CHAPTER 1

Introduction

This thesis will be examining the C^*-algebras generated by semigroups of partial isometries. There is an extensive literature examining the C^*-algebras generated by groups of unitaries and semigroups of isometries. Through the next several sections we will introduce the results that have come before which provide a road map for our theory. We begin by stating results about unitary representations of groups and then stating results about the isometric representations of semigroups introduced by Nica. We end with a brief discussion of partial isometric representations of semigroups and an overview of the thesis structure and major results.

1.1. Group algebras

For a long time mathematicians have been interested in the C^*-algebras generated by representations of groups. For a discrete group G with identity e, a unitary representation of G into a unital C^*-algebra A is a map $U : G \to A$ such that U preserves the group structure in the following sense: for $g, h \in G$ we have U_g is unitary, $U_g U_h = U_{gh}$, $U_e = 1$ where e is the identity of G and $U_g^* = U_{g^{-1}}$. Note that all products of the form $U_{g_1} U_{g_2}^* U_{g_3} \dots U_{g_n}$ may be simplified to U_h for some $h \in G$. A group algebra of a group representation $U : G \to A$ is the C^*-subalgebra of A generated by $\{U_g : g \in G\}$. There are two specific group algebras that we are most interested in: the reduced algebra and the universal algebra.

The reduced group algebra $C_r^*(G)$ is concretely defined. Let $\{\epsilon_h : h \in G\}$ be the usual orthonormal basis for $\ell^2(G)$. Then $\lambda : G \to B(\ell^2(G))$ defined by $\lambda_g \epsilon_h = \epsilon_{gh}$ is a group representation. Let $C_r^*(G)$ be the C^*-subalgebra of $B(\ell^2(G))$ generated by $\{\lambda_g : g \in G\}$.

The universal algebra of G, $C^*(G)$, is characterized abstractly (see [18, §7.1.5]): Let G be a group. There exists a C^*-algebra generated by unitaries $\{u_g : g \in G\}$, $C^*(G)$, that is universal for representations of G in the following sense: for any group representation $U : G \to A$ there exists a unital homomorphism $\phi_U : C^*(G) \to A$ such that $\phi_U(u_g) = U_g$. Further the pair $(C^*(G), \{u_g\})$ is unique up to isomorphism.

By the universal property of $C^*(G)$ there is a homomorphism $\pi_r : C^*(G) \to C_r^*(G)$ such that $\pi_r(u_g) = \lambda_g$. The homomorphism π_r is faithful if and only if G is an amenable group ([**8**],[**18**, Theorem 7.3.9]). Amenability is a concept that shows up all over mathematics when dealing with groups. We will not concern ourselves here with the precise definition of amenability, merely noting some other equivalent conditions: a group G is amenable if and only if $C^*(G)$ is nuclear [**17**, Theorem 2]. A group G is amenable if and only if the canonical trace τ on $C^*(G)$ characterised by

$$\tau(u_g) = \begin{cases} 1 & g = e \\ 0 & \text{otherwise} \end{cases}$$

is faithful for positive elements, i.e. if $a \in C^*(G)$, then $\tau(a^*a) = 0$ implies $a = 0$ (see Lemma 5.14).

1.2. C^*-algebras generated by semigroups of isometries

Having examined the C^*-algebras generated by groups, a natural next step is to consider the C^*-algebras generated by semigroups. Let P be a unital subsemigroup of a group G. Throughout, all semigroups will be assumed to contain the group identity which we denote e.[1] An isometric representation of P on a unital C^*-algebra A is a map, $W : P \to A$, such that W_p is an isometry for all $p \in P$, $W_e = 1$ where e is the identity of G, and $W_p W_q = W_{pq}$ for all $p, q \in P$.

For any $p \in P$, if $p^{-1} \in P$ then W_p is a unitary and $W_p^* = W_{p^{-1}}$. To avoid representations with automatically unitary operators we restrict our attention to semigroups with no units except the identity: $P \cap P^{-1} = \{e\}$. The pair (G, P) gives a partial order on G: $x \leq y$ if $x^{-1}y \in P$. The theory follows the road map we outlined above for group algebras: we define a representation of semigroups, find a concrete example, construct a universal algebra and then find a notion of amenability for semigroups which characterizes when the universal and concrete algebras are isomorphic. This is also the path we will follow when we consider semigroups of partial isometries.

1.2.1. Coburn's theorem. Consider the semigroup $\mathbb{N}$ under addition. For any isometric representation W of $\mathbb{N}$ we have $W_n = W_{1+1+...+1} = (W_1)^n$. Thus any C^*-algebra generated by an isometric representation of $\mathbb{N}$ is a C^*-algebra generated by a single isometry, first studied by Coburn in [**4**]. Let $\{\epsilon_n : n \in \mathbb{N}\}$ be the usual orthonormal basis for $\ell^2(\mathbb{N})$ and let S be the unilateral shift on $\ell^2(\mathbb{N})$ such that $S\epsilon_n =$

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[1]These are more usually called monoids however we follow Nica's precedent in [**16**].

ϵ_{n+1}. Coburn found that any C^*-algebra generated by a single non-unitary isometry is isomorphic to $C^*(S)$, the C^*-algebra generated by the unilateral shift S.

A formulation of Coburn's result in terms of semigroup representations would say:

THEOREM ([**4**]). *$C^*(S)$ is universal for isometric representations of $\mathbb{N}$ in the following sense: if $W : \mathbb{N} \to A$ is an isometric representation, then there exists a homomorphism $\phi_W : C^*(S) \to A$ such that $\phi_W(S) = W_1$. Further, ϕ_W is faithful if and only if $1 - W_1 W_1^* \neq 0$.*

Another useful property of $C^*(S)$ is that the set $K = \{S^m S^{*n} : m, n \in \mathbb{N}\}$ is linearly independent and that span K is a dense subalgebra of $C^*(S)$. The simplification of products into a product of just two terms holds for all isometric representations of $\mathbb{N}$. Let $W : \mathbb{N} \to A$ be an isometric representation. Observe that any product of the form $W_m^* W_n$ collapses to either W_{n-m} or W_{m-n}^*. Let $n_i \in \mathbb{N}$. Then any product of the form $W_{n_1} W_{n_2}^* W_{n_3} W_{n_4}^* \ldots W_{n_m}$ may be simplified to $W_p W_q^*$ for some $p, q \in \mathbb{N}$. We can also evaluate products of the range projections: $W_m W_m^* W_n W_n^* = W_{\max\{m,n\}} W_{\max\{m,n\}}^*$.

1.2.2. Quasi-lattice ordered groups. Following the example of groups, a natural example of a semigroup representation is the map $S : P \to B(\ell^2(P))$ defined by $S_p \epsilon_x = \epsilon_{px}$. The representation S is a natural generalisation of the unilateral shift for $\mathbb{N}$. Straightforward calculation shows that, for all $p \in P$,

$$S_p S_p^* \epsilon_x = \begin{cases} \epsilon_x & \text{if } p^{-1}x \in P \Leftrightarrow p \leq x \\ 0 & \text{otherwise.} \end{cases}$$

For $p, q \in P$ we have

$$S_p S_p^* S_q S_q^* \epsilon_x = \begin{cases} \epsilon_x & \text{if } p \leq x \text{ and } q \leq x \\ 0 & \text{otherwise.} \end{cases}$$

If p and q have no common upper bound in P then $S_p S_p^* S_q S_q^* = 0$. For certain semigroups we can write the product $S_p S_p^* S_q S_q^*$ as another $S_y S_y^*$ for some $y \in P$. Consider the semigroup $\mathbb{N}^2$. Any pair $(a, b), (c, d) \in \mathbb{N}^2$ has a natural least upper bound, $(a, b) \vee (c, d) = (\max\{a, c\}, \max\{b, d\})$. In particular, for $S : \mathbb{N}^2 \to B(\ell^2(\mathbb{N}^2))$ we have

$$(S_{(a,b)} S_{(a,b)}^*)(S_{(c,d)} S_{(c,d)}^*) = S_{(\max\{a,c\}, \max\{b,d\})} S_{(\max\{a,c\}, \max\{b,d\})}^*.$$

From the example of $\mathbb{N}^2$, it makes sense to restrict our attention to semigroups with natural least upper bounds. In [**16**] Nica introduced a class of partially ordered

3

groups called quasi-lattice ordered groups. A partially ordered group (G, P) is *quasi-lattice ordered* if every finite subset of G with a common upper bound in P has a least common upper bound in P. For a pair $x, y \in G$ we denote the least upper bound of x and y in P by $x \vee y$.

Let (G, P) be a quasi-lattice ordered group. We say an isometric representation $W : P \to A$ is Nica-covariant if, for all $p, q \in P$,

$$(1.1) \qquad W_p W_p^* W_q W_q^* = \begin{cases} W_{p \vee q} W_{p \vee q}^* & \text{if } p \vee q < \infty \\ 0 & \text{otherwise.} \end{cases}$$

For any quasi-lattice ordered group (G, P) the isometric representation $S : P \to B(\ell^2(P))$ is covariant.

Using the covariance condition we can show that, for all $p, q \in P$,

$$W_p^* W_q = \begin{cases} W_{(p^{-1}(p \vee q))} W_{q^{-1}(p \vee q)}^* & \text{if } p \vee q < \infty \\ 0 & \text{otherwise.} \end{cases}$$

In particular for $n_i \in P$ any product of the form $W_{n_1} W_{n_2}^* W_{n_3} W_{n_4}^* \dots W_{n_m}$ may be simplified to $W_p W_q^*$ for some $p, q \in P$.

1.2.3. Amenability for quasi-lattice ordered groups.
As with the group algebra there are two specific algebras that Nica associated with a quasi-lattice ordered group, the reduced algebra and the universal algebra.

The reduced algebra $C_r^*(G, P)$ is concretely defined. Let $C_r^*(G, P)$ be the C^*-subalgebra of $B(\ell^2(P))$ generated by the shift analogues $\{S_p : p \in P\}$. (See [**16**, §2.4].)

The universal algebra $C^*(G, P)$ is characterized abstractly. By [**16**, §4.1] there exists a C^*-algebra generated by isometries $\{w_p : p \in P\}$, $C^*(G, P)$, that is universal for covariant isometric representations of P in the following sense: for any covariant isometric representation $W : P \to A$ there exists a unital homomorphism $\phi_W : C^*(G, P) \to A$ such that $\phi_W(w_p) = W_p$. Further the pair $(C^*(G, P), \{w_p\})$ is unique up to isomorphism.

As was the case with group algebras we are interested in when the reduced and universal algebras are isomorphic. This requires a notion of amenability for quasi-lattice ordered groups. There are two equivalent amenability definitions given in the literature. The first, stated by Nica [**16**, §4.2], states that a quasi-lattice ordered group (G, P) is amenable if the homomorphism $\pi_r : C^*(G, P) \to C_r^*(G, P)$ is faithful. In the same paper he noted an equivalent condition: there exists a conditional expectation $E :$

$C^*(G, P) \to \overline{\text{span}}\{w_p w_p^* : p \in P\}$ onto the diagonal algebra, and (G, P) is amenable if and only if E is faithful for positive elements. In many ways this conditional expectation plays the same role for amenability of quasi-lattice ordered groups that the trace does in the amenability of groups. Laca and Raeburn [**11**] took this second condition as their definition of amenability because it is easier to check. When we consider partial isometric representations we will follow Laca and Raeburn's definition of amenability.

1.2.4. Recognition theorems for amenable quasi-lattice ordered groups. Nica proved that if (G, P) is a quasi-lattice ordered group and G is an amenable group then (G, P) is amenable. However, he also proved in [**16**, §5] that the free group on n generators $(\mathbb{F}_n, \mathbb{F}_n^+)$ is an amenable quasi-lattice ordered group. Since $\mathbb{F}_n$ is not amenable we are then left with the problem of deciding when quasi-lattice ordered groups are amenable. Laca and Raeburn introduced the notion of a controlled map: a controlled map between two quasi-lattice ordered groups (G, P) and (K, Q) is an order preserving homomorphism $\phi : G \to K$ that preserves the least upper bound structure. By [**11**, Proposition 6.6], if there is a controlled map from (G, P) to (K, Q) and K is an amenable group, then (G, P) is an amenable quasi-lattice ordered group. Li extended this result in [**14**, Corollary 8.3] and proved that if K is amenable, then $C^*(G, P)$ is a nuclear C^*-algebra.

1.2.5. Other semigroups of isometries. Since Nica published [**16**] there have been many studies examining semigroups of partial isometries. Several have relaxed the conditions of their semigroups. For example, Li in [**13**] and [**14**] studied the isometric representations of left-cancellative semigroups. Brownlowe, Larsen and Stammeier [**2**] and Starling [**22**] have examined the C^*-algebras associated to semigroups with a relaxed upper bound structure: right LCM semigroups. A semigroup P is right LCM if it is left cancellative and for any $p, q \in P$ the intersection of principal right ideals $pP \cap qP$ is either empty or of the form rP for some $r \in P$. In particular, this r may not be unique since these semigroups are permitted to have nontrivial units.

1.3. Partial isometries

There are many interesting examples of C^*-algebras generated by partial isometries, including graph algebras and the C^*-algebras associated to inverse semigroups. In this thesis we consider the C^*-algebras generated by semigroups of partial isometries. While there has been some study of semigroups of partial isometries this has mostly been confined to totally ordered groups such as those studied by Lindiarni and Raeburn

[**15**] and Kwasniewski and Lebedev [**10**]. An operator T on a Hilbert space H is a partial isometry if, for all $h \in (\ker T)^\perp$, $\|Th\| = \|h\|$. We say T is a power partial isometry if, for all $n \in \mathbb{N}$, T^n is a partial isometry.

PROPOSITION 1.1. [**19**, Proposition A.4] *Let T be a bounded operator on a Hilbert space H. The following are equivalent.*

    (1) T is a partial isometry;

    (2) $TT^*T = T$;

    (3) TT^* is a projection;

    (4) T^* is a partial isometry;

    (5) $T^*TT^* = T^*$;

    (6) T^*T is a projection.

If so, TT^* is the projection onto $\operatorname{range} T$ and T^*T is the projection onto $(\ker T)^\perp$.

For C^*-algebras we use the algebraic relation $TT^*T = T$ to define a partial isometry. All isometries are partial isometries (and are power partial isometries) as are their adjoints. The main example of a non-isometric partial isometry is the truncated shift. Let $\{e_i : 1 \le i \le n\}$ be an orthonormal basis for $\mathbb{C}^n$. The truncated shift $J_n : \mathbb{C}^n \to \mathbb{C}^n$ is defined by

$$J_n e_i = \begin{cases} e_{i+1} & \text{if } i < n \\ 0 & \text{if } i = n. \end{cases}$$

Let P be a unital semigroup and A be a unital C^*-algebra. A map $W : P \to A$ is a partial isometric representation if, for all $p, q \in P$, W_p is a partial isometry, $W_p W_q = W_{pq}$ and $W_e = 1$ where e is the identity of G.

There are two immediate observations from this definition. The first observation is that all compositions of the W_p are also partial isometries which is a significant restriction. In general, compositions of partial isometries are not themselves partial isometries. By [**6**, Lemma 2] the composition of two partial isometries ST is a partial isometry if and only if the projections S^*S and TT^* commute.

The second is that for any $p \in P$ and any $n \in \mathbb{N}$ we have $W_p^n = W_{p^n}$ is a partial isometry. Thus all of the W_p are power partial isometries. By Halmos and Wallen [**6**, Theorem 1] power partial isometries are unitarily equivalent to the direct sum of a unitary operator, copies of the unilateral shift S on $\ell^2(\mathbb{N})$, copies of S^*, and copies of the truncated shifts J_n for each $n \in \mathbb{N}$. We will not be using this observation in our proofs but it does illustrate that the maps we consider we still have a large amount of structure.

1.3.1. Analogue of Coburn's theorem. Once again the case $P = \mathbb{N}$ has been well studied, a partial isometric representation $W : \mathbb{N} \to A$ being the positive powers of a single power partial isometry. Hancock and Raeburn [7] proved an analogue of the Coburn theorem for a C^*-algebra generated by a single power partial isometry. Let

$$J = \oplus_{n=1}^{\infty} J_n : \oplus_{n=1}^{\infty} \mathbb{C}^n \to \oplus_{n=1}^{\infty} \mathbb{C}^n.$$

In [7, Theorem 1.3] Hancock and Raeburn found that $C^*(J)$ was universal for C^*-algebras generated by a single power partial isometry. That is: for any C^*-algebra A generated by a power partial isometry T there is a homomorphism $\phi : C^*(J) \to A$ such that $\phi(J) = T$. As with Coburn's theorem we can rewrite Hancock and Raeburns's result to say that $C^*(J)$ is universal for partial isometric representations of $\mathbb{N}$.

There are two algebraic properties of a single power partial isometry T that serve as a guide of what to expect for general semigroups. First, the range and source projections compose nicely

$$T^m T^{*m} T^n T^{*n} = T^{\max\{m,n\}} T^{* \max\{m,n\}}$$

$$T^{*m} T^m T^{*n} T^n = T^{* \max\{m,n\}} T^{\max\{m,n\}}.$$

Second, every product of the form $T^{n_1} T^{*n_2} T^{n_3} \ldots T^{n_k}$ may be simplified to a product of the form $T^p T^{*q} T^r$ where $p, r \le q$.[2]

1.3.2. Doubly quasi-lattice Ordered semigroups. As we mentioned above, the standard example of a power partial isometry is the truncated shift. So, just as Nica generalised the unilateral shift for semigroups, we will attempt to generalise the truncated shift. Let (G, P) be a quasi-lattice ordered group and let A be a subset of P. Let $\{\epsilon_x : x \in A\}$ be the usual orthonormal basis for $\ell^2(A)$. Naively we define $J^A : P \to B(\ell^2(A))$ by

$$J_p^A \epsilon_x = \begin{cases} \epsilon_{px} & \text{if } px \in A \\ 0 & \text{otherwise.} \end{cases}$$

For all $p \in P$, the operator J_p^A is a partial isometry. However, J^A does not in general preserve the semigroup multiplication. The map J^A is a partial isometric representation if and only if, for every $a, b \in A$, the set $\{x \in P : xa^{-1} \in P, \text{ and } bx^{-1} \in P\} \subseteq A$.

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[2]We have made a choice here. We could just as easily have simplified our product to the form $T^{*p} T^q T^{*r}$ with $p, r \le q$. If I was starting this project from scratch I would have chosen this as our convention as certain calculations become easier to parse and are more intuitive.

There exists a right-invariant partial order: $x \leq_r y$ if $yx^{-1} \in P$. We see that we can rewrite

$$\{x \in P : xa^{-1} \in P, \text{ and } bx^{-1} \in P\} = \{x \in P : a \leq_r x \leq_r b\}.$$

In other words, if J^A is a representation, then A must have no "gaps" in the right-invariant partial order. This result led us to the realization that when considering partial isometries we needed to consider both partial orders on (G, P). To distinguish them we write $\leq_l$ for the left-invariant partial order and $\leq_r$ for the right-invariant order.

Fix $a \in P$ and define $I_a := \{x \in P : x \leq_r a\}$. Consider the map $J^a : P \to B(\ell^2(I_a))$ defined by

$$J^a_p \epsilon_x = \begin{cases} \epsilon_{px} & \text{if } px \in I_a \\ 0 & \text{otherwise.} \end{cases}$$

In particular, in Lemma 2.12 we will show that the range and source projections relate to the left and right partial orders respectively:

$$J^a_q J^{a*}_q \epsilon_x = \begin{cases} J^a_q \epsilon_{q^{-1}x} = \epsilon_x & \text{if } q \leq_l x \\ 0 & \text{otherwise} \end{cases}$$

$$J^{a*}_q J^a_q \epsilon_x = \begin{cases} J^{a*}_q \epsilon_{qx} = \epsilon_x & \text{if } qx \leq_r a \\ 0 & \text{otherwise.} \end{cases}$$

Following Nica's example we insist that any finite set that has a common upper bound in P with respect to the left or right order has a least upper bound with respect to that order. In other words (G, P) is quasi-lattice ordered with respect to both left and right partial orders. We call such a (G, P) doubly quasi-lattice ordered.

We say that a partial isometric representation W is covariant if

$$W_p W_p^* W_q W_q^* = \begin{cases} W_{p \vee_l q} W_{p \vee_l q}^* & \text{if } p \vee_l q < \infty \\ 0 & \text{otherwise} \end{cases}$$

$$W_p^* W_p W_q^* W_q = \begin{cases} W_{p \vee_r q}^* W_{p \vee_r q} & \text{if } p \vee_r q < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Now that we have set up our basic definitions we can outline the thesis.

1.4. Overview of the thesis

Chapter 2. We begin by presenting the basic definitions, properties and examples we will use throughout the thesis. We start this chapter by defining doubly quasi-lattice ordered groups and proving basic properties of these groups. We also present examples of doubly quasi-lattice ordered groups, with discussion of particular interesting properties. We present several constructions that preserve doubly quasi-lattice order, such as direct products and free products.

Having defined doubly quasi-lattice ordered groups we next define representations of these semigroups by partial isometries. We then present a generalization of the truncated shift for arbitrary doubly quasi-lattice ordered groups in Lemma 2.12. Then we prove basic properties and manipulations. In particular we prove, in Lemma 2.13, that any product of partial isometric representations $W_{p_1} W_{p_2}^* W_{p_3} \ldots W_{p_n}$ is either zero or simplifies to a product of just three terms $W_p W_q^* W_r$. We can also write down multiplication and adjoint operations for these triples (Lemma 2.14). We end the chapter with our first main result: defining the reduced algebra. In our case the reduced algebra $C_{\mathrm{ts}}^*(G, P, P^{\mathrm{op}})$ is generated by a direct sum of truncated shift analogues in Definition 2.15.

Chapter 3. In this chapter we use the properties of partial isometric representations we have established to construct the universal algebra $C^*(G, P, P^{\mathrm{op}})$ in Theorem 4.8. The universal algebra $C^*(G, P, P^{\mathrm{op}})$ is generated by partial isometries $\{v_p : p \in P\}$ and has the universal property: for every covariant partial isometric representation $W : P \to A$ there exists a homomorphism $\phi_W : C^*(G, P, P^{\mathrm{op}}) \to A$ such that $\phi_W(v_p) = W_p$. The proof is a straightforward construction of a $*$-algebra that satisfies the algebraic structure of covariant partial isometric representations and is then completed with respect to a norm defined by supremum over representations of the $*$-algebra.

Chapter 4. With our universal algebra constructed we now wish to know when a representation of the universal algebra is faithful. It turns out that faithfulness depends on an amenability condition very similar to that of Nica and Laca and Raeburn. In Definition 4.7 we define amenability for a doubly quasi-lattice ordered group (G, P) in terms of a conditional expectation onto a diagonal subalgebra. This allows us to state our third major result, Theorem 4.8, which describes when a given representation of the universal algebra is faithful.

Chapter 5. This chapter is mostly background results about tensor products of C^*-algebras and group algebras. We set up the results we need to prove recognition

theorems for amenable doubly quasi-lattice ordered groups. The main result of this chapter is the construction of conditional expectations for concrete and abstract C^*-algebras. We will be reusing these results through the next two chapters.

Chapter 6. This chapter is devoted to a discussion of the recognitions theorems for amenability. We begin by proving Theorem 6.2 which states that $\pi_J : C^*(G, P, P^{\mathrm{op}}) \to C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$ is faithful if and only if (G, P) is amenable. This result demonstrates that our definition of "amenability" is analogous to Nica's original definition and to amenability of group algebras. We prove that amenability is a property of semigroups preserved under semigroup isomorphisms in Theorem 6.4. We finally prove Theorem 6.6 which states that if (G, P) is a doubly quasi-lattice ordered group and G is an amenable group then (G, P) is an amenable doubly quasi-lattice ordered group.

Chapter 7. In this chapter we prove our last major result, a stronger recognition theorem for amenable doubly quasi-lattice ordered groups. A controlled map between two doubly quasi-lattice ordered groups (G, P) and (K, Q) is an order preserving homomorphism $\phi : G \to K$ that preserves the least upper bound structure. Theorem 7.7 states that if K is an amenable group and there is a controlled map from (G, P) to (K, Q) then (G, P) is an amenable doubly quasi-lattice ordered group and $C^*(G, P)$ is a nuclear C^*-algebra. This theorem allows us to show that our examples are amenable. We can also construct new examples of amenable doubly quasi-lattice ordered groups from free and direct products of known examples. In addition, Theorem 6.6 appears as a special case of Theorem 7.7.

CHAPTER 2

Definitions, examples and basic properties

Let P be a unital subsemigroup of a discrete group G such that $P \cap P^{-1} = \{e\}$ where e is the identity of G. There is a partial order on G defined by $x \leq y$ if $x^{-1}y \in P$. Some authors prefer an equivalent formulation: $x \leq y \Leftrightarrow y \in xP$. The order $\leq$ is left-invariant in the sense that $x \leq y$ implies $zx \leq zy$ for every $z \in G$. The partial order is determined by the pair (G, P) of a group G and its subsemigroup P.

DEFINITION 2.1 ([**16**, Definition 2.1]). A partially ordered group (G, P) is *quasi-lattice ordered* if every finite subset of G with a common upper bound in P has a least common upper bound in P.

There are several equivalent conditions for a pair (G, P) to be quasi-lattice ordered given in Lemma 2.3.

To handle the structure of a semigroup of partial isometries we must have a second partial order. Consider the opposite group $(G^{\mathrm{op}}, P^{\mathrm{op}})$ where $G^{\mathrm{op}} = G$ and $P^{\mathrm{op}} = P$ with operation $x \cdot_{\mathrm{op}} y = yx$. This group has a left-invariant partial order defined by $x \leq y$ if $x^{-1} \cdot_{\mathrm{op}} y \in P^{\mathrm{op}}$.

DEFINITION 2.2. A partially ordered group (G, P) is said to be *doubly quasi-lattice ordered* if (G, P), and its opposite group $(G^{\mathrm{op}}, P^{\mathrm{op}})$, are quasi-lattice ordered groups.

To avoid dealing with two distinct group operations, one on G and one on G^{op}, we will instead define two partial orders on (G, P) one of which will be left-invariant, derived from (G, P), and the other right-invariant, derived from $(G^{\mathrm{op}}, P^{\mathrm{op}})$.

- Define the partial order $\leq_l$ by $x \leq_l y$ if $x^{-1}y \in P$.
- Define the partial order $\leq_r$ by $x \leq_r y$ if $yx^{-1} \in P$ i.e. if $x^{-1} \cdot_{\mathrm{op}} y \in P^{\mathrm{op}}$.

NOTATION. If x and y have a common upper bound in P under $\leq_l$, then their least common upper bound will be denoted by $x \vee_l y$. Similarly if x and y have a common upper bound in P under $\leq_r$, then their least common upper bound will be denoted by $x \vee_r y$. To simplify notation we introduce the symbol ∞ and say $x \vee_l y = \infty$ or $x \vee_r y = \infty$ when x and y have no common upper bound in P under the respective

partial order. We will also use ∞ as a shorthand: we say $x \vee_l y < \infty$ if x and y have a common left upper bound $x \vee_l y$ in P.

2.1. Properties of doubly quasi-lattice ordered groups

In 2002 Crisp and Laca established a set of equivalent conditions for quasi-lattice ordered groups in [**5**, Lemma 7].

LEMMA 2.3 ([**5**, Lemma 7]). *For a partially ordered group (G, P) the following statements are equivalent:*

(1) (G, P) *is a quasi-lattice ordered group.*
(2) *Every finite set in G with a common left upper bound in P has a least left common upper bound in P.*
(3) *Every element of G having a left upper bound in P has a least left upper bound in P.*
(4) *If $x \in PP^{-1}$, then there exist a pair of elements $a, b \in P$ with $x = ab^{-1}$ and such that for every $u, v \in P$ with $ab^{-1} = uv^{-1}$, one has $a \leq_l u$ and $b \leq_l v$. (The pair a, b is clearly unique.)*
(5) *Every pair u, v of elements in P has a greatest lower bound $u \wedge_r v$ with respect to the right-invariant partial order on G.*
(6) *If $x \in PP^{-1}$ then there exist a pair of elements $a, b \in P$ with $x = ab^{-1}$ and such that $a \wedge_r b = 1$.*

Assuming that (1)-(6) hold and given $x \in PP^{-1}$ there is in fact a unique pair $a, b \in P$ satisfying statement (6), being precisely the pair a, b of statement (4).

While Lemma 2.3 is stated for the left-invariant partial order the proof is easily adapted to the right-partial order.

LEMMA 2.4 ([**5**, Lemma 7]). *For a partially ordered group (G, P) the following statements are equivalent:*

(1) $(G^{\mathrm{op}}, P^{\mathrm{op}})$ *is a quasi-lattice ordered group.*
(2) *Every finite set in G with a common right upper bound in P has a least right common upper bound in P.*
(3) *Every element of G having a right upper bound in P has a least right upper bound in P.*
(4) *If $x \in P^{-1}P$ then there exist a pair of elements $a, b \in P$ with $x = a^{-1}b$ and such that for every $u, v \in P$ with $a^{-1}b = u^{-1}v$, one has $a \leq_r u$ and $b \leq_r v$. (The pair a, b is clearly unique.)*

(5) Every pair u, v of elements in P has a greatest lower bound $u \wedge_l v$ with respect to the left-invariant partial order on G.

(6) If $x \in P^{-1}P$ then there exist a pair of elements $a, b \in P$ with $x = a^{-1}b$ and such that $a \wedge_l b = 1$.

Assuming that (1)-(6) hold and given $x \in P^{-1}P$ there is in fact a unique pair $a, b \in P$ satisfying statement (6), being precisely the pair a, b of statement (4).

The equivalent conditions of Lemma 2.3 give easier ways to check when a group is quasi-lattice ordered. Which test is most efficient depends on the group. One interesting consequence of Lemmas 2.3 and 2.4 is that we can write a definition of a doubly quasi-lattice ordered group that only involves one of the partial orders:

COROLLARY 2.5. *Let (G, P) be a left and right partially ordered group. The following are equivalent:*

(1) (G, P) is doubly quasi-lattice ordered.

(2) Every element of G having a left upper bound in P has a least left upper bound in P, and every pair u, v of elements in P has a greatest lower bound $u \wedge_l v$ with respect to the left-invariant partial order on G.

(3) Every element of G having a right upper bound in P has a least right upper bound in P. Every pair u, v of elements in P has a greatest lower bound $u \wedge_r v$ with respect to the right-invariant partial order on G.

PROOF. We will show that $(1) \Leftrightarrow (2)$ and then $(1) \Leftrightarrow (3)$ follows by a symmetric argument.

$(1) \Rightarrow (2)$. Suppose that (G, P) is a doubly quasi-lattice ordered group. Then (G, P) is a quasi-lattice ordered group and so Lemma 2.3(3) states that every element of G that has a left upper bound in P has a least left upper bound in P. But $(G^{\mathrm{op}}, P^{\mathrm{op}})$ is also a quasi-lattice ordered group and so Lemma 2.4(5) states that any pair in P has a greatest lower bound $u \wedge_l v$ with respect to the left-invariant partial order on G.

$(2) \Rightarrow (1)$. Suppose that every element of G having a left upper bound in P has a least left upper bound in P, and every pair u, v of elements in P has a greatest lower bound $u \wedge_l v$ with respect to the left-invariant partial order on G. Then, by 2.3(3) (G, P) is quasi-lattice ordered and by 2.4(5) $(G^{\mathrm{op}}, P^{\mathrm{op}})$ is quasi-lattice ordered. Thus (G, P) is doubly quasi-lattice ordered. $\square$

The least upper bounds have nice algebraic structure which will be useful in many proofs later in this thesis.

LEMMA 2.6. *Let (G, P) be a doubly quasi-lattice ordered group.*

(1) *The left least upper bound $\vee_l$ is associative: for all $g, h, k \in G$ which have a common left upper bound $(g \vee_l h) \vee_l k = g \vee_l (h \vee_l k)$.*

(2) *The right least upper bound $\vee_r$ is associative: for all $g, h, k \in G$ which have a common right upper bound $(g \vee_r h) \vee_r k = g \vee_r (h \vee_r k)$.*

PROOF. (1). We will show that $(g \vee_l h) \vee_l k \leq_l g \vee_l (h \vee_l k)$ and $g \vee_l (h \vee_l k) \leq_l (g \vee_l h) \vee_l k$. We have $g, h, k \leq_l g \vee_l (h \vee_l k)$ and hence $g \vee_l (h \vee_l k)$ is an upper bound for g and h. Therefore $g \vee_l h \leq_l g \vee_l (h \vee_l k)$ and so $g \vee_l (h \vee_l k)$ is a common upper count for k and $g \vee_l h$. Thus $(g \vee_l h) \vee_l k \leq_l g \vee_l (h \vee_l k)$. The second inequality follows by a similar argument.

The proof of (2) follows by symmetry. $\qquad\square$

LEMMA 2.7. *Let (G, P) be a doubly quasi-lattice ordered group. Let $q \in P$ and $g \in G$.*

(1) *The following are equivalent:*
   (a) $q \vee_l g < \infty$;
   (b) *for all $p \in P$ $pq \vee_l pg < \infty$;*
   (c) *there exists $p \in P$ such that $pq \vee_l pg < \infty$.*
   In particular, for all $p \in P$, $pq \vee_l pg = p(q \vee_l g)$.

(2) *The following are equivalent:*
   (a) $q \vee_r g < \infty$;
   (b) *for all $p \in P$ $qp \vee_r gp < \infty$;*
   (c) *there exists $p \in P$ such that $qp \vee_r gp < \infty$.*
   In particular, for all $p \in P$, $qp \vee_r gp = (q \vee_r g)p$.

PROOF. We will prove (1) first and then (2) follows by symmetry.

$(a) \Rightarrow (b)$. Let $p, q \in P$ and $g \in G$. Suppose that $q \vee_l g < \infty$. We will show that $pq \vee_l pg$ exists and that $pq \vee_l pg \leq_l p(q \vee_l g)$. Observe that $q \leq_l q \vee_l g$ and $g \leq_l q \vee_l g$. By left-invariance of the partial order we have $pq \leq_l p(q \vee_l g)$ and $pg \leq_l p(q \vee_l g)$. Hence $p(q \vee_l g)$ is a left upper bound for pq and pg in P. Thus $pq \vee_l pg$ exists and $pq \vee_l pg \leq_l p(q \vee_l g)$.

$(b) \Rightarrow (c)$ is trivial.

$(c) \Rightarrow (a)$. Let $p, q \in P$ and $g \in G$ and suppose that there exists $p \in P$ such that $pq \vee_l pg < \infty$. We know $pq \leq_l pq \vee_l pg$ and $pg \leq_l pq \vee_l pg$. By left-invariance $q \leq_l p^{-1}(pq \vee_l pg)$ and $g \leq_l p^{-1}(pq \vee_l pg)$. Since $q \in P$ we know that $p^{-1}(pq \vee_l pg) \in P$

and hence that $p^{-1}(pq \vee_l pg)$ is a common upper bound in P for q and g. Hence $q \vee_l g$ exists and $q \vee_l g \leq_l p^{-1}(pq \vee_l pg)$.

To prove that for all $p \in P$, $pq \vee_l pg = p(q \vee_l g)$ first observe that both upper bounds exist or are ∞ at the same time therefore this equality makes sense even when the upper bound does not exist. Suppose that $q \vee_l g < \infty$. As we observed above $q \vee_l g \leq_l p^{-1}(pq \vee_l pg)$. By left invariance we see that $p(q \vee_l g) \leq_l pq \vee_l pg$. In addition we also showed that $pq \vee_l pg \leq_l p(q \vee_l g)$. Thus $pq \vee_l pg = p(q \vee_l g)$. $\square$

REMARK. It is important in the proof of Lemma 2.7 that $q \in P$. As a counter example consider $(\mathbb{Z}, \mathbb{N})$. Since the least upper bound must be in $\mathbb{N}$ we have $-3 \vee -2 = 0$. Then we see that $(-3+1) \vee (-2+1) = -2 \vee -1 = 0$ but $1 + (-3 \vee -2) = 1 + 0 = 1$.

Any partially ordered finite set has a minimal element. This fact will be crucial to many proofs so we state and prove it below.

LEMMA 2.8. *Let $(X, \leq)$ be a partially ordered set. Any finite subset $\{x_i : 1 \leq i \leq n\} \subseteq X$ has an element x_j that is minimal in the sense that $x_i \leq x_j$ implies $x_i = x_j$.*

PROOF. We will prove this lemma by induction on the size of the subset n. For $n = 1$ the sole element x_1 is trivially minimal as $x_1 \leq x_1$ and $x_1 = x_1$.

Suppose that the result is true for $n = k$. Now consider $n = k + 1$ and the subset $\{x_1, \ldots, x_{k+1}\}$. The k-element subset $\{x_1, \ldots, x_k\}$ has a minimal element x_j by our induction hypothesis. Now we can compare the element x_{k+1} to x_j. There are 3 possibilities: $x_j \leq x_{k+1}$, $x_{k+1} \leq x_j$, or x_j and x_{k+1} are not comparable.

If $x_j \leq x_{k+1}$ and $x_j \neq x_{k+1}$ then x_j is still minimal for the subset $\{x_1, \ldots, x_k, x_{k+1}\}$.

If $x_{k+1} \leq x_j$ and $x_j \neq x_{k+1}$ then we claim x_{k+1} is minimal. To prove this claim suppose that there is some element x_l such that $x_l \leq x_{k+1}$. Transitivity of the partial order implies that $x_l \leq x_j$. But x_j is minimal for $\{x_1, \ldots, x_k\}$ and hence $x_j = x_l$ and we have a contradiction.

If x_j and x_{k+1} are not comparable then x_j is still minimal.

Thus the subset $\{x_1, \ldots, x_k, x_{k+1}\}$ has a minimal element. By the principle of mathematical induction, every finite subset has a minimal element. $\square$

2.2. Examples of doubly quasi-lattice ordered groups

With some notable exceptions most standard examples of quasi-lattice ordered groups that appear in the literature are also doubly quasi-lattice ordered. We outline these examples and give several methods for constructing new doubly quasi-lattice ordered groups by combining known examples. We end this section by proving that

$(\mathbb{Q} \rtimes \mathbb{Q}_+^*, \mathbb{N} \rtimes \mathbb{N}^\times)$ is doubly quasi-lattice ordered and that the left and right orders are significantly different in an interesting way.

EXAMPLES. (1). For all abelian groups $G^{\text{op}} = G$ and hence any abelian quasi-lattice ordered group is also *doubly* quasi-lattice ordered with the two partial orders coinciding.

(2). For any doubly quasi-lattice ordered group (G, P) its opposite group $(G^{\text{op}}, P^{\text{op}})$ is also a doubly quasi-lattice ordered group.

(3). $(\mathbb{Z}, \mathbb{N})$ is doubly quasi-lattice ordered with both left and right orders given by the usual order on $\mathbb{Z}$. For $m, n \in \mathbb{Z}$ we have $m \vee_l n = m \vee_r n = \max\{m, n, 0\}$. (We include the 0 to ensure that the maximum is in $\mathbb{N}$.)

(4). For all $n \in \mathbb{N}$ the partially ordered group $(\mathbb{Z}^n, \mathbb{N}^n)$ is doubly quasi-lattice ordered. Both left and right partial orders are given by $(x_1, x_2, \ldots, x_n) \leq (x_1, x_2, \ldots, x_n)$ if $x_i \leq y_i$ for all $1 \leq i \leq n$. Every pair $x, y \in \mathbb{Z}^n$ has a common upper bound in $\mathbb{N}^n$ and $x \vee y = (\max\{x_1, y_1, 0\}, \max\{x_2, y_2, 0\}, \ldots, \max\{x_n, y_n, 0\})$.

