Analogues of Leavitt path algebras for higher-rank graphs

Yosafat Eka Prasetya Pangalela

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Abstract

Directed graphs and their higher-rank analogues provide an intuitive framework to study a class of C^*-algebras which we call graph algebras. The theory of graph algebras has been developed by a number of researchers and also influenced other branches of mathematics: Leavitt path algebras and Cohn path algebras, to name just two.

Leavitt path algebras for directed graphs were developed independently by two groups of mathematicians using different approaches. One group, which consists of Ara, Goodearl and Pardo, was motivated to give an algebraic framework of graph algebras. Meanwhile, the motivation of the other group, which consists of Abrams and Aranda Pino, is to generalise Leavitt's algebras, in which the name Leavitt comes from. Later, Abrams and now with Mesyan introduced the notion of Cohn path algebras for directed graphs. Interestingly, both Leavitt path algebras and Cohn path algebras for directed graphs can be viewed as algebraic analogues of C^*-algebras of directed graphs.

In 2013, Aranda Pino, J. Clark, an Huef and Raeburn introduced a higher-rank version of Leavitt path algebras which we call Kumjian-Pask algebras. At their first appearance, Kumjian-Pask algebras were only defined for row-finite higher-rank graphs with no sources. Clark, Flynn and an Huef later extended the coverage by also considering locally convex row-finite higher-rank graphs. On the other hand, Cohn path algebras for higher rank graphs still remained a mystery.

This thesis has two main goals. The first aim is to introduce Kumjian-Pask algebras for a class of higher-rank graphs called finitely-aligned higher-rank graphs. This type of higher-rank graph covers both row-finite higher-rank

graphs with no sources and locally convex row-finite higher-rank graphs. Therefore, we give a generalisation of the existing Kumjian-Pask algebras. We also establish the graded uniqueness theorem and the Cuntz-Krieger uniqueness theorem for Kumjian-Pask algebras of finitely-aligned higher-rank graphs.

The second aim is to introduce a higher-rank analogue of Cohn path algebras. We then study the relationship between Kumjian-Pask algebras and Cohn path algebras and use this to investigate properties of Cohn path algebras. Finally, we establish a uniqueness theorem for Cohn path algebras.

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Contents

Chapter 1

Introduction

Leavitt path algebras have received a great deal of interest in recent years, and can be approached from many different points of view [3, 10, 15]. Graph C^*-algebras have a special place in the study of Leavitt path algebras because many results about Leavitt path algebras were motivated by previous results of graph C^*-algebras. This includes how to generalise Leavitt path algebras to a bigger class of graphs.

Meanwhile, Cohn path algebras were introduced in [6, 8]. Since Cohn path algebras are obtained from the Leavitt path algebras by omitting one of the so-called Cuntz-Krieger relations, these algebras can be viewed as an algebraic analogue of Toeplitz algebras of directed graphs.

Since the theory of graph C^*-algebras and Toeplitz algebras have been successfully extended to higher-rank graphs [29, 39], mathematicians have started wondering about higher-rank versions of Leavitt path algebras, called Kumjian-Pask algebras, and higher-rank versions of Cohn path algebras. These are the motivations of this thesis. We extend the definition of Kumjian-Pask algebras of [11, 16] to finitely aligned higher-rank graphs. We also introduce Cohn path algebras of higher-rank graphs. We then study the properties of both algebras, especially related to uniqueness theorems, and the relationship between the algebras.

We begin with a brief history of the development of the theory of graph C^*-alegbras (Section 1.1 and Section 1.2) and Leavitt path algebras (Section 1.3 and Section 1.4). We give more details about uniqueness theorems because one of our aims is to establish uniqueness theorems for our new Kumjian-Pask algebras and Cohn path algebras.

1.1 C^*-algebras and Toeplitz algebras for graphs

A *directed graph* $E = (E^0, E^1, r, s)$ consists of countable sets of vertices E^0, edges E^1 and functions $r, s : E^1 \to E^0$, which map edges to their range and source, respectively. C^*-algebras of directed graphs, also known as *graph algebras*, were introduced in [31] to extend the Cuntz-Krieger algebras of $\{0,1\}$-matrices of [21]. In [31], Kumjian, Pask, Raeburn and Renault used a groupoid approach to study C^*-algebras of *locally finite* graphs in which both $r^{-1}(v)$ and $s^{-1}(v)$ are finite and nonempty for all $v \in E^0$.

In [30], for a locally finite graph E, Kumjian, Pask and Raeburn introduced a *Cuntz-Krieger E-family* to be a family which consists of projections $\{P_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$ in a C^*-algebra B satisfying $S_e^* S_e = P_{s(e)}$ for all $e \in E^1$ and the *Cuntz-Krieger relation*:

$$P_v = \sum_{e \in r^{-1}(v)} S_e S_e^* \text{ for all } v \in E^0. \tag{1.1.1}$$

They then defined the graph C^*-algebra $C^*(E)$ to be the C^*-algebra generated by the universal Cuntz-Krieger E-family $\{s_e, p_v\}$.

In [12], Bates, Pask, Raeburn and Szymański extended the coverage of C^*-algebras of directed graphs by introducing row-finite graphs and their C^*-algebras. A graph E is *row-finite*[1] if $r^{-1}(v)$ is finite for all $v \in E^0$; so $s^{-1}(v)$ may be infinite for some $v \in E^0$, and then E is not locally finite and also might have *sources*; that is, vertices which do not receive any edges. In general, the generalisation from locally finite to row-finite graphs required no modification in relations, except is that we do not impose (1.1.1) when v is a source.

For a row-finite graph E, the path space E^* consists of all paths $\lambda = e_1 e_2 \dots e_n$ where $|\lambda| := n \in \mathbb{N}$, and $e_i \in E^1$ satisfies $s(e_{i-1}) = r(e_i)$ for all $2 \leq i \leq n$. For $\lambda = e_1 e_2 \dots e_n \in E^*$, we write s_λ to denote $s_{e_1} s_{e_2} \cdots s_{e_n}$, and write $s(\lambda)$ for $s(e_n)$. A path $\lambda = e_1 e_2 \dots e_n \in E^*$ is a *cycle* if $|\lambda| \geq 1$, $r(\lambda) = s(\lambda)$ and $s(e_i) \neq s(e_j)$ for $i \neq j$. An edge e is an *entrance* to the cycle λ if there exists i such that $r(e) = r(e_i)$ and $e \neq e_i$. The authors of [12] then showed that for every row-finite graph E,

$$C^*(E) = \overline{\operatorname{span}} \left\{ s_\lambda s_\mu^* : \lambda, \mu \in E^*, s(\lambda) = s(\mu) \right\}.$$

Write $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Then for every $z \in \mathbb{T}$, the family $\{z s_e, p_v\}$ is also a Cuntz-Krieger E-family which generates $C^*(E)$, and hence, the universal property of $C^*(E)$ gives a homomorphism $\gamma_z : C^*(E) \to C^*(E)$ such that $\gamma_z(s_e) = z s_e$ and

[1] We use the paths convention of [43] because we view the collection of paths as a category.

$\gamma_z(p_v) = p_v$ for all $e \in E^1$ and $v \in E^0$. Furthermore, they showed that $\gamma_z \in \operatorname{Aut} C^*(E)$ for all $z \in \mathbb{T}$ and then γ is a strongly continuous action of $\mathbb{T}$ on $C^*(E)$, called the *gauge action*. The next theorem is a generalisation of the gauge-invariant uniqueness theorem for the Cuntz-Krieger algebras of $\{0,1\}$-matrices [28, Theorem 2.3].

Theorem 1.1.1 ([12, Theorem 2.1]: The gauge-invariant uniqueness theorem)**.** *Suppose that E is a row-finite graph, that $\{S_e, P_v : e \in E^1, v \in E^0\}$ is a Cuntz-Krieger E-family in a C^*-algebra B, and that $\pi_{S,P} : C^*(E) \to B$ is the homomorphism such that $\pi_{S,P}(s_e) = S_e$ and $\pi_{S,P}(p_v) = P_v$ for all $e \in E^1$ and $v \in E^0$. Suppose that each P_v is nonzero and that there is a strongly continuous action β of $\mathbb{T}$ on $C^*(\{S_e, P_v : e \in E^1, v \in E^0\})$ such that $\beta_z \circ \pi_{S,P} = \pi_{S,P} \circ \gamma_z$ for $z \in \mathbb{T}$. Then $\pi_{S,P}$ is faithful.*

The authors of [12] also introduced another uniqueness theorem, called the Cuntz-Krieger uniqueness theorem. Unlike Theorem 1.1.1, which applies to any row-finite graphs, the Cuntz-Krieger uniqueness theorem only applies to row-finite graphs which satisfy *Condition (L)*: every cycle has an entrance. Now we state the uniqueness theorem as follows:

Theorem 1.1.2 ([12, Theorem 3.1]: The Cuntz-Krieger uniqueness theorem)**.** *Suppose that E is a row-finite graph satisfying Condition (L) and that $\{S_e, P_v : e \in E^1, v \in E^0\}$ and $\{T_e, Q_v : e \in E^1, v \in E^0\}$ are two Cuntz-Krieger E-families in which all the projections P_v and Q_v are nonzero. Then there is an isomorphism ϕ of $C^*(\{S_e, P_v : e \in E^1, v \in E^0\})$ onto $C^*(\{T_e, Q_v : e \in E^1, v \in E^0\})$ such that $\phi(S_e) = T_e$ and $\phi(P_v) = Q_v$ for all $e \in E^1$ and $v \in E^0$.*

When we consider graphs that are not row finite, we need to modify the the definition of a Cuntz-Krieger E-family: for when $|r^{-1}(v)| = \infty$, the right hand side of (1.1.1) is an infinite sum. The solution to this problem was offered by Fowler and Raeburn in [25]. To see how the problem was overcome, we recall Toeplitz algebras of Hilbert bimodules of [25], which cover Toeplitz algebras of arbitrary graphs. For a directed graph E, a *Toeplitz-Cuntz-Krieger E-family* consists of projections $\{Q_v : v \in E^0\}$ and partial isometries $\{T_e : e \in E^1\}$ in a C^*-algebra B satisfying $T_e^* T_e = Q_{s(e)}$ for all $e \in E^1$ and

$$Q_v \geq \sum_{e \in F} T_e T_e^* \text{ for all } v \in E^0 \text{ and finite } F \subseteq r^{-1}(v).^2 \tag{1.1.2}$$

[2] For a, b in a C^*-algebra B, we say $a \geq b$ if $a - b = c^*c$ for some $c \in B$, see [36]. For (1.1.2), we choose $a = Q_v$, $b = \sum_{e \in F} T_e T_e^*$ and $c = Q_v - \sum_{e \in F} T_e T_e^*$.

Meanwhile, the *Toeplitz algebra* $TC^*(E)$ is the C^*-algebra generated by the universal Toeplitz-Cuntz-Krieger E-family $\{t_e, q_v\}$. Every Cuntz-Krieger E-family is a Toeplitz-Cuntz-Krieger E-family.

Like graph algebras, Toeplitz algebras also have a uniqueness theorem.

Theorem 1.1.3 ([25, Theorem 4.1]: The uniqueness theorem for Toeplitz algebras)**.** *Suppose that E is an arbitrary graph and that $\{T_e, Q_v : e \in E^1, v \in E^0\}$ is a Toeplitz-Cuntz-Krieger E-family. Suppose that $\phi_{T,Q}$ is a representation of $TC^*(E)$ such that $\phi_{T,Q}(t_e) = T_e$ and $\phi_{T,Q}(q_v) = Q_v$ for all $e \in E^1$ and $v \in E^0$. Suppose that each Q_v is nonzero and*

$$Q_v \neq \sum_{e \in F} T_e T_e^* \text{ for all } v \in E^0 \text{ and finite } F \subseteq r^{-1}(v).$$

Then $\phi_{T,Q}$ is faithful.

Fowler, Laca and Raeburn [23] introduced C^*-algebras for arbitrary graphs by modifying (1.1.1) into

$$\begin{aligned} P_v &\geq \sum_{e \in F} S_e S_e^* \text{ for all } v \in E^0 \text{ and finite } F \subseteq r^{-1}(v), \text{ and} \\ P_v &= \sum_{e \in r^{-1}(v)} S_e S_e^* \text{ when } 0 < \left|r^{-1}(v)\right| < \infty. \end{aligned} \tag{1.1.3}$$

They also generalised the gauge-invariant uniqueness thoerem (Theorem 1.1.1) and the Cuntz-Krieger uniqueness theorem (Theorem 1.1.2) to arbitrary graphs.

There are interesting relationship between graph algebras and Toeplitz algebras for arbitrary graphs. We have mentioned that every Cuntz-Krieger E-family is a Toeplitz-Cuntz-Krieger E-family. In fact, $C^*(E)$ is the quotient of $TC^*(E)$ by the ideal I generated by

$$\left\{t_v - \sum_{e \in r^{-1}(v)} q_e q_e^* : v \in E^0 \text{ where } r^{-1}(v) \text{ is finite}\right\}$$

where $\{t_e, q_v\}$ is the universal Toeplitz-Cuntz-Krieger E-family. Furthermore, if q denotes the quotient map of $TC^*(E)$ onto $TC^*(E)/I = C^*(E)$, then the family $\{S_e, P_v\}$ defined by

$$S_e := q(t_e) \text{ and } P_v := q(q_v) \text{ for all } v \in E^0 \text{ and } e \in E^1$$

is isomorphic to the universal Cuntz-Krieger E-family. Another interesting result is that for every arbitrary graph E, there exists a graph TE (denoted E_V in [35]) such that the Toeplitz algebra of E is isomorphic to the Cuntz-Krieger algebra of TE [35, Theorem 3.7].

1.2 C^*-algebras and Toeplitz algebras for higher-rank graphs

In [29], Kumjian and Pask introduced *higher-rank graphs* or *k-graphs* and their C^*-algebras. They generalise the higher-rank Cuntz-Krieger algebras of Robertson and Steger [47] in the same way that graph algebras generalise the Cuntz-Krieger algebras of [21].

For a positive integer k, a *k-graph* $\Lambda = (\Lambda^0, \Lambda, r, s)$ is a countable small category Λ with a functor $d : \Lambda \to \mathbb{N}^k$, called the *degree map*, satisfying the *factorisation property*: for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\upsilon$ and $d(\mu) = m$, $d(\nu) = n$. For $m, n \in \mathbb{N}^k$, the expression $m \vee n$ denotes their coordinate-wise maximum and $m \wedge n$ their coordinate-wise minimum. We write Λ^n for the set $d^{-1}(n)$ of *paths with degree n*. The degree is the higher-rank analogue of the length n of a path $e_1 \ldots e_n$ in a directed graph E. We also regard elements of Λ^0 as *vertices*. For $v \in \Lambda^0$, $\lambda \in \Lambda$ and $F \subseteq \Lambda$, we define $vF := \{\mu \in F : r(\mu) = v\}$. For detailed discussion, see Section 2.2.

For a directed graph E, the path space E^* is a 1-graph. It satisfies the factorisation property since, for $\lambda = e_1 \ldots e_n \in E^*$ and $0 < m < n$, the paths $\mu = e_1 \ldots e_m$ and $\nu = e_{m+1} \ldots e_n$ are the only paths such that $\lambda = \mu\upsilon$ and $d(\mu) = m$, $d(\nu) = n - m$.

A k-graph Λ is *row-finite* if for every $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, the set $v\Lambda^n$ is finite. A vertex $v \in \Lambda^0$ is a *source* if there exists $n \in \mathbb{N}^k$ such that $v\Lambda^n = \emptyset$. For a row-finite higher-rank graph Λ with no sources, a family $\{T_\lambda : \lambda \in \Lambda\}$ of partial isometries in a C^*-algebra B is a *Cuntz-Krieger Λ-family* if it satisfies:

(i) $\{T_v : v \in \Lambda^0\}$ is a set of mutually orthogonal projections;

(ii) $T_\lambda T_\mu = T_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$;

(iii) $T_\lambda^* T_\lambda = T_{s(\lambda)}$ for all $\lambda \in \Lambda$; and

(iv) $T_v = \sum_{\lambda \in v\Lambda^n} T_\lambda T_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

Condition (iv) is called the *Cuntz-Krieger relation* and as for directed graphs, $C^*(\Lambda)$ is defined to be the C^*-algebra generated by a universal Cuntz-Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$. Kumjian and Pask showed that $C^*(\Lambda)$ is the closed span of elements in the form $t_\lambda t_\mu^*$ where $s(\lambda) = s(\mu)$ [29, Lemma 3.1] and proved that $C^*(\Lambda)$ carries a strongly continuous gauge action γ of $\mathbb{T}^k$. Using this gauge action, they generalised Theorem 1.1.1 as follows:

Theorem 1.2.1 ([29, Theorem 3.4]: The gauge-invariant uniqueness theorem)**.** *Suppose that Λ is a row-finite k-graph with no sources. Suppose that $\pi : C^*(\Lambda) \to B$ is a homomorphism and that $\beta : \mathbb{T}^k \to \operatorname{Aut} B$ is an action such that $\pi \circ \gamma_z = \beta_z \circ \pi$ for all $z \in \mathbb{T}^k$. Then $\pi(t_v) \neq 0$ for all $v \in \Lambda^0$ if and only if π is faithful.*

Inspired by the analysis of graph algebras in [31], which uses a groupoid approach, Kumjian and Pask also used groupoids to investigate C^*-algebras for k-graphs. They established a higher-rank analogue of Condition (L) and Theorem 1.1.2. To state their result, we need some notation. For $k \in \mathbb{N}$ and $n \in (\mathbb{N} \cup \{\infty\})^k$, the category

$$\Omega_{k,n} := \left\{(p, q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leq q \leq n\right\}$$

is a k-graph with objects $\left\{m \in \mathbb{N}^k : p \leq n\right\}$, range map $r(p, q) = p$, source map $s(p, q) = q$, and degree map $d(p, q) = q - p$ [40, Example 2.2]. Kumjian and Pask then defined

$$\Lambda^\infty := \{x : \Omega_{k,\infty} \to \Lambda : x \text{ is a degree preseving functor}\}$$

and for $n \in \mathbb{N}^k$, wrote $\sigma^n : \Lambda^\infty \to \Lambda^\infty$ for the shift map determined by $\sigma^n(x(m)) = x(n + m)$. Their analogue Condition (L) says that a k-graph Λ satisfies *Condition (A)* if for every $v \in \Lambda^0$, there exists $x \in v\Lambda^\infty$ such that $\sigma^m(x) \neq \sigma^n(x)$ for all $m \neq n \in \mathbb{N}^k$ [29, Definition 3.1]. They then generalised Theorem 1.1.2 as follows:

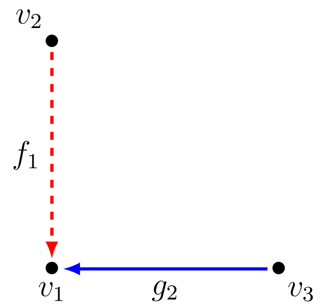
Theorem 1.2.2 ([29, Theorem 4.6]: The Cuntz-Krieger uniqueness theorem)**.** *Suppose that Λ is a row-finite graph with no sources. Suppose that Λ satisfies Condition (A) and that $\pi : C^*(\Lambda) \to B$ is a homomorphism. Then $\pi(t_v) \neq 0$ for all $v \in \Lambda^0$ if and only if π is faithful.*

In [40] Raeburn, Sims and Yeend expanded the theory of C^*-algebras of higher-rank graphs by considering "locally convex" row-finite higher-rank graphs. A k-graph Λ is *locally convex* if for all $v \in \Lambda^0$, $i, j \in \{1, \ldots, k\}$ with $i \neq j$, $\lambda \in v\Lambda^{e_i}$ and $\mu \in v\Lambda^{e_j}$, both $s(\lambda)\Lambda^{e_j}$ and $s(\mu)\Lambda^{e_i}$ are non-empty. Every row-finite higher-rank graph with no sources is locally convex; but locally convex row-finite higher-rank graphs may have sources. For example, the 2-graph

$$\begin{tikzcd}
v_2 \arrow[d, dashed, red, "f_1"'] & v_4 \arrow[l, blue, "g_1"'] \arrow[d, dashed, red, "f_2"] \\
v_1 & v_3 \arrow[l, blue, "g_2"]
\end{tikzcd}$$

is locally convex even though v_1, v_2, v_3 and v_4 are all sources.

For locally convex k-graphs, one only has to make a minor adjustment to the Cuntz-Krieger relation (see [40, Proposition 3.11]). Even for row-finite k-graphs, if they are not locally convex, it is difficult to see what an appropriate Cuntz-Krieger relation (iv) might be. For example, the 2-graph



is problematic (see [40, Example A.1]).

In [39] Raeburn and Sims found a new Cuntz-Krieger relation which works for *finitely aligned* higher-rank graphs; that is, higher-rank graphs such that for all $\lambda, \mu \in \Lambda$, the set

$$\Lambda^{\min}(\lambda, \mu) := \{(\rho, \tau) \in \Lambda \times \Lambda : \lambda\rho = \mu\tau, d(\lambda\rho) = d(\lambda) \vee d(\mu)\} \tag{1.2.1}$$

is finite (possibly empty). The finitely aligned higher-rank graphs include the row-finite higher rank graphs and also some graphs that fail to be row-finite.

For a finitely aligned higher-rank graph Λ, a family of partial isometries $\{Q_\lambda : \lambda \in \Lambda\}$ in a C^*-algebra B is a *Toeplitz-Cuntz-Krieger* Λ*-family* if it satisfies

(TCK1) $\{Q_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;

(TCK2) $Q_\lambda Q_\mu = Q_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$;

(TCK3) $Q_\lambda^* Q_\mu = \sum_{(\rho,\tau)\in\Lambda^{\min}(\lambda,\mu)} Q_\rho Q_\tau^*$ for all $\lambda, \mu \in \Lambda$; and

$$Q_v \geq \sum_{\lambda \in F} Q_\lambda Q_\lambda^* \text{ for all } v \in \Lambda^0, n \in \mathbb{N}^k, \text{ and finite } F \subseteq v\Lambda^n. \tag{1.2.2}$$

The *Toeplitz algebra* $TC^*(\Lambda)$ is defined to be the C^*-algebra generated by a universal Toeplitz-Cuntz-Krieger Λ-family $\{q_\lambda : \lambda \in \Lambda\}$. As we can see, (TCK1) and (TCK2) are same as Condition (i) and (ii), respectively. Meanwhile, (TCK3) generalises Condition (iii); and (1.2.2) generalises (1.1.2) for the directed graph setting[3].

[3] By [41, Lemma 2.7 (iii)], any family of partial isometries satisfying (TCK1-3) holds (1.2.2). Hence (1.2.2) can be omitted to define a Toeplitz-Cuntz-Krieger Λ-family.

The main result of [39] was a higher-rank version of Theorem 1.1.3 [39, Theorem 8.1]. We restate the uniqueness theorem as follows. For a discussion about how we get the following theorem from the original version of [39, Theorem 8.1], see [37, Remark 2.3].

Theorem 1.2.3 (The uniqueness theorem for Toeplitz algebras)**.** *Suppose that* Λ *is a finitely aligned* k*-graph. Suppose that* $\phi : TC^*(\Lambda) \to B$ *is a homomorphism such that for all* $v \in \Lambda^0$ *and finite sets* $F_i \subseteq v\Lambda^{e_i}$*,*

$$\prod_{i=1}^{k} \Big(\phi(q_v) - \sum_{\lambda \in F_i} \phi(q_\lambda) \phi(q_\lambda^*) \Big) \gneqq 0$$

(where this includes $\phi(q_v) \gneqq 0$ *if* $v\Lambda = \{v\}$*). Then* $\phi : TC^*(\Lambda) \to B$ *is injective.*

Theorem 1.2.3 indicates how to define the Cuntz-Krieger algebra of a finitely aligned k-graph. The solution was then implemented by Raeburn, Sims and now with Yeend in [41]. They said that a set $E \subseteq v\Lambda$ is *exhaustive* if for every $\mu \in v\Lambda$, there exists $\lambda \in E$ such that $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$. Then a *Cuntz-Krieger* Λ*-family* for finitely aligned higher-rank graphs is a family of partial isometries $\{T_\lambda : \lambda \in \Lambda\}$ in a C^*-algebra B satisfying (TCK1-3) and

(CK) $\qquad \displaystyle\prod_{\lambda \in E} (T_v - T_\lambda T_\lambda^*) = 0$ for all $v \in \Lambda^0$ and finite exhaustive $E \subseteq v\Lambda$.

Condition (CK) is the *Cuntz-Krieger relation.* In Appendix B of [41], Raeburn, Sims and Yeend showed that in directed graphs, (CK) is equivalent to (1.1.1), and (CK) is the Cuntz-Krieger relation for locally convex higher-rank graphs. Raeburn, Sims and Yeend also generalised the gauge-invariant uniqueness theorem and the Cuntz-Krieger uniqueness theorem to finitely aligned k-graphs (Theorem 4.2 and Theorem 4.5 of [41], respectively).

As for directed graphs, higher-rank graph C^*-algebras and higher-rank graph Toeplitz algebras also have an interesting relationship. For example, every finitely aligned higher-rank graph Λ, $C^*(\Lambda)$ is a quotient of its Toeplitz algebra $TC^*(\Lambda)$. Recently, we showed that for every row-finite higher-rank graph Λ, there is a higher-rank graph $T\Lambda$ such that the Toeplitz algebra of Λ is the C^*-algebra of $T\Lambda$ [37, Theorem 4.1]. This generalises Muhly and Tomforde's [35, Theorem 3.7] and Sims's [51, Lemma 3.5] for directed graphs. Proposition 3.4 of [37] shows that the k-graph $T\Lambda$ is always *aperiodic* in the sense of [33], and the Cuntz-Krieger uniqueness theorem of [33] always applies to $T\Lambda$. This helps explain why no hypothesis on Λ is required in Theorem 1.2.3.

1.3 Leavitt path algebras and Cohn path algebras for graphs

Leavitt path algebras for graphs were developed independently by two groups of mathematicians. The first group, which consists of Ara, Goodearl and Pardo, was motivated by the K-theory of graph algebras [42]. They introduced Leavitt path algebras [10] in order to answer analogous K-theoretic questions about the algebraic Cuntz-Krieger algebras of [7]. On the other hand, Abrams and Aranda Pino introduced Leavitt path algebras in [3] to generalise Leavitt's algebras, specifically the algebras $L_K(1,n)$ of [32]. Both groups defined *Leavitt path algebra* $L_K(E)$ for a row-finite graph E and a field K.

For $e \in E^1$, we call e^* a *ghost path* (e^* is a formal symbol) and we define $(E^1)^* := \{e^* : e \in E^1\}$. We also extend r and s to be defined on $(E^1)^*$ by $r(e^*) := s(e)$ and $s(e^*) := r(e)$. For a row-finite graph E and field K, $L_K(E)$ is the associative K-algebra generated by the set $\{p_v : v \in E^0\}$ and $\{s_e, s_{e^*} : e \in E^1\}$ subject to the following relations:

(V) $p_v p_w = \delta_{v,w} p_v$ for all $v, w \in E^0$,

(E1) $s_e p_{s(e)} = p_{r(e)} s_e = s_e$ for all $e \in E^1$,

(E2) $p_{s(e)} s_{e^*} = s_{e^*} p_{r(e)} = s_{e^*}$ for all $e \in E^1$,

(CK1) $s_{e^*} s_f = \delta_{e,f} p_{s(e)}$ for all $e, f \in E^1$,

(CK2) $p_v = \sum_{e \in r^{-1}(v)} s_e s_{e^*}$ for every $v \in E^0$ which is not a source.

At a glance, we can see the above conditions are algebraic version of the definition of Cuntz-Krieger E-families and so (CK1-2) are the Cuntz-Krieger relations. The only difference is that Leavitt path algebras are K-algebras and not C^*-algebras; and hence elements need not have adjoints. Abrams and Aranda Pino showed that $L_K(E)$ is a $\mathbb{Z}$-graded algebra, with grading induced by

$$L_K(E)_n := \operatorname{span}_K\{s_{e_{i_1}} \dots s_{e_{i_p}} s_{e^*_{j_1}} \dots s_{e^*_{j_q}} : p+q > 0, e_{i_s} \in E^1, e^*_{jt} \in \left(E^1\right)^*, p-q = n\}$$

for $n \neq 0$. The definition of Leavitt path algebras was later expanded by Abrams and Aranda Pino to arbitrary graphs [4]. As for graph algebras, the modification imposes (CK2) only for $v \in E^0$ where $r^{-1}(v)$ is finite and nonempty.

In [54], Tomforde showed that the Leavitt path algebra of [4] is the algebra generated by the universal family satisfying (V), (E1), (E2), (CK1) and (CK2). Tomforde then showed that every Leavitt path algebra is a $\mathbb{Z}$-graded algebra, and formulated a uniqueness theorem as follows:

Theorem 1.3.1 ([54, Theorem 4.8]: The graded uniqueness theorem)**.** *Suppose that E is a graph and that K is a field. Suppose that $L_K(E)$ is the associated Leavitt path algebra with the usual $\mathbb{Z}$-grading. If A is a $\mathbb{Z}$-graded ring and $\pi : L_K(E) \to A$ is a graded ring homomorphism with $\pi(p_v) \neq 0$ for all $v \in E^0$, then π is injective.*

An interesting consequence of this uniqueness theorem is that for $K = \mathbb{C}$, the Leavitt path algebra $L_{\mathbb{C}}(E)$ sits densely inside the graph algebra $C^*(E)$ [54, Theorem 7.3]. Furthemore, this theorem is an algebraic analogue of the gauge-invariant uniqueness theorem for graph algebras.

As for graph algebras, there is a Cuntz-Krieger uniqueness theorem for the Leavitt path algebras of graphs which satisfy Condition (L) [54].

Theorem 1.3.2 ([54, Theorem 6.8]: The Cuntz-Krieger uniqueness theorem)**.** *Suppose that E is a graph which satisfies Condition (L) and that K is a field. Suppose that $L_K(E)$ is the associated Leavitt path algebra. If $\pi : L_K(E) \to A$ is a ring homomorphism with $\pi(p_v) \neq 0$ for all $v \in E^0$, then π is injective.*

Tomforde then improved his result in [55] by replacing a field K with a commutative ring R with 1. His Leavitt path algebras over rings also have universal property, a graded uniqueness theorem [55, Theorem 5.3] and a Cuntz-Krieger uniqueness theorem [55, Theorem 6.5].

For a graph E and a field K, the *Cohn path algebra* $C_K(E)$ was defined and investigated in [6] and [8], generalising the algebras $U_{1,n}$ studied by Cohn in [20]. The algebra $C_K(E)$ is the associative K-algebra generated by the set $\{t_v : v \in E^0\}$ and $\{q_e, q_{e^*} : e \in E^1\}$ satisfying (V), (E1),(E2) and (CK1). The Leavitt path algebra $L_K(E)$ is the quotient of $C_K(E)$ by the ideal generated by

$$\Big\{t_v - \sum_{e \in r^{-1}(v)} q_e q_{e^*} : v \in E^0 \text{ where } r^{-1}(v) \text{ is finite}\Big\}.$$

Furthermore, for every graph E, the graph TE (denoted $E(X)$ in [2]), which satisfies $TC^*(E) \cong C^*(TE)$, also satisfies $C_K(E) \cong L_K(TE)$ [2, Theorem 1.5.18]. However, mathematicians did not study the universal property or the uniqueness theorem for Cohn path algebras.

Up to this point, the properties of both Leavitt path algebras and Cohn path algebras have mirrored graph algebras and Toeplitz algebras, respectively. So it is natural to say that Leavitt path algebras and Cohn path algebras are algebraic versions of graph algebras and Toeplitz algebras, respectively.

1.4 Kumjian-Pask algebras for higher-rank graphs

Aranda Pino, J. Clark, an Huef and Raeburn defined and studied a higher-rank analogue of Leavitt path algebras, which they called Kumjian-Pask algebras [11]. They focused on row-finite k-graphs with no sources. They first introduced ghost paths λ^* for $\lambda \in \Lambda$ (with $v^* := v$ for $v \in \Lambda^0$), and defined $G(\Lambda) := \{\lambda^* : \lambda \in \Lambda\}$. As for Leavitt path algebras, they also extended the range and source maps r and s to be defined on $G(\Lambda)$ by $r(\lambda^*) := s(\lambda)$ and $s(\lambda^*) := r(\lambda)$. For a row-finite k-graph Λ with no sources and commutative ring R with 1, a *Kumjian-Pask* Λ*-family* $\{S_\lambda, S_{\mu^*} : \lambda, u \in \Lambda\}$ in an R-algebra A consists of $S : \Lambda \cup G\left(\Lambda^{\neq 0}\right) \to A$ such that:

(i) $\{S_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal idempotents;

(ii) for $\lambda, \mu \in \Lambda$ with $s(\lambda) = r(\mu)$, we have $S_\lambda S_\mu = S_{\lambda\mu}$ and $S_{\mu^*} S_{\lambda^*} = S_{(\lambda\mu)^*}$;

(iii) $S_{\lambda^*} S_\mu = \delta_{\lambda,\mu} S_{s(\lambda)}$ for all $\lambda, \mu \in \Lambda$ with $d(\lambda) = d(\mu)$; and

(iv) $S_v = \sum_{\lambda \in v\Lambda^n} S_\lambda S_{\lambda^*}$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

The *Kumjian-Pask algebra* $\mathrm{KP}_R(\Lambda)$ is the R-algebra generated by the universal Kumjian-Pask Λ-family and there is a $\mathbb{Z}^k$-grading on $\mathrm{KP}_R(\Lambda)$ satisfying

$$\mathrm{KP}_R(\Lambda)_n := \mathrm{span}_R\{s_\lambda s_{\mu^*} : \lambda, \mu \in \Lambda \text{ and } d(\lambda) - d(\mu) = n\}$$

[11, Theorem 3.4]. The authors of [11] then formulated a uniqueness theorem as follows:

Theorem 1.4.1 ([11, Theorem 4.1]: The graded uniqueness theorem)**.** *Suppose that* Λ *is a row-finite graph with no sources, that* R *is a commutative ring with* 1 *and that* A *is a* $\mathbb{Z}^k$*-graded ring. If* $\pi : \mathrm{KP}_R(\Lambda) \to A$ *is a* $\mathbb{Z}^k$*-graded ring homomorphism such that* $\pi(rs_v) \neq 0$ *for all* $v \in \Lambda^0$ *and* $r \in R \setminus \{0\}$*, then* π *is injective.*

As for Leavitt path algebras, the graded uniqueness theorem for $R = \mathbb{C}$ implies that the Kumjian-Pask algebra $\mathrm{KP}_{\mathbb{C}}(\Lambda)$ is isomorphic to a dense subset of $C^*(\Lambda)$.

Apart from the graded uniqueness theorem, we also have a Cuntz-Krieger uniqueness theorem for Kumjian-Pask algebras. To state this uniqueness theorem, the authors of [11] used an aperiodicity condition which is equivalent to our previous Conditon (A) [46, Lemma 3.2].

Theorem 1.4.2 ([11, Theorem 4.7]: The Cuntz-Krieger uniqueness theorem)**.** *Suppose that Λ is a row-finite graph with no sources which satisfies Condition (A), and that R is a commutative ring with 1 and A a ring. If $\pi : \mathrm{KP}_R(\Lambda) \to A$ is a ring homomorphism such that $\pi(rs_v) \neq 0$ for all $v \in \Lambda^0$ and $r \in R\setminus\{0\}$, then π is injective.*

As for higher-rank graph C^*-algebras, Kumjian-Pask algebras were defined for locally convex row-finite k-graphs by Clark, Flynn and an Huef in [16][4]. They also showed the universal property and proved uniqueness theorems for their Kumjian-Pask algebras.

1.5 Main results of this thesis

The main results of this thesis extend the definition of Kumjian-Pask algebras to finitely aligned k-graphs over a commutative ring with 1 (Chapter 3), and introduce Cohn path algebras for k-graphs over a commutative ring with 1 (Chapter 4). These results are taken from joint work with my supervisor Clark [18, 19].

By exploiting the similarity between Cuntz-Krieger algebras and Leavitt path algebras (and their generalisation), we arrive at a candidate for the Kumjian-Pask algebras of finitely aligned k-graphs. We then need to show that this candidate has a graded uniqueness theorem and a Cuntz-Krieger uniqueness theorem. This is the first main innovation of this thesis.

For our graded uniqueness theorem (Theorem 3.2.1), we use an algebraic version of the C^*-algebra proof of the gauge-invariant uniqueness theorem for finitely aligned k-graph C^*-algebras [41, Theorem 4.2]. However, this method does not work to show the Cuntz-Krieger uniqueness theorem and so we use a groupoid model. We first introduce the boundary-path groupoid $\mathcal{G}_\Lambda$ of Λ (see Example 2.8.3). Then using the graded uniqueness theorem, we show that the Kumjian-Pask algebra $\mathrm{KP}_R(\Lambda)$ is isomorphic to the Steinberg algebra $A_R(\mathcal{G}_\Lambda)$ (Proposition 3.4.1). We then use this isomorphism to apply results about Steinberg algebras to Kumjian-Pask algebras.

In Section 3.5 and Section 3.6, we study the relationship between properties of Λ and properties of $\mathcal{G}_\Lambda$. In Proposition 3.5.1, we show that a finitely aligned higher-rank

[4]The graded uniqueness theorem and the Cuntz-Krieger uniqueness theorem of [16] required the homomorphism π to be an R-algebra homomorphism. However, this hypothesis can be relaxed to be a ring homomorphism and no changes in the proofs are required.

graph Λ is *aperiodic* if and only if the boundary-path groupoid $\mathcal{G}_\Lambda$ is *effective.* In Proposition 3.6.1, we show that a higher-rank graph Λ is *cofinal* if and only if $\mathcal{G}_\Lambda$ is *minimal.*

In Section 3.7, we use the Cuntz-Krieger theorem for Steinberg algebras [14, Theorem 3.2] to prove the Cuntz-Krieger uniqueness theorem for Kumjian-Pask algebras (Theorem 3.7.1). Since the Cuntz-Krieger theorem for Steinberg algebras only applies to effective groupoids, the aperiodicity condition of Λ ensures that we can apply the uniqueness theorem.

In Section 3.8, we give necessary and sufficient conditions for $\mathrm{KP}_R(\Lambda)$ to be basically simple in Theorem 3.8.3 and simple in Theorem 3.8.4. We show these results by applying the characterisation of basic simplicity and simplicity of the Steinberg algebra $A_R(\mathcal{G}_\Lambda)$ (see Theorem 4.1 and Corollary 4.6 of [14]).

Note that our proofs of the uniqueness theorems for Kumjian-Pask algebras are different from those used for Kumjian-Pask algebras for row-finite k-graphs with no sources [11] and locally convex row-finite k-graphs [16]. Hence this thesis gives new proofs of the uniqueness theorems in [11, 16].

We then turn our attention to row-finite k-graphs with no sources. On this class of k-graphs, we introduce a Cohn path Λ-family and study its properties in Section 4.1.

The next main achievement of this thesis is to establish a uniqueness theorem for Cohn path algebras (Theorem 4.2.1). Our strategy is to follow the analysis of [37]. In that paper, the author shows that for every row-finite higher-rank graph Λ, there exists a higher-rank graph $T\Lambda$ such that the Toeplitz algebra of Λ is isomorphic to the C^*-algebra of $T\Lambda$. Here we show that the Cohn path algebra of Λ is isomorphic to the Kumjian-Pask algebra of $T\Lambda$ (Theorem 4.2.16(a)). We then apply the Cuntz-Krieger uniqueness theorem for Kumjian-Pask algebras (Theorem 3.7.1) to prove Theorem 4.2.1. Another consequence of this isomorphism is that every Cohn path algebra is $\mathbb{Z}^k$-graded (Theorem 4.2.16(b)).