(5). Consider the group of additive rationals $\mathbb{Q}$ and its subsemigroup $\mathbb{Q}^+ = \mathbb{Q} \cap [0, \infty)$. Then $(\mathbb{Q}, \mathbb{Q}^+)$ is doubly quasi-lattice ordered (in fact totally ordered). Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for all $a \in \mathbb{Q}^+ \backslash \{0\}$ the set $I_a := \{x \in \mathbb{Q}^+ : x \leq a\}$ is countably infinite. For most other standard examples of doubly quasi-lattice ordered groups, (G, P), the set $I_a := \{x \in P : x \leq_r a\}$ is finite for all $a \in P$. Sets of the form I_a will become useful later when defining examples of partial isometric representations. It is useful to have a straightforward example where these sets are infinite, as a reality check of hypotheses.

(6). Let $\mathbb{F}_n$ be the free group with n generators $\{a_i : 1 \leq i \leq n\}$ and let $\mathbb{F}_n^+$ be the semigroup generated by $\{a_i : 1 \leq i \leq n\} \cup \{e\}$. To see that $(\mathbb{F}_n, \mathbb{F}_n^+)$ is a quasi-lattice ordered group we use Lemma 2.3(4). Every $z \in \mathbb{F}_n^+ (\mathbb{F}_n^+)^{-1}$ has unique reduced word

$$ z = a_{i_1} a_{i_2} \ldots a_{i_m} a_{j_n}^{-1} \ldots a_{j_2}^{-1} a_{j_1}^{-1} $$

such that $a_{i_m} \neq a_{j_n}$. Suppose $p, q \in F_n^+$ such that $z = pq^{-1}$. Since the reduced word is unique we must have $a_{i_1} a_{i_2} \ldots a_{i_m} \leq_l p$ and $a_{j_1} a_{j_2} \ldots a_{j_n} \leq_l q$. Thus, by Lemma 2.3(4), $(\mathbb{F}_n, \mathbb{F}_n^+)$ is quasi-lattice ordered. By symmetry the same argument shows that $(\mathbb{F}_n^{\text{op}}, (\mathbb{F}_n^+)^{\text{op}})$ is quasi-lattice ordered and hence $(\mathbb{F}_n, \mathbb{F}_n^+)$ is doubly quasi-lattice ordered.

Let $t \in \mathbb{F}_n^+$ and write $t = a_{n_1} a_{n_2} a_{n_3} \ldots a_{n_k}$. If $x \in \mathbb{F}_n^+$ and $x \leq_l t$ then the word of x must be an initial segment of t. Thus the set $\{x \in \mathbb{F}_n^+ : x \leq_l t\}$ can be rewritten as

$$ \{e \lessapprox_l a_{n_1} \lessapprox_l a_{n_1} a_{n_2} \lessapprox_l \ldots \lessapprox_l a_{n_1} a_{n_2} a_{n_3} \ldots a_{n_{k-1}} \lessapprox_l a_{n_1} a_{n_2} \ldots a_{n_{k-1}} a_{n_k} = t\}. $$

Thus $\{x \in \mathbb{F}_n^+ : x \leq_l t\}$ is totally ordered. For all $x, y \in \mathbb{F}_n^+$, if $x \vee_l y < \infty$ then either $x \leq_l y$ or $y \leq_l x$. Similarly, for all $x, y \in \mathbb{F}_n^+$, if $x \vee_r y < \infty$ then either $x \leq_r y$ or $y \leq_r x$.

(7). An $n \times n$ matrix $M = (m_{ij})$ is a Coxeter matrix if $m_{ij} = m_{ji} \in \{2, 3, \ldots\} \cup \{\infty\}$ for $i \neq j$ and $m_{ii} = 1$. Let $\langle a_i a_j \rangle^{m_{ij}}$ denote the word $a_i a_j a_i \ldots$ with alternating a_i and a_j of length m_{ij}. The Artin group A_M associated to M is the group on n generators $\{a_i : 1 \leq i \leq n\}$ with presentation

$$\langle \{a_i : 1 \leq i \leq n\} | \langle a_i a_j \rangle^{m_{ij}} = \langle a_j a_i \rangle^{m_{ji}} \rangle$$

If $m_{ij} = \infty$ then we impose no relation on a_i and a_j. So, for example, $\langle ab \rangle^5 = \langle ba \rangle^5$ means $ababa = babab$. We say an Artin group is *right-angled* if all the m_{ij} are either 2 or ∞. (In other words each pair of generators either commutes or has no relation.)

Let A_M be a right-angled Artin group and let A_M^+ be the subsemigroup of A_M generated by $\{a_i : 1 \leq i \leq n\}$. Then (A_M, A_M^+) is quasi-lattice ordered by [**5**, §5]. The opposite group A_M^{op} is also a right-angled Artin group and hence $(A_M^{\mathrm{op}}, (A_M^+)^{\mathrm{op}})$ is also quasi-lattice ordered. Thus (A_M, A_M^+) is doubly quasi-lattice ordered.

(8). Let $c, d \in \mathbb{N} \backslash \{0\}$. The Baumslag-Solitar group is the groups $\mathrm{BS}(c, d)$ with presentation

$$\mathrm{BS}(c, d) = \langle \{x, t\} : tx^c = x^d t \rangle.$$

Let $\mathrm{BS}(c, d)^+$ be the subsemigroup of $\mathrm{BS}(c, d)$ generated by $\{x, t\}$. Spielberg showed in [**21**, Theorem 2.11] that $(\mathrm{BS}(c, d), \mathrm{BS}(c, d)^+)$ is quasi-lattice ordered. Note that

$$\mathrm{BS}(c, d)^{\mathrm{op}} = \langle \{x, t\} : t \cdot_{\mathrm{op}} x^c = x^d \cdot_{\mathrm{op}} t \rangle = \langle \{x, t\} : x^c t = t x^d \rangle = \mathrm{BS}(d, c).$$

Thus $(\mathrm{BS}(c, d)^{\mathrm{op}}, (\mathrm{BS}(c, d)^+)^{\mathrm{op}})$ is quasi-lattice ordered and $(\mathrm{BS}(c, d), \mathrm{BS}(c, d)^+)$ is doubly quasi-lattice ordered.

(9). Spielberg also considered Baumslag-Solitar groups with negative coefficients. In [**21**, Theorem 2.12] he showed that $(\mathrm{BS}(c, -d), \mathrm{BS}(c, -d)^+)$ is quasi-lattice ordered if and only if $c = 1$. This result gives us a class of quasi-lattice ordered groups that are not doubly quasi-lattice ordered. Suppose d is positive and not equal to 1. Then $(\mathrm{BS}(1, -d), \mathrm{BS}(1, -d)^+)$ is quasi-lattice ordered. However,

$$\mathrm{BS}(1, -d)^{\mathrm{op}} = \mathrm{BS}(-d, 1) = \mathrm{BS}(d, -1),$$

and by assumption $d \neq 1$. Thus $(\mathrm{BS}(1, -d)^{\mathrm{op}}, (\mathrm{BS}(1, -d)^+)^{\mathrm{op}})$ is not quasi-lattice ordered.

There are several methods of constructing new doubly quasi-lattice ordered groups out of the known examples. The doubly quasi-lattice ordered properties are preserved under direct products, free products and semidirect products.

EXAMPLES. (1). Suppose (G, P) and (K, Q) are doubly quasi-lattice ordered groups. The direct product $(G \times K, P \times Q)$ is also a doubly quasi-lattice ordered group.

$$(g_1, k_1) \leq_l (g_2, k_2) \Leftrightarrow g_1 \leq_l g_2 \text{ and } k_1 \leq_l k_2$$

$$(g_1, k_1) \leq_r (g_2, k_2) \Leftrightarrow g_1 \leq_r g_2 \text{ and } k_1 \leq_r k_2.$$

Two elements $(g_1, k_1), (g_2, k_2) \in G \times K$ have a common left upper bound in $P \times Q$ if and only if $g_1 \vee_l g_2 < \infty$ and $k_1 \vee_l k_2 < \infty$. Then $(g_1, k_1) \vee_l (g_2, k_2) = (g_1 \vee_l g_2, k_1 \vee_l k_2)$. Similarly, if $g_1 \vee_r g_2 < \infty$ and $k_1 \vee_r k_2 < \infty$ then $(g_1, k_1) \vee_r (g_2, k_2) = (g_1 \vee_r g_2, k_1 \vee_r k_2)$.

For example, $(\mathbb{Z}^n, \mathbb{N}^n)$ is the direct product of n copies of $(\mathbb{Z}, \mathbb{N})$,

$$(\mathbb{Z}^n, \mathbb{N}^n) = (\prod_{i=1}^{n} \mathbb{Z}, \prod_{i=1}^{n} \mathbb{N}).$$

(2). Let $\{(G_i, P_i) : i \in I\}$ be a family of doubly quasi-lattice ordered groups. Let $G^* = *_{i \in I} G_i$ be the free product of the G_i and let $P^* = *_{i \in I} P_i$. Then (G^*, P^*) is a doubly quasi-lattice ordered group. The elements of P^* have reduced form $x = x_{1,i_1} x_{2,i_2} \ldots x_{m,i_m}$ where $x_{k,i_k} \in P_{i_k}$ and $i_k \neq i_{k+1}$ for all $k \leq m$. For any two elements $x, y \in P^*$ we write

$$x = x_{1,i_1} x_{2,i_2} \ldots x_{m,i_m}$$

$$y = y_{1,j_1} y_{2,j_2} \ldots y_{n,j_n}.$$

We have $x \leq_l y$ if and only if $m \leq n$, $x_{k,i_k} = y_{k,j_k}$ for $k < m$, $i_m = j_m$ and $x_{m,i_m} \leq_l y_{m,j_m}$.

Consider an element $z \in PP^{-1}$ of the form $z = x_{1,i_1} x_{2,i_2} \ldots x_{m,i_m} y_{n,j_n}^{-1} \ldots y_{1,j_1}^{-1}$ where $x_{k,i_k} \in P_{i_k}$ and $y_{l,j_l} \in P_{j_l}$. After initial cancellations z can be reduced to the form $z = x_{1,i_1} x_{2,i_2} \ldots x_{k,i_k} y_{l,j_l}^{-1} \ldots y_{1,j_1}^{-1}$ for some $k \leq m$ and $l \leq n$. If $i_k \neq j_l$ then no further cancellation is possible and we have a unique reduced form. It is then easy to see that if $z = pq^{-1}$ then $x_{1,i_1} x_{2,i_2} \ldots x_{k,i_k} \leq_l p$ and $y_{1,j_1} \ldots y_{l,j_l} \leq_l q$. If $i_k = j_l$ then $x_{k,i_k} y_{l,j_l}^{-1} \in P_{i_k} P_{i_k}^{-1}$. By Lemma 2.3(4), there exist left-minimal $a, b \in P_{i_k}$ such that $ab^{-1} = x_{k,i_k} y_{l,j_l}^{-1}$. If $z = pq^{-1}$, then $x_{1,i_1} x_{2,i_2} \ldots x_{k,i_{k-1}} a \leq_l p$ and $y_{1,j_1} \ldots y_{l,j_{l-1}} b \leq_l q$. Thus (G^*, P^*) is a quasi-lattice ordered group. By a symmetric argument we see that $((G^*)^{\mathrm{op}}, (P^*)^{\mathrm{op}})$ is also quasi-lattice ordered.

We can now describe the least upper bounds. Let $x, y \in P^*$ and write $x = x_{1,i_1} x_{2,i_2} \ldots x_{m,i_m}$ and $y = y_{1,j_1} y_{2,j_2} \ldots y_{n,j_n}$. Suppose that $x \vee_l y < \infty$. Suppose,

without loss of generality, that $m \leq n$. Write $x \vee_l y = z = z_{1,l_1} \ldots z_{n,l_n}$. Since $y \leq_l z$ we see that $y_{k,j_k} = z_{k,l_k}$ for $k < n$ and $y_{n,j_n} \leq_l z_{n,l_n}$. Similarly $x \leq_l z$ we see that $x_{k,i_k} = z_{k,l_k}$ for $k < m$ and $x_{m,i_m} \leq_l z_{m,l_m}$. If $m < n$, then $x \leq_l y$. If $m = n$ then z_{m,l_m} is a common upper bound for x_{m,i_m} and y_{m,j_m} so $z_{m,l_m} = x_{m,i_m} \vee_l y_{m,j_m}$.

Similarly, Let $x, y \in P^*$ and write $x = x_{1,i_1} x_{2,i_2} \ldots x_{m,i_m}$ and $y = y_{1,j_1} y_{2,j_2} \cdots y_{n,j_n}$. Suppose that $x \vee_r y < \infty$. Suppose, without loss of generality, that $m \leq n$. Either $x \leq_r y$ and $x \vee_r y = y$ or $m = n$ and $x_{k,i_k} = y_{k,i_k}$ for $k > 1$ and

$$x \vee_r y = (x_{1,i_1} \vee_r y_{1,j_1}) x_{2,i_2} x_{3,i_3} \ldots x_{m,i_m}.$$

(3). Let (G, P) and (K, Q) be doubly quasi-lattice ordered groups and let $\alpha : K \to$ $\text{Aut}(G)$ be a homomorphism such that, for all $k \in K$, the automorphism α_k fixes P. Then the semidirect product $(G \rtimes_\alpha K, P \times Q)$ is doubly quasi-lattice ordered. Note that for $r, s \in G$ and $x, y \in K$ we have

$$(r, x)^{-1}(s, y) = (\alpha_{x^{-1}}(r^{-1}), x^{-1})(s, y)$$
$$= (\alpha_{x^{-1}}(r^{-1}) \alpha_{x^{-1}}(s), x^{-1} y)$$
$$= (\alpha_{x^{-1}}(r^{-1} s), x^{-1} y).$$

Since α fixes P it follows that $(\alpha_{x^{-1}}(r^{-1} s), x^{-1} y) \in P \times Q$ if and only if $r^{-1} s \in P$ and $x^{-1} y \in Q$. Thus $(r, x) \leq_l (s, y)$ if and only if $r \leq_l s$ and $x \leq_l y$. Since it inherits the same left order as $(G \times K, P \times Q)$ it follows that $(G \rtimes_\alpha K, P \times Q)$ is quasi-lattice ordered. Since the left order is preserved for any $(p_1, q_1), (p_2, q_2) \in P \times Q$ we have $(p_1, q_1) \wedge_l (p_2, q_2) = (p_1 \wedge_l p_2, q_1 \wedge_l q_2)$. Thus, by Lemma 2.4(5) $((G \rtimes_\alpha K)^{\text{op}}, (P \times Q)^{\text{op}})$ is quasi-lattice ordered. We can write down the right partial order,

$$(r, x) \leq_r (t, z) \Leftrightarrow (t, z)(r, x)^{-1} \in P \times Q$$
$$\Leftrightarrow (t \alpha_{zx^{-1}}(r^{-1}), zx^{-1}) \in P \times Q$$
$$\Leftrightarrow \alpha_{zx^{-1}}(r) \leq_r t \text{ and } x \leq_r z$$
$$\Leftrightarrow \alpha_{x^{-1}}(r) \leq_r \alpha_{z^{-1}}(t) \text{ and } x \leq_r z$$

If $(r, x), (s, y) \in G \rtimes_\alpha K$, then $(r, x) \vee_r (s, y) < \infty$ if and only if $x \vee_r y < \infty$ and $\alpha_{x^{-1}}(r) \vee_r \alpha_{y^{-1}}(s) < \infty$. Further,

$$(r, x) \vee_r (s, y) = (\alpha_{x \vee_r y}(\alpha_{x^{-1}}(r) \vee_r \alpha_{y^{-1}}(s)), x \vee_r y).$$

Thus, while the left order is preserved, the right order is twisted by α.

One of the more interesting examples of a doubly quasi-lattice ordered group is the affine semigroup over the natural numbers that we define below. As we will see, the

19

left and right partial orders have very different structures. Under the right order every pair of elements has a least upper bound whereas under the left order many pairs have no common upper bound.

DEFINITION 2.9. Let $\mathbb{Q} \rtimes \mathbb{Q}_+^*$ denote the semidirect product of the additive rationals $\mathbb{Q}$ and the positive multiplicative rationals $\mathbb{Q}^*$ with operations defined by

$$(r, x)(s, y) = (r + xs, xy) \text{ for } r, s \in \mathbb{Q} \text{ and } x, y \in \mathbb{Q}_+^*$$

and

$$(r, x)^{-1} = (-x^{-1}r, x^{-1}) \text{ for } r \in \mathbb{Q} \text{ and } x \in \mathbb{Q}_+^*.$$

Laca and Raeburn proved in [**12**, Proposition 2.2] that $(\mathbb{Q} \rtimes \mathbb{Q}_+^*, \mathbb{N} \rtimes \mathbb{N}^\times)$ is a quasi-lattice ordered group with left partial order:

$$(r, x) \leq_l (s, y) \Leftrightarrow (r, x)^{-1}(s, y) \in \mathbb{N} \rtimes \mathbb{N}^\times$$

$$\Leftrightarrow (-x^{-1}r + x^{-1}s, x^{-1}y) \in \mathbb{N} \rtimes \mathbb{N}^\times.$$

(Note that in this case, the automorphism does not fix $\mathbb{N}$ so we cannot apply example (3) above.) To prove that $(\mathbb{Q} \rtimes \mathbb{Q}_+^*, \mathbb{N} \rtimes \mathbb{N}^\times)$ is *doubly* quasi-lattice ordered we will now show that $((\mathbb{Q} \rtimes \mathbb{Q}_+^*)^{\mathrm{op}}, (\mathbb{N} \rtimes \mathbb{N}^\times)^{\mathrm{op}})$ is also quasi-lattice ordered. The partial ordering on $((\mathbb{Q} \rtimes \mathbb{Q}_+^*)^{\mathrm{op}}, (\mathbb{N} \rtimes \mathbb{N}^\times)^{\mathrm{op}})$ is given by:

$$(r, x) \leq_r (s, y) \Leftrightarrow (r, x)^{-1} \cdot_{\mathrm{op}} (s, y) \in (\mathbb{N} \rtimes \mathbb{N}^\times)^{\mathrm{op}}$$

(2.1) $$\Leftrightarrow (s - yx^{-1}r, yx^{-1}) \in (\mathbb{N} \rtimes \mathbb{N}^\times)^{\mathrm{op}}$$

PROPOSITION 2.10. *The pair* $((\mathbb{Q} \rtimes \mathbb{Q}_+^*)^{\mathrm{op}}, (\mathbb{N} \rtimes \mathbb{N}^\times)^{\mathrm{op}})$ *is a quasi-lattice ordered group.*

PROOF. By Lemma 2.3(3) it suffices to show that every element $(r, x) \in (\mathbb{Q} \rtimes \mathbb{Q}_+^*)^{\mathrm{op}}$ with an upper bound in $(\mathbb{N} \rtimes \mathbb{N}^\times)^{\mathrm{op}}$ has a least upper bound in $(\mathbb{N} \rtimes \mathbb{N}^\times)^{\mathrm{op}}$. In fact we can show more: that *every* $(r, x) \in (\mathbb{Q} \rtimes \mathbb{Q}_+^*)^{\mathrm{op}}$ has a least upper bound in $(\mathbb{N} \rtimes \mathbb{N}^\times)^{\mathrm{op}}$.

Let $(r, x) \in (\mathbb{Q} \rtimes \mathbb{Q}_+^*)^{\mathrm{op}}$. Write $r = \frac{p}{q}$ with $p \in \mathbb{Z}$, $q \in \mathbb{N}^\times$ and p, q coprime. Write $x = \frac{a}{b}$ with $a, b \in \mathbb{N}$ and a, b coprime. Let $m = \max\left\{0, \frac{bp}{\gcd(q,b)}\right\}$. We claim that $\left(m, \frac{aq}{\gcd(q,b)}\right)$ is an upper bound for (r, x) in $(\mathbb{N} \rtimes \mathbb{N}^\times)^{\mathrm{op}}$ and further it is the least such.

First, let us demonstrate that $\left(m, \frac{aq}{\gcd(q,b)}\right)$ is an upper bound for (r, x). By (2.1) we must show $m - \frac{aq}{\gcd(q,b)}x^{-1}r \in \mathbb{N}$ and $\frac{aq}{\gcd(q,b)}x^{-1} \in \mathbb{N}^\times$. Compute:

$$m - \frac{aq}{\gcd(q,b)}x^{-1}r = m - \frac{aq}{\gcd(q,b)}\frac{b}{a}\frac{p}{q} = m - \frac{bp}{\gcd(q,b)} = \begin{cases} 0 & \text{if } m = \frac{bp}{\gcd(q,b)} \\ -\frac{bp}{\gcd(q,b)} & \text{if } m = 0. \end{cases}$$

20

Note that if $m = 0$ then $p \leq 0$ and hence $-\frac{bp}{\gcd(q,b)} \in \mathbb{N}$. In either case $m - \frac{bp}{\gcd(q,b)} \in \mathbb{N}$. Also compute:

$$\frac{aq}{\gcd(q,b)}x^{-1} = \frac{aq}{\gcd(q,b)}\frac{b}{a} = \frac{bq}{\gcd(q,b)} \in \mathbb{N}^{\times}.$$

Thus $\left(m, \frac{aq}{\gcd(q,b)}\right)$ is an upper bound for (r, x).

Second we show that $\left(m, \frac{aq}{\gcd(q,b)}\right)$ is the *least* upper bound in $(\mathbb{N} \rtimes \mathbb{N}^{\times})^{\mathrm{op}}$. Let $(k, c) \in (\mathbb{N} \rtimes \mathbb{N}^{\times})^{\mathrm{op}}$ be an upper bound for (r, x). We must show that $\left(m, \frac{aq}{\gcd(q,b)}\right) \leq_r (k, c)$ i.e. that $k - c(\frac{\gcd(q,b)}{aq})m$ and $c\frac{\gcd(q,b)}{aq} \in \mathbb{N}^{\times}$.

Consider $k - c(\frac{\gcd(q,b)}{aq})m$. If $m = 0$ then $k - c(\frac{\gcd(q,b)}{aq})m = k \in \mathbb{N}$. If $m = \frac{bp}{\gcd(q,b)} > 0$ then

$$k - c\left(\frac{\gcd(q,b)}{aq}\right)m = k - c\left(\frac{\gcd(q,b)}{aq}\right)\frac{bp}{\gcd(q,b)} = k - c\frac{b}{a}\frac{p}{q} = k - cx^{-1}r.$$

But $(r, x) \leq_r (k, c)$ and so, by (2.1), we have $k - cx^{-1}r \in \mathbb{N}$. Hence $k - c(\frac{\gcd(q,b)}{aq})m \in \mathbb{N}$.

To prove that $c\frac{\gcd(q,b)}{aq} \in \mathbb{N}^{\times}$ we must know more about c. We claim that c is a multiple of $\frac{aq}{\gcd(q,b)}$ and hence that $c\frac{\gcd(q,b)}{aq} \in \mathbb{N}^{\times}$. Since $cx^{-1} = \frac{cb}{a} \in \mathbb{N}^{\times}$ it follows c is a multiple of a. Thus we may write $c = \gamma a$ with $\gamma \in \mathbb{N}$.

Since $k - cx^{-1}r \in \mathbb{N}$ and $k \in \mathbb{N}$ it follows that $cx^{-1}r \in \mathbb{Z}$. Now consider

$$cx^{-1}r = \gamma a\frac{b}{a}\frac{p}{q} = \frac{\gamma bp}{q} = \frac{\gamma bp}{q} \cdot \frac{\frac{1}{\gcd(q,b)}}{\frac{1}{\gcd(q,b)}} = \frac{\gamma\frac{b}{\gcd(q,b)}p}{\frac{q}{\gcd(q,b)}} \in \mathbb{Z}.$$

We know that p and q are coprime and that $\frac{b}{\gcd(q,b)}$ and $\frac{q}{\gcd(q,b)}$ are coprime. Thus γ must be a multiple of $\frac{q}{\gcd(q,b)}$ and so c is a multiple of $\frac{aq}{\gcd(q,b)}$. We have shown that $c\frac{\gcd(q,b)}{aq} \in \mathbb{N}^{\times}$ and hence that $\left(m, \frac{aq}{\gcd(q,b)}\right) \leq_r (k, c)$. Hence $\left(m, \frac{aq}{\gcd(q,b)}\right)$ is the least upper bound for (r, x) in $(\mathbb{N} \rtimes \mathbb{N}^{\times})^{\mathrm{op}}$.

Thus $((\mathbb{Q} \rtimes \mathbb{Q}_+^*)^{\mathrm{op}}, (\mathbb{N} \rtimes \mathbb{N}^{\times})^{\mathrm{op}})$ is a quasi-lattice ordered group. $\qquad\square$

REMARK. We have shown that $((\mathbb{Q} \rtimes \mathbb{Q}_+^*)^{\mathrm{op}}, (\mathbb{N} \rtimes \mathbb{N}^{\times})^{\mathrm{op}})$ is a quasi-lattice ordered group but we can go further. Since every element of $(\mathbb{Q} \rtimes \mathbb{Q}_+^*)^{\mathrm{op}}$ has a least common upper bound it follows that every pair $(m, a), (n, b) \in (\mathbb{N} \rtimes \mathbb{N}^{\times})^{\mathrm{op}}$ has a least common right upper bound in $(\mathbb{N} \rtimes \mathbb{N}^{\times})^{\mathrm{op}}$, namely

$$(m, a) \vee_r (n, b) = \left(\mathrm{lcm}(a, b)\max\{a^{-1}m, b^{-1}n\}, \mathrm{lcm}(a, b)\right).$$

The observation that every pair of elements $(m, a), (n, b) \in (\mathbb{N} \rtimes \mathbb{N}^{\times})^{\mathrm{op}}$ has a least common upper bound stands in sharp contrast to the behaviour of the partial order on $(\mathbb{Q} \rtimes \mathbb{Q}_+^*, \mathbb{N} \rtimes \mathbb{N}^{\times})$. By [**12**, Remark 2.3], a pair of elements $(m, a), (n, b) \in \mathbb{N} \rtimes \mathbb{N}^{\times}$

has a least common upper bound if and only if $(m + a\mathbb{N}) \cap (n + b\mathbb{N})$ is nonempty. In particular

$$(m, a) \vee_l (n, b) = \begin{cases} \infty & \text{if } (m + a\mathbb{N}) \cap (n + b\mathbb{N}) = \emptyset \\ \big(l, \text{lcm}(a, b)\big) & \text{if } (m + a\mathbb{N}) \cap (n + b\mathbb{N}) \neq \emptyset \end{cases}$$

where l is the minimum element in $(m + a\mathbb{N}) \cap (n + b\mathbb{N})$.

This example demonstrates that the two partial orders of a doubly quasi-lattice ordered group may be very different and that we are getting new information by considering both partial orders.

2.3. Covariant partial isometric representations

Let T be a bounded operator on a Hilbert space H. Then T is a *partial isometry* if $\|Th\| = \|h\|$ for all $h \in (\ker T)^\perp$. Equivalently, T is a partial isometry if and only if $TT^*T = T$. Then T^*T is the orthogonal projection onto $(\ker T)^\perp$, and TT^* is the orthogonal projection onto the range of T. In an arbitrary C^*-algebra we say $a \in A$ is a partial isometry if $aa^*a = a$, i.e. if the representation of a on a Hilbert space is a partial isometry. (This definition also holds on a $*$-algebra.)

DEFINITION 2.11. Let A be a unital C^*-algebra and let (G, P) be a doubly quasi-lattice ordered group. A *representation of P by partial isometries* is a map $W : P \to A$ such that W_p is a partial isometry for all $p \in P$, $W_e = 1$ where e is the identity of G and $W_x W_y = W_{xy}$ for $x, y \in P$. A representation is *covariant* if it satisfies

$$(2.2) \qquad W_x W_x^* W_y W_y^* = \begin{cases} W_{x \vee_l y} W_{x \vee_l y}^* & \text{if } x \vee_l y < \infty. \\ 0 & \text{otherwise.} \end{cases}$$

$$(2.3) \qquad W_x^* W_x W_y^* W_y = \begin{cases} W_{x \vee_r y}^* W_{x \vee_r y} & \text{if } x \vee_r y < \infty. \\ 0 & \text{otherwise.} \end{cases}$$

From here on we will refer to these representations as *covariant partial isometric representations*.

Let W be a covariant partial isometric representation. Let $x, y \in P$. The covariance conditions (2.2) and (2.3) ensure that the range projections $W_x W_x^*$ and $W_y W_y^*$ commute as do the source projections $W_x^* W_x$ and $W_y^* W_y$. Given two partial isometries S and T [6, Lemma 2] states that their composition ST is a partial isometry if and only if S^*S and TT^* commute. The product $W_x W_y = W_{xy}$ is a partial isometry and hence, $W_x^* W_x$ and $W_y W_y^*$ commute. Therefore, for any covariant partial isometric representation

22

$W : P \to A$, the set $\{W_x W_x^* : x \in P\} \cup \{W_y^* W_y : y \in P\}$ is a family of commuting projections.

Since we write $x \vee y = \infty$ if the least upper bound $x \vee y$ does not exist, we will use the convention $W_\infty = 0$. Thus we can always write $W_x W_x^* W_y W_y^* = W_{x \vee_l y} W_{x \vee_l y}^*$ and $W_x^* W_x W_y^* W_y = W_{x \vee_r y}^* W_{x \vee_r y}$. With this convention we can simplify our covariance relations into an equivalent form better suited for calculations. Note that W_y is a partial isometry so $W_y = W_y W_y^* W_y$. We compute:

$$
\begin{aligned}
W_x^* W_y &= (W_x^* W_x W_x^*)(W_y W_y^* W_y) \\
&= W_x^* W_{x \vee_l y} W_{x \vee_l y}^* W_y \quad \text{(by (2.2))} \\
&= W_x^* W_x W_{x^{-1}(x \vee_l y)} W_{x^{-1}(x \vee_l y)}^* W_x^* W_y \\
&= W_{x^{-1}(x \vee_l y)} W_{x^{-1}(x \vee_l y)}^* W_x^* W_x W_x^* W_y \quad \text{(projections commute)} \\
&= W_{x^{-1}(x \vee_l y)} W_{x^{-1}(x \vee_l y)}^* W_x^* W_y \\
&= W_{x^{-1}(x \vee_l y)} W_{x \vee_l y}^* W_y.
\end{aligned}
$$

Thus we have

(2.4) $$ W_x^* W_y = W_{x^{-1}(x \vee_l y)} W_{x \vee_l y}^* W_y. $$

A similar argument using (2.3) shows that

(2.5) $$ W_x W_y^* = W_x W_{x \vee_r y}^* W_{(x \vee_r y)y^{-1}}. $$

NOTATION. In (2.4) and (2.5) there is an abuse of notation, namely that if $x \vee_l y = \infty$ then the product $x^{-1}(x \vee_l y) = x^{-1}\infty$ is undefined. This does not cause problems: if $x \vee_l y = \infty$ then $W_x^* W_{x \vee_l y} W_{x \vee_l y}^* W_y = 0$ and hence in (2.4) $W_x^* W_y = 0$. We impose the convention that if $x \vee_l y = \infty$ then for all $g \in G$ we write $g(x \vee_l y) = \infty = (x \vee_l y)g$ and $W_{g(x \vee_l y)} = 0$. Similarly for the right order: we let $g(x \vee_r y) = \infty = (x \vee_r y)g$ and $W_{g(x \vee_r y)} = 0$ whenever $x \vee_r y = \infty$.

We now have two equivalent covariance conditions. We use them interchangeably: (2.2) and (2.3) are easier to check and give a clearer picture of what is going on, while (2.4) and (2.5) are easier to use in calculations.

2.4. Examples of covariant partial isometric representations

EXAMPLES. (1). Consider $(\mathbb{Z}, \mathbb{N})$. Let T be a partial isometry on a Hilbert space H such that, for all $n \in \mathbb{N}$, T^n is a partial isometry. We say T is a power partial isometry. The map $W : \mathbb{N} \to B(H)$ defined by $W_n = T^n$ and $W_0 = T^0 = 1$, is

a covariant partial isometric representation of $\mathbb{N}$. A simple calculation shows that $W_m W_n = T^m T^n = T^{m+n} = W_{m+n}$. Also we have covariance:

$$W_m W_m^* W_n W_n^* = T^m T^{*m} T^n T^{*n}$$

$$= \begin{cases} T^m T^{*m-n} T^{*n} T^n T^{*n} & \text{if } m \geq n \\ T^m T^{*m} T^m T^{n-m} T^{*n} & \text{if } n > m \end{cases}$$

$$= \begin{cases} T^m T^{*m-n} T^{*n} & \text{if } m \geq n \\ T^m T^{n-m} T^{*n} & \text{if } n > m \end{cases}$$

$$= \begin{cases} T^m T^{*m} & \text{if } m \geq n \\ T^n T^{*n} & \text{if } n > m \end{cases}$$

$$= T^{\max\{m,n\}} T^{*\max\{m,n\}}$$

$$= W_{\max\{m,n\}} W_{\max\{m,n\}}^*.$$

Similarly, $W_m^* W_m W_n^* W_n = W_{\max\{m,n\}}^* W_{\max\{m,n\}}$. Thus W is covariant.

Let T be the truncated shift on $\mathbb{C}^n$ such that

$$T(x_1, x_2, \ldots, x_{n-1}, x_n) = (0, x_1, x_2, \ldots, x_{n-1}).$$

Written in terms of the usual basis elements $\{e_i : 1 \leq i \leq n\}$ for $\mathbb{C}^n$ we see that

$$T e_i = \begin{cases} e_{i+1} & \text{if } i < n \\ 0 & \text{otherwise.} \end{cases}$$

The truncated shift T is a power partial isometry and is one of the standard examples. Halmos and Wallen proved [**6**, Theorem 1] that every power partial isometry is composed of a direct sum of a unitary and copies of the unilateral shift, the backwards unilateral shift and copies of the truncated shifts on $\mathbb{C}^n$ for all $n \in \mathbb{N}$.

(2). Let T_1, T_2 be a pair of star-commuting power partial isometries on a Hilbert space H, that is $T_1 T_2 = T_2 T_1$ and $T_1^* T_2 = T_2 T_1^*$. The map $W : \mathbb{N}^2 \to B(H)$ given by $W_{(m,n)} = T_1^m T_2^n$ is a covariant partial isometric representation of $\mathbb{N}^2$. Since T_1, T_2 star-commute their composition is also a power partial isometry. To see W is covariant compute:

$$W_{(m,n)} W_{(m,n)}^* W_{(p,q)} W_{(p,q)}^* = T_1^m T_2^n T_2^{*n} T_1^{*m} T_1^p T_2^q T_2^{*q} T_1^{*p}$$

$$= T_1^m T_1^{*m} T_1^p T_1^{*p} T_2^n T_2^{*n} T_2^q T_2^{*q}$$

$$= T_1^{\max\{m,p\}} T_1^{*\max\{m,p\}} T_2^{\max\{n,q\}} T_2^{*\max\{n,q\}}$$

$$= T_1^{\max\{m,p\}} T_2^{\max\{n,q\}} T_1^{*\max\{m,p\}} T_2^{*\max\{n,q\}}$$

$$= W_{(m,n)\vee_l(p,q)} W_{(m,n)\vee_l(p,q)}^*.$$

The two partial orders are the same, hence we also have

$$W_{(m,n)}^* W_{(m,n)} W_{(p,q)}^* W_{(p,q)} = W_{(m,n)\vee_r(p,q)}^* W_{(m,n)\vee_r(p,q)}$$

and (2.3) holds.

(3). Our definition of covariance and Nica's original definition in [**16**] (see (1.1)) for isometric representations are very similar. In general Nica's covariant isometric representations are not covariant in our sense. An isometric representation of P cannot be covariant (in the sense of Definition 2.11) unless $p \vee_r q < \infty$ for all $p, q \in P$. As a counterexample suppose that $W : P \to A$ is an isometric representation and that $p, q \in P$ such that $p \vee_r q = \infty$. Since W_p, W_q are isometries we have $W_p^* W_p W_q^* W_q = 1 \neq 0$. So W is not covariant in the sense of (2.3).

Since the truncated shift on $\mathbb{C}^n$ is so central to the study of power partial isometries it makes sense for us to attempt to find an analogous example for general doubly quasi-lattice ordered groups:

LEMMA 2.12. *Let* (G, P) *be a doubly quasi-lattice ordered group. Fix* $a \in P$ *and define* $I_a := \{x \in P : x \leq_r a\}$. *The map* $J^a : P \to B(\ell^2(I_a))$ *defined by*

$$J_p^a \delta_x = \begin{cases} \delta_{px} & \text{if } px \in I_a \\ 0 & \text{otherwise} \end{cases}$$

is a covariant partial isometric representation.

PROOF. Observe that J_p^a is isometric on $\text{span}\{\delta_x : px \in I_a\}$ and 0 otherwise. Thus J_p^a is a partial isometry for all $p \in P$. To show that J^a is a representation of P we must show that $J_e^a = 1$ and $J_p^a J_q^a = J_{pq}^a$.

Fix $x \in I_a$. Compute $J_e^a \delta_x = \delta_{ex} = \delta_x$. Thus $J_e^a = 1$.

Second, let $p, q \in P$ and compute:

$$J_p^a J_q^a \delta_x = \begin{cases} J_p^a \delta_{qx} & \text{if } qx \in I_a \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \delta_{pqx} & \text{if } qx \in I_a \text{ and } pqx \in I_a \\ 0 & \text{otherwise.} \end{cases}$$

However $pqx \in I_a$ implies $pqx \leq_r a$ which implies $a(pqx)^{-1} \in P$. Then $a(pqx)^{-1}p = a(qx)^{-1} \in P$, so $qx \leq_r a$. Thus $pqx \in I_a$ implies $qx \in I_a$. Therefore:

$$J_p^a J_q^a \delta_x = \begin{cases} \delta_{pqx} & \text{if } pqx \in I_a \\ 0 & \text{otherwise.} \end{cases}$$

$$= J_{pq}^a \delta_x.$$

We claim that the adjoint J_p^{a*} satisfies

$$J_p^{a*} \delta_x = \begin{cases} \delta_{p^{-1}x} & \text{if } p \leq_l x \\ 0 & \text{otherwise.} \end{cases}$$

Let $y \in I_a$. Then consider the inner product:

$$(J_p^a \delta_x \mid \delta_y) = \begin{cases} (\delta_{px} \mid \delta_y) & \text{if } px \in I_a \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{if } y = px \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{if } p^{-1}y = x \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} (\delta_x \mid \delta_{p^{-1}y}) & \text{if } p^{-1}y \in I_a \\ 0 & \text{otherwise.} \end{cases}$$

Now it follows that $J_p^{a*} \delta_x = \delta_{p^{-1}x}$ if $p^{-1}x \in I_a$. Note that for all $x \in I_a$ and $p \in P$, $p^{-1}x \leq_r a$. Thus $p^{-1}x \in I_a$ if and only if $p \leq_l x$. We have thus proved our claim.

To show J^a is a covariant representation we start by computing the range and source projections:

$$J_q^a J_q^{a*} \delta_x = \begin{cases} J_q^a \delta_{q^{-1}x} = \delta_x & \text{if } q \leq_l x \\ 0 & \text{otherwise} \end{cases}$$

$$J_q^{a*} J_q^a \delta_x = \begin{cases} J_q^{a*} \delta_{qx} = \delta_x & \text{if } qx \leq_r a \\ 0 & \text{otherwise.} \end{cases}$$

Using the above projections, compute:

$$J_p^a J_p^{a*} J_q^a J_q^{a*} \delta_x = \begin{cases} J_p^a J_p^{a*} \delta_x & \text{if } q \leq_l x \\ 0 & \text{otherwise} \end{cases}$$

26

$$
= \begin{cases} \delta_x & \text{if } q \leq_l x \text{ and } p \leq_l x \\ 0 & \text{otherwise.} \end{cases}
$$

If $q \leq_l x$ and $p \leq_l x$, then x is an upper bound for p and q and so $p \vee_l q$ exists and $p \vee_l q \leq x$. Conversely, if $p \vee_l q \leq x$ then $p, q \leq_l x$. Thus:

$$
J_p^a J_p^{a*} J_q^a J_q^{a*} \delta_x = \begin{cases} \delta_x & \text{if } p \vee_l q \leq_l x \\ 0 & \text{otherwise} \end{cases}
$$

$$
= J_{p \vee_l q}^a J_{p \vee_l q}^{a*} \delta_x.
$$

Similarly,

$$
J_p^{a*} J_p^a J_q^{a*} J_q^a \delta_x = \begin{cases} J_p^{a*} J_p^a \delta_x & \text{if } qx \leq_r a \\ 0 & \text{otherwise} \end{cases}
$$

$$
= \begin{cases} \delta_x & \text{if } px \leq_r a \text{ and } qx \leq_r a \\ 0 & \text{otherwise} \end{cases}
$$

By right-invariance $px \leq_r a \Leftrightarrow p \leq_r ax^{-1}$ and $qx \leq_r a \Leftrightarrow q \leq_r ax^{-1}$. Thus:

$$
J_p^{a*} J_p^a J_q^{a*} J_q^a \delta_x = \begin{cases} \delta_x & \text{if } p \leq_r ax^{-1} \text{ and } q \leq_r ax^{-1} \\ 0 & \text{otherwise} \end{cases}
$$

Since $x \leq_r a$, we know $ax^{-1} \in P$. If $p \leq_r ax^{-1}$ and $q \leq_r ax^{-1}$, then ax^{-1} is a right upper bound for p and q. Hence $p \vee_r q$ exists and $p \vee_r q \leq_r ax^{-1}$. Conversely, if $p \vee_r q \leq_r ax^{-1}$, then $p \leq_r ax^{-1}$ and $q \leq_r ax^{-1}$. Thus:

$$
J_p^{a*} J_p^a J_q^{a*} J_q^a \delta_x = \begin{cases} \delta_x & \text{if } p \vee_r q \leq_r ax^{-1} \\ 0 & \text{otherwise} \end{cases}
$$

$$
= J_{p \vee_r q}^{a*} J_{p \vee_r q}^a \delta_x
$$

Thus J^a is a covariant partial isometric representation of P. $\qquad \square$

We will show later, in Lemma 2.16, that the family of partial isometric representations $\{J^a : P \to B(\ell^2(I_a)) : a \in P\}$ of Lemma 2.12 is well behaved and has very useful properties.

2.5. Properties of covariant partial isometric representations

The covariance conditions, (2.4) and (2.5), are extremely powerful and allow us to simplify any product of $\{W_p\} \cup \{W_p^*\}$ into a product of just three terms and to write down a formula for the multiplication of these terms.

LEMMA 2.13. *Let (G, P) be a doubly quasi-lattice ordered group and let $W : P \to A$ be a covariant partial isometric representation. Any product of the form $W_{n_1} W_{n_2}^* W_{n_3} W_{n_4}^* \ldots W_{n_m}$ where $n_i \in P$, is either 0 or of the form $W_p W_q^* W_r$ for some $p, q, r \in P$ satisfying $p \leq_r q$ and $r \leq_l q$.*

PROOF. We will prove this lemma by induction on the length of the product m. By adding W_e and W_e^* to the product we may assume that $m \geq 3$. We begin by proving that the result holds for $m = 3$.

Fix $a, b, c \in P$. Now compute:

$$W_a W_b^* W_c = W_a W_{a \vee_r b}^* W_{(a \vee_r b)b^{-1}} W_c \quad \text{by (2.5)}$$
$$= W_a W_{a \vee_r b}^* W_{(a \vee_r b)b^{-1}c}$$

Recall that by convention, if $a \vee_r b = \infty$, then $W_{a \vee_r b}^* = 0$. To prevent the equation from becoming unreadable let $x = a \vee_r b$ and $y = (a \vee_r b)b^{-1}c$. Then:

$$W_a W_b^* W_c = W_a W_x^* W_y$$
$$= W_a W_{x^{-1}(x \vee_l y)} W_{(x \vee_l y)}^* W_y \quad \text{by (2.4)}$$
$$= W_{ax^{-1}(x \vee_l y)} W_{(x \vee_l y)}^* W_y$$

So we take $p = ax^{-1}(x \vee_l y)$, $q = (x \vee_l y)$ and $r = y$. Examining the indices we see immediately that $r \leq_l q$. To see $p \leq_r q$ we compute:

$$qp^{-1} = (x \vee_l y)[ax^{-1}(x \vee_l y)]^{-1} = (x \vee_l y)(x \vee_l y)^{-1}xa^{-1} = xa^{-1} = (a \vee_r b)a^{-1}$$

Thus $qp^{-1} = (a \vee_r b)a^{-1} \in P$ and hence $p \leq_r q$. Thus any product $W_a W_b^* W_c$ is either zero or can be written as $W_p W_q^* W_r$ with $p \leq_r q$ and $r \leq_l q$ and the result holds for $m = 3$. Now suppose that the result holds for $m = k$ and consider $m = k + 1$. We have two cases to consider: if $k + 1$ is odd or even which determines whether the product ends in an adjoint.

If $k + 1$ is odd, then the product ends in $W_{n_{k+1}}$. By assumption $W_{n_1} \ldots W_{n_k}^* W_{n_{k+1}}$ is either zero or may be simplified to $W_p W_q^* W_r W_{n_{k+1}}$ by our assumption. We may then simplify again to see $W_p W_q^* W_r W_{n_{k+1}} = W_p W_q^* W_{rn_{k+1}}$. We have a product of length 3 which may be simplified to the desired form as above.

If $k+1$ is even, then the product ends in $W^*_{n_{k+1}}$. We simplify the first k terms $W_{n_1}W^*_{n_2}W_{n_3}W^*_{n_4}\ldots W^*_{n_k}$ which gives 0 or $W_pW^*_qW_r$. Then consider $W_pW^*_qW_rW^*_{n_{k+1}}$. Compute:

$$W_pW^*_qW_rW^*_{n_{k+1}} = W_pW^*_qW_rW^*_{r\vee_r n_{k+1}}W_{(r\vee_r n_{k+1})n_{k+1}^{-1}}. \quad \text{(by (2.5))}$$

$$= W_pW^*_{r^{-1}q}(W^*_rW_rW^*_r)W^*_{(r\vee_r n_{k+1})r^{-1}}W_{(r\vee_r n_{k+1})n_{k+1}^{-1}}$$

$$= W_pW^*_{r^{-1}q}W^*_rW^*_{(r\vee_r n_{k+1})r^{-1}}W_{(r\vee_r n_{k+1})n_{k+1}^{-1}} \quad (W^*_rW_rW^*_r = W^*_r)$$

$$= W_pW^*_{(r\vee_r n_{k+1})r^{-1}q}W_{(r\vee_r n_{k+1})n_{k+1}^{-1}}.$$

So we have reduced the product to one of length 3 which we can put into the desired form. By induction the proof is complete. $\square$

LEMMA 2.14. *Let (G,P) be a doubly quasi-lattice ordered group. Let $W : P \to A$ be a covariant partial isometric representation. Let $a,b,c,p,q,r \in P$ such that $a \leq_r b$, $c \leq_l b$, $p \leq_r q$ and $r \leq_l q$. Then*

(1) $(W_aW^*_bW_c)(W_pW^*_qW_r) = W_{ab^{-1}(b\vee_l cp)}W^*_{(cp\vee_r q)(cp)^{-1}(b\vee_l cp)}W_{(cp\vee_r q)q^{-1}r}$

(2) $r^{-1}q \leq_r q$, $qp^{-1} \leq_l q$ *and* $(W_pW^*_qW_r)^* = W_{r^{-1}q}W^*_qW_{qp^{-1}}$.

PROOF. (1). Compute:

$$(W_aW^*_bW_c)(W_pW^*_qW_r) = W_aW^*_b(W_{cp}W^*_q)W_r$$

$$= W_aW^*_b(W_{cp}W^*_{(cp\vee_r q)}W_{(cp\vee_r q)q^{-1}})W_r \quad \text{(by (2.5))}$$

$$= W_a(W^*_bW_{cp})W^*_{(cp\vee_r q)}W_{(cp\vee_r q)q^{-1}r}$$

$$= W_a(W_{b^{-1}(b\vee_l cp)}W^*_{(b\vee_l cp)}W_{cp})W^*_{(cp\vee_r q)}W_{(cp\vee_r q)q^{-1}r} \quad \text{(by (2.4))}$$

$$= W_{ab^{-1}(b\vee_l cp)}W^*_{(cp)^{-1}(b\vee_l cp)}(W^*_{cp}W_{cp}W^*_{cp})W^*_{(cp\vee_r q)(cp)^{-1}}W_{(cp\vee_r q)q^{-1}r}$$

$$= W_{ab^{-1}(b\vee_l cp)}W^*_{(cp)^{-1}(b\vee_l cp)}W^*_{cp}W^*_{(cp\vee_r q)(cp)^{-1}}W_{(cp\vee_r q)q^{-1}r}$$

$$= W_{ab^{-1}(b\vee_l cp)}W^*_{(cp\vee_r q)(cp)^{-1}(b\vee_l cp)}W_{(cp\vee_r q)q^{-1}r}.$$

(2) Compute:

$$(W_pW^*_qW_r)^* = W^*_rW_qW^*_p$$

$$= W_{r^{-1}(r\vee_l q)}W^*_{r\vee_l q}W_qW^*_p \quad \text{by (2.4)}$$

$$= W_{r^{-1}q}W^*_qW_qW^*_p \quad (r\vee_l q = q)$$

$$= W_{r^{-1}q}W^*_qW_qW^*_{q\vee_r p}W_{(q\vee_r p)p^{-1}} \quad \text{by (2.5)}$$

$$= W_{r^{-1}q}W^*_qW_qW^*_qW_{qp^{-1}} \quad (q\vee_r p = q)$$

$$= W_{r^{-1}q}W^*_qW_{qp^{-1}} \qquad\qquad \square$$

29

Since $q(r^{-1}q)^{-1} = qq^{-1}r = r \in P$ we have $r^{-1}q \leq_r q$. Similarly, $qp^{-1} \leq_l q$ because $(qp^{-1})^{-1}q = pq^{-1}q = p \in P$.

In the previous two lemmas we have established the standard form for products of covariant partial isometric representations and how these triple product multiply. In Lemma 2.16 we will show that there is a partial isometric representation J of P such that the set $\{J_p J_q^* J_r : p \leq_r q, r \leq_l q\}$ is linearly independent. This fact is crucial in our construction of the universal algebra in the next chapter.

DEFINITION 2.15. Let (G, P) be a doubly quasi-lattice ordered group. Define $J : P \to B(\bigoplus_{a \in P} \ell^2(I_a))$ by $J_p := \bigoplus_{a \in P} J_p^a$. Define the C^*-algebra $C_{\mathrm{ts}}^*(G, P, P^{\mathrm{op}})$ to be the C^*-subalgebra of $B(\bigoplus_{a \in P} \ell^2(I_a))$ generated by $\{J_p : p \in P\}$.

We choose the notation $C_{\mathrm{ts}}^*(G, P, P^{\mathrm{op}})$ advisedly. When we construct our universal algebra in Chapter 3, we will call it $C^*(G, P, P^{\mathrm{op}})$ to stay consistent with Nica's notation of $C^*(G, P)$ for the universal algebra generated by isometric representations in [16]. We have added P^{op} to distinguish the two algebras and to emphasise that we have two partial orders and are dealing with partial isometries. The C^*-algebra $C_{\mathrm{ts}}^*(G, P, P^{\mathrm{op}})$ is a concretely defined C^*-algebra of operators on a particular Hilbert space, and we regard it as a "reduced algebra". In our proofs $C_{\mathrm{ts}}^*(G, P, P^{\mathrm{op}})$ takes a similar role to $C_r^*(G) \subseteq B(\ell^2(G))$ for group algebras. Here the subscript ts stands for truncated shift, as the J_p^a were constructed to generalise the properties of the truncated shift on $\mathbb{C}^n$. In fact for $(\mathbb{Z}, \mathbb{N})$, $C_{\mathrm{ts}}^*(\mathbb{Z}, \mathbb{N}, \mathbb{N}^{\mathrm{op}})$ and $C^*(\mathbb{Z}, \mathbb{N}, \mathbb{N}^{\mathrm{op}})$ are both isomorphic to the C^*-algebra $C^*(J)$ studied by Hancock and Raeburn in [7, Theorem 1.3].

LEMMA 2.16. *Let (G, P) be a doubly quasi-lattice ordered group. The set*

$$S := \{J_p J_q^* J_r : p, q, r \in P, p \leq_r q, r \leq_l q\}$$

is linearly independent and $\operatorname{span} S$ *is a dense unital $*$-subalgebra of* $C_{\mathrm{ts}}^*(G, P, P^{\mathrm{op}})$.

PROOF. Suppose, aiming for a contradiction, that S is linearly dependent. Then there exists a set of nonzero complex numbers $\{\lambda_i : 1 \leq i \leq n\}$ and a set of distinct triples $\{(p_i, q_i, r_i) : 1 \leq i \leq n, p_i \leq_r q_i, r_i \leq_l q_i\}$ with $(p_i, q_i, r_i) \neq (p_j, q_j, r_j)$ for $i \neq j$, such that

$$\sum_{i=1}^n \lambda_i J_{p_i} J_{q_i}^* J_{r_i} = 0.$$

Since $J_p = \bigoplus_{a \in P} J_p^a$ it follows that $\sum_{i=1}^n \lambda_i J_{p_i}^a J_{q_i}^{a*} J_{r_i}^a = 0$ for all $a \in P$. We will select a particular $a \in P$ where a contradiction will fall out nicely. By Lemma 2.8 every partially ordered finite set has a minimal element. Thus the set $\{r_i : 1 \leq i \leq n\}$

must have some element r_k that is right-minimal in the following sense: if $r_j \leq_r r_k$ then $r_j = r_k$. In the set $\{q_i : 1 \leq i \leq n, r_i = r_k\}$ there exists some element q_h that is left-minimal in the following sense: if $q_j \leq_l q_h$ then $q_j = q_h$. We will show that choosing $a = q_h$ gives a contradiction.

Consider $\sum_{i=1}^n \lambda_i J_{p_i} J_{q_i}^* J_{r_i}$ acting on $\ell^2(I_{q_h})$. Since $r_k = r_h \leq_l q_h$ it follows that $r_k^{-1} q_h \in I_{q_h}$ so we can focus on the unit vector $\delta_{r_k^{-1} q_h}$.

Now we compute $J_{p_i}^{q_h} J_{q_i}^{q_h *} J_{r_i}^{q_h} \delta_{r_k^{-1} q_h}$. We have

$$
J_{r_i}^{q_h} \delta_{r_k^{-1} q_h} = \begin{cases} \delta_{r_i r_k^{-1} q_h} & \text{if } r_i r_k^{-1} q_h \leq_r q_h \Leftrightarrow q_h q_h^{-1} r_k r_i^{-1} \in P \Leftrightarrow r_k r_i^{-1} \in P \\ 0 & \text{otherwise.} \end{cases}
$$

But $r_k r_i^{-1} \in P$ implies that $r_i \leq_r r_k$, which only occurs if $r_i = r_k$ due to the minimality of r_k. Thus we have

$$
J_{p_i}^{q_h} J_{q_i}^{q_h *} J_{r_i}^{q_h} \delta_{r_k^{-1} q_h} = \begin{cases} J_{p_i}^{q_h} J_{q_i}^{q_h *} \delta_{r_i r_k^{-1} q_h} & \text{if } r_i = r_k \\ 0 & \text{otherwise} \end{cases}
$$

$$
= \begin{cases} J_{p_i}^{q_h} J_{q_i}^{q_h *} \delta_{q_h} & \text{if } r_i = r_k \\ 0 & \text{otherwise.} \end{cases}
$$

Now we consider $J_{q_i}^{q_h *} \delta_{q_h}$. We have $J_{q_i}^{q_h *} \delta_{q_h} = \delta_{q_i^{-1} q_h}$ if $q_i \leq_l q_h$ and zero otherwise. But q_h is left-minimal so $q_i \leq_l q_h$ if and only if $q_i = q_h$. Hence we have

$$
J_{p_i}^{q_h} J_{q_i}^{q_h *} J_{r_i}^{q_h} \delta_{r_k^{-1} q_h} = \begin{cases} J_{p_i}^{q_h} \delta_{q_i^{-1} q_h} & \text{if } r_i = r_k \text{ and } q_i = q_h \\ 0 & \text{otherwise} \end{cases}
$$

$$
= \begin{cases} J_{p_i}^{q_h} \delta_e & \text{if } r_i = r_k \text{ and } q_i = q_h \\ 0 & \text{otherwise} \end{cases}
$$

$$
= \begin{cases} \delta_{p_i} & \text{if } r_i = r_k \text{ and } q_i = q_h \\ 0 & \text{otherwise.} \end{cases}
$$

Now we compute:

$$
\sum_{i=1}^n \lambda_i J_{p_i}^{q_h} J_{q_i}^{q_h *} J_{r_i}^{q_h} \delta_{r_k^{-1} q_h} = \sum_{i \in \{i : r_i = r_k, q_i = q_h\}} \lambda_i \delta_{p_i} = 0.
$$

By our initial assumption the triples (p_i, q_i, r_i) are distinct. Hence all the p_i such that $r_i = r_k$ and $q_i = q_h$ must be distinct. Since the unit vectors δ_x are linearly independent it follows that $\lambda_i = 0$ for all i such that $r_i = r_k$ and $q_i = q_h$, contradicting our

assumption that the λ_i are nonzero. Thus $S = \{J_p J_q^* J_r : p, q, r \in P, p \leq_r q, r \leq_l q\}$ is linearly independent.

Since $C_{\mathrm{ts}}^*(G, P, P^{\mathrm{op}})$ is generated by $\{J_p : p \in P\}$, $\mathrm{span}\{J_{n_1} J_{n_2}^* J_{n_3} J_{n_4}^* \dots : n \in P\}$ is a dense unital $*$-subalgebra of $C_{\mathrm{ts}}^*(G, P, P^{\mathrm{op}})$. By Lemma 2.13, any product of the form $J_{n_1} J_{n_2}^* J_{n_3} J_{n_4}^* \dots$ may be written as $J_p J_q^* J_r$ where $p \leq_r q$ and $r \leq_l q$. Thus the span S is equal to $\mathrm{span}\{J_{n_1} J_{n_2}^* J_{n_3} J_{n_4}^* \dots : n_i \in P\}$ and hence is a dense unital $*$-subalgebra of $C_{\mathrm{ts}}^*(G, P, P^{\mathrm{op}})$. $\qquad\square$

CHAPTER 3

Constructing a universal C^*-algebra

In this chapter we construct a C^*-algebra which is universal for covariant partial isometric representations.

THEOREM 3.1. *Let (G, P) be a doubly quasi-lattice ordered group. Then there is a unital C^*-algebra $C^*(G, P, P^{\text{op}})$ generated by partial isometries $\{v_p : p \in P\}$ such that $v : p \mapsto v_p$ is a covariant partial isometric representation of P and that $C^*(G, P, P^{\text{op}})$ has the following universal property: for every covariant partial isometric representation $W : P \to A$ there is a unital homomorphism $\phi_W : C^*(G, P, P^{\text{op}}) \to A$ such that $\phi_W(v_p) = W_p$.*

The pair $(C^*(G, P, P^{\text{op}}), \{v_p\})$ is unique up to isomorphism: for every pair $(C, \{w_p\})$ satisfying this universal property there exists an isomorphism $\phi_w : C^*(G, P, P^{\text{op}}) \to C$ such that $\phi_w(v_p) = w_p$.

In other words, $C^*(G, P, P^{\text{op}})$ is universal for covariant partial isometric representations. As mentioned in the previous chapter we use the notation $C^*(G, P, P^{\text{op}})$ to stay consistent with Nica's use of $C^*(G, P)$ while adding the P^{op} to emphasise that we are working with two quasi-lattice orders.

To construct the C^*-algebra $C^*(G, P, P^{\text{op}})$ of Theorem 3.1, we construct a normed $*$-algebra generated by partial isometries, and then complete it.

Our candidate for such a normed $*$-algebra $\mathcal{A}$ is the vector space over $\mathbb{C}$ with basis

$$\{v_{p,q,r} : q \in P, e \leq_r p \leq_r q, e \leq_l r \leq_l q\}.$$

Recall that an element a of a $*$-algebra is a partial isometry if $aa^*a = a$. Using this we can define a covariant partial isometric representation into a $*$-algebra. Let $\mathcal{B}$ be a unital $*$-algebra. A *partial isometric representation into a $*$-algebra* is a map $w : P \to \mathcal{B}$ such that w_p is a partial isometry for all $p \in P$, $w_e = 1$ where e is the identity of G and $w_x w_y = w_{xy}$ for $x, y \in P$. A representation is *covariant* if it satisfies

$$w_x w_x^* w_y w_y^* = \begin{cases} w_{x \vee_l y} w_{x \vee_l y}^* & \text{if } x \vee_l y < \infty. \\ 0 & \text{otherwise.} \end{cases}$$

$$w_x^* w_x w_y^* w_y = \begin{cases} w_{x \vee_r y}^* w_{x \vee_r y} & \text{if } x \vee_r y < \infty. \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 3.2. *Define a multiplication on $\mathcal{A}$ by*

$$v_{a,b,c} v_{p,q,r} = \begin{cases} v_{ab^{-1}(b \vee_l cp),(cp \vee_r q)(cp)^{-1}(b \vee_l cp),(cp \vee_r q)q^{-1}r} & \text{if } cp \vee_r q < \infty \text{ and } b \vee_l cp < \infty. \\ 0 & \text{otherwise.} \end{cases}$$

This multiplication is associative.

This multiplication is constructed to mimic the multiplication of covariant partial isometric representations as shown in Lemma 2.14. (Here $v_{p,q,r}$ corresponds to $W_p W_q^* W_r$ for some covariant partial isometric representation W.)

PROOF. We apply Van der Waerden's method: consider the map $\Psi : \mathcal{A} \to C_{\text{ts}}^*(G, P, P^{\text{op}})$ defined by $\Psi(\sum c_{p,q,r} v_{p,q,r}) = \sum c_{p,q,r} J_p J_q^* J_r$. We will show that Ψ is injective and $\Psi(\alpha)\Psi(\beta) = \Psi(\alpha\beta)$ for all $\alpha, \beta \in \mathcal{A}$. Then we can borrow the associativity of $C_{\text{ts}}^*(G, P, P^{\text{op}})$ to prove the multiplication defined on $\mathcal{A}$ is associative.

Suppose that $\Psi(\sum c_{p,q,r} v_{p,q,r}) = \Psi(\sum d_{p,q,r} v_{p,q,r})$. Computing, we see that

$$\sum c_{p,q,r} J_p J_q^* J_r = \sum d_{p,q,r} J_p J_q^* J_r.$$

By Lemma 2.16, the set $\{J_p J_q^* J_r\}$ is linearly independent and hence $c_{p,q,r} = d_{p,q,r}$ for each triple p, q, r. Thus $\sum c_{p,q,r} v_{p,q,r} = \sum d_{p,q,r} v_{p,q,r}$ and Ψ is injective. By Lemma 2.14 we have:

$$\Psi(v_{a,b,c})\Psi(v_{p,q,r}) = (J_a J_b^* J_c)(J_p J_q^* J_r)$$

$$= \begin{cases} (J_{ab^{-1}(b \vee_l cp)} J_{(cp \vee_r q)(cp)^{-1}(b \vee_l cp)}^* J_{(cp \vee_r q)q^{-1}r}) & \text{if } cp \vee_r q, b \vee_l cp < \infty. \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \Psi(v_{ab^{-1}(b \vee_l cp),(cp \vee_r q)(cp)^{-1}(b \vee_l cp),(cp \vee_r q)q^{-1}r}) & \text{if } cp \vee_r q, b \vee_l cp < \infty. \\ 0 & \text{otherwise.} \end{cases}$$

$$= \Psi(v_{a,b,c} v_{p,q,r}).$$

It follows immediately that $\Psi(\alpha)\Psi(\beta) = \Psi(\alpha\beta)$ for all $\alpha, \beta \in \mathcal{A}$. Therefore Ψ is injective and preserves multiplication. Now we can use Ψ to prove that multiplication on $\mathcal{A}$ is associative.

Fix $\alpha, \beta, \gamma \in \mathcal{A}$ and consider $(\alpha\beta)\gamma$. Now:

$$\Psi((\alpha\beta)\gamma) = \Psi(\alpha\beta)\Psi(\gamma)$$

$$= (\Psi(\alpha)\Psi(\beta))\Psi(\gamma).$$

Multiplication in $C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$ is associative so:

$$
\begin{aligned}
(\Psi(\alpha)\Psi(\beta))\Psi(\gamma) &= \Psi(\alpha)(\Psi(\beta)\Psi(\gamma)) \\
&= \Psi(\alpha)\Psi(\beta\gamma) \\
&= \Psi(\alpha(\beta\gamma)).
\end{aligned}
$$

Thus $\Psi((\alpha\beta)\gamma) = \Psi(\alpha(\beta\gamma))$. By the injectivity of Ψ, we have $(\alpha\beta)\gamma = \alpha(\beta\gamma)$. Therefore multiplication in $\mathcal{A}$ is associative. $\qquad\square$

Define a map $a \mapsto a^*$ by

$$
\Big(\sum_{\substack{e\leq_r p\leq_r q \\ e\leq_l r\leq_l q}} c_{p,q,r} v_{p,q,r} \Big)^* = \sum_{\substack{e\leq_r p\leq_r q \\ e\leq_l r\leq_l q}} \overline{c_{p,q,r}} v_{r^{-1}q,q,qp^{-1}}.
$$

In particular, $(v_{p,q,r})^* = v_{r^{-1}q,q,qp^{-1}}$. This again mimics the adjoints of partial isometric representations as shown in Lemma 2.14. The map $a \mapsto a^*$ is conjugate linear and is an involution since

$$
((v_{p,q,r})^*)^* = (v_{r^{-1}q,q,qp^{-1}})^* = v_{pq^{-1}q,q,qq^{-1}r} = v_{p,q,r}.
$$

Thus $\mathcal{A}$ is a $*$-algebra.

We claim that $v : P \to \mathcal{A}$ defined $v_p = v_{p,p,p}$ is a covariant partial isometric representation into a $*$-algebra. To see that $v_e = v_{e,e,e}$ is the identity for $\mathcal{A}$ fix $p, q, r \in P$ with $p \leq_r q$ and $r \leq_l q$. Compute:

$$
v_{e,e,e} v_{p,q,r} = v_{ee^{-1}(e\vee_l ep),(ep\vee_r q)(ep)^{-1}(e\vee_l ep),(ep\vee_r q)q^{-1}r} = v_{p,qp^{-1}p,qq^{-1}r} = v_{p,q,r}
$$

and similarly $v_{p,q,r} v_{e,e,e} = v_{p,q,r}$. Thus $v_{e,e,e}$ is the identity.

Fix $p, q \in P$. We have $p \leq_l pq$ and $q \leq_r pq$ thus $p \vee_l pq = pq$ and $q \vee_r pq = pq$. So we can compute:

$$
\begin{aligned}
v_p v_q &= v_{p,p,p} v_{q,q,q} \\
&= v_{pp^{-1}(p\vee_l pq),(pq\vee_r q)(pq)^{-1}(p\vee_l pq),(pq\vee_r q)q^{-1}q} \\
&= v_{pp^{-1}pq,pq(pq)^{-1}pq,pqq^{-1}q} \qquad (p \leq_l pq \text{ and } q \leq_r pq) \\
&= v_{pq,pq,pq} \\
&= v_{pq}.
\end{aligned}
$$

Thus $v_p v_q = v_{pq}$ and v preserves the semigroup multiplication. To see that v_p is a partial isometry:

$$v_p v_p^* v_p = v_{p,p,p}(v_{p,p,p})^* v_{p,p,p}$$

$$= v_{p,p,p} v_{e,p,e} v_{p,p,p}$$

$$= v_{pp^{-1}(p\vee_l pe),(pe\vee_r p)(pe)^{-1}(p\vee_l pe),(pe\vee_r p)p^{-1}e} v_{p,p,p}$$

$$= v_{p,(p\vee_r p),(p\vee_r p)p^{-1}} v_{p,p,p}$$

$$= v_{p,p,e} v_{p,p,p}$$

$$= v_{pp^{-1}(p\vee_l ep),(ep\vee_r p)(ep)^{-1}(p\vee_l ep),(ep\vee_r p)p^{-1}p}$$

$$= v_{p,p,p}$$

$$= v_p.$$

Thus v_p is a partial isometry for all $p \in P$. To prove covariance, observe that $v_p v_p^* = v_p v_p^* v_e = v_{p,p,e}$ and similarly $v_p^* v_p = v_{e,p,p}$. We compute:

$$v_p v_p^* v_q v_q^* = v_{p,p,e} v_{q,q,e}$$

$$= v_{pp^{-1}(p\vee_l eq),(eq\vee_r q)q^{-1}e^{-1}(p\vee_l eq),(eq\vee_r q)q^{-1}e}$$

$$= v_{(p\vee_l q),qq^{-1}(p\vee_l q),qq^{-1}}$$

$$= v_{(p\vee_l q),(p\vee_l q),e}$$

$$= v_{(p\vee_l q)} v_{(p\vee_l q)}^*.$$

By a similar argument

$$v_p^* v_p v_q^* v_q = v_{e,p,p} v_{e,q,q} = v_{e,(p\vee_r q),(p\vee_r q)} = v_{(p\vee_r q)}^* v_{(p\vee_r q)}.$$

We claim that $\{v_p = v_{p,p,p} : p \in P\}$ generates $\mathcal{A}$ as a $*$-algebra. From our definition of $\mathcal{A}$ as a vector space we see that $\mathcal{A} = \text{span}\{v_{p,q,r} : p,q,r \in P, p \leq_r q, r \leq_l q\}$. Thus it will suffice to show that $v_{p,q,r} = v_p v_q^* v_r = v_{p,p,p}(v_{q,q,q})^* v_{r,r,r}$. Fix $p,q,r \in P$ with $p \leq_r q$ and $r \leq_l q$. We compute:

$$v_p v_q^* v_r = v_{p,p,p}(v_{q,q,q})^* v_{r,r,r}$$

$$= v_{p,p,p} v_{e,q,e} v_{r,r,r}$$

$$= v_{pp^{-1}(p\vee_l pe),(pe\vee_r q)(pe)^{-1}(p\vee_l pe),(pe\vee_r q)q^{-1}e} v_{r,r,r}$$

$$= v_{p,(p\vee_r q),(p\vee_r q)q^{-1}} v_{r,r,r}$$

$$= v_{p,q,e} v_{r,r,r}$$

$$= v_{pq^{-1}(q\vee_l er),(er\vee_r r)(er)^{-1}(q\vee_l er),(er\vee_r r)r^{-1}r}$$

36

$$= v_{p,q,r}$$

Thus $\mathcal{A}$ is generated by $\{v_{p,p,p} : p \in P\}$. For ease of reading we will write $v_{p,p,p} = v_p$, and then $v_{p,q,r} = v_p v_q^* v_r$.

Next, we want to define a norm on $\mathcal{A}$ using representations of $\mathcal{A}$. We need to know that appropriate representations exist.

LEMMA 3.3. *Let (G, P) be a doubly quasi-lattice ordered group and let $v : p \mapsto v_p$ be the covariant partial isometric representation constructed in Section 3.*

(1) *Let π be a unital $*$-representation of $\mathcal{A}$ on a Hilbert space H. Then $\pi(v_p)$ is a partial isometry and there exists a covariant partial isometric representation $W : P \to B(H)$ such that $W_p = \pi(v_p)$.*

(2) *Let W be a covariant partial isometric representation $W : P \to B(H)$. Then there exists a unital $*$-representation $\pi_W : \mathcal{A} \to B(H)$ such that $\pi_W(v_p) = W_p$.*

PROOF. (1). Suppose π is a unital representation of $\mathcal{A}$ on a Hilbert space H. We showed above that for all $p, q \in P$, v_p is a partial isometry, $v_e = 1$, $v_p v_q = v_{pq}$ and $v : P \to \mathcal{A}$ is covariant. Thus the map $W : P \to B(H)$ defined by $W_p = \pi(v_p)$ is a covariant partial isometric representation.

(2). Suppose W is a covariant partial isometric representation $W : P \to B(H)$. Define a map $\pi_W : \mathcal{A} \to B(H)$ by

$$\pi_W \left(\sum_{\substack{e \leq_r p \leq_r q \\ e \leq_l r \leq_l q}} c_{p,q,r} v_{p,q,r} \right) = \sum_{\substack{e \leq_r p \leq_r q \\ e \leq_l r \leq_l q}} c_{p,q,r} W_p W_q^* W_r.$$

Then π_W preserves addition and scalar multiplication. Multiplication and involution in $\mathcal{A}$ were defined to mimic the multiplication and adjoint of partial isometric representations, thus π_W also preserves multiplication and involution. In particular $\pi_W(v_p) = \pi_W(v_{p,p,p}) = W_p W_p^* W_p = W_p$ and $\pi_W(v_e) = W_e = 1$. Hence π_W is a unital $*$-representation. $\square$

Now we have the representations to define a norm on $\mathcal{A}$. For all $a \in \mathcal{A}$ define

$$\|a\| := \sup\{\|\pi(a)\| : \pi \text{ is a unital } *\text{-representation of } \mathcal{A} \text{ on a Hilbert space } H\}.$$

To see that this candidate for the norm is finite, fix a unital $*$-representation π of $\mathcal{A}$ on a Hilbert space H. Observe that $\pi(v_{p,q,r}) = W_p W_q^* W_r$ for some partial isometric representation W. All partial isometries have norm less than or equal to 1, and thus

$\|W_p W_q^* W_r\| \leq 1$. Hence

$$\left\|\pi\left(\sum c_{p,q,r} v_{p,q,r}\right)\right\| = \left\|\sum c_{p,q,r} W_p W_q^* W_r\right\| \leq \sum |c_{p,q,r}| \left\|W_p W_q^* W_r\right\| \leq \sum |c_{p,q,r}|.$$

So $\sum |c_{p,q,r}|$ is an upper bound for the norm of $\sum c_{p,q,r} v_{p,q,r}$ and so our candidate for the norm is well defined. Note that there is no issue taking the supremum over this collection as the $\{\|\pi(a)\|\} \subset \mathbb{R}$. Apart from showing $\|a\| = 0 \Rightarrow a = 0$ verifying the norm axioms is straightforward, and we omit it.

Suppose $\|a\| = 0$. Then $\|\pi(a)\| = 0$ for all π and hence $\pi(a) = 0$ for all π. Let $J : P \to B(\bigoplus_{a \in P} \ell^2(I_a))$ be the covariant partial isometric representation of Definition 2.15. By Lemma 3.3(2) there is a unital $*$-representation π_J such that $\pi_J(v_p) = J_p$ for all $p \in P$. Write $a = \sum\limits_{\substack{e \leq_r p \leq_r q \\ e \leq_l r \leq_l q}} c_{p,q,r} v_{p,q,r}$. Then

$$0 = \pi_J(a) = \sum_{\substack{e \leq_r p \leq_r q \\ e \leq_l r \leq_l q}} c_{p,q,r} J_p J_q^* J_r.$$

Since $\{J_p J_q^* J_r : p, q, r \in P, p \leq_r q, r \leq_l q\}$ is linearly independent by Lemma 2.16, each $c_{p,q,r} = 0$ and hence $a = 0$. Thus $\|a\| = 0$ implies $a = 0$. Hence $\mathcal{A}$ is a normed $*$-algebra and we may complete $\mathcal{A}$ with respect to that norm to obtain a C^*-algebra $C^*(G, P, P^{\mathrm{op}})$ generated by partial isometries $\{v_p : p \in P\}$.

PROOF OF THEOREM 3.1. Take $C^*(G, P, P^{\mathrm{op}})$ to be the C^*-algebra constructed above. Then $C^*(G, P, P^{\mathrm{op}})$ is generated by the partial isometries $\{v_p : p \in P\}$. Suppose that A is a unital C^*-algebra and that $W : P \to A$ is a covariant partial isometric representation. Choose a faithful unital representation $\pi : A \to B(H)$. Now define $V : P \to B(H)$ by $V_p := \pi(W_p)$. Then V is a covariant partial isometric representation of P.

By Lemma 3.3 there is a unital representation $\pi_V : \mathcal{A} \to B(H)$ such that $\pi_V(v_p) = V_p$. For all $a \in \mathcal{A}$,

$$\|\pi_V(a)\| \leq \sup\{\|\pi(a)\| : \pi \text{ is a unital } *\text{-representation of } \mathcal{A}\} = \|a\|.$$

so π_V is bounded and extends to a representation of $C^*(G, P, P^{\mathrm{op}})$. With an abuse of notation we continue to use π_V for the representation of $C^*(G, P, P^{\mathrm{op}})$ on H.

Now let $\phi_W := \pi^{-1} \circ \pi_V$. This is a well-defined homomorphism from $C^*(G, P, P^{\mathrm{op}})$ to A since π is faithful and range $\pi_V \subseteq$ range π. In particular

$$\phi_W(v_p) = \pi^{-1} \circ \pi_V(v_p) = \pi^{-1}(V_p) = \pi^{-1}(\pi(W_p)) = W_p.$$

Thus ϕ_W has the required properties.

To prove uniqueness, suppose that C is a unital C^*-algebra generated by a set of partial isometries $\{w_p : p \in P\}$ such that $w : p \mapsto w_p$ is a covariant partial isometric representation, and that the pair $(C, \{w_p\})$ has the universal property. Then the universal property of $(C, \{w_p\})$ gives a homomorphism $\psi_v : C \to C^*(G, P, P^{\mathrm{op}})$ such that $\psi_v(w_p) = v_p$. However, the $w : P \to C$ is a covariant partial isometric representation so the universal property of $C^*(G, P, P^{\mathrm{op}})$ means that there is a homomorphism $\phi_w : C^*(G, P, P^{\mathrm{op}}) \to C$ such that $\phi_w(v_p) = w_p$. Further, ϕ_w is an inverse for ψ_v and hence ϕ_w is an isomorphism. Thus the pair $(C^*(G, P, P^{\mathrm{op}}), \{v_p\})$ is unique up to isomorphism. $\qquad\square$

REMARK. It is important to note that $C^*(G, P, P^{\mathrm{op}})$ is not the same as $C^*(G, P)$ that Nica constructed in [**16**, §4.2]. (See §1.2.3.) As we mentioned in §2.4, covariant isometric representations are not in general covariant *partial* isometric representations. When they are, i.e. when every pair in P has a common right upper bound in P, then $C^*(G, P)$ is a quotient of $C^*(G, P, P^{\mathrm{op}})$. For example, $(\mathbb{Z}, \mathbb{N})$ is lattice ordered in both left and right orders. Therefore the universal covariant isometric representation of $w : \mathbb{N} \to C^*(\mathbb{Z}, \mathbb{N})$ is a covariant isometric representation. Thus there is a homomorphism $\phi_w : C^*(\mathbb{Z}, \mathbb{N}, \mathbb{N}^{\mathrm{op}}) \to C^*(\mathbb{Z}, \mathbb{N})$. This ϕ_w is a surjection, however, it is not faithful. As shown in Lemma 2.16 $v_1^* v_1 - v_2^* v_2 \neq 0$ however,

$$\phi_w(v_1^* v_1 - v_2^* v_2) = w_1^* w_1 - w_2^* w_2 = 1 - 1 = 0.$$

Thus $C^*(G, P, P^{\mathrm{op}})$is a new, larger algebra associated to a given doubly quasi-lattice ordered group.