Finally, we discuss examples and applications in Section 4.3. We explicitly demonstrate the relationship between Cohn path algebras and Toeplitz algebras (Proposition 4.3.1), and show that our Cohn algebras can be realised as Steinberg algebras (Proposition 4.3.4).

Chapter 2

Preliminaries

In this chapter, we give a more detailed discussion of k-graphs, graded rings and groupoids. We also give examples in order to help readers to get a better understanding.

A *directed graph* E consists of a countable vertex set E^0, a countable edge set E^1, and range and source maps $r, s : E^1 \to E^0$ which indicate the direction of the edges.

For a positive integer k, we view $\mathbb{N}^k$ as an additive semigroup with identity 0. For $n \in \mathbb{N}^k$, we write $n = (n_1, \ldots, n_k)$. For $m, n \in \mathbb{N}^k$, we write $m \leq n$ to mean $m_i \leq n_i$ for $1 \leq i \leq k$, and we use the expression $m \vee n$ for their coordinate-wise maximum, and $m \wedge n$ for their coordinate-wise minimum. We also write e_i for the usual basis elements in $\mathbb{N}^k$.

2.1 Basics of categories

A k-graph and a groupoid are defined in terms of category theory, so in this section, we establish basic notations and definitions of categories. We use the definitions of [34, Chapter 1] and [43, Chapter 10].

For our purposes, a *category*[1] $\mathcal{C}$ consists of two sets $\mathcal{C}^0$ and $\mathcal{C}^*$, two functions $r, s : \mathcal{C}^* \to \mathcal{C}^0$, a partially defined product $(a, b) \mapsto ab$ from

$$\{(a, b) \in \mathcal{C}^* \times \mathcal{C}^* : s(a) = r(b)\}$$

to $\mathcal{C}^*$, and distinguished elements $\{\iota_v \in \mathcal{C}^* : v \in \mathcal{C}^0\}$ such that

- $r(ab) = r(a)$ and $s(ab) = s(b)$ for all $a, b \in \mathcal{C}^*$;

[1] In category-theory books, our notion of category is called a *small category*.

- $(ab)\,c = a\,(bc)$ when $s\,(a) = r\,(b)$ and $s\,(b) = r\,(c)$ for all $a, b, c \in \mathcal{C}^*$;

- $r\,(\iota_v) = v = r\,(\iota_v)$ and $\iota_v a = a, b\iota_v = b$ when $r\,(a) = v$ and $s\,(b) = v$.

We call the elements of $\mathcal{C}^0$ the *objects* of the category; and the elements of $\mathcal{C}^*$ the *morphisms* of $\mathcal{C}$. For $a \in \mathcal{C}^*$, we call $r\,(a)$ and $s\,(a)$ the *codomain* and *domain* of a, respectively. The function $(a, b) \mapsto ab$ is called *composition*, and for $v \in \mathcal{C}^0$, ι_v is called the *identity morphism on the object* v. We also say that a category $\mathcal{C}$ is countable if $\mathcal{C}^*$ is countable.

For example, we can view the semigroup $(\mathbb{N}^k, +)$ as a category with a single object o, morphisms $\mathbb{N}^k$, $r\,(n) = o = s\,(n)$ for all $n \in \mathbb{N}^k$, $\iota_o = 0$ and operation $(m, n) \mapsto m + n$ [49, Example 2.1.2].

Given categories $\mathcal{C}$ and $\mathcal{B}$, a *functor*[2] $T : \mathcal{C} \to \mathcal{B}$ consists of two functions (both denoted T): $T : \mathcal{C}^0 \to \mathcal{B}^0$ and $T : \mathcal{C}^* \to \mathcal{B}^*$ which satisfy:

- $s\,(T\,(a)) = T\,(s\,(a))$ and $r\,(T\,(a)) = T\,(r\,(a))$ for all $a \in \mathcal{C}^*$;

- $T\,(\iota_v) = \iota_{T(v)}$ for all $v \in \mathcal{C}^0$;

- $T\,(a)\,T\,(b) = T\,(ab)$ for all $a, b \in \mathcal{C}^*$ with $s\,(a) = r\,(b)$.

The motivating example is:

Example 2.1.1. Let $E = (E^0, E^1, r, s)$ be a directed graph. For a positive integer n, we set

$$E^n := \{\mu_1\mu_2 \cdots \mu_n : n \in \mathbb{N}, \mu_i \in E^1, r\,(\mu_{i+1}) = s\,(\mu_i) \text{ for all } i \leq n-1\}$$

and define $r, s : E^n \to E^0$ by

$$s\,(\mu) = s\,(\mu_n) \text{ and } r\,(\mu) = r\,(\mu_1)$$

for $\mu = \mu_1\mu_2 \cdots \mu_n \in E^n$. We define composition as follows: for $\mu_1 \cdots \mu_n, \gamma_1 \cdots \gamma_m \in E^*$ with $s\,(\mu_n) = r\,(\gamma_1)$,

$$(\mu_1 \cdots \mu_n, \gamma_1 \cdots \gamma_m) \mapsto \mu_1 \cdots \mu_n\gamma_1 \cdots \gamma_m$$

$$(r\,(\mu_1)\,, \mu_1 \cdots \mu_n) \mapsto \mu_1 \cdots \mu_n \text{ and } (\mu_1 \cdots \mu_n, s\,(\mu_n)) \mapsto \mu_1 \cdots \mu_n.$$

Finally, for $v \in E^0$, let v be its identity morphism. Then $E^* := \bigcup_{n\in\mathbb{N}} E^n$ is a category with the set of objects E^0 and the set of morphisms E^*. The map $d : E^* \to (\mathbb{N}, +)$ which satisfies $d\,(v) = 0$ for $v \in E^0 \subseteq E^*$ and $d\,(\mu_1 \cdots \mu_n) = n$ for $\mu_1 \cdots \mu_n \in E^* \setminus E^0$, is a functor, called the *degree functor*.

[2]As for Footnote 2.1, our notion of functor is a *covariant functor*.

2.2 Basic definitions of higher-rank graphs

A *higher-rank graph* or *k-graph* Λ is a countable category Λ with a functor d from Λ to $\mathbb{N}^k = (\mathbb{N}^k, +)$, called the *degree map*, which satisfies the *factorisation property*:

> for every morphism λ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique morphisms μ, ν such that $\lambda = \mu v$ and $d(\mu) = m, d(\nu) = n$.

We then write $\lambda(0, m)$ for μ and $\lambda(m, m+n)$ for ν. For every morphism λ, we have $d(\lambda) \in \mathbb{N}^k$ and call that λ is a *finite path*. We discuss *infinite paths* in Section 2.5.

To simplify the notation, from now on, we write Λ for the set Λ^* of morphisms of Λ and identify the objects with the identity morphism.

Example 2.2.1 ([40, Example 2.2(ii)]). Let $k \in \mathbb{N}$ and $n \in (\mathbb{N} \cup \{\infty\})^k$. We define

$$\Omega_{k,n}^0 := \left\{p \in \mathbb{N}^k : p \leq n\right\},$$

$$\Omega_{k,n} := \left\{(p, q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leq q \leq n\right\},$$

$$s(p, q) := q,\ r(p, q) := p, \text{ and } \iota_p := (p, p).$$

Then $\Omega_{k,n}$ is a countable category. We define $d(p, q) = q - p$, and then $\Omega_{k,n}$ becomes a k-graph.

Since d is a functor and $\mathbb{N}^k$ is a category with one object O, then for $v \in \Lambda^0$, we have $d(\iota_v) = 0$. Take $v \in \Lambda^0$ and $\lambda \in \Lambda$ with $d(\lambda) = 0$ and $r(\lambda) = v$. Since $d(\lambda) = 0 + 0$, then by the factorisation property, there exist unique $\mu, \nu \in \Lambda$ such that $d(\mu) = d(\nu) = 0$ and $\mu\nu = \lambda$. Then (μ, ν) is either (λ, ι_v) or (ι_v, λ); and the uniqueness of factorisations ensures $\lambda = \iota_v$. So

$$\{\lambda \in \Lambda : d(\lambda) = 0\} = \left\{\iota_v : v \in \Lambda^0\right\}. \tag{2.2.1}$$

Now for $n \in \mathbb{N}^k$, we write

$$\Lambda^n := \{\lambda \in \Lambda : d(\lambda) = n\}$$

and so we call its elements *paths with degree n*. This is consistent notation when $n = 0$ (2.2.1). In particular, we call elements of Λ^0 *vertices*. We also use the term *edge* to denote a path $e \in \Lambda^{e_i}$ where $1 \leq i \leq k$, and write

$$\Lambda^1 := \bigcup_{1 \leq i \leq k} \Lambda^{e_i}$$

for the set of edges. For $\lambda \in \Lambda$, we call $s(\lambda)$ and $r(\lambda)$ the *source and range of* λ, respectively. For $v \in \Lambda^0$, $\lambda \in \Lambda$ and $E \subseteq \Lambda$, we define

$$\begin{aligned} vE &:= \{\mu \in E : r(\mu) = v\}, \\ \lambda E &:= \{\lambda\mu \in \Lambda : \mu \in E, r(\mu) = s(\lambda)\}, \\ E\lambda &:= \{\mu\lambda \in \Lambda : \mu \in E, s(\mu) = r(\lambda)\}. \end{aligned}$$

When $k = 1$, for every directed graph $E = (E^0, E^1, r, s)$, the category E^* as in Example 2.1.1 is a 1-graph.

2.3 Visualising higher-rank graphs using skeletons

To study a k-graph, it is helpful to depict it in terms of its *skeleton*, which is the graph with vertex set Λ^0, edge set $\bigcup_{i=1}^k \Lambda^{e_i}$, range and source maps inherited from Λ, and with the edges of different degrees distinguished using k different colours. Note that skeletons are *k-coloured graphs* [26].

Let $\mathbb{F}_k$ be the free semigroup on k-generators $\{c_1, \ldots, c_k\}$. A *k-coloured graph* is a graph E together with a map $c : E^1 \to \{c_1, \ldots, c_k\}$, which we extend to a functor $c : E^* \to \mathbb{F}_k^+$ by setting

$$c(\mu_1 \cdots \mu_n) := c(\mu_1) \cdots c(\mu_n) \text{ and } c(v) := 0$$

for all $\mu_1 \cdots \mu_n \in E^* \setminus E^0$ and $v \in E^0$.

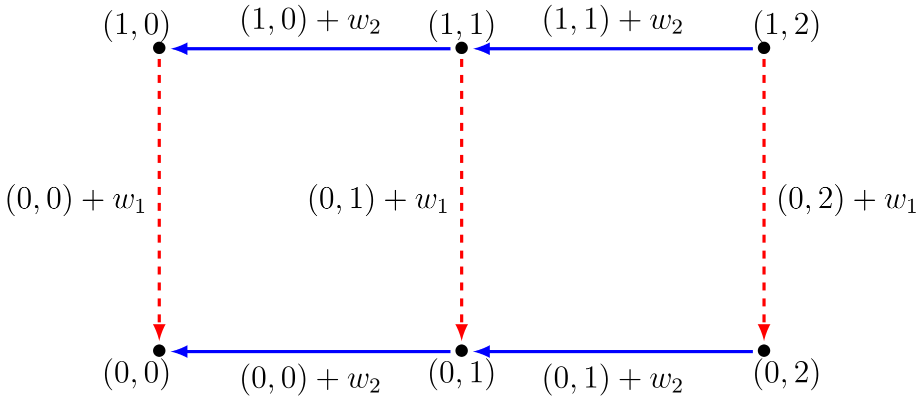
The following example describes a k-coloured graph which will be used to study k-graphs. In this example, $m + w_i$ is a formal symbol to denote an edge of colour c_i pointing from $m + e_i$ to m.

Example 2.3.1 ([26, Example 3.1]). For $n \in (\mathbb{N} \cup \{\infty\})^k$, we define a coloured graph $E_{k,n}$ by

$$E_{k,n}^0 := \left\{m \in \mathbb{N}^k : m \leq n\right\}, \; E_{k,n}^1 := \left\{m + w_i : m, m + e_i \in E_{k,n}^0\right\},$$

$$r(m + w_i) := m, \; s(m + w_i) = m + e_i \text{ and } c(m + w_i) = c_i.$$

For example, we draw $E_{2,(1,2)}$:



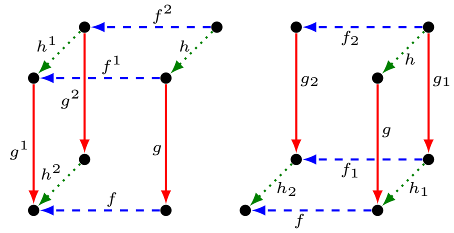
A *graph morphism* φ from a graph E to a graph F is a pair of maps $\varphi^0 : E^0 \to F^0$ and $\varphi^1 : E^1 \to F^1$ such that $r(\varphi^1(e)) = \varphi^0(r(e))$ and $s(\varphi^1(e)) = \varphi^0(s(e))$ for all $e \in E^1$. To simplify the notation, we write φ for each φ^0 and φ^1. A *coloured-graph morphism* is a graph morphism which preserves colour.

Now suppose that E is a k-coloured graph. For distinct $i, j \in \{1, \dots, k\}$, an $\{i,j\}$-*square* (or just a *square*) in a k-coloured graph E is a coloured-graph morphism $\varphi : E_{k,e_i+e_j} \to E$. If $\varphi : E_{k,n} \to E$ is a coloured-graph morphism and ϕ is a square in E, then ϕ *occurs in* φ if there exists $m \in \mathbb{N}^k$ such that $\phi(x) = \varphi(x+m)$ for all $x \in E_{k,e_i+e_j}$.

A *complete collection of squares* is a collection $\mathcal{C}$ of squares in E such that for each $fg \in E^*$ with $c(f) = c_i$, $c(g) = c_j$ and $i \neq j$, there exists a unique $\varphi \in \mathcal{C}$ such that $\varphi(w_i) = f$ and $\varphi(e_i + w_j) = g$. We then write $\varphi(w_i)\varphi(e_i+w_j) \sim \varphi(w_j)\varphi(e_j+w_i)$. So for every c_ic_j-coloured path $x \in E^*$, there is a unique c_jc_i-coloured path $y \in E^*$ such that $x \sim y$. We also say a complete collection of squares $\mathcal{C}$ is *associative* if for every path fgh in E such that f, g, h are edges of distinct colour, the edges $f_1, f_2, g_1, g_2, h_1, h_2$ and $f^1, f^2, g^1, g^2, h^1, h^2$ determined by

$$fg \sim g^1 f^1,\ f^1 h \sim h^1 f^2, \text{ and } g^1 h^1 \sim h^2 g^2$$

$$gh \sim h_1 g_1,\ f h_1 \sim h_2 f_1, \text{ and } f_1 g_1 \sim g_2 f_2$$



satisfy $f^2 = f_2, g^2 = g_2$ and $h^2 = h_2$. If $\mathcal{C}$ is a complete and associative collection of squares in E, we say that a coloured-graph morphism $\varphi : E_{k,n} \to E$ is $\mathcal{C}$-*compatible* if every square occuring in φ belongs to $\mathcal{C}$.

Now we construct a k-graph from a k-coloured graph as follows:

Notation 2.3.2 ([26, Notation 4.3])**.** Let E be a k-coloured graph and let $\mathcal{C}$ be a complete collection of squares in E that is associative. For each $n \in \mathbb{N}^k$, we write $\Lambda^n_{(E,\mathcal{C})}$ for the set of all $\mathcal{C}$-compatible coloured-graph morphisms $E_{k,n} \to E$. Let

$$\Lambda_{(E,\mathcal{C})} := \bigcup_{n \in \mathbb{N}^k} \Lambda^n_{(E,\mathcal{C})}.$$

Let $d : \Lambda_{(E,\mathcal{C})} \to \mathbb{N}^k$ and $r, s : \Lambda_{(E,\mathcal{C})} \to \Lambda^{(0)}_{(E,\mathcal{C})}$ be as defined in Example 2.3.1. For $v \in E^0$, we define $\varphi_v : E_{k,0} \to E$ by $\varphi_v(0) = v$; and for $1 \leq i \leq k$ and $f \in E^1$ with $c(f) = c_i$, we define $\varphi_f : E_{k,e_i} \to E$ by $\varphi_f(0) = r(f)$, $\varphi_f(e_i) = s(f)$ and $\varphi_f(0 + w_i) = f$.

Theorem 1.9 of [52] says that for a k-coloured graph E and a complete collection of squares $\mathcal{C}$ in E, there is a unique k-graph $\Lambda = \Lambda_{(E,\mathcal{C})}$ such that $\Lambda^{e_i} = c^{-1}(c_i)$ for every i and

$$fg = g'f' \text{ in } \Lambda \text{ if and only if } fg \sim g'f' \text{ in } E$$

for every $fg \in \Lambda^{e_i+e_j}$ with distinct $i, j \in \{1, \dots, k\}$. We then call E the *skeleton of* Λ. For further discussion about the construction of the reverse direction, which is constructing the skeleton from a k-graph, see [26, Definition 4.1].

An interesting consequence of [52, Theorem 1.9] is that the factorisation property of paths in $\Lambda^{e_i+e_j}$ for all distinct $i, j \in \{1, \dots, k\}$ determines the factorisation property of all paths in Λ. Hence, to depict a k-graph, it suffices to give its skeleton and its complete collection of squares that is associative.

For example, the skeleton of the 2-graph $\Omega_{2,(1,2)}$ as in Example 2.2.1 is the 2-coloured graph $E_{2,(1,2)}$ as in Example 2.3.1 where

$$(m + w_1)(m + e_1 + w_2) = (m + w_2)(m + e_2 + w_1)$$

for $m \in \{(0,0), (0,1)\}$.

2.4 Row-finite and finitely aligned higher-rank graphs

We briefly introduced higher-rank graphs in Section 1.2. In this section, we give further discussion and examples to help readers get better understanding.

Let Λ be a k-graph and $\lambda, \mu \in \Lambda$. Then $\tau \in \Lambda$ is a *minimal common extension* of λ and μ if

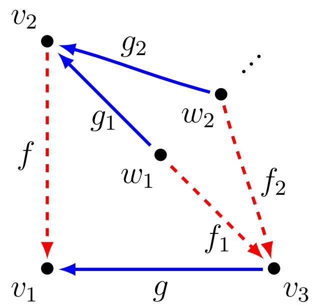
$$d(\tau) = d(\lambda) \vee d(\mu), \ \tau(0, d(\lambda)) = \lambda \text{ and } \tau(0, d(\mu)) = \mu.$$

Let $\mathrm{MCE}(\lambda, \mu)$ denote the set of all minimal common extensions of λ and μ. The map $(\rho, \tau) \mapsto \lambda\rho$ is a bijection of the set $\Lambda^{\min}(\lambda, \mu)$ of (1.2.1) onto $\mathrm{MCE}(\lambda, \mu)$, with inverse given by $\lambda\rho \mapsto (\rho, (\lambda\rho)(d(\mu), d(\lambda\rho)))$.

For $E \subseteq \Lambda$ and $\lambda \in \Lambda$, we write

$$\mathrm{Ext}(\lambda; E) := \bigcup_{\mu \in E} \{\rho : (\rho, \tau) \in \Lambda^{\min}(\lambda, \mu)\}.$$

Example 2.4.1. Consider the 2-graph Λ with skeleton



where $fg_i = gf_i$ for all positive integers i, dashed edges have degree $(1, 0)$ and solid edges have degree $(0, 1)$. Then

$$\mathrm{MCE}(f, g) = \{fg_i : i \in \mathbb{N} \setminus \{0\}\} \text{ and } \Lambda^{\min}(f, g) = \{(g_i, f_i) : i \in \mathbb{N} \setminus \{0\}\}.$$

We also have

$$\mathrm{Ext}(f; \{f\}) = \{v_2\}, \; \mathrm{Ext}(f; \{g\}) = \{g_i : i \in \mathbb{N} \setminus \{0\}\}, \text{ and } \mathrm{Ext}(f; \{gf_1\}) = \{g_1\}.$$

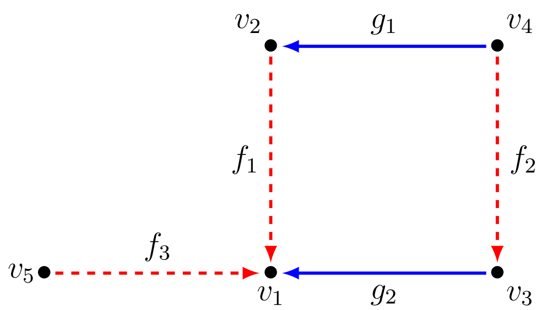
Now recall from page 8 that a set $E \subseteq v\Lambda$ is *exhaustive* if for every $\lambda \in v\Lambda$, there exists $\mu \in E$ such that $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$. Alternatively, we also could say that $E \subseteq v\Lambda$ is exhaustive if for every $\lambda \in v\Lambda$, $\mathrm{Ext}(\lambda; E) \neq \emptyset$.

Next we define

$$\mathrm{FE}(\Lambda) := \bigcup_{v \in \Lambda^0} \Big\{E \subseteq v\Lambda \setminus \{v\} : E \text{ is finite and exhaustive}\Big\}.$$

For $E \in \mathrm{FE}(\Lambda)$, we write $r(E)$ for the vertex v which satisfies $E \subseteq v\Lambda$.

Example 2.4.2. Consider the 2-graph Λ which has skeleton



where $f_1 g_1 = g_2 f_2$, dashed edges have degree $(1,0)$ and solid edges have degree $(0,1)$. First note that since $\Lambda^{\min}(\lambda, f_3) \neq \emptyset$ for all $\lambda \in v_1\Lambda \backslash \{v_1, f_3\}$, then for every $E \in v_1 \operatorname{FE}(\Lambda)$, E must contain f_3. Furthermore, since $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$ for all $\lambda \in v_1\Lambda \backslash \{f_3\}$ and $\mu \in \{f_1, g_2, f_1 g_1\}$, then for every $E \in v_1 \operatorname{FE}(\Lambda)$, E also must contain at least one of $f_1, g_2, f_1 g_1$. Then

$$v_1 \operatorname{FE}(\Lambda) = \left\{ \begin{array}{c} \{f_1, f_3\}, \{g_2, f_3\}, \{f_1, g_2, f_3\}, \{f_1 g_1, f_3\}, \\ \{f_1, f_1 g_1, f_3\}, \{g_2, f_1 g_1, f_3\}, \{f_1, g_2, f_1 g_1, f_3\} \end{array} \right\},$$

$$v_2 \operatorname{FE}(\Lambda) = \{\{g_1\}\}, \ v_3 \operatorname{FE}(\Lambda) = \{\{f_2\}\}, \text{ and } v_4 \operatorname{FE}(\Lambda) = \emptyset.$$

We say that Λ is *finitely aligned* if $\operatorname{MCE}(\lambda, \mu)$ is finite (possibly empty) for all $\lambda, \mu \in \Lambda$. Since there is a bijection between $\Lambda^{\min}(\lambda, \mu)$ and $\operatorname{MCE}(\lambda, \mu)$, then we also could say that Λ is finitely aligned if $\Lambda^{\min}(\lambda, \mu)$ is finite (possibly empty) for all $\lambda, \mu \in \Lambda$.

For a directed graph E and paths $\lambda, \mu \in E^*$, we have

$$\operatorname{MCE}(\lambda, \mu) = \begin{cases} \{\lambda\} & \text{if } \lambda(0, d(\mu)) = \mu, \\ \{\mu\} & \text{if } \mu(0, d(\lambda)) = \lambda, \\ \emptyset & \text{otherwise;} \end{cases}$$

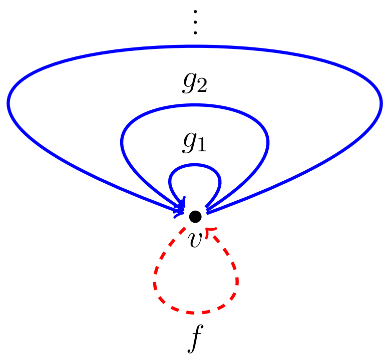
and so $|\operatorname{MCE}(\lambda, \mu)|$ is either 0 or 1. Therefore, every 1-graph is finitely aligned.

Recall from page 5 that a k-graph Λ is *row-finite* if $v\Lambda^n$ is finite for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. For all $\lambda, \mu \in \Lambda$, we have

$$\left|\Lambda^{\min}(\lambda, \mu)\right| = |\operatorname{MCE}(\lambda, \mu)| \leq \left|r(\lambda)\Lambda^{d(\lambda) \vee d(\mu)}\right|.$$

Hence every row-finite k-graph Λ is finitely aligned. On the other hand, a finitely aligned k-graph Λ is not necessarily row-finite.

Example 2.4.3. Consider the 2-graph Λ with skeleton



and $fg_i = g_i f$ for all positive integers i, the dashed edge has degree $(1,0)$ and solid edges have degree $(0,1)$. The 2-graph Λ has infinitely many edges g_i. It is not row-finite because $|v\Lambda^{(0,1)}| = \infty$. On the other hand, for $\lambda, \mu \in \Lambda$, $\left|\Lambda^{\min}(\lambda, \mu)\right|$ is either 0 or 1, and hence Λ is finitely aligned.

Following [29, Definition 1.4], a k-graph Λ has *no sources* if $v\Lambda^n$ is nonempty for every $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

Now consider the 2-graph Λ in Example 2.4.2. Since v_5 does not receive edges of degree $(0,1)$, v_5 is a source of Λ. Furthermore, Λ fails to be locally-convex (see page 6) since $e_3 \in v_1\Lambda^{(1,0)}, f_2 \in v_1\Lambda^{(0,1)}$ but $s(e_3)\Lambda^{(0,1)} = \emptyset$. On the other hand, Λ is row-finite and hence is finitely aligned.

Next consider the 2-graph Λ as Example 2.4.1. Since $\left|\Lambda^{\min}(e, f)\right| = \infty$, Λ is not finitely aligned.

To summarise, finitely aligned k-graphs generalise both row-finite k-graphs with no sources and locally convex row-finite k-graphs, but not every k-graph is finitely aligned.

2.5 The boundary-path space and the path space

Throughout this section, we suppose that Λ is a finitely aligned k-graph. For a positive integer k and $n \in (\mathbb{N} \cup \{\infty\})^k$, we consider the k-graph $\Omega_{k,n}$ of Example 2.2.1.

We define

$$W_\Lambda := \bigcup_{n \in (\mathbb{N}\cup\{\infty\})^k} \{x : \Omega_{k,n} \to \Lambda : x \text{ is a degree preseving functor}\}$$

and call W_Λ the *path space* of Λ. For any finite path $\lambda \in \Lambda$, we can view λ as an element of W_Λ by viewing it as a degree preserving functor from $\Omega_{k,d(\lambda)}$ to Λ. This map is well-defined by the unique factorisation property. Thus W_Λ contains all finite and inifinite paths of Λ.

For example, for the k-graph Λ of Example 2.4.2, $W_\Lambda = \Lambda^{e_1} \cup \Lambda^{e_2} \cup \Lambda^{e_1+e_2}$ where we identify elements of W_Λ with their image in Λ, and then W_Λ is the set of all finite paths of Λ. For the k-graph Λ of Example 2.4.3, W_Λ also contains infinite paths, such as $ef_1f_2f_3\cdots$.

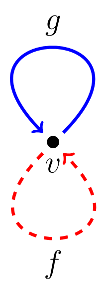
Finite paths and infinite paths are fundamentally different objects and so, it is not obvious to compose them. Webster showed how to compose finite and infinite paths.

Proposition 2.5.1 ([56, Proposition 3.0.11])**.** *Suppose that Λ is a k-graph. Suppose $\lambda \in \Lambda$ and $x \in W_\Lambda$ satisfies $r(x) = s(\lambda)$. Then there exists a unique k-graph morphism*

$\lambda x : \Omega_{k,d(\lambda)+d(x)} \to \Lambda$ *such that* $(\lambda x)(0, d(\lambda)) = \lambda$ *and* $(\lambda x)(d(\lambda), d(\lambda) + n) = x(0, n)$ *for all* $n \leq d(x)$.

Now suppose $x \in W_\Lambda$. For $n \in \mathbb{N}^k$ and $n \leq d(x)$, the path $\sigma^n x$ is defined by $\sigma^n x(0, m) = x(n, n+m)$ for all $m \leq d(x) - n$. We also write $x(n)$ for the vertex $x(n, n)$. Then the range of the path x is the vertex $r(x) := x(0)$.

Example 2.5.2. Consider the 2-graph Λ which has skeleton



where $fg = gf$, the dashed edge has degree $(1, 0)$ and the solid edge has degree $(0, 1)$. Choose the path $x = fgfg\ldots$, then

$$\sigma^{(3,0)}(x) = gggfgfg\ldots \text{ and } \sigma^{(1,1)}(x) = fgfg\ldots.$$

As in [40, Definition 3.1], for $n \in \mathbb{N}^k$ we define

$$\Lambda^{\leq n} := \{\lambda \in \Lambda : d(\lambda) \leq n \text{ and } d(\lambda)_i < n_i \text{ imply } s(\lambda)\Lambda^{e_i} = \emptyset\}. \tag{2.5.1}$$

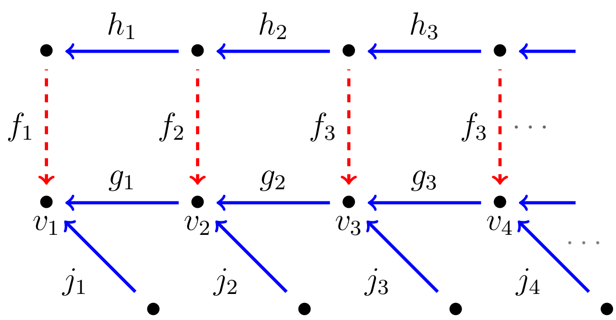
Note that $v\Lambda^{\leq n} \neq \emptyset$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. This is because v is contained in $v\Lambda^{\leq n}$ whenever $v\Lambda^{\leq n}$ has no non-trivial paths of degree less than or equal to n. If Λ has no sources, then for $n \in \mathbb{N}^k$, we have $\Lambda^{\leq n} = \Lambda^n$. However, in general, we have $\Lambda^n \subseteq \Lambda^{\leq n}$ and the two can be different, see the following example.

Consider the k-graph Λ in Example 2.4.2. Because there is no path with degree $(2, 0)$, then $\Lambda^{(2,0)} = \emptyset$, $\Lambda^{\leq(2,0)} = \Lambda^{(1,0)}$ and hence $\Lambda^{(2,0)} \subsetneq \Lambda^{\leq(2,0)}$. Here $\Lambda^{\leq(1,1)} = \Lambda^{(1,1)}$.

Following [24, Definition 5.10], we say that $x \in W_\Lambda$ is a *boundary path* of Λ if for every $m \in \mathbb{N}^k$ with $m \leq d(x)$ and $E \in x(m)\mathrm{FE}(\Lambda)$, there exists $\lambda \in E$ such that $x(m, m + d(\lambda)) = \lambda$. We write $\partial\Lambda$ for the set of all boundary paths. Note that for $v \in \Lambda^0$, $v\partial\Lambda$ is nonempty [24, Lemma 5.15].

Remark 2.5.3. For locally convex graphs, the set $\Lambda^{\leq\infty}$ (as defined in [40, Definition 3.14]) is the same as $\partial\Lambda$ [57, Proposition 2.12]. However, more generally, $\Lambda^{\leq\infty} \subseteq \partial\Lambda$, and the two can be different (see [57, Example 2.11] as follows).

Example 2.5.4 ([57, Example 2.11]). Suppose that Λ is the 2-graph with the skeleton



where $f_i h_i = g_i f_{i+1}$ for all positive integers i, dashed edges have degree $(1,0)$ and solid edges have degree $(0,1)$. Consider the path $x = g_1 g_2 \cdots$. We claim that $x \in \partial\Lambda$. Fix a positive integer m and $E \in v_m \operatorname{FE}(\Lambda)$. We have to show that there exists $\lambda \in E$ such that $x\,(m, m + d\,(\lambda)) = \lambda$. Since E is exhaustive, for each $n \geq m$, there exists $\lambda_n \in E$ such that $\operatorname{MCE}(\lambda_n, g_m \cdots g_{n-1} j_n) \neq \emptyset$. Since E is finite, it can not contain $g_m \cdots g_{n-1} j_n$ for every $n \geq m$, so it must contain $g_m \cdots g_p$ for some $p \in \mathbb{N}$. So $x((0,m),(0,p)) = g_m \cdots g_p$ belongs to E and $x \in \partial\Lambda$. On the other hand, $x \notin \Lambda^{\leq\infty}$ (see [57, Example 2.11]).

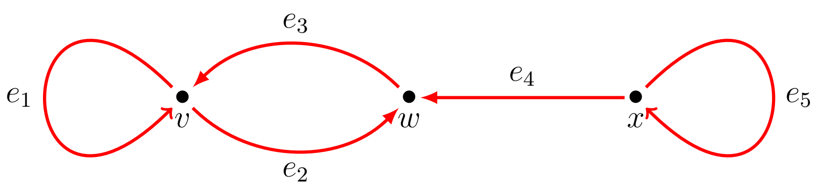
Take $x \in \partial\Lambda$. For $n \in \mathbb{N}^k$ and $n \leq d\,(x)$, the path $\sigma^n x$ belongs to $\partial\Lambda$ [24, Lemma 5.13(1)]. Meanwhile, for $\lambda \in \Lambda x\,(0)$, we also have $\lambda x \in \partial\Lambda$ [24, Lemma 5.13(2)].

2.6 Aperiodic higher-rank graphs

The aperiodic k-graphs form a class of k-graphs for which the Cuntz-Krieger uniqueness theorems for Cuntz-Krieger algebras and Kumjian-Pask algebras hold (see Chapter 1). There are several versions of the aperiodicity condition that appear in the literature. In this section, we discuss the aperiodicity condition that we use in this thesis. We also give some equivalent formulations of this condition (Proposition 2.6.3).

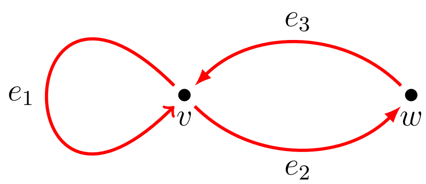
For a directed graph E, the aperiodicity condition is also called *Condition (L)* and says that every cycle has an entrance (see page 3).

Example 2.6.1. Consider the following graph E:



e_1, e_2e_3, e_3e_2 and e_5 are cycles; $e_3e_2e_1$ and $e_2e_1e_2e_1$ are closed paths but are not cycles, because they visit v twice. Every cycle e_2e_3, e_3e_2 and e_1 has an entrance; for example, e_1 is an entry to e_3e_2; and e_3 is an entry to e_1. However, the cycle e_5 has no entrance; so E does not satisfy Condition (L).

However, the following graph



satisfies Condition (L).

For more general k-graphs, we use Condition (B′) of [24] as the aperiodicity condition in this thesis. Now we state the definition.

Definition 2.6.2. Let Λ be a finitely aligned k-graph. We say that Λ is *aperiodic* if for all $v \in \Lambda^0$, there is $x \in v\partial\Lambda$ such that $\lambda, \mu \in \Lambda v$ and $\lambda \neq \mu$ imply $\lambda x \neq \mu x$.

In the literature, there are several equivalent formulations of the aperiodicity condition. So in the following proposition, we state alternative formulations of aperiodicity that we use in this thesis. See also the discussion in Remark 2.6.4.

Proposition 2.6.3. *Let Λ be a finitely aligned k-graph. Then the following statements are equivalent:*

(a) Λ *is aperiodic in the sense of Definition 2.6.2.*

(b) Λ *satisfies Condition (A) of [24].*

(c) For every pair of distinct paths $\lambda, \mu \in \Lambda$ *with* $s(\lambda) = s(\mu)$*, there exists* $\eta \in s(\lambda)\Lambda$ *such that* $\mathrm{MCE}(\lambda\eta, \mu\eta) \neq \emptyset$.

(d) Λ *has* no local periodicity*; that is, for every* $v \in \Lambda^0$ *and every* $n \neq m \in \mathbb{N}^k$*, there exists* $x \in v\partial\Lambda$ *such that either* $d(x) \ngeq n \vee m$ *or* $\sigma^n x \neq \sigma^m x$.

Proof. Shotwell showed that (a) and (b) are equivalent [48, Proposition 2.11]. On the other hand, Lewin and Sims proved that (b) is equivalent to both (c) and (d) [33, Proposition 3.6]. □

Remark 2.6.4. For row-finite k-graphs with no sources, Condition (A) in Proposition 2.6.3(b) is Kumjian and Pask's Condition (A), referred to on page 6.

2.7 Graded rings

Suppose that G is an additive abelian group. A ring A is *G-graded* if there are additive subgroups $\{A_g : g \in G\}$ satisfying:

(2.7.1) $$A = \bigoplus\nolimits_{g\in G} A_g \text{ and for } g, h \in G,\ A_g A_h \subseteq A_{g+h}.$$

We call $\{A_g : g \in G\}$ a *G-grading* of A, and elements of the subgroup A_g are called *homogenous elements of degree g*. For every $a \in A\backslash\{0\}$, there exist unique $\{a_g\}_{g\in G}$ such that each $a_g \in A_g\backslash\{0\}$ and $a = \sum_{g\in G} a_g$.

For example, consider the Laurent polynomial ring $A := K[x, x^{-1}]$ over a field K. For $n \in \mathbb{Z}$, we define $A_n := \{kx^n : k \in K\}$, and then $\{A_n : n \in \mathbb{Z}\}$ is a $\mathbb{Z}$-grading of A.

Let A be a G-graded ring. An ideal I of A is *G-graded* if $\{I \cap A_g : g \in G\}$ is a grading of I. In other words, if $a \in I$ and $a = \sum_{g\in G} a_g$ with every $a_g \in A_g$, then every a_g also belongs to I.

Finally, if A and B are G-graded rings, we say that a homomorphism $\pi : A \to B$ is *G-graded* if $\pi(A_g) \subseteq B_g$ for all $g \in G$.

2.8 Groupoids

A groupoid $\mathcal{G}$ is a small category in which every morphism has an inverse. We write $\mathcal{G}^{(0)}$, called the *unit space*, to denote the set of objects $\mathcal{G}^0$. We also write and $\mathcal{G}$ for set of morphisms $\mathcal{G}^*$. We then call the codomain and domain functions r and s as the *range* and *source maps*, respectively. We also write $\mathcal{G}^{(2)}$ for the set of elements $(a, b) \in \mathcal{G} \times \mathcal{G}$ with $s(a) = r(b)$. For $A, B \subseteq \mathcal{G}$ we write

$$AB := \left\{ab : a \in A, b \in B, (a, b) \in \mathcal{G}^{(2)}\right\}.$$

For example, every group G is a groupoid with one object. Its elements are the morphisms.

We say that $\mathcal{G}$ is a *topological groupoid* if $\mathcal{G}$ is endowed with a topology such that composition and inversion on $\mathcal{G}$ are continuous. A groupoid $\mathcal{G}$ is *étale* if $\mathcal{G}$ is a topological groupoid and the source map s is a local homeomorphism. In this case, r is also a local homeomorphism. An open set $U \subseteq \mathcal{G}$ is an *open bisection*[3] if s and r restricted to U are homeomorphisms into $\mathcal{G}^{(0)}$. Finally, $\mathcal{G}$ is *ample*[4] if $\mathcal{G}$ has a basis of compact open bisections.