3.1. Universal algebra of $(G^{\mathrm{op}}, P^{\mathrm{op}})$

As we noted in Chapter 2, for any doubly quasi-lattice ordered group (G, P), its opposite group $(G^{\mathrm{op}}, P^{\mathrm{op}})$ is also doubly quasi-lattice ordered. Therefore our construction of the universal algebra for (G, P) also applies to $(G^{\mathrm{op}}, P^{\mathrm{op}})$. For Nica's covariant isometric representations, the two universal algebras $C^*(G, P)$ and $C^*(G^{\mathrm{op}}, P^{\mathrm{op}})$ are distinct (for a discussion of the isometric case see [**3**, Remark 7.5]). However, this distinction occurs because $C^*(G, P)$ only captures the left least upper bound structure and $C^*(G^{\mathrm{op}}, P^{\mathrm{op}})$ only captures the right least upper bound structure. As we showed in Lemma 2.10 the left and right least upper bound structure of a doubly quasi-lattice ordered group can be very different.

When we look at $C^*(G, P, P^{\mathrm{op}})$ we have already included all the information from the two partial orders and so the switch to $(G^{\mathrm{op}}, P^{\mathrm{op}})$ is just a switch of labelling. As

we show in Lemma 3.4, every covariant partial isometric representation of P^{op} is the adjoint of a covariant representation of P.

NOTATION. Let (G, P) be a doubly quasi-lattice ordered group. Then $(G^{\mathrm{op}}, P^{\mathrm{op}})$ must also be a doubly quasi-lattice ordered group. We need new notation for the left and right orders on $(G^{\mathrm{op}}, P^{\mathrm{op}})$. Let $\leq_{l,\mathrm{op}}$ be the left-invariant partial order on $(G^{\mathrm{op}}, P^{\mathrm{op}})$ and let $\leq_{r,\mathrm{op}}$ be the right-invariant partial order. It is easy to see that these "new" orders are just the right and left orders respectively on (G, P):

$$x \leq_{l,\mathrm{op}} y \Leftrightarrow x^{-1} \cdot_{\mathrm{op}} y \in P \Leftrightarrow yx^{-1} \in P \Leftrightarrow x \leq_r y$$

$$x \leq_{r,\mathrm{op}} y \Leftrightarrow y \cdot_{\mathrm{op}} x^{-1} \in P \Leftrightarrow x^{-1}y \in P \Leftrightarrow x \leq_l y$$

Similarly, $(G^{\mathrm{op}}, P^{\mathrm{op}})$ reverses the least upper bound structure: for all $x, y \in P$ we have $x \vee_{l,\mathrm{op}} y = x \vee_r y$ and $x \vee_{r,\mathrm{op}} y = x \vee_l y$.

LEMMA 3.4. *Let (G, P) be a doubly quasi-lattice ordered group and let A be a unital C^*-algebra. Let $W : P \to A$ be a map and define a map $R^W : P^{\mathrm{op}} \to A$ by $R_p^W := W_p^*$. The map W is a covariant partial isometric representation of P if and only if R^W is a covariant partial isometric representation of P^{op}.*

PROOF. Suppose that W is a covariant partial isometric representation of P. Since W_p is a partial isometry for all $p \in P$, so is W_p^*. It follows that $R_p^W = W_p^*$ is a partial isometry for all $p \in P^{\mathrm{op}}$. We have $R_e^W = W_e^* = 1$. Fix $p, q \in P^{\mathrm{op}}$ and compute:

$$R_p^W R_q^W = W_p^* W_q^* = (W_q W_p)^* = W_{qp}^* = R_{qp}^W = R_{p \cdot_{\mathrm{op}} q}^W.$$

Thus R^W preserves the semigroup multiplication in P^{op} and hence is a partial isometric representation. To show that R^W is covariant, we compute:

$$R_p^W R_p^{W*} R_q^W R_q^{W*} = W_p^* W_p W_q^* W_q = W_{p \vee_r q}^* W_{p \vee_r q} = R_{p \vee_r q}^W R_{p \vee_r q}^{W*} = R_{p \vee_{l,\mathrm{op}} q}^W R_{p \vee_{l,\mathrm{op}} q}^{W*}$$

$$R_p^{W*} R_p^W R_q^{W*} R_q^W = W_p W_p^* W_q W_q^* = W_{p \vee_l q} W_{p \vee_l q}^* = R_{p \vee_l q}^{W*} R_{p \vee_l q}^W = R_{p \vee_{r,\mathrm{op}} q}^{W*} R_{p \vee_{r,\mathrm{op}} q}^W.$$

Thus $R^W : P^{\mathrm{op}} \to A$ is covariant partial isometric representation of P^{op}.

To prove the reverse implication we use a symmetric argument. Suppose that $R^W : P^{\mathrm{op}} \to A$ defined by $R_p^W := W_p^*$ is a covariant partial isometric representation of P^{op}. By the proof of the first part we know that $R^{R^W} : (P^{\mathrm{op}})^{\mathrm{op}} \to A$ defined by $R_p^{R^W} = (R_p^W)^*$ is a covariant partial isometric representation of $(P^{\mathrm{op}})^{\mathrm{op}}$. However, $(P^{\mathrm{op}})^{\mathrm{op}} = P$ and, for all $p \in P$, we have $R_p^{R^W} = (R_p^W)^* = (W_p^*)^* = W_p$. Thus $R^{R^W} = W$ and so W is a covariant partial isometric representation of P. $\qquad\square$

COROLLARY 3.5. *Let (G, P) be a doubly quasi-lattice ordered group.*

(1) *There exists a unital C^*-algebra $C^*(G^{\mathrm{op}}, P^{\mathrm{op}}, P)$ generated by partial isometries $\{w_p : p \in P^{\mathrm{op}}\}$ such that $w : p \mapsto w_p$ is a covariant partial isometric representation that has the following property: for every covariant partial isometric representation $W : P^{\mathrm{op}} \to A$ there is a unital homomorphism $\phi_W : C^*(G^{\mathrm{op}}, P^{\mathrm{op}}, P) \to A$ such that $\phi_W(w_p) = W_p$.*

(2) *There is an isomorphism $\Phi_{\mathrm{op}} : C^*(G, P, P^{\mathrm{op}}) \to C^*(G^{\mathrm{op}}, P^{\mathrm{op}}, P)$ such that $\Phi(v_p) = w_p^*$.*

PROOF. (1). $(G^{\mathrm{op}}, P^{\mathrm{op}})$ is a doubly quasi-lattice ordered group and hence (1) follows immediately from Theorem 3.1.

(2). The map $w : P^{\mathrm{op}} \to C^*(G^{\mathrm{op}}, P^{\mathrm{op}}, P)$ is a covariant partial isometric representation of P^{op}. Hence, by Lemma 3.4, $R^w : P \to C^*(G^{\mathrm{op}}, P^{\mathrm{op}}, P)$ defined $R_p^w = w_p^*$ is a covariant partial isometric representation of $(P^{\mathrm{op}})^{\mathrm{op}} = P$. By Theorem 3.1, there is a unital homomorphism

$$\Phi_{\mathrm{op}} : C^*(G, P, P^{\mathrm{op}}) \to C^*(G^{\mathrm{op}}, P^{\mathrm{op}}, P)$$

such that $\Phi_{\mathrm{op}}(v_p) = w_p^*$. To show that Φ_{op} is an isomorphism we construct its inverse. By Lemma 3.4, map $R^v : P^{\mathrm{op}} \to C^*(G, P, P^{\mathrm{op}})$ defined $R_P^v = v_p^*$ is a covariant partial isometric representation of P^{op}. Hence, by part (1), there is a unital homomorphism

$$\phi_v : C^*(G^{\mathrm{op}}, P^{\mathrm{op}}, P) \to C^*(G, P, P^{\mathrm{op}})$$

such that $\phi_v(w_p) = v_p^*$. Then $\phi_v \circ \Phi_{\mathrm{op}}(v_p) = \phi_v(w_p^*) = (v_p^*)^* = v_p$ and $\Phi_{\mathrm{op}} \circ \phi_v(w_p) = w_p$. Thus ϕ_v is the inverse of Φ_{op}. Hence Φ_{op} is an isomorphism. $\square$

Faithful representations of $C^*(G, P, P^{\mathrm{op}})$

In this chapter we seek to answer the question: given a doubly quasi-lattice ordered group (G, P) and a covariant partial isometric representation $W : P \to A$, when is the corresponding homomorphism $\pi_W : C^*(G, P, P^{\mathrm{op}}) \to A$ of Theorem 3.1 faithful?

Let (G, P) be a doubly quasi-lattice ordered group. Then $(G \times G^{\mathrm{op}}, P \times P^{\mathrm{op}})$ is quasi-lattice ordered. In particular, $(G \times G^{\mathrm{op}}, P \times P^{\mathrm{op}})$ has a partial order defined by $(x_1, x_2) \leq (y_1, y_2)$ if $x_1 \leq_l y_1$ and $x_2 \leq_r y_2$. Further, if $x_1 \vee_l y_1 < \infty$ and $x_2 \vee_r y_2 < \infty$ then $(x_1, x_2) \vee (y_1, y_2) = (x_1 \vee_l y_1, x_2 \vee_r y_2)$. (Note that we are only considering one partial order on $(G \times G^{\mathrm{op}}, P \times P^{\mathrm{op}})$, and so we use it without subscript.) For a finite subset $K \subset P \times P^{\mathrm{op}}$, let $\vee K$ be the least common upper bound of K in $P \times P^{\mathrm{op}}$.

LEMMA 4.1. *Let (G, P) be a doubly quasi-lattice ordered group and let $W : P \to A$ be a covariant partial isometric representation. Then*

$$\{L^W_{(x_1,x_2)} := W_{x_1} W^*_{x_1} W^*_{x_2} W_{x_2} : (x_1, x_2) \in P \times P^{\mathrm{op}}\}$$

is a family of projections satisfying $L^W_{(e,e)} = 1$ where e is the identity of G and $L^W_x L^W_y = L^W_{x \vee y}$ for $x, y \in P \times P^{\mathrm{op}}$.

PROOF. Since W is a covariant partial isometric representation the projections $W_{x_1} W^*_{x_1}$ and $W^*_{x_2} W_{x_2}$ commute and hence each of the L^W_x is a projection. As with covariant representations, we use the convention $L^W_\infty = 0$. We have

$$L^W_{(e,e)} = W_e W^*_e W^*_e W_e = 1.$$

Fix $x, y \in P \times P^{\mathrm{op}}$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$, and compute:

$$\begin{aligned}
L^W_x L^W_y &= W_{x_1} W^*_{x_1} W^*_{x_2} W_{x_2} W_{y_1} W^*_{y_1} W^*_{y_2} W_{y_2} \\
&= W_{x_1} W^*_{x_1} W_{y_1} W^*_{y_1} W^*_{x_2} W_{x_2} W^*_{y_2} W_{y_2} \\
&= W_{x_1 \vee_l y_1} W^*_{x_1 \vee_l y_1} W^*_{x_2 \vee_r y_2} W_{x_2 \vee_r y_2} \\
&= L^W_{x \vee y}. \qquad\qquad \square
\end{aligned}$$

DEFINITION 4.2. Let (G, P) be a doubly quasi-lattice ordered group. A function $L : P \times P^{\mathrm{op}} \to A$ *sees all projections* if $L(x)$ is a projection for all $x \in P \times P^{\mathrm{op}}$ and for

every finite set $F \subset P \times P^{\text{op}}$ and $x \notin F$ such that x is a lower bound for F, we have $\prod_{y \in F}(L(x) - L(y)) \neq 0$.

A covariant partial isometric representation $W : P \to A$ *sees all projections* if the map $L^W : \times P^{\text{op}} \to A$ defined by $L^W((x_1, x_2)) = L^W_{(x_1, x_2)}$ where $L^W_{(x_1, x_2)} = W_{x_1} W^*_{x_1} W^*_{x_2} W_{x_2}$ sees all projections.

In the next lemma we show that "W sees all projections" is a necessary condition for π_W to be faithful.

LEMMA 4.3. *Let (G, P) be a doubly quasi-lattice ordered group.*

(1) *Let $J : P \to B(\bigoplus_{a \in P} \ell^2(I_a))$ be the covariant partial isometric representation of Definition 2.15. Then J sees all projections.*

(2) *Let $v : P \to C^*(G, P, P^{\text{op}})$ be the universal covariant partial isometric representation. Then v sees all projections.*

(3) *Let $\pi : C^*(G, P, P^{\text{op}}) \to A$ be a faithful homomorphism. Then $W : P \to A$ such that $W_p = \pi(v_p)$ sees all projections.*

PROOF. (1). Let $F \subset P \times P^{\text{op}}$ be a finite set with lower bound $x \notin F$. We see that

$$\prod_{y \in F}(L^J_x - L^J_y) = L^J_x + \sum_{K \in \mathcal{P}(F)} (-1)^{|K|} \prod_{y \in K} L^J_y$$
$$= L^J_x + \sum_{K \in \mathcal{P}(F)} (-1)^{|K|} L^J_{\vee K}.$$

Since x is a lower bound for F, $\vee K \neq x$ for all $K \subseteq F$. Therefore we have a linear combination of distinct projections of the form $L^J_z = J_{z_1} J^*_{z_2 z_1} J_{z_2}$. By Lemma 2.16 the set $\{J_p J^*_q J_r : p \leq_r q, r \leq_l q\}$ is linearly independent. So $\prod_{y \in F}(L^J_x - L^J_y)$ cannot be 0. Thus J sees all projections.

(2). By the universal property of $C^*(G, P, P^{\text{op}})$ there is a representation $\pi_J : C^*(G, P, P^{\text{op}}) \to C^*_{\text{ts}}(G, P, P^{\text{op}})$ such that $\pi_J(v_p) = J_p$. From part (1) J sees all projections. Therefore v must also see all projections.

(3). We proved in (2) that $v : P \to C^*(G, P, P^{\text{op}})$ sees all projections. Thus, for every finite set $F \subset P \times P^{\text{op}}$ and $x \in P \times P^{\text{op}} \backslash F$ such that x is a lower bound for F, we have $\prod_{y \in F}(L^v_x - L^v_y) \neq 0$. Since π is faithful we see that

$$\prod_{y \in F}(L^W_x - L^W_y) = \pi\left(\prod_{y \in F}(L^v_x - L^v_y)\right) \neq 0.$$

Thus W sees all projections. $\qquad\square$

From Lemma 4.3(3) it follows that W seeing all projections is a necessary condition for the corresponding representation π_W to be faithful. However, we need another condition to complete the statement of our theorem. We take a moment to define a conditional expectation on our universal algebra. This will allow us to define a property of doubly quasi-lattice ordered groups that we shall call *amenability*. We will justify our use of this term after we have defined it.

DEFINITION 4.4. Let A a be a unital C^*-algebra and let B be a subalgebra of A. A completely positive norm-decreasing linear map $E : A \to B$ is a *conditional expectation* if, for all $a \in A$ and $b, c \in B$, $E(bac) = bE(a)c$. A conditional expectation is *faithful for positive elements* if $E(a^*a) = 0$ for $a \in C^*(G, P, P^{\text{op}})$ implies $a = 0$.

We quote a theorem of Tomiyama which gives a more useful equivalent condition.

THEOREM 4.5 ([**23**, Theorem 1][**1**, Theorem II.6.10.2]). *Let A be a unital C^*-algebra and let B be a subalgebra of A. Suppose that $E : A \to B$ is a norm-decreasing linear map. Then E is a conditional expectation if and only if E is an idempotent with norm 1.*

PROPOSITION 4.6. *Let (G, P) be a doubly quasi-lattice ordered group. Then there is a unique norm-decreasing linear map $E : C^*(G, P, P^{\text{op}}) \to C^*(G, P, P^{\text{op}})$ such that*

$$E\left(\sum_{i=1}^{n} \lambda_i v_{p_i} v_{q_i}^* v_{r_i}\right) = \sum_{q_i = r_i p_i} \lambda_i v_{p_i} v_{q_i}^* v_{r_i}$$

and E is a conditional expectation onto $\overline{\text{span}}\{v_p v_{rp}^* v_r : p, r \in P\}$.

We require several lemmas for the proof of Proposition 4.6 and so we will defer the proof until Section 4.2. We can now define a concept of amenability for doubly quasi-lattice ordered groups and state the theorem that motivates this chapter.

DEFINITION 4.7. We say a doubly quasi-lattice ordered group (G, P) is *amenable* if $E : C^*(G, P, P^{\text{op}}) \to C^*(G, P, P^{\text{op}})$ as described in Proposition 4.6, is faithful for positive elements.

THEOREM 4.8. *Let (G, P) be an amenable doubly quasi-lattice ordered group and let $W : P \to A$ be a covariant partial isometric representation. Further, let π_W be the corresponding homomorphism of $C^*(G, P, P^{\text{op}})$. If W sees all projections then π_W is faithful.*

REMARK. We justify our use of the term "amenable" on several fronts. First, it follows the precedent set by Nica in [**16**, §4.2] for an analogous property of quasi-lattice ordered groups (for example [**11**, Definition 3.4] and [**5**, Definition 15]). Second, we will prove in Theorem 6.2 that amenability is a necessary and sufficient condition for the universal C^*-algebra $C^*(G, P, P^{\mathrm{op}})$ to be isomorphic to the reduced C^*-algebra $C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$. Third, we will prove in Proposition 6.6 if (G, P) is a doubly quasi-lattice ordered group and G is amenable as a group, then (G, P) is amenable as a doubly quasi-lattice ordered group. Fourth, the expectation E is analogous to the canonical trace τ on the universal group C^*-algebra $C^*(G)$, which is faithful for positive elements if and only if G is amenable as a group (see Lemma 5.14).

The rest of this chapter is devoted to the proof of Theorem 4.8. We divide the work into two sections. The first section examines the diagonal subalgebra of $C^*(G, P, P^{\mathrm{op}})$ generated by the projections L^v_x. This diagonal subalgebra is also the range of E. The second section covers the proof of Proposition 4.6.

4.1. The projections of $C^*(G, P, P^{\mathrm{op}})$

In this section we examine the diagonal subalgebra $\overline{\mathrm{span}}\{v_p v^*_{rp} v_r : p, r\}$ generated by the range and source projections $v_p v^*_{rp} v_r = L^v_{(p,r)}$. We will show in Proposition 4.10 that π_W is faithful when restricted to $\overline{\mathrm{span}}\{v_p v^*_{rp} v_r : p, r\}$ if and only if W sees all projections. We start by proving a restatement of [**11**, Lemma 1.4] which allows us to write down the norm of a linear combination of projections.

LEMMA 4.9. [**11**, Lemma 1.4] *Let (G, P) be a doubly quasi-lattice ordered group and $W : P \to A$ be a covariant partial isometric representation. Suppose that W sees all projections. Then for every finite set $K \subseteq P \times P^{\mathrm{op}}$ we have*

$$\Big\| \sum_{x \in K} \lambda_x L^W_x \Big\| = \max\Big\{ \Big| \sum_{x \in K, x \leq y} \lambda_x \Big| : y \in P \times P^{\mathrm{op}} \Big\}.$$

PROOF. Fix a finite set $K \subseteq P \times P^{\mathrm{op}}$. We begin by decomposing the identity into a linear combination of the L^W_x. We have

$$1 = \prod_{x \in K} (L^W_x + (1 - L^W_x)) = \sum_{B \subseteq K} \Big(\prod_{x \in B} L^W_x \prod_{y \in K \setminus B} (1 - L^W_y) \Big)$$

$$= \sum_{B \subseteq K} L^W_{\vee B} \prod_{y \in K \setminus B} (1 - L^W_y)$$

$$= \sum_{B \subseteq K} \prod_{y \in K \setminus B} (L^W_{\vee B} - L^W_{y \vee B}).$$

46

We will now show that we can write $\sum_{x\in K}\lambda_x L_x^W$ as a finite linear combination of mutually orthogonal projections. We have

$$\sum_{x\in K}\lambda_x L_x^W = \sum_{x\in K}\lambda_x L_x^W\Big(\sum_{B\subseteq K}\prod_{y\in K\setminus B}(L_{\vee B}^W - L_{y\vee B}^W)\Big)$$

$$= \sum_{x\in K}\Big(\sum_{B\subseteq K}\lambda_x\prod_{y\in K\setminus B}L_x^W(L_{\vee B}^W - L_{y\vee B}^W)\Big)$$

(4.1)
$$= \sum_{x\in K}\Big(\sum_{B\subseteq K}\lambda_x\prod_{y\in K\setminus B}(L_{x\vee B}^W - L_{x\vee y\vee B}^W)\Big).$$

The term $\prod_{y\in K\setminus B}(L_{x\vee B}^W - L_{x\vee y\vee B}^W)$ is zero if $x\notin B$ because then $x\in K\setminus B$ and $L_{x\vee B}^W - L_{x\vee x\vee B}^W = 0$. If $x\in B$ then $x\le\vee B$ so $L_{x\vee B}^W - L_{x\vee y\vee B}^W = L_{\vee B}^W - L_{y\vee B}^W$ for all $y\in K\setminus B$. Thus we may reverse the order of summation in (4.1) to get:

$$\sum_{x\in K}\lambda_x L_x^W = \sum_{B\subseteq K}\sum_{x\in B}\lambda_x\prod_{y\in K\setminus B}(L_{\vee B}^W - L_{y\vee B}^W)$$

We claim that the $\{Q_B := \prod_{y\in K\setminus B}(L_{\vee B}^W - L_{y\vee B}^W) : B\subseteq K\}$ are mutually orthogonal. Suppose that $B, C\subseteq K$ and $B\ne C$. Without loss of generality assume that there exists $z\in C\setminus B$ (otherwise use $B\setminus C$). Then

$$Q_B Q_C = Q_B(L_{\vee B}^W - L_{z\vee B}^W)L_{\vee C}^W Q_C$$

$$= Q_B(L_{B\vee C}^W - L_{z\vee B\vee C}^W)Q_C$$

$$= Q_B(L_{B\vee C}^W - L_{B\vee C}^W)Q_C$$

$$= 0.$$

Thus $\{Q_B = \prod_{y\in K\setminus B}(L_{\vee B}^W - L_{y\vee B}^W) : B\subseteq K\}$ are mutually orthogonal projections. Therefore we have written $\sum_{x\in K}\lambda_x L_x^W$ as a finite linear combination of mutually orthogonal projections

$$\sum_{x\in K}\lambda_x L_x^W = \sum_{B\subseteq K}\Big(\sum_{x\in B}\lambda_x\Big)\prod_{y\in K\setminus B}(L_{\vee B}^W - L_{y\vee B}^W).$$

Thus

$$\Big\|\sum_{x\in K}\lambda_x L_x^W\Big\| = \max\Big\{\Big|\sum_{x\in B}\lambda_x\Big| : B\subseteq K \text{ and } Q_B\ne 0\Big\}.$$

We claim $Q_B\ne 0$ if and only if there is a $y\in P\times P^{\mathrm{op}}$ such that $B = \{x\in K : x\le y\}$. First, suppose that $B = \{x\in K : x\le y\}$ for some $y\in P\times P^{\mathrm{op}}$. Then B has a common upper bound so $\vee B < \infty$ and $F = \{z\vee B : z\in K\setminus B\}$ is a finite set with $\vee B\notin F$ and $\vee B$ is a lower bound for F. Thus $\prod_{y\in K\setminus B}(L_{\vee B}^W - L_{y\vee B}^W) = \prod_{z\in F}(L_{\vee B}^W - L_z^W)\ne 0$ by our assumption that W sees all projections.

47

Second, suppose that there is no $y \in P \times P^{\mathrm{op}}$ such that $B = \{x \in K : x \leq y\}$. There are two subcases: either B has no common upper bound, in which case $\vee B = \infty$ and $L_{\vee B}^W = 0$. Hence $L_{\vee B}^W - L_{y \vee B}^W$ for all $y \in K \backslash B$ since $L_{y \vee B}^W \leq L_{\vee B}^W$. Or the second subcase: B has an upper bound in $P \times P^{\mathrm{op}}$ but there exists $x \in K \backslash B$ such that $x \leq \vee B$. Then $L_{\vee B}^W - L_{x \vee B}^W = L_{\vee B}^W - L_{\vee B}^W = 0$. Thus we have proved our claim that $Q_B \neq 0$ if and only if there is a $y \in P \times P^{\mathrm{op}}$ such that $B = \{x \in K : x \leq y\}$.

Therefore

$$\{B \subseteq K : Q_B \neq 0\} = \{B \subseteq K : y \in P \times P^{\mathrm{op}} \text{ and } B = \{x \in K : x \leq y\}\}.$$

Now applying this to the norm we calculated above

$$
\begin{aligned}
\|\sum_{x \in K} \lambda_x L_x^W\| &= \max\{|\sum_{x \in B} \lambda_x| : B \subseteq K \text{ and } Q_B \neq 0\} \\
&= \max\{|\sum_{x \in B} \lambda_x| : y \in P \times P^{\mathrm{op}} \text{ and } B = \{x \in K : x \leq y\}\} \\
&= \max\{|\sum_{x \in K, x \leq y} \lambda_x| : y \in P \times P^{\mathrm{op}}\}.
\end{aligned}
$$

Since K is finite, there are at most $|\mathcal{P}(K)|$ possible values for $|\sum_{x \in K, x \leq y} \lambda_x|$ and hence the maximum is well-defined. Thus

$$\left\|\sum_{x \in K} \lambda_x L_x^W\right\| = \max\left\{\left|\sum_{x \in K, x \leq y} \lambda_x\right| : y \in P \times P^{\mathrm{op}}\right\}. \quad \square$$

LEMMA 4.10. *Let (G, P) be a doubly quasi-lattice ordered group and let $W : P \to A$ be a covariant partial isometric representation. Then $\pi_W : C^*(G, P, P^{\mathrm{op}}) \to A$ is faithful on $\overline{\mathrm{span}}\{v_p v_{rp}^* v_r : p, r \in P\}$ if and only if W sees all projections.*

PROOF. First, suppose that π_W is faithful on $\overline{\mathrm{span}}\{v_p v_{rp}^* v_r : p, r \in P\}$. We will show that W sees all projections. Fix a finite set $F \subset P \times P^{\mathrm{op}}$ and $x \notin F$ such that x is a lower bound for F. By Lemma 4.3, $v : P \to C^*(G, P, P^{\mathrm{op}})$ sees all projections so we have $\prod_{y \in F}(L_x^v - L_y^v) \neq 0$. Further,

$$\prod_{y \in F}(L_x^v - L_y^v) \in \overline{\mathrm{span}}\{v_p v_{rp}^* v_r : p, r \in P\}.$$

Since π_W is faithful on $\overline{\mathrm{span}}\{v_p v_{rp}^* v_r : p, r \in P\}$ we have

$$0 \neq \pi_W\left(\prod_{y \in F}(L_x^v - L_y^v)\right) = \prod_{y \in F}(L_x^W - L_y^W).$$

Thus W sees all projections.

Second, suppose that W sees all projections. We will show that π_W is isometric on $\overline{\operatorname{span}}\{v_p v_{rp}^* v_r : p, r \in P\}$. Observe that $v_p v_{rp}^* v_r = L_{(p,r)}^v$ and hence

$$\overline{\operatorname{span}}\{v_p v_{rp}^* v_r : p, r \in P\} = \overline{\operatorname{span}}\{L_x^v : x \in P \times P^{\mathrm{op}}\}.$$

Let K be a finite set in $P \times P^{\mathrm{op}}$. Since v and W see all projections we can apply Lemma 4.9 to see that

$$\|\sum_{x \in K} \lambda_x L_x^v\| = \max\{|\sum_{x \in K, x \leq y} \lambda_x| : y \in P \times P^{\mathrm{op}}\} = \|\sum_{x \in K} \lambda_x L_x^W\|.$$

Hence

$$\|\pi_W(\sum_{x \in K} \lambda_x L_x^v)\| = \|\sum_{x \in K} \lambda_x L_x^W\| = \|\sum_{x \in K} \lambda_x L_x^v\|.$$

Thus π_W is isometric on $\operatorname{span}\{v_p v_{rp}^* v_r : p, r \in P\}$ which is dense in $\overline{\operatorname{span}}\{v_p v_{rp}^* v_r : p, r \in P\}$. Hence π_W is faithful on $\overline{\operatorname{span}}\{v_p v_{rp}^* v_r : p, r \in P\}$. $\qquad\square$

4.2. Construction of E and proof of Proposition 4.6

In this section we put together the three lemmas we need to prove Proposition 4.6. We first show how the projections L_x^W interact with W_p and W_p^*.

LEMMA 4.11. *Let (G, P) be a doubly quasi-lattice ordered group and let $W : P \to A$ be a covariant partial isometric representation.*

(1) For all $p \in P$ we have

$$W_p L_{(x_1, x_2)}^W = L_{(px_1, (p \vee_r x_2)p^{-1})}^W W_p \ \text{and} \ L_{(x_1, x_2)}^W W_p = W_p L_{(p^{-1}(p \vee_l x_1), x_2 p)}^W.$$

(2) For all $p \in P$ we have

$$W_p^* L_{(x_1, x_2)}^W = L_{(p^{-1}(p \vee_l x_1), x_2 p)}^W W_p^* \ \text{and} \ L_{(x_1, x_2)}^W W_p^* = W_p^* L_{(px_1, (p \vee_r x_2)p^{-1})}^W.$$

PROOF. (1). Fix $p \in P$ and $x = (x_1, x_2) \in P \times P^{\mathrm{op}}$. To compute $W_p L_x^W$ we use the partial isometry identity to rewrite W_p as $W_p W_p^* W_p$ and perform the computation using Lemma 2.14(1). Compute:

$$W_p L_x^W = W_p W_p^* W_p W_{x_1} W_{x_2 x_1}^* W_{x_2}$$
$$= W_{pp^{-1}(p \vee_l px_1)} W_{(x_2 x_1 \vee_r px_1)(px_1)^{-1}(p \vee_l px_1)}^* W_{(x_2 x_1 \vee_r px_1)(x_2 x_1)^{-1} x_2}$$

Observe that $p \leq_l px_1$ so $p \vee_l px_1 = px_1$. By Lemma 2.7(2) $(x_2 x_1 \vee_r px_1) = (x_2 \vee_r p)x_1$. Thus we simplify:

$$W_p L_x^W = W_{px_1} W_{(x_2 \vee_r p)p^{-1} px_1}^* W_{x_2 \vee_r p}$$

49

Since $p \leq_r x_2 \vee_r p$ we can write $W_{(x_2 \vee_r p)} = W_{(x_2 \vee_r p)p^{-1}} W_p$. Thus:

$$W_p L_x^W = W_{px_1} W^*_{(x_2 \vee_r p)p^{-1}px_1} W_{(x_2 \vee_r p)p^{-1}} W_p$$
$$= L^W_{(px_1, (p \vee_r x_2)p^{-1})} W_p.$$

To prove the second relation we compute:

$$L_x^W W_p = W_{x_1} W^*_{x_2 x_1} W_{x_2} W_p W_p^* W_p$$
$$= W_{(x_1(x_2 x_1)^{-1}(x_2 x_1 \vee_l x_2 p))} W^*_{(p \vee_r x_2 p)(x_2 p)^{-1}(x_2 x_1 \vee_l x_2 p)} W_{(p \vee_r x_2 p)p^{-1}p}$$
$$= W_{(x_1 \vee_l p)} W^*_{x_2 p p^{-1}(x_1 \vee_l p)} W_{x_2 p}$$
$$= W_p W_{p^{-1}(x_1 \vee_l p)} W^*_{x_2 p p^{-1}(x_1 \vee_l p)} W_{x_2 p}$$
$$= W_p L^W_{(p^{-1}(p \vee_l x_1), x_2 p)}.$$

Thus the proof of (1) is complete.

(2). We prove (2) by passing to the adjoint and applying (1):

$$W_p^* L^W_{(x_1, x_2)} = (L^W_{(x_1, x_2)} W_p)^* = (W_p L^W_{(p^{-1}(p \vee_l x_1), x_2 p)})^* = L^W_{(p^{-1}(p \vee_l x_1), x_2 p)} W_p^*$$

$$L^W_{(x_1, x_2)} W_p^* = (W_p L^W_{(x_1, x_2)})^* = (L^W_{(px_1, (p \vee_r x_2)p^{-1})} W_p)^* = W_p^* L^W_{(px_1, (p \vee_r x_2)p^{-1})}.$$

Thus the proof is complete. $\qquad\square$

LEMMA 4.12. *Let (G, P) be a doubly quasi-lattice ordered group and let $W : P \to A$ be a covariant partial isometric representation that sees all projections. Let $\sum_{i=1}^n \lambda_i W_{p_i} W^*_{q_i} W_{r_i}$ be a fixed finite sum such that $p_i \leq_r q_i$ and $r_i \leq_l q_i$. For all $y \in P \times P^{\mathrm{op}}$ there exists a nonzero projection Q_y that satisfies*

$$Q_y \Big(\sum_{i=1}^n \lambda_i W_{p_i} W^*_{q_i} W_{r_i} \Big) Q_y = \Big(\sum_{\substack{q_i = r_i p_i \\ (p_i, r_i) \leq y}} \lambda_i \Big) Q_y.$$

PROOF. Let $\sum_{i=1}^n \lambda_i W_{p_i} W^*_{q_i} W_{r_i}$ be a fixed finite sum and fix $y = (y_1, y_2) \in P \times P^{\mathrm{op}}$. We can write down our candidate for Q_y. For each i such that $r_i p_i \neq q_i$ let

$$a_i := ((p_i \vee_l p_i q_i^{-1} r_i y_1), (q_i p_i^{-1} \vee_r y_2 r_i^{-1} q_i p_i^{-1}))$$

$$b_i := ((r_i^{-1} q_i \vee_l r_i^{-1} q_i p_i^{-1} y_1), (r_i \vee_r y_2 p_i q_i^{-1} r_i)).$$

It is not guaranteed that these upper bounds exist, however the proof still works if the a_i or b_i are ∞. Now we define

(4.2)
$$d_i := \begin{cases} a_i & \text{if } a_i \nleq y \\ b_i & \text{if } a_i \leq y. \end{cases}$$

Then write

$$Q_y = L_y^W \Big(\prod_{q_i = r_i p_i, (p_i, r_i) \not\leq y} (L_y^W - L_{y \vee (p_i, r_i)}^W) \Big) \Big(\prod_{q_i \neq r_i p_i} (L_y^W - L_{y \vee d_i}^W) \Big).$$

(We are not worried if the $y \vee (p_i, r_i) = \infty$ or $y \vee d_i = \infty$ because $L_\infty^W = 0$ by convention. In that case $L_y^W - L_\infty^W = L_y^W$.)

We claim $d_i \not\leq y$ for all i such that $q_i \neq r_i p_i$. There are two cases to consider: either $a_i \not\leq y$ or $a_i \leq y$. If $a_i \not\leq y$ then $d_i = a_i$ and $d_i \not\leq y$. Now suppose $a_i \leq y$. Then $d_i = b_i$ and we suppose, aiming for a contradiction, that $b_i \leq y$. Since $a_i, b_i \leq y$ we must have

$$(r_i^{-1} q_i \vee_l r_i^{-1} q_i p_i^{-1} y_1) \leq_l y_1 \text{ and } (p_i \vee_l p_i q_i^{-1} r_i y_1) \leq_l y_1.$$

Thus $r_i^{-1} q_i p_i^{-1} y_1 \leq_l y_1$ and $p_i q_i^{-1} r_i y_1 \leq_l y_1$. By left-invariance we see

$$r_i^{-1} q_i p_i^{-1} y_1 \leq_l y_1 \Leftrightarrow y_1 \leq_l p_i q_i^{-1} r_i y_1.$$

Hence $y_1 = p_i q_i^{-1} r_i y_1$ and so $q_i = r_i p_i$, giving a contradiction. Thus $d_i \not\leq y$.

We need to show Q_y is nonzero. The set

$$F = \{y \vee (p_i, r_i) : q_i = r_i p_i, (p_i, r_i) \not\leq y\} \cup \{y \vee d_i : q_i \neq r_i p_i\}$$

is finite, y is a lower bound for F and $y \notin F$. We can write $Q_y = \prod_{z \in F} (L_y^W - L_z^W)$. Since W sees all projections, Q_y is nonzero.

We now want to see how Q_y interacts with a single $W_{p_i} W_{q_i}^* W_{r_i}$ so we fix an i. There are three cases: (1) $q_i = r_i p_i$ and $(p_i, r_i) \leq y$; (2) $q_i = r_i p_i$ and $(p_i, r_i) \not\leq y$; (3) $q_i \neq r_i p_i$.

(1). Suppose $q_i = r_i p_i$ and $(p_i, r_i) \leq y$. Then

$$W_{p_i} W_{q_i}^* W_{r_i} = W_{p_i} W_{r_i p_i}^* W_{r_i} = L_{(p_i, r_i)}^W$$

and so

$$Q_y W_{p_i} W_{q_i}^* W_{r_i} Q_y = Q_y L_y^W W_{p_i} W_{q_i}^* W_{r_i} Q_y = Q_y L_y^W L_{(p_i, r_i)}^W Q_y = Q_y L_y^W Q_y = Q_y.$$

(2). Suppose $q_i = r_i p_i$ and $(p_i, r_i) \not\leq y$. Then $W_{p_i} W_{q_i}^* W_{r_i} = L_{(p_i, r_i)}^W$ and

$$\begin{aligned} Q_y W_{p_i} W_{q_i}^* W_{r_i} Q_y &= Q_y (L_y^W - L_{y \vee (p_i, r_i)}^W) W_{p_i} W_{q_i}^* W_{r_i} Q_y \\ &= Q_y (L_y^W - L_{y \vee (p_i, r_i)}^W) L_{(p_i, r_i)}^W Q_y \\ &= Q_y (L_{y \vee (p_i, r_i)}^W - L_{y \vee (p_i, r_i)}^W) Q_y \\ &= 0 \end{aligned}$$

(3). Suppose that $q_i \neq r_i p_i$. There are two subcases for the two values of d_i, either $d_i = a_i$ or $d_i = b_i$. First, suppose that $d_i = a_i$. Then we can compute, using Lemma 4.11(1) and (2):

$$
\begin{aligned}
W_{p_i} W_{q_i}^* W_{r_i} L_y^W &= W_{p_i} W_{q_i}^* L_{(r_i y_1, (r_i \vee_r y_2) r_i^{-1})}^W W_{r_i} \\
&= W_{p_i} W_{q_i}^* L_{(r_i y_1, (e \vee_r y_2 r_i^{-1}))}^W W_{r_i} \\
&= W_{p_i} L_{(q_i^{-1}(q_i \vee_l r_i y_1), (e \vee_r y_2 r_i^{-1}) q_i)}^W W_{q_i}^* W_{r_i} \\
&= W_{p_i} L_{(q_i^{-1}(q_i \vee_l r_i y_1), (q_i \vee_r y_2 r_i^{-1} q_i))}^W W_{q_i}^* W_{r_i} \\
&= L_{(p_i q_i^{-1}(q_i \vee_l r_i y_1), (p_i \vee_r (q_i \vee_r y_2 r_i^{-1} q_i)) p_i^{-1})}^W W_{p_i} W_{q_i}^* W_{r_i} \\
&= L_{((p_i \vee_l p_i q_i^{-1} r_i y_1), (q_i \vee_r y_2 r_i^{-1} q_i) p_i^{-1})}^W W_{p_i} W_{q_i}^* W_{r_i} \quad (p_i \leq_r q_i) \\
&= L_{((p_i \vee_l p_i q_i^{-1} r_i y_1), (q_i p_i^{-1} \vee_r y_2 r_i^{-1} q_i p_i^{-1}))}^W W_{p_i} W_{q_i}^* W_{r_i} \\
&= L_{d_i}^W W_{p_i} W_{q_i}^* W_{r_i}.
\end{aligned}
$$

Then

$$
\begin{aligned}
Q_y W_{p_i} W_{q_i}^* W_{r_i} Q_y &= Q_y (L_y^W - L_{y \vee d_i}^W) W_{p_i} W_{q_i}^* W_{r_i} L_y^W Q_y \\
&= Q_y (L_y^W - L_{y \vee d_i}^W) L_{d_i}^W W_{p_i} W_{q_i}^* W_{r_i} Q_y \\
&= Q_y (L_{y \vee d_i}^W - L_{y \vee d_i}^W) W_{p_i} W_{q_i}^* W_{r_i} Q_y = 0.
\end{aligned}
$$

Second, suppose that $d_i = b_i$. Then we compute:

$$
\begin{aligned}
L_y^W W_{p_i} W_{q_i}^* W_{r_i} &= W_{p_i} L_{(p_i^{-1}(p_i \vee_l y_1), y_2 p_i)}^W W_{q_i}^* W_{r_i} \\
&= W_{p_i} W_{q_i}^* L_{(q_i p_i^{-1}(p_i \vee_l y_1), (q_i \vee_r y_2 p_i) q_i^{-1})}^W W_{r_i} \\
&= W_{p_i} W_{q_i}^* L_{((q_i \vee_l q_i p_i^{-1} y_1), (e \vee_r y_2 p_i q_i^{-1}))}^W W_{r_i} \\
&= W_{p_i} W_{q_i}^* W_{r_i} L_{(r_i^{-1}(r_i \vee_l q_i \vee_l q_i p_i^{-1} y_1), (e \vee_r y_2 p_i q_i^{-1}) r_i)}^W \\
&= W_{p_i} W_{q_i}^* W_{r_i} L_{((r_i^{-1} q_i \vee_l r_i^{-1} q_i p_i^{-1} y_1), (r_i \vee_r y_2 p_i q_i^{-1} r_i))}^W \quad (r_i \leq_l q_i) \\
&= W_{p_i} W_{q_i}^* W_{r_i} L_{d_i}^W.
\end{aligned}
$$

Then

$$
\begin{aligned}
Q_y W_{p_i} W_{q_i}^* W_{r_i} Q_y &= Q_y L_y^W W_{p_i} W_{q_i}^* W_{r_i} (L_y^W - L_{y \vee d_i}^W) Q_y \\
&= Q_y W_{p_i} W_{q_i}^* W_{r_i} L_{d_i}^W (L_y^W - L_{y \vee d_i}^W) Q_y \\
&= Q_y W_{p_i} W_{q_i}^* W_{r_i} (L_{y \vee d_i}^W - L_{y \vee d_i}^W) Q_y = 0.
\end{aligned}
$$

Thus we see that

$$Q_y W_{p_i} W_{q_i}^* W_{r_i} Q_y = \begin{cases} Q_y & \text{if } q_i = r_i p_i \text{ and } (p_i, r_i) \le y \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$Q_y(\sum_{i=1}^{n} \lambda_i W_{p_i} W_{q_1}^* W_{r_i})Q_y = \Big(\sum_{\substack{q_i = r_i p_i \\ (p_i, r_i) \le y}} \lambda_i \Big) Q_y$$

and the proof is complete. □

LEMMA 4.13. *Let (G, P) be a doubly quasi-lattice ordered group and let $W : P \to A$ be a covariant partial isometric representation. If W sees all projections then, for $p_i \le_r q_i$ and $r_i \le_l q_i$,*

$$\|\sum_{i=1}^{n} \lambda_i W_{p_i} W_{q_i}^* W_{r_i}\| \ge \|\sum_{r_i p_i = q_i} \lambda_i W_{p_i} W_{q_i}^* W_{r_i}\|.$$

PROOF. First observe that

$$\sum_{r_i p_i = q_i} \lambda_i W_{p_i} W_{q_i}^* W_{r_i} = \sum_{r_i p_i = q_i} \lambda_i L_{(p_i, r_i)}^W.$$

Since W sees all projections, we can apply Lemma 4.9, to see

$$\|\sum_{r_i p_i = q_i} \lambda_i L_{(p_i, r_i)}^W\| = \max\{|\sum_{\substack{q_i = r_i p_i \\ (p_i, r_i) \le z}} \lambda_i| : z \in P \times P^{\mathrm{op}}\}.$$

Let y be an element of $P \times P^{\mathrm{op}}$ that attains this maximum i.e.

$$|\sum_{(p_i, r_i) \le y} \lambda_i| = \max\{|\sum_{\substack{q_i = r_i p_i \\ (p_i, r_i) \le z}} \lambda_i| : z \in P \times P^{\mathrm{op}}\}.$$

(This element y is not necessarily unique.) By Lemma 4.12 there exists a nonzero projection Q_y such that

(4.3) $$Q_y(\sum_{i=1}^{n} \lambda_i W_{p_i} W_{q_i}^* W_{r_i})Q_y = \Big(\sum_{\substack{q_i = r_i p_i \\ (p_i, r_i) \le y}} \lambda_i \Big) Q_y$$

Since projections have norm ≤ 1 we observe:

$$\|\sum_{i=1}^{n} \lambda_i W_{p_i} W_{q_i}^* W_{r_i}\| \ge \|Q_y(\sum_{i=1}^{n} \lambda_i W_{p_i} W_{q_i}^* W_{r_i})Q_y\|$$

$$= \|\Big(\sum_{\substack{q_i = r_i p_i \\ (p_i, r_i) \le y}} \lambda_i \Big) Q_y\|$$

53

$$= |\sum_{\substack{q_i=r_ip_i \\ (p_i,r_i)\leq y}} \lambda_i|$$

By our choice of y we have $|\sum_{(p_i,r_i)\leq y} \lambda_i| = \max\{|\sum_{\substack{q_i=r_ip_i \\ (p_i,r_i)\leq z}} \lambda_i| : z \in P \times P^{\mathrm{op}}\}$. So:

$$\|\sum_{i=1}^n \lambda_i W_{p_i} W_{q_i}^* W_{r_i}\| \geq \max\{|\sum_{\substack{q_i=r_ip_i \\ (p_i,r_i)\leq z}} \lambda_i| : z \in P \times P^{\mathrm{op}}\}$$

$$= \|\sum_{r_ip_i=q_i} \lambda_i L_{(p_i,r_i)}^W\|$$

$$= \|\sum_{r_ip_i=q_i} \lambda_i W_{p_i} W_{q_i}^* W_{r_i}\|.$$

Thus $\|\sum_{i=1}^n \lambda_i W_{p_i} W_{q_i}^* W_{r_i}\| \geq \|\sum_{r_ip_i=q_i} \lambda_i W_{p_i} W_{q_i}^* W_{r_i}\|$ and the proof is complete. $\square$

We can now prove Proposition 4.6 using Lemma 4.13.

PROOF OF PROPOSITION 4.6. Define a map E on $\mathrm{span}\{v_p v_q^* v_r : p \leq_r q, r \leq_l q\}$ by

$$E(\sum_{i=1}^n \lambda_i v_{p_i} v_{q_i}^* v_{r_i}) = \sum_{q_i=r_ip_i} \lambda_i v_{p_i} v_{q_i}^* v_{r_i}.$$

By Lemma 4.3(2) v sees all projections and hence we can apply Lemma 4.13 to see

$$\|\sum_{i=1}^n \lambda_i v_{p_i} v_{q_i}^* v_{r_i}\| \geq \|\sum_{q_i=r_ip_i} \lambda_i v_{p_i} v_{q_i}^* v_{r_i}\| = \|E(\sum_{i=1}^n \lambda_i v_{p_i} v_{q_i}^* v_{r_i})\|.$$

Hence E is norm-decreasing. The $*$-subalgebra $\mathrm{span}\{v_p v_q^* v_r : p \leq_r q, r \leq_l q\}$ is dense in $C^*(G, P, P^{\mathrm{op}})$, so we may extend E to a norm-decreasing linear map E on $C^*(G, P, P^{\mathrm{op}})$. It is clear that E is onto $\mathrm{span}\{v_p v_{rp}^* v_r : p, r \in P\}$ and hence, by continuity, onto $\overline{\mathrm{span}}\{v_p v_{rp}^* v_r : p, r \in P\}$.

To show that E is a conditional expectation it suffices, by Theorem 4.5, to show that E is an idempotent. Fix $a \in \mathrm{span}\{v_p v_{rp}^* v_r : p, r \in P\}$ and write $a = \sum_{i=1}^n \lambda_i v_{p_i} v_{q_i}^* v_{r_i}$. We compute:

$$E(E(a)) = E(E(\sum_{i=1}^n \lambda_i v_{p_i} v_{q_i}^* v_{r_i})) = E(\sum_{q_i=r_ip_i} \lambda_i v_{p_i} v_{q_i}^* v_{r_i}) = \sum_{q_i=r_ip_i} \lambda_i v_{p_i} v_{q_i}^* v_{r_i}.$$

So $E(E(a)) = E(a)$. By continuity this relation extends to all $b \in C^*(G, P, P^{\mathrm{op}})$. Thus E is an idempotent and hence is a conditional expectation by Theorem 4.5.

To prove E is unique, suppose that there is another bounded, linear map F on $C^*(G, P, P^{\mathrm{op}})$ such that

$$F(\sum_{i=1}^{n} \lambda_i v_{p_i} v_{q_i}^* v_{r_i}) = \sum_{q_i = r_i p_i} \lambda_i v_{p_i} v_{q_i}^* v_{r_i}.$$

As we noted above, the $*$-subalgebra $\mathrm{span}\{v_p v_q^* v_r : p \leq_r q, r \leq_l q\}$ is dense in $C^*(G, P, P^{\mathrm{op}})$. For all $a \in \mathrm{span}\{v_p v_q^* v_r : p \leq_r q, r \leq_l q\}$ we have $E(a) = F(a)$. So E and F are bounded linear maps that agree on a dense subalgebra and hence $E = F$. So E is indeed unique. $\qquad\square$

We can now prove Theorem 4.8.

PROOF OF THEOREM 4.8. Let $a \in C^*(G, P, P^{\mathrm{op}})$ such that $\pi_W(a) = 0$. To show π_W is faithful we will prove that $a = 0$.

We claim that, for all $b \in C^*(G, P, P^{\mathrm{op}})$, we have $\|\pi_W(E(b))\| \leq \|\pi_W(b)\|$. By continuity, it will suffice to prove this for $b \in \mathrm{span}\{v_p v_q^* v_r : p \leq_r q, r \leq_l q\}$. Compute

$$\|\pi_W(E(\sum_{i=1}^{n} \lambda_i v_{p_i} v_{q_i}^* v_{r_i}))\| = \|\pi_W(\sum_{q_i = r_i p_i} \lambda_i v_{p_i} v_{q_i}^* v_{r_i})\|$$

$$= \|\sum_{q_i = r_i p_i} \lambda_i W_{p_i} W_{q_i}^* W_{r_i}\|$$

$$\leq \|\sum_{i=1}^{n} \lambda_i W_{p_i} W_{q_i}^* W_{r_i}\| \quad \text{(by Lemma 4.13)}$$

$$= \|\pi_W(\sum_{i=1}^{n} \lambda_i v_{p_i} v_{q_i}^* v_{r_i})\|.$$

Thus for all $b \in C^*(G, P, P^{\mathrm{op}})$, we have $\|\pi_W(E(b))\| \leq \|\pi_W(b)\|$.

Since $\pi_W(a) = 0$ it follows that $\pi_W(a^* a) = 0$. So $\|\pi_W(E(a^* a))\| \leq \|\pi_W(a^* a)\| = 0$, and hence $\pi_W(E(a^* a)) = 0$. We know that $E(a^* a) \in \overline{\mathrm{span}}\{v_p v_{rp}^* v_r : p, r \in P\}$. Further, by Lemma 4.10, π_W is faithful on $\overline{\mathrm{span}}\{v_p v_{rp}^* v_r : p, r \in P\}$. Thus

(4.4) $$\pi_W(E(a^* a)) = 0 \Rightarrow E(a^* a) = 0.$$

By assumption, (G, P) is amenable, that is, E is faithful for positive elements. Hence $E(a^* a) = 0$ implies $a = 0$. Thus π_W is faithful. $\qquad\square$

CHAPTER 5

Necessary background and constructing conditional expectations

The main results of this chapter are Lemma 5.18 and Lemma 5.15 which detail two methods for constructing conditional expectations. Given the central role that conditional expectations play in our definition of amenability these are central to proofs in the next two chapters. These constructions require considerable set up, most of which we will use again anyway. In particular, we need results about tensor products, group algebras and amenable groups. Rather than constantly referring to results printed elsewhere, we take a moment and dedicate the first three sections of this chapter to going over the constructions and quoting the relevant results so we can refer back.

5.1. Tensor products of C^*-algebras

We will be using tensor products of C^*-algebras extensively, so we will take a moment to go over the key points we will use. We start by defining the tensor product of vector spaces. These results are taken from [**20**, Appendix B].

Let U, V be vector spaces. The tensor product of U and V is the vector space $U \odot V$ together with a bilinear map $T : (u, v) \mapsto u \otimes v$ such that, for any bilinear map $B : U \times V \to Z$ there is a unique linear map $L : U \odot V \to Z$ satisfying $L \circ T(u, v) = B(u, v)$. One can also think of $U \odot V$ as the vector space spanned by $\{u \otimes v : u \in U, v \in V\}$ where the addition of the $u \otimes v$ is bilinear, for example:

$$\lambda_1(u_1 \otimes v) + \lambda_2(u_2 \otimes v) = (\lambda_1 u_1 + \lambda_2 u_2) \otimes v$$

and

$$\lambda(u \otimes v) = (\lambda u) \otimes v = u \otimes (\lambda v).$$

If U and V are Hilbert spaces then, by [**20**, Lemma 2.59], there is a natural inner product on $U \odot V$ satisfying

$$(u_1 \otimes v_1 \mid u_2 \otimes v_2) = (u_1 \mid u_2)(v_1 \mid v_2).$$

We can complete $U \odot V$ in the norm induced by this inner product to get a Hilbert space $U \otimes V$. For Hilbert spaces there is a natural choice of norm, this is not the

case for C^*-algebras. (We use $\odot$ for purely algebraic tensor products and $\otimes$ for tensor products that have been completed in some norm.)

Let A and B be C^*-algebras. Taking the algebraic tensor product of A and B there is, by [20, Lemma B.1], a unique $*$-algebra structure on $A \odot B$ given by

$$(a \otimes b)(c \otimes d) = (ac \otimes bd) \text{ and } (a \otimes b)^* = a^* \otimes b^*.$$

To get a C^*-algebra we need to apply a norm to $A \odot B$ and complete it. There is, in general, no unique norm on $A \odot B$. However, there are two particular norms that are useful in different situations: the spatial norm and the maximal norm.

5.1.1. The spatial norm. The spatial norm is defined by representing $A \odot B$ on the tensor product of Hilbert spaces. Suppose that H and K are Hilbert spaces and that $S \in B(H)$ and $T \in B(K)$. By [20, Lemma B.2] there is a unique bounded operator $S \hat{\otimes} T$ on $H \otimes K$ such that $S \hat{\otimes} T(h \otimes k) = Sh \otimes Tk$ and $\|S \hat{\otimes} T\| = \|S\| \|T\|$. Further, by [20, Lemma B.3], there is an injective $*$-homomorphism $\iota : B(H) \odot B(K) \to B(H \otimes K)$ such that $\iota(S \otimes T)(h \otimes k) = S \hat{\otimes} T(h \otimes k)$. So we can identify $B(H) \odot B(K)$ with a subalgebra of $B(H \otimes K)$ and take $B(H) \otimes_c B(K)$ to be the closure of $B(H) \odot B(K)$.

So suppose that A and B are arbitrary C^*-algebras. Let $\pi : A \to B(H_\pi)$ and $\eta : B \to B(H_\eta)$ be faithful nondegenerate representations. The map $(a, b) \mapsto \pi(a) \otimes \eta(b) \in B(H_\pi \otimes H_\eta)$ is bilinear. Hence, by the universal property of the algebraic tensor product, there exists a linear map $\pi \otimes \eta : A \odot B \to B(H_\pi \otimes H_\eta)$ characterized by $\pi \otimes \eta(a \otimes b) = \pi(a) \otimes \eta(b)$. Since π and η are faithful, the map $\pi \otimes \eta$ is injective on $A \odot B$. We now have a norm on $A \odot B$: for $t \in A \odot B$ let $\|t\|_{\pi,\eta} = \|\pi \otimes \eta(t)\|_{B(H_\pi \otimes H_\eta)}$. By [20, Remark B.4] the completion of $A \odot B$ in this norm is isomorphic to $\pi(A) \otimes_c \eta(B)$. By [20, Theorem B.9], the norm $\|\cdot\|_{\pi,\eta}$ does not depend on our choice of π and η and so we write $\|\cdot\|_{\pi,\eta} = \|\cdot\|_{\min}$.

DEFINITION 5.1. If A and B are C^*-algebras, then the norm $\|\cdot\|_{\min}$ is called the *minimal norm* (or *spatial norm*) on $A \odot B$. The completion of $A \odot B$ with respect to $\|\cdot\|_{\min}$ is denoted, $A \otimes_{\min} B$, and is called the *minimal tensor product*. (Or *spatial tensor product*.)

We call $\|\cdot\|_{\min}$ the minimal norm because for all C^*-norms $\|\cdot\|_\alpha$ on $A \odot B$ we have $\|a \otimes b\| = \|a\| \|b\|$ and $\|t\|_\alpha \geq \|t\|_{\min}$ for all $t \in A \odot B$. (See [20, Theorem B.38]). The main result we use going forward regarding the minimal tensor is that it preserves injective homomorphisms and representations:

PROPOSITION 5.2 ([**20**, Proposition B.13]). *Suppose that $\phi : A \to C$ and $\psi : B \to D$ are homomorphisms between C^*-algebras. Then there is a unique homomorphism $\phi \otimes \psi : A \otimes_{\min} B \to C \otimes_{\min} D$ such that $\phi \otimes \psi(a \otimes b) = \phi(a) \otimes \psi(b)$ for all $a \in A$ and $b \in B$. If ϕ and ψ are injective, then so is $\phi \otimes \psi$.*

5.1.2. The maximal tensor product. There is also a biggest norm on $A \odot B$, the maximal norm. This is the norm defined by taking the supremum over all other C^*-norms on $A \odot B$.

$$\|t\|_{\max} = \sup\{\|t\|_\gamma : \|\cdot\|_\gamma \text{ is a } C^*\text{-norm on } A \odot B\}.$$

This supremum is well-defined by [**20**, Proposition B.25].

DEFINITION 5.3. The completion of $A \odot B$ in $\|\cdot\|_{\max}$ is C^*-algebra called the *maximal tensor product* of A and B, and denoted by $A \otimes_{\max} B$.

The maximal tensor product, $A \otimes_{\max} B$, is universal for representations of A and B with commuting ranges.

THEOREM 5.4 ([**20**, Theorem B.27]). *Suppose A and B are unital C^*-algebras. Then there are unital homomorphisms $i_A : A \to A \otimes_{\max} B$ and $i_B : B \to A \otimes_{\max} B$ such that*

(1) $i_A(a)i_B(b) = i_B(b)i_A(a) = a \otimes b$ for $a \in A$, $b \in B$;
(2) if ϕ and ψ are representations of A and B with commuting ranges, then there is a representation $\phi \otimes_{\max} \psi$ of $A \otimes_{\max} B$ such that

$$\phi \otimes_{\max} \psi(i_A(a)i_B(b)) = \phi(a)\psi(b) \text{ for } a \in A, b \in B.$$

(3) $A \otimes_{\max} B = \overline{\operatorname{span}}\{i_A(a)i_B(b) : a \in A \text{ and } b \in B\}$.

In this thesis, we only consider unital C^*-algebras and so we don't need to consider the multiplier algebra $M(A \otimes_{\max} B)$, that appear in the more general statement of Theorem 5.4. The maximal tensor product has an analogue of Proposition 5.2, however, the maximal tensor product does not necessarily preserve injectivity.

LEMMA 5.5 ([**20**, Lemma B.31]). *Suppose that $\phi : A \to C$ and $\psi : B \to D$ are homomorphisms between C^*-algebras. Then there is a unique homomorphism $\phi \otimes \psi : A \otimes_{\max} B \to C \otimes_{\max} D$ such that $\phi \otimes_{\max} \psi(a \otimes b) = \phi(a) \otimes \psi(b)$ for all $a \in A$ and $b \in B$.*

5.1.3. Nuclear C^*-algebras. We have two different completions of $A \odot B$ with properties that are useful in different situations (as well as many intermediate completions). Obviously we would like to be able to use both sets of results. Fortunately, there is a large class of C^*-algebras where the two norms are the same.

DEFINITION 5.6. We say that a C^*-algebra A is *nuclear* if, for every C^*-algebra B, $A \odot B$ has only one C^*-norm. In particular $\| \cdot \|_{\min} = \| \cdot \|_{\max}$.

When we prove nuclearity in Chapter 7 we will use an equivalent condition: A is nuclear if and only if the canonical homomorphism from $A \otimes_{\max} B$ to $A \otimes_{\min} B$ is injective for every C^*-algebra B. Fortunately, it suffices to check only *unital* C^*-algebras B (see [20, Lemma B.42]).

EXAMPLES.
- All commutative C^*-algebras are nuclear. [20, Proposition B.43]
- Let H be a Hilbert space with countable orthonormal basis. Then $\mathcal{K}(H)$ the set of compact operators on H is a nuclear C^*-algebra.

5.2. States and tensor products

We will construct conditional expectations from states on tensor products, so we collect the results we use. We prove two that we were unable to find a suitable reference for. These results do not feature, beyond the construction of conditional expectations.

PROPOSITION 5.7 ([20, Proposition A.5]). *For every state τ on a C^*-algebra A, the GNS-construction gives a nondegenerate representation π_τ of A on a Hilbert space H_τ.*

PROPOSITION 5.8 ([20, Proposition A.6]). *If ρ is a state on a C^*-algebra A, then there is a unit vector h_ρ in H_ρ which is cyclic for π_ρ and satisfies $\rho(a) = (\pi_\rho(a)h_\rho \mid h_\rho)$ for all $a \in A$. Conversely, if h is a unit cyclic vector for a representation $\pi : A \to B(H_\pi)$, then $\tau : a \mapsto (\pi(a)h \mid h)$ is a state on A, and the map $a \mapsto \pi(a)h$ induces a unitary isomorphism U of H_τ onto H_π such that $\pi(a) = U\pi_\tau(a)U^*$ for all $a \in A$.*

LEMMA 5.9. *Let A and B be unital C^*-algebras and let $f : A \to \mathbb{C}$ and $g : B \to \mathbb{C}$ be states.*
- *(1) Then there exists a norm decreasing linear map $\mathrm{id} \otimes g : A \otimes_{\min} B \to A$ such that $\mathrm{id} \otimes g(a \otimes b) = g(b)a$.*
- *(2) Then there exists a norm decreasing linear map $f \otimes \mathrm{id} : A \otimes_{\min} B \to B$ such that $f \otimes \mathrm{id}(a \otimes b) = f(a)b$.*

PROOF. We prove (1) and then (2) follows a symmetrical argument. (1). The map from $A \times B$ to A defined by $(a, b) \mapsto g(b)a$ is bilinear. Thus by the universal property of the algebraic tensor product there exists a unique linear map $\mathrm{id} \otimes g : A \odot B \to A$ such that $\mathrm{id} \otimes g(a \otimes b) = g(b)a$. We wish to extend this map to the complete minimal tensor product. To do this it will suffice to show that $\mathrm{id} \otimes g$ is bounded in the spatial norm.

Choose a faithful representation π_A of A on a Hilbert space K. By Proposition 5.8 there exist a representation π_g of B on a Hilbert space H_g and a cyclic unit vector $h_g \in H_g$ such that $g(b) = (\pi_g(b)h_g|h_g)$ for all $b \in B$. By Proposition 5.2, there is a unique representation $\pi_A \otimes \pi_g : A \otimes_{\min} B \to B(K \otimes H_g)$ satisfying $\pi_A \otimes \pi_g(a \otimes b) = \pi_A(a) \otimes \pi_g(b)$. To show $\mathrm{id} \otimes g$ is norm-decreasing we will prove that for all $c \in A \odot B$ $\pi_A(\mathrm{id} \otimes g(c))$ is a bounded operator on K with $\|\pi_A(\mathrm{id} \otimes g(c))\| \leq \|c\|$.

Write $c \in A \odot B$ as $c = \sum_{i=1}^{n} a_i \otimes b_i$. Consider the map $\beta_c : K \times K \to \mathbb{C}$ given by

$$\beta_c(j, k) := (\pi_A(\mathrm{id} \otimes g(c))j|k).$$

We claim β_c is a bounded sesquilinear form.

Fix $j, k \in K$. Compute:

$$\beta_c(j, k) = \left(\pi_A \left(\mathrm{id} \otimes g \left(\sum_{i=1}^{n} a_i \otimes b_i \right) \right) j \Big| k \right) = \sum_{i=1}^{n} g(b_i)(\pi_A(a_i)j|k)$$

The inner product is sesquilinear and the $\pi_A(a_i)$ are bounded linear operators on K. Hence β_c is sesquilinear. To show β_c is bounded, compute:

$$|\beta_c(j, k)| = \left| \sum_{i=1}^{n} g(b_i)(\pi_A(a_i)j|k) \right|$$

$$= \left| \sum_{i=1}^{n} (\pi_g(b_i)h_g|h_g)(\pi_A(a_i)j|k) \right|$$

$$|\beta_c(j, k)| = \left| \left(\pi_A \otimes \pi_g \left(\sum_{i=1}^{n} a_i \otimes b_i \right)(j \otimes h_g)|k \otimes h_g \right) \right|$$

$$\leq \left\| \pi_A \otimes \pi_g \left(\sum_{i=1}^{n} a_i \otimes b_i \right) \right\| \|j \otimes h_g\| \|k \otimes h_g\|$$

Note that $\|h_g\| = 1$, therefore $\|j \otimes h_g\| = \|j\|$ and $\|k \otimes h_g\| = \|k\|$.

$$|\beta_c(j, k)| \leq \left\| c \right\| \|j\| \|k\|$$

Thus β_c is bounded. Hence there exists, by [**9**, Theorem 3.8-4], a bounded operator T_c on K such that $\beta_c(j, k) = (T_c j|k)$ and $\|T_c\| \leq \|c\|$. In particular note that $T_c =$

$\pi_A(\mathrm{id} \otimes g(c))$ and, $\|\pi_A(\mathrm{id} \otimes g(c))\| = \|T_c\| \leq \|c\|$. Thus $\mathrm{id} \otimes g$ is bounded in the spatial norm and hence may be extended to the complete minimal tensor product $A \otimes_{\min} B$. $\quad\square$

LEMMA 5.10 ([**20**, Corollary B.12]). *Let A and B be unital C^*-algebras and let $f : A \to \mathbb{C}$ and $g : B \to \mathbb{C}$ be states. There exists a unique state $f \otimes g : A \otimes_{\min} B \to \mathbb{C}$ such that $f \otimes g(a \otimes b) = f(a)g(b)$.*

LEMMA 5.11. *Let A and B be unital C^*-algebras and let $f : A \to \mathbb{C}$ and $g : B \to \mathbb{C}$ be states. Then $f \circ (\mathrm{id} \otimes g) = f \otimes g = g \circ (f \otimes \mathrm{id})$.*

PROOF. We will show the first equality and the second will follow by symmetry. Both f and $\mathrm{id} \otimes g$ are bounded and linear and hence their composition is likewise bounded and linear. The state $f \otimes g$ is also bounded. Hence it will suffice to show that these two functions agree on the dense subspace spanned by elementary tensors. Let $a \in A$ and $b \in B$. Then

$$f \circ (\mathrm{id} \otimes g)(a \otimes b) = f(g(b)a) = f(a)g(b) = f \otimes g(a \otimes b). \quad\square$$

5.3. Group algebras and amenable groups

For a group G, a unitary representation of G into a unital C^*-algebra A is a map $U : G \to A$ such that U preserves the group structure in the following sense: for $g, h \in G$ we have $U_g U_h = U_{gh}$, $U_e = 1$ where e is the identity of G, and $U_g^* = U_{g^{-1}}$. The group algebra of a group representation $U : G \to A$ is the C^*-subalgebra of A generated by $\{U_g : g \in G\}$. There are two specific group algebras that we are interested in: the reduced algebra and the universal algebra.

The reduced group algebra $C_r^*(G)$ is concretely defined. Let $\{\epsilon_h : h \in G\}$ be the usual orthonormal basis for $\ell^2(G)$. Then $\lambda : G \to B(\ell^2(G))$ defined by $\lambda_g \epsilon_h = \epsilon_{gh}$ is a group representation. Let $C_r^*(G)$ be the C^*-subalgebra of $B(\ell^2(G))$ generated by $\{\lambda_g : g \in G\}$.

The universal algebra of G is characterized abstractly:

THEOREM ([**18**, §7.1.5]). *Let G be a discrete group. There exists a C^*-algebra $C^*(G)$ generated by unitaries $\{u_g : g \in G\}$ such that $u : g \mapsto u_g$ is a unitary representation of G, which is universal for unitary representations of G in the following sense: for any unitary representation $U : G \to A$ there exists a unital homomorphism $\pi_U : C^*(G) \to A$ such that $\phi_U(u_g) = U_g$. Further the pair $(C^*(G), \{u_g\})$ is unique up to isomorphism.*

By the universal property of $C^*(G)$ there is a homomorphism $\pi_r : C^*(G) \to C_r^*(G)$ such that $\pi_r(u_g) = \lambda_g$. We call π_r the left regular representation of G on $\ell^2(G)$.

Let G be a group and consider $\ell^\infty(G)$. We write $1_G \in \ell^\infty(G)$ for the indicator function in G. For $f \in \ell^\infty(G)$ and $g \in G$, let $g \cdot f$ be defined by $(g \cdot f)(h) = f(g^{-1}h)$ for all $h \in G$.

DEFINITION 5.12. A discrete group G is *amenable* if G admits a left-invariant mean: that is, there exists a bounded linear functional $\mu : \ell^\infty(G) \to \mathbb{R}$, such that $\mu(1_G) = 1$, and $\mu(f) = \mu(g \cdot f)$ for all $f \in \ell^\infty(G)$ and $g \in G$.[1]

This definition is not particularly useful for our purposes. So we state three equivalent conditions that are more illuminating. (These will also serve as a model for the properties we would like our amenable doubly quasi-lattice ordered groups to have.)

THEOREM 5.13. *Let G be a discrete group. The following are equivalent:*

(1) G is amenable.
(2) $\pi_r : C^*(G) \to C_r^*(G)$ is faithful.
(3) $C^*(G)$ is nuclear.
(4) The canonical trace $\tau : C^*(G) \to \mathbb{C}$ is faithful for positive elements: if $\tau(a^*a) = 0$ then $a = 0$.

$(1) \Leftrightarrow (2)$ is due to Hulanicki [**8**] (see also [**18**, Theorem 7.3.9]). $(1) \Leftrightarrow (3)$ is due to Paterson [**17**, Theorem 2]. We have been unable to find a suitable reference for $(1) \Leftrightarrow (4)$ but it is a known result. We prove it below:

LEMMA 5.14. *Let $C_r^*(G)$ be the reduced group algebra of G, let $\pi_r : C^*(G) \to C_r^*(G)$ be the corresponding homomorphism of the universal group algebra $C^*(G)$ and let e be the identity of G. Then $\tau : C^*(G) \to \mathbb{C}$ defined by $\tau(a) := (\pi_r(a)\epsilon_e \mid \epsilon_e)$ is a tracial state. In particular, G is amenable if and only if τ is faithful for positive elements.*

PROOF. To show that τ is a state we must show that it is a positive linear functional and that it has norm 1. By the linearity of π_r and the first coordinate of the inner product, we have that τ is linear. To show that τ is positive, fix $a \in C^*(G)$. Compute

$$\tau(a^*a) = (\pi_r(a^*a)\epsilon_e \mid \epsilon_e) = (\pi_r(a)\epsilon_e \mid \pi_r(a)\epsilon_e) = \|\pi_r(a)\epsilon_e\|^2 \geq 0.$$

To see that τ is bounded with $\|\tau\| \leq 1$ we note

$$|\tau(a)| = |(\pi_r(a)\epsilon_e \mid \epsilon_e)| \leq \|\pi_r(a)\|\|\epsilon_e\|^2 \leq \|a\|.$$

---

[1] I am required by longstanding tradition to mention that the word "amenable" was introduced by Mahlon M. Day as a pun: a group is a-*mean*-able if you can put a *mean* on it.

Further $\tau(u_e) = 1$ and hence τ has norm 1. Thus τ is a state.

Now we must show that τ has the tracial property: $\tau(ab) = \tau(ba)$. We will show this by considering the dense subspace $\mathrm{span}\{u_g : g \in G\}$. Fix $g, h \in G$. Then

$$\tau(u_g u_h) = \tau(u_{gh}) = (\pi_r(u_{gh})\epsilon_e \mid \epsilon_e) = (\lambda_{gh}\epsilon_e \mid \epsilon_e) = (\epsilon_{gh} \mid \epsilon_e) = \begin{cases} 1 & \text{if } g = h^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

Similarly

$$\tau(u_h u_g) = \begin{cases} 1 & \text{if } g = h^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

Thus we see, by linearity that for all a, b in the dense subspace $\mathrm{span}\{u_g : g \in G\}$ $\tau(ab) = \tau(ba)$. This relation then extends to the entire C^*-algebra. Thus τ is a tracial state.

Now we can show that G is amenable if and only if τ is faithful for positive elements. First, suppose G is an amenable group. Let $b \in C^*(G)$ such that $\tau(b^*b) = 0$. By the tracial property $\tau(b^*b) = \tau(b^*bu_g u_g^*) = \tau(u_g^* b^* b u_g)$ for all $g \in G$. Then $0 = \tau(b^*b) = \|\pi_r(bu_g)\epsilon_e\|^2$. Thus

$$0 = \|\pi_r(bu_g)\epsilon_e\| = \|\pi_r(b)\lambda_g \epsilon_e\| = \|\pi_r(b)\epsilon_g\|.$$

Hence $\pi_r(b) = 0$. Since G is amenable π_r is faithful and hence $b = 0$. Thus τ is faithful for positive elements.

Second, aiming for a contradiction, suppose that G is not amenable. Then π_r is not faithful and there must exist $b \in C^*(G)$ such that $b \neq 0$ and $\pi_r(b) = 0$. Then $\tau(b^*b) = (\pi_r(b^*b)\epsilon_e \mid \epsilon_e) = (0 \mid \epsilon_e) = 0$. Thus τ is not faithful for positive elements and we have proved the contrapositive. So G is amenable if and only if τ is faithful for positive elements. $\square$

5.4. Coactions and conditional expectations

We defined amenability of a doubly quasi-lattice ordered group (G, P) in terms of a conditional expectation on $C^*(G, P, P^{\mathrm{op}})$. Thus when proving that a given (G, P) is amenable we need ways to construct conditional expectations that are known to be faithful. There are two methods for constructing conditional expectations depending on whether we are working with concrete or abstract C^*-algebras. We first work with concrete C^*-algebras:

5.4.1. Concrete C^*-algebras. The next lemma is proved in full generality as we will use it several times in different situations.

LEMMA 5.15. *Let H and K be Hilbert spaces. Suppose that I is some index set and $\{e_i : i \in I\}$ is an orthonormal basis for K. Then there exists a faithful conditional expectation $\Delta : B(H \otimes K) \to B(H \otimes K)$ such that, for all $T \in B(H \otimes K)$, $h, h' \in H$ and $i \in I$, we have*

$$(\Delta(T)(h \otimes e_i) \mid h' \otimes e_i) = (T(h \otimes e_i) \mid h' \otimes e_i).$$

PROOF. Let $T \in B(H \otimes K)$. For each $i \in I$ consider the form

$$B_i^T(h, h') = (T(h \otimes e_i) \mid h' \otimes e_i).$$

The form B_i^T is bounded by $\|T\|$ and is sesquilinear by the sesquilinearity of the inner product. Therefore by [**9**, Theorem 3.8-4] there exists a bounded operator $F_i^T : H \to H$ such that $(F_i^T(h) \mid h') = (T(h \otimes e_i) \mid h' \otimes e_i)$ and $\|F_i^T\| \leq \|T\|$.

For all $i \in I$ let 1_i be the projection onto $\mathrm{span}\{e_i\}$. The map $F_i^T \otimes 1_i$ is a bounded operator on $H \otimes \mathrm{span}\{e_i\}$. So we define

$$\Delta(T) := \oplus_{i \in I} F_i^T \otimes 1_i.$$

Then $\Delta(T)$ is a bounded operator on $\oplus_{i \in I} H \otimes \mathrm{span}\{e_i\} = H \otimes K$ with norm

$$\|\Delta(T)\| = \sup\{\|F_i^T \otimes 1_i\| : i \in I\} \leq \|T\|.$$

In addition, we can compute:

$$
\begin{aligned}
(\Delta(T)(h \otimes e_i) \mid h' \otimes e_i) &= (\oplus_{j \in I} F_j^T \otimes 1_j(h \otimes e_i) \mid h' \otimes e_i) \\
&= (F_i^T \otimes 1_i(h \otimes e_i) \mid h' \otimes e_i) \\
&= (F_i^T h \mid h')(e_i \mid e_i) \\
&= (F_i^T h \mid h') \\
&= (T(h \otimes e_i) \mid h' \otimes e_i).
\end{aligned}
$$

It is now left to show that Δ is a conditional expectation and that Δ is faithful for positive elements.

To prove that Δ is a conditional expectation, it suffices, by Theorem 4.5, to show that Δ is an idempotent with norm 1. We know Δ is norm-decreasing and $\Delta(1) = 1$, hence we have $\|\Delta\| = 1$. To show Δ is an idempotent we fix $T \in B(H \otimes K)$. We

must show $\Delta(\Delta(T)) = \Delta(T)$. By construction $\Delta(T) := \oplus_{i \in I} F_i^T \otimes 1_i$ so we only need to show that $F_i^T = F_i^{\Delta(T)}$. We compute:

$$B_i^{\Delta(T)}(h, h') = (\oplus_{j \in I} F_j^T \otimes 1_j(h \otimes e_i) \mid h' \otimes e_i) = (F_i^T \otimes 1_i(h \otimes e_i) \mid h' \otimes e_i) = (F_i^T h \mid h').$$

This $B_i^{\Delta(T)}$ is a bounded sesquilinear form and hence there exists a unique bounded operator $F_i^{\Delta(T)}$ such that $(F_i^{\Delta(T)} h \mid h') = B_i^{\Delta(T)}(h, h')$. However, $(F_i^{\Delta(T)} h \mid h') = (F_i^T h \mid h')$. By uniqueness $F_i^{\Delta(T)} = F_i^T$ and $\Delta(\Delta(T)) = \Delta(T)$.