Example 2.8.1. For a finitely aligned k-graph Λ, the *path groupoid* $\mathcal{TG}_\Lambda$ from [58, Definition 3.4] is defined as follows. Write

$$\Lambda *_s \Lambda := \{(\lambda, \mu) \in \Lambda \times \Lambda : s(\lambda) = s(\mu)\}$$

and recall the path space W_Λ from Section 2.5. The objects of $\mathcal{TG}_\Lambda$ are

$$\mathcal{TG}_\Lambda^{(0)} := W_\Lambda.$$

The morphisms are

$$\begin{aligned}\mathcal{TG}_\Lambda := \{(\lambda z, d(\lambda) - d(\mu), \mu z) \in W_\Lambda \times \mathbb{Z}^k \times W_\Lambda :& \\ (\lambda, \mu) \in \Lambda *_s \Lambda, z \in s(\lambda) W_\Lambda\}& \\ = \{(x, m, y) \in W_\Lambda \times \mathbb{Z}^k \times W_\Lambda : \text{there exists } p, q \in \mathbb{N}^k \text{ such that}& \\ p \le d(x), q \le d(y), p - q = m \text{ and } \sigma^p x = \sigma^q y\}.&\end{aligned}$$

The range and source maps are given by $r(x, m, y) := x$ and $s(x, m, y) := y$, composition is defined by

$$((x_1, m_1, y_1), (y_1, m_2, y_2)) \mapsto (x_1, m_1 + m_2, y_2),$$

and inversion is given by $(x, m, y) \mapsto (y, -m, x)$.

[3]Open bisections are sometimes referred to as either *slices* or *open $\mathcal{G}$-sets*, see for example [22].

[4]If $\mathcal{G}$ is ample, then $\mathcal{G}$ is locally compact and étale. In fact, $\mathcal{G}$ is Hausdorff ample if and only if $\mathcal{G}$ is locally compact, Hausdorff and étale with totally disconnected unit space (see [22, Proposition 4.1]).

Next we show how to realise $\mathcal{TG}_\Lambda$ as a topological groupoid. For each pair $(\lambda, \mu) \in \Lambda *_s \Lambda$ and finite subset G of $s(\lambda)\Lambda$, we write

$$TZ_\Lambda(\lambda) := \lambda W_\Lambda,$$

$$TZ_\Lambda(\lambda \backslash G) := TZ_\Lambda(\lambda) \backslash \Big(\bigcup_{\nu \in G} TZ_\Lambda(\lambda\nu) \Big),$$

$$TZ_\Lambda(\lambda *_s \mu) := \{(x, d(\lambda) - d(\mu), y) \in \mathcal{TG}_\Lambda : x \in TZ_\Lambda(\lambda), y \in TZ_\Lambda(\mu) \text{ and } \sigma^{d(\lambda)}x = \sigma^{d(\mu)}y\},$$

and

$$TZ_\Lambda(\lambda *_s \mu \backslash G) := TZ_\Lambda(\lambda *_s \mu) \backslash \Big(\bigcup_{\nu \in G} TZ_\Lambda(\lambda\nu *_s \mu\nu) \Big).$$

Theorem 3.16 of [58] says that the sets $TZ_\Lambda(\lambda *_s \mu \backslash G)$ form a basis of compact open bisections for a second-countable Hausdorff topology on $\mathcal{TG}_\Lambda$ under which it is an ample groupoid. Further, the sets $TZ_\Lambda(\lambda \backslash G)$ form a basis of compact open sets for $\mathcal{TG}_\Lambda^{(0)}$ [58, Lemma 3.8].

Remark 2.8.2. We think of $\mathcal{TG}_\Lambda^0 = W_\Lambda$ as a subset of $\mathcal{TG}_\Lambda$ under the correspondence $x \mapsto (x, 0, x)$.

Example 2.8.3 ([58, Definition 4.8]). Let Λ be a finitely aligned k-graph, let $\mathcal{TG}_\Lambda$ be the path groupoid of Λ as in Example 2.8.1, and let $\partial\Lambda$ be the boundary-path space as in Section 2.5. The set $\partial\Lambda$ is nonempty [58, Proposition 4.3], closed in $\mathcal{TG}_\Lambda^{(0)}$ [58, Proposition 4.4], and an invariant subset of $\mathcal{TG}_\Lambda^{(0)}$ [58, Proposition 4.7]. Hence

$$\mathcal{G}_\Lambda := \mathcal{TG}_\Lambda|_{\partial\Lambda}$$

is a closed subgroupoid, called the *boundary-path groupoid of* Λ. The groupoid $\mathcal{G}_\Lambda$ is second-countable Hausdorff because $\mathcal{TG}_\Lambda$ is second-countable Hausdorff.

For $(\lambda, \mu) \in \Lambda *_s \Lambda$ and finite subset $G \subseteq s(\lambda)\Lambda$, we write

$$Z_\Lambda(\lambda \backslash G) := TZ_\Lambda(\lambda \backslash G) \cap \mathcal{G}_\Lambda \text{ and } Z_\Lambda(\lambda *_s \mu \backslash G) := TZ_\Lambda(\lambda *_s \mu \backslash G) \cap \mathcal{G}_\Lambda.$$

Since the sets $TZ_\Lambda(\lambda *_s \mu \backslash G)$ of Example 2.8.1 form a basis of compact open bisections for $\mathcal{TG}_\Lambda$, and since $\mathcal{G}_\Lambda$ is a closed subset of $\mathcal{TG}_\Lambda$, the sets $Z_\Lambda(\lambda *_s \mu \backslash G)$ also form a basis of compact open bisections for $\mathcal{G}_\Lambda$. Thus $\mathcal{G}_\Lambda$ is also ample. The sets $Z_\Lambda(\lambda \backslash G)$ also form a basis of compact open sets for $\mathcal{G}_\Lambda^{(0)}$.

Remark 2.8.4. A number of notes relating to this example:

(i) We think of $\mathcal{G}_\Lambda^{(0)} = \partial\Lambda$ as a subset of $\mathcal{G}_\Lambda$ under the correspondence $x \mapsto (x, 0, x)$.

(ii) In Section 2.5, we define $Z_\Lambda\left(\lambda \backslash G\right)$ and $Z_\Lambda\left(\lambda *_s \mu \backslash G\right)$ where G is finite. However, if G is exhaustive, then $Z_\Lambda\left(\lambda \backslash G\right)$ and $Z_\Lambda\left(\lambda *_s \mu \backslash G\right)$ are empty sets. Thus our definitions make sure that both $Z_\Lambda\left(\lambda \backslash G\right)$ and $Z_\Lambda\left(\lambda *_s \mu \backslash G\right)$ are non-empty.

(iii) $\mathcal{G}_\Lambda$ is also étale (see Footnote 2.8).

2.9 Steinberg algebras

Steinberg algebras were introduced in [53], and are algebraic analogues of groupoid C^*-algebras of [44]. Various mathematicians have studied the relationship between Steinberg algebras and Leavitt path algebras, see for example [15, 17]. We give a brief introduction to Steinberg algebras.

Suppose that $\mathcal{G}$ is a Hausdorff ample groupoid and R is a commutative ring with 1. As in [14, Section 2.2], the Steinberg algebra[5] associated to $\mathcal{G}$ is

$$A_R\left(\mathcal{G}\right) := \{f : \mathcal{G} \to R : f \text{ is locally constant and has compact support}\};$$

addition and scalar multiplication are defined pointwise, and multiplication is given to be the convolution

$$\left(f \star g\right)(a) := \sum_{r(a)=r(b)} f\left(b\right) g\left(b^{-1}a\right).$$

For compact open bisections U and V, the characteristic function 1_U belongs to $A_R(\mathcal{G})$, and

$$1_U \star 1_V = 1_{UV}$$

(see [53, Proposition 4.3]).

Let Γ be a discrete group with identity e, and suppose that $c : \mathcal{G} \to \Gamma$ is a *continuous cocycle*: $c\left(a\right) c\left(b\right) = c\left(ab\right)$ for $a, b \in \mathcal{G}$. For $n \in \Gamma$, we write $\mathcal{G}_n := c^{-1}\left(n\right)$. Then Proposition 3.6 of [15] says that the subsets

$$A_R(\mathcal{G})_n := \{f \in A_R(\mathcal{G}) : \operatorname{supp}\left(f\right) \subseteq \mathcal{G}_n\}$$

form a Γ-grading of $A_R(\mathcal{G})$.

[5]In [53, Definition 4.1], Steinberg writes $R\mathcal{G}$ to denote $A_R(\mathcal{G})$. In some references, for example [15], Steinberg algebras are defined on locally compact, Hausdorff, *étale* groupoids with totally disconnected space. However, this kind of groupoids is equivalent to Hausdorff ample groupoids [15, Lemma 2.1].

We say that a subset $S \subseteq \mathcal{G}$ is *n-graded* if S is also a subset of $\mathcal{G}_n$. For $n \in \Gamma$, we write $B_n^{\mathrm{co}}(\mathcal{G})$ for the collection of all n-graded compact open bisections of $\mathcal{G}$. We also write

$$B_*^{\mathrm{co}}(\mathcal{G}) := \bigcup_{n\in\Gamma} B_n^{\mathrm{co}}(\mathcal{G}).$$

The following proposition explains how an element of $A_R(\mathcal{G})$ can be written in terms of a sum of elements of $B_*^{\mathrm{co}}(\mathcal{G})$.

Proposition 2.9.1 ([14, Lemma 2.2])**.** *Suppose that $\mathcal{G}$ is a Hausdorff ample groupoid, that Γ is a discrete group and that $c : \mathcal{G} \to \Gamma$ is a continuous cocycle. Then every $f \in A_R(\mathcal{G})$ has the form*

$$f = \sum_{B\in F} a_B 1_B$$

where each $a_B \in R$ and F is a finite set of mutually disjoint elements of $B_*^{\mathrm{co}}(\mathcal{G})$.

2.10 Effective groupoids

There is a powerful uniqueness theorem for Steinberg algebras, called the Cuntz-Krieger uniqueness theorem (see Theorem 2.10.2). It was proved in [15] for Steinberg algebras over the complex numbers, and generalised by Clark and Edie-Michell to Steinberg algebras over commutative ring with 1 [14]. This uniqueness theorem only holds for *effective* groupoids, so this class of groupoids is an analogue of the class of aperiodic k-graphs. In fact, there is a relationship between aperiodic k-graphs and effective groupoids, see Proposition 3.5.1 and Remark 3.5.3.

Let $\mathcal{G}$ be a locally compact Hausdorff étale groupoid. We define the *isotropy groupoid* of $\mathcal{G}$ by

$$\mathrm{Iso}(\mathcal{G}) := \{a \in \mathcal{G} : s(a) = r(a)\}.$$

We say that $\mathcal{G}$ is *effective* if the interior of $\mathrm{Iso}(\mathcal{G})$ is $\mathcal{G}^{(0)}$. This definition is from [13, Lemma 3.1] which states several equivalent characterisations of effective groupoids.

Remark 2.10.1. Renault showed that for second countable groupoids, $\mathcal{G}$ is effective if and only if $\mathcal{G}$ is *topologically principal* in that the set of units with trivial isotropy is dense in $\mathcal{G}^{(0)}$ [44, Proposition 3.6]. Since for every finitely aligned k-graph Λ, the groupoids $\mathcal{T}\mathcal{G}_\Lambda$ and $\mathcal{G}_\Lambda$ are also second-countable, the notions of effective and topogically principal are interchangeable in this setting. For other notions of effective groupoids, see Remark 3.5.3.

We state the Cuntz-Krieger uniqueness theorem for Steinberg algebras as follows.

Theorem 2.10.2 ([14, Theorem 3.2]). *Suppose that $\mathcal{G}$ is an effective Hausdorff ample groupoid and that R is a commutative ring with 1. Suppose that $\phi : A_R(\mathcal{G}) \to A$ is an R-algebra homomorphism such that* $\ker(\phi) \neq 0$. *Then there is a nonempty compact open subset $K \subseteq \mathcal{G}^{(0)}$ and $r \in R \setminus \{0\}$ such that $\phi(r1_K) = 0$.*

Chapter 3

Kumjian-Pask algebras

Suppose that Λ is a finitely aligned higher-rank graph and that R is a commutative ring with 1. In this chapter, we introduce Kumjian-Pask Λ-families and study their properties. We also establish the graded uniqueness theorem (Theorem 3.2.1) and the Cuntz-Krieger uniqueness theorem for Kumjian-Pask algebras (Theorem 3.7.1).

In general, the material in this chapter is taken from a joint paper with my supervisor Clark in [18]. However, we give more background and details in Section 3.2 than in the paper [18, Section 4].

3.1 Kumjian-Pask Λ-families

In this section, we define Kumjian-Pask Λ-families for finitely aligned higher-rank graphs. These include the Kumjian-Pask Λ-families of [11, 16] (see Section 1.4).

Suppose that Λ is a finitely aligned k-graph and R is a commutative ring with identity 1. For $\lambda \in \Lambda$, we call λ^* a *ghost path* (λ^* is a formal symbol). We write

$$G(\Lambda) := \{\lambda^* : \lambda \in \Lambda\}.$$

For $v \in \Lambda^0$, we define $v^* := v$. We extend r and s to be defined on $G(\Lambda)$ by

$$r(\lambda^*) := s(\lambda) \text{ and } s(\lambda^*) := r(\lambda).$$

We define composition on $G(\Lambda)$ by setting $\lambda^*\mu^* = (\mu\lambda)^*$ for $\lambda, \mu \in \Lambda$, and write $G(\Lambda^{\neq 0})$ for the set of ghost paths that are not vertices. Note that the factorisation property of Λ induces a similar factorisation property on $G(\Lambda)$.

Definition 3.1.1. A *Kumjian-Pask Λ-family* $\{S_\lambda, S_{\mu^*} : \lambda, u \in \Lambda\}$ in an R-algebra A consists of a function $S : \Lambda \cup G(\Lambda^{\neq 0}) \to A$ such that:

(KP1) $\{S_v : v \in \Lambda^0\}$ are mutually orthogonal idempotents;

(KP2) for $\lambda, \mu \in \Lambda$ with $s(\lambda) = r(\mu)$, we have $S_\lambda S_\mu = S_{\lambda\mu}$ and $S_{\mu^*} S_{\lambda^*} = S_{(\lambda\mu)^*}$;

(KP3) $S_{\lambda^*} S_\mu = \sum_{(\rho,\tau)\in\Lambda^{\min}(\lambda,\mu)} S_\rho S_{\tau^*}$ for all $\lambda, \mu \in \Lambda$; and

(KP4) $\prod_{\lambda\in E} \left(S_{r(E)} - S_\lambda S_{\lambda^*}\right) = 0$ for all $E \in \mathrm{FE}(\Lambda)$.

Remark 3.1.2. A number of aspects of these relations are worth commenting on:

(i) In previous references about Leavitt path algebras and Kumjian-Pask algebras, people usually distinguish the vertex idempotents as "P_v" (for example, see [1, 3, 4, 10, 11, 16, 54, 55]). We do not follow this convention because we do not want to make additional unnecessary cases in proofs.

(ii) (KP2) in [11, 16] has more relations to check. However, using our notational convention, those relations are equivalent to our (KP2).

(iii) The restriction to finitely aligned k-graphs is necessary for the sum in (KP3) to make sense (see [39]).

(iv) In (KP3), we interpret the empty sum as 0, so $S_{\lambda^*} S_\mu = 0$ whenever $\Lambda^{\min}(\lambda, \mu) = \emptyset$. On the other hand, by taking $\lambda = \mu$, we get $S_{\lambda^*} S_\lambda = S_{s(\lambda)}$.

(v) (KP3-4) have been changed from those in [11, Definition 3.1] and [16, Definition 3.1] to take into consideration k-graphs that are not locally convex. For further discussion, see Appendix A of [41].

The following lemma establishes some useful properties of families satisfying (KP1-3).

Proposition 3.1.3 ([18, Proposition 3.3])**.** *Suppose that Λ is a finitely aligned k-graph, that R is a commutative ring with* 1*, and that $\{S_\lambda, S_{\mu^*} : \lambda, u \in \Lambda\}$ is a family satisfying (KP1-3) in an R-algebra A.*

(a) We have $S_\lambda S_{\lambda^*} S_\mu S_{\mu^*} = \sum_{\lambda\rho\in\mathrm{MCE}(\lambda,\mu)} S_{\lambda\rho} S_{(\lambda\rho)^*}$ for $\lambda, \mu \in \Lambda$; and $\{S_\lambda S_{\lambda^*} : \lambda \in \Lambda\}$ is a commuting family.

(b) The subalgebra generated by $\{S_\lambda, S_{\mu^*} : \lambda, u \in \Lambda\}$ is

$$\mathrm{span}_R\{S_\lambda S_{\mu^*} : \lambda, u \in \Lambda, s(\lambda) = s(\mu)\}.$$

(c) *For $n \in \mathbb{N}^k$ and $\lambda, \mu \in \Lambda^{\leq n}$ (see 2.5.1), we have $S_{\lambda^*} S_\mu = \delta_{\lambda,\mu} S_{s(\lambda)}$.*

(d) *Suppose that $rS_v \neq 0$ for all $r \in R\setminus\{0\}$, that $v \in \Lambda^0$ and that $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu)$. If $r \in R\setminus\{0\}$ and $G \subseteq s(\lambda)\Lambda$ is finite non-exhaustive, then*

$$rS_\lambda \neq 0 \text{ and } rS_\lambda \Big(\prod_{\nu \in G} \left(S_{s(\lambda)} - S_\nu S_{\nu^*} \right) \Big) S_{\mu^*} \neq 0.$$

Proof. To show (a), we take $\lambda, \mu \in \Lambda$ and then

$$\begin{aligned} S_\lambda S_{\lambda^*} S_\mu S_{\mu^*} &= S_\lambda \Big(\sum_{(\rho,\tau) \in \Lambda^{\min}(\lambda,\mu)} S_\rho S_{\tau^*} \Big) S_{\mu^*} = \sum_{(\rho,\tau) \in \Lambda^{\min}(\lambda,\mu)} S_{\lambda\rho} S_{(\mu\tau)^*} \\ &= \sum_{(\rho,\tau) \in \Lambda^{\min}(\lambda,\mu)} S_{\lambda\rho} S_{(\lambda\rho)^*} = \sum_{\lambda\rho \in \mathrm{MCE}(\lambda,\mu)} S_{\lambda\rho} S_{(\lambda\rho)^*} \\ &= \sum_{\mu\tau \in \mathrm{MCE}(\lambda,\mu)} S_{\mu\tau} S_{(\mu\tau)^*} = S_\mu S_{\mu^*} S_\lambda S_{\lambda^*}. \end{aligned}$$

Next we show (b). For $\lambda, u \in \Lambda$, we have $S_\lambda S_{\mu^*} = S_\lambda S_{s(\lambda)} S_{s(\mu)} S_{\mu^*}$ by (KP2), and by (KP1), $S_\lambda S_{\mu^*} \neq 0$ implies $s(\lambda) = s(\mu)$. On the other hand, for $\lambda_1, u_1, \lambda_2, u_2 \in \Lambda$, we have

$$\begin{aligned} \left(S_{\lambda_1} S_{\mu_1^*} \right) \left(S_{\lambda_2} S_{\mu_2^*} \right) &= S_{\lambda_1} \Big(\sum_{(\rho,\tau) \in \Lambda^{\min}(\mu_1,\lambda_2)} S_\rho S_{\tau^*} \Big) S_{\mu_2^*} \text{ (by (KP3))} \\ &= \sum_{(\rho,\tau) \in \Lambda^{\min}(\mu_1,\lambda_1)} S_{\lambda_1\rho} S_{(\mu_2\tau)^*} \text{ (by (KP2))}. \end{aligned}$$

To show (c), we take $\lambda, \mu \in \Lambda^{\leq n}$. Suppose that $S_{\lambda^*} S_\mu \neq 0$. By (KP3), there exists $(\rho,\tau) \in \Lambda^{\min}(\lambda,\mu)$ such that $\lambda\rho = \mu\tau$ and $d(\lambda\rho) \leq n$. Since $\lambda, \mu \in \Lambda^{\leq n}$, we have $\rho = s(\lambda) = \tau$, and hence $\lambda = \mu$.

Finally, we show (d). Take $r \in R\setminus\{0\}$ and $\lambda \in \Lambda$. Suppose for contradiction that $rS_\lambda = 0$. Then

$$0 = S_{\lambda^*}(rS_\lambda) = rS_{\lambda^*} S_\lambda = rS_{s(\lambda)},$$

which contradicts with $rS_v \neq 0$ for all $r \in R\setminus\{0\}$ and $v \in \Lambda^0$. Hence $rS_\lambda \neq 0$.

Now take $r \in R\setminus\{0\}$, $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu)$ and finite non-exhaustive $G \subseteq s(\lambda)\Lambda$. Suppose for contradiction that

$$rS_\lambda \Big(\prod_{\nu \in G} \left(S_{s(\lambda)} - S_\nu S_{\nu^*} \right) \Big) S_{\mu^*} = 0.$$

Since G is non-exhaustive, then there exists $\gamma \in s(\lambda)\Lambda$ such that $\Lambda^{\min}(\nu,\gamma) = \emptyset$ for every $\nu \in G$. By (KP3), we get $S_{\nu^*} S_\gamma = 0$ for $\nu \in G$. Therefore

$$0 = \Big(rS_\lambda \Big(\prod_{\nu \in G} \left(S_{s(\lambda)} - S_\nu S_{\nu^*} \right) \Big) S_{\mu^*} \Big) S_{\mu\gamma}$$

$$
\begin{aligned}
&= rS_\lambda \Big(\prod_{\nu \in G} \left(S_{s(\lambda)} - S_\nu S_{\nu^*} \right) \Big) S_\gamma \\
&= rS_\lambda S_\gamma = rS_{\lambda\gamma},
\end{aligned}
$$

which contradicts $rS_{\lambda\gamma} \neq 0$. Hence $rS_\lambda \Big(\prod_{\nu \in G} \left(S_{s(\lambda)} - S_\nu S_{\nu^*} \right) \Big) S_{\mu^*} \neq 0$. □

Remark 3.1.4. Since $\{S_\lambda S_{\lambda^*} : \lambda \in \Lambda\}$ is a commuting family, Proposition 3.1.3(a) implies that for all $\lambda \in \Lambda$ and $G \subseteq r(\lambda)\Lambda$, the order of multiplication does not matter and $\prod_{\mu \in G} (S_\lambda S_{\lambda^*} - S_\mu S_{\mu^*})$ is well-defined. In particular, $\prod_{\nu \in G} \left(S_{s(\lambda)} - S_\nu S_{\nu^*} \right)$ of Proposition 3.1.3(d) is also well-defined.

Remark 3.1.5. For $n \in \mathbb{N}^k$, we have $\Lambda^n \subseteq \Lambda^{\leq n}$. Hence Proposition 3.1.3(c) also implies that for $n \in \mathbb{N}^k$ and $\lambda, \mu \in \Lambda^n$, we have $S_{\lambda^*} S_\mu = \delta_{\lambda,\mu} S_{s(\lambda)}$.

Remark 3.1.6. Suppose that $rS_v \neq 0$ for all $r \in R \backslash \{0\}$, $v \in \Lambda^0$ and that $\lambda, \mu \in \Lambda$ have $s(\lambda) = s(\mu)$. The contrapositive of Proposition 3.1.3(d) says: if $r \in R$, $G \subseteq s(\lambda)\Lambda$ is finite, and $rS_\lambda \Big(\prod_{\nu \in G} \left(S_{s(\lambda)} - S_\nu S_{\nu^*} \right) \Big) S_{\mu^*} = 0$, then either $r = 0$ or G is exhaustive.

Now we give an example of a Kumjian-Pask Λ-family in an algebra of endomorphisms.

Proposition 3.1.7 ([18, Proposition 3.6])**.** *Suppose that Λ is a finitely aligned k-graph and that R is a commutative ring with 1. Suppose that $\mathbb{F}_R(\partial\Lambda)$ is the free module with basis the boundary path space. Then for every $v \in \Lambda^0$ and $\lambda, \mu \in \Lambda \backslash \Lambda^0$, there exist endomorphisms $S_v, S_\lambda, S_{\mu^*} : \mathbb{F}_R(\partial\Lambda) \to \mathbb{F}_R(\partial\Lambda)$ such that for $x \in \partial\Lambda$,*

$$
S_v(x) = \begin{cases} x & \textit{if } r(x) = v \\ 0 & \textit{otherwise,} \end{cases}
$$

$$
S_\lambda(x) = \begin{cases} \lambda x & \textit{if } s(\lambda) = r(x) \\ 0 & \textit{otherwise,} \end{cases}
$$

$$
S_{\mu^*}(x) = \begin{cases} \sigma^{d(\mu)} x & \textit{if } x(0, d(\mu)) = \mu \\ 0 & \textit{otherwise.} \end{cases}
$$

The set $\{S_\lambda, S_{\mu^*} : \lambda, u \in \Lambda\}$ is a Kumjian-Pask Λ-family in the R-algebra $\operatorname{End}(\mathbb{F}_R(\partial\Lambda))$ *with $rS_v \neq 0$ for all $r \in R \backslash \{0\}$ and $v \in \Lambda^0$.*

Proof. Take $v \in \Lambda^0$ and $\lambda, \mu \in \Lambda \backslash \Lambda^0$. Note that for $x \in \partial\Lambda$ and $m \leq d(x)$, we have $\sigma^m x \in \partial\Lambda$. Define functions f_v, f_λ, and $f_{\mu^*} : \partial\Lambda \to \mathbb{F}_R(\partial\Lambda)$ by

$$
f_v(x) = \begin{cases} x & \text{if } r(x) = v \\ 0 & \text{otherwise,} \end{cases}
$$

$$f_\lambda(x) = \begin{cases} \lambda x & \text{if } s(\lambda) = r(x) \\ 0 & \text{otherwise,} \end{cases}$$

$$f_{\mu^*}(x) = \begin{cases} \sigma^{d(\mu)} x & \text{if } x(0, d(\mu)) = \mu \\ 0 & \text{otherwise.} \end{cases}$$

The universal property of free modules gives endomorphisms

$$S_v, S_\lambda, S_{\mu^*} : \mathbb{F}_R(\partial\Lambda) \to \mathbb{F}_R(\partial\Lambda)$$

extending f_v, f_λ, and f_{μ^*}.

We claim that $\{S_\lambda, S_{\mu^*} : \lambda, u \in \Lambda\}$ is a Kumjian-Pask Λ-family. To see (KP1), take $v \in \Lambda^0$ and $x \in \partial\Lambda$. Then $S_v^2(x) = x = S_v(x)$ if $r(x) = v$, and $S_v^2(x) = 0 = S_v(x)$ otherwise. Hence $S_v^2 = S_v$. Now take $v, w \in \Lambda^0$ with $v \neq w$ and $x \in \partial\Lambda$. Since $x \in w\partial\Lambda$ implies $x \notin v\partial\Lambda$, we have $S_v S_w(x) = 0$ for $x \in \partial\Lambda$ and $S_v S_w = 0$.

To show (KP2), take $\lambda, \mu \in \Lambda$ with $s(\lambda) = r(\mu)$. Then for $x \in s(\mu)\partial\Lambda$, we have $\mu x \in s(\lambda)\partial\Lambda$. Then $S_\lambda S_\mu(x) = \lambda\mu x = S_{\lambda\mu}(x)$ if $x \in s(\mu)\partial\Lambda$, and $S_\lambda S_\mu(x) = 0 = S_{\lambda\mu}(x)$ otherwise. Hence $S_\lambda S_\mu = S_{\lambda\mu}$. Meanwhile, for $x \in r(\lambda)\partial\Lambda$ with $x(0, d(\lambda\mu)) = \lambda\mu$, we have $d(\lambda\mu) \leq d(x)$ and $\sigma^{d(\lambda\mu)} x \in s(\mu)\partial\Lambda$. Furthermore, $x(0, d(\lambda\mu)) = \lambda\mu$ also implies that $x(0, d(\lambda)) = \lambda$, and then $d(\lambda) \leq d(x)$ and $\sigma^{d(\lambda)} x \in s(\lambda)\partial\Lambda$. Hence

$$S_{\mu^*} S_{\lambda^*}(x) = S_{\mu^*}\sigma^{d(\lambda)} x = \sigma^{d(\lambda)+d(\mu)} x = \sigma^{d(\lambda\mu)} x = S_{(\lambda\mu)^*}(x)$$

if $x(0, d(\lambda\mu)) = \lambda\mu$, and $S_{\mu^*} S_{\lambda^*}(x) = 0 = S_{(\lambda\mu)^*}(x)$ otherwise. Therefore $S_{\mu^*} S_{\lambda^*} = S_{(\lambda\mu)^*}$.

To see (KP3), we take $\lambda, \mu \in \Lambda$. If $r(\lambda) \neq r(\mu)$, then $S_{\lambda^*} S_\mu = 0$ and $\Lambda^{\min}(\lambda, \mu) = \emptyset$, as required. Suppose $r(\lambda) = r(\mu)$. We have

$$S_{\lambda^*} S_\mu(x) = \begin{cases} (\mu x)(d(\lambda), d(\mu x)) & \text{if } x \in s(\mu)\partial\Lambda \text{ and } (\mu x)(0, d(\lambda)) = \lambda \\ 0 & \text{otherwise.} \end{cases}$$

Take $x \in s(\mu)\partial\Lambda$. Note that $s(\mu) = r(\tau)$ for $(\rho, \tau) \in \Lambda^{\min}(\lambda, \mu)$. First suppose that $(\mu x)(0, d(\lambda)) \neq \lambda$. Then for $(\rho, \tau) \in \Lambda^{\min}(\lambda, \mu)$,

$$(\mu x)(0, d(\lambda\rho)) \neq \lambda\rho \text{ and } (\mu x)(0, d(\mu\tau)) \neq \mu\tau.$$

Hence $x(0, d(\tau)) \neq \tau$ and $S_\rho S_{\tau^*}(x) = S_\rho(0) = 0$, so that

$$\sum_{(\rho,\tau)\in\Lambda^{\min}(\lambda,\mu)} S_\rho S_{\tau^*}(x) = 0.$$

So suppose that $(\mu x)(0, d(\lambda)) = \lambda$. Since $(\mu x)(0, d(\lambda)) = \lambda$ and $(\mu x)(0, d(\mu)) = \mu$, there exists $\tau \in s(\mu)\Lambda$ such that $(\rho, \tau) \in \Lambda^{\min}(\lambda, \mu)$ and $(\mu x)(0, d(\mu\tau)) = \mu\tau$. Then $x(0, d(\tau)) = \tau$. The factorisation property implies that there is only one such τ. Hence for $(\rho', \tau') \in \Lambda^{\min}(\lambda, \mu)$ with $(\rho', \tau') \neq (\rho, \tau)$, we have $S_{\rho'} S_{\tau'^*}(x) = 0$. Since we also have $x(0, d(\tau)) = \tau$, we can conclude:

$$\begin{aligned}
S_\rho S_{\tau^*}(x) = S_\rho(x(d(\tau), d(x))) &= \rho[x(d(\tau), d(x))] \\
&= \rho[(\mu x)(d(\mu\tau), d(\mu x))] \\
&= \rho[(\mu x)(d(\lambda\rho), d(\mu x))] \;\; (\text{since } \mu\tau = \lambda\rho) \\
&= (\mu x)(d(\lambda), d(\mu x))
\end{aligned}$$

and

$$\sum_{(\rho', \tau') \in \Lambda^{\min}(\lambda, \mu)} S_\rho S_{\tau^*}(x) = S_\rho S_{\tau^*}(x) = (\mu x)(d(\lambda), d(\mu x)) = S_{\lambda^*} S_\mu(x),$$

as required.

Finally, we show (KP4). Take $E \in \mathrm{FE}(\Lambda)$ and $x \in r(E)\partial\Lambda$. Since $E \in x(0)\mathrm{FE}(\Lambda)$ and x is a boundary path, there exists $\lambda \in E$ such that $x(0, d(\lambda)) = \lambda$. Then

$$\begin{aligned}
\left(S_{r(E)} - S_\lambda S_{\lambda^*}\right)(x) &= S_{r(E)}(x) - S_\lambda S_{\lambda^*}(x) \\
&= x - S_\lambda(x(d(\lambda), d(x))) \\
&= x - x = 0.
\end{aligned}$$

Hence

$$\left(\prod_{\lambda \in E} \left(S_{r(E)} - S_\lambda S_{\lambda^*}\right)\right)(x) = 0$$

for $x \in r(E)\partial\Lambda$, and $\prod_{\lambda \in E}\left(S_{r(E)} - S_\lambda S_{\lambda^*}\right) = 0$.

Thus $\{S_\lambda, S_{\mu^*} : \lambda, u \in \Lambda\}$ is a Kumjian-Pask Λ-family, as claimed. Since each $v\partial\Lambda$ is nonempty, we have $rS_v \neq 0$ for all $r \in R \setminus \{0\}$ and $v \in \Lambda^0$. □

Using a different construction of a Kumjian-Pask Λ-family, we show that there is an R-algebra which is universal for Kumjian-Pask Λ-families.

Theorem 3.1.8 ([18, Theorem 3.7]). *Suppose that Λ is a finitely aligned k-graph and that R is a commutative ring with 1.*

(a) There is a universal R-algebra $\mathrm{KP}_R(\Lambda)$ generated by a Kumjian-Pask Λ-family $\{s_\lambda, s_{\mu^*} : \lambda, u \in \Lambda\}$ such that: whenever $\{S_\lambda, S_{\mu^*} : \lambda, u \in \Lambda\}$ is a Kumjian-Pask Λ-family in an R-algebra A, there exists a unique ring homomorphism $\pi_S : \mathrm{KP}_R(\Lambda) \to A$ such that $\pi_S(s_\lambda) = S_\lambda$ and $\pi_S(s_{\mu^*}) = S_{\mu^*}$ for $\lambda, \mu \in \Lambda$.

(b) We have $rs_v \neq 0$ for all $r \in R\setminus\{0\}$ and $v \in \Lambda^0$.

(c) The subsets

$$\mathrm{KP}_R(\Lambda)_n := \mathrm{span}_R\{s_\lambda s_{\mu^*} : \lambda, \mu \in \Lambda, d(\lambda) - d(\mu) = n\}$$

forms a $\mathbb{Z}^k$-grading of $\mathrm{KP}_R(\Lambda)$.

Proof. We use an argument similar to [11, Theorem 3.4] and [16, Theorem 3.7]. To show (a), we use the free algebra $\mathbb{F}_R(w(X))$ on the set $w(X)$ of words on $X := \Lambda \cup G(\Lambda^{\neq 0})$. Let I be the ideal of $\mathbb{F}_R(w(X))$ generated by the elements:

(i) $\{vw - \delta_{v,w}v : v, w \in \Lambda^0\}$,

(ii) $\{\lambda - \mu\nu, \lambda^* - \nu^*\mu^* : \lambda, \mu, \nu \in \Lambda \text{ and } \lambda = \mu\nu\}$,

(iii) $\{\lambda^*\mu - \sum_{(\rho,\tau)\in\Lambda^{\min}(\lambda,\mu)} \rho\tau^* : \lambda, \mu \in \Lambda\}$, and

(iv) $\{\prod_{\lambda\in E}(r(E) - \lambda\lambda^*) : E \in \mathrm{FE}(\Lambda)\}$.

We define $\mathrm{KP}_R(\Lambda) := \mathbb{F}_R(w(X))/I$ and write $q : \mathbb{F}_R(w(X)) \to \mathbb{F}_R(w(X))/I$ for the quotient map. Define $s_\lambda := q(\lambda)$ for $\lambda \in \Lambda$, and $s_{\mu^*} := q(\mu^*)$ for $\mu^* \in G(\Lambda^{\neq 0})$. Then $\{s_\lambda, s_{\mu^*} : \lambda \in \Lambda, \mu^* \in G(\Lambda^{\neq 0})\}$ is a Kumjian-Pask Λ-family in $\mathrm{KP}_R(\Lambda)$.

To show the universal property, suppose that $\{S_\lambda, S_{\mu^*} : \lambda, u \in \Lambda\}$ is a Kumjian-Pask Λ-family in an R-algebra A. Define $f : X \to A$ by $f(\lambda) := S_\lambda$ for $\lambda \in \Lambda$, and $f(\mu^*) := S_{\mu^*}$ for $\mu^* \in G(\Lambda^{\neq 0})$. The universal property of $\mathbb{F}_R(w(X))$ gives an unique R-algebra homomorphism $\phi : \mathbb{F}_R(w(X)) \to A$ such that $\phi|_X = f$. Since $\{S_\lambda, S_{\mu^*} : \lambda, u \in \Lambda\}$ is a Kumjian-Pask Λ-family, then $I \subseteq \ker(\phi)$. Thus there exists an R-algebra homomorphism $\pi_S : \mathrm{KP}_R(\Lambda) \to A$ such that $\pi_S \circ q = \phi$. The homomorphism π_S is unique since the elements in X generate $\mathbb{F}_R(w(X))$ as an algebra. We also have $\pi_S(s_\lambda) = S_\lambda$ for $\lambda \in \Lambda$ and $\pi_S(s_{\mu^*}) = S_{\mu^*}$ for $\mu^* \in G(\Lambda^{\neq 0})$, as required.

To show (b), let $\{S_\lambda, S_{\mu^*} : \lambda, u \in \Lambda\}$ be the Kumjian-Pask Λ-family of Proposition 3.1.7. Then $rS_v \neq 0$ for $v \in \Lambda^0$. Since $\pi_S(rs_v) = rS_v \neq 0$ for all $r \in R\setminus\{0\}$ and $v \in \Lambda^0$, we have $rs_v \neq 0$ for all $r \in R\setminus\{0\}$ and $v \in \Lambda^0$.

Next we show (c). We first extend the degree map to $w(X)$ by $d(w) := \sum_{i=1}^{|w|} d((w_i))$ for $w \in w(X)$. By [11, Proposition 2.7], $\mathbb{F}_R(w(X))$ is $\mathbb{Z}^k$-graded by the subgroups

$$\mathbb{F}_R(w(X))_n := \Big\{ \sum_{w\in w(X)} r_w w : r_w \neq 0 \text{ implies } d(w) = n \Big\}.$$

We claim that the ideal I defined in (a) is a graded ideal. It suffices to show that I is generated by elements in $\mathbb{F}_R(w(X))_n$ for some $n \in \mathbb{Z}^k$. Since $d(v) = 0$ for $v \in \Lambda^0$, then the generators in (i) belong to $\mathbb{F}_R(w(X))_0$. If $\lambda = \mu\nu$ in Λ, then $\lambda - \mu\nu$ belongs to $\mathbb{F}_R(w(X))_{d(\lambda)}$ and $\lambda^* - \nu^*\mu^*$ belongs to $\mathbb{F}_R(w(X))_{-d(\lambda)}$. For $\lambda, \mu \in \Lambda$ and $(\rho, \tau) \in \Lambda^{\min}(\lambda, \mu)$, we have

$$d(\rho) - d(\tau) = (d(\lambda) \vee d(\mu) - d(\lambda)) - (d(\lambda) \vee d(\mu) - d(\mu)) = -d(\lambda) + d(\mu)$$

and then the generators in (iii) belong to $\mathbb{F}_R(w(X))_{-d(\lambda)+d(\mu)}$. Finally, a word $\lambda\lambda^*$ has degree 0 and then the generators in (iv) belong to $\mathbb{F}_R(w(X))_0$. Thus I is a graded ideal.

Since I is graded, $\mathrm{KP}_R(\Lambda) = \mathbb{F}_R(w(X))/I$ is graded by the subgroups

$$(\mathbb{F}_R(w(X))/I)_n := \operatorname{span}_R\{q(w) : w \in w(X), d(w) = n\}.$$

By Proposition 3.1.3(b), we have $\mathrm{KP}_R(\Lambda) = \operatorname{span}_R\{s_\lambda s_{\mu^*} : \lambda, u \in \Lambda, s(\lambda) = s(\mu)\}$. We have to show that

$$\mathrm{KP}_R(\Lambda)_n := \operatorname{span}_R\{s_\lambda s_{\mu^*} : \lambda, u \in \Lambda, d(\lambda) - d(\mu) = n\} = (\mathbb{F}_R(w(X))/I)_n.$$

Take $\lambda, u \in \Lambda$ with $d(\lambda) - d(\mu) = n$. Then $s_\lambda s_{\mu^*} = q(\lambda)q(\mu^*) = q(\lambda\mu^*)$ and $d(\lambda\mu^*) = d(\lambda) - d(\mu) = n$. Hence $s_\lambda s_{\mu^*} \in (\mathbb{F}_R(w(X))/I)_n$, and $\mathrm{KP}_R(\Lambda)_n \subseteq (\mathbb{F}_R(w(X))/I)_n$.

Before proving that $(\mathbb{F}_R(w(X))/I)_n \subseteq \mathrm{KP}_R(\Lambda)_n$, we establish the following claim:

Claim 3.1.9. *Let $X := \Lambda \cup G(\Lambda^{\neq 0})$ and $q : \mathbb{F}_R(w(X)) \to \mathrm{KP}_R(\Lambda)$ be the quotient map. Then for $w \in w(X)$, we have $q(w) \in \mathrm{KP}_R(\Lambda)_{d(w)}$.*

Proof of Claim 3.1.9. We modify the proof of [11, Lemma 3.5] and [16, Lemma 3.8] using our version of (KP3). We prove the claim by induction on $|w|$. For $|w| = 0$, we have $w \in \Lambda^0$. Then $q(w) = s_v = s_v s_{v^*}$ and $d(v) - d(v) = 0$, so $q(w) \in \mathrm{KP}_R(\Lambda)_{d(w)}$.

For $|w| = 1$, we have two possibilities. If $w = \lambda$ for $\lambda \in \Lambda$, then $q(w) = s_\lambda = s_\lambda s_{s(\lambda)^*}$, $d(\lambda) - d(s(\lambda)) = d(\lambda)$, and $q(w) \in \mathrm{KP}_R(\Lambda)_{d(w)}$. If $w = \lambda^*$ for $\lambda \in \Lambda$, then $q(w) = s_{\lambda^*} = s_{s(\lambda)} s_{\lambda^*}$, $d(s(\lambda)) - d(\lambda) = -d(\lambda) = d(\lambda^*)$, and $q(w) \in \mathrm{KP}_R(\Lambda)_{d(w)}$.

For $|w| = 2$, we have four possibilities: $w = \lambda\mu^*$, $w = \lambda\mu$, $w = \mu^*\lambda^*$, or $w = \lambda^*\mu$. For the first three possibilities, we have

$$\begin{aligned}
&q(\lambda\mu^*) = s_\lambda s_{\mu^*} \text{ and } d(\lambda) - d(\mu) = d(\lambda\mu^*), \\
&q(\lambda\mu) = s_{\lambda\mu} s_{s(\mu)^*} \text{ and } d(\lambda\mu) - d(s(\mu)) = d(\lambda\mu), \\
&q(\mu^*\lambda^*) = s_{s(\mu)} s_{(\lambda\mu)^*} \text{ and } d(s(\mu)) - d((\lambda\mu)^*) = d(\mu^*\lambda^*),
\end{aligned}$$

as required. So suppose that $w = \lambda^*\mu$. By (KP3), we have

$$q(\lambda^*\mu) = s_{\lambda^*}s_\mu = \sum_{(\rho,\tau)\in\Lambda^{\min}(\lambda,\mu)} s_\rho s_{\tau^*}.$$

For $(\rho,\tau) \in \Lambda^{\min}(\lambda,\mu)$, we have $\lambda\rho = \mu\tau$ and $d(w) = d(\mu) - d(\lambda) = d(\rho) - d(\rho)$, so $q(w) \in \mathrm{KP}_R(\Lambda)_{d(w)}$.

Now suppose that $n \geq 2$ and $q(y) \in \mathrm{KP}_R(\Lambda)_{d(y)}$ for every word y with $|y| \leq n$. Let w be a word with $|w| = n+1$ and $q(w) \neq 0$. If w contains a subword $w_i w_{i+1} = \lambda\mu$, then λ and μ are composable in Λ since otherwise $q(\lambda\mu) = 0$. Now let w' be the word obtained from w by replacing $w_i w_{i+1}$ with the single path $\lambda\mu$, and then

$$q(w) = s_{w_1}\cdots s_{w_{i-1}} s_\lambda s_\mu s_{w_{i+2}} s_{w_{n+1}} = s_{w_1}\cdots s_{w_{i-1}} s_{\lambda\mu} s_{w_{i+2}} s_{w_{n+1}} = q(w').$$

Since $|w'| = n$ and $d(w') = d(w)$, the inductive hypothesis implies that $q(w) \in \mathrm{KP}_R(\Lambda)_{d(w)}$. A similar argument shows $q(w) \in \mathrm{KP}_R(\Lambda)_{d(w)}$ whenever w contains a subword $w_i w_{i+1} = \mu^*\lambda^*$.

So suppose that w contains no subword of the form $\lambda\mu$ or $\mu^*\lambda^*$. Since $|w| \geq 3$, either $w_1 w_2$ or $w_2 w_3$ has the form $\lambda^*\mu$. By (KP3), we write $q(w)$ as a sum of terms $q(y^i)$ with $|y^i| = n+1$ and $d(y^i) = d(w)$. Since $|w| \geq 3$, each nonzero summand $q(y^i)$ contains a factor of the form $s_\gamma s_\rho$ or one of the form $s_{\tau^*}s_{\gamma^*}$. Then the previous argument shows that every $q(y^i) \in \mathrm{KP}_R(\Lambda)_{d(w)}$ and $q(w) \in \mathrm{KP}_R(\Lambda)_{d(w)}$, as required.

$\square$ Claim 3.1.9

Every element of $(\mathbb{F}_R(w(X))/I)_n$ has the form $q(w)$ with $w \in w(X)$ and $d(w) = n$, which belongs to $\mathrm{KP}_R(\Lambda)_n$ by the claim. Thus $(\mathbb{F}_R(w(X))/I)_n \subseteq \mathrm{KP}_R(\Lambda)_n$, as required. $\square$

Definition 3.1.10. Suppose that $\{S_\lambda, S_{\mu^*} : \lambda, u \in \Lambda\}$ is the Kumjian-Pask Λ-family in the R-algebra $\mathrm{End}(\mathbb{F}_R(\partial\Lambda))$ of Proposition 3.1.7. We call the R-algebra homomorphism $\pi_S : \mathrm{KP}_R(\Lambda) \to \mathrm{End}(\mathbb{F}_R(\partial\Lambda))$ obtained from Theorem 3.1.8(a) the *boundary path representation of* $\mathrm{KP}_R(\Lambda)$.

We say more about the boundary path representation of $\mathrm{KP}_R(\Lambda)$ in Section 3.7.

3.2 The graded uniqueness theorem

Throughout this section, Λ is a finitely aligned k-graph and R is a commutative ring with identity 1.

Both [11] and [16] contain two uniqueness theorems for Kumjian-Pask algebras (see Section 1.4). The graded-uniqueness theorem has no hypothesis on the graph, and is an analogue of the gauge-invariant uniqueness theorem for k-graph C^*-algebras [41, Theorem 4.2]. The Cuntz-Krieger uniqueness theorem only applies to "aperiodic" k-graphs. As the name suggests, this is an analogue to the Cuntz-Krieger uniqueness theorem for k-graph C^*-algebras [41, Theorem 4.5]. For an overview of the uniqueness theorems for k-graph C^*-algebras, see Section 1.2.

In this section, we establish a graded-uniqueness theorem for Kumjian-Pask algebras of finitely aligned k-graphs (Theorem 3.2.1). We shall discuss a Cuntz Krieger uniqueness theorem in Section 3.7.

Theorem 3.2.1 ([18, Theorem 4.1]: The graded uniqueness theorem)**.** *Suppose that Λ is a finitely aligned k-graph, that R is a commutative ring with 1, and that A is a $\mathbb{Z}^k$-graded R-algebra. Suppose that $\pi : \mathrm{KP}_R(\Lambda) \to A$ is a $\mathbb{Z}^k$-graded ring homomorphism such that $\pi(rs_v) \neq 0$ for all $r \in R\backslash\{0\}$ and $v \in \Lambda^0$. Then π is injective.*

Our proof of Theorem 3.2.1 is based on the proof of the gauge-invariant uniqueness theorem for C^*-algebras [41, Theorem 4.2]. Although the argument is rather technical, the work in [49] for C^*-algebras carry over to the algebraic setting without change as the ring elements will not feature. We check the details below. However, we omit the proofs of the graph theoretic results as they follow exactly as in [39, 41, 49].

Remark 3.2.2. The work in [49] is for Toeplitz-Cuntz-Krieger families. So the result also applies to Cuntz-Krieger families.We do not change the arguments when carrying over to the algebraic setting.

We divide the arguments into four subsections. We establish some preliminary notation and results in Subsection 3.2.1. We then introduce a subalgebra $M^s_{\Pi E}$ which is closed under multiplication (Subsection 3.2.2) and identify its matrix units $\Theta(s)^{\Pi E}_{\lambda,\mu}$ (Subsection 3.2.3). In Lemma 3.2.21, we show that the homomorphism π of Theorem 3.2.1 is injective on each matrix unit $\Theta(s)^{\Pi E}_{\lambda,\mu}$; and so also on each subalgebra $M^s_{\Pi E}$. We then prove the graded uniqueness theorem.

3.2.1 Orthogonalising range projection

We introduce a set E that is closed under taking minimal common extensions (see Proposition 3.2.6) and establish preliminary results related to such a set. This is an algebraic version of Section 3.3 of [49] and hence we follow the arguments there.

Definition 3.2.3. Suppose that Λ is a finitely aligned k-graph, that E is a finite subset of Λ, and that R is a commutative ring with 1. Suppose that $\{S_\lambda, S_{\mu^*} : \lambda, u \in \Lambda\}$ is a Kumjian-Pask Λ-family in an R-algebra A. For $\lambda \in E$, we define

$$Q(S)^E_\lambda := S_\lambda S_{\lambda^*} \prod_{\substack{\lambda\nu \in E \\ d(\nu)\neq 0}} \left(S_\lambda S_{\lambda^*} - S_{\lambda\nu} S_{(\lambda\nu)^*}\right).$$

Remark 3.2.4. Note that by Remark 3.1.4, $Q(S)^E_\lambda$ is well-defined for $\lambda \in E$. For $\lambda \in E$, we have

$$\left(Q(S)^E_\lambda\right)^2 = Q(S)^E_\lambda \text{ and } Q(S)^E_\lambda (S_\lambda S_{\lambda^*}) = Q(S)^E_\lambda = (S_\lambda S_{\lambda^*}) Q(S)^E_\lambda.$$

Remark 3.2.5. Our $Q(S)^E_\lambda$ is an algebraic analogue of [49, Definition 3.3.1]. In general, $Q(S)^E_\lambda$ of [49] is different from Raeburn and Sims' $Q(S)^E_\lambda$ of [39]. However, both $Q(S)^E_\lambda$ share the same properties as we shall see in this subsection.

The main aim of this subsection is to prove the following proposition, which is an algebraic analogue of [49, Proposition 3.3.3].

Proposition 3.2.6. *Suppose that Λ is a finitely aligned k-graph and that R is a commutative ring with 1. Suppose that $\{S_\lambda, S_{\mu^*} : \lambda, u \in \Lambda\}$ is a Kumjian-Pask Λ-family in an R-algebra A. Suppose that $E \subseteq \Lambda$ is finite and non-empty and that E is closed under taking minimal common extensions in the sense that*

$$\lambda, \mu \in E \Rightarrow \mathrm{MCE}(\lambda, \mu) \subseteq E. \tag{3.2.1}$$

Then $\{Q(S)^E_\lambda : \lambda \in E\}$ is a collection of mutually orthogonal (possibly zero) idempotents such that for all $v \in r(E)$,

$$\left(S_v \prod_{\lambda \in vE} (S_v - S_\lambda S_{\lambda^*})\right) + \sum_{\lambda \in vE} Q(S)^E_\lambda = S_v. \tag{3.2.2}$$

To prove Proposition 3.2.6, we first consider the case when E consists of paths with fixed range $v \in \Lambda^0$, for some vertex v in E, as stated in the following lemma. This is an algebraic analogue of Lemma 3.3.4 of [49].

Lemma 3.2.7. *Suppose that Λ is a finitely aligned k-graph and that R is a commutative ring with 1. Suppose that $v \in \Lambda^0$ and that E is a finite subset of $v\Lambda$ containing v. Suppose that E satisfies (3.2.1) and that $\{S_\lambda, S_{\mu^*} : \lambda, u \in \Lambda\}$ is a Kumjian-Pask Λ-family in an R-algebra A. Then $\{Q(S)^E_\lambda : \lambda \in E\}$ is a collection of mutually orthogonal (possibly zero) idempotents such that*

$$\sum_{\lambda \in E} Q(S)^E_\lambda = S_v.$$

In the rest of this subsection, we prove Lemma 3.2.7, and then use the lemma to deduce Proposition 3.2.6.

We prove Lemma 3.2.7 by induction on the cardinality of E. The problem is that when we remove a path λ from a set E satisfying (3.2.1), the new set $E\backslash\{\lambda\}$ may not satisfy (3.2.1). Therefore instead of working on any finite set $E \subseteq \Lambda$, we work on a set $\vee E$ which contains E and satisfies (3.2.1). Now we construct $\vee E$ as follows:

Definition 3.2.8 ([39, Definition 8.3])**.** Suppose that Λ is a finitely aligned k-graph and that $E \subseteq \Lambda$ is finite. We then define

$$\mathrm{MCE}(E) := \left\{\lambda \in \Lambda : d(\lambda) = \bigvee_{\mu\in E} d(\mu) \text{ and } \lambda(0, d(\mu)) = \mu \text{ for all } \mu \in E\right\}$$

and

$$\vee E := \bigcup_{G\subseteq E} \mathrm{MCE}(G).$$

Lemma 3.2.9 ([39, Lemma 8.4])**.** *Suppose that Λ is a finitely aligned k-graph. Suppose that $v \in \Lambda^0$ and that $E \subseteq v\Lambda$ is a finite set which contains v. Then*

(a) $E \subseteq \vee E$;

(b) $\vee E$ is finite;

(c) $F \subseteq \vee E$ implies $\mathrm{MCE}(F) \subseteq \vee E$; and

(d) $\lambda \in \vee E$ implies $d(\lambda) \leq \bigvee_{\mu\in E} d(\mu)$.

Remark 3.2.10. If $E \subseteq \Lambda$ already satisfies (3.2.1), then an induction on $|G|$ shows that $G \subseteq E$ implies $\mathrm{MCE}(G) \subseteq E$, and then $\vee E \subseteq E$. Hence by Lemma 3.2.9(a), we have $\vee E = E$.

Now we establish two lemmas that are needed in the proof of Lemma 3.2.7.

Lemma 3.2.11 ([39, Lemma 8.7])**.** *Suppose that Λ is a finitely aligned k-graph and that $v \in \Lambda^0$. Suppose that $E \subseteq v\Lambda$ is a finite set which contains v and that $\lambda \in E\backslash\{v\}$. Let $F := E\backslash\{\lambda\}$. Then for every $\gamma \in \vee E\backslash \vee F$, there exists a unique $\mu_\gamma \in \vee F$ such that*

(i) $d(\gamma) \geq d(\mu_\gamma)$ and $\gamma(0, d(\mu_\gamma)) = \mu_\gamma$; and

(ii) if $\mu \in \vee F$ and $\gamma(0, d(\mu)) = \mu$, then $d(\mu) \leq d(\mu_\gamma)$ and $\mu_\gamma(0, d(\mu)) = \mu$.

Furthermore, for all $\gamma \in \vee E \setminus \vee F$, $\gamma \in \mathrm{MCE}(\mu_\gamma, \lambda)$.

Lemma 3.2.12. *Suppose that Λ is a finitely aligned k-graph, that $v \in \Lambda^0$, and that R is a commutative ring with 1. Suppose that $E \subseteq v\Lambda$ is a finite set which contains v, and that $\lambda \in E\setminus\{v\}$. Set $F := E\setminus\{\lambda\}$. Suppose also that $\{S_\lambda, S_{\mu^*} : \lambda, u \in \Lambda\}$ is a Kumjian-Pask Λ-family in an R-algebra A, that $\gamma \in \vee E \setminus \vee F$ and that $\mu_\gamma \in \vee F$ is the path of Lemma 3.2.11. Then*

(a) $Q(S)^{\vee E}_{\gamma} = Q(S)^{\vee F}_{\mu_\gamma} Q(S)^{\vee E}_{\gamma}$;

(b) If $\gamma\nu \in \vee E$ with $d(\nu) \neq 0$, then $Q(S)^{\vee F}_{\mu_\gamma} S_{\gamma\nu} S_{(\gamma\nu)^*} = 0$; and

(c) $Q(S)^{\vee E}_{\gamma} = Q(S)^{\vee F}_{\mu_\gamma} S_\gamma S_{\gamma^*}$.

Proof. We follow the C^*-algebraic argument of [39, Lemma 8.8]. First we show part (a). Since $\gamma(0, d(\mu_\gamma)) = \mu_\gamma$, then by Proposition 3.1.3(a), $S_\gamma S_{\gamma^*} S_{\mu_\gamma} S_{\mu_\gamma^*} = S_\gamma S_{\gamma^*}$ and

$$
\begin{aligned}
Q(S)^{\vee F}_{\mu_\gamma} Q(S)^{\vee E}_{\gamma} &= \Big(S_{\mu_\gamma} S_{\mu_\gamma^*} \prod_{\substack{\mu_\gamma\nu \in \vee F \\ d(\nu)\neq 0}} \big(S_{\mu_\gamma} S_{\mu_\gamma^*} - S_{\mu_\gamma\nu} S_{(\mu_\gamma\nu)^*}\big)\Big) Q(S)^{\vee E}_{\gamma} \\
&= \Big(S_{\mu_\gamma} S_{\mu_\gamma^*} \prod_{\substack{\mu_\gamma\nu \in \vee F \\ d(\nu)\neq 0}} \big(S_{\mu_\gamma} S_{\mu_\gamma^*} - S_{\mu_\gamma\nu} S_{(\mu_\gamma\nu)^*}\big)\Big) (S_\gamma S_{\gamma^*}) Q(S)^{\vee E}_{\gamma} \\
&\quad \text{(since } (S_\gamma S_{\gamma^*})^2 = S_\gamma S_{\gamma^*} \text{ and then } S_\gamma S_{\gamma^*} Q(S)^{\vee E}_{\gamma} = Q(S)^{\vee E}_{\gamma}\text{)} \\
&= \Big((S_{\mu_\gamma} S_{\mu_\gamma^*} S_\gamma S_{\gamma^*}) \prod_{\substack{\mu_\gamma\nu \in \vee F \\ d(\nu)\neq 0}} \big(S_{\mu_\gamma} S_{\mu_\gamma^*} - S_{\mu_\gamma\nu} S_{(\mu_\gamma\nu)^*}\big)\Big) Q(S)^{\vee E}_{\gamma} \\
&\quad \text{(by Proposition 3.1.3(a))} \\
&= S_\gamma S_{\gamma^*} \Big(\prod_{\substack{\mu_\gamma\nu \in \vee F \\ d(\nu)\neq 0}} \big(S_{\mu_\gamma} S_{\mu_\gamma^*} - S_{\mu_\gamma\nu} S_{(\mu_\gamma\nu)^*}\big)\Big) Q(S)^{\vee E}_{\gamma}
\end{aligned}
$$

since $S_\gamma S_{\gamma^*} S_{\mu_\gamma} S_{\mu_\gamma^*} = S_\gamma S_{\gamma^*}$. Hence it suffices to show that

$$S_\gamma S_{\gamma^*} \big(S_{\mu_\gamma} S_{\mu_\gamma^*} - S_{\mu_\gamma\nu} S_{(\mu_\gamma\nu)^*}\big) Q(S)^{\vee E}_{\gamma} = Q(S)^{\vee E}_{\gamma} \tag{3.2.3}$$

for all $\mu_\gamma\nu \in G$ with $d(\nu) \neq 0$.

Take $\mu_\gamma\nu \in \vee F$ with $d(\nu) \neq 0$ and note that

$$S_\gamma S_{\gamma^*} \big(S_{\mu_\gamma} S_{\mu_\gamma^*} - S_{\mu_\gamma\nu} S_{(\mu_\gamma\nu)^*}\big) = S_\gamma S_{\gamma^*} - \sum_{\gamma\tau \in \mathrm{MCE}(\gamma, \mu_\gamma\nu)} S_{\gamma\tau} S_{(\gamma\tau)^*} \tag{3.2.4}$$

(by Proposition 3.1.3(a))

$$= \prod_{\gamma\tau \in \mathrm{MCE}(\gamma, \mu_\gamma \nu)} \Big(S_\gamma S_{\gamma^*} - S_{\gamma\tau} S_{(\gamma\tau)^*} \Big).$$

We claim that for $\gamma\tau \in \mathrm{MCE}(\gamma, \mu_\gamma \nu)$, we have $\gamma\tau \in \vee E$ and $d(\tau) \neq 0$. Take $\gamma\tau \in \mathrm{MCE}(\gamma, \mu_\gamma \nu)$. Then $(\gamma\tau)(0, d(\mu_\gamma \nu)) = \mu_\gamma \nu \in \vee F$ and by Lemma 3.2.11(ii), $d(\mu_{\gamma\tau}) \geq d(\mu_\gamma \nu) > d(\mu_\gamma)$ (the last inequality holds because $d(\nu) \neq 0$). If $\gamma\tau = \gamma$, then $\mu_{\gamma\tau} = \mu_\gamma$ and $d(\mu_{\gamma\tau}) = d(\mu_\gamma)$, which contradicts $d(\mu_{\gamma\tau}) > d(\mu_\gamma)$. So $\gamma\tau \neq \gamma$ and $d(\tau) \neq 0$. Since $\mu_\gamma \nu \in (\vee F \cap \vee E)$ and $\gamma \in \vee E$, Lemma 3.2.9(c) gives $\gamma\tau \in \mathrm{MCE}(\gamma, \mu_\gamma \nu) \subseteq \vee E$ as claimed.

Thus each factor in (3.2.4) is a factor in $Q(S)^{\vee E}_{\gamma}$, and then we have (3.2.3), as required.

Now we show part (b). Take $\gamma\nu \in \vee E$ with $d(\nu) \neq 0$. We have to show $Q(S)^{\vee G}_{\mu_\gamma} S_{\gamma\nu} S_{(\gamma\nu)^*} = 0$. Suppose for contradiction that $\mu_{\gamma\nu} = \mu_\gamma$. Then

$$\begin{aligned} d(\gamma\nu) &= d(\lambda) \vee d(\mu_{\gamma\nu}) \ \ (\text{since } \gamma\nu \in \mathrm{MCE}(\mu_{\gamma\nu}, \lambda) \ \text{ by Lemma 3.2.11}) \\ &= d(\lambda) \vee d(\mu_\gamma) \ \ (\text{since } \mu_{\gamma\nu} = \mu_\gamma) \\ &= d(\gamma) \ \ (\text{since } \gamma \in \mathrm{MCE}(\mu_\gamma, \lambda) \ \text{ by Lemma 3.2.11}) \end{aligned}$$

which contradicts $d(\nu) \neq 0$. So $\mu_{\gamma\nu} \neq \mu_\gamma$. Since $(\gamma\nu)(0, d(\mu_\gamma)) = \gamma(0, d(\mu_\gamma)) = \mu_\gamma \in \vee F$, Lemma 3.2.11(ii) gives $d(\mu_{\gamma\nu}) \geq d(\mu_\gamma)$; since $\mu_{\gamma\nu} \neq \mu_\gamma$, we have $\mu_{\gamma\nu} = \mu_\gamma \tau$ for some τ with $d(\tau) \neq 0$. Since $\mu_{\gamma\nu} \in \vee F$, then

(3.2.5)

$$\begin{aligned} &Q(S)^{\vee F}_{\mu_\gamma} S_{\gamma\nu} S_{(\gamma\nu)^*} \\ &\quad = \Big(S_{\mu_\gamma} S_{\mu_\gamma^*} \prod_{\substack{\mu_\gamma \nu \in \vee F \\ d(\nu) \neq 0}} \big(S_{\mu_\gamma} S_{\mu_\gamma^*} - S_{\mu_\gamma \nu} S_{(\mu_\gamma \nu)^*} \big) \Big) S_{\gamma\nu} S_{(\gamma\nu)^*} \\ &\quad = S_{\mu_\gamma} S_{\mu_\gamma^*} \Big(\prod_{\substack{\mu_\gamma \nu \in \vee F \setminus \{\mu_\gamma \tau\} \\ d(\nu) \neq 0}} \big(S_{\mu_\gamma} S_{\mu_\gamma^*} - S_{\mu_\gamma \nu} S_{(\mu_\gamma \nu)^*} \big) \Big) \big(S_{\mu_\gamma} S_{\mu_\gamma^*} - S_{\mu_\gamma \tau} S_{(\mu_\gamma \tau)^*} \big) S_{\gamma\nu} S_{(\gamma\nu)^*} \\ &\quad = S_{\mu_\gamma} S_{\mu_\gamma^*} \Big(\prod_{\substack{\mu_\gamma \nu \in \vee F \setminus \{\mu_\gamma \tau\} \\ d(\nu) \neq 0}} \big(S_{\mu_\gamma} S_{\mu_\gamma^*} - S_{\mu_\gamma \nu} S_{(\mu_\gamma \nu)^*} \big) \Big) \big(S_{\mu_\gamma} S_{\mu_\gamma^*} - S_{\mu_{\gamma\nu}} S_{\mu_{\gamma\nu}^*} \big) S_{\gamma\nu} S_{(\gamma\nu)^*} \end{aligned}$$

since $\mu_{\gamma\nu} = \mu_\gamma \tau$. Since $(\gamma\nu)(0, d(\mu_\gamma)) = \mu_\gamma$ and $(\gamma\nu)(0, d(\mu_{\gamma\nu})) = \mu_{\gamma\nu}$, Proposition 3.1.3(a) gives

$$\big(S_{\mu_\gamma} S_{\mu_\gamma^*} - S_{\mu_{\gamma\nu}} S_{\mu_{\gamma\nu}^*} \big) S_{\gamma\nu} S_{(\gamma\nu)^*} = S_{\gamma\nu} S_{(\gamma\nu)^*} - S_{\gamma\nu} S_{(\gamma\nu)^*} = 0$$

and (3.2.5) becomes $Q(S)^{\vee F}_{\mu_\gamma} S_{\gamma\nu} S_{(\gamma\nu)^*} = 0$, as required.

Finally we show part (c). Note that

$$
\begin{aligned}
(3.2.6) \qquad Q(S)^{\vee E}_{\gamma} &= Q(S)^{\vee F}_{\mu_\gamma} Q(S)^{\vee E}_{\gamma} \text{ (by part (a))} \\
&= Q(S)^{\vee F}_{\mu_\gamma} S_\gamma S_{\gamma^*} \prod_{\substack{\gamma\nu \in \vee E \\ d(\nu)\neq 0}} \left(S_\gamma S_{\gamma^*} - S_{\gamma\nu} S_{(\gamma\nu)^*}\right) \\
&= Q(S)^{\vee F}_{\mu_\gamma} \prod_{\substack{\gamma\nu \in \vee E \\ d(\nu)\neq 0}} \left(S_\gamma S_{\gamma^*} - S_{\gamma\nu} S_{(\gamma\nu)^*}\right)
\end{aligned}
$$

since $\gamma(0, d(\mu_\gamma)) = \mu_\gamma$. Thus $S_{\mu_\gamma} S_{\mu_\gamma^*} S_\gamma S_{\gamma^*} = S_\gamma S_{\gamma^*}$ and $Q(S)^{\vee F}_{\mu_\gamma} S_\gamma S_{\gamma^*} = Q(S)^{\vee F}_{\mu_\gamma}$. By part (b), (3.2.6) becomes $Q(S)^{\vee E}_{\gamma} = Q(S)^{\vee F}_{\mu_\gamma} S_\gamma S_{\gamma^*}$, as required. □

Now we are ready to prove Lemma 3.2.7.

Proof of Lemma 3.2.7. Since E satisfies (3.2.1), Remark 3.2.10 gives $\vee E = E$. So it suffices to show that if E is a finite subset of $v\Lambda$ such that $v \in E$, then $Q(S)^{\vee E}_{\lambda} Q(S)^{\vee E}_{\mu} = \delta_{\lambda,\mu} Q(S)^{\vee E}_{\lambda}$ and $\sum_{\lambda \in \vee E} Q(S)^{\vee E}_{\lambda} = S_v$. Note that if $\lambda = \mu$, then we already have $\left(Q(S)^{\vee E}_{\lambda}\right)^2 = Q(S)^{\vee E}_{\lambda}$ from Remark 3.2.4.

So take $\lambda, \mu \in \vee E$ with $\lambda \neq \mu$ and we have to show $Q(S)^{\vee E}_{\lambda} Q(S)^{\vee E}_{\mu} = 0$. First suppose that $d(\lambda) = d(\mu)$. Note that by Remark 3.2.4,

$$
(3.2.7) \qquad Q(S)^{\vee E}_{\lambda} (S_\lambda S_{\lambda^*}) = Q(S)^{\vee E}_{\lambda} \text{ and } Q(S)^{\vee E}_{\mu} (S_\mu S_{\mu^*}) = Q(S)^{\vee E}_{\mu}.
$$

Then by Proposition 3.1.3(a),

$$
Q(S)^{\vee E}_{\lambda} Q(S)^{\vee E}_{\mu} (S_\lambda S_{\lambda^*} S_\mu S_{\mu^*}) = Q(S)^{\vee E}_{\lambda} Q(S)^{\vee E}_{\mu}
$$

and by Remark 3.1.5, $S_\lambda S_{\lambda^*} S_\mu S_{\mu^*} = 0$ (using $d(\lambda) = d(\mu)$). Hence $Q(S)^{\vee E}_{\lambda} Q(S)^{\vee E}_{\mu} = 0$, as required.

Next suppose that $d(\lambda) \neq d(\mu)$. Then $d(\lambda) \vee d(\mu)$ is strictly larger that at least one of $d(\lambda)$ and $d(\mu)$. Without loss of generality, we assume that $d(\lambda) \vee d(\mu) > d(\lambda)$. Then $\gamma \in \mathrm{MCE}(\lambda, \mu)$ implies that $\gamma = \lambda\nu$ with $d(\nu) \neq 0$ and $\gamma \in \vee E$ (since $\lambda, \mu \in \vee E$ and Lemma 3.2.9(c)). Now from (3.2.7) and Proposition 3.1.3(a), we have

(3.2.8)

$$
\begin{aligned}
Q(S)^{\vee E}_{\lambda} Q(S)^{\vee E}_{\mu} &= (S_\mu S_{\mu^*}) Q(S)^{\vee E}_{\lambda} Q(S)^{\vee E}_{\mu} \\
&= (S_\mu S_{\mu^*} S_\lambda S_{\lambda^*}) \Big(\prod_{\substack{\lambda\nu \in \vee E \\ d(\nu)\neq 0}} \left(S_\lambda S_{\lambda^*} - S_{\lambda\nu} S_{(\lambda\nu)^*}\right) \Big) Q(S)^{\vee E}_{\mu} \\
&= \Big(\sum_{\lambda\rho \in \mathrm{MCE}(\lambda,\mu)} S_{\lambda\rho} S_{(\lambda\rho)^*} \Big) \Big(\prod_{\substack{\lambda\nu \in \vee E \\ d(\nu)\neq 0}} \left(S_\lambda S_{\lambda^*} - S_{\lambda\nu} S_{(\lambda\nu)^*}\right) \Big) Q(S)^{\vee E}_{\mu}
\end{aligned}
$$

by Proposition 3.1.3(a). Now note that for all $\lambda\rho \in \mathrm{MCE}(\lambda, \mu)$, we have

$$S_{\lambda\rho}S_{(\lambda\rho)^*}\left(S_\lambda S_{\lambda^*} - S_{\lambda\rho}S_{(\lambda\rho)^*}\right) = S_{\lambda\rho}S_{(\lambda\rho)^*} - S_{\lambda\rho}S_{(\lambda\rho)^*} = 0$$

and then

$$S_{\lambda\rho}S_{(\lambda\rho)^*}\prod_{\substack{\lambda\nu\in\vee E\\ d(\nu)\neq 0}}\left(S_\lambda S_{\lambda^*} - S_{\lambda\nu}S_{(\lambda\nu)^*}\right) = 0$$

because $\lambda\rho \in \vee E$ and $d(\rho) \neq 0$. So (3.2.8) implies $Q(S)_\lambda^{\vee E} Q(S)_\mu^{\vee E} = 0$.

Next we show that $\sum_{\lambda\in\vee E} Q(S)_\lambda^{\vee E} = S_v$. We prove this by induction on $|E|$. First suppose $|E| = 1$. Since $v \in E$ by assumption, then $\vee E = E = \{v\}$ and $\sum_{\lambda\in\vee E} Q(S)_\lambda^{\vee E} = Q(S)_v^{\{v\}} = S_v$.

Now suppose that $\sum_{\lambda\in\vee G} Q(S)_\lambda^{\vee G} = S_v$ for all $|G| \leq n-1$ for some $n \geq 2$. Suppose $|E| = n$ and we have to show $\sum_{\lambda\in\vee E} Q(S)_\lambda^{\vee E} = S_v$. Since $|E| \geq 2$, then there exists $\lambda \in E\backslash\{v\}$. Define $F := E\backslash\{v\}$. For $\gamma \in \vee F$, we have

$$\begin{aligned}(3.2.9)\qquad Q(S)_\gamma^{\vee E} &= S_\gamma S_{\gamma^*}\prod_{\substack{\gamma\nu\in\vee E\\ d(\nu)\neq 0}}\left(S_\gamma S_{\gamma^*} - S_{\gamma\nu}S_{(\gamma\nu)^*}\right)\\ &= S_\gamma S_{\gamma^*}\prod_{\substack{\gamma\nu\in\vee F\\ d(\nu)\neq 0}}\left(S_\gamma S_{\gamma^*} - S_{\gamma\nu}S_{(\gamma\nu)^*}\right)\prod_{\substack{\gamma\tau\in\vee E\backslash\vee F\\ d(\tau)\neq 0}}\left(S_\gamma S_{\gamma^*} - S_{\gamma\tau}S_{(\gamma\tau)^*}\right).\end{aligned}$$

We claim that for $\gamma\tau \in \vee E\backslash\vee F$ with $\mu_{\gamma\tau} \neq \gamma$, the factor $S_\gamma S_{\gamma^*} - S_{\gamma\tau}S_{(\gamma\tau)^*}$ from (3.2.9) can be deleted without changing the result of the product. Take $\gamma\tau \in \vee E\backslash\vee F$ with $\mu_{\gamma\tau} \neq \gamma$. Since $\gamma \in \vee F$ and $(\gamma\tau)(0, d(\gamma)) = \gamma$, then by Lemma 3.2.11(ii), $\mu_{\gamma\tau} = \gamma\rho$ for some ρ with $d(\rho) > 0$. Since $\gamma\rho = \mu_{\gamma\tau} \in \vee F$, then $S_\gamma S_{\gamma^*} - S_{\mu_{\gamma\tau}}S_{\mu_{\gamma\tau}^*}$ is a factor in $Q(S)_\gamma^{\vee E}$. Now from Lemma 3.2.11(i)) and Proposition 3.1.3(a), we have

$$\left(S_\gamma S_{\gamma^*} - S_{\mu_{\gamma\tau}}S_{\mu_{\gamma\tau}^*}\right)\left(S_\gamma S_{\gamma^*} - S_{\gamma\tau}S_{(\gamma\tau)^*}\right) = S_\gamma S_{\gamma^*} - S_{\gamma\tau}S_{(\gamma\tau)^*}$$

since $(\gamma\tau)(0, d(\mu_{\gamma\tau})) = \mu_{\gamma\tau}$. Hence

$$\left(S_\gamma S_{\gamma^*} - S_{\gamma\tau}S_{(\gamma\tau)^*}\right) Q(S)_\gamma^{\vee E} = Q(S)_\gamma^{\vee E}$$

and we can delete the factors $S_\gamma S_{\gamma^*} - S_{\gamma\tau}S_{(\gamma\tau)^*}$ from (3.2.9), as claimed. Therefore we can rewrite (3.2.9) as

$$\begin{aligned}(3.2.10)\qquad Q(S)_\gamma^{\vee E} &= S_\gamma S_{\gamma^*}\prod_{\substack{\gamma\nu\in\vee F\\ d(\nu)\neq 0}}\left(S_\gamma S_{\gamma^*} - S_{\gamma\nu}S_{(\gamma\nu)^*}\right)\prod_{\substack{\gamma\tau\in\vee E\backslash\vee F\\ \mu_{\gamma\tau}=\gamma}}\left(S_\gamma S_{\gamma^*} - S_{\gamma\tau}S_{(\gamma\tau)^*}\right)\\ &= Q(S)_\gamma^{\vee F}\prod_{\substack{\gamma\tau\in\vee E\backslash\vee F\\ \mu_{\gamma\tau}=\gamma}}\left(S_\gamma S_{\gamma^*} - S_{\gamma\tau}S_{(\gamma\tau)^*}\right).\end{aligned}$$

Now note that for $\gamma\tau \in \vee E \backslash \vee F$, we have $\gamma\tau \in \operatorname{MCE}(\mu_{\gamma\tau}, \lambda)$ by Lemma 3.2.11. So for $\gamma\tau \in \vee E \backslash \vee F$ with $\mu_{\gamma\tau} = \gamma$, we have $\gamma\tau \in \operatorname{MCE}(\gamma, \lambda)$ and then $d(\gamma\tau) = d(\gamma) \vee d(\lambda)$. So if $\gamma\tau, \gamma\tau' \in \vee E \backslash \vee F$ such that $\mu_{\gamma\tau} = \gamma = \mu_{\gamma\tau'}$, then $d(\mu_{\gamma\tau}) = d(\mu_{\gamma\tau'})$ and so

$$S_{\gamma\tau} S_{(\gamma\tau)^*} S_{\gamma\tau'} S_{(\gamma\tau')^*} = \delta_{\tau,\tau'} S_{\gamma\tau} S_{(\gamma\tau)^*}$$

by Remark 3.1.5. Hence

$$\prod_{\substack{\gamma\tau \in \vee E \backslash \vee F \\ \mu_{\gamma\tau} = \gamma}} \left(S_\gamma S_{\gamma^*} - S_{\gamma\tau} S_{(\gamma\tau)^*} \right) = S_\gamma S_{\gamma^*} - \sum_{\substack{\gamma\tau \in \vee E \backslash \vee F \\ \mu_{\gamma\tau} = \gamma}} S_{\gamma\tau} S_{(\gamma\tau)^*},$$

and we can rewrite (3.2.10) as

$$Q(S)_\gamma^{\vee E} = Q(S)_\gamma^{\vee F} \Big(S_\gamma S_{\gamma^*} - \sum_{\substack{\gamma\tau \in \vee E \backslash \vee F \\ \mu_{\gamma\tau} = \gamma}} S_{\gamma\tau} S_{(\gamma\tau)^*} \Big).$$

This holds for all $\gamma \in \vee F$, and hence

$$\begin{aligned}
\sum_{\tau \in \vee E} Q(S)_\tau^{\vee E} &= \sum_{\gamma \in \vee F} Q(S)_\gamma^{\vee E} + \sum_{\nu \in \vee E \backslash \vee F} Q(S)_\nu^{\vee E} \\
&= \sum_{\gamma \in \vee F} Q(S)_\gamma^{\vee F} \Big(S_\gamma S_{\gamma^*} - \sum_{\substack{\gamma\tau \in \vee E \backslash \vee F \\ \mu_{\gamma\tau} = \gamma}} S_{\gamma\tau} S_{(\gamma\tau)^*} \Big) + \sum_{\nu \in \vee E \backslash \vee F} Q(S)_\nu^{\vee E} \\
&= \sum_{\gamma \in \vee F} Q(S)_\gamma^{\vee F} \Big(S_\gamma S_{\gamma^*} - \sum_{\substack{\gamma\tau \in \vee E \backslash \vee F \\ \mu_{\gamma\tau} = \gamma}} S_{\gamma\tau} S_{(\gamma\tau)^*} \Big) + \sum_{\gamma \in \vee F} \sum_{\substack{\nu \in \vee E \backslash \vee F \\ \mu_\nu = \gamma}} Q(S)_\nu^{\vee E},
\end{aligned}$$

since for $\nu \in \vee E \backslash \vee F$, there exists a unique $\mu_\gamma \in \vee F$ (see Lemma 3.2.11). Hence

$$\begin{aligned}
\sum_{\tau \in \vee E} Q(S)_\tau^{\vee E} &= \sum_{\gamma \in \vee F} \Big(Q(S)_\gamma^{\vee F} \Big(S_\gamma S_{\gamma^*} - \sum_{\substack{\gamma\tau \in \vee E \backslash \vee F \\ \mu_{\gamma\tau} = \gamma}} S_{\gamma\tau} S_{(\gamma\tau)^*} \Big) + \sum_{\substack{\nu \in \vee E \backslash \vee F \\ \mu_\nu = \gamma}} Q(S)_\nu^{\vee E} \Big) \\
&= \sum_{\gamma \in \vee F} \Big(Q(S)_\gamma^{\vee F} \Big(S_\gamma S_{\gamma^*} - \sum_{\substack{\gamma\tau \in \vee E \backslash \vee F \\ \mu_{\gamma\tau} = \gamma}} S_{\gamma\tau} S_{(\gamma\tau)^*} \Big) + \sum_{\substack{\nu \in \vee E \backslash \vee F \\ \mu_\nu = \gamma}} Q(S)_{\mu_\nu}^{\vee F} S_\nu S_{\nu^*} \Big)
\end{aligned}$$

by Lemma 3.2.12(c). Furthermore,

$$\begin{aligned}
\sum_{\tau \in \vee E} Q(S)_\tau^{\vee E} &= \sum_{\gamma \in \vee F} Q(S)_\gamma^{\vee F} \Big(\Big(S_\gamma S_{\gamma^*} - \sum_{\substack{\gamma\tau \in \vee E \backslash \vee F \\ \mu_{\gamma\tau} = \gamma}} S_{\gamma\tau} S_{(\gamma\tau)^*} \Big) + \sum_{\substack{\nu \in \vee E \backslash \vee F \\ \mu_\nu = \gamma}} S_\nu S_{\nu^*} \Big) \\
&= \sum_{\gamma \in \vee F} Q(S)_\gamma^{\vee F} S_\gamma S_{\gamma^*} \\
&= \sum_{\gamma \in \vee F} Q(S)_\gamma^{\vee F} \text{ (by Remark 3.2.4)} \\
&= S_v
\end{aligned}$$

by the inductive hypothesis. $\square$

Proof of Proposition 3.2.6. We follow the argument Proposition 3.5 of [41]. By Remark 3.2.4, we already have $\left(Q\left(S\right)_{\lambda}^{\vee E}\right)^{2}=Q\left(S\right)_{\lambda}^{\vee E}$. So it suffices to show

$$Q\left(S\right)_{\lambda}^{\vee E}Q\left(S\right)_{\mu}^{\vee E}=0 \text{ for } \lambda\neq\mu,$$

and (3.2.2). Take $\lambda,\mu\in E$ with $\lambda\neq\mu$. If $r\left(\lambda\right)\neq r\left(\mu\right)$, then $S_{r(\lambda)}S_{r(\mu)}=0$ and so $Q\left(S\right)_{\lambda}^{\vee E}Q\left(S\right)_{\mu}^{\vee E}=0$. On the other hand, if $r\left(\lambda\right)=r\left(\mu\right)$, we also have $Q\left(S\right)_{\lambda}^{\vee E}Q\left(S\right)_{\mu}^{\vee E}=0$ by Lemma 3.2.7, as required.