To see Δ is faithful for positive elements we suppose there exists $S \in B(H \otimes K)$ such that $\Delta(S^*S) = 0$. Then for all $h \in H$ and $i \in I$

$$
\begin{aligned}
(\Delta(S^*S)(h \otimes e_i) \mid h \otimes e_i) &= (S^*S(h \otimes e_i) \mid h \otimes e_i) \\
&= (S(h \otimes e_i) \mid S(h \otimes e_i)) \\
&= \|S(h \otimes e_i)\|^2 = 0.
\end{aligned}
$$

Hence $S(h \otimes e_i) = 0$ for all $h \in H$ and $i \in I$. Therefore $S = 0$ and hence Δ is faithful for positive elements. Hence Δ is a faithful conditional expectation. $\qquad \square$

5.4.2. Abstract algebras. If we are working with an abstract C^*-algebra and do not have an appropriate orthonormal basis we can construct a conditional expectation on an arbitrary C^*-algebra A using a coaction of a group G on A.

DEFINITION 5.16. Let G be a discrete group and let A be a unital C^*-algebra. Let

$$\delta_G : C^*(G) \to C^*(G) \otimes_{\min} C^*(G)$$

be the comultiplication characterised by $\delta_G(u_g) = u_g \otimes u_g$ for all $g \in G$. A *coaction* of G on A is a homomorphism $\delta : A \to A \otimes_{\min} C^*(G)$ such that

$$(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta_G) \circ \delta.$$

The *fixed-point algebra* of δ is the subalgebra $A^\delta := \{a \in A : \delta(a) = a \otimes 1\}$ of A. We say that δ is *nondegenerate* if δ is unital (i.e. a nondegenerate homomorphism) and $\delta(A)(1 \otimes C^*(G)) = A \otimes_{\min} C^*(G)$.

We will use an injective coaction δ to construct a conditional expectation

$$\Psi_\delta := (\mathrm{id} \otimes \tau) \circ \delta.$$

However, we need two relations for the proof:

LEMMA 5.17. *Let A be a unital C^*-algebra, let G be a discrete group, let*

$$\delta_G : C^*(G) \to C^*(G) \otimes_{\min} C^*(G)$$

be the comultiplication of G, let τ be the trace on $C^*(G)$ and let δ be a coaction of G on A. Then, for all $x \in A \otimes_{\min} C^*(G)$, we have

$$(5.1) \qquad (\mathrm{id}_{A \otimes_{\min} C^*(G)} \otimes \tau) \circ (\mathrm{id} \otimes \delta_G)(x) = (\mathrm{id} \otimes \tau(x)) \otimes 1$$

$$(5.2) \qquad \delta \circ (\mathrm{id} \otimes \tau)(x) = \mathrm{id}_{A \otimes_{\min} C^*(G)} \otimes \tau \circ (\delta \otimes \mathrm{id})(x).$$

PROOF. (1). The set $\mathrm{span}\{u_g : g \in G\}$ is dense in $C^*(G)$. Hence

$$\mathrm{span}\{c \otimes u_g : c \in A, g \in G\}$$

is dense in $A \otimes_{\min} C^*(G)$. So it will suffice, by linearity and continuity, to show that

$$(\mathrm{id}_{A \otimes_{\min} C^*(G)} \otimes \tau) \circ (\mathrm{id} \otimes \delta_G)(c \otimes u_g) = (\mathrm{id} \otimes \tau(c \otimes u_g)) \otimes 1.$$

Fix $c \in A$ and $g \in G$ and compute:

$$
\begin{aligned}
(\mathrm{id}_{A \otimes_{\min} C^*(G)} \otimes \tau) \circ (\mathrm{id} \otimes \delta_G)(c \otimes u_g) &= (\mathrm{id}_{A \otimes_{\min} C^*(G)} \otimes \tau)(c \otimes \delta_G(u_g)) \\
&= (\mathrm{id}_{A \otimes_{\min} C^*(G)} \otimes \tau)(c \otimes (u_g \otimes u_g)) \\
&= \tau(u_g)(c \otimes u_g).
\end{aligned}
$$

For $g \in G$, the trace $\tau(u_g) = 0$ unless $g = e$. Also u_e is the identity in $C^*(G)$ and $\tau(u_e) = 1$. Thus:

$$
(\mathrm{id}_{A \otimes_{\min} C^*(G)} \otimes \tau) \circ (\mathrm{id} \otimes \delta_G)(c \otimes u_g) =
\begin{cases}
c \otimes 1 & \text{if } g = e \\
0 & \text{otherwise}
\end{cases}
$$

$$
\begin{aligned}
&= (\tau(u_g)c) \otimes 1 \\
&= (\mathrm{id} \otimes \tau(c \otimes u_g)) \otimes 1.
\end{aligned}
$$

Now (5.1) follows.

(2). To prove (5.2), it suffices to prove the relation for elementary tensors. Fix $c \in A$, $d \in C^*(G)$ and compute:

$$
\begin{aligned}
\delta \circ (\mathrm{id} \otimes \tau)(c \otimes d) &= \tau(d)\delta(c) \\
&= (\mathrm{id}_{A \otimes_{\min} C^*(G)} \otimes \tau)(\delta(c) \otimes d) \\
&= (\mathrm{id}_{A \otimes_{\min} C^*(G)} \otimes \tau) \circ (\delta \otimes \mathrm{id})(c \otimes d).
\end{aligned}
$$

Now (5.2) follows. $\qquad \square$

Now we can apply (5.1) and (5.2) to construct a conditional expectation for a given injective coaction.

LEMMA 5.18. *Let A be a unital C^*-algebra and let G be a discrete group. Let $\delta : A \to A \otimes_{\min} C^*(G)$ be an injective, unital coaction and let τ be the trace of Lemma 5.14. Then*

$$\Psi := (\mathrm{id} \otimes \tau) \circ \delta$$

is a conditional expectation of A onto A^δ. If G is an amenable group then Ψ is faithful.

PROOF. Both $\mathrm{id} \otimes \tau$ and δ are linear and norm decreasing, and thus Ψ is also linear and norm decreasing.

We first show that range $\Psi = A^\delta$. Fix $a \in A^\delta$. Since A^δ is the fixed-point algebra of δ, we have $\delta(a) = a \otimes 1$. Now compute:

(5.3) $$\Psi(a) = (\mathrm{id} \otimes \tau) \circ \delta(a) = \mathrm{id} \otimes \tau(a \otimes 1) = \tau(1)a = a.$$

Thus $\Psi(a) = a$ and so $A^\delta \subseteq$ range Ψ_δ.

To show the reverse inclusion, range $\Psi \subseteq A^\delta$, we fix $b \in A$ and prove that $\delta(\Psi(b)) = \Psi(b) \otimes 1$. Compute:

$$\delta(\Psi(b)) = \delta \circ (\mathrm{id} \otimes \tau) \circ \delta(b)$$
$$= (\mathrm{id}_{A \otimes_{\min} C^*(G)} \otimes \tau) \circ (\delta \otimes \mathrm{id}) \circ \delta(b). \quad \text{(by (5.2))}$$

The coaction identity gives

$$\delta(\Psi(b)) = (\mathrm{id}_{A \otimes_{\min} C^*(G)} \otimes \tau) \circ (\mathrm{id} \otimes \delta_G) \circ \delta(b)$$
$$= \mathrm{id} \otimes \tau(\delta(b)) \otimes 1 \quad \text{(by (5.1))}$$
$$= \Psi(b) \otimes 1.$$

Hence $\Psi(b) \in A^\delta$ and range $\Psi = A^\delta$.

To show Ψ is a conditional expectation it suffices, by Theorem 4.5, to show that Ψ is an idempotent with norm 1. As we showed in (5.3), $\Psi(a) = a$ for all $a \in A^\delta$. Further, range $\Psi = A^\delta$ and so $\Psi(\Psi(b)) = \Psi(b)$ for all $b \in A$. Thus Ψ is an idempotent. We know that Ψ is norm-decreasing and hence $\|\Psi\|_{\mathrm{op}} \leq 1$. Compute:

$$\Psi(1) = \mathrm{id} \otimes \tau(\delta(1)) = \mathrm{id} \otimes \tau(1 \otimes 1) = 1,$$

and it follows that $\|\Psi\|_{\mathrm{op}} = 1$. Thus Ψ is a conditional expectation.

Now suppose that G is an amenable group. We will prove Ψ is faithful in the sense that $\Psi(b^*b) = 0$ implies $b = 0$. Suppose $b \in A$ satisfies $\Psi(b^*b) = 0$. Let f be an arbitrary state on A. Then, applying Lemma 5.11, we see

$$0 = f(\Psi(b^*b))$$
$$= f \circ (\mathrm{id} \otimes \tau) \circ \delta(b^*b)$$
$$= (f \otimes \tau) \circ \delta(b^*b) \quad \text{(by Lemma 5.11)}$$
$$= \tau \circ (f \otimes \mathrm{id}) \circ \delta(b^*b)$$

By Lemma 5.14, τ is faithful for positive elements because G is amenable. Hence $(f \otimes \mathrm{id}) \circ \delta(b^*b) = 0$. This in turn implies that for all states f on A and states g on $C^*(G)$,

$$g \circ (f \otimes \mathrm{id}) \circ \delta(b^*b) = (f \otimes g) \circ \delta(b^*b) = 0.$$

To see that $\delta(b^*b) = 0$, let $\pi_1 : A \to H_1$ and $\pi_2 : C^*(G) \to H_2$ be faithful representations. By Proposition 5.2 $\pi_1 \otimes \pi_2$ is a faithful representation of $A \otimes_{\min} C^*(G)$ on $B(H_1 \otimes H_2)$. For every pair of unit vectors $h \in H_1$, $k \in H_2$ there exists a state $f_h \otimes f_k$ on $A \otimes_{\min} C^*(G)$ defined by

$$f_h \otimes f_k(a) = (\pi_1 \otimes \pi_2(a)(h \otimes k) \mid h \otimes k).$$

Thus, for every pair of unit vectors $h \in H_1$, $k \in H_2$,

$$0 = f_h \otimes f_k(\delta(b^*b))$$
$$= (\pi_1 \otimes \pi_2(\delta(b^*b))(h \otimes k) \mid h \otimes k)$$
$$= (\pi_1 \otimes \pi_2(\delta(b))(h \otimes k) \mid \pi_1 \otimes \pi_2(\delta(b))h \otimes k)$$
$$= \|\pi_1 \otimes \pi_2(\delta(b))(h \otimes k)\|^2$$

Hence $\pi_1 \otimes \pi_2(\delta(b^*b)) = 0$. Since $\pi_1 \otimes \pi_2$ is faithful, $\delta(b^*b)) = 0$. The injectivity of δ implies $b = 0$. Hence Ψ is faithful for positive elements. $\qquad\square$

CHAPTER 6

Recognition theorems for amenable doubly quasi-lattice ordered groups

In this chapter we will outline several recognition theorems for amenable doubly quasi-lattice groups. These are not the most powerful theorems available, and we will prove a more complete recognition theorem in the next chapter. However, these results give useful insights into the nature of amenability.

In Theorem 6.2, we prove that (G, P) is amenable if and only if the representation $\pi_J : C^*(G, P, P^{\mathrm{op}}) \to C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$ is faithful.

In Proposition 6.4, we first prove that the universal algebra generated by (G, P) is determined by the semigroup P, and second, that amenability is a property of P preserved under semigroup isomorphism.

Third, we prove Theorem 6.6, which states that if (G, P) is a doubly quasi-lattice ordered group and G is an amenable group, then (G, P) is amenable in the doubly quasi-lattice ordered sense.

6.1. Amenability and $C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$

As we proved in Lemma 4.3(1), J sees all projections. Therefore, Theorem 4.8 implies that if (G, P) is amenable, then $\pi_J : C^*(G, P, P^{\mathrm{op}}) \to C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$ is faithful. We will now show that the converse is true. To do this we will construct a faithful conditional expectation on $C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$ using Lemma 5.15.

LEMMA 6.1. *There is a faithful conditional expectation Δ on $C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$ such that*

$$\Delta(J_p J_q^* J_r) = \begin{cases} J_p J_q^* J_r & \text{if } q = rp \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. For $b \in P$ let $\{\epsilon_a : a \in P, a \leq_r b\}$ be the usual orthonormal basis for $\ell^2(I_b)$. Let $\epsilon_{a,b}$ denote the orthonormal basis element $\epsilon_a \in \ell^2(I_b)$ viewed in $\bigoplus_{b \in P} \ell^2(I_b)$. Then $\{\epsilon_{a,b} : a, b \in P, a \leq_r b\}$ is an orthonormal basis for $\bigoplus_{b \in P} \ell^2(I_b)$. We now apply Lemma 5.15 with $H = \mathbb{C}$ and $K = \oplus_{b \in P} \ell^2(I_b)$ to get a faithful conditional expectation Δ such

that

$$(\Delta(T)(\epsilon_{a,b})|\epsilon_{a,b}) = (T(\epsilon_{a,b})|\epsilon_{a,b})$$

for all $T \in B(\oplus_{b \in P} \ell^2(I_b))$. To see that

$$\Delta(J_p J_q^* J_r) = \begin{cases} J_p J_q^* J_r & \text{if } q = rp \\ 0 & \text{otherwise} \end{cases}$$

we compute:

$$(\Delta(J_p J_q^* J_r)(\epsilon_{a,b})|\epsilon_{a,b}) = (J_p J_q^* J_r(\epsilon_{a,b})|\epsilon_{a,b})$$

$$= \begin{cases} (\epsilon_{pq^{-1}ra,b}|\epsilon_{a,b}) & \text{if } ra \leq_r b \text{ and } q \leq_l ra \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} 1 & \text{if } rp = q \text{ and } ra \leq_r b \text{ and } q \leq_l ra \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} (J_p J_q^* J_r(\epsilon_{a,b})|\epsilon_{a,b}) & \text{if } rp = q \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\Delta(J_p J_q^* J_r) = \begin{cases} J_p J_q^* J_r & \text{if } q = rp \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

THEOREM 6.2. *Let (G, P) be a doubly quasi-lattice ordered group. Then*

$$\pi_J : C^*(G, P, P^{\mathrm{op}}) \to C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$$

is faithful if and only if (G, P) is amenable.

PROOF. First, suppose that (G, P) is amenable. We know J sees all projections, by Lemma 4.3(1). Thus we may apply Theorem 4.8 to see that π_J is faithful.

Second, we suppose that $\pi_J : C^*(G, P, P^{\mathrm{op}}) \to C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$ is faithful. By Lemma 6.1, there exists a faithful conditional expectation Δ on $C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$ such that

$$\Delta(J_p J_q^* J_r) = \begin{cases} J_p J_q^* J_r & \text{if } q = rp \\ 0 & \text{otherwise.} \end{cases}$$

Recall that E is the conditional expectation on $C^*(G, P, P^{\mathrm{op}})$ of Proposition 4.6. We claim that $\pi_J \circ E = \Delta \circ \pi_J$. Fix $p, q, r \in P$ such that $p \leq_r q$ and $r \leq_l q$. Compute:

$$\pi_J \circ E(v_p v_q^* v_r) = \begin{cases} \pi_J(v_p v_q^* v_r) & \text{if } q = rp \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} J_p J_q^* J_r & \text{if } q = rp \\ 0 & \text{otherwise.} \end{cases}$$

$$= \Delta(J_p J_q^* J_r)$$

$$= \Delta \circ \pi_J(v_p v_q^* v_r).$$

By linearity and continuity this relation extends to the whole of $C^*(G, P, P^{\mathrm{op}})$ and hence $\pi_J \circ E = \Delta \circ \pi_J$.

Now suppose that $a \in C^*(G, P, P^{\mathrm{op}})$ such that $E(a^*a) = 0$. Then

$$0 = \pi_J \circ E(a^*a) = \Delta \circ \pi_J(a^*a).$$

However, Δ is faithful for positive elements and hence $\pi_J(a) = 0$. By assumption, π_J is faithful and hence $a = 0$. Thus E is faithful for positive elements and (G, P) is amenable. $\qquad\square$

6.2. Amenability is a property of semigroups

One observation that is perhaps obscured by our notation is that the universal algebra $C^*(G, P, P^{\mathrm{op}})$ associated to a doubly quasi-lattice ordered group is uniquely determined by the semigroup P and does not depend on the group G. Similarly, the amenability of (G, P) is determined by P. (Our constructions make use of the group properties of G so it is still important that P is imbedded in a group.)

We first prove that a semigroup isomorphism is automatically order preserving and preserves the least upper bound structure.

LEMMA 6.3. *Let (G, P) and (K, Q) be doubly quasi-lattice ordered groups. Suppose there is a semigroup isomorphism $\phi : P \to Q$. Then ϕ is left and right order preserving. In particular, for $x, y \in P$,*

(1) $x \vee_l y < \infty$ if and only if $\phi(x) \vee_l \phi(y) < \infty$. If $x \vee_l y < \infty$ then $\phi(x \vee_l y) = \phi(x) \vee_l \phi(y)$;

(2) $x \vee_r y < \infty$ if and only if $\phi(x) \vee_r \phi(y) < \infty$. If $x \vee_r y < \infty$ then $\phi(x \vee_r y) = \phi(x) \vee_r \phi(y)$.

PROOF. We first show that ϕ is left and right order preserving. Suppose that $x, y \in P$ and $x \leq_l y$. Then $x^{-1}y \in P$ and $\phi(x^{-1}y) \in Q$. We claim that $\phi(x)^{-1}\phi(y) = \phi(x^{-1}y)$ and hence that $\phi(x) \leq_l \phi(y)$. To prove our claim compute:

$$\phi(y) = \phi(xx^{-1}y) = \phi(x)\phi(x^{-1}y).$$

Since $\phi(x) \in Q \subset K$, there exists an inverse $\phi(x)^{-1} \in K$. Then

$$\phi(x)^{-1}\phi(y) = \phi(x)^{-1}\phi(x)\phi(x^{-1}y) = \phi(x^{-1}y).$$

Thus $\phi(x)^{-1}\phi(y) \in Q$. So $x \leq_l y$ implies $\phi(x) \leq_l \phi(y)$ and ϕ preserves the left order. A similar argument shows that ϕ preserves the right order.

To show that ϕ preserves the least upper bound structure, suppose that $x, y \in P$ and $x \vee_l y < \infty$. We will show that $\phi(x) \vee_l \phi(y)$ exists. Since ϕ is order preserving it follows that $\phi(x), \phi(y) \leq_l \phi(x \vee_l y)$. Thus $\phi(x), \phi(y)$ have a common upper bound in Q. Hence $\phi(x) \vee_l \phi(y)$ exists and $\phi(x) \vee_l \phi(y) \leq_l \phi(x \vee_l y)$. To show the other direction suppose that $\phi(x) \vee_l \phi(y) < \infty$. Observe that $\phi^{-1} : Q \to P$ is also a semigroup isomorphism and hence also order preserving. Thus $\phi^{-1}(\phi(x) \vee_l \phi(y))$ is an upper bound for $x = \phi^{-1}(\phi(x))$ and $y = \phi^{-1}(\phi(y))$. Hence $x \vee_l y$ exists and $x \vee_l y \leq_l \phi^{-1}(\phi(x) \vee_l \phi(y))$. Thus

$$\phi(x \vee_l y) \leq_l \phi(\phi^{-1}(\phi(x) \vee_l \phi(y))) = \phi(x) \vee_l \phi(y).$$

Hence $x \vee_l y < \infty$ if and only if $\phi(x) \vee_l \phi(y) < \infty$ and $\phi(x \vee_l y) = \phi(x) \vee_l \phi(y)$ whenever $x \vee_l y < \infty$. A similar argument holds for the right order. □

PROPOSITION 6.4. *Let (G, P) and (K, Q) be doubly quasi-lattice ordered groups. Let $\{v_p : p \in P\}$ and $\{w_q : q \in Q\}$ be the generating elements of $C^*(G, P, P^{\text{op}})$ and $C^*(K, Q, Q^{\text{op}})$ respectively. Suppose there is a semigroup isomorphism $\phi : P \to Q$. Then:*

(1) There exists an isomorphism $\pi_\phi : C^*(G, P, P^{\text{op}}) \to C^*(K, Q, Q^{\text{op}})$ such that $\pi_\phi(v_p) = w_{\phi(p)}$.

(2) (G, P) is amenable if and only if (K, Q) is amenable.

PROOF. (1). We will first use the universal property of $C^*(G, P, P^{\text{op}})$ to get a candidate for π_ϕ and then prove that it is an isomorphism. We claim that $T : P \to C^*(K, Q, Q^{\text{op}})$ defined by $T_p = w_{\phi(p)}$ is a covariant partial isometric representation of P. Fix $p, q \in P$. Since ϕ is a semigroup isomorphism we have

$$T_p T_q = w_{\phi(p)} w_{\phi(q)} = w_{\phi(pq)} = T_{pq}$$

74

and $T_e = w_{\phi(e_G)} = w_{e_Q} = 1$. Hence T is a partial isometric representation. To show that T is covariant we use Lemma 6.3(1) which states that $x \vee_l y < \infty$ if and only if $\phi(x) \vee_l \phi(y) < \infty$ and $\phi(x \vee_l y) = \phi(x) \vee_l \phi(y)$ whenever $x \vee_l y < \infty$. Thus

$$
\begin{aligned}
T_p T_p^* T_q T_q^* &= w_{\phi(p)} w_{\phi(p)}^* w_{\phi(q)} w_{\phi(q)}^* \\
&= \begin{cases} w_{\phi(p) \vee_l \phi(q)} w_{\phi(p) \vee_l \phi(q)}^* & \text{if } \phi(p) \vee_l \phi(q) < \infty \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} w_{\phi(p \vee_l q)} w_{\phi(p \vee_l q)}^* & \text{if } p \vee_l q < \infty \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} T_{p \vee_l q} T_{p \vee_l q}^* & \text{if } p \vee_l q < \infty \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Similarly,

$$
T_p^* T_p T_q^* T_q = \begin{cases} T_{p \vee_r q}^* T_{p \vee_r q} & \text{if } p \vee_r q < \infty \\ 0 & \text{otherwise.} \end{cases}
$$

Hence T is covariant. Thus, by Theorem 3.1 there exists a homomorphism $\pi_\phi : C^*(G, P, P^{\mathrm{op}}) \to C^*(K, Q, Q^{\mathrm{op}})$ such that $\pi_\phi(v_p) = T_p = w_{\phi(p)}$. Now we must prove that this π_ϕ is an isomorphism.

Since $\phi^{-1} : Q \to P$ is an isomorphism the argument above gives a homomorphism $\pi_{\phi^{-1}} : C^*(K, Q, Q^{\mathrm{op}}) \to C^*(G, P, P^{\mathrm{op}})$ such that $\pi_{\phi^{-1}}(w_q) = v_{\phi^{-1}(q)}$. In particular $\pi_{\phi^{-1}}$ is an inverse for π_ϕ since

$$
\pi_{\phi^{-1}}\big(\pi_\phi(v_p)\big) = \pi_{\phi^{-1}}\big(w_{\phi(p)}\big) = v_{\phi^{-1}(\phi(p))} = v_p
$$

and

$$
\pi_\phi\big(\pi_{\phi^{-1}}(w_q)\big) = \pi_\phi\big(v_{\phi^{-1}(q)}\big) = w_q.
$$

Thus π_ϕ is an isomorphism from $C^*(G, P, P^{\mathrm{op}})$ to $C^*(K, Q, Q^{\mathrm{op}})$.

(2) By symmetry it suffices to show that if (K, Q) is amenable then (G, P) is amenable. Let E_Q and E_P be the conditional expectations of Proposition 4.6 on $C^*(K, Q, Q^{\mathrm{op}})$ and $C^*(G, P, P^{\mathrm{op}})$, respectively. We claim that

$$
E_P = \pi_\phi^{-1} \circ E_Q \circ \pi_\phi.
$$

Computing on spanning elements we see:

$$
\pi_\phi^{-1} \circ E_Q \circ \pi_\phi(v_p v_q^* v_r) = \pi_\phi^{-1} \circ E_Q\big(w_{\phi(p)} w_{\phi(q)}^* w_{\phi(r)}\big)
$$

$$
= \begin{cases} \pi_\phi^{-1}(w_{\phi(p)} w^*_{\phi(q)} w_{\phi(r)}) & \text{if } \phi(rp) = \phi(q) \\ 0 & \text{otherwise} \end{cases}
$$

$$
= \begin{cases} \pi_\phi^{-1}(w_{\phi(p)} w^*_{\phi(q)} w_{\phi(r)}) & \text{if } rp = q \\ 0 & \text{otherwise} \end{cases}
$$

$$
= \begin{cases} v_p v^*_q v_r & \text{if } rp = q \\ 0 & \text{otherwise} \end{cases}
$$

$$
= E_P(v_p v^*_q v_r).
$$

Since (K, Q) is amenable, E_Q is faithful for positive elements. Since π_ϕ^{-1} and π_ϕ are faithful, it follows that E_P must also be faithful for positive elements. Thus (G, P) is amenable. So we have shown that (G, P) is amenable if and only if (K, Q) is amenable and the proof is complete. $\qquad\square$

6.3. Amenable groups

In this section we will justify our use of the term amenable by showing that if G is an amenable group, then (G, P) is amenable as a doubly quasi-lattice group. The converse is not true. As we will show in the next chapter there are many non-amenable groups that have amenable doubly quasi-lattice ordered groups.

We will construct an injective coaction, then we can apply Lemma 7.2 to construct a faithful conditional expectation that matches up with the conditional expectation E of Proposition 4.6.

LEMMA 6.5. *Let (G, P) be a doubly quasi-lattice ordered group. There is an injective, nondegenerate coaction $\delta : C^*(G, P, P^{\mathrm{op}}) \to C^*(G, P, P^{\mathrm{op}}) \otimes_{\min} C^*(G)$ such that $\delta(v_p) = v_p \otimes u_p$.*

PROOF. First let us construct a candidate for δ. We claim that $W : P \to C^*(G, P, P^{\mathrm{op}}) \otimes_{\min} C^*(G)$, defined by $W_p = v_p \otimes u_p$, is a covariant partial isometric representation. Unitaries are partial isometries and hence each W_p is a partial isometry. Observe $W_e = v_e \otimes u_e = 1 \otimes 1$ and for $p, q \in P$ we have $W_p W_q = v_p v_q \otimes u_p u_q = v_{pq} \otimes u_{pq}$. To prove W is covariant, fix $x, y \in P$ and compute:

$$
W_x W^*_x W_y W^*_y = v_x v^*_x v_y v^*_y \otimes u_x u^*_x u_y u^*_y
$$

$$
= v_{x \vee_l y} v^*_{x \vee_l y} \otimes 1
$$

$$= \begin{cases} v_{x \vee_l y} v^*_{x \vee_l y} \otimes u_{x \vee_l y} u^*_{x \vee_l y} & \text{if } x \vee_l y < \infty \\ 0 \otimes 1 & \text{otherwise} \end{cases}$$

$$= \begin{cases} W_{x \vee_l y} W^*_{x \vee_l y} & \text{if } x \vee_l y < \infty \\ 0 \otimes 1 & \text{otherwise.} \end{cases}$$

Thus $W_x W^*_x W_y W^*_y = W_{x \vee_l y} W^*_{x \vee_l y}$. Similarly, $W^*_x W_x W^*_y W_y = W^*_{x \vee_r y} W_{x \vee_r y}$. Thus W is a covariant partial isometric representation of P. Hence, by Theorem 3.1, there exists a homomorphism $\delta : C^*(G, P, P^{\mathrm{op}}) \to C^*(G, P, P^{\mathrm{op}}) \otimes_{\min} C^*(G)$ such that $\delta(v_p) = v_p \otimes u_p$.

We must now show that δ is coassociative, injective and non-degenerate. We will show that δ is coassociative by considering the generators v_p. Fix $p \in P$. Compute:

$$(\delta \otimes \mathrm{id}) \circ \delta(v_p) = (\delta \otimes \mathrm{id})(v_p \otimes u_p)$$
$$= \delta(v_p) \otimes u_p$$
$$= v_p \otimes u_p \otimes u_p$$
$$= v_p \otimes \delta_G(u_p)$$
$$= (\mathrm{id} \otimes \delta_G) \circ \delta(v_p).$$

Since $(\delta \otimes \mathrm{id}) \circ \delta$ and $(\mathrm{id} \otimes \delta_G) \circ \delta$ are continuous homomorphisms which agree on generators, δ is coassociative.

To show that δ is injective we will show that a faithful representation π may be written as a composition of δ and another representation. Choose a faithful representation $\pi : C^*(G, P, P^{\mathrm{op}}) \to B(H)$ and let ϵ be the trivial representation on $\mathbb{C}$ such that $\epsilon(u_g) = 1$ for all $g \in G$. By Proposition 5.2 there exists a homomorphism

$$\pi \otimes \epsilon : C^*(G, P, P^{\mathrm{op}}) \otimes_{\min} C^*(G) \to B(H) \otimes \mathbb{C} = B(H).$$

We claim $\pi = \pi \otimes \epsilon \circ \delta$. Again we need compute only on the generators v_p. Compute:

$$(\pi \otimes \epsilon) \circ \delta(v_p) = (\pi \otimes \epsilon)(v_p \otimes u_p) = \pi(v_p).$$

Now suppose $\delta(a) = 0$ for some $a \in C^*(G, P, P^{\mathrm{op}})$. Then $(\pi \otimes \epsilon) \circ \delta(a) = 0 = \pi(a)$, and thus $a = 0$ since π is faithful. Hence δ is injective.

To prove that δ is nondegenerate we must show that

$$\delta(C^*(G, P, P^{\mathrm{op}}))(1 \otimes C^*(G)) = C^*(G, P, P^{\mathrm{op}}) \otimes_{\min} C^*(G).$$

It suffices to show that we can produce the spanning elements $v_p v^*_q v_r \otimes u_g$. Compute

$$\delta(v_p v^*_q v_r)(1 \otimes u_{(pq^{-1}r)^{-1}g}) = v_p v^*_q v_r \otimes u_{pq^{-1}r}(1 \otimes u_{(pq^{-1}r)^{-1}g}) = v_p v^*_q v_r \otimes u_g.$$

Thus δ is nondegenerate.

We claim that δ has fixed-point algebra $C^*(G, P, P^{\mathrm{op}})^\delta = \overline{\mathrm{span}}\{v_p v_q^* v_r : rp = q\}$. Fix $p, q, r \in P$ such that $p \leq_r q$ and $r \leq_l q$. Then consider

$$\delta(v_p v_q^* v_r) = v_p v_q^* v_r \otimes u_p u_q^* u_r = v_p v_q^* v_r \otimes u_{pq^{-1}r}.$$

Then $\delta(v_p v_q^* v_r) = v_p v_q^* v_r \otimes 1$ if and only if $rp = q$. Thus

$$C^*(G, P, P^{\mathrm{op}})^\delta = \overline{\mathrm{span}}\{v_p v_q^* v_r : rp = q\}. \quad \square$$

PROPOSITION 6.6. *Let (G, P) be a doubly quasi-lattice ordered group. If G is an amenable group then (G, P) is amenable.*

PROOF. By Lemma 6.5 there is a coaction $\delta : C^*(G, P, P^{\mathrm{op}}) \to C^*(G, P, P^{\mathrm{op}}) \otimes C^*(G)$ characterised by $\delta(v_p) = v_p \otimes u_p$. By Lemma 5.18 there exists a conditional expectation

$$\Psi_\delta = (\mathrm{id} \otimes \tau) \circ \delta : C^*(G, P, P^{\mathrm{op}}) \to C^*(G, P, P^{\mathrm{op}})^\delta.$$

Since G is amenable Ψ_δ is faithful for positive elements. Now we can compute:

$$\begin{aligned}
\Psi_\delta(v_p v_q^* v_r) &= (\mathrm{id} \otimes \tau) \circ \delta(v_p v_q^* v_r) \\
&= (\mathrm{id} \otimes \tau)(v_p v_q^* v_r \otimes u_{pq^{-1}r}) \\
&= \begin{cases} v_p v_q^* v_r & \text{if } q = rp \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

In particular Ψ_δ agrees with the conditional expectation E of Proposition 4.6 on the dense subspace $\mathrm{span}\{v_p v_q^* v_r : p \leq_r q, r \leq_l q\}$. Thus Ψ_δ is the conditional expectation E is faithful for positive elements. So (G, P) is amenable. $\quad \square$

REMARK. We will see in the next chapter that Proposition 6.6 appears as a special case of Theorem 7.7. However, Proposition 6.6 is an important point in its own right and is much easier to prove.

Amenability of (G, P) and the Nuclearity of $C^*(G, P, P^{\mathrm{op}})$

In this chapter we aim to prove a much stronger recognition theorem for amenable doubly quasi-lattice ordered groups. We do this by setting up a group homomorphism with certain properties from one doubly quasi-lattice ordered group (G, P) to another (K, Q) where K is an amenable group. Doubly quasi-lattice ordered groups with such a homomorphism are amenable and give nuclear C^*-algebras.

7.1. Conditional expectations on tensor products

To prove nuclearity we will consider the tensor products $A \otimes_{\max} C^*(G, P, P^{\mathrm{op}})$ and $A \otimes_{\min} C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$. We begin by constructing conditional expectations on these tensor products. We will use Lemma 5.18 to construct a conditional expectation on $A \otimes_{\max} C^*(G, P, P^{\mathrm{op}})$. We must first construct a suitable coaction on $A \otimes_{\max} C^*(G, P, P^{\mathrm{op}})$.

LEMMA 7.1. *Let (G, P) be a doubly quasi-lattice ordered group and let A be a unital C^*-algebra. Suppose that there exists a group K and a homomorphism $\phi : G \to K$. Then there exists an injective nondegenerate coaction*

$$\delta_\phi : A \otimes_{\max} C^*(G, P, P^{\mathrm{op}}) \to A \otimes_{\max} \left(C^*(G, P, P^{\mathrm{op}}) \otimes_{\min} C^*(K) \right)$$

characterised by $\delta_\phi(a \otimes v_p) = a \otimes v_p \otimes u_{\phi(p)}$ for all $a \in A$ and $p \in P$.

PROOF. We will construct a homomorphism

$$\pi_K : C^*(G, P, P^{\mathrm{op}}) \to C^*(G, P, P^{\mathrm{op}}) \otimes_{\min} C^*(K),$$

and then show that $\delta_\phi := \mathrm{id} \otimes \pi_K$ is well-defined and has the properties we want. We claim there exists a homomorphism π_K such that $\pi_K(v_p) = v_p \otimes u_{\phi(p)}$. To see this, we will show that $W : P \to C^*(G, P, P^{\mathrm{op}}) \otimes_{\min} C^*(K)$, defined by $W_p = v_p \otimes u_{\phi(p)}$, is a covariant partial isometric representation and then use the properties of the universal algebra to get π_K. Unitaries are partial isometries and hence W is partial isometric. Observe that

$$W_e = v_e \otimes u_{\phi(e)} = 1 \otimes 1,$$

$$W_p W_q = v_p v_q \otimes u_{\phi(p)} u_{\phi(q)} = v_{pq} \otimes u_{\phi(pq)} \text{ for all } p, q \in P.$$

To prove W is covariant, fix $x, y \in P$ and compute:

$$W_x W_x^* W_y W_y^* = v_x v_x^* v_y v_y^* \otimes u_{\phi(x)} u_{\phi(x)}^* u_{\phi(y)} u_{\phi(y)}^*$$

$$= \begin{cases} v_{x \vee_l y} v_{x \vee_l y}^* \otimes 1 & \text{if } x \vee_l y < \infty \\ 0 \otimes 1 & \text{otherwise} \end{cases}$$

$$= \begin{cases} v_{x \vee_l y} v_{x \vee_l y}^* \otimes v_{\phi(x \vee_l y)} v_{\phi(x \vee_l y)}^* & \text{if } x \vee_l y < \infty \\ 0 & \text{otherwise} \end{cases}$$

$$= W_{x \vee_l y} W_{x \vee_l y}^*.$$

Similarly $W_x^* W_x W_y^* W_y = W_{x \vee_r y}^* W_{x \vee_r y}$. Thus W is a covariant partial isometric representation of P. Hence, by Theorem 3.1, there exists a homomorphism

$$\pi_K : C^*(G, P, P^{\mathrm{op}}) \to C^*(G, P, P^{\mathrm{op}}) \otimes_{\max} C^*(K)$$

such that $\pi_K(v_p) = v_p \otimes u_{\phi(p)}$.

Since $\mathrm{id} : A \to A$ and $\pi_K : C^*(G, P, P^{\mathrm{op}}) \to C^*(G, P, P^{\mathrm{op}}) \otimes_{\min} C^*(K)$ are both homomorphisms, Lemma 5.5 states that there exists a unique homomorphism

$$\mathrm{id} \otimes \pi_K : A \otimes_{\max} C^*(G, P, P^{\mathrm{op}}) \to A \otimes_{\max} (C^*(G, P, P^{\mathrm{op}}) \otimes_{\min} C^*(K))$$

which acts on the generators of $A \otimes_{\max} C^*(G, P, P^{\mathrm{op}})$ as

$$\mathrm{id} \otimes \pi_K(a \otimes v_p v_q^* v_r) = a \otimes v_p v_q^* v_r \otimes u_{\phi(pq^{-1}r)}.$$

Let $\delta_\phi := \mathrm{id} \otimes \pi_K$. Since δ_ϕ is unital it is nondegenerate as a homomorphism. To complete the proof we must show that δ_ϕ is injective, satisfies the comultiplicative property and is nondegenerate.

To show that δ_ϕ is injective we will show that a faithful representation π may be written as a composition of δ_ϕ and another representation. Choose a faithful representation $\pi : A \otimes_{\max} C^*(G, P, P^{\mathrm{op}}) \to B(H)$ and let $\epsilon : C^*(K) \to \mathbb{C}$ be the trivial representation on $\mathbb{C}$ such that $\epsilon(u_k) = 1$ for all $k \in K$. By Proposition 5.2 there exists a homomorphism

$$\pi \otimes \epsilon : A \otimes_{\max} C^*(G, P, P^{\mathrm{op}}) \otimes_{\min} C^*(G) \to B(H) \otimes \mathbb{C} = B(H).$$

By Theorem 5.4 there exists a homomorphism

$$\lambda : A \otimes_{\max} (C^*(G, P, P^{\mathrm{op}}) \otimes_{\min} C^*(K)) \to (A \otimes_{\max} C^*(G, P, P^{\mathrm{op}})) \otimes_{\min} C^*(K)$$

such that $\lambda(a \otimes v_p \otimes u_k) = a \otimes v_p \otimes u_k$. We claim $\pi = (\pi \otimes \epsilon) \circ \lambda \circ \delta_\phi$. We need compute only on the generators $a \otimes v_p$. Compute:

$$(\pi \otimes \epsilon) \circ \lambda \circ \delta_\phi(a \otimes v_p) = (\pi \otimes \epsilon) \circ \lambda(a \otimes v_p \otimes u_{\phi(p)}) = (\pi \otimes \epsilon)(a \otimes v_p \otimes u_{\phi(p)}) = \pi(a \otimes v_p).$$

Now suppose $\delta_\phi(a) = 0$ for some $a \in A \otimes_{\max} C^*(G, P, P^{\mathrm{op}})$. Then $(\pi \otimes \epsilon) \circ \delta_\phi(a) = 0 = \pi(a)$ thus $a = 0$ since π is faithful. Hence δ_ϕ is injective.

To prove that δ_ϕ is nondegenerate we must show that

$$\delta_\phi(A \otimes_{\max} C^*(G, P, P^{\mathrm{op}}))(1 \otimes C^*(K)) = (A \otimes_{\max} C^*(G, P, P^{\mathrm{op}})) \otimes_{\min} C^*(K).$$

It suffices to show that we can produce the spanning elements $a \otimes v_p v_q^* v_r \otimes u_k$. Compute:

$$\delta_\phi(a \otimes v_p v_q^* v_r)(1 \otimes u_{\phi(pq^{-1}r)^{-1}k}) = a \otimes v_p v_q^* v_r \otimes u_{\phi(pq^{-1}r)}(1 \otimes u_{\phi(pq^{-1}r)^{-1}k}) = a \otimes v_p v_q^* v_r \otimes u_k.$$

Thus δ_ϕ is nondegenerate.