To establish (3.2.2), we claim that for $v\in r\left(E\right)$,

$$E \text{ satisfies (3.2.1) if and only if } E\cup\{v\} \text{ satisfies (3.2.1).} \tag{3.2.11}$$

First suppose that E satisfies (3.2.1). For every $\lambda\in E$, we have either $\operatorname{MCE}\left(\lambda,v\right)=\{\lambda\}$ (if $\lambda\in vE$) or $\operatorname{MCE}\left(\lambda,v\right)=\emptyset$ (if $\lambda\notin vE$), so $E\cup\{v\}$ also satisfies (3.2.1). On the other hand, if $E\cup\{v\}$ satisfies (3.2.1), then E also satisfies (3.2.1) since there are no paths $\lambda,\mu\in E\backslash\{v\}$ which satisfy $\operatorname{MCE}\left(\lambda,\mu\right)=\{v\}$. Thus we have (3.2.11).

Now fix $v\in r\left(E\right)$. To prove (3.2.2), we consider two cases where $v\in E$ and $v\notin E$. If $v\in E$, then $\prod_{\lambda\in vE}\left(S_{v}-S_{\lambda}S_{\lambda^{*}}\right)=0$ and the left hand side of (3.2.2) is $\sum_{\lambda\in vE}Q\left(S\right)_{\lambda}^{E}$ where E satisfies (3.2.1) and $v\in E$.

On the other hand, suppose $v\notin E$. We define $F:=E\cup\{v\}$ and by claim (3.2.11), F also satisfies (3.2.1). We also have

$$Q\left(S\right)_{\lambda}^{F}=Q\left(S\right)_{\lambda}^{E\cup\{v\}}=Q\left(S\right)_{\lambda}^{E} \text{ for all } \lambda\in vE,$$

and

$$Q\left(S\right)_{v}^{F}=\prod_{\lambda\in vE}\left(S_{v}-S_{\lambda}S_{\lambda^{*}}\right).$$

So

$$\left(S_{v}\prod_{\lambda\in vE}\left(S_{v}-S_{\lambda}S_{\lambda^{*}}\right)\right)+\sum_{\lambda\in vE}Q\left(S\right)_{\lambda}^{E}=Q\left(S\right)_{v}^{F}+\sum_{\lambda\in vE}Q\left(S\right)_{\lambda}^{F}=\sum_{\lambda\in vF}Q\left(S\right)_{\lambda}^{F}.$$

Hence the left hand side of (3.2.2) is $\sum_{\lambda\in vF}Q\left(S\right)_{\lambda}^{F}$, where F satisfies (3.2.1) and contains v.

Therefore it suffices to show that

$$\sum_{\lambda\in vE}Q\left(S\right)_{\lambda}^{E}=S_{v}$$

where E satisfies (3.2.1) and $v\in E$. However, this is true by Lemma 3.2.7. The conclusion follows. $\square$

A direct consequence of Proposition 3.2.6 is:

Corollary 3.2.13. *Suppose that Λ is a finitely aligned k-graph and that R is a commutative ring with 1. Suppose that $E \subseteq \Lambda$ is a finite set which satisfies (3.2.1). Suppose that $\{S_\lambda, S_{\mu^*} : \lambda, u \in \Lambda\}$ is a Kumjian-Pask Λ-family in an R-algebra A. Then for each $\mu \in E$, we have*

$$S_\mu S_{\mu^*} = \sum_{\mu\nu \in E} Q\,(S)^E_{\mu\nu}\,.$$

Proof. Proposition 3.2.6 tells us that

$$S_\mu S_{\mu^*} = S_\mu S_{\mu^*}\Big(\Big(\prod_{\lambda \in r(\mu)E} \big(S_{r(\mu)} - S_\lambda S_{\lambda^*}\big)\Big) + \sum_{\lambda \in r(\mu)E} Q\,(S)^E_\lambda\Big). \tag{3.2.12}$$

For all ν with $\mu\nu \in E$, we have

$$\begin{aligned}
S_\mu S_{\mu^*} Q\,(S)^E_{\mu\nu} &= S_\mu S_{\mu^*} S_{\mu\nu} S_{\mu\nu^*} \prod_{\substack{\mu\nu\gamma \in E \\ d(\gamma) \neq 0}} \big(S_{\mu\nu} S_{(\mu\nu)^*} - S_{\mu\nu\gamma} S_{(\mu\nu\gamma)^*}\big) \\
&= S_{\mu\nu} S_{\mu\nu^*} \prod_{\substack{\mu\nu\gamma \in E \\ d(\gamma) \neq 0}} \big(S_{\mu\nu} S_{(\mu\nu)^*} - S_{\mu\nu\gamma} S_{(\mu\nu\gamma)^*}\big) \quad \text{(by Proposition 3.1.3(a))} \\
&= Q\,(S)^E_{\mu\nu}\,.
\end{aligned}$$

So by (3.2.12), it suffices to show that

(a) $S_\mu S_{\mu^*} \prod_{\lambda \in r(\mu)E} \big(S_{r(\mu)} - S_\lambda S_{\lambda^*}\big) = 0$, and

(b) $S_\mu S_{\mu^*} Q\,(S)^E_\lambda = 0$ for all $\lambda \in E \backslash \mu\Lambda$.

First we show part (a). Since $\mu \in r(\mu)\,E$ then

$$\begin{aligned}
S_\mu S_{\mu^*} \prod_{\lambda \in r(\mu)E} \big(S_{r(\mu)} - S_\lambda S_{\lambda^*}\big) &= S_\mu S_{\mu^*} \big(S_{r(\mu)} - S_\mu S_{\mu^*}\big) \prod_{\lambda \in r(\mu)E\backslash\{\mu\}} \big(S_{r(\mu)} - S_\lambda S_{\lambda^*}\big) \\
&= \big(S_\mu S_{\mu^*} - S_\mu S_{\mu^*}\big) \prod_{\lambda \in r(\mu)E\backslash\{\mu\}} \big(S_{r(\mu)} - S_\lambda S_{\lambda^*}\big) \\
&= 0,
\end{aligned}$$

as required.

Next we show (b). Take $\lambda \in E \backslash \mu\Lambda$. If $\mathrm{MCE}\,(\mu, \lambda) = \emptyset$, then $S_\mu S_{\mu^*} S_\lambda S_{\lambda^*} = 0$ (by Proposition 3.1.3(a)) and

$$S_\mu S_{\mu^*} Q\,(S)^E_\lambda = S_\mu S_{\mu^*} S_\lambda S_{\lambda^*} \prod_{\substack{\lambda\nu \in E \\ d(\nu) \neq 0}} \big(S_\lambda S_{\lambda^*} - S_{\lambda\nu} S_{(\lambda\nu)^*}\big) = 0.$$

So suppose $\mathrm{MCE}(\mu,\lambda)\neq\emptyset$. Then by Proposition 3.1.3(a),

$$S_\mu S_{\mu^*} Q(S)^E_\lambda = \sum_{\lambda\rho\in\mathrm{MCE}(\lambda,\mu)} \Big(S_{\lambda\rho} S_{(\lambda\rho)^*} \prod_{\substack{\lambda\nu\in E \\ d(\nu)\neq 0}} \left(S_\lambda S_{\lambda^*} - S_{\lambda\nu} S_{(\lambda\nu)^*} \right) \Big).$$

Take $\lambda\rho\in\mathrm{MCE}(\lambda,\mu)$. Since $\lambda,\mu\in E$ and E satisfies (3.2.1), then $\lambda\rho\in E$. Furthermore, since $\lambda\notin\mu\Lambda$, then $d(\rho)\neq 0$ and

$$\begin{aligned}
&S_{\lambda\rho} S_{(\lambda\rho)^*} \prod_{\substack{\lambda\nu\in E \\ d(\nu)\neq 0}} \left(S_\lambda S_{\lambda^*} - S_{\lambda\nu} S_{(\lambda\nu)^*} \right) \\
&\quad = S_{\lambda\rho} S_{(\lambda\rho)^*} \left(S_\lambda S_{\lambda^*} - S_{\lambda\rho} S_{(\lambda\rho)^*} \right) \prod_{\substack{\lambda\nu\in E\setminus\{\lambda\rho\} \\ d(\nu)\neq 0}} \left(S_\lambda S_{\lambda^*} - S_{\lambda\nu} S_{(\lambda\nu)^*} \right) \\
&\quad = \left(S_{\lambda\rho} S_{(\lambda\rho)^*} - S_{\lambda\rho} S_{(\lambda\rho)^*} \right) \prod_{\substack{\lambda\nu\in E\setminus\{\lambda\rho\} \\ d(\nu)\neq 0}} \left(S_\lambda S_{\lambda^*} - S_{\lambda\nu} S_{(\lambda\nu)^*} \right) \\
&\quad\quad \text{(by Proposition3.1.3(a))}
\end{aligned}$$

which equals 0. The conclusion follows. □

3.2.2 Subalgebras of the core

For a finite set $E\subseteq\Lambda$, we want to identify a finite set ΠE containing E such that $\mathrm{span}\{S_\lambda S_{\mu^*} : \lambda,\mu\in\Pi E, d(\lambda)=d(\mu)\}$ is closed under multiplication for every Kumjian-Pask Λ-family $\{S_\lambda, S_{\mu^*} : \lambda, u\in\Lambda\}$ in an R-algebra A (Proposition 3.2.18). We follow the C^*-algebraic argument of [41, Section 3].

Lemma 3.2.14 ([41, Lemma 3.2])**.** *Suppose that Λ is a finitely aligned k-graph and that $E\subseteq\Lambda$ is finite. Then there exists a finite set $F\subseteq\Lambda$ which contains E and satisfies*

(3.2.13) $\lambda,\mu,\sigma\in F$ *with* $d(\lambda)=d(\mu)$ *and* $s(\lambda)=s(\mu)$ *implies* $\lambda\,\mathrm{Ext}(\mu;\{\sigma\})\subseteq F$.

Remark 3.2.15. Condition (3.2.13) is equivalent to

$$\begin{aligned}
\lambda,\mu,\rho,\tau \in F,\ d(\lambda)=d(\mu),\ d(\rho)=d(\tau),\ s(\lambda)=s(\mu), \text{ and } s(\rho)=s(\tau) \\
\text{imply } \left\{\lambda\alpha,\tau\beta : (\alpha,\beta)\in\Lambda^{\min}(\mu,\rho)\right\}\subseteq F
\end{aligned}$$

which is Condition (3.1) of [41].

Now note that for a family of sets satisfying (3.2.13), their intersection also satisfies (3.2.13), so we make the following definition.

Definition 3.2.16. Suppose that Λ is a finitely aligned k-graph. For every finite set $E \subseteq \Lambda$, we define ΠE to be the smallest set containing E and satisfying (3.2.13); that is

$$\Pi E := \bigcap \{F \subseteq \Lambda : E \subseteq F \text{ and } F \text{ satisfies (3.2.13)}\}.$$

We also write $\Pi E \times_{d,s} \Pi E$ for the set $\{(\lambda, \mu) \in \Pi E \times \Pi E : d(\lambda) = d(\mu), s(\lambda) = s(\mu)\}$.

Remark 3.2.17. A number of aspects of these sets are worth commenting on:

(i) ΠE is finite.

(ii) ΠE satisfies (3.2.13). If we choose $\lambda = \mu$, Condition (3.2.13) becomes "$\lambda, \sigma \in \Pi E$ implies $\mathrm{MCE}(\lambda, \sigma) \subseteq \Pi E$". Hence ΠE also satisfies (3.2.1) and $\vee(\Pi E) = \Pi E$.

(iii) Suppose that $(\lambda, \mu) \in \Pi E \times_{d,s} \Pi E$. If $\nu \in s(\lambda)\Lambda$ such that $\mu\nu \in \Pi E$, then set $\sigma = \mu\nu$ in (3.2.13) and we get $\lambda\nu \in \Pi E$. By symmetry, $\lambda\nu \in \Pi E$ also implies $\mu\nu \in \Pi E$. Hence for $\nu \in s(\lambda)\Lambda$,

$$\lambda\nu \in \Pi E \iff \mu\nu \in \Pi E.$$

Now we state the main result of this subsection as follows:

Proposition 3.2.18. *Suppose that Λ is a finitely aligned k-graph and that R is a commutative ring with 1. Suppose that $\{S_\lambda, S_{\mu^*} : \lambda, u \in \Lambda\}$ is a Kumjian-Pask Λ-family in an R-algebra A. For every finite set $E \subseteq \Lambda$, the set*

$$M^S_{\Pi E} := \operatorname{span}_R \{S_\lambda S_{\mu^*} : (\lambda, \mu) \in \Pi E \times_{d,s} \Pi E\}$$

is closed under multiplication.

Proof. Take $(\lambda, \mu), (\rho, \tau) \in \Pi E \times_{d,s} \Pi E$. Then

$$\begin{aligned} S_\lambda S_{\mu^*} S_\rho S_{\tau^*} &= S_\lambda \Big(\sum_{(\nu,\gamma) \in \Lambda^{\min}(\mu,\rho)} S_\nu S_{\gamma^*} \Big) S_{\tau^*} \text{ (by (KP3))} \\ &= \sum_{(\nu,\gamma) \in \Lambda^{\min}(\mu,\rho)} S_{\lambda\nu} S_{(\tau\gamma)^*}. \end{aligned}$$

Suppose $(\nu, \gamma) \in \Lambda^{\min}(\mu, \rho)$. Then

$$\begin{aligned} d(\lambda\nu) &= d(\mu\nu) \text{ (since } d(\lambda) = d(\mu)\text{)} \\ &= d(\rho\gamma) \text{ (since } (\nu,\gamma) \in \Lambda^{\min}(\mu,\rho)\text{)} \\ &= d(\tau\gamma) \text{ (since } d(\rho) = d(\tau)\text{)}. \end{aligned}$$

By Remark 3.2.17(iii), both $\lambda\nu$ and $\tau\gamma$ belong to ΠE, and so $S_{\lambda\nu} S_{(\tau\gamma)^*} \in M^S_{\Pi E}$. Hence $S_\lambda S_{\mu^*} S_\rho S_{\tau^*}$ belongs to $M^S_{\Pi E}$, as required. $\square$

3.2.3 Identifying matrix units

In this subsection, for a finite set $E \subseteq \Lambda$, we identify a collection of nonzero matrix units for $M^S_{\Pi E}$, and investigate their properties.

Suppose that $E \subseteq \Lambda$ is finite and $\{S_\lambda, S_{\mu^*} : \lambda, u \in \Lambda\}$ is a Kumjian-Pask Λ-family in an R-algebra A. For $(\lambda, \mu) \in \Pi E \times_{d,s} \Pi E$, we define

$$\Theta\,(S)^{\Pi E}_{\lambda,\mu} := Q\,(S)^{\Pi E}_{\lambda}\, S_\lambda S_{\mu^*}.$$

The aim of this subsection is to show that the $\Theta\,(S)^{\Pi E}_{\lambda,\mu}$ are matrix units for $M^S_{\Pi E}$, as stated in the following proposition.

Lemma 3.2.19. *Suppose that Λ is a finitely aligned k-graph and that R is a commutative ring with 1. Suppose that $\{S_\lambda, S_{\mu^*} : \lambda, u \in \Lambda\}$ is a Kumjian-Pask Λ-family in an R-algebra A. Suppose that $E \subseteq \Lambda$ is finite. For $(\lambda, \mu), (\rho, \tau) \in \Pi E \times_{d,s} \Pi E$, we have*

(a) $\Theta\,(S)^{\Pi E}_{\lambda,\mu}\Theta\,(S)^{\Pi E}_{\rho,\tau} = \delta_{\mu,\rho}\Theta\,(S)^{\Pi E}_{\lambda,\tau}$, *and*

(b) $S_\lambda S_{\mu^*} = \sum_{\lambda\nu\in\Pi E}\Theta\,(S)^{\Pi E}_{\lambda\nu,\mu\nu}$.

The subalgebra $M^S_{\Pi E}$ is spanned by the set $\{\Theta\,(S)^{\Pi E}_{\lambda,\mu} : (\lambda, \mu) \in \Pi E \times_{d,s} \Pi E\}$.

To prove Lemma 3.2.19, we need to prove the following lemma first.

Lemma 3.2.20. *Suppose that Λ is a finitely aligned k-graph, and that R is a commutative ring with 1. Suppose that $\{S_\lambda, S_{\mu^*} : \lambda, u \in \Lambda\}$ is a Kumjian-Pask Λ-family in an R-algebra A. Suppose that $E \subseteq \Lambda$ is finite. For $(\lambda, \mu) \in \Pi E \times_{d,s} \Pi E$, we have*

$$\Theta\,(S)^{\Pi E}_{\lambda,\mu} = S_\lambda\Big(\prod_{\substack{\lambda\nu\in\Pi E\\ d(\nu)\neq 0}}\big(S_{s(\lambda)} - S_\nu S_{\nu^*}\big)\Big)S_{\mu^*} = S_\lambda S_{\mu^*} Q\,(S)^{\Pi E}_{\mu}.$$

Proof. We adapt some ideas from [41, Lemma 3.10]. Take $(\lambda, \mu) \in \Pi E \times_{d,s} \Pi E$. Then

$$\begin{aligned}\Theta\,(S)^{\Pi E}_{\lambda,\mu} &= Q\,(S)^{\Pi E}_{\lambda}\, S_\lambda S_{\mu^*}\\ &= S_\lambda S_{\lambda^*}\Big(\prod_{\substack{\lambda\nu\in\Pi E\\ d(\nu)\neq 0}}\big(S_\lambda S_{\lambda^*} - S_{\lambda\nu} S_{(\lambda\nu)^*}\big)\Big)S_\lambda S_{\mu^*}\\ &= S_\lambda S_{\lambda^*}\Big(\prod_{\substack{\lambda\nu\in\Pi E\\ d(\nu)\neq 0}}\left(S_\lambda S_{\lambda^*} - S_\lambda S_\nu S_{\nu^*} S_{\lambda^*}\right)\Big)S_\lambda S_{\mu^*}\\ &= S_\lambda S_{\lambda^*}\Big(\prod_{\substack{\lambda\nu\in\Pi E\\ d(\nu)\neq 0}}\big(S_\lambda\left(S_{s(\lambda)} - S_\nu S_{\nu^*}\right) S_{\lambda^*}\big)\Big)S_\lambda S_{\mu^*}.\end{aligned} \tag{3.2.14}$$

Since $S_{\lambda^*}S_\lambda = S_{s(\lambda)}$ and $S_{s(\lambda)}\left(S_{s(\lambda)} - S_\nu S_{\nu^*}\right) = S_{s(\lambda)} - S_\nu S_{\nu^*}$ for all $\nu \in s(\lambda)\Lambda$, so all $S_{\lambda^*}S_\lambda$ which occur between terms in (3.2.14) can be deleted and we have

$$\begin{aligned}\Theta(S)^{\Pi E}_{\lambda,\mu} &= S_\lambda S_{\lambda^*} S_\lambda \Big(\prod_{\substack{\lambda\nu\in\Pi E\\ d(\nu)\neq 0}} \left(S_{s(\lambda)} - S_\nu S_{\nu^*}\right) \Big) S_{\lambda^*} S_\lambda S_{\mu^*} \\ &= S_\lambda \Big(\prod_{\substack{\lambda\nu\in\Pi E\\ d(\nu)\neq 0}} \left(S_{s(\lambda)} - S_\nu S_{\nu^*}\right) \Big) S_{\mu^*},\end{aligned}$$

which proves the left equation. A similar argument gives

$$S_\lambda S_{\mu^*} Q(S)^{\Pi E}_\mu = S_\lambda \Big(\prod_{\substack{\lambda\nu\in\Pi E\\ d(\nu)\neq 0}} \left(S_{s(\lambda)} - S_\nu S_{\nu^*}\right) \Big) S_{\mu^*},$$

as required. $\square$

Proof of Lemma 3.2.19. Take $(\lambda,\mu), (\rho,\tau) \in \Pi E \times_{d,s} \Pi E$. Then by Lemma 3.2.20, we have

$$\begin{aligned}\Theta(S)^{\Pi E}_{\lambda,\mu}\,\Theta(S)^{\Pi E}_{\rho,\tau} &= S_\lambda S_{\mu^*} Q(S)^{\Pi E}_\mu\, \Theta(S)^{\Pi E}_{\rho,\tau} \\ &= S_\lambda S_{\mu^*} Q(S)^{\Pi E}_\mu\, Q(S)^{\Pi E}_\rho\, S_\rho S_{\tau^*} \text{ (by the definition of } \Theta(S)^{\Pi E}_{\rho,\tau}) \\ &= \delta_{\mu,\sigma}\, S_\lambda S_{\mu^*} Q(S)^{\Pi E}_\mu\, S_\mu S_{\tau^*} \text{ (by Proposition 3.2.6)} \\ &= \delta_{\mu,\sigma}\, Q(S)^{\Pi E}_\lambda\, S_\lambda S_{\mu^*} S_\mu S_{\tau^*} \text{ (by Lemma 3.2.20)} \\ &= \delta_{\mu,\sigma}\, Q(S)^{\Pi E}_\lambda\, S_\lambda S_{s(\mu)} S_{\tau^*} \\ &= \delta_{\mu,\sigma}\, Q(S)^{\Pi E}_\lambda\, S_\lambda S_{\tau^*} \text{ (since } s(\lambda) = s(\mu)) \\ &= \delta_{\mu,\rho} \Theta(S)^{\Pi E}_{\lambda,\tau},\end{aligned}$$

which proves (a).

Next we show part (b) using an argument like that of [41, Lemma 3.11]. Note that

$$\begin{aligned}S_\lambda S_{\mu^*} &= S_\lambda S_{\mu^*} S_\mu S_{\mu^*} \\ &= S_\lambda S_{\mu^*} \sum_{\mu\nu\in\Pi E} Q(S)^{\Pi E}_{\mu\nu} \text{ (by Corollary 3.2.13)} \\ &= S_\lambda S_{\mu^*} \sum_{\mu\nu\in\Pi E} \Big(S_{\mu\nu} S_{(\mu\nu)^*} \prod_{\substack{\mu\nu\gamma\in\Pi E\\ d(\gamma)\neq 0}} \left(S_{\mu\nu} S_{(\mu\nu)^*} - S_{\mu\nu\gamma} S_{(\mu\nu\gamma)^*}\right) \Big) \\ &= S_\lambda S_{\mu^*} \sum_{\mu\nu\in\Pi E} \Big(S_{\mu\nu} S_{(\mu\nu)^*} \prod_{\substack{\mu\nu\gamma\in\Pi E\\ d(\gamma)\neq 0}} \left(S_{\mu\nu} S_{(\mu\nu)^*} - S_{\mu\nu} S_\gamma S_{\gamma^*} S_{(\mu\nu)^*}\right) \Big) \\ &= S_\lambda S_{\mu^*} \sum_{\mu\nu\in\Pi E} \Big(S_{\mu\nu} S_{(\mu\nu)^*} \prod_{\substack{\mu\nu\gamma\in\Pi E\\ d(\gamma)\neq 0}} \left(S_{\mu\nu} \left(S_{s(\mu\nu)} - S_\gamma S_{\gamma^*}\right) S_{(\mu\nu)^*}\right) \Big).\end{aligned} \tag{3.2.15}$$

Since $S_{(\mu\nu)^*}S_{\mu\nu} = S_{s(\mu\nu)}$, then we rewite (3.2.15) as

$$
\begin{aligned}
(3.2.16) \qquad S_\lambda S_{\mu^*} &= S_\lambda S_{\mu^*} \sum_{\mu\nu\in\Pi E} \Big(S_{\mu\nu} S_{(\mu\nu)^*} \Big(S_{\mu\nu} \Big(\prod_{\substack{\mu\nu\gamma\in\Pi E \\ d(\gamma)\neq 0}} \left(S_{s(\mu\nu)} - S_\gamma S_{\gamma^*} \right) \Big) S_{(\mu\nu)^*} \Big) \Big) \\
&= S_\lambda S_{\mu^*} \sum_{\mu\nu\in\Pi E} \Big(S_{\mu\nu} \Big(\prod_{\substack{\mu\nu\gamma\in\Pi E \\ d(\gamma)\neq 0}} \left(S_{s(\mu\nu)} - S_\gamma S_{\gamma^*} \right) \Big) S_{(\mu\nu)^*} \Big) \\
&\quad (\text{since } S_{(\mu\nu)^*}S_{\mu\nu} = S_{s(\mu\nu)}) \\
&= \sum_{\mu\nu\in\Pi E} \Big(S_\lambda S_{\mu^*} S_{\mu\nu} \Big(\prod_{\substack{\mu\nu\gamma\in\Pi E \\ d(\gamma)\neq 0}} \left(S_{s(\mu\nu)} - S_\gamma S_{\gamma^*} \right) \Big) S_{(\mu\nu)^*} \Big) \\
&= \sum_{\mu\nu\in\Pi E} \Big(S_\lambda S_{\mu^*} S_\mu S_\nu \Big(\prod_{\substack{\mu\nu\gamma\in\Pi E \\ d(\gamma)\neq 0}} \left(S_{s(\mu\nu)} - S_\gamma S_{\gamma^*} \right) \Big) S_{(\mu\nu)^*} \Big) \\
&= \sum_{\mu\nu\in\Pi E} \Big(S_{\lambda\nu} \Big(\prod_{\substack{\mu\nu\gamma\in\Pi E \\ d(\gamma)\neq 0}} \left(S_{s(\mu\nu)} - S_\gamma S_{\gamma^*} \right) \Big) S_{(\mu\nu)^*} \Big)
\end{aligned}
$$

since $S_{\mu^*}S_\mu = S_{s(\mu)}$. By Lemma 3.2.20, the last line of (3.2.16) equals $\sum_{\mu\nu\in\Pi E} \Theta(S)^{\Pi E}_{\lambda\nu,\mu\nu}$, and we get the desired result.

Since $M^S_{\Pi E}$ is generated by elements in the form $S_\lambda S_{\mu^*}$ where $(\lambda,\mu) \in \Pi E \times_{d,s} \Pi E$, and since each $S_\lambda S_{\mu^*}$ can be written as the sum of $\Theta(S)^{\Pi E}_{\lambda\nu,\mu\nu}$, $M^S_{\Pi E}$ is spanned by the set $\{\Theta(S)^{\Pi E}_{\lambda,\mu} : (\lambda,\mu) \in \Pi E \times_{d,s} \Pi E\}$. □

3.2.4 Proof of the graded uniqueness theorem

Now we establish the last technical results before proving Theorem 3.2.1. The key ingredient is to prove the injectivity on $M^S_{\Pi E}$ (Lemma 3.2.21) and on $\mathrm{KP}_R(\Lambda)_0 := \mathrm{span}_R\{s_\lambda s_{\mu^*} : d(\lambda) = d(\mu)\}$ (Theorem 3.2.23).

First recall from Remark 3.2.17(iii) that for a finite set $E \subseteq \Lambda$ and $(\lambda,\mu) \in \Pi E \times_{d,s} \Pi E$ and $\nu \in s(\lambda)\Lambda$, $\lambda\nu \in \Pi E$ if and only if $\mu\nu \in \Pi E$. Hence

$$\{\nu \in s(\lambda)\Lambda : d(\nu) \neq 0, \lambda\nu \in \Pi E\} = \{\nu \in s(\lambda)\Lambda : d(\nu) \neq 0, \mu\nu \in \Pi E\}.$$

We denote this set by $T(\lambda)$. Note that since $\lambda T(\lambda) \subseteq \Pi E$ and ΠE is finite (Remark 3.2.17(i)), then $T(\lambda)$ is also finite.

Lemma 3.2.21 ([18, Lemma 4.3])**.** *Suppose that Λ is a finitely aligned k-graph, that R is a commutative ring with 1 and that $E \subseteq \Lambda$ is finite. Suppose that $\pi : \mathrm{KP}_R(\Lambda) \to A$ is a ring homomorphism such that $\pi(rs_v) \neq 0$ for all $r \in R\setminus\{0\}$ and $v \in \Lambda^0$. Let $(\lambda,\mu) \in \Pi E \times_{d,s} \Pi E$. Then the following conditions are equivalent:*

(a) $\pi\Big(\Theta\,(s)^{\Pi E}_{\lambda,\mu}\Big) = 0.$

(b) $\Theta\,(s)^{\Pi E}_{\lambda,\mu} = 0.$

(c) $T\,(\lambda)$ *is exhaustive.*

Furthermore, for $r \in R\backslash\{0\}$ *we have*

$$\pi\Big(r\Theta\,(s)^{\Pi E}_{\lambda,\mu}\Big) = 0 \text{ if and only if } r\Theta\,(s)^{\Pi E}_{\lambda,\mu} = 0 \tag{3.2.17}$$

and π *is injective on* $M^s_{\Pi E}$.

Proof. To show the three equivalent conditions, we first prove the following claim.

Claim 3.2.22. *Suppose that* $\phi : \mathrm{KP}_R(\Lambda) \to A$ *is a ring homomorphism such that* $\phi\,(rs_v) \neq 0$ *for all* $r \in R\backslash\{0\}$ *and* $v \in \Lambda^0$. *Then for* $(\lambda, \mu) \in \Pi E \times_{d,s} \Pi E$,

$$\phi\Big(\Theta\,(s)^{\Pi E}_{\lambda,\mu}\Big) = 0 \text{ if and only if } T\,(\lambda) \text{ is exhaustive.}$$

Proof of Claim 3.2.22. We modify the argument of Proposition 3.13 of [41]. Suppose that $T\,(\lambda)$ is non-exhaustive. We have to show $\phi\Big(\Theta\,(s)^{\Pi E}_{\lambda,\mu}\Big)$ is nonzero. Since $T\,(\lambda)$ is non-exhaustive, we choose $\xi \in s\,(\lambda)\,\Lambda$ such that $\Lambda^{\min}(\xi, \mu) = \emptyset$ for all $\mu \in T\,(\lambda)$. It suffices to show that

$$\phi\Big(s_{\lambda\xi}s_{(\lambda\xi)^*}\Theta\,(s)^{\Pi E}_{\lambda,\mu}\,s_\mu s_{\lambda^*}\Big) = \phi\Big(s_{\lambda\xi}s_{(\lambda\xi)^*}\Big) \tag{3.2.18}$$

since the right hand side is nonzero (by the hypothsesis and Proposition 3.1.3(d) with $G = \emptyset$). Then

(3.2.19)

$$\begin{aligned}
s_{\lambda\xi}s_{(\lambda\xi)^*}\Theta\,(s)^{\Pi E}_{\lambda,\mu}\,s_\mu s_{\lambda^*} &= s_{\lambda\xi}s_{(\lambda\xi)^*}\Big(Q\,(s)^{\Pi E}_{\lambda}\,s_\lambda s_{\mu^*}\Big)s_\mu s_{\lambda^*} \text{ (by the definition of } \Theta\,(s)^{\Pi E}_{\lambda,\mu}\text{)}\\
&= s_{\lambda\xi}s_{(\lambda\xi)^*}Q\,(s)^{\Pi E}_{\lambda}\,s_\lambda s_{\lambda^*} \text{ (since } s_{\mu^*}s_\mu = s_{s(\mu)}\text{)}\\
&= s_{\lambda\xi}s_{(\lambda\xi)^*}\Big(s_\lambda s_{\lambda^*}\prod_{\nu\in T(\lambda)}\big(s_\lambda s_{\lambda^*} - s_{\lambda\nu}s_{(\lambda\nu)^*}\big)\Big)s_\lambda s_{\lambda^*}\\
&\text{(by the definition of } Q\,(s)^{\Pi E}_{\lambda}\text{)}\\
&= s_{\lambda\xi}s_{(\lambda\xi)^*}\Big(s_\lambda s_{\lambda^*}s_\lambda s_{\lambda^*}\prod_{\nu\in T(\lambda)}\big(s_\lambda s_{\lambda^*} - s_{\lambda\nu}s_{(\lambda\nu)^*}\big)\Big)
\end{aligned}$$

since $\{s_\lambda s_{\lambda^*} : \lambda \in \Lambda\}$ is a commuting family (Proposition 3.1.3(a)). Proposition 3.1.3(a) says $(s_\lambda s_{\lambda^*})^2 = s_\lambda s_{\lambda^*}$ and $s_{\lambda\xi}s_{(\lambda\xi)^*} = s_\lambda s_{\lambda^*}s_{\lambda\xi}s_{(\lambda\xi)^*}$. Then we rewrite (3.2.19) as

$$s_{\lambda\xi}s_{(\lambda\xi)^*}\Theta\,(s)^{\Pi E}_{\lambda,\mu}\,s_\mu s_{\lambda^*} = s_{\lambda\xi}s_{(\lambda\xi)^*}\prod_{\nu\in T(\lambda)}\big(s_\lambda s_{\lambda^*} - s_{\lambda\nu}s_{(\lambda\nu)^*}\big).$$

Now note that for $\nu \in T(\lambda)$, we have $\Lambda^{\min}(\xi,\mu) = \emptyset$ and $\Lambda^{\min}(\lambda\xi,\lambda\mu) = \emptyset$. By Proposition 3.1.3(a), we have $s_{(\lambda\xi)^*}s_{\lambda\nu} = 0$ and $s_{\lambda\xi}s_{(\lambda\xi)^*}s_{\lambda\nu}s_{(\lambda\nu)^*} = 0$. This implies (3.2.18).

Next we suppose that $T(\lambda)$ is exhaustive, and show that $\phi\Big(\Theta(s)^{\Pi E}_{\lambda,\mu}\Big) = 0$. Note that $T(\lambda)$ is also finite and belongs to $\mathrm{FE}(\Lambda)$, so

$$\prod_{\nu\in T(\lambda)} \left(s_{s(\lambda)} - s_\nu s_{\nu^*}\right) = 0. \tag{3.2.20}$$

On the other hand, by Lemma 3.2.20, we have

$$\phi\Big(\Theta(s)^{\Pi E}_{\lambda,\mu}\Big) = \phi\Big(s_\lambda\Big(\prod_{\nu\in T(\lambda)} \left(s_{s(\lambda)} - s_\nu s_{\nu^*}\right)\Big)s_{\mu^*}\Big),$$

which equals 0 by (3.2.20), as required. □ Claim 3.2.22

By Claim 3.2.22 with ϕ the identity homomorphism, we get (b)⇔(c). Meanwhile, choose $\phi = \pi$ and the claim also gives us (a)⇔(c). Hence (a), (b) and (c) are three equivalent conditions.

Now take $(\lambda,\mu) \in \Pi E \times_{d,s} \Pi E$ and $r \in R\setminus\{0\}$. We have to show (3.2.17). If $r\Theta(s)^{\Pi E}_{\lambda,\mu} = 0$, we trivially have $\pi\Big(r\Theta(s)^{\Pi E}_{\lambda,\mu}\Big) = 0$. So suppose $\pi\Big(r\Theta(s)^{\Pi E}_{\lambda,\mu}\Big) = 0$. By Remark 3.1.6, $\pi\Big(r\Theta(s)^{\Pi E}_{\lambda,\mu}\Big) = 0$ implies that $T(\lambda)$ is exhaustive (since $r \neq 0$). Since $T(\lambda)$ is exhaustive, $\Theta(s)^{\Pi E}_{\lambda,\mu} = 0$ by (c)⇒(b). So $r\Theta(s)^{\Pi E}_{\lambda,\mu} = 0$, as required.