To prove comultiplicativity we must show that

$$(\delta_\phi \otimes \mathrm{id}_{C^*(K)}) \circ (\delta_\phi) = \mathrm{id}_{A \otimes C^*(G,P,P^{\mathrm{op}})} \otimes \delta_K \circ \delta_\phi.$$

Again it suffices to calculate on the generators:

$$\begin{aligned}
(\delta_\phi \otimes \mathrm{id}_{C^*(K)}) \circ (\delta_\phi)(a \otimes v_p v_q^* v_r) &= ((\delta_\phi) \otimes \mathrm{id}_{C^*(K)})(a \otimes v_p v_q^* v_r \otimes u_{\phi(pq^{-1}r)}) \\
&= \delta_\phi(a \otimes v_p v_q^* v_r) \otimes \mathrm{id}_{C^*(K)}(u_{\phi(pq^{-1}r)}) \\
&= a \otimes v_p v_q^* v_r \otimes u_{\phi(pq^{-1}r)} \otimes u_{\phi(pq^{-1}r)} \\
&= a \otimes v_p v_q^* v_r \otimes \delta_K(u_{\phi(pq^{-1}r)}) \\
&= \mathrm{id} \otimes \delta_K(a \otimes v_p v_q^* v_r \otimes u_{\phi(pq^{-1}r)}) \\
&= (\mathrm{id}_{A \otimes C^*(G,P,P^{\mathrm{op}})} \otimes \delta_K) \circ (\delta_\phi)(a \otimes v_p v_q^* v_r).
\end{aligned}$$

Hence $((\delta_\phi) \otimes \mathrm{id}_{C^*(K)}) \circ (\delta_\phi) = (\mathrm{id}_{A \otimes C^*(G,P,P^{\mathrm{op}})} \otimes \delta_K) \circ (\delta_\phi)$ and so comultiplicativity holds. Hence δ_ϕ is a coaction. $\qquad\square$

We can now construct a conditional expectation Ψ_K on $A \otimes_{\max} C^*(G, P, P^{\mathrm{op}})$ using Lemma 7.1.

LEMMA 7.2. *Let (G, P) be a doubly quasi-lattice ordered group and let A be a unital C^*-algebra. Suppose that K is a group and that there is a homomorphism $\phi : G \to K$. Then there exists a conditional expectation*

$$\Psi_K : A \otimes_{\max} C^*(G, P, P^{\mathrm{op}}) \to \overline{\mathrm{span}}\{a \otimes v_p v_q^* v_r : a \in A, \phi(q) = \phi(rp)\}$$

such that

$$\Psi_K(a \otimes v_p v_q^* v_r) = \begin{cases} a \otimes v_p v_q^* v_r & \text{if } \phi(q) = \phi(rp) \\ 0 & \text{otherwise.} \end{cases}$$

If K is amenable then Ψ_K is faithful.

PROOF. By Lemma 7.1, there exists a faithful coaction

$$\delta_\phi : A \otimes_{\max} C^*(G, P, P^{\mathrm{op}}) \to A \otimes_{\max} (C^*(G, P, P^{\mathrm{op}}) \otimes C^*(K)).$$

Let τ be the trace on $C^*(K)$ of Lemma 5.14. By Lemma 5.18, the map $\Psi_K := (\mathrm{id}_{A \otimes C^*(G,P,P^{\mathrm{op}})}) \otimes \tau \circ (\delta_\phi)$ is a conditional expectation. In particular,

$$
\begin{aligned}
\Psi_K(a \otimes v_p v_q^* v_r) &= (\mathrm{id}_{A \otimes C^*(G,P,P^{\mathrm{op}})}) \otimes \tau \circ (\delta_\phi)(a \otimes v_p v_q^* v_r) \\
&= (\mathrm{id}_{A \otimes C^*(G,P,P^{\mathrm{op}})}) \otimes \tau(a \otimes v_p v_q^* v_r \otimes u_{\phi(pq^{-1}r)}) \\
&= \begin{cases} a \otimes v_p v_q^* v_r & \text{if } \phi(q) = \phi(rp) \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

We claim that

$$\text{range } \Psi_K = (A \otimes_{\max} C^*(G, P, P^{\mathrm{op}}))^{\delta_\phi} = \overline{\text{span}}\{a \otimes v_p v_q^* v_r : \phi(q) = \phi(rp)\}.$$

We see that $\overline{\text{span}}\{a \otimes v_p v_q^* v_r : \phi(q) = \phi(rp)\} \subseteq \text{range } \Psi_K$. To show the reverse inclusion, fix $b \in \text{range } \Psi_K$ and $\epsilon > 0$. We can approximate b as a linear combination of $a \otimes v_p v_q^* v_r$, i.e. there exists $\sum_{i=1}^n a_i \otimes v_{p_i} v_{q_i}^* v_{r_i}$ such that $\|b - \sum_{i=1}^n a_i \otimes v_{p_i} v_{q_i}^* v_{r_i}\| < \epsilon$. Since Ψ_K is linear and norm-decreasing, we have

$$
\begin{aligned}
\epsilon &> \left\| \Psi_K \left(b - \sum_{i=1}^n a_i \otimes v_{p_i} v_{q_i}^* v_{r_i} \right) \right\| \\
&\geq \left\| \Psi_K(b) - \Psi_K \left(\sum_{i=1}^n a_i \otimes v_{p_i} v_{q_i}^* v_{r_i} \right) \right\| \\
&\geq \left\| b - \sum_{\{i : \mu(r_i p_i) = \mu(q_i)\}} a_i \otimes v_{p_i} v_{q_i}^* v_{r_i} \right\|
\end{aligned}
$$

Thus $b \in \overline{\text{span}}\{a \otimes v_p v_q^* v_r : \phi(q) = \phi(rp)\}$. So range $\Psi_K = \overline{\text{span}}\{a \otimes v_p v_q^* v_r : \phi(q) = \phi(rp)\}$ as claimed. Thus Ψ_K has the desired form. Further, if K is an amenable group then Lemma 5.18 states that Ψ_K is faithful. $\qquad\square$

We construct a faithful conditional expectation on $A \otimes_{\min} C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$ using Lemma 5.15.

LEMMA 7.3. *Let A be a unital C^*-algebra and let (G, P) be a doubly quasi-lattice ordered group. There is a faithful conditional expectation Δ on $A \otimes_{\min} C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$ such that*

$$\Delta(a \otimes J_p J_q^* J_r) = \begin{cases} a \otimes J_p J_q^* J_r & \text{if } q = rp \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Let η be a faithful representation of A on a Hilbert space H. We can embed $A \otimes_{\min} C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$ faithfully as bounded operators on $H \otimes (\oplus_{b \in P} \ell^2(I_b))$ by Proposition 5.2. Let $\{\epsilon_{a,b} : a, b \in P, a \leq_r b\}$ be the usual orthonormal basis for $\oplus_{b \in P} \ell^2(I_b)$. We apply Lemma 5.15 to get a faithful conditional expectation Δ such that

$$(\Delta(T)(h \otimes \epsilon_{a,b}) | h' \otimes \epsilon_{a,b}) = (T(h \otimes \epsilon_{a,b}) | h' \otimes \epsilon_{a,b})$$

for all $T \in B(H \otimes (\oplus_{b \in P} \ell^2(I_b)))$, all $h, h' \in H$ and $a, b \in P$ with $a \leq_r b$.

It remains to show that Δ has the property

$$\Delta(a \otimes J_p J_q^* J_r) = \begin{cases} a \otimes J_p J_q^* J_r & \text{if } q = rp \\ 0 & \text{otherwise.} \end{cases}$$

We compute:

$$
\begin{aligned}
(\Delta(a \otimes J_p J_q^* J_r)(h \otimes \epsilon_{a,b}) | h' \otimes \epsilon_{a,b}) &= (a \otimes J_p J_q^* J_r(h \otimes \epsilon_{a,b}) | h' \otimes \epsilon_{a,b}) \\
&= (ah | h')(J_p J_q^* J_r \epsilon_{a,b} | \epsilon_{a,b}) \\
&= \begin{cases} (ah | h')(\epsilon_{pq^{-1}ra,b} | \epsilon_{a,b}) & \text{if } ra \leq_r b \text{ and } q \leq_l ra \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} (ah | h') & \text{if } rp = q,\ ra \leq_r b \text{ and } q \leq_l ra \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} (ah | h')(J_p J_q^* J_r \epsilon_{a,b} | \epsilon_{a,b}) & \text{if } rp = q \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} (a \otimes J_p J_q^* J_r(h \otimes \epsilon_{a,b}) | h' \otimes \epsilon_{a,b}) & \text{if } rp = q \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Thus $\Delta(a \otimes J_p J_q^* J_r) = \begin{cases} a \otimes J_p J_q^* J_r & \text{if } q = rp \\ 0 & \text{otherwise.} \end{cases}$ The conditional expectation on $A \otimes_{\min} C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$ is the restriction of Δ to $A \otimes_{\min} C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$. $\qquad \square$

7.2. Controlled maps of doubly quasi-lattice ordered groups

We have adapted our definition of a "controlled map" in Definition 7.4 from [**11**, Proposition 6.6].

DEFINITION 7.4. Suppose that (G, P) and (K, Q) are doubly quasi-lattice ordered groups. A *controlled map* $\phi : (G, P) \to (K, Q)$ is a group homomorphism $\phi : G \to K$ such that:

(1) $\phi(P) \subseteq Q$
(2) for all $x, y \in P$ satisfying $x \vee_l y < \infty$ we have $\phi(x) \vee_l \phi(y) = \phi(x \vee_l y)$, and
   $\phi(x) = \phi(y) \Rightarrow x = y$
(3) for all $x, y \in P$ satisfying $x \vee_r y < \infty$ we have $\phi(x) \vee_r \phi(y) = \phi(x \vee_r y)$, and
   $\phi(x) = \phi(y) \Rightarrow x = y.$

Condition (1) implies that a controlled map is order preserving for both left and right orders: if $x \leq_l y$ then $x^{-1}y \in P$ so $\phi(x)^{-1}\phi(y) = \phi(x^{-1}y) \in Q$ and hence $\phi(x) \leq_l \phi(y)$. Similarly $x \leq_r y$ implies $\phi(x) \leq_r \phi(y)$.

Conditions (2) and (3) are stronger than order preserving. Consider $(\mathbb{Z}^2, \mathbb{N}^2)$ and the homomorphism $\phi : (\mathbb{Z}^2, \mathbb{N}^2) \to (\mathbb{Z}, \mathbb{N})$ defined by $\phi((m, n)) = m + n$ for all $(m, n) \in \mathbb{Z}^2$. This homomorphism satisfies $\phi(\mathbb{N}^2) \subseteq \mathbb{N}$ and is order preserving, however it does not satisfy $\phi(x) \vee_l \phi(y) = \phi(x \vee_l y)$. Consider $(1, 0)$ and $(0, 1)$. We have $(1, 0) \vee_l (0, 1) = (1, 1)$ but $\phi(1, 0) \vee_l \phi(0, 1) = 1 \vee_l 1 = 1$ and $\phi((1, 0) \vee_l (0, 1)) = \phi(1, 1) = 2$.

The notion of a controlled map for quasi-lattice ordered groups was introduced by Laca and Raeburn [**11**, Proposition 6.6][1] (see also Crisp and Laca in [**5**, Proposition 16]). Li adapted Laca-Raeburn's proof in [**14**, Corollary 8.2] to show that if (G, P) has a controlled map into an amenable group, then the universal algebra generated by covariant isometric representations of (G, P) is nuclear.

In this section we will prove that if there exists a doubly quasi-lattice ordered group (k, Q) with K an amenable group and a controlled map $\phi : (G, P) \to (K, Q)$, then (G, P) is amenable and $C^*(G, P, P^{\mathrm{op}})$ is nuclear. The proofs that (G, P) is amenable and that $C^*(G, P, P^{\mathrm{op}})$ is nuclear are very similar in structure so we shall prove them both via Proposition 7.5. This approach follows the method of Li in [**14**, Corollary 8.2].

Let A be a unital C^*-algebra. There exist homomorphisms $\mathrm{id} : A \to A \otimes_{\min} C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$ such that $\mathrm{id}(a) = a \otimes 1$ for all $a \in A$ and $\pi_J^{\otimes} : C^*(G, P, P^{\mathrm{op}}) \to A \otimes_{\min}$

---

[1]Note that there is a mistake in the statement of [**11**, Proposition 6.6]: the final line should read "If $\mathcal{G}$ is amenable then (G, P) is amenable".

$C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$ such that $\pi_J^{\otimes}(v_p) = 1 \otimes J_p$. By Theorem 5.4 there exists a homomorphism $\mathrm{id} \otimes \pi_J : A \otimes_{\max} C^*(G, P, P^{\mathrm{op}}) \to A \otimes_{\min} C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$ such that $\mathrm{id} \otimes \pi_J(a \otimes v_p) = a \otimes J_p$.

PROPOSITION 7.5. *Let (G, P) be a doubly quasi-lattice ordered group and let A be a unital C^*-algebra.*

(1) *There exists a conditional expectation Ψ on $A \otimes_{\max} C^*(G, P, P^{\mathrm{op}})$ such that*

$$\Psi(a \otimes v_p v_q^* v_r) = \begin{cases} a \otimes v_p v_q^* v_r & \textit{if } q = rp \\ 0 & \textit{otherwise.} \end{cases}$$

(2) *Suppose there exists a doubly quasi-lattice ordered group (K, Q) with K an amenable group and a controlled map $\phi : (G, P) \to (K, Q)$. Then:*
   (a) *The conditional expectation Ψ is faithful for positive elements.*
   (b) *The homomorphism $\mathrm{id} \otimes \pi_J : A \otimes_{\max} C^*(G, P, P^{\mathrm{op}}) \to A \otimes_{\min} C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$ is an isomorphism.*

REMARK. Note that we have defined $\mathrm{id} \otimes \pi_J$ from the maximal tensor product to the minimal tensor product. This will become important when we want to prove that the canonical homomorphism $A \otimes_{\max} C^*(G, P, P^{\mathrm{op}}) \to A \otimes_{\min} C^*(G, P, P^{\mathrm{op}})$ is faithful and hence that $C^*(G, P, P^{\mathrm{op}})$ is nuclear.

The proof of Proposition 7.5 requires Lemma 7.6. The proof of Lemma 7.6 is quite involved so we will state the result here and defer the proof to Section 7.3.

LEMMA 7.6. *Let (G, P) be a doubly quasi-lattice ordered group. Suppose that (K, Q) is a doubly quasi-lattice ordered group with a controlled map $\phi : (G, P) \to (K, Q)$. Then $\mathrm{id} \otimes \pi_J : A \otimes_{\max} C^*(G, P, P^{\mathrm{op}}) \to A \otimes_{\min} C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$ is faithful when restricted to $\overline{\mathrm{span}}\{a \otimes v_p v_q^* v_r : a \in A, \phi(q) = \phi(rp)\}$.*

We now apply Lemma 7.6 to the proof of Proposition 7.5.

PROOF OF PROPOSITION 7.5. (1). Consider the identity homomorphism, $\mathrm{id} : G \to G$. By Lemma 7.2, there exists a conditional expectation

$$\Psi = \Psi_G : A \otimes_{\max} C^*(G, P, P^{\mathrm{op}}) \to \overline{\mathrm{span}}\{a \otimes v_p v_q^* v_r : a \in A, q = rp\}$$

such that

$$\Psi(a \otimes v_p v_q^* v_r) = \begin{cases} a \otimes v_p v_q^* v_r & \textit{if } q = rp \\ 0 & \textit{otherwise.} \end{cases}$$

(2a). Suppose that $\phi : (G, P) \to (K, Q)$ is a controlled map and that K is an amenable group. We must show that the conditional expectation Ψ is faithful for

positive elements. By Lemma 7.2, there is a faithful conditional expectation Ψ_K such that

$$\Psi_K(a \otimes v_p v_q^* v_r) = \begin{cases} a \otimes v_p v_q^* v_r & \text{if } \phi(q) = \phi(rp) \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $\Psi = \Psi \circ \Psi_K$. To prove our claim we calculate on the spanning elements to see:

$$\Psi \circ \Psi_K(a \otimes v_p v_q^* v_r) = \begin{cases} \Psi(a \otimes v_p v_q^* v_r) & \text{if } \phi(q) = \phi(rp) \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} a \otimes v_p v_q^* v_r & \text{if } q = rp \text{ and } \phi(q) = \phi(rp) \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} a \otimes v_p v_q^* v_r & \text{if } q = rp \\ 0 & \text{otherwise.} \end{cases}$$

$$= \Psi(a \otimes v_p v_q^* v_r).$$

So we have proved our claim. Since Ψ_K is faithful for positive elements, and conditional expectations are positive maps, and $\Psi = \Psi \circ \Psi_K$ it suffices to show that Ψ is faithful when restricted to

$$\text{range } \Psi_K = \overline{\text{span}}\{a \otimes v_p v_q^* v_r : a \in A, \phi(q) = \phi(rp)\}.$$

By Lemma 7.3 there is a faithful conditional expectation Δ on $A \otimes_{\min} C^*_{\text{ts}}(G, P, P^{\text{op}})$ such that

$$\Delta(a \otimes J_p J_q^* J_r) = \begin{cases} a \otimes J_p J_q^* J_r & \text{if } q = rp \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $\Delta \circ \text{id} \otimes \pi_J = \text{id} \otimes \pi_J \circ \Psi$. Fix $a \in A$ and $p, q, r \in P$ such that $p \leq_r q$ and $r \leq_l q$ and compute:

$$\text{id} \otimes \pi_J \circ \Psi(a \otimes v_p v_q^* v_r) = \begin{cases} \text{id} \otimes \pi_J(a \otimes v_p v_q^* v_r) & \text{if } rp = q \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} a \otimes J_p J_q^* J_r & \text{if } rp = q \\ 0 & \text{otherwise} \end{cases}$$

$$= \Delta(a \otimes J_p J_q^* J_r)$$

$$= \Delta \circ \text{id} \otimes \pi_J(a \otimes v_p v_q^* v_r).$$

86

Thus $\Delta \circ \mathrm{id} \otimes \pi_J = \mathrm{id} \otimes \pi_J \circ \Psi$. By Lemma 7.6 $\mathrm{id} \otimes \pi_J$ is faithful when restricted to $\overline{\mathrm{span}}\{a \otimes v_p v_q^* v_r : \phi(rp) = \phi(q)\}$. It follows that $\Delta \circ \mathrm{id} \otimes \pi_J$ is faithful for positive elements in $\overline{\mathrm{span}}\{a \otimes v_p v_q^* v_r : \phi(rp) = \phi(q)\}$. Thus Ψ must also be faithful for positive elements when restricted to $\overline{\mathrm{span}}\{a \otimes v_p v_q^* v_r : a \in A, \phi(q) = \phi(rp)\}$. Hence Ψ is a faithful conditional expectation of $A \otimes_{\max} C^*(G, P, P^{\mathrm{op}})$ onto $\overline{\mathrm{span}}\{a \otimes v_p v_q^* v_r : q = rp\}$ and our proof of (2a) is complete.

(2b). Now we can prove that $\mathrm{id} \otimes \pi_J : A \otimes_{\max} C^*(G, P, P^{\mathrm{op}}) \to A \otimes_{\min} C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$ is faithful. Suppose that $b \in A \otimes_{\max} C^*(G, P, P^{\mathrm{op}})$ and $\mathrm{id} \otimes \pi_J(b) = 0$. Then $\mathrm{id} \otimes \pi_J(b^*b) = 0$ and so $\Delta \circ \mathrm{id} \otimes \pi_J(b^*b) = 0$. Now we see that $\mathrm{id} \otimes \pi_J \circ \Psi(b^*b) = 0$. Since $\mathrm{id} \otimes \pi_J$ is faithful on $\overline{\mathrm{span}}\{a \otimes v_p v_q^* v_r : \phi(rp) = \phi(q)\}$ which contains the range of Ψ, it follows that $\Psi(b^*b) = 0$. But Ψ is faithful for positive elements, so $b = 0$. Thus $\mathrm{id} \otimes \pi_J$ is faithful. $\qquad\square$

REMARK. It is interesting that the properties of the controlled map $\phi : (G, P) \to (K, Q)$ have two independent roles in the proof of Proposition 7.5(2). First, in the construction of Ψ_K in Lemma 7.2 we use that ϕ is a homomorphism and that Ψ_K is faithful if K is amenable. The construction still works if K is not associated to a doubly quasi-lattice ordered group. Second, Lemma 7.6 states that $\mathrm{id} \otimes \pi_J$ is faithful when restricted to $\overline{\mathrm{span}}\{a \otimes v_p v_q^* v_r : \phi(q) = \phi(rp), a \in A\}$ and relies only on the properties of ϕ as a controlled map. Thus the restriction of $\mathrm{id} \otimes \pi_J$ is faithful even if K is not amenable. This split is analogous to the roles that "W sees all projections" and "(G, P) is amenable" play in the proofs in Chapter 4.

Now that we have completed our proof of Proposition 7.5 it is easy to state and prove the theorem that motivates this chapter:

THEOREM 7.7. *Let (G, P) be a doubly quasi-lattice ordered group. Suppose that (K, Q) is a doubly quasi-lattice ordered group with a controlled map $\phi : (G, P) \to (K, Q)$. If K is an amenable group then (G, P) is amenable and $C^*(G, P, P^{\mathrm{op}})$ is nuclear.*

PROOF. To prove that (G, P) is amenable we apply Proposition 7.5 with $A = \mathbb{C}$. Then Ψ as described in Proposition 7.5(1) is the conditional expectation E of Proposition 4.6 and, by Proposition 7.5(2a), Ψ is faithful for positive elements. Thus (G, P) is amenable.

To show $C^*(G, P, P^{\mathrm{op}})$ is nuclear, we fix a unital C^*-algebra A, and show that the canonical homomorphism from $A \otimes_{\max} C^*(G, P, P^{\mathrm{op}})$ to $A \otimes_{\min} C^*(G, P, P^{\mathrm{op}})$ is an isomorphism. Since (G, P) is amenable, Theorem 6.2 states that $\pi_J : C^*(G, P, P^{\mathrm{op}}) \to$

$C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$ is an isomorphism. It follows that π_J^{-1} exists and is also an isomorphism. Hence, by Proposition 5.2,

$$\mathrm{id} \otimes \pi_J^{-1} : A \otimes_{\min} C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}}) \to A \otimes_{\min} C^*(G, P, P^{\mathrm{op}})$$

exists and is faithful. Since π_J^{-1} is surjective, $\mathrm{id} \otimes \pi_J^{-1}$ is surjective and hence is an isomorphism. In addition, by Proposition 7.5(2b), $\mathrm{id} \otimes \pi_J$ from $A \otimes_{\max} C^*(G, P, P^{\mathrm{op}})$ to $A \otimes_{\min} C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$ is an isomorphism. Hence

$$\mathrm{id} \otimes \pi_J^{-1} \circ \mathrm{id} \otimes \pi_J : A \otimes_{\max} C^*(G, P, P^{\mathrm{op}}) \to A \otimes_{\min} C^*(G, P, P^{\mathrm{op}})$$

is the composition of two isomorphisms. Computing on generators we see that

$$\mathrm{id} \otimes \pi_J^{-1} \circ \mathrm{id} \otimes \pi_J (a \otimes v_p v_q^* v_r) = \mathrm{id} \otimes \pi_J^{-1}(a \otimes J_p J_q^* J_r) = a \otimes v_p v_q^* v_r.$$

Thus $\mathrm{id} \otimes \pi_J^{-1} \circ \mathrm{id} \otimes \pi_J$ is the canonical homomorphism from $A \otimes_{\max} C^*(G, P, P^{\mathrm{op}})$ to $A \otimes_{\min} C^*(G, P, P^{\mathrm{op}})$. Hence $C^*(G, P, P^{\mathrm{op}})$ is nuclear. □

We devote the next section to the proof of Lemma 7.6. In the final section of this chapter we give examples of amenable doubly quasi-lattice ordered groups with nuclear C^*-algebras.

7.3. Proof of Lemma 7.6

In this section we prove Lemma 7.6, which states that $\mathrm{id} \otimes \pi_J$ is faithful when restricted to $\overline{\mathrm{span}}\{a \otimes v_p v_q^* v_r : \phi(q) = \phi(rp), a \in A\}$. The proofs in this section rely on the order preserving properties of the controlled map and not on the amenability of K. We first prove some basic properties of the controlled map.

LEMMA 7.8. *Let (G, P) be a doubly quasi-lattice ordered group. Suppose that (K, Q) is a doubly quasi-lattice ordered group with a controlled map $\phi : (G, P) \to (K, Q)$. For all $x, y \in G$:*

    (1) If $x \vee_l y < \infty$ then $\phi(x) \leq_l \phi(y) \Rightarrow x \leq_l y$.
    (2) If $x \vee_r y < \infty$ then $\phi(x) \leq_r \phi(y) \Rightarrow x \leq_r y$.

PROOF. (1). Let $x, y \in G$ such that $x \vee_l y < \infty$. Suppose that $\phi(x) \leq_l \phi(y)$. Then $\phi(x) \vee_l \phi(y) = \phi(y)$. By the definition of a controlled map $\phi(x \vee_l y) = \phi(x) \vee_l \phi(y)$. Therefore $\phi(y^{-1}(x \vee_l y)) = \phi(y)^{-1}\phi(x \vee_l y) = e_K$. Since $y^{-1}(x \vee_l y) \in P$ we have $e_G \vee_l y^{-1}(x \vee_l y) = y^{-1}(x \vee_l y)$. Thus, since $\phi(e_G) = e_K = \phi(y^{-1}(x \vee_l y))$, we have $e_G = y^{-1}(x \vee_l y)$ by the properties of the controlled map. Hence $y = x \vee_l y$ and $x \leq_l y$.

The proof of (2) follows from (1) by symmetry. □

We need to break $\overline{\text{span}}\{a \otimes v_p v_q^* v_r : \phi(q) = \phi(rp), a \in A\}$ down into more manageable chunks. We will partition $\overline{\text{span}}\{a \otimes v_p v_q^* v_r : \phi(q) = \phi(rp), a \in A\}$ into subalgebras, and prove that $\text{id} \otimes \pi_J$ is isometric when restricted to these subalgebras. We can then show that $\text{id} \otimes \pi_J$ is isometric on the direct limit of these smaller subalgebras. Since we have many variables we will build our subalgebras slowly. We use this direct limit procedure twice: in Lemma 7.10 and again in Lemma 7.12. We then finally prove Lemma 7.6.

We start with our smallest subalgebra: let $q \in P$ and $(s, t) \in Q \times Q^{\text{op}}$ such that $ts = \phi(q)$. Define

$$\mathcal{D}_{(s,t)}^q := \overline{\text{span}}\{a \otimes v_p v_q^* v_r : a \in A, (\phi(p), \phi(r)) = (s, t)\}.$$

Let $\{\epsilon_{a,b} : a, b \in P, a \leq_r b\}$ be the usual orthonormal basis for $\oplus_{b \in P} \ell^2(I_b)$. Let $P_{(s,t)}^q$ denote the projection onto

$$H_{(s,t)}^q = \overline{\text{span}}\{\epsilon_{a,q} : a \leq_r q, \phi(a) = s\}.$$

LEMMA 7.9. *Let (G, P) be a doubly quasi-lattice ordered group and let A be a unital C^*-algebra. Suppose that (K, Q) is a doubly quasi-lattice ordered group with a controlled map $\phi : (G, P) \to (K, Q)$. Let $q \in P$ and $(s, t) \in Q \times Q^{\text{op}}$ such that $ts = \phi(q)$. Then $\mathcal{D}_{(s,t)}^q$ is a subalgebra and $\text{id} \otimes \pi_J$ is isometric on $\mathcal{D}_{(s,t)}^q$. In particular, for $x \in \mathcal{D}_{(s,t)}^q$, the map*

$$x \mapsto 1 \otimes P_{(s,t)}^q (\text{id} \otimes \pi_J)(x) 1 \otimes P_{(s,t)}^q$$

is an isomorphism of $\mathcal{D}_{(s,t)}^q$ onto $A \otimes \mathcal{K}(H_{(s,t)}^q)$.

PROOF. This result relies on the properties of the truncated shift J so we will prove it as a result on $C^*(G, P, P^{\text{op}})$ and then extend it using the properties of the tensor product to a result about $A \otimes_{\max} C^*(G, P, P^{\text{op}})$. Let $q \in P$ and $(s, t) \in Q \times Q^{\text{op}}$. Let

$$\mathcal{J}_{(s,t)}^q := \overline{\text{span}}\{v_p v_q^* v_r : (\phi(p), \phi(r)) = (s, t)\}.$$

We first prove that $\mathcal{J}_{(s,t)}^q$ is a subalgebra, and then show that $A \otimes \mathcal{J}_{(s,t)}^q = \mathcal{D}_{(s,t)}^q$. Since A is a C^*-algebra it then follows immediately that $\mathcal{D}_{(s,t)}^q$ is a subalgebra. So we will show that $\mathcal{J}_{(s,t)}^q$ is closed under multiplication and adjoints. Let $p, , r, x, , z \in P$ such that $(\phi(p), \phi(r)) = (\phi(x), \phi(z)) = (s, t)$. Now consider

$$(7.1) \qquad (v_p v_q^* v_r)(v_x v_q^* v_z) = v_{pq^{-1}(q \vee_l rx)} v_{(q \vee_r rx)(rx)^{-1}(q \vee_l rx)}^* v_{(q \vee_r rx)q^{-1}z}.$$

If either of the upper bounds, $q \vee_l rx$ or $q \vee_r rx$ don't exist then (7.1) is zero. So we suppose $q \vee_l rx$ and $q \vee_r rx$ both exist. Since $\phi(q) = ts = \phi(rx)$ the properties of the controlled map imply that $q = rx$. Thus we can rewrite (7.1) as

$$(v_p v_q^* v_r)(v_x v_q^* v_z) = \begin{cases} v_p v_q^* v_z & \text{if } r^{-1}q = x \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\mathcal{J}_{(s,t)}^q$ is closed under multiplication.

To show that $\mathcal{J}_{(s,t)}^q$ is closed under adjoints consider $(v_p v_q^* v_r)^* = v_{r^{-1}q} v_q^* v_{qp^{-1}}$. Then $\phi(r^{-1}q) = \phi(r)^{-1}\phi(q) = t^{-1}ts = s$. Similarly $\phi(qp^{-1}) = t$. Thus $\mathcal{J}_{(s,t)}^q$ is closed under adjoints and is a subalgebra.

To see that $A \otimes \mathcal{J}_{(s,t)}^q = \mathcal{D}_{(s,t)}^q$, observe that for every generating element $a \otimes v_p v_q^* v_r$ with $(\phi(p), \phi(r)) = (s,t)$ we have $a \otimes v_p v_q^* v_r \in A \otimes \mathcal{J}_{(s,t)}^q$. Thus $\mathcal{D}_{(s,t)}^q \subseteq A \otimes \mathcal{J}_{(s,t)}^q$. Similarly, $A \otimes \mathcal{J}_{(s,t)}^q \subseteq \mathcal{D}_{(s,t)}^q$ and so $A \otimes \mathcal{J}_{(s,t)}^q = \mathcal{D}_{(s,t)}^q$. It then follows that $\mathcal{D}_{(s,t)}^q$ is a subalgebra.

We claim that for $j \in \mathcal{J}_{(s,t)}^q$ the map $\chi : j \mapsto P_{(s,t)}^q \pi_J(j) P_{(s,t)}^q$ is an isomorphism between $\mathcal{J}_{(s,t)}^q$ and the compact operators $\mathcal{K}(H_{(s,t)}^q)$ on $H_{(s,t)}^q$. First we will show that the spanning elements of $\pi_J(\mathcal{J}_{(s,t)}^q)$ are rank-one operators on $H_{(s,t)}^q$. Let $p, r, a, b \in P$ such that, $a \leq_r b$ $\phi(p) = s$ and $\phi(r) = t$. Compute:

$$(7.2) \quad P_{(s,t)}^q \pi_J(v_p v_q^* v_r) P_{(s,t)}^q \epsilon_{a,b} = \begin{cases} P_{(s,t)}^q J_p J_q^* J_r \epsilon_{a,q} & \text{if b=q} \\ 0 & \text{otherwise} \end{cases}$$

$$(7.3) \qquad\qquad = \begin{cases} P_{(s,t)}^q \epsilon_{pq^{-1}ra,q} & \text{if } ra \leq_r q \text{ and } q \leq_l ra \text{ and } b = q. \\ 0 & \text{otherwise.} \end{cases}$$

By assumption $\phi(q) = ts = \phi(ra)$. If $ra \leq_r q$, then $ra \vee_r q = q$. So $\phi(q) = \phi(ra)$ implies that $q = ra$. It then follows that $\epsilon_{pq^{-1}ra,q} = \epsilon_{p,q}$. Further $P_{(s,t)}^q \epsilon_{p,q} = \epsilon_{(p,q)}$. Thus we may rewrite (7.2) as

$$P_{(s,t)}^q \pi_J(v_p v_q^* v_r) P_{(s,t)}^q \epsilon_{a,b} = \begin{cases} \epsilon_{p,q} & \text{if } b = q \text{ and } a = r^{-1}q. \\ 0 & \text{otherwise.} \end{cases}$$

So $P_{(s,t)}^q \pi_J(v_p v_q^* v_r) P_{(s,t)}^q$ is the rank one operator $(\cdot \mid \epsilon_{r^{-1}q,q})\epsilon_{p,q}$.

Second we show that the spanning elements of $\mathcal{J}_{(s,t)}^q$ behave as matrix units on $H_{(s,t)}^q$. Let $v_p v_q^* v_r, v_x v_q^* v_z \in \mathcal{J}_{(s,t)}^q$. From above we have:

$$(v_p v_q^* v_r)(v_x v_q^* v_z) = \begin{cases} v_p v_q^* v_z & \text{if } x = r^{-1}q \\ 0 & \text{otherwise.} \end{cases}$$

Thus $v_p v_q^* v_r$ and $v_x v_q^* v_z$ behave as the matrix units $E_{p,r^{-1}q}$ and $E_{x,z^{-1}q}$ respectively in the sense that

$$(v_p v_q^* v_r)(v_x v_q^* v_z) = E_{p,r^{-1}q} E_{x,z^{-1}q} = \delta_{r^{-1}q,x} E_{p,z^{-1}q} = \delta_{r^{-1}q,x} v_p v_q^* v_z.$$

Hence χ gives a correspondence between the system of matrix units in $\mathcal{J}_{(s,t)}^q$ and the system of matrix units associated with the canonical orthonormal basis in $H_{(s,t)}^q$. To see that χ preserves multiplication observe that

$$P_{(s,t)}^q \pi_J(v_x v_q^* v_z) P_{(s,t)}^q \pi_J(v_p v_q^* v_r) P_{(s,t)}^q \epsilon_{a,b} =$$

$$= \begin{cases} P_{(s,t)}^q \pi_J(v_x v_q^* v_z) P_{(s,t)}^q \epsilon_{p,q} & \text{if } b = q \text{ and } a = r^{-1}q. \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \epsilon_{x,q} & \text{if } b = q, \ a = r^{-1}q \text{ and } p = z^{-1}q. \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} P_{(s,t)}^q \pi_J(v_x v_q^* v_r) P_{(s,t)}^q \epsilon_{a,b} & \text{if } b = q, \ a = r^{-1}q \text{ and } p = z^{-1}q. \\ 0 & \text{otherwise.} \end{cases}$$

$$= P_{(s,t)}^q \pi_J(v_x v_q^* v_z v_p v_q^* v_r) P_{(s,t)}^q \epsilon_{a,b}$$

As a consequence the map $j \mapsto P_{(s,t)}^q \pi_J(v_p v_q^* v_r) P_{(s,t)}^q$ is an isomorphism between $\mathcal{J}_{(s,t)}^q$ and $\mathcal{K}(H_{(s,t)}^q)$ the compact operators on $H_{(s,t)}^q$.

Since $\mathcal{J}_{(s,t)}^q$ is isomorphic to the compact operators on a separable Hilbert space it is nuclear. Thus $A \otimes_{\max} \mathcal{J}_{(s,t)}^q = A \otimes_{\min} \mathcal{J}_{(s,t)}^q$. Define

Both $\mathrm{id} : A \to A$ and $\chi : \mathcal{J}_{(s,t)}^q \to C_{ts}^*(G, P, P^{\mathrm{op}})$ are injective, and hence Proposition 5.2 states that there exists a unique injective homomorphism $\mathrm{id} \otimes \chi : A \otimes \mathcal{J}_{(s,t)}^q \to A \otimes_{\min} C_{ts}^*(G, P, P^{\mathrm{op}})$. Further, we have $A \otimes \mathcal{J}_{(s,t)}^q = \mathcal{D}_{(s,t)}^q$. In particular, $\mathrm{id} \otimes \chi(x) = 1 \otimes P_{(s,t)}^q(\mathrm{id} \otimes \pi_J)(x) 1 \otimes P_{(s,t)}^q$. Thus the map $x \mapsto 1 \otimes P_{(s,t)}^q(\mathrm{id} \otimes \pi_J)(x) 1 \otimes P_{(s,t)}^q$ is an isomorphism of $\mathcal{D}_{(s,t)}^q$ onto $A \otimes \mathcal{K}(H_{(s,t)}^q)$. $\qquad\square$

Now we work up to a larger subalgebra: let $(s,t) \in Q \times Q^{\mathrm{op}}$. Define

$$\mathcal{D}_{(s,t)} := \overline{\mathrm{span}}\{a \otimes v_p v_q^* v_r : a \in A, (\phi(p), \phi(r)) = (s,t), \phi(q) = \phi(rp)\}.$$

Let $\{\epsilon_{a,b} : a, b \in P, a \leq_r b\}$ be the usual orthonormal basis for $\oplus_{b \in P} \ell^2(I_b)$. For each $(s,t) \in Q \times Q^{\mathrm{op}}$, let $P_{(s,t)}$ denote the projection onto

$$H_{(s,t)} = \overline{\mathrm{span}}\{\epsilon_{a,b} : a, b \in P, a \leq_r b, \phi(a) = s, \phi(b) = ts\}.$$

Let $\mathcal{I}_{(s,t)}$ denote the collection of all finite subsets $I \subset \phi^{-1}(\{ts\}) \cap P$. For each $I \in \mathcal{I}_{(s,t)}$ let $\mathcal{D}^I_{(s,t)}$ be

(7.4) $$\mathcal{D}^I_{(s,t)} = \overline{\mathrm{span}}\{a \otimes v_p v_q^* v_r : a \in A, (\phi(p), \phi(r)) = (s,t), q \in I\}.$$

In particular, each $\mathcal{D}^I_{(s,t)}$ is the finite span of $\mathcal{D}^q_{(s,t)}$ such that $q \in I$.