Next we show that π is injective on $M^s_{\Pi E}$. Take $a \in M^s_{\Pi E}$ such that $\pi(a) = 0$. We have to show $a = 0$. Since $a \in M^s_{\Pi E}$ and the $M^s_{\Pi E}$ are matrix units (Lemma 3.2.19), we write $a = \sum_{(\lambda,\mu)\in F} r_{\lambda,\mu}\Theta(s)^{\Pi E}_{\lambda,\mu}$ where $F \subseteq \Pi E \times_{d,s} \Pi E$, $r_{\lambda,\mu} \in R$ and $\Theta(s)^{\Pi E}_{\lambda,\mu} \neq 0$. If $T(\lambda)$ is exhaustive for some $(\lambda,\mu) \in F$, then by (c)⇒(b), $\Theta(s)^{\Pi E}_{\lambda,\mu} = 0$, which contradicts $\Theta(s)^{\Pi E}_{\lambda,\mu} \neq 0$. So $T(\lambda)$ is non-exhaustive for all $(\lambda,\mu) \in F$. Since $\pi(a) = 0$, then for $(\rho,\tau) \in F$, we have

$$\begin{aligned}
0 &= \pi\Big(\Theta(s)^{\Pi E}_{\rho,\rho}\Big)\pi(a)\,\pi\Big(\Theta(s)^{\Pi E}_{\tau,\tau}\Big)\\
&= \pi\Big(\Theta(s)^{\Pi E}_{\rho,\rho}\Big)\pi\Big(\sum_{(\lambda,\mu)\in F} r_{\lambda,\mu}\Theta(s)^{\Pi E}_{\lambda,\mu}\Big)\pi\Big(\Theta(s)^{\Pi E}_{\tau,\tau}\Big)\\
&= r_{\rho,\tau}\pi\Big(\Theta(s)^{\Pi E}_{\rho,\tau}\Big) = r_{\rho,\tau}\Theta(\pi(s))^{\Pi E}_{\rho,\tau} \text{ (by Lemma 3.2.19).}
\end{aligned}$$

By Remark 3.1.6, $r_{\rho,\tau}\Theta(\pi(s))^{\Pi E}_{\rho,\tau} = 0$ implies that $r_{\rho,\tau} = 0$ (since $T(\rho)$ is non-exhaustive). Therefore $a = 0$ and π is injective on $M^s_{\Pi E}$. □

A direct consequence of Lemma 3.2.21 is:

Theorem 3.2.23 ([18, Theorem 4.4]). *Suppose that Λ is a finitely aligned k-graph and that R is a commutative ring with 1. Suppose that $\pi : \mathrm{KP}_R(\Lambda) \to A$ is a ring homomorphism such that $\pi(rs_v) \neq 0$ for all $r \in R\backslash\{0\}$ and $v \in \Lambda^0$. Then π is injective on $\mathrm{KP}_R(\Lambda)_0$.*

Proof. Take $a \in \mathrm{KP}_R(\Lambda)_0$ such that $\pi(a) = 0$. We have to show $a = 0$. Write $a = \sum_{(\lambda,\mu)\in F} r_{\lambda,\mu} s_\lambda s_{\mu^*}$ with $d(\lambda) = d(\mu)$ for $(\lambda,\mu) \in F$. Define $E := \{\lambda, \mu : (\lambda,\mu) \in F\}$ and then $a \in M^s_{\Pi E}$. Since π is injective on $M^s_{\Pi E}$ by Lemma 3.2.21, $a = 0$. □

We need one final lemma for the proof of Theorem 3.2.1.

Lemma 3.2.24 ([18, Lemma 4.5]). *Suppose that Λ is a finitely aligned k-graph and that R is a commutative ring with 1. Suppose that I is a graded ideal of $\mathrm{KP}_R(\Lambda)$. Then I is generated as an ideal by the set $I_0 := I \cap \mathrm{KP}_R(\Lambda)_0$.*

Proof. We generalise the argument of [55, Lemma 5.1]. Take $n \in \mathbb{Z}^k$ and write $n = n_1 - n_2$ such that $n_1, n_2 \in \mathbb{N}^k$ and $|n_1 + n_2|$ as minimum as possible. We show that $I_n := I \cap \mathrm{KP}_R(\Lambda)_n$ is contained in $\mathrm{KP}_R(\Lambda)_{n_1} I_0 \mathrm{KP}_R(\Lambda)_{n_2}$. Now take $a \in I_n$ and write $a = \sum_{(\lambda,\mu)\in F} r_{\lambda,\mu} s_\lambda s_{\mu^*}$. Note that $d(\lambda) - d(\mu) = n$ for $(\lambda,\mu) \in F$. Since $n = n_1 - n_2$ with $n_1, n_2 \in \mathbb{N}^k$ and $|n_1 + n_2|$ as minimum as possible, then for every $(\lambda,\mu) \in F$, $d(\lambda) \geq n_1$ and $d(\mu) \geq n_2$, so by the factorisation property, there exist $\lambda_1, \lambda_2, \mu_1, \mu_2$ such that

$$\lambda = \lambda_1\lambda_2,\ \mu = \mu_1\mu_2,\ d(\lambda_1) = n_1,\ d(\mu_1) = n_2, \text{ and } d(\lambda_2) = d(\mu_2).$$

Hence

$$a = \sum_{(\lambda,\mu)\in F} r_{\lambda_1\lambda_2,\mu_1\mu_2} s_{\lambda_1}\left(s_{\lambda_2} s_{\mu_2^*}\right) s_{\mu_1^*}.$$

Take $(\alpha,\beta) \in F$ and write $\alpha = \alpha_1\alpha_2$ and $\beta = \beta_1\beta_2$. Note that for $\nu, \gamma \in \Lambda$ with $d(\nu) = d(\gamma)$, Remark 3.1.5 gives $s_{\nu^*}s_\gamma = 0$ for $\nu \neq \gamma$. Then

$$\begin{aligned} s_{\alpha_1^*} a s_{\beta_1} &= \sum_{(\lambda,\mu)\in F} r_{\lambda_1\lambda_2,\mu_1\mu_2}\left(s_{\alpha_1^*}s_{\lambda_1}\right)\left(s_{\lambda_2}s_{\mu_2^*}\right)\left(s_{\mu_1^*}s_{\beta_1}\right) \\ &= \sum_{\{(\lambda,\mu)\in F:\lambda_1=\alpha_1,\mu_1=\beta_1\}} r_{\alpha_1\lambda_2,\beta_1\mu_2} s_{\lambda_2}s_{\mu_2^*} \end{aligned}$$

since $d(\alpha_1) = n_1 = d(\lambda_1)$ and $d(\beta_1) = n_2 = d(\mu_1)$ for $(\lambda,\mu) \in F$. Since $a \in I$, we have $s_{\alpha_1^*} a s_{\beta_1} \in I$. Since $d(\lambda_2) = d(\mu_2)$ for $(\alpha_1\lambda_2, \beta_1\mu_2) \in F$, we have $s_{\alpha_1^*} a s_{\beta_1} \in \mathrm{KP}_R(\Lambda)_0$. Hence

$$\sum_{\{(\lambda,\mu)\in F:\lambda_1=\alpha_1,\mu_1=\beta_1\}} r_{\alpha_1\lambda_2,\beta_1\mu_2} s_{\lambda_2}s_{\mu_2^*} = s_{\alpha_1^*} a s_{\beta_1} \in I_0$$

and

$$\sum_{\{(\lambda,\mu)\in F:\lambda_1=\alpha_1,\mu_1=\beta_1\}} r_{\alpha_1\lambda_2,\beta_1\mu_2} s_{\alpha_1\lambda_2} s_{(\beta_1\mu_2)^*} = s_{\alpha_1}\left(s_{\alpha_1^*} a s_{\beta_1}\right) s_{\beta_1^*} \in \mathrm{KP}_R(\Lambda)_{n_1} I_0 \mathrm{KP}_R(\Lambda)_{n_2}.$$

Therefore

$$\begin{aligned} a &= \sum_{(\lambda,\mu)\in F} r_{\lambda_1\lambda_2,\mu_1\mu_2} s_{\lambda_1\lambda_2} s_{(\mu_1\mu_2)^*} \\ &= \sum_{\{(\alpha_1,\beta_1):(\alpha,\beta)\in F\}} \left(\sum_{\{(\lambda,\mu)\in F:\lambda_1=\alpha_1,\mu_1=\beta_1\}} r_{\alpha_1\lambda_2,\beta_1\mu_2} s_{\alpha_1\lambda_2} s_{(\beta_1\mu_2)^*} \right) \end{aligned}$$

also belongs to $\mathrm{KP}_R(\Lambda)_{n_1} I_0 \mathrm{KP}_R(\Lambda)_{n_2}$, and $I_n \subseteq \mathrm{KP}_R(\Lambda)_{n_1} I_0 \mathrm{KP}_R(\Lambda)_{n_2}$.

Now since I is a graded ideal and $I = \bigoplus_{n\in\mathbb{Z}^k} I_n$, we have that I is generated as an ideal by I_0. □

Proof of Theorem 3.2.1. Because π is graded, $\ker\pi$ is a graded ideal. By Lemma 3.2.24, the ideal $\ker\pi$ is generated by the set $\ker\pi \cap \mathrm{KP}_R(\Lambda)_0$. Thus it suffices to show that $\pi|_{\mathrm{KP}_R(\Lambda)_0} : \mathrm{KP}_R(\Lambda)_0 \to A$ is injective. This follows from Theorem 3.2.23. □

3.3 Relationship with higher-rank graph algebras

We recall from page 8 that for a finitely aligned k-graph Λ, a *Cuntz-Krieger Λ-family* is a collection $\{T_\lambda : \lambda \in \Lambda\}$ of partial isometries in a C^*-algebra B satisfying (TCK1-3) and (CK). There exists a universal C^*-algebra $C^*(\Lambda)$ generated by the universal Cuntz-Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$.

The main result of this section generalises Proposition 7.3 of [11] as follows.

Proposition 3.3.1 ([18, Proposition 4.6])**.** *Suppose that Λ is a finitely aligned k-graph. Suppose that $\{s_\lambda, s_{\mu^*} : \lambda, u \in \Lambda\}$ is the universal Kumjian-Pask Λ-family over $\mathbb{C}$ and that $\{t_\lambda : \lambda \in \Lambda\}$ is the universal Cuntz-Krieger Λ-family. Define*

$$A := \mathrm{span}_{\mathbb{C}}\left\{t_\lambda t_\mu^* : \lambda, \mu \in \Lambda\right\}.$$

Then there is an isomorphism $\pi_t : \mathrm{KP}_{\mathbb{C}}(\Lambda) \to A$ such that $\pi_t(s_\lambda) = t_\lambda$ and $\pi_t(s_{\mu^*}) = t_\mu^*$ for $\lambda, u \in \Lambda$. In particular, $\mathrm{KP}_{\mathbb{C}}(\Lambda)$ is isomorphic to a dense subalgebra of $C^*(\Lambda)$.

We shall use the graded uniqueness theorem to prove Proposition 3.3.1. So the first step is to look for a candidate for the $\mathbb{Z}^k$-grading of $\mathrm{span}_{\mathbb{C}}\left\{t_\lambda t_\mu^* : \lambda, \mu \in \Lambda\right\}$ as stated in the following lemma.

Lemma 3.3.2. *The subspaces*

$$A_n := \operatorname{span}_{\mathbb{C}} \left\{ t_\lambda t_\mu^* : \lambda, \mu \in \Lambda, d(\lambda) - d(\mu) = n \right\}$$

form a $\mathbb{Z}^k$-grading of $\operatorname{span}_{\mathbb{C}} \left\{ t_\lambda t_\mu^* : \lambda, \mu \in \Lambda \right\}$.

The proof of this lemma uses the *gauge action* from page 5.

Proof of Lemma 3.3.2. We generalise the argument of [11, Lemma 7.4]. First we show $A_n A_m \subseteq A_{n+m}$ for $m, n \in \mathbb{Z}^k$. Take $m, n \in \mathbb{Z}^k$. Note that for $\lambda, \mu, \rho, \tau \in \Lambda$ with $d(\lambda) - d(\mu) = n$ and $d(\rho) - d(\tau) = m$, (TCK3) gives

$$\begin{aligned} t_\lambda t_\mu^* t_\rho t_\tau^* &= t_\lambda \Big(\sum_{(\mu', \rho') \in \Lambda^{\min}(\mu, \rho)} t_{\mu'} t_{\rho'}^* \Big) t_\tau^* \\ &= \sum_{(\mu', \rho') \in \Lambda^{\min}(\mu, \rho)} t_{\lambda\mu'} t_{\tau\rho'}^*. \end{aligned}$$

For $(\mu', \rho') \in \Lambda^{\min}(\mu, \rho)$, we have

$$\begin{aligned} d(\lambda\mu') - d(\tau\rho') &= d(\lambda) + d(\mu') - d(\tau) - d(\rho') \\ &= d(\lambda) + (d(\mu) \vee d(\rho) - d(\mu)) \\ &\quad - d(\tau) - (d(\mu) \vee d(\rho) - d(\rho)) \\ &= (d(\lambda) - d(\mu)) - (d(\tau) - d(\rho)) \\ &= n + m. \end{aligned}$$

Hence $t_\lambda t_\mu^* t_\rho t_\tau^* \in A_{n+m}$, and $A_n A_m \subseteq A_{n+m}$.

Next we show that $A = \bigoplus_{n \in \mathbb{Z}^k} A_n$. Since each spanning element $t_\lambda t_\mu^*$ belongs to $A_{d(\lambda) - d(\mu)}$, every element a of A can be written as a finite sum $\sum_n a_n$ with $a_n \in A_n$. To see that the A_n are independent, we suppose that $a_n \in A_n$ and $\sum_n a_n = 0$. We have to show $a_n = 0$ for all $n \in \mathbb{Z}^k$. Now recall that

$$\int_{\mathbb{T}^k} z^m dz = \begin{cases} 1 & \text{if } m = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.1}$$

Hence for $m \in \mathbb{Z}^k$, we have

(3.3.2)

$$\begin{aligned} \int_{\mathbb{T}^k} z^{-m} \gamma_z \left(t_\lambda t_\mu^* \right) dz &= \int_{\mathbb{T}^k} z^{-m} \gamma_z (t_\lambda) \gamma_z \left(t_\mu^* \right) dz = \int_{\mathbb{T}^k} z^{-m} \left(z^{d(\lambda)} t_\lambda \right) \left(z^{-d(\mu)} t_\mu^* \right) dz \\ &= \int_{\mathbb{T}^k} z^{-m+d(\lambda)-d(\mu)} \left(t_\lambda t_\mu^* \right) dz = t_\lambda t_\mu^* \int_{\mathbb{T}^k} z^{-m+d(\lambda)-d(\mu)} dz \end{aligned}$$

$$= \begin{cases} t_\lambda t_\mu^* & \text{if } m = d(\lambda) - d(\mu) \\ 0 & \text{otherwise.} \end{cases}$$

Each a_n can be written as $\sum_{(\lambda,\mu)\in F} c_{n,\lambda,\mu} t_\lambda t_\mu^*$, where F is finite. We also have $c_{n,\lambda,\mu} \in \mathbb{C}$ and $d(\lambda) - d(\mu) = n$ for each $(\lambda, \mu) \in F$. So

(3.3.3)

$$\begin{aligned}
\int_{\mathbb{T}^k} z^{-m} \gamma_z(a_n)\, dz &= \int_{\mathbb{T}^k} z^{-m} \gamma_z \Big(\sum_{(\lambda,\mu)\in F} c_{n,\lambda,\mu} t_\lambda t_\mu^* \Big) dz \\
&= \int_{\mathbb{T}^k} z^{-m} \Big(\sum_{(\lambda,\mu)\in F} c_{n,\lambda,\mu} \gamma_z\left(t_\lambda t_\mu^*\right) \Big) dz \text{ (since } \gamma_z \text{ is linear)} \\
&= \sum_{(\lambda,\mu)\in F} c_{n,\lambda,\mu} \Big(\int_{\mathbb{T}^k} z^{-m} \gamma_z\left(t_\lambda t_\mu^*\right) dz \Big) \text{ (since the integral is linear)} \\
&= \begin{cases} \sum_{(\lambda,\mu)\in F} c_{n,\lambda,\mu} t_\lambda t_\mu^* & \text{if } m = d(\lambda) - d(\mu) \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} a_n & \text{if } m = n \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Since $\sum_n a_n = 0$, (3.3.3) gives

$$0 = \int_{\mathbb{T}^k} z^{-n} \gamma_z \Big(\sum_n a_n \Big) dz = \sum_n \int_{\mathbb{T}^k} z^{-n} \gamma_z(a_n)\, dz = a_n.$$

The conclusion follows. □

Proof of Proposition 3.3.1. Since $\{t_\lambda : \lambda \in \Lambda\}$ satisfies (TCK1-3) and (CK), the family $\left\{t_\lambda, t_\mu^* : \lambda, \mu \in \Lambda\right\}$ also satisfies (KP1-4) and is a Kumjian-Pask Λ-family in $C^*(\Lambda)$. Thus the universal property of $\mathrm{KP}_{\mathbb{C}}(\Lambda)$ gives a homomorphism π_t from $\mathrm{KP}_{\mathbb{C}}(\Lambda)$ onto the dense subalgebra A of $C^*(\Lambda)$.

Next we show the injectivity of π_t. By Theorem 3.2.1, it suffices to show that π_t is a $\mathbb{Z}^k$-graded ring homomorphism. However, this follows from Lemma 3.3.2. □

3.4 The boundary-path groupoid and its Steinberg algebra

In [17], Clark and Sims show that the Leavitt path algebra of an arbitrary 1-graph E is isomorphic to a Steinberg algebra. In this section, we generalise their result by showing that for the Kumjian-Pask algebra of a finitely aligned k-graph Λ is isomorphic to a Steinberg algebra (Proposition 3.4.1).

Recall the boundary-path groupoid $\mathcal{G}_\Lambda$ of a finitely aligned k-graph Λ from Example 2.8.3. We generalise [15, Proposition 4.3] as follows:

Proposition 3.4.1 ([18, Proposition 5.4])**.** *Suppose that Λ is a finitely aligned k-graph and that $\mathcal{G}_\Lambda$ is its boundary-path groupoid as defined in Example 2.8.3. Suppose that R is a commutative ring with 1. Then there is an isomorphism $\pi_T : \mathrm{KP}_R(\Lambda) \to A_R(\mathcal{G}_\Lambda)$ such that $\pi_T(s_\lambda) = 1_{Z_\Lambda(\lambda *_s s(\lambda))}$ and $\pi_T(s_{\mu^*}) = 1_{Z_\Lambda(s(\mu) *_s \mu)}$ for $\lambda, u \in \Lambda$.*

To show the surjectivity of π_T in Proposition 3.4.1, we establish the following two lemmas, which show that the characteristic function of a compact open set in $\mathcal{G}_\Lambda$ can be written as a sum of elements in the form $1_{Z_\Lambda(\lambda *_s \mu \setminus G)}$.

Lemma 3.4.2 ([18, Lemma 5.5])**.** *Suppose that Λ is a finitely aligned k-graph. Suppose that $(\lambda, \mu), (\lambda', \mu') \in \Lambda *_s \Lambda$, $G \subseteq s(\lambda)\Lambda$, and that $G' \subseteq s(\lambda')\Lambda$. Define $F := \Lambda^{\min}(\lambda, \lambda') \cap \Lambda^{\min}(\mu, \mu')$. Then*
(*)

$$Z_\Lambda(\lambda *_s \mu\setminus G) \cap Z_\Lambda(\lambda' *_s \mu'\setminus G') = \bigsqcup_{(\gamma,\gamma')\in F} Z_\Lambda(\lambda\gamma *_s \mu'\gamma'\setminus [\mathrm{Ext}(\gamma; G) \cup \mathrm{Ext}(\gamma'; G')]).$$

Proof. We generalise the argument of [17, Example 3.2] for 1-graphs. First we show that the collection

$$\{Z_\Lambda(\lambda\gamma *_s \mu'\gamma'\setminus [\mathrm{Ext}(\gamma; G) \cup \mathrm{Ext}(\gamma'; G')]) : (\gamma, \gamma') \in F\}$$

is disjoint. It suffices to show that the collection

$$\{Z_\Lambda(\lambda\gamma *_s \mu'\gamma') : (\gamma, \gamma') \in F\}$$

is disjoint. Suppose for contradiction that there exist $(\gamma, \gamma'), (\gamma'', \gamma''') \in F$ such that $(\gamma, \gamma') \neq (\gamma'', \gamma''')$ and $V := Z_\Lambda(\lambda\gamma *_s \mu'\gamma') \cap Z_\Lambda(\lambda\gamma'' *_s \mu'\gamma''') \neq \emptyset$. Note that if $\gamma = \gamma''$, then

$$\begin{aligned} \lambda'\gamma' &= \lambda\gamma \text{ (since } (\gamma, \gamma') \in \Lambda^{\min}(\lambda, \lambda')\text{)} \\ &= \lambda\gamma'' \text{ (since } \gamma = \gamma''\text{)} \\ &= \lambda'\gamma''' \text{ (since } (\gamma'', \gamma''') \in \Lambda^{\min}(\lambda, \lambda')\text{)} \end{aligned}$$

and $\gamma' = \gamma'''$ by the factorisation property, which contradicts $(\gamma, \gamma') \neq (\gamma'', \gamma''')$. The same argument shows that $\gamma' = \gamma'''$ implies $\gamma = \gamma''$. Hence $\gamma \neq \gamma''$ and $\gamma' \neq \gamma'''$.

Meanwhile, since $(\gamma,\gamma'),(\gamma'',\gamma''')\in F$, then $d(\gamma)=d(\gamma'')$ and $d(\gamma')=d(\gamma''')$. Take $(x,m,y)\in V$. Then $x\in Z_\Lambda(\lambda\gamma)$ and $x\in Z_\Lambda(\lambda\gamma'')$. Since $d(\gamma)=d(\gamma'')$, then $d(\lambda\gamma)=d(\lambda\gamma'')$ and $\gamma=x(d(\lambda),d(\lambda\gamma))=x(d(\lambda),d(\lambda\gamma''))=\gamma''$, which contradicts $\gamma\neq\gamma''$. Hence the collection $\{Z_\Lambda(\lambda\gamma *_s \mu'\gamma'):(\gamma,\gamma')\in F\}$ is disjoint, and so is

$$\{Z_\Lambda(\lambda\gamma *_s \mu'\gamma' \setminus [\mathrm{Ext}(\gamma;G)\cup\mathrm{Ext}(\gamma';G')]):(\gamma,\gamma')\in F\}.$$

Now we show the right inclusion of (*). Write

$$U:=Z_\Lambda(\lambda *_s \mu\setminus G)\cap Z_\Lambda(\lambda' *_s \mu'\setminus G')$$

and take $(x,m,y)\in U$. We show $(x,m,y)\in Z_\Lambda(\lambda\gamma *_s \mu'\gamma'\setminus[\mathrm{Ext}(\gamma;G)\cup\mathrm{Ext}(\gamma';G')])$ for some $(\gamma,\gamma')\in F$. Because $x\in Z_\Lambda(\lambda)$ and $x\in Z_\Lambda(\lambda')$, then $d(x)\geq d(\lambda)\vee d(\lambda')$ and there exists $(\gamma,\gamma')\in\Lambda^{\min}(\lambda,\lambda')$ such that

$$x\in Z_\Lambda(\lambda\gamma). \tag{3.4.1}$$

Using a similar argument, there exists $(\gamma'',\gamma''')\in\Lambda^{\min}(\mu,\mu')$ such that

$$y\in Z_\Lambda(\mu\gamma''). \tag{3.4.2}$$

We claim that $\gamma=\gamma''$ and $\gamma'=\gamma'''$. To see this, note that $m=d(\lambda)-d(\mu)=d(\lambda')-d(\mu')$ and

$$\begin{aligned} d(\gamma)&=d(\lambda)\vee d(\lambda')-d(\lambda)=(d(\mu)+m)\vee(d(\mu')+m)-(d(\mu)+m)\\ &=(d(\mu)\vee d(\mu'))+m-(d(\mu)+m)=d(\mu)\vee d(\mu')-d(\mu)=d(\gamma''). \end{aligned}$$

Since $(x,m,y)\in Z_\Lambda(\lambda *_s \mu\setminus G)$, then $\sigma^{d(\lambda)}x=\sigma^{d(\mu)}y$ and

$$\gamma=\left(\sigma^{d(\lambda)}x\right)(0,d(\gamma))=\left(\sigma^{d(\mu)}y\right)(0,d(\gamma'))=\gamma''.$$

Using a similar argument, we also get $\gamma'=\gamma'''$ proving the claim.

Next we show that $(x,m,y)\in Z_\Lambda(\lambda\gamma *_s \mu'\gamma')$. By (3.4.1) and (3.4.2), we have $x\in Z_\Lambda(\lambda\gamma)$ and $y\in Z_\Lambda(\mu\gamma'')$. Since $\gamma=\gamma''$, $\gamma'=\gamma'''$, $(\gamma'',\gamma''')\in\Lambda^{\min}(\mu,\mu')$, then $\mu\gamma''=\mu\gamma=\mu'\gamma'$ and $y\in Z_\Lambda(\mu'\gamma')$. On the other hand, since $(x,m,y)\in Z_\Lambda(\lambda *_s \mu\setminus G)$, then $\sigma^{d(\lambda)}x=\sigma^{d(\mu)}y$ and

$$\sigma^{d(\lambda\gamma)}x=\sigma^{d(\mu\gamma)}y=\sigma^{d(\mu'\gamma')}y$$

since $\mu\gamma=\mu'\gamma'$. Since $m=d(\lambda)-d(\mu)=d(\lambda\gamma)-d(\mu'\gamma')$, then $(x,m,y)\in Z_\Lambda(\lambda\gamma *_s \mu'\gamma')$, as required.

Finally we show that $(x,m,y) \notin Z_\Lambda(\lambda\gamma\nu *_s \mu'\gamma'\nu)$ for all $\nu \in \operatorname{Ext}(\gamma;G) \cup \operatorname{Ext}(\gamma';G')$. Suppose for a contradiction that there exists $\nu \in \operatorname{Ext}(\gamma;G) \cup \operatorname{Ext}(\gamma';G')$ such that $(x,m,y) \in Z_\Lambda(\lambda\gamma\nu *_s \mu'\gamma'\nu)$. Without loss of generality, suppose $\nu \in \operatorname{Ext}(\gamma;G)$. Then there exists $\nu' \in G$ such that $\gamma\nu \in Z_\Lambda(\nu')$. Since $x \in Z_\Lambda(\lambda\gamma\nu)$, $y \in Z_\Lambda(\mu'\gamma'\nu) = Z_\Lambda(\mu\gamma\nu)$, and $\gamma\nu \in Z_\Lambda(\nu')$, then $x \in Z_\Lambda(\lambda\nu')$ and $y \in Z_\Lambda(\mu\nu')$ where $\nu' \in G$. This contradicts $(x,m,y) \in Z_\Lambda(\lambda *_s \mu \setminus G)$. Hence

$$(x,m,y) \in Z_\Lambda(\lambda\gamma *_s \mu'\gamma' \setminus [\operatorname{Ext}(\gamma;G) \cup \operatorname{Ext}(\gamma';G')])$$

and

$$U \subseteq \bigsqcup_{(\gamma,\gamma')\in F} Z_\Lambda(\lambda\gamma *_s \mu'\gamma' \setminus [\operatorname{Ext}(\gamma;G) \cup \operatorname{Ext}(\gamma';G')]).$$

Now we show the left inclusion of (*). Take $(\gamma,\gamma') \in F$ and

$$(x,m,y) \in Z_\Lambda(\lambda\gamma *_s \mu'\gamma' \setminus [\operatorname{Ext}(\gamma;G) \cup \operatorname{Ext}(\gamma';G')]). \tag{3.4.3}$$

We show (x,m,y) belongs to both $Z_\Lambda(\lambda *_s \mu \setminus G)$ and $Z_\Lambda(\lambda' *_s \mu' \setminus G')$. Without loss of generality, it suffices to show $(x,m,y) \in Z_\Lambda(\lambda *_s \mu \setminus G)$. First we show that $(x,m,y) \in Z_\Lambda(\lambda *_s \mu)$. Note that we have $\mu\gamma = \mu'\gamma'$ and $m = d(\lambda\gamma) - d(\mu'\gamma') = d(\lambda) - d(\mu)$. On the other hand, $(x,m,y) \in Z_\Lambda(\lambda\gamma *_s \mu'\gamma')$ also implies $x \in Z_\Lambda(\lambda\gamma)$ and $y \in Z_\Lambda(\mu'\gamma') = Z_\Lambda(\mu\gamma)$. Furthermore,

$$\begin{aligned}
\sigma^{(\lambda)}x &= [x(d(\lambda), d(\lambda\gamma))]\left[\sigma^{(\lambda\gamma)}x\right] \\
&= \gamma\left[\sigma^{(\lambda\gamma)}x\right] \text{ (since } x(d(\lambda), d(\lambda\gamma)) = \gamma) \\
&= \gamma[\sigma^{(\mu'\gamma')}y] \text{ (since } \sigma^{(\lambda\gamma)}x = \sigma^{(\mu'\gamma')}y) \\
&= [y(d(\mu), d(\mu\gamma))][\sigma^{(\mu'\gamma')}y] \text{ (since } y(d(\mu), d(\mu\gamma)) = \gamma) \\
&= [y(d(\mu), d(\mu\gamma))][\sigma^{(\mu\gamma)}y] \text{ (since } \mu\gamma = \mu'\gamma') \\
&= \sigma^{(\mu)}y
\end{aligned}$$

and then $(x,m,y) \in Z_\Lambda(\lambda *_s \mu)$, as required.

To complete the proof, we have to show $(x,m,y) \notin Z_\Lambda(\lambda\nu *_s \mu\nu)$ for all $\nu \in G$. Suppose for contradiction that there exists $\nu \in G$ such that $(x,m,y) \in Z_\Lambda(\lambda\nu *_s \mu\nu)$. In particular, $x \in Z_\Lambda(\lambda\nu)$. Since $x \in Z_\Lambda(\lambda\gamma)$ and $x \in Z_\Lambda(\lambda\nu)$, then there exists $\nu' \in \operatorname{Ext}(\gamma;\{\nu\})$ such that $x \in Z_\Lambda(\lambda\gamma\nu')$. Hence

$$\begin{aligned}
\sigma^{(\lambda\gamma\nu')}x &= \sigma^{(\mu\gamma\nu')}y \text{ (since } \sigma^{(\lambda)}x = \sigma^{(\mu)}y) \\
&= \sigma^{(\mu'\gamma'\nu')}y \text{ (since } \mu\gamma = \mu'\gamma'),
\end{aligned}$$

$$(3.4.4) \qquad \begin{aligned}\left(\sigma^{(\mu)}y\right)(0,d(\gamma\nu')) &= \left(\sigma^{(\lambda)}x\right)(0,d(\gamma\nu')) \text{ (since } \sigma^{(\lambda)}x=\sigma^{(\mu)}y)\\ &= x(d(\lambda),d(\lambda\gamma\nu'))\\ &= \gamma\nu' \text{ (since } x\in Z_\Lambda(\lambda\gamma\nu')\text{)},\end{aligned}$$

and

$$\begin{aligned} y(0,d(\mu'\gamma'\nu')) &= y(0,d(\mu\gamma\nu')) \text{ (since } \mu\gamma=\mu'\gamma')\\ &= \mu\gamma\nu' \text{ (by (3.4.4))}\\ &= \mu'\gamma'\nu' \text{ (since } \mu\gamma=\mu'\gamma').\end{aligned}$$

Furthermore,

$$\begin{aligned} d(\lambda\gamma\nu')-d(\mu'\gamma'\nu') &= d(\lambda\gamma)-d(\mu'\gamma')\\ &= d(\lambda\gamma)-d(\mu\gamma) \text{ (since } \mu\gamma=\mu'\gamma')\\ &= d(\lambda)-d(\mu)=m.\end{aligned}$$

Hence $(x,m,y)\in Z_\Lambda(\lambda\gamma\nu' *_s \mu'\gamma'\nu')$ for some $\nu'\in \operatorname{Ext}(\gamma;\{\nu\})\subseteq \operatorname{Ext}(\gamma;G)$, which contradicts (3.4.3). The conclusion follows. $\square$

Lemma 3.4.3 ([18, Lemma 5.6])**.** *Suppose that Λ is a finitely aligned k-graph. Suppose that for $i\in\{1,\ldots,n\}$, $(\lambda_i,\mu_i)\in\Lambda *_s \Lambda$ and $G_i\subseteq s(\lambda_i)\Lambda$. Suppose that $\{Z_\Lambda(\lambda_i *_s \mu_i\setminus G_i)\}_{i=1}^n$ is a finite collection of compact open bisection sets and*

$$U := \bigcup_{i=1}^n Z_\Lambda(\lambda_i *_s \mu_i\setminus G_i).$$

Then

$$1_U\in \operatorname{span}_R\left\{1_{Z_\Lambda(\lambda *_s \mu\setminus G)} : (\lambda,\mu)\in\Lambda *_s\Lambda, G\subseteq s(\lambda)\Lambda\right\}.$$

Proof. The lemma is trivial for $n=1$. Now let $n=2$ and $F:=\Lambda^{\min}(\lambda_1,\lambda_2)\cap\Lambda^{\min}(\mu_1,\mu_2)$. If $F=\emptyset$, then

$$1_U = 1_{Z_\Lambda(\lambda_1 *_s \mu_1\setminus G_1)} + 1_{Z_\Lambda(\lambda_2 *_s \mu_2\setminus G_2)}.$$

Otherwise, Proposition 3.4.2 gives

$$1_U = 1_{Z_\Lambda(\lambda_1 *_s \mu_1\setminus G_1)} + 1_{Z_\Lambda(\lambda_2 *_s \mu_2\setminus G_2)} - \sum_{(\gamma_1,\gamma_2)\in F} 1_{Z_{\gamma_1,\gamma_2}}$$

where $Z_{\gamma_1,\gamma_2} := Z_\Lambda(\lambda_1\gamma_1 *_s \mu_2\gamma_2\setminus \operatorname{Ext}(\gamma_1;G_1)\cup \operatorname{Ext}(\gamma_2;G_2))$. For $n\geq 3$, by using the inclusion-exclusion principle and de Morgan's law, 1_U can be written as a sum of elements in the form $1_{Z_\Lambda(\lambda *_s \mu\setminus G)}$. $\square$

Proof of Proposition 3.4.1. Define $T_\lambda := 1_{Z_\Lambda(\lambda *_s s(\lambda))}$ and $T^*_\mu := 1_{Z_\Lambda(s(\mu) *_s \mu)}$ for $\lambda, \mu \in \Lambda$. Then by [24, Theorem 6.13] (or [58, Example 7.1]), $\{T_\lambda : \lambda, u \in \Lambda\}$ satisfies (TCK1-3) and (CK). Thus $\left\{T_\lambda, T^*_\mu : \lambda, u \in \Lambda\right\}$ is a Kumjian-Pask Λ-family in $A_R(\mathcal{G}_\Lambda)$. Hence there exists a homomorphism $\pi_T : \mathrm{KP}_R(\Lambda) \to A_R(\mathcal{G}_\Lambda)$ such that $\pi_T(s_\lambda) = T_\lambda$ and $\pi_T(s_{\mu^*}) = T^*_\mu$ for $\lambda, \mu \in \Lambda$ by Theorem 3.1.8(a).

To see that π_T is injective, first we show that π_T is graded. Take $\lambda, \mu \in \Lambda$. Then $s_\lambda s_{\mu^*} \in \mathrm{KP}_R(\Lambda)_{d(\lambda)-d(\mu)}$ and

$$\pi_T(s_\lambda s_{\mu^*}) = 1_{Z_\Lambda(\lambda *_s \mu)} = 1_{\{(x, d(\lambda)-d(\mu), y) : (\lambda,\mu) \in \Lambda *_s \Lambda, z \in s(\lambda)\partial\Lambda\}} \in A_R(\mathcal{G}_\Lambda)_{d(\lambda)-d(\mu)}.$$

Since for every $n \in \mathbb{Z}^k$, $\mathrm{KP}_R(\Lambda)_n$ is spanned by elements in the form $s_\lambda s_{\mu^*}$ (Theorem 3.1.8(c)), then for $n \in \mathbb{Z}^k$, $\pi_T(\mathrm{KP}_R(\Lambda)_n) \subseteq A_R(\mathcal{G}_\Lambda)_n$ and π_T is graded. Since π_T is graded and $\pi_T(rs_v) = r1_{Z_\Lambda(v *_s v)} \neq 0$ for all $r \in R \setminus \{0\}$ and $v \in \Lambda^0$, Theorem 3.2.1 implies that π_T is injective.

Finally we show the surjectivity of π_T. Take $f \in A_R(\mathcal{G}_\Lambda)$. By Proposition 2.9.1, f can be written as $\sum_{U \in F} a_U 1_U$ where $a_U \in R$, each U is in the form $\bigcup_{i=1}^n Z_\Lambda(\lambda_i *_s \mu_i \setminus G_i)$ for some $n \in \mathbb{N}$, and F is a finite set of mutually disjoint open sets. Hence to show $f \in \mathrm{im}(\pi_T)$, it suffices to show

$$1_U \in \mathrm{im}(\pi_T)$$

where $U := \bigcup_{i=1}^n Z_\Lambda(\lambda_i *_s \mu_i \setminus G_i)$ for some $n \in \mathbb{N}$ and collection $\{Z_\Lambda(\lambda_i *_s \mu_i \setminus G_i)\}_{i=1}^n$. By Lemma 3.4.3, 1_U can be written as the sum of elements of the form $1_{Z_\Lambda(\lambda *_s \mu \setminus G)}$. On the other hand, for $(\lambda, \mu) \in \Lambda *_s \Lambda$ and finite $G \subseteq s(\lambda)\Lambda$, we have

(3.4.5)

$$\begin{aligned} T_\lambda \Big(\prod_{\nu \in G} \left(T_{s(\lambda)} - T_\nu T_{\nu^*}\right) \Big) T_{\mu^*} &= 1_{Z_\Lambda(\lambda *_s s(\lambda))} \Big(\prod_{\nu \in G} \left(1_{Z_\Lambda(s(\lambda) *_s s(\lambda))} - 1_{Z_\Lambda(\nu *_s \nu)}\right) \Big) 1_{Z_\Lambda(s(\mu) *_s \mu)} \\ &= 1_{Z_\Lambda(\lambda *_s s(\lambda))} \Big(\prod_{\nu \in G} \left(1_{Z_\Lambda(s(\lambda) *_s s(\lambda) \setminus \{\nu\})}\right) \Big) 1_{Z_\Lambda(s(\mu) *_s \mu)} \\ &= 1_{Z_\Lambda(\lambda *_s s(\lambda))} \Big(1_{\prod_{\nu \in G} Z_\Lambda(s(\lambda) *_s s(\lambda) \setminus \{\nu\})} \Big) 1_{Z_\Lambda(s(\mu) *_s \mu)} \\ &= 1_{Z_\Lambda(\lambda *_s s(\lambda))} \Big(1_{Z_\Lambda(s(\lambda) *_s s(\lambda) \setminus G)} \Big) 1_{Z_\Lambda(s(\mu) *_s \mu)} \\ &= 1_{Z_\Lambda(\lambda *_s \mu \setminus G)} \end{aligned}$$

since $s(\lambda) = s(\mu)$. Hence $1_{Z_\Lambda(\lambda *_s \mu \setminus G)}$ belongs to $\mathrm{im}(\pi_T)$, and then so does 1_U, as required. Therefore π_T is surjective and is an isomorphism. $\square$

Remark 3.4.4. Finitely aligned k-graphs include 1-graphs and row-finite k-graphs with no sources (see Section 2.4). In these cases, the boundary path groupoid $\mathcal{G}_\Lambda$ of Example

2.8.3 coincides with $\mathcal{G}_E$ of [17] and $\mathcal{G}_\Lambda$ of [15]. Thus we have generalised Example 3.2 of [17] and Proposition 4.3 of [15]. For locally convex row-finite k-graphs, our construction gives a Steinberg algebra model of the Kumjian-Pask algebras of [16].

3.5 Relation between aperiodic higher-rank graphs and effective groupoids

In this section and Section 3.6, we investigate the relationship between a k-graph Λ and the boundary-path groupoid $\mathcal{G}_\Lambda$ constructed in Example 2.8.3. We expect the Cuntz-Krieger uniqueness theorem (Theorem 3.7.1) to apply only to *aperiodic* finitely aligned k-graphs. On the other hand, *effective* groupoids (definition below) are needed in the hypothesis of the Cuntz-Krieger uniqueness theorem for Steinberg algebras (Theorem 3.7.2). In this section, our main result is Proposition 3.5.1 which says that a finitely aligned k-graph Λ is aperiodic if and only if the boundary-path groupoid $\mathcal{G}_\Lambda$ is effective.

Recall from Definition 2.6.2 that a boundary path x is *aperiodic* if for all $\lambda, \mu \in \Lambda r(x)$, $\lambda \neq \mu$ implies $\lambda x \neq \mu x$. We then say a finitely aligned k-graph Λ is *aperiodic* if for each $v \in \Lambda^0$, there exists an aperiodic boundary path x with $r(x) = v$.

Next let $\mathcal{G}$ be a topological groupoid. Recall from Section 2.10 that $\mathcal{G}$ is *effective* if the interior of

$$\operatorname{Iso}(\mathcal{G}) := \{a \in \mathcal{G} : s(a) = r(a)\}$$

is $\mathcal{G}^{(0)}$. See [13, Lemma 3.1] for some equivalent characterisations.

Proposition 3.5.1 ([18, Proposition 6.3]). *Suppose that Λ is a finitely aligned k-graph. Then Λ is aperiodic if and only if the boundary-path groupoid $\mathcal{G}_\Lambda$ is effective.*

Proof. First suppose that Λ is aperiodic. We trivially have that $\mathcal{G}_\Lambda^{(0)}$ is contained in the interior of $\operatorname{Iso}(\mathcal{G}_\Lambda)$. Now we show the reverse inclusion. Take an interior point a of $\operatorname{Iso}(\mathcal{G}_\Lambda)$. Then there exits $Z_\Lambda(\lambda *_s \mu \setminus G)$ such that $Z_\Lambda(\lambda *_s \mu \setminus G) \subseteq \operatorname{Iso}(\mathcal{G}_\Lambda)$ and $a \in Z_\Lambda(\lambda *_s \mu \setminus G)$. We show $\lambda = \mu$.

Note that since $a \in Z_\Lambda(\lambda *_s \mu \setminus G)$, $Z_\Lambda(\lambda *_s \mu \setminus G)$ is not empty, and G is not exhaustive (see Remark 2.8.4(ii)). Hence there exists $\nu \in s(\lambda)\Lambda$ such that $\Lambda^{\min}(\nu, \gamma) = \emptyset$ for $\gamma \in G$. Because Λ is aperiodic, there exists a aperiodic boundary path $x \in s(\nu)\partial\Lambda$.

We claim that the boundary path νx is also aperiodic. Suppose for contradiction that there exist $\lambda', \mu' \in \Lambda r(\nu x)$ such that $\lambda' \neq \mu'$ and

$$\lambda'(\nu x) = \mu'(\nu x). \tag{3.5.1}$$

Since $\lambda', \mu', \nu \in \Lambda$, by uniqueness in the factorisation property, $\lambda' \neq \mu'$ implies $\lambda'\nu \neq \mu'\nu$. Now because x is aperiodic, $\lambda'\nu \neq \mu'\nu$ implies $\lambda'\nu x \neq \mu'\nu x$, which contradicts (3.5.1). Hence νx is aperiodic, as claimed.

Since $\lambda\nu x \in Z_\Lambda(\lambda) \setminus Z_\Lambda(\lambda\gamma)$ and $\mu\nu x \in Z_\Lambda(\mu) \setminus Z_\Lambda(\mu\gamma)$ for $\gamma \in G$, we have

$$(\lambda\nu x, d(\lambda) - d(\mu), \mu\nu x) \in Z_\Lambda(\lambda *_s \mu \setminus G).$$

Thus $Z_\Lambda(\lambda *_s \mu \setminus G) \subseteq \mathrm{Iso}(\mathcal{G}_\Lambda)$, and hence $\lambda\nu x = \mu\nu x$. Since νx is aperiodic, we have $\lambda(\nu x) = \mu(\nu x)$ which implies $\lambda = \mu$. Therefore $\mathcal{G}_\Lambda$ is effective.

Now suppose that Λ is not aperiodic. Then there exists $v \in \Lambda^0$ such that for all boundary paths $x \in v\partial\Lambda$, x is not aperiodic.

Claim 3.5.2. *For $x \in v\partial\Lambda$, we have $x\mathcal{G}_\Lambda x \neq \{x\}$.*

Proof of Claim 3.5.2. Take $x \in v\partial\Lambda$. Since x is not aperiodic, there exist $\lambda, \mu \in \Lambda r(x)$ such that $\lambda \neq \mu$ and $\lambda x = \mu x$. If $d(\lambda) = d(\mu)$, then

$$\lambda = (\lambda x)(0, d(\lambda)) = (\mu x)(0, d(\mu)) = \mu,$$

which contradicts $\lambda \neq \mu$.

So suppose $d(\lambda) \neq d(\mu)$. Note that for $1 \leq i \leq k$ such that $d(\lambda)_i \neq d(\mu)_i$, we have $d(x)_i = \infty$ (since $\lambda x = \mu x$). Hence

$$((d(\lambda) \vee d(\mu)) - d(\lambda)) \vee ((d(\lambda) \vee d(\mu)) - d(\mu)) \leq d(x).$$

Write $p := (d(\lambda) \vee d(\mu)) - d(\lambda)$ and $q := (d(\lambda) \vee d(\mu)) - d(\mu)$, and note that $p \neq q$. Since $\lambda x = \mu x$,

$$\begin{aligned} \sigma^p x = \sigma^p\left(\sigma^{d(\lambda)}(\lambda x)\right) &= \sigma^{d(\lambda)\vee d(\mu)}(\lambda x) \\ &= \sigma^{d(\lambda)\vee d(\mu)}(\mu x) \\ &= \sigma^q\left(\sigma^{d(\mu)}(\mu x)\right) = \sigma^q x. \end{aligned}$$

This implies that $(x, p-q, x) \in \mathcal{G}_\Lambda \setminus \mathcal{G}_\Lambda^{(0)}$ and $x\mathcal{G}_\Lambda x \neq \{x\}$. □ Claim 3.5.2

Since $x\mathcal{G}_\Lambda x \neq \{x\}$ for all $x \in v\partial\Lambda$,

$$Z_\Lambda(v) \cap \{z \in \mathcal{G}_\Lambda^{(0)} : z\mathcal{G}_\Lambda z = \{z\}\} = \emptyset$$

and $\{z \in \mathcal{G}_\Lambda^{(0)} : z\mathcal{G}_\Lambda z = \{z\}\}$ is not dense in $\mathcal{G}_\Lambda^{(0)}$. Since $\mathcal{G}_\Lambda$ is locally compact, second-countable, Hausdorff and étale (see Remark 2.8.4(iii)), [44, Proposition 3.6(b)] implies that $\mathcal{G}_\Lambda$ is not effective, as required. □

Remark 3.5.3. For a finitely aligned k-graph Λ, the following five conditions are equivalent:

(a) $\mathcal{G}_\Lambda$ is effective;

(b) $\mathcal{G}_\Lambda$ is *topologically principal*: the set of units with trivial isotropy is dense in $\mathcal{G}^{(0)}$;

(c) $\mathcal{G}_\Lambda$ satisfies Condition (1) of Theorem 5.1 of [45];

(d) Λ has *no local periodicity* as defined in [48];

(e) Λ is aperiodic.

In [44, Proposition 3.6], Renault shows that for a locally compact, second-countable, Hausdorff, étale $\mathcal{G}$, $\mathcal{G}$ is effective if and only if it is topologically principal. Since the boundary-path groupoid $\mathcal{G}_\Lambda$ is locally compact, second-countable, Hausdorff and étale, we have (a)$\Leftrightarrow$(b). In [58, Theorem 5.2], Yeend proves (b)$\Leftrightarrow$(c). (Note that Yeend uses "*essentially free*" to mean "topologically principal".) Lemma 5.6 of [45] gives (c)$\Leftrightarrow$(d). Finally, (d)$\Leftrightarrow$(e) follows from [48, Proposition 2.11].

3.6 Relation between cofinal higher-rank graphs and minimal groupoids

In this section, we show that a finitely aligned k-graph Λ is cofinal if and only if the boundary-path groupoid $\mathcal{G}_\Lambda$ is minimal (Proposition 3.6.1). We use this to study the simplicity of Kumjian-Pask algebras in Section 3.8.

Recall from [50, Definition 8.4] that a k-graph Λ is *cofinal* if for all $v \in \Lambda^0$ and $x \in \partial\Lambda$, there exists $n \leq d(x)$ such that $v\Lambda x(n) \neq \emptyset$.

In a groupoid $\mathcal{G}$, a subset $U \subseteq \mathcal{G}^{(0)}$ is *invariant* if $s(a) \in U$ implies $r(a) \in U$ for all $a \in \mathcal{G}$. Note that U is invariant if and only if $\mathcal{G}^{(0)} \backslash U$ is invariant. A topological groupoid $\mathcal{G}$ is *minimal* if $\mathcal{G}^{(0)}$ has no nontrivial open invariant subsets. Equivalently, $\mathcal{G}$ is minimal if for each $x \in \mathcal{G}^{(0)}$, the orbit $[x] := s(x\mathcal{G})$ is dense in $\mathcal{G}^{(0)}$.

Proposition 3.6.1 ([18, Proposition 7.1])**.** *Suppose that Λ is a finitely aligned k-graph. Then Λ is cofinal if and only if the boundary-path groupoid $\mathcal{G}_\Lambda$ is minimal.*

Proof. Suppose that Λ is cofinal. Take $x \in \mathcal{G}_\Lambda^{(0)}$. We have to show that $[x]$ is dense in $\mathcal{G}_\Lambda^{(0)}$. Take a nonempty open set $Z_\Lambda(\lambda \backslash G)$ and we claim that $Z_\Lambda(\lambda \backslash G) \cap [x] \neq \emptyset$. Since

$Z_\Lambda(\lambda\backslash G)$ is nonempty, G is not exhaustive (see Remark 2.8.4(i)). Then there exists $\nu \in s(\lambda)\Lambda$ such that $\Lambda^{\min}(\nu,\gamma) = \emptyset$ for $\gamma \in G$. Now consider the vertex $s(\lambda\nu)$ and the boundary path x. Since Λ is cofinal, there exists $n \leq d(x)$ such that $s(\lambda\nu)\Lambda x(n) \neq \emptyset$. Take $\mu \in s(\lambda\nu)\Lambda x(n)$. Because x is a boundary path, so is $\sigma^n x$. Hence

$$y := \lambda\nu\mu[\sigma^n x]$$

is also a boundary path and $y \in Z_\Lambda(\lambda)$. Since $\Lambda^{\min}(\nu,\gamma) = \emptyset$ for $\gamma \in G$, we have $y \notin Z_\Lambda(\lambda\gamma)$ for $\gamma \in G$. Hence $y \in Z_\Lambda(\lambda\backslash G)$.

On the other hand, since $y = \lambda\nu\mu[\sigma^n x]$, we have $(x, n - d(\lambda\nu\mu), y) \in \mathcal{G}_\Lambda$ and $y \in [x]$. Therefore $Z_\Lambda(\lambda\backslash G) \cap [x] \neq \emptyset$. Thus $[x]$ is dense in $\mathcal{G}_\Lambda^{(0)}$ and $\mathcal{G}_\Lambda$ is minimal.

Suppose that Λ is not cofinal. Then there exist $v \in \Lambda^0$ and $x \in \partial\Lambda$ such that for all $n \leq d(x)$, we have $v\Lambda x(n) = \emptyset$. We claim $Z_\Lambda(v) \cap [x] = \emptyset$. Suppose for contradiction that $Z_\Lambda(v) \cap [x] \neq \emptyset$. Take $y \in Z_\Lambda(v) \cap [x]$. Because $y \in [x]$, there exists $p, q \in \mathbb{N}^k$ such that $(x, p - q, y) \in \mathcal{G}_\Lambda$. This implies $\sigma^p x = \sigma^q y$. Thus $r(y) = v$ and $\sigma^p x = \sigma^q y$ imply that $y(0,q)$ belongs to $v\Lambda x(p)$, which contradict $v\Lambda x(p) = \emptyset$. Therefore $Z_\Lambda(v) \cap [x] = \emptyset$, as claimed, and $[x]$ is not dense in $\mathcal{G}_\Lambda^{(0)}$. Thus $\mathcal{G}_\Lambda$ is not minimal. □

3.7 The Cuntz-Krieger uniqueness theorem

Throughout this section, Λ is a finitely aligned k-graph and R is a commutative ring with identity 1.

Theorem 3.7.1 ([18, Theorem 8.1]: The Cuntz-Krieger uniqueness theorem)**.** *Suppose that Λ is an aperiodic finitely aligned k-graph and that R is a commutative ring with 1. Suppose that $\pi : \mathrm{KP}_R(\Lambda) \to A$ is a ring homomorphism such that $\pi(rs_v) \neq 0$ for all $r \in R\backslash\{0\}$ and $v \in \Lambda^0$. Then π is injective.*

We show Theorem 3.7.1 using the Cuntz-Krieger uniqueness theorem for Steinberg algebras Theorem 2.10.2. First we verify an alternative formulation of the Cuntz-Krieger uniqueness theorem for Steinberg algebras.

Theorem 3.7.2 ([18, Theorem 8.2])**.** *Suppose that $\mathcal{G}$ is an effective Hausdorff ample groupoid and that R is a commutative ring with 1. Suppose that $\mathcal{B}$ is a basis of compact open bisections for the topology on $\mathcal{G}$. Suppose that $\phi : A_R(\mathcal{G}) \to A$ is a ring homomorphism such that* $\ker(\phi) \neq \{0\}$*. Then there exist $r \in R\backslash\{0\}$ and $B \in \mathcal{B}$ such that $B \subseteq \mathcal{G}^{(0)}$ and $\phi(r1_B) = 0$.*

Proof. Since $\ker(\phi) \neq 0$, then by [14, Theorem 3.2], there exist $r \in R\setminus\{0\}$ and a nonempty compact open subset $K \subseteq \mathcal{G}^{(0)}$ such that $\phi(r1_K) = 0$. Since K is open, there exists $B \in \mathcal{B}$ such that $B \subseteq K$. Hence $B \subseteq \mathcal{G}^{(0)}$ and

$$0 = \phi(r1_K)\phi(1_B) = \phi(r1_{KB}) = \phi(r1_{K\cap B}) = \phi(r1_B).$$

□

Proof of Theorem 3.7.1. First note that $\mathcal{G}_\Lambda$ is a Hausdorff and ample groupoid, and is effective by Proposition 3.5.1. Thus it satisfies the hypothesis of Theorem 3.7.2. Now recall the isomorphism $\pi_T : \mathrm{KP}_R(\Lambda) \to A_R(\mathcal{G}_\Lambda)$ of Proposition 3.4.1. Then $\pi_T(s_\lambda) = 1_{Z_\Lambda(\lambda *_s s(\lambda))}$ and $\pi_T(s_{\mu^*}) = 1_{Z_\Lambda(s(\mu) *_s \mu)}$ for $\lambda, u \in \Lambda$. Define $\phi := \pi \circ \pi_T^{-1}$. To show the injectivity of π, it suffices to show that ϕ is injective. Suppose for contradiction that ϕ is not injective. By Theorem 3.7.2, there exist $r \in R\setminus\{0\}$ and $Z_\Lambda(\lambda\setminus G)$ such that $\phi\left(r1_{Z_\Lambda(\lambda\setminus G)}\right) = 0$. Since $1_{Z_\Lambda(\lambda\setminus G)} = 1_{Z_\Lambda(\lambda *_s \lambda\setminus G)}$ (Remark 2.8.4(i)), then the argument of (3.4.5) gives

$$\phi\left(r1_{Z_\Lambda(\lambda\setminus G)}\right) = \pi\Big(rs_\lambda\Big(\prod_{\nu\in G}\left(s_{s(\lambda)} - s_\nu s_{\nu^*}\right)\Big)s_{\lambda^*}\Big),$$

and then

$$\pi\Big(rs_\lambda\Big(\prod_{\nu\in G}\left(s_{s(\lambda)} - s_\nu s_{\nu^*}\right)\Big)s_{\lambda^*}\Big) = 0. \tag{3.7.1}$$

On the other hand, since $\pi(rs_v) \neq 0$ for all $r \in R\setminus\{0\}$ and $v \in \Lambda^0$, and G is finite non-exhaustive, then Proposition 3.1.3(d) implies that

$$\pi\Big(rs_\lambda\Big(\prod_{\nu\in G}\left(s_{s(\lambda)} - s_\nu s_{\nu^*}\right)\Big)s_{\lambda^*}\Big) \neq 0,$$

which contradicts (3.7.1). The conclusion follows. □

One application of Theorem 3.7.1 is:

Corollary 3.7.3 ([18, Corollary 8.3])**.** *Suppose that Λ is a finitely aligned k-graph and that R is a commutative ring with 1. Then Λ is aperiodic if and only if the boundary-path representation $\pi_S : \mathrm{KP}_R(\Lambda) \to \mathrm{End}(\mathbb{F}_R(\partial\Lambda))$ is injective.*

To show Corollary 3.7.3, we establish some results and notation.

Following [48, Definition 2.3], for a finitely aligned k-graph Λ, we say Λ that has *no local periodicity* if for every $v \in \Lambda^0$ and every $n \neq m \in \mathbb{N}^k$, there exists $x \in v\partial\Lambda$ such that either $d(x) \not\geq n \vee m$ or $\sigma^n x \neq \sigma^m x$. If no local aperiodicity fails at $v \in \Lambda^0$, then there are $n \neq m \in \mathbb{N}^k$ such that $\sigma^n x = \sigma^m x$ for all $x \in v\partial\Lambda$. In this case, we say Λ has *local periodicity n, m at $v \in \Lambda^0$*.

Lemma 3.7.4 ([48, Lemma 2.9])**.** *Suppose that Λ is a finitely aligned k-graph which has local periodicity n, m at $v \in \Lambda^0$. Then $d(x) \geq n \vee m$ and $\sigma^n x = \sigma^m x$ for every $x \in v\partial\Lambda$. Fix $x \in v\partial\Lambda$ and set $\mu = x(0, m)$, $\alpha = x(m, m \vee n)$, and $\nu = (\mu\alpha)(0, n)$. Then $\mu\alpha y = \nu\alpha y$ for every $y \in s(\alpha)\partial\Lambda$.*

Proof. Take $y \in s(\alpha)\partial\Lambda$ and define $z := \mu\alpha y$. Because Λ has local periodicity n, m at $v \in \Lambda^0$, then $\sigma^n z = \sigma^m z$. Now note that

$$\begin{aligned} z(0, n) &= (\mu\alpha y)(0, n) \\ &= (\mu\alpha)(0, n) \ \ (\text{since } n \leq d(\mu\alpha)) \\ &= \nu \end{aligned}$$

and $z = \nu[\sigma^n z]$. Since $d(\mu) = m$, we deduce that

$$\sigma^n z = \sigma^m z = (\mu\alpha y)(0, m) = \alpha y,$$

and

$$\mu\alpha y = z = \nu[\sigma^n z] = \nu\alpha y,$$

as required. □

Proof of Corollary 3.7.3. Suppose that Λ is aperiodic. By Proposition 3.1.7, we have $\pi_S(rs_v) \neq 0$ for all $r \in R\backslash\{0\}$ and $v \in \Lambda^0$. Since Λ is aperiodic, Theorem 3.7.1 implies that π_S is injective.

Next suppose that Λ is not aperiodic. We follow the argument of [11, Lemma 5.9]. Since Λ is not aperiodic, by [48, Proposition 2.11] (see also Proposition 2.6.3), there exist $v \in \Lambda^0$ and $n \neq m \in \mathbb{N}^k$ such that Λ has local periodicity n, m at $v \in \Lambda^0$. Let μ, ν, α be as in Lemma 3.7.4 and define $a := s_{\mu\alpha}s_{(\mu\alpha)^*} - s_{\nu\alpha}s_{(\mu\alpha)^*}$. We claim that $a \in \ker(\pi_S)\backslash\{0\}$.

First we show that $a \neq 0$. Suppose for contradiction that $a = 0$. Then $s_{\mu\alpha}s_{(\mu\alpha)^*} = s_{\nu\alpha}s_{(\mu\alpha)^*}$. Note that $d\left(s_{\mu\alpha}s_{(\mu\alpha)^*}\right) = d(\mu\alpha) - d(\mu\alpha) = 0$ and

$$d\left(s_{\nu\alpha}s_{(\mu\alpha)^*}\right) = d(\nu\alpha) - d(\mu\alpha) = d(\nu) + d(\alpha) - d(\mu) - d(\alpha) = n - m \neq 0.$$

Hence $s_{\mu\alpha}s_{(\mu\alpha)^*} = s_{\nu\alpha}s_{(\mu\alpha)^*} = 0$. Thus $0 = s_{(\mu\alpha)^*}\left(s_{\mu\alpha}s_{(\mu\alpha)^*}\right)s_{\mu\alpha} = s_{s(\mu\alpha)}^2 = s_{s(\mu\alpha)}$, which contradicts Theorem 3.1.8(b). Hence $a \neq 0$.

Now we show that $a \in \ker(\pi_S)$. We take $y \in \partial\Lambda$, and have to show $\pi_S(a)(y) = 0$. Recall that $\pi_S(s_\lambda) = S_\lambda$ and $\pi_S(s_{\mu^*}) = S_{\mu^*}$ where

$$S_\lambda(y) = \begin{cases} \lambda y & \text{if } s(\lambda) = r(y) \\ 0 & \text{otherwise,} \end{cases} \quad \text{and } S_{\mu^*}(y) = \begin{cases} \sigma^{d(\mu)}y & \text{if } y(0, d(\mu)) = \mu \\ 0 & \text{otherwise.} \end{cases}$$

First suppose $y(0,d(\mu\alpha)) \neq \mu\alpha$. Then $S_{(\mu\alpha)^*}(y) = 0$ and $\pi_S(a)(y) = S_{\mu\alpha}S_{(\mu\alpha)^*}(y) - S_{\nu\alpha}S_{(\mu\alpha)^*}(y) = 0$. Suppose $y(0,d(\mu\alpha)) = \mu\alpha$. Then

$$\pi_S(a)(y) = (S_{\mu\alpha} - S_{\nu\alpha})\left(\sigma^{d(\mu\alpha)}y\right).$$

Since $y \in \partial\Lambda$, then $\sigma^{d(\mu\alpha)}y \in s(\alpha)\partial\Lambda$ and Lemma 3.7.4 gives $\mu\alpha\left(\sigma^{d(\mu\alpha)}y\right) = \nu\alpha\left(\sigma^{d(\mu\alpha)}y\right)$. Hence $\pi_S(a)(y) = 0$. Thus $a \in \ker(\pi_S)\setminus\{0\}$, as claimed, and π_S is not injective. □

3.8 Basic simplicity and simplicity

As in [55], an ideal I in $\mathrm{KP}_R(\Lambda)$ is *basic* if whenever $r \in R\setminus\{0\}$, $v \in \Lambda^0$, and $rs_v \in I$ imply $s_v \in I$. We say $\mathrm{KP}_R(\Lambda)$ is *basically simple* if its only basic ideals are $\{0\}$ and $\mathrm{KP}_R(\Lambda)$.

In this section, we investigate necessary and sufficient conditions for $\mathrm{KP}_R(\Lambda)$ to be basically simple (Theorem 3.8.3) and to be simple (Theorem 3.8.4). We show that both results follow from characterisations of basic simplicity and simplicity for Steinberg algebras (see Theorem 3.8.1 and Theorem 3.8.2). Therefore we state necessary and sufficient conditions for the Steinberg algebra $A_R(\mathcal{G})$ to be basically simple and to be simple in the following two theorems.

Theorem 3.8.1 ([14, Theorem 4.1])**.** *Suppose that $\mathcal{G}$ is a Hausdorff ample groupoid and that R is a commutative ring with 1. Then $A_R(\mathcal{G})$ is basically simple if and only if $\mathcal{G}$ is effective and minimal.*

Theorem 3.8.2 ([14, Corollary 4.6])**.** *Suppose that $\mathcal{G}$ is a Hausdorff ample groupoid and that R is a commutative ring with 1. Then $A_R(\mathcal{G})$ is simple if and only if R is a field and $\mathcal{G}$ is effective and minimal.*

Now we are ready to prove our results.

Theorem 3.8.3 ([18, Theorem 9.3])**.** *Suppose that Λ is a finitely aligned k-graph and that R is a commutative ring with 1. Then $\mathrm{KP}_R(\Lambda)$ is basically simple if and only if Λ is aperiodic and cofinal.*

Proof. First suppose that $\mathrm{KP}_R(\Lambda)$ is basically simple. By Proposition 3.4.1, $A_R(\mathcal{G}_\Lambda)$ is also basically simple and then by Theorem 3.8.1, $\mathcal{G}_\Lambda$ is effective and minimal. On the other hand, $\mathcal{G}_\Lambda$ is effective implies that Λ is aperiodic (Proposition 3.5.1), and $\mathcal{G}_\Lambda$ is minimal implies that Λ is cofinal (Proposition 3.6.1). The conclusion follows.

Next suppose that Λ is aperiodic and cofinal. By Proposition 3.5.1 and Proposition 3.6.1, $\mathcal{G}_\Lambda$ is effective and minimal and then Theorem 3.8.1 implies that $A_R(\mathcal{G}_\Lambda)$ is basically simple. Since $A_R(\mathcal{G}_\Lambda)$ is isomorphic to $\mathrm{KP}_R(\Lambda)$ (Proposition 3.4.1), $\mathrm{KP}_R(\Lambda)$ is also basically simple, as required. $\square$

Theorem 3.8.4 ([18, Theorem 9.4])**.** *Suppose that Λ is a finitely aligned k-graph and that R is a commutative ring with* 1*. Then* $\mathrm{KP}_R(\Lambda)$ *is simple if and only if R is a field and Λ is aperiodic and cofinal.*

Proof. First suppose that $\mathrm{KP}_R(\Lambda)$ is simple. Then $\mathrm{KP}_R(\Lambda)$ is also basically simple, and Theorem 3.8.3 implies that Λ is aperiodic and cofinal. On the other hand, since $\mathrm{KP}_R(\Lambda)$ is simple, then by Proposition 3.4.1, $A_R(\mathcal{G}_\Lambda)$ is also simple, and Theorem 3.8.2 implies that R is a field, as required.

Next suppose that R is a field and Λ is aperiodic and cofinal. By Proposition 3.5.1 and Proposition 3.6.1, $\mathcal{G}_\Lambda$ is effective and minimal. Hence Theorem 3.8.2 implies that $A_R(\mathcal{G}_\Lambda)$ is simple and by Proposition 3.4.1, so is $\mathrm{KP}_R(\Lambda)$. $\square$

Chapter 4

Cohn path algebras

Suppose that Λ is a row-finite higher-rank graph with no sources and that R is a commutative ring with 1. In this chapter, we introduce a Cohn path Λ-family for row-finite higher-rank graphs with no sources and study its properties. We show there is a universal Cohn path algebra $\mathrm{C}_R(\Lambda)$ (Proposition 4.1.5). We also establish a uniqueness theorem for Cohn path algebras (Theorem 4.2.1), and then give examples of Cohn path algebras in Section 4.3.

The material in this chapter is taken from a joint work with my supervisor Clark in [19].

4.1 Cohn Λ-families

Throughout this section, suppose that Λ is a row-finite k-graph with no sources and that R is a commutative ring with 1. Recall from Chapter 3 that for $\lambda \in \Lambda$, λ^* is a *ghost path*, and $v^* := v$ for $v \in \Lambda^0$. We also write $G(\Lambda)$ for the set of ghost paths and define r and s on $G(\Lambda)$ by

$$r(\lambda^*) := s(\lambda) \text{ and } s(\lambda^*) := r(\lambda).$$

We also define composition in $G(\Lambda)$ by setting $\lambda^*\mu^* = (\mu\lambda)^*$ for $\lambda, \mu \in \Lambda$ with $s(\mu) = r(\lambda)$, we write $G\left(\Lambda^{\neq 0}\right) := \{\lambda^* : \lambda \in \Lambda \setminus \Lambda^0\}$. We also recall from Chapter 2 that Λ^1 denotes the set of edges of Λ.

Definition 4.1.1. A *Cohn Λ-family* $\{T_\lambda, T_{\mu^*} : \lambda, \mu \in \Lambda\}$ in an R-algebra A consists of a function $T : \Lambda \cup G\left(\Lambda^{\neq 0}\right) \to A$ such that:

(CP1) $\{T_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal idempotents;

(CP2) for $\lambda, \mu \in \Lambda$ with $s(\lambda) = r(\mu)$, we have $T_\lambda T_\mu = T_{\lambda\mu}$ and $T_{\mu^*} T_{\lambda^*} = T_{(\lambda\mu)^*}$;

(CP3) $T_{\lambda^*} T_\mu = \sum_{(\nu,\gamma)\in\Lambda^{\min}(\lambda,\mu)} T_\nu T_{\gamma^*}$ for all $\lambda, \mu \in \Lambda$.

Remark 4.1.2. (i) In a 1-graph E, people usually write $\{v, e, e^* : v \in E^0, e \in E^1\}$ instead of $\{T_\lambda, T_{\mu^*} : \lambda, \mu \in E^*\}$ (see [1, 3, 5, 6, 9]). We do not use this notation because we want to distinguish between the paths in E and the corresponding elements of the algebra A.

(ii) Since Λ is row-finite, $\left|\Lambda^{\min}(\lambda, \mu)\right|$ is finite and the sum in (CP3) makes sense. We also interpret any empty sums as 0, and hence $\Lambda^{\min}(\lambda, \mu) = \emptyset$ implies $T_{\lambda^*} T_\mu = 0$.

Since (CP1-3) are the same as (KP1-3) of Chapter 3, Proposition 3.1.3 also applies to Cohn Λ-families as follows.

Proposition 4.1.3 ([19, Proposition 3.3])**.** *Suppose that Λ is a row-finite k-graph with no sources, that R is a commutative ring with 1, and that $\{T_\lambda, T_{\mu^*} : \lambda, \mu \in \Lambda\}$ is a Cohn Λ-family in an R-algebra A. Then*

(a) $T_\lambda T_{\lambda^*} T_\mu T_{\mu^*} = \sum_{\lambda\nu\in \mathrm{MCE}(\lambda,\mu)} T_{\lambda\nu} T_{(\lambda\nu)^*}$ for $\lambda, \mu \in \Lambda$, and $\{T_\lambda T_{\lambda^*} : \lambda \in \Lambda\}$ is a commuting family.

(b) The subalgebra generated by $\{T_\lambda, T_{\mu^*} : \lambda, \mu \in \Lambda\}$ is

$$\mathrm{span}_R\{T_\lambda T_{\mu^*} : \lambda, \mu \in \Lambda, s(\lambda) = s(\mu)\}.$$

Now we give an example of a Cohn Λ-family. We use this example later to study properties of "the universal Cohn Λ-family" (Theorem 4.1.5). Here W_Λ denotes the *path space* of a k-graph Λ (see Section 2.5).

Proposition 4.1.4 ([19, Proposition 3.4])**.** *Suppose that Λ is a row-finite k-graph with no sources and that R is a commutative ring with 1. Suppose that $\mathbb{F}_R(W_\Lambda)$ is the free R-module with basis W_Λ. Then there exists a Cohn Λ-family $\{T_\lambda, T_{\mu^*} : \lambda, \mu \in \Lambda\}$ in the R-algebra* $\mathrm{End}(\mathbb{F}_R(W_\Lambda))$ *such that for $v \in \Lambda^0$, $\lambda, \mu \in \Lambda$ and $x \in W_\Lambda$, we have*

$$T_v(x) = \begin{cases} x & \text{if } r(x) = v \\ 0 & \text{otherwise,} \end{cases}$$

$$T_\lambda(x) = \begin{cases} \lambda x & \text{if } s(\lambda) = r(x) \\ 0 & \text{otherwise,} \end{cases}$$

$$T_{\mu^*}(x) = \begin{cases} \sigma^{d(\mu)}x & \textit{if } x(0,d(\mu)) = \mu \\ 0 & \textit{otherwise.} \end{cases}$$

Further, we have $rT_v \neq 0$ *and* $r\prod_{e\in v\Lambda^1}(T_v - T_eT_e^*) \neq 0$ *for all* $r \in R\backslash\{0\}$ *and* $v \in \Lambda^0$.

Proof. We modify the construction of the infinite-path representation in [11]. Take $v \in \Lambda^0$ and $\lambda, \mu \in \Lambda\backslash\Lambda^0$. Define functions f_v, f_λ, and $f_{\mu^*} : W_\Lambda \to \mathbb{F}_R(W_\Lambda)$ by

$$f_v(x) = \begin{cases} x & \text{if } r(x) = v \\ 0 & \text{otherwise,} \end{cases}$$

$$f_\lambda(x) = \begin{cases} \lambda x & \text{if } s(\lambda) = r(x) \\ 0 & \text{otherwise,} \end{cases}$$

$$f_{\mu^*}(x) = \begin{cases} \sigma^{d(\mu)}x & \text{if } x(0,d(\mu)) = \mu \\ 0 & \text{otherwise.} \end{cases}$$

By the universal property of free modules, there exist nonzero endomorphisms $S_v, S_\lambda, S_{\mu^*}$ of $\mathbb{F}_R(W_\Lambda)$ extending f_v, f_λ, and f_{μ^*}.

We claim that $\{T_\lambda, T_{\mu^*} : \lambda, u \in \Lambda\}$ is a Cohn Λ-family. First we show (CP1). Take $v \in \Lambda^0$. Then $T_v^2(x) = x = T_v(x)$ if $r(x) = v$, and $T_v^2(x) = 0 = T_v(x)$ otherwise. Hence $T_v^2 = T_v$. Now take $v, w \in \Lambda^0$ with $v \neq w$. Then $x \in wW_\Lambda$ implies $x \notin vW_\Lambda$. Thus $T_vT_w(x) = 0$ for every $x \in W_\Lambda$, and $T_vT_w = 0$.

Next we show (CP2). Take $\lambda, \mu \in \Lambda$ with $s(\lambda) = r(\mu)$. Then $T_\lambda T_\mu(x) = \lambda\mu x = T_{\lambda\mu}(x)$ if $x \in s(\mu)W_\Lambda$, and $T_\lambda T_\mu(x) = 0 = T_{\lambda\mu}(x)$ otherwise. Hence $T_\lambda T_\mu = T_{\lambda\mu}$. On the other hand, if $x(0,d(\lambda\mu)) = \lambda\mu$, we have

$$T_{\mu^*}T_{\lambda^*}(x) = T_{\mu^*}\sigma^{d(\lambda)}x = \sigma^{d(\lambda)+d(\mu)}x = \sigma^{d(\lambda\mu)}x = T_{(\lambda\mu)^*}(x);$$

otherwise $T_{\mu^*}T_{\lambda^*}(x) = 0 = T_{(\lambda\mu)^*}(x)$. Therefore $T_{\mu^*}T_{\lambda^*} = T_{(\lambda\mu)^*}$.

Next we show (CP3). Take $\lambda, \mu \in \Lambda$. If $r(\lambda) \neq r(\mu)$, then $T_{\lambda^*}T_\mu = 0$ and $\Lambda^{\min}(\lambda,\mu) = \emptyset$, as required. Suppose $r(\lambda) = r(\mu)$. We have

$$T_{\lambda^*}T_\mu(x) = \begin{cases} (\mu x)(d(\lambda), d(\mu x)) & \text{if } x \in s(\mu)W_\Lambda \text{ and } (\mu x)(0,d(\lambda)) = \lambda; \\ 0 & \text{otherwise.} \end{cases}$$

Take $x \in s(\mu)W_\Lambda$. Note that $s(\mu) = r(\gamma)$ for $(\nu,\gamma) \in \Lambda^{\min}(\lambda,\mu)$. First suppose $(\mu x)(0,d(\lambda)) \neq \lambda$. Then for $(\nu,\gamma) \in \Lambda^{\min}(\lambda,\mu)$,

$$(\mu x)(0,d(\lambda\nu)) \neq \lambda\nu \text{ and } (\mu x)(0,d(\mu\gamma)) \neq \mu\gamma.$$

Hence $x(0,d(\gamma)) \neq \gamma$ and $T_\nu T_{\gamma^*}(x) = T_\nu(0) = 0$. Therefore

$$\sum_{(\nu,\gamma)\in\Lambda^{\min}(\lambda,\mu)} T_\nu T_{\gamma^*}(x) = 0.$$

Next suppose $(\mu x)(0,d(\lambda)) = \lambda$. Since $(\mu x)(0,d(\lambda)) = \lambda$ and $(\mu x)(0,d(\mu)) = \mu$, then there is $\gamma \in s(\mu)\Lambda$ such that $(\nu,\gamma) \in \Lambda^{\min}(\lambda,\mu)$ and $(\mu x)(0,d(\mu\gamma)) = \mu\gamma$. Therefore $x(0,d(\gamma)) = \gamma$. Hence for $(\nu',\gamma') \in \Lambda^{\min}(\lambda,\mu)$ such that $(\nu',\gamma') \neq (\nu,\gamma)$, we have $T_{\nu'}T_{\gamma'^*}(x) = 0$. Since $x(0,d(\gamma)) = \gamma$, then

$$\begin{aligned}
T_\nu T_{\gamma^*}(x) &= T_\nu(x(d(\gamma),d(x))) = \nu[x(d(\gamma),d(x))] \\
&= \nu[(\mu x)(d(\mu\gamma),d(\mu x))] \\
&= \nu[(\mu x)(d(\lambda\gamma),d(\mu x))] \ \text{(since } \mu\gamma = \lambda\nu) \\
&= (\mu x)(d(\lambda),d(\mu x))
\end{aligned}$$

and

$$\sum_{(\nu',\gamma';)\in\Lambda^{\min}(\lambda,\mu)} T_{\nu'}T_{\gamma'^*}(x) = T_\nu T_{\gamma^*}(x) = (\mu x)(d(\lambda),d(\mu x)) = T_{\lambda^*}T_\mu(x),$$

as required. Thus $\{T_\lambda, T_{\mu^*} : \lambda, u \in \Lambda\}$ is a Cohn Λ-family, as claimed.

Finally we take $r \in R\backslash\{0\}$ and $v \in \Lambda^0$ and show that $r\prod_{e\in v\Lambda^1}(T_v - T_eT_e^*) \neq 0$ and $rT_v \neq 0$. Then $v \in W_\Lambda$. Hence $rT_v(v) = rv$ and $rT_v \neq 0$. On the other hand, for $e \in v\Lambda^1$, we have $T_e^*(v) = 0$ and then

$$r\prod_{e\in v\Lambda^1}(T_v - T_eT_e^*)(v) = rT_v(v) = rv.$$

Hence $r\prod_{e\in v\Lambda^1}(T_v - T_eT_e^*) \neq 0$, as required. □

Next we show that there is an R-algebra which is universal for Cohn Λ-families.

Theorem 4.1.5 ([19, Theorem 3.5]). *Suppose that Λ is a row-finite k-graph with no sources and that R is a commutative ring with 1.*

(a) There is a universal R-algebra $\mathrm{C}_R(\Lambda)$ generated by a Cohn Λ-family $\{t_\lambda, t_{\mu^*} : \lambda, \mu \in \Lambda\}$ such that if $\{T_\lambda, T_{\mu^*} : \lambda, \mu \in \Lambda\}$ is a Cohn Λ-family in an R-algebra A, then there exists a unique R-algebra homomorphism $\phi_T : \mathrm{C}_R(\Lambda) \to A$ such that $\phi_T(t_\lambda) = T_\lambda$ and $\phi_T(t_{\mu^*}) = T_{\mu^*}$ for $\lambda, \mu \in \Lambda$.