LEMMA 7.10. *Let (G, P) be a doubly quasi-lattice ordered group and let A be a C^*-algebra. Suppose that (K, Q) is a doubly quasi-lattice ordered group with a controlled map $\phi : (G, P) \to (K, Q)$. For each $(s,t) \in Q \times Q^{\mathrm{op}}$ the set $\mathcal{I}_{(s,t)}$ is directed, ordered by inclusion. Further, for each $I \in \mathcal{I}$, the set $\mathcal{D}^I_{(s,t)}$ is a C^*-subalgebra of $A \otimes_{\max} C^*(G, P, P^{\mathrm{op}})$ and $\{\mathcal{D}^I_{(s,t)} : I \in \mathcal{I}_{(s,t)}\}$ is an inductive system with limit*

$$\overline{\cup_{I \in \mathcal{I}_{(s,t)}} \mathcal{D}^I_{(s,t)}} = \mathcal{D}_{(s,t)}.$$

PROOF. Fix $(s,t) \in Q \times Q^{\mathrm{op}}$. For any $I_1, I_2 \in \mathcal{I}_{(s,t)}$ we have $I_1 \cup I_2 \in \mathcal{I}_{(s,t)}$. Hence $\mathcal{I}_{(s,t)}$ is directed, ordered by inclusion. Now we fix $I \in \mathcal{I}_{(s,t)}$. We first prove that $\mathcal{D}^I_{(s,t)}$ is a subalgebra. Since A is a C^*-algebra it will suffice to show that $\mathrm{span}\{v_p v_q^* v_r : (\phi(p), \phi(r)) = (s,t), q \in I\}$ is closed under multiplication and adjoints. Let $p, q, r, x, y, z \in P$ such that $q, y \in I$ and $(\phi(p), \phi(r)) = (\phi(x), \phi(z)) = (s,t)$. Now consider

(7.5) $$(v_p v_q^* v_r)(v_x v_y^* v_z) = v_{pq^{-1}(q \vee_l rx)} v^*_{(y \vee_r rx)(rx)^{-1}(q \vee_l rx)} v_{(y \vee_r rx)y^{-1}z}.$$

If either of the upper bounds, $q \vee_l rx$ or $y \vee_r rx$ don't exist then (7.5) is zero. So we suppose $q \vee_l rx$ and $y \vee_r rx$ both exist. Since $\phi(q) = \phi(y) = ts = \phi(rx)$ the properties of the controlled map imply that $q = rx$ and $y = rx$. Thus we can rewrite (7.5) as

$$(v_p v_q^* v_r)(v_x v_y^* v_z) = \begin{cases} v_p v_q^* v_z & \text{if } q = y \text{ and } r^{-1}q = x \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\mathcal{D}^I_{(s,t)}$ is closed under multiplication.

To show that $\mathcal{D}^I_{(s,t)}$ is closed under adjoints consider $(v_p v_q^* v_r)^* = v_{r^{-1}q} v_q^* v_{qp^{-1}}$. Then $\phi(r^{-1}q) = \phi(r)^{-1}\phi(q) = t^{-1}ts = s$. Similarly $\phi(qp^{-1}) = t$. Thus $\mathcal{D}^I_{(s,t)}$ is closed under adjoints and is a subalgebra.

We now show that $\overline{\cup_{I \in \mathcal{I}_{(s,t)}} \mathcal{D}^I_{(s,t)}} = \mathcal{D}_{(s,t)}$. For all $I \in \mathcal{I}_{(s,t)}$ we have $\mathcal{D}^I_{(s,t)} \subseteq \mathcal{D}_{(s,t)}$. Since $\mathcal{D}_{(s,t)}$ is closed we have

$$\overline{\cup_{I \in \mathcal{I}_{(s,t)}} \mathcal{D}^I_{(s,t)}} \subseteq \mathcal{D}_{(s,t)}.$$

To show the reverse inclusion fix $T \in \overline{\text{span}}\{a \otimes v_p v_q^* v_r : (\phi(p), \phi(r)) = (s,t), \phi(q) = \phi(rp), a \in A\}$. Then write T as a limit of finite sums of the form

$$T_n = \sum_{i=1}^{l_n} \lambda_{i,n} a_{i,n} \otimes v_{p_{i,n}} v_{q_{i,n}}^* v_{r_{i,n}},$$

where, for all i, $(\phi(p_{i,n}), \phi(r_{i,n})) = (s,t)$ and $\phi(q_{i,n}) = \phi(r_{i,n} p_{i,n})$. For each n we can construct a finite set $I_n := \{q_{i,n} : 1 \leq i \leq l_n\} \in \mathcal{I}_{(s,t)}$. Thus each $T_n \in \cup_{I \in \mathcal{I}_{(s,t)}} \mathcal{D}_{(s,t)}^I$. It follows that the limit $T \in \overline{\cup_{I \in \mathcal{I}_{(s,t)}} \mathcal{D}_{(s,t)}^I}$ and hence $\overline{\cup_{I \in \mathcal{I}_{(s,t)}} \mathcal{D}_{(s,t)}^I} = \mathcal{D}_{(s,t)}$. $\square$

LEMMA 7.11. *Let (G, P) be a doubly quasi-lattice ordered group and let A be a C^*-algebra. Suppose that (K, Q) is a doubly quasi-lattice ordered group with a controlled map $\phi : (G, P) \to (K, Q)$. Let $(s,t) \in Q \times Q^{\mathrm{op}}$. Then $\mathrm{id} \otimes \pi_J$ is isometric on $\mathcal{D}_{(s,t)}$. Further, for all $T \in \mathcal{D}_{(s,t)}$,*

$$\|\mathrm{id} \otimes \pi_J(T) 1 \otimes P_{(s,t)}\| = \|T\|.$$

PROOF. By Lemma 7.12 $\overline{\cup_{I \in \mathcal{I}_{(s,t)}} \mathcal{D}_{(s,t)}^I} = \mathcal{D}_{(s,t)}$. So, to prove that $\mathrm{id} \otimes \pi_J$ is isometric on $\mathcal{D}_{(s,t)}$, it suffices to prove that $\mathrm{id} \otimes \pi_J$ is isometric on $\mathcal{D}_{(s,t)}^I$ for each $I \in \mathcal{I}_{(s,t)}$.

Fix $I \in \mathcal{I}_{(s,t)}$. Suppose there exists $T \in \mathcal{D}_{(s,t)}^I$ such that $\mathrm{id} \otimes \pi_J(T) = 0$. We will consider $\mathcal{D}_{(s,t)}^I$ as the span of all subspaces $\mathcal{D}_{(s,t)}^q$ such that $q \in I$. For each $q \in I$ and each $n \in \mathbb{N}$ let $T_{q,n} \in \mathcal{D}_{(s,t)}^q$ and write T as the limit of a sequence of finite sums:

(7.6) $$T = \lim_{n \to \infty} \sum_{q \in I} T_{q,n}.$$

Let $q_0 \in I$ and consider the projection $1 \otimes P_{(s,t)}^{q_0}$.

We claim that

(7.7) $$\left\| \mathrm{id} \otimes \pi_J \left(\sum_{q \in I} T_{q,n} \right) 1 \otimes P_{(s,t)}^{q_0} \right\| = \|T_{q_0,n}\|.$$

To prove our claim we will examine the behaviour of the $\pi_J(v_p v_q^* v_r) = J_p J_q^* J_r$ on range $P_{(s,t)}^{q_0} = H_{(s,t)}^{q_0}$. Fix $p, q, r \in P$ such that $q \in I$, $(\phi(p), \phi(r)) = (s,t)$. Fix $a \in P$ such that $a \leq_r q_0$ and $\phi(a) = s$. Compute

$$J_p J_q^* J_r \epsilon_{a,q_0} = \begin{cases} \epsilon_{pq^{-1}ra,q_0} & \text{if } ra \leq_r q_0 \text{ and } q \leq_l ra. \\ 0 & \text{otherwise.} \end{cases}$$

If $ra \leq_r q_0$ and $q \leq_l ra$ then $ra \vee_l q_0 = q_0$ and $q \vee_r ra = ra$. Since $\phi(q) = \phi(ra)$ and $\phi(q_0) = \phi(ra)$ the properties of the controlled map imply that $q = q_0 = ra$. Thus

$J_p J_q^* J_r P_{(s,t)}^{q_0} = 0$ unless $q = q_0$. Thus

$$\left\| \mathrm{id} \otimes \pi_J \left(\sum_{q \in I} T_{q,n} \right) 1 \otimes P_{(s,t)}^{q_0} \right\| = \| \mathrm{id} \otimes \pi_J(T_{q_0,n}) 1 \otimes P_{(s,t)}^{q_0} \|.$$

Further, in Lemma 7.9, we proved that $\mathrm{id} \otimes \pi_J$ is an isomorphism between $\mathcal{D}_{(s,t)}^{q_0}$ and $A \otimes \mathcal{K}(H_{(s,t)}^{q_0})$. Since range $P_{(s,t)}^{q_0} = H_{(s,t)}^{q_0}$ we have $\| \mathrm{id} \otimes \pi_J(T_{q_0,n}) 1 \otimes P_{(s,t)}^{q_0} \| = \| T_{q_0,n} \|$. Thus we have proved our claim.

By our earlier assumption, $\mathrm{id} \otimes \pi_J(T) = 0$ and so as $n \to \infty$

$$\left\| \mathrm{id} \otimes \pi_J \left(\sum_{q \in I} T_{q,n} \right) 1 \otimes P_{(s,t)}^{q_0} \right\| \to 0.$$

By our claim in (7.7),

$$\left\| \mathrm{id} \otimes \pi_J \left(\sum_{q \in I} T_{q,n} \right) 1 \otimes P_{(s,t)}^{q_0} \right\| = \| T_{q_0,n} \|$$

and hence $\| T_{q_0,n} \| \to 0$ as $n \to \infty$. Thus, for all $q \in I$, we have $\| T_{q,n} \| \to 0$. Since I is finite we see that

$$T = \lim_{n \to \infty} \sum_{q \in I \setminus \{q_0\}} T_{q,n} = 0.$$

Thus $\mathrm{id} \otimes \pi_J$ is isometric on $\mathcal{D}_{(s,t)}^I$ for all $I \in \mathcal{I}_{(s,t)}$. Thus $\mathrm{id} \otimes \pi_J$ is isometric on the direct limit $\overline{\cup_{I \in \mathcal{I}_{(s,t)}} \mathcal{D}_{(s,t)}^I} = \mathcal{D}_{(s,t)}$.

We now show that $\| \mathrm{id} \otimes \pi_J(T) 1 \otimes P_{(s,t)} \| = \| T \|$ for all $T \in \mathcal{D}_{(s,t)}$. We will consider the action of $\pi_J(v_p v_q^* v_r)$ on $H_{(s,t)}$. Let $p, q, r, a, b \in P$ such that $\phi(p) = \phi(a) = s$, $\phi(r) = t$ and $\phi(q) = \phi(b) = ts$. Then $\epsilon_{a,b} \in H_{(s,t)}$. Compute:

$$(7.8) \qquad \pi_J(v_p v_q^* v_r) \epsilon_{a,b} = J_p J_q^* J_r \epsilon_{a,b} = \begin{cases} \epsilon_{pq^{-1}ra,b} & \text{if } ra \leq_r b \text{ and } q \leq_l ra. \\ 0 & \text{otherwise.} \end{cases}$$

If $ra \leq_r b$ then $ra \vee_r b = b$. We know that $\phi(b) = \phi(ra)$ and so, by the properties of the controlled map, $b = ra$. Similarly, if $q \leq_l ra$ then $\phi(ra) = \phi(q)$ implies that $q = ra$. Rearranging gives $a = r^{-1}q$. Thus we may rewrite (7.8) as

$$J_p J_q^* J_r \epsilon_{a,b} = \begin{cases} \epsilon_{p,q} & \text{if } a = r^{-1}q \text{ and } q = b. \\ 0 & \text{otherwise.} \end{cases}$$

So $\pi_J(v_p v_q^* v_r) = J_p J_q^* J_r$ is the rank one operator $(\cdot \mid \epsilon_{r^{-1}q,q}) \epsilon_{p,q}$ on $H_{(s,t)}$. Therefore the restriction of $\mathrm{id} \otimes \pi_J(T)$ to range $1 \otimes H_{(s,t)}$ is isometric. Hence

$$\| \mathrm{id} \otimes \pi_J(T) 1 \otimes P_{(s,t)} \| = \| \mathrm{id} \otimes \pi_J(T) \|.$$

94

We already showed that $\mathrm{id} \otimes \pi_J$ is isometric on $\mathcal{D}_{(s,t)}$. Thus $\|\mathrm{id} \otimes \pi_J(T) 1 \otimes P_{(s,t)}\| = \|T\|$ for all $T \in \mathcal{D}_{(s,t)}$ as desired.

$\square$

We will now repeat this process once more to build $\overline{\mathrm{span}}\{a \otimes v_p v_q^* v_r : \phi(q) = \phi(rp), a \in A\}$ out of finite unions of $\mathcal{D}_{(s,t)}$. Let $\mathcal{F}$ denote the collection of all finite subsets $F \subset Q \times Q^{\mathrm{op}}$ such that $x \vee y \in F$ whenever $x, y \in F$ and $x \vee y < \infty$. For each $F \in \mathcal{F}$ let $\mathcal{D}_F$ be

(7.9) $$\mathcal{D}_F = \overline{\mathrm{span}}\{a \otimes v_p v_q^* v_r : a \in A, (\phi(p), \phi(r)) \in F, \phi(q) = \phi(rp)\}.$$

LEMMA 7.12. *Let (G, P) be a doubly quasi-lattice ordered group and let A be a C^*-algebra. Suppose that (K, Q) is a doubly quasi-lattice ordered group with a controlled map $\phi : (G, P) \to (K, Q)$. Then the set $\mathcal{F}$ is directed, ordered by inclusion. Further, for each $F \in \mathcal{F}$, the set $\mathcal{D}_F$ is a C^*-subalgebra of $A \otimes_{\max} C^*(G, P, P^{\mathrm{op}})$ and $\{\mathcal{D}_F : F \in \mathcal{F}\}$ is an inductive system with limit*

$$\overline{\cup_{F \in \mathcal{F}} \mathcal{D}_F} = \overline{\mathrm{span}}\{a \otimes v_p v_q^* v_r : \phi(q) = \phi(rp), a \in A\}.$$

PROOF. We first prove that $\mathcal{D}_F$ is a subalgebra. Since A is a C^*-algebra it will suffice to show that $\mathrm{span}\{v_p v_q^* v_r : (\phi(p), \phi(r)) \in F, \phi(q) = \phi(rp)\}$ is closed under multiplication and adjoints. Let $p, q, r, x, y, z \in P$ such that $\phi(rp) = \phi(q)$, $\phi(zx) = \phi(y)$ and $(\phi(p), \phi(r)), (\phi(x), \phi(z)) \in F$. Now consider

(7.10) $$(v_p v_q^* v_r)(v_x v_y^* v_z) = v_{pq^{-1}(q \vee_l rx)} v_{(y \vee_r rx)(rx)^{-1}(q \vee_l rx)}^* v_{(y \vee_r rx)y^{-1}z}.$$

If either of the upper bounds $q \vee_l rx$ or $y \vee_r rx$ don't exist, then (7.10) is zero. So we suppose that $q \vee_l rx$ and $y \vee_r rx$ both exist. We must show that

$$\phi((y \vee_r rx)y^{-1}zpq^{-1}(q \vee_l rx)) = \phi((y \vee_r rx)(rx)^{-1}(q \vee_l rx))$$

and that

$$(\phi(pq^{-1}(q \vee_l rx)), \phi((y \vee_r rx)y^{-1}z)) \in F.$$

We know $\phi(y^{-1}z) = \phi(x)^{-1}$ and $\phi(pq^{-1}) = \phi(r)^{-1}$, hence we can compute:

$$\phi((y \vee_r rx)y^{-1}zpq^{-1}(q \vee_l rx)) = \phi((y \vee_r rx)x^{-1}r^{-1}(q \vee_l rx))$$
$$= \phi((y \vee_r rx)(rx)^{-1}(q \vee_l rx)).$$

We must show that $(\phi(pq^{-1}(q \vee_l rx)), \phi((y \vee_r rx)y^{-1}z)) \in F$. We compute:

$$\phi(pq^{-1}(q \vee_l rx)) = \phi(p)\phi(q)^{-1}\phi(q \vee_l rx)$$
$$= \phi(p)\phi(q)^{-1}[\phi(q) \vee_l \phi(rx)]$$

$$= \phi(p)\phi(rp)^{-1}[\phi(rp) \vee_l \phi(rx)] \quad (\phi(q) = \phi(rp))$$
$$= \phi(r)^{-1}[\phi(rp) \vee_l \phi(rx)]$$
$$= [\phi(r)^{-1}\phi(rp) \vee_l \phi(r)^{-1}\phi(rx)]$$
$$= \phi(p) \vee_l \phi(x)$$

Similarly, $\phi((y \vee_r rx)y^{-1}z) = \phi(z) \vee_r \phi(r)$. Since F is closed under $\vee$ it follows that $(\phi(p), \phi(r)) \vee (\phi(x), \phi(z)) = (\phi(p) \vee_l \phi(x), \phi(r) \vee_r \phi(z)) \in F$. Thus $\mathcal{D}_F$ is closed under multiplication.

To show that $\mathcal{D}_F$ is closed under adjoints consider $(v_p v_q^* v_r)^* = v_{r^{-1}q} v_q^* v_{qp^{-1}}$. Then $\phi(r^{-1}q) = \phi(r)^{-1}\phi(q) = \phi(r)^{-1}\phi(rp) = \phi(p)$. Similarly $\phi(qp^{-1}) = \phi(r)$. Hence

$$(\phi(r^{-1}q), \phi(qp^{-1})) = (\phi(p), \phi(r)).$$

In addition $\phi(qp^{-1})\phi(r^{-1}q) = \phi(rp) = \phi(q)$. Thus $\mathcal{D}_F$ is closed under adjoints and is a subalgebra.

To prove that $\overline{\cup_{F \in \mathcal{F}} \mathcal{D}_F} = \overline{\text{span}}\{a \otimes v_p v_q^* v_r : \phi(q) = \phi(rp), a \in A\}$ observe that for all $F \in \mathcal{F}$ we have $\mathcal{D}_F \subseteq \overline{\text{span}}\{a \otimes v_p v_q^* v_r : \phi(q) = \phi(rp), a \in A\}$ and hence, since $\overline{\text{span}}\{a \otimes v_p v_q^* v_r : \phi(q) = \phi(rp), a \in A\}$ is closed we have

$$\overline{\cup_{F \in \mathcal{F}} \mathcal{D}_F} \subseteq \overline{\text{span}}\{a \otimes v_p v_q^* v_r : \phi(q) = \phi(rp), a \in A\}.$$

To show the reverse inclusion fix $T \in \overline{\text{span}}\{a \otimes v_p v_q^* v_r : \phi(q) = \phi(rp), a \in A\}$. Then write T as a limit of finite sums of the form

$$T_n = \sum_{i=1}^{l_n} \lambda_{i,n} a_{i,n} \otimes v_{p_{i,n}} v_{q_{i,n}}^* v_{r_{i,n}}$$

where, for all i, $\phi(r_{i,n} p_{i,n}) = \phi(q_{i,n})$. For each n we can construct a finite set by taking the closure under $\vee$ of $\{(\phi(p_{i,n}), \phi(r_{i,n})) : 1 \leq i \leq l_n\}$ to get a finite $\vee$-closed set $F_n \in \mathcal{F}$. Thus each $T_n \in \cup_{F \in \mathcal{F}} \mathcal{D}_F$. It follows that the limit $T \in \overline{\cup_{F \in \mathcal{F}} \mathcal{D}_F}$. Hence

$$\overline{\text{span}}\{a \otimes v_p v_q^* v_r : \phi(q) = \phi(rp), a \in A\} \subseteq \overline{\cup_{F \in \mathcal{F}} \mathcal{D}_F}.$$

$\square$

We finally have all the pieces to prove Lemma 7.6 and complete the proof of Proposition 7.2.

PROOF OF LEMMA 7.6. We must prove that $\text{id} \otimes \pi_J$ is faithful on $\overline{\text{span}}\{a \otimes v_p v_q^* v_r : \phi(q) = \phi(rp), a \in A\}$. By Lemma 7.12

$$\overline{\text{span}}\{a \otimes v_p v_q^* v_r : \phi(q) = \phi(rp), a \in A\} = \overline{\cup_{F \in \mathcal{F}} \mathcal{D}_F}.$$

So it suffices to prove that $\mathrm{id} \otimes \pi_J$ is isometric on $\mathcal{D}_F$ for each $F \in \mathcal{F}$.

Let $F \in \mathcal{F}$. Suppose there exists $T \in \mathcal{D}_F$ such that $\mathrm{id} \otimes \pi_J(T) = 0$. We will consider $\mathcal{D}_F$ as the closed span of all subspaces $\mathcal{D}_{(s,t)}$ such that $(s,t) \in F$. Let $T_{(s,t),n} \in \mathcal{D}_{(s,t)}$ for each $n \in \mathbb{N}$ and write T as the limit of a sequence of finite sums:

$$(7.11) \qquad T = \lim_{n \to \infty} \sum_{(s,t) \in F} T_{(s,t),n}.$$

We will show that for each $(s,t) \in F$ the sequence $T_{(s,t),n} \to 0$. Since F is finite, by Lemma 2.8, it has a minimal element (s_0, t_0). Without loss of generality we may assume there exists $p, r \in P$ such that $(\phi(p), \phi(r)) = (s_0, t_0)$. (If there is not then $\mathcal{D}_{(s_0,t_0)}$ is empty. Hence $\mathcal{D}_F = \mathcal{D}_{F \setminus \{(s_0,t_0)\}}$ and we may discard (s_0, t_0) and move to the next minimal element.) Consider the projection $1 \otimes P_{(s_0,t_0)}$ on $A \otimes_{\min} C^*_{\mathrm{ts}}(G, P, P^{\mathrm{op}})$.

Claim: We claim that

$$(7.12) \qquad \left\| \mathrm{id} \otimes \pi_J \left(\sum_{(s,t) \in F} T_{(s,t),n} \right) 1 \otimes P_{(s_0,t_0)} \right\| = \| T_{(s_0,t_0),n} \|.$$

We will prove the claim of (7.12) at the end of the proof. So assume (7.12). By our earlier assumption, $\mathrm{id} \otimes \pi_J(T) = 0$ and so as $n \to \infty$

$$\left\| \mathrm{id} \otimes \pi_J \left(\sum_{(s,t) \in F} T_{(s,t),n} \right) 1 \otimes P_{(s_0,t_0)} \right\| \to 0.$$

However, by our claim in (7.12),

$$\left\| \mathrm{id} \otimes \pi_J \left(\sum_{(s,t) \in F} T_{(s,t),n} \right) 1 \otimes P_{(s_0,t_0)} \right\| = \| T_{(s_0,t_0),n} \|$$

and hence $\| T_{(s_0,t_0),n} \| \to 0$ as $n \to \infty$.

So we may remove $\{T_{(s_0,t_0),n}\}$ from the sum without changing the limit. Thus

$$T = \lim_{n \to \infty} \sum_{(s,t) \in F \setminus \{(s_0,t_0)\}} T_{(s,t),n}.$$

We know that F is finite. So we can repeat the above argument at most $|F|$ times to see that T is 0. Thus $\mathrm{id} \otimes \pi_J$ is isometric on $\mathcal{D}_F$ for all $F \in \mathcal{F}$. Thus $\mathrm{id} \otimes \pi_J$ is isometric on the direct limit $\overline{\cup_{F \in \mathcal{F}} \mathcal{D}_F}$. Hence $\mathrm{id} \otimes \pi_J$ is faithful when restricted to $\overline{\mathrm{span}}\{a \otimes v_p v_q^* v_r : \phi(q) = \phi(rp), a \in A\}$.

Proof of Claim: To complete the proof we must prove the claim of (7.12). We must show that $1 \otimes P_{(s_0,t_0)}$ sends all the $T_{(s,t),n}$ to zero unless $q = q_0$, $s = s_0$ and $t = t_0$. We will examine the behaviour of the $J_p J_q^* J_r = \pi_J(v_p v_q^* v_r)$ on range $P_{(s_0,t_0)} = H_{(s_0,t_0)}$.

Fix $(s,t) \in F$ and $p, q, r \in P$ such that $\phi(q) = ts$, $\phi(p) = s$ and $\phi(r) = t$. Fix $a \in P$ such that $a \leq_r q_0$ and $\phi(a) = s_0$. Compute

$$J_p J_q^* J_r \epsilon_{a,q_0} = \begin{cases} \epsilon_{pq^{-1}ra, q_0} & \text{if } ra \leq_r q_0 \text{ and } q \leq_l ra \\ 0 & \text{otherwise.} \end{cases}$$

We break this into two subcases: (1) $s \neq s_0$, (2) $s = s_0$ and $t \neq t_0$.

(1). Suppose that $s \neq s_0$. Then, since s_0 is minimal, it follows that $s \nleq_l s_0$. Then $\phi(p) \nleq_l \phi(a)$ which implies that $\phi(rp) \nleq_l \phi(ra)$ by left-invariance. We then have $\phi(rp) = \phi(q) \nleq_l \phi(ra)$ and hence $q \nleq_l ra$. So $s \nleq_l s_0$ implies $q \nleq_l ra$ and hence $J_p J_q^* J_r \epsilon_{a,q_0} = 0$. Thus $\text{id} \otimes \pi_J(v_p v_q^* v_r) 1 \otimes P_{(s_0,t_0)} = 0$ if $s \neq s_0$.

(2). Suppose $s = s_0$ and $t \neq t_0$. Then $t \nleq_r t_0$ and hence $ts_0 \nleq_r t_0 s_0 = \phi(q_0)$ by right-invariance. Then $\phi(ra) \nleq_r \phi(q_0)$ and hence $ra \nleq_r q_0$. So $J_p J_q^* J_r \epsilon_{a,q_0} = 0$. Thus $\text{id} \otimes \pi_J(v_p v_q^* v_r) 1 \otimes P_{(s_0,t_0)} = 0$ if $s = s_0$ and $t \neq t_0$.

Thus we have shown that

$$\left\| \text{id} \otimes \pi_J \left(\sum_{(s,t) \in F} T_{(s,t),n} \right) 1 \otimes P_{(s_0,t_0)} \right\| = \left\| \text{id} \otimes \pi_J(T_{(s_0,t_0),n}) 1 \otimes P_{(s_0,t_0)} \right\|.$$

By Lemma 7.11 $\text{id} \otimes \pi_J$ is isometric on $\mathcal{D}_{(s_0,t_0)}$ and for all $T_{(s_0,t_0),n} \in \mathcal{D}_{(s_0,t_0)}$ we have

$$\left\| \text{id} \otimes \pi_J(T_{(s_0,t_0),n}) 1 \otimes P_{(s_0,t_0)} \right\| = \left\| T_{(s_0,t_0),n} \right\|.$$

Thus we have proved our claim and the proof of Lemma 7.6 is complete. $\qquad\square$

7.4. Examples of amenable doubly quasi-lattice ordered groups

In this section we will give examples of amenable doubly quasi-lattice ordered groups. To do this we will construct controlled maps into amenable groups and apply Theorem 7.7.

EXAMPLES. (1). Suppose (G, P) is a doubly quasi-lattice ordered group and G is an amenable group. The identity homomorphism $\iota : G \to G$ is a controlled map. By Theorem 7.7 (G, P) is an amenable doubly quasi-lattice ordered group and $C^*(G, P, P^{\text{op}})$ is nuclear. This example shows that Theorem 6.6 is actually a special case of Theorem 7.7.

(2). For all $n \in \mathbb{N}$ the additive group $\mathbb{Z}^n$ is abelian and hence amenable. Thus $(\mathbb{Z}^n, \mathbb{N}^n)$ is an amenable doubly quasi-lattice ordered group and $C^*(\mathbb{Z}^n, \mathbb{N}^n, \mathbb{N}^{n\text{op}})$ is nuclear.

(3). The semidirect product $\mathbb{Q} \rtimes \mathbb{Q}^*$ of Proposition 2.10 is an amenable group. Thus $(\mathbb{Q} \rtimes \mathbb{Q}_+^*, \mathbb{N} \rtimes \mathbb{N}^\times)$ is amenable and $C^*(\mathbb{Q} \rtimes \mathbb{Q}_+^*, \mathbb{N} \rtimes \mathbb{N}^\times, (\mathbb{N} \rtimes \mathbb{N}^\times)^{\text{op}})$ is nuclear.

For examples without an amenable underlying group the proofs get more involved. We begin by proving that the free group on n generators $(\mathbb{F}_n, \mathbb{F}_n^+)$ is an amenable doubly quasi-lattice ordered group. This is a useful example because there are multiple straightforward controlled maps into different amenable groups. We will focus of the abelianization map $\psi : \mathbb{F}_n \to \mathbb{Z}^n$ and the length map $\theta : \mathbb{F}_n \to \mathbb{Z}$.

LEMMA 7.13. *Let $\mathbb{F}_n$ be the free group on n generators $\{a_i : 1 \leq i \leq n\}$, and let $\mathbb{F}_n^+$ be the subsemigroup generated by $\{a_i : 1 \leq i \leq n\} \cup \{e\}$. Then*

  (1) *The length map $\theta : \mathbb{F}_n \to \mathbb{Z}$ defined by $\theta(a_i) = 1$ for each $i \leq n$ is a controlled map $\theta : (\mathbb{F}_n, \mathbb{F}_n^+) \to (\mathbb{Z}, \mathbb{N})$;*

  (2) *Let $\{e_i : 1 \leq i \leq\}$ be the usual generators of the additive group $\mathbb{Z}^n$. The abelianization map $\psi : \mathbb{F}_n \to \mathbb{Z}^n$ defined by $\psi(a_i) = e_i$ for each $i \leq n$ is a controlled map $\psi : (\mathbb{F}_n, \mathbb{F}_n^+) \to (\mathbb{Z}^n, \mathbb{N}^n)$;*

  (3) *$(\mathbb{F}_n, \mathbb{F}_n^+)$ is amenable and $C^*(\mathbb{F}_n, \mathbb{F}_n^+, (\mathbb{F}_n^+)^{\mathrm{op}})$ is nuclear.*

PROOF. (1). The map θ is a homomorphism. For every $x \in \mathbb{F}_n^+$, we can write x as a sequence of generators a_i with positive powers, or the identity. Thus we have $\theta(x) \in \mathbb{N}$ and hence θ is order preserving. We must show that θ is a controlled map. Suppose that $x, y \in \mathbb{F}_n^+$ and $x \vee_l y < \infty$. Recall from Section 2.2 that if $x \vee_l y < \infty$, then x, y are comparable. So we may assume, without loss of generality, that $x \leq_l y$ and $x \vee_l y = y$. Since θ is order preserving we have $\theta(x) \leq_l \theta(y)$. Thus $\theta(y) \vee_l \theta(x) = \theta(y) = \theta(x \vee_l y)$.

Suppose that $x \vee_l y < \infty$ and $\theta(x) = \theta(y)$. Then x and y have the same length and are comparable. Thus $x = y$. The arguments for $\vee_r$ follow by symmetry. Thus θ is a controlled map.

(2). The abelianization map $\psi : \mathbb{F}_n \to \mathbb{Z}^n$ is a homomorphism and effectively counts how many of each generator a_i appear in a word of $\mathbb{F}_n$. For every $x \in \mathbb{F}_n^+$ we write x as a sequence of generators with positive power or the identity so we have $\psi(x) \in \mathbb{N}$. Since any pair x, y has a common upper bound in either the left or right order if and only if they are comparable, the same argument from (1) follows. Hence ψ is a controlled map.

(3). From (1) and (2) we see that $(\mathbb{F}_n, \mathbb{F}_n^+)$ has two natural controlled maps into amenable groups. Thus Theorem 7.7 implies that $(\mathbb{F}_n, \mathbb{F}_n^+)$ is an amenable doubly quasi-lattice ordered group and $C^*(\mathbb{F}_n, \mathbb{F}_n^+, (\mathbb{F}_n^+)^{\mathrm{op}})$ is nuclear. $\square$

LEMMA 7.14. *Let $(G_1, P_1), (G_2, P_2), (G_3, P_3)$ be doubly quasi-lattice ordered groups. Suppose that $\phi_1 : (G_1, P_1) \to (G_2, P_2)$ and $\phi_2 : (G_2, P_2) \to (G_3, P_3)$ are controlled maps. Then $\phi_2 \circ \phi_1 : (G_1, P_1) \to (G_3, P_3)$ is a controlled map.*

PROOF. We can immediately see that $\phi_2 \circ \phi_1(P_1) \subseteq \phi_2(P_2) \subseteq P_3$, and hence $\phi_2 \circ \phi_1$ is order preserving. Suppose that $x, y \in P$ and $x \vee_l y < \infty$. By the properties of the controlled map we see

$$\phi_2 \circ \phi_1(x \vee_l y) = \phi_2(\phi_1(x) \vee_l \phi_1(y)) = \phi_2 \circ \phi_1(x) \vee_l \phi_2 \circ \phi_1(y).$$

If $\phi_2 \circ \phi_1(x) = \phi_2 \circ \phi_1(y)$ then we have $\phi_1(x) = \phi_1(y)$ and hence $x = y$. A symmetric argument shows that $x \vee_r y < \infty$ then $\phi_2 \circ \phi_1(\vee_r y) = \phi_2 \circ \phi_1(x) \vee_r \phi_2 \circ \phi_1(y)$ and $\phi_2 \circ \phi_1(x) = \phi_2 \circ \phi_1(y)$ implies $x = y$. □

PROPOSITION 7.15. *Let I be an index set. Let $\{(G_i, P_i) : i \in I\}$ and $\{(K_i, Q_i) : i \in I\}$ be families of doubly quasi-lattice ordered groups. Suppose that for each $i \in I$ the group K_i is amenable and there exists a controlled map $\phi_i : (G_i, P_i) \to (K_i, Q_i)$.*

(1) Let $G^\times = \prod_{i \in I} G_i$ and $P^\times = \prod_{i \in I} P_i$. Then $(G^\times, P^\times)$ is an amenable doubly quasi-lattice ordered group and $C^*(G^\times, P^\times, (P^\times)^{\mathrm{op}})$ is nuclear.

(2) Let $G^* = *_{i \in I} G_i$ and $P^* = *_{i \in I} P_i$. Then (G^*, P^*) is an amenable doubly quasi-lattice ordered group and $C^*(G^*, P^*, (P^*)^{\mathrm{op}})$ is nuclear.

PROOF. (1). Let $K^\times = \prod_{i \in I} K_i$ and let $Q^\times = \prod_{i \in I} Q_i$. Consider the homomorphism ϕ from $G^\times = \prod_{i \in I} G_i$ to $K^\times = \prod_{i \in I} K_i$ defined by $\phi(x)_i = \phi_i(x_i)$. Each of the ϕ_i is a controlled map, hence $\phi(P^\times) \subseteq Q^\times$. Thus ϕ is order preserving. Suppose that $x, y \in P$ and $x \vee_l y < \infty$. Then, by the properties of the direct product, for each $i \in I$ we have $x_i \vee_l y_i < \infty$ and $(x \vee_l y)_i = x_i \vee_l y_i$. Since each ϕ_i is a controlled map we have

$$\phi(x \vee_l y)_i = \phi_i(x_i \vee_l y_i) = \phi_i(x_i) \vee_l \phi_i(y_i) = \phi(x)_i \vee_l \phi(y)_i.$$

Therefore $\phi(x \vee_l y) = \phi(x) \vee_l \phi(y)$.

Suppose that $x \vee_l y < \infty$ and $\phi(x) = \phi(y)$. For each $i \in I$ we have $x_i \vee_l y_i < \infty$ and $\phi_i(x_i) = \phi(x)_i = \phi(y)_i = \phi_i(y_i)$. By the properties of the controlled map $\phi_i(x_i) = \phi_i(y_i)$ implies that $x_i = y_i$ for all i and hence $x = y$. A symmetric argument shows that if $x \vee_r y < \infty$ then $\phi(x \vee_r y) = \phi(x) \vee_r \phi(y)$ and $\phi(x) = \phi(y)$ implies $x = y$. Therefore ϕ is a controlled map. The direct product of amenable groups is amenable, so it follows that $K^\times$ is amenable. We may apply Theorem 7.7 to see that $(G^\times, P^\times)$ is an amenable doubly quasi-lattice ordered group and $C^*(G^\times, P^\times, (P^\times)^{\mathrm{op}})$ is nuclear.

(2). Consider the homomorphism ψ from $G^* = *_{i \in I} G_i$ to $G^\times = \prod_{i \in I} G_i$ that takes each $x \in G_{i_k}$ in the free product to the same x in the direct product. We will show that ψ is a controlled map. Note that $G^\times$ is not in general amenable. However, we can then compose ψ with the controlled map ϕ of (1) to get an appropriate controlled

map into an amenable group. It is straightforward to see that $\psi(P^*) \subseteq P^\times$ and hence that ψ is order preserving.

Suppose that $x, y \in P^*$ and $x \vee_l y < \infty$. Write

$$x = x_{1,i_1} x_{2,i_2} x_{3,i_3} \ldots x_{m,i_m}$$

$$y = y_{1,j_1} y_{2,j_2} \ldots y_{n,j_n}.$$

Without loss of generality, suppose that $m \leq n$. If $m < n$ then $x \lneq_l y$ in which case $\psi(x) \lneq_l \psi(y)$ and $\psi(x \vee_l y) = \psi(y) = \psi(x) \vee_l \psi(y)$. If $m = n$ then $x_{k,i_k} = y_{k,j_k}$ for $k < m$, $i_m = j_m$ and $x_{m,i_m} \vee_l y_{m,i_m} < \infty$. Then

$$x \vee_l y = x_{1,i_1} x_{2,i_2} \ldots x_{m-1,i_{m-1}} (x_{m,i_m} \vee_l y_{m,j_m}).$$

Since $x_{m,i_m}, y_{m,j_m} \in P_{i_m}$ we have $\psi(x_{m,i_m} \vee_l y_{m,j_m}) = \psi(x_{m,i_m}) \vee_l \psi(y_{m,j_m}) \in P_{i_m}$. Thus we can compute:

$$
\begin{aligned}
\psi(x \vee_l y) &= \psi(x_{1,i_1} x_{2,i_2} \ldots x_{m-1,i_{m-1}} (x_{m,i_m} \vee_l y_{m,j_m})) \\
&= \psi(x_{1,i_1} x_{2,i_2} \ldots x_{m-1,i_{m-1}})(\psi(x_{m,i_m}) \vee_l \psi(y_{m,j_m})) \\
&= \psi(x_{1,i_1} x_{2,i_2} \ldots x_{m-1,i_{m-1}})\psi(x_{m,i_m}) \vee_l \psi(x_{1,i_1} x_{2,i_2} \ldots x_{m-1,i_{m-1}})\psi(y_{m,j_m}) \\
&= \psi(x) \vee_l \psi(y).
\end{aligned}
$$

Suppose that $x \vee_l y < \infty$ and $\psi(x) = \psi(y)$. Then $m = n$. Since $x_{k,i_k} = y_{k,j_k}$ for $k < m$, it follows by left cancellation that $\psi(x_{m,i_m}) = \psi(y_{m,j_m})$. Since $x_{m,i_m}, y_{m,j_m} \in P_{i_m}$ we have $x_{m,i_m} = y_{m,j_m}$. Thus $x = y$. By a symmetric argument we see that if $x \vee_r y < \infty$ then $\psi(x \vee_r y) = \psi(x) \vee_r \psi(y)$ and $\psi(x) = \psi(y)$ implies $x = y$. Thus ψ is a controlled map.

Now let $\phi : (G^\times, P\times) \to (K^\times, P^\times)$ be the controlled map of (1) and consider $\phi \circ \psi : (G^*, P^*) \to (K^\times, P^\times)$. By Lemma 7.14 the composition of two controlled maps is a controlled map. Thus $\phi \circ \psi$ is a controlled map into an amenable group. We may apply Theorem 7.7 to see that (G^*, P^*) is an amenable doubly quasi-lattice ordered group and $C^*(G^*, P^*, (P^*)^{\mathrm{op}})$ is nuclear. $\qquad \square$

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