(b) For all $r \in R\backslash\{0\}$ and $v \in \Lambda^0$, we have $rt_v \neq 0$ and $r\prod_{e\in v\Lambda^1}(t_v - t_et_e^*) \neq 0$.

Proof. Let $X := \Lambda \cup G\left(\Lambda^{\neq 0}\right)$ and $\mathbb{F}_R(w(X))$ be the free algebra on the set $w(X)$ of words on X. Let I be the ideal of $\mathbb{F}_R(w(X))$ generated by elements of the following sets:

(i) $\{vw - \delta_{v,w}v : v, w \in \Lambda^0\}$,

(ii) $\{\lambda - \mu\nu, \lambda^* - \nu^*\mu^* : \lambda, \mu, \nu \in \Lambda \text{ and } \lambda = \mu\nu\}$ and

(iii) $\{\lambda^*\mu - \sum_{(\nu,\gamma)\in\Lambda^{\min}(\lambda,\mu)} \nu\gamma^* : \lambda, \mu \in \Lambda\}$.

Set $\mathrm{C}_R(\Lambda) := \mathbb{F}_R(w(X))/I$ and write $q : \mathbb{F}_R(w(X)) \to \mathbb{F}_R(w(X))/I$ for the quotient map. Define $t_\lambda := q(\lambda)$ for $\lambda \in \Lambda$, and $t_{\mu^*} := q(\mu^*)$ for $\mu^* \in G\left(\Lambda^{\neq 0}\right)$. Then $\{t_\lambda, t_{\mu^*} : \lambda, \mu \in \Lambda\}$ is a Cohn Λ-family in $\mathrm{C}_R(\Lambda)$.

Now suppose that $\{T_\lambda, T_{\mu^*} : \lambda, u \in \Lambda\}$ is a Cohn Λ-family in an R-algebra A. Define $f : X \to A$ by $f(\lambda) := T_\lambda$ for $\lambda \in \Lambda$, and $f(\mu^*) := T_{\mu^*}$ for $\mu^* \in G\left(\Lambda^{\neq 0}\right)$. By the universal property of $\mathbb{F}_R(w(X))$, there exists a unique R-algebra homomorphism $\pi : \mathbb{F}_R(w(X)) \to A$ such that $\pi|_X = f$. Since $\{T_\lambda, T_{\mu^*} : \lambda, u \in \Lambda\}$ is a Cohn Λ-family, then $I \subseteq \ker(\pi)$. Thus there exists an R-algebra homomorphism $\phi_T : \mathrm{C}_R(\Lambda) \to A$ such that $\phi_T \circ q = \pi$. The homomorphism ϕ_T is unique since the element in X generate $\mathbb{F}_R(w(X))$ as an algebra, and we have $\phi_T(t_\lambda) = \phi_T(q(\lambda)) = \pi(\lambda) = T_\lambda$ and $\phi_T(t_{\mu^*}) = T_{\mu^*}$ for $\lambda, \mu \in \Lambda$, as required.

For (b), suppose that $\{T_\lambda, T_{\mu^*} : \lambda, u \in \Lambda\}$ is the Cohn Λ-family as in Proposition 4.1.4. Take $r \in R\backslash\{0\}$ and $v \in \Lambda^0$. We have $rT_v \neq 0$ and $r\Big(\prod_{e\in v\Lambda^1}(T_v - T_eT_e^*)\Big) \neq 0$. Since $\phi_T(rt_v) = rT_v \neq 0$ and

$$\phi\Big(r\prod_{e\in v\Lambda^1}(t_v - t_et_e^*)\Big) = r\prod_{e\in v\Lambda^1}(T_v - T_eT_e^*) \neq 0,$$

then $rt_v \neq 0$ and $r\prod_{e\in v\Lambda^1}(t_v - t_et_e^*) \neq 0$. $\square$

4.2 The uniqueness theorem for Cohn path algebras

In this section, we establish a uniqueness theorem for Cohn path algebras (Theorem 4.2.1). This uniqueness theorem can be viewed as an algebraic analogue of the uniqueness theorem for Toeplitz algebras (Theorem 1.2.3). Our uniqueness theorem for Cohn path algebras does not require any hypothesis on the k-graph and thus applies generally.

Theorem 4.2.1 ([19, Theorem 4.1]: The uniqueness theorem for Cohn path algebras)**.** *Suppose that Λ is a row-finite k-graph with no sources and that R is a commutative ring with* 1*. Suppose that $\phi : \mathrm{C}_R(\Lambda) \to A$ is a ring homomorphism such that*

$$\phi(rt_v) \neq 0 \text{ and } \phi\Big(r \prod_{e \in v\Lambda^1} (t_v - t_e t_{e^*})\Big) \neq 0$$

for all $r \in R \backslash \{0\}$ and $v \in \Lambda^0$. Then ϕ is injective.

The rest of this section is devoted to proving Theorem 4.2.1. To help readers follow our proofs, we divide the arguments into three subsections. In Subsection 4.2.1, we recall the Kumjian-Pask Λ-families of Chapter 3 and study some of their properties. In Subsection 4.2.2, we recall the k-graph $T\Lambda$ of [37] and investigate the Kumjian-Pask algebra of $T\Lambda$. Finally, in Subsection 4.2.3, we show that every Cohn path algebra is isomorphic to the Kumjian-Pask algebra (Theorem 4.2.16). Once we have this isomorphism, we show that Theorem 4.2.1 is a consequence of the Cuntz-Krieger uniqueness theorem for Kumjian-Pask algebras (Theorem 3.7.1).

4.2.1 Kumjian-Pask algebras

Suppose that Λ is a row-finite k-graph. Recall from Chapter 3 that a *Kumjian-Pask Λ-family* $\{S_\lambda, S_{\mu^*} : \lambda, \mu \in \Lambda\}$ in an R-algebra A is a family which satisfies (CP1-3) and

(KP) $\prod_{\lambda \in E} (S_v - S_\lambda S_{\lambda^*}) = 0$ for all $v \in \Lambda^0$ and finite exhaustive $E \subseteq v\Lambda$.

Remark 4.2.2. We are careful not to say that a Kumjian-Pask Λ-family is a Cohn Λ-family which satisfies (KP). This is because in Definition 4.1.1, we define Cohn Λ-family of row-finite k-graphs with no sources; however, the above definition of Kumjian-Pask Λ-family allows for more general row-finite k-graphs (in particular, to k-graphs with sources). We will need this level of generality later on.

The following proposition establishes the relationship between the Kumjian-Pask algebra $\mathrm{KP}_R(\Lambda)$ and the Cohn path algebra $\mathrm{C}_R(\Lambda)$.

Proposition 4.2.3. *Suppose that Λ is a row-finite k-graph with no sources and that R is a commutative ring with* 1*. Then $\mathrm{KP}_R(\Lambda)$ is a nontrivial quotient of $\mathrm{C}_R(\Lambda)$, and $\mathrm{C}_R(\Lambda)$ is not simple.*

Proof. It suffices to show that the set $v\Lambda^{e_i}$ is finite exhaustive for all $v \in \Lambda^0$ and $1 \leq i \leq k$. To prove this, we use the argument of [41, Lemma B.2]. Note that each

$v\Lambda^{e_i}$ is finite (since Λ is row-finite) and nonempty (since Λ has no sources). Take $v \in \Lambda^0$ and $1 \leq i \leq k$. Take $\lambda \in v\Lambda$. If $\lambda = v$, then for every $e \in v\Lambda^{e_i}$, we have $\Lambda^{\min}(\lambda, e) \neq \emptyset$. Otherwise, suppose $\lambda \in v\Lambda \backslash \{v\}$. Since Λ has no sources, there exists $e \in s(\lambda)\Lambda^{e_i}$. Thus $(\lambda e)(0, d(e)) \in v\Lambda^{e_i}$ and $\Lambda^{\min}(\lambda, (\lambda e)(0, d(e))) \neq \emptyset$. Therefore $v\Lambda^{e_i}$ is exhaustive. The conclusion follows. □

The following proposition will be useful to simplify calculations in Kumjian-Pask algebras. It essentially gives an alternate formulation of (KP).

Proposition 4.2.4 ([19, Proposition 4.7])**.** *Suppose that Λ is a row-finite k-graph and that R is a commutative ring with 1. Suppose that $\{S_\lambda, S_{\mu^*} : \lambda, \mu \in \Lambda\}$ is a Cohn Λ-family in an R-algebra A. Then*

$$\{S_\lambda, S_{\mu^*} : \lambda, \mu \in \Lambda\}$$

is a Kumjian-Pask Λ-family if and only if

$$\prod_{e \in E} (S_v - S_e S_{e^*}) = 0$$

for all $v \in \Lambda^0$ and exhaustive $E \subseteq v\Lambda^1$.

Before proving Proposition 4.2.4, we establish the following helper lemma.

Lemma 4.2.5 ([19, Lemma 4.8])**.** *Suppose that Λ is a row-finite k-graph and that R is a commutative ring with 1. Suppose that $\{S_\lambda, S_{\mu^*} : \lambda, \mu \in \Lambda\}$ is a Cohn Λ-family in an R-algebra A. Suppose $v \in \Lambda^0$, $\lambda \in v\Lambda$ and $E \subseteq s(\lambda)\Lambda$ is finite and satisfies $\prod_{\nu \in E}\left(S_{s(\lambda)} - S_\nu S_{\nu^*}\right) = 0$. Then*

$$S_v - S_\lambda S_{\lambda^*} = \prod_{\nu \in E}\left(S_v - S_{\lambda\nu} S_{(\lambda\nu)^*}\right).$$

Proof. We follow the C^*-algebraic argument of [41, Lemma C.7]. For $\nu \in s(\lambda)\Lambda$, we have

$$(S_v - S_\lambda S_{\lambda^*})\left(S_v - S_{\lambda\nu} S_{(\lambda\nu)^*}\right) = S_v - S_\lambda S_{\lambda^*};$$

so

$$(S_v - S_\lambda S_{\lambda^*}) \prod_{\nu \in E}\left(S_v - S_{\lambda\nu} S_{(\lambda\nu)^*}\right) = S_v - S_\lambda S_{\lambda^*}.$$

On the other hand,

$$(S_v - S_\lambda S_{\lambda^*}) \prod_{\nu \in E}\left(S_v - S_{\lambda\nu} S_{(\lambda\nu)^*}\right)$$

$$
\begin{aligned}
&= S_v \prod_{\nu\in E} \left(S_v - S_{\lambda\nu}S_{(\lambda\nu)^*}\right) - S_\lambda S_{\lambda^*} \prod_{\nu\in E} \left(S_v - S_{\lambda\nu}S_{(\lambda\nu)^*}\right) \\
&= \prod_{\nu\in E} \left(S_v - S_{\lambda\nu}S_{(\lambda\nu)^*}\right) - \prod_{\nu\in E} \left(S_\lambda S_{\lambda^*} - S_{\lambda\nu}S_{(\lambda\nu)^*}\right) \\
&= \prod_{\nu\in E} \left(S_v - S_{\lambda\nu}S_{(\lambda\nu)^*}\right) - S_\lambda \Big(\prod_{\nu\in E} \left(S_{s(\lambda)} - S_\nu S_{\nu^*}\right) \Big) S_{\lambda^*} \\
&= \prod_{\nu\in E} \left(S_v - S_{\lambda\nu}S_{(\lambda\nu)^*}\right)
\end{aligned}
$$

since $\prod_{\nu\in E}\left(S_{s(\lambda)} - S_\nu S_{\nu^*}\right) = 0$ by the hypothesis. The conclusion follows. $\square$

Proof of Proposition 4.2.4. We use a similar argument to the C^*-algebraic version in [41, Proposition C.3]. If $\{S_\lambda, S_{\mu^*} : \lambda, \mu \in \Lambda\}$ is a Kumjian-Pask Λ-family, then it satisfies $\prod_{e\in E}(S_v - S_e S_{e^*}) = 0$ for all $v \in \Lambda^0$ and exhaustive set $E \subseteq v\Lambda^1$. Now we show the reverse implication. First for $E \subseteq \Lambda$, we write

$$
I(E) := \bigcup_{i=1}^{k} \{\lambda(0, e_i) : \lambda \in E, d(\lambda)_i > 0\} \text{ and } L(E) := \sum_{i=1}^{k} \max_{\lambda\in E} d(\lambda)_i .
$$

We have to show that $\prod_{\lambda\in E}(S_v - S_\lambda S_{\lambda^*}) = 0$ for all $v \in \Lambda^0$ and exhaustive set $E \subseteq v\Lambda$. We show this by induction on $L(E)$. If $L(E) = 1$, then $E \subseteq v\Lambda^1$ for some $v \in \Lambda^0$ and $\prod_{e\in E}(S_v - S_e S_{e^*}) = 0$ by assumption.

Now fix $l \geq 1$ and suppose that $\prod_{\lambda\in F}(S_v - S_\lambda S_{\lambda^*}) = 0$ for all $v \in \Lambda^0$ and exhaustive sets $F \subseteq v\Lambda$ with $L(F) \leq l$. Take $v \in \Lambda^0$ and an exhaustive set $E \subseteq v\Lambda$ with $L(E) = l + 1$. If $v \in E$, then $\prod_{\lambda\in E}(S_v - S_\lambda S_{\lambda^*}) = 0$. So suppose $v \notin E$. Note that $I(E) \subseteq v\Lambda^1$. Since E is exhaustive, then by [41, Lemma C.6], $I(E)$ is also exhaustive. So

$$
I(E) \subseteq v\Lambda^1 \text{ is exhaustive.} \tag{4.2.1}
$$

Take $e \in I(E)$ and by [41, Lemma C.5], $\operatorname{Ext}(e; E)$ is exhaustive. By [41, Lemma C.8], $L(\operatorname{Ext}(e; E)) < L(E) = l + 1$ and then $L(\operatorname{Ext}(e; E)) \leq l$. So by the inductive hypothesis, $\prod_{\nu\in \operatorname{Ext}(e;E)}\left(S_{s(e)} - S_\nu S_{\nu^*}\right) = 0$ and then by Lemma 4.2.5, we get

$$
S_v - S_e S_{e^*} = \prod_{\nu\in \operatorname{Ext}(e;E)} \left(S_v - S_{e\nu}S_{(e\nu)^*}\right). \tag{4.2.2}
$$

Now note that for $\nu \in \operatorname{Ext}(e; E)$, there exists $\lambda \in E$ with $e\nu = \lambda\lambda'$, and then

$$
(S_v - S_\lambda S_{\lambda^*})\left(S_v - S_{e\nu}S_{(e\nu)^*}\right) = S_v - S_\lambda S_{\lambda^*}.
$$

Hence

$$\begin{aligned}\prod_{\lambda\in E}(S_v - S_\lambda S_{\lambda^*}) &= \Big(\prod_{\lambda\in E}(S_v - S_\lambda S_{\lambda^*})\Big)\Big(\prod_{e\in I(E)}\prod_{\nu\in \mathrm{Ext}(e;E)}\big(S_v - S_{e\nu}S_{(e\nu)^*}\big)\Big)\\ &= \Big(\prod_{\lambda\in E}(S_v - S_\lambda S_{\lambda^*})\Big)\Big(\prod_{e\in I(E)}(S_v - S_e S_{e^*})\Big) \text{ (by (4.2.2))}\\ &= \Big(\prod_{\lambda\in E}(S_v - S_\lambda S_{\lambda^*})\Big)(0) \text{ (by (4.2.1) and the inductive hypothesis)}\\ &= 0,\end{aligned}$$

as required. □

4.2.2 The k-graph $T\Lambda$ and Kumjian-Pask $T\Lambda$-families

Suppose that Λ is a row-finite k-graph. In this subsection, we recall the k-graph $T\Lambda$ of [37, Proposition 3.1]. Interestingly, the k-graph $T\Lambda$ is always aperiodic (Proposition 4.2.10). We also study the properties of a Kumjian-Pask $T\Lambda$-family (Lemma 4.2.13).

Proposition 4.2.6 ([37, Proposition 3.1])**.** *Suppose that $\Lambda = (\Lambda, d, r, s)$ is a row-finite k-graph. Then define sets $T\Lambda^0$ and $T\Lambda$ as follows:*

$$T\Lambda^0 := \left\{\alpha(v) : v\in\Lambda^0\right\} \cup \left\{\beta(v) : v\Lambda^1 \neq \emptyset\right\},$$

$$T\Lambda := \{\alpha(\lambda) : \lambda\in\Lambda\} \cup \left\{\beta(\lambda) : \lambda\in\Lambda, s(\lambda)\Lambda^1 \neq \emptyset\right\}.$$

Define functions $r, s : T\Lambda\backslash T\Lambda^0 \to T\Lambda^0$ by

$$\begin{aligned} r(\alpha(\lambda)) = \alpha(r(\lambda)),\ s(\alpha(\lambda)) = \alpha(s(\lambda)),\\ r(\beta(\lambda)) = \alpha(r(\lambda)),\ s(\beta(\lambda)) = \beta(s(\lambda))\end{aligned}$$

(r, s are the identity on $T\Lambda^0$). We also define a partially defined product $(\tau,\omega)\mapsto\tau\omega$ from

$$\{(\tau,\omega)\in T\Lambda\times T\Lambda : s(\tau) = r(\omega)\}$$

to $T\Lambda$, where

$$(\alpha(\lambda), \alpha(\mu)) \mapsto \alpha(\lambda\mu)$$

$$(\alpha(\lambda), \beta(\mu)) \mapsto \beta(\lambda\mu)$$

and a function $d : T\Lambda \to \mathbb{N}^k$ where

$$d(\alpha(\lambda)) = d(\beta(\lambda)) = d(\lambda).$$

Then $(T\Lambda, d)$ is a k-graph.

Proof. We prove that $T\Lambda$ is a countable category. Note that $T\Lambda$ is countable since Λ is countable.

Take η, τ, ω in $T\Lambda$ where $s(\eta) = r(\tau)$ and $s(\tau) = r(\omega)$. We have to show $s(\tau\omega) = s(\omega)$, $r(\tau\omega) = r(\tau)$, and $(\eta\tau)\omega = \eta(\tau\omega)$. If one of τ, ω is a vertex then we are done. So assume otherwise, and we have $\eta = \alpha(\lambda)$, $\tau = \alpha(\mu)$, and ω is either $\alpha(\nu)$ or $\beta(\nu)$ for some paths λ, μ, ν in Λ. In both cases, we always have $s(\lambda) = r(\mu)$, $s(\mu) = r(\nu)$, and $(\lambda\mu)\nu = \lambda(\mu\nu)$. If $\omega = \alpha(\nu)$, we have

$$\begin{aligned} s(\tau\omega) &= s(\alpha(\mu)\alpha(\nu)) = s(\alpha(\mu\nu)) \\ &= \alpha(s(\mu\nu)) = \alpha(s(\nu)) = s(\alpha(\nu)) = s(\omega), \end{aligned}$$

$$\begin{aligned} r(\tau\omega) &= r(\alpha(\mu)\alpha(\nu)) = r(\alpha(\mu\nu)) \\ &= \alpha(r(\mu\nu)) = \alpha(r(\mu)) = r(\alpha(\mu)) = r(\tau), \end{aligned}$$

and

$$\begin{aligned} (\eta\tau)\omega &= \big(\alpha(\lambda)\alpha(\mu)\big)\alpha(\nu) = \alpha(\lambda\mu)\alpha(\nu) = \alpha((\lambda\mu)\nu) \\ &= \alpha(\lambda(\mu\nu)) = \alpha(\lambda)\alpha(\mu\nu) = \alpha(\lambda)\big(\alpha(\mu)\alpha(\nu)\big) = \eta(\tau\omega). \end{aligned}$$

On the other hand, if $\omega = \beta(\nu)$, then

$$\begin{aligned} s(\tau\omega) &= s(\alpha(\mu)\beta(\nu)) = s(\beta(\mu\nu)) \\ &= \beta(s(\mu\nu)) = \beta(s(\nu)) = s(\beta(\nu)) = s(\omega), \end{aligned}$$

$$\begin{aligned} r(\tau\omega) &= r(\alpha(\mu)\beta(\nu)) = r(\beta(\mu\nu)) \\ &= \alpha(r(\mu\nu)) = \alpha(r(\mu)) = r(\alpha(\mu)) = r(\tau), \end{aligned}$$

and

$$\begin{aligned} (\eta\tau)\omega &= \big(\alpha(\lambda)\alpha(\mu)\big)\beta(\nu) = \alpha(\lambda\mu)\beta(\nu) = \beta((\lambda\mu)\nu) \\ &= \beta(\lambda(\mu\nu)) = \alpha(\lambda)\beta(\mu\nu) = \alpha(\lambda)\big(\alpha(\mu)\beta(\nu)\big) = \eta(\tau\omega). \end{aligned}$$

Thus $T\Lambda$ is a countable category.

We show that d is a functor. Both $T\Lambda$ and $\mathbb{N}^k$ are categories. First take an object $x \in T\Lambda^0$, then $d(x) = 0$ is an object in category $\mathbb{N}^k$. For morphisms $\tau, \omega \in T\Lambda$ with $s(\tau) = r(\omega)$, a cases analysis gives

$$d(\tau\omega) = d(\tau) + d(\omega).$$

Hence d is a functor.

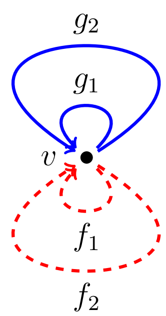
To show that d satisfies the factorisation property, take $\omega \in T\Lambda$ and $m, n \in \mathbb{N}^k$ such that $d(\omega) = m + n$. By definition, ω is either $\alpha(\lambda)$ or $\beta(\lambda)$ for some path λ in Λ. In both cases, there exist paths μ, ν in Λ such that $\lambda = \mu\nu$, $d(\mu) = m$, and $d(\nu) = n$. Then, we have $d(\alpha(\mu)) = m$, $d(\alpha(\nu)) = d(\beta(\nu)) = n$, and ω is either equal to $\alpha(\mu)\alpha(\nu)$ or $\alpha(\mu)\beta(\nu)$. Therefore there is a factorisation.

Now we show that the factorisation is unique. Suppose $\omega = \alpha(\mu)\alpha(\nu) = \alpha(\mu')\alpha(\nu')$ where $d(\alpha(\mu)) = d(\alpha(\mu'))$ and $d(\alpha(\nu)) = d(\alpha(\nu'))$. We consider paths $\lambda = \mu\nu$ and $\lambda' = \mu'\nu'$. Since $\alpha(\lambda) = \omega = \alpha(\lambda')$, then $\lambda = \lambda'$. This implies $\mu = \mu'$ and $\nu = \nu'$ based on the uniquness of factorisation in Λ. Then $\alpha(\mu) = \alpha(\mu')$ and $\alpha(\nu) = \alpha(\nu')$. For the case $\omega = \alpha(\mu)\beta(\nu)$, we get the same result by using the same argument. The conclusion follows. $\square$

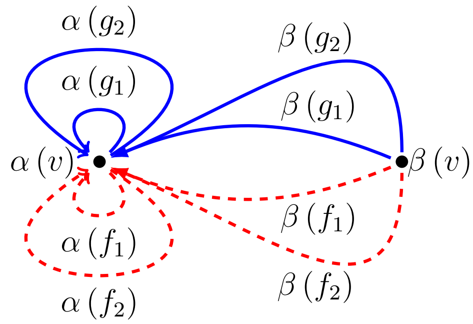
Remark 4.2.7. For a directed graph E (that is, for $k = 1$), the graph TE was constructed by Muhly and Tomforde [35, Definition 3.6] (denoted E_V), and by Sims [51, Section 3] (denoted $\widetilde{E}$). Our notation follows that of Sims because we want to distinguish between paths in $T\Lambda$ (denoted $\alpha(\lambda)$ and $\beta(\lambda)$) and those in Λ (denoted λ).

Remark 4.2.8. Every vertex $\beta(v)$ satisfies $\beta(v)T\Lambda^1 = \emptyset$. If Λ has a vertex v which receives edges e, f with $d(e) \neq d(f)$, then there is no edge $g \in \beta(s(e))T\Lambda^{d(f)}$ (or $g \in \alpha(s(e))T\Lambda^{d(f)}$ if $s(e)\Lambda = \emptyset$), and hence $T\Lambda$ is not locally convex.

Example 4.2.9. Consider the 2-graph Λ which has skeleton



where $f_i g_j = g_i f_j$ for all $i, j \in \{1, 2\}$, dashed edges have degree $(1, 0)$ and solid edges have degree $(0,1)$. Then the 2-graph $T\Lambda$ has skeleton



where $\alpha(f_i)\alpha(g_j) = \alpha(g_i)\alpha(f_j)$ and $\alpha(f_i)\beta(g_j) = \alpha(g_i)\beta(f_j)$ for all $i,j \in \{1,2\}$, dashed edges have degree $(1,0)$ and solid edges have degree $(0,1)$.

The following lemma gives some properties of the k-graph $T\Lambda$.

Proposition 4.2.10 ([37, Proposition 3.5]). *Let Λ be a row-finite k-graph and $T\Lambda$ be the k-graph as in Proposition 4.2.6. Then,*

(a) $T\Lambda$ is row-finite.

(b) $T\Lambda$ is aperiodic.

Proof. To show part (a), take $x \in T\Lambda^0$. If $x = \beta(v)$ for some $v \in \Lambda^0$, then $xT\Lambda^1 = \emptyset$ by Remark 4.2.8. Suppose $x = \alpha(v)$ for some $v \in \Lambda^0$. If $v\Lambda^1 = \emptyset$, then $xT\Lambda^1 = \emptyset$. Otherwise, for $1 \leq i \leq k$ such that $v\Lambda^{e_i} \neq \emptyset$, we have

$$|xT\Lambda^{e_i}| \leq 2\,|v\Lambda^{e_i}|,$$

which is finite.

Next we show part (b). Take $\tau, \omega \in T\Lambda$ such that $\tau \neq \omega$ and $s(\tau) = s(\omega)$. Proposition 2.6.3 says that it suffices to show that there exists $\eta \in s(\tau)\Lambda$ such that $\operatorname{MCE}(\tau\eta, \omega\eta) \neq \emptyset$. If $s(\tau) = \beta(v)$ for some $v \in \Lambda^0$, then choose $\eta = \beta(v)$ and $\operatorname{MCE}(\tau\eta, \omega\eta) = \emptyset$. So suppose $s(\tau) = \alpha(v)$ for some $v \in \Lambda^0$. If $v\Lambda^1 = \emptyset$, then choose $\eta = \alpha(v)$ and $\operatorname{MCE}(\tau\eta, \omega\eta) = \emptyset$. Suppose $v\Lambda^1 \neq \emptyset$. Take $e \in v\Lambda^1$. If $s(e)\Lambda^1 = \emptyset$, then choose $\eta = \alpha(e)$ and $\operatorname{MCE}(\tau\eta, \omega\eta) = \emptyset$. Otherwise, we have $s(e)\Lambda^1 \neq \emptyset$. Then choose $\eta = \beta(e)$ and $\operatorname{MCE}(\tau\eta, \omega\eta) = \emptyset$. Hence $T\Lambda$ is aperiodic. $\square$

Remark 4.2.11. Since $T\Lambda$ is row-finite, $T\Lambda$ is also finitely aligned.

The next proposition characterises exhaustive sets of $T\Lambda$.

Proposition 4.2.12 ([19, Proposition 4.9]). *Suppose that Λ is a row-finite k-graph with no sources. Then for every $\alpha(v) \in T\Lambda^0$, the only exhaustive set contained in $\alpha(v)T\Lambda^1$ is $\alpha(v)T\Lambda^1$ itself.*

Proof. Fix an exhaustive set $E \subseteq \alpha(v)T\Lambda^1$. We have to show $E = \alpha(v)T\Lambda^1$. Since E is exhaustive, for $\beta(e) \in \alpha(v)T\Lambda^1$, there exists an edge $\tau_e \in E$ such that $T\Lambda^{\min}(\beta(e), \tau_e) \neq \emptyset$. Since $s(\beta(e))T\Lambda = \{s(\beta(e))\}$, then $\mathrm{MCE}(\beta(e), \tau_e) = \{\beta(e)\}$. Hence $\tau_e = \beta(e)$ because both τ_e and $\beta(e)$ are edges. Thus $\beta(e) \in E$ and E contains $\beta(v\Lambda^1)$.

Now we claim $\alpha(v\Lambda^1) \subseteq E$. Suppose for a contradiction that there exist $1 \leq i \leq k$ and $e \in v\Lambda^{e_i}$ such that $\alpha(e) \notin E$. Since Λ has no sources, there exists an edge $f \in s(e)\Lambda^{e_i}$. Now consider the path $\tau = \alpha(e)\beta(f)$. This is a path with degree $2e_i$ whose range at $\alpha(v)$ and $s(\tau)T\Lambda = \{\beta(s(f))\}$. Since E is exhaustive, there exists $\omega \in E$ such that $T\Lambda^{\min}(\tau, \omega) \neq \emptyset$. Since τ is a path with length $2e_i$ and $s(\tau)T\Lambda = \{\beta(s(f))\}$, then ω is either equal to τ or $\alpha(e)$. Since $\alpha(e) \notin E$, then $\tau = \omega \in E$, which contradicts that E only contains edges. The conclusion follows. $\square$

A consequence of Proposition 4.2.12 is the following:

Lemma 4.2.13 ([19, Lemma 4.10]). *Let Λ be a row-finite k-graph with no sources and R be a commutative ring with 1. Suppose that $\{S_\tau, S_{\omega^*} : \tau, \omega \in T\Lambda\}$ is a Cohn $T\Lambda$-family in an R-algebra A. Then the collection is a Kumjian-Pask $T\Lambda$-family if and only if for every $\alpha(v) \in T\Lambda^0$,*

$$\prod_{g \in \alpha(v)T\Lambda^1} \left(S_{\alpha(v)} - S_g S_{g^*}\right) = 0.$$

Proof. If $x = \beta(v)$, then $\beta(v)T\Lambda = \{\beta(v)\}$ and there is no exhaustive set contained in $xT\Lambda^1$. On the other hand, if $x = \alpha(v)$, by Proposition 4.2.12, the only exhaustive set contained in $\alpha(v)T\Lambda^1$ is $\alpha(v)T\Lambda^1$. Therefore by Proposition 4.2.4, $\{S_\tau, S_{\omega^*} : \tau, \omega \in T\Lambda\}$ is a Kumjian-Pask $T\Lambda$-family if and only if for every $\alpha(v) \in T\Lambda^0$, we have $\prod_{g \in \alpha(v)T\Lambda^1} \left(S_{\alpha(v)} - S_g S_{g^*}\right) = 0$, as required. $\square$

4.2.3 Relationship between Cohn Λ-families and Kumjian-Pask $T\Lambda$-families

In this subsection, we start out by investigating the relationship between Cohn Λ-families and Kumjian-Pask $T\Lambda$-families (Theorem 4.2.16). Once we have this, we are then ready to prove Theorem 4.2.1.

First we establish some stepping stone results (Lemma 4.2.14 and Lemma 4.2.15).

Lemma 4.2.14 ([19, Lemma 4.11])**.** *Suppose that $\{T_\lambda, T_{\mu^*} : \lambda, \mu \in \Lambda\}$ is a Cohn Λ-family in an R-algebra A. For $v \in \Lambda^0$, define*

$$F_{T,v} := T_v - \prod_{e \in v\Lambda^1} (T_v - T_e T_{e^*}).$$

Then

(a) For $v \in \Lambda^0$, we have

$$F_{T,v} = F_{T,v}^2 \text{ and } T_v - F_{T,v} = (T_v - F_{T,v})^2.$$

(b) For every $v, w \in \Lambda^0$ with $v \neq w$, we have

$$F_{T,w} F_{T,v} = 0 = F_{T,v} F_{T,w} \text{ and } T_w F_{T,v} = 0 = F_{T,v} T_w.$$

(c) For $v \in \Lambda^0$ and $\lambda \in v\Lambda \setminus \{v\}$, we have

$$T_v F_{T,v} = F_{T,v} = F_{T,v} T_v,$$

$$F_{T,v} T_\lambda = T_\lambda \text{ and } T_{\lambda^*} F_{T,v} = T_{\lambda^*}.$$

(d) Furthermore, $F_{T,v} \neq 0$ for all $r \in R\setminus\{0\}$ and $v \in \Lambda^0$ if and only if $T_v \neq 0$ for all $r \in R\setminus\{0\}$ and $v \in \Lambda^0$.

Proof. First we show (a). Take $v \in \Lambda^0$. Note that $(T_v - T_e T_{e^*})^2 = (T_v - T_e T_{e^*})$ for $e \in v\Lambda^1$. Hence

$$(T_v - F_{T,v})^2 = \prod_{e \in v\Lambda^1} (T_v - T_e T_{e^*})^2 = \prod_{e \in v\Lambda^1} (T_v - T_e T_{e^*}) = T_v - F_{T,v}$$

and

$$F_{T,v}^2 = \Big(T_v - \prod_{e \in v\Lambda^1} (T_v - T_e T_{e^*})\Big)^2 = T_v - \prod_{e \in v\Lambda^1} (T_v - T_e T_{e^*}) = F_{T,v}.$$

To show (b), we take $v, w \in \Lambda^0$ with $v \neq w$. Then $T_w T_v = 0$ and $T_w T_e = 0$ for all $e \in v\Lambda^1$. Hence

$$T_w F_{T,v} = T_w \Big(T_v - \prod_{e \in v\Lambda^1} (T_v - T_e T_{e^*})\Big) = 0$$

and by using a similar argument, we also get $F_{T,v} T_w = 0$, as required. On the other hand, we also have

$$F_{T,w} F_{T,v} = \Big(T_w - \prod_{f \in w\Lambda^1} (T_w - T_f T_{f^*})\Big)\Big(T_v - \prod_{e \in v\Lambda^1} (T_v - T_e T_{e^*})\Big) = 0$$

and a similar argument also applies to get $F_{T,v}F_{T,w} = 0$.

Next we show (c). We take $v \in \Lambda^0$. Then

$$T_v F_{T,v} = T_v\Big(T_v - \prod_{e\in v\Lambda^1} (T_v - T_e T_{e^*})\Big) = T_v - \prod_{e\in v\Lambda^1} (T_v - T_e T_{e^*}) = F_{T,v}$$

and since $T_v = T_{v^*}$, then by using a similar argument, we also get $F_{T,v}T_v = F_{T,v}$.

Take $\lambda \in v\Lambda\backslash\{v\}$. Then there exists $f \in v\Lambda^1$ such that $ff' = \lambda$. This implies $T_{f^*}T_\lambda = T_{f'}$ and

$$(T_v - T_f T_{f^*})\, T_\lambda = T_\lambda - T_f T_{f^*} T_\lambda = T_\lambda - T_f T_{f'} = T_\lambda - T_\lambda = 0.$$

Hence

$$\Big(\prod_{e\in v\Lambda^1} (T_v - T_e T_{e^*})\Big) T_\lambda = 0$$

and

$$F_{T,v}T_\lambda = \Big(T_v - \prod_{e\in v\Lambda^1} (T_v - T_e T_{e^*})\Big) T_\lambda = T_\lambda.$$

By a similar argument, we get $T_{\lambda^*}F_{T,v} = T_{\lambda^*}$.

To show (d). First suppose that there exists $r \in R\backslash\{0\}$ and $v \in \Lambda^0$ with $rT_v = 0$. Then $rT_e = rT_vT_e = 0$ for all $e \in v\Lambda^1$, and then $rF_{T,v} = rT_v - r\prod_{e\in v\Lambda^1} (T_v - T_e T_{e^*}) = 0$.

For the reverse implication, suppose that there exists $r \in R\backslash\{0\}$ and $v \in \Lambda^0$ with $rF_{T,v} = 0$. Take $f \in v\Lambda^1$. Then

$$T_f T_{f^*} (T_v - T_f T_{f^*}) = T_f T_{f^*} - T_f (T_{f^*} T_f) T_{f^*} = T_f T_{f^*} - T_f T_{f^*} = 0. \tag{4.2.3}$$

Since $r(f) = v$, (4.2.3) gives

$$\begin{aligned} rT_f T_{f^*} = rT_f T_{f^*} T_v &= rT_f T_{f^*}\Big(F_{T,v} + \prod_{e\in v\Lambda^1} (T_v - T_e T_{e^*})\Big) \\ &= rT_f T_{f^*} \prod_{e\in v\Lambda^1} (T_v - T_e T_{e^*}) = 0. \end{aligned}$$

Therefore

$$rT_f = rT_f T_{s(f)} = rT_f (T_{f^*} T_f) = (rT_f T_{f^*})\, T_f = (0)\, T_f = 0$$

and then

$$rT_{s(f)} = rT_{f^*} T_f = T_{f^*} (rT_f) = T_{f^*} (0) = 0.$$

□

Lemma 4.2.15 ([19, Lemma 4.12]). *Suppose that Λ is a row-finite k-graph with no sources and that R is a commutative ring with 1. Suppose that $\{T_\lambda, T_{\mu^*} : \lambda, \mu \in \Lambda\}$ is a Cohn Λ-family in an R-algebra A. For $\tau, \omega \in T\Lambda$, define*

$$S_\tau := \begin{cases} T_\lambda F_{T,s(\lambda)} & \text{if} \quad \tau = \alpha(\lambda) \\ T_\lambda \left(T_{s(\lambda)} - F_{T,s(\lambda)}\right) & \text{if} \quad \tau = \beta(\lambda), \end{cases}$$

$$S_{\omega^*} := \begin{cases} F_{T,s(\mu)} T_{\mu^*} & \text{if} \quad \omega = \alpha(\mu) \\ \left(T_{s(\mu)} - F_{T,s(\mu)}\right) T_{\mu^*} & \text{if} \quad \omega = \beta(\mu). \end{cases}$$

Then

(a) $\{S_\tau, S_{\omega^*} : \tau, \omega \in T\Lambda\}$ is a Kumjian-Pask $T\Lambda$-family.

(b) Suppose that $rT_v \neq 0$ and $r\prod_{e\in v\Lambda^1}(T_v - T_eT_e^*) \neq 0$ for all $r \in R\backslash\{0\}$ and $v \in \Lambda^0$. Suppose that $\pi_S : \mathrm{KP}_R(T\Lambda) \to A$ is the R-algebra homomorphism such that $\pi_S(s_\tau) = S_\tau$ and $\pi_S(s_{\omega^*}) = S_{\omega^*}$ for $\tau, \omega \in T\Lambda$. Then π_S is injective.

Proof. Now we show (a). First we show that $\{S_\tau, S_{\omega^*} : \tau, \omega \in T\Lambda\}$ satisfies (CP1). Take $x \in T\Lambda^0$. We have to show $S_x = S_{x^*} = S_x^2$. Note that $S_x = F_{T,v}$ if $x = \alpha(v)$; and $S_x = T_v - F_{T,v}$, otherwise. In both cases, by Lemma 4.2.14(a), we have $S_x = S_{x^*} = S_x^2$, as required.

Take $x, y \in T\Lambda^0$ with $x \neq y$. We have to show $S_xS_y = 0$. Since S_x is either $F_{T,v}$ or $T_v - F_{T,v}$, and S_y is either $F_{T,w}$ or $T_w - F_{T,w}$, then Lemma 4.2.14(b) tells that $x \neq y$ implies $S_xS_y = 0$. Therefore $\{S_\tau, S_{\omega^*} : \tau, \omega \in T\Lambda\}$ satisfies (CP1).

We show that $\{S_\tau, S_{\omega^*} : \tau, \omega \in T\Lambda\}$ satisfies (CP2). Take $\tau, \omega \in T\Lambda$ where $s(\tau) = r(\omega)$. We have to show $S_\tau S_\omega = S_{\tau\omega}$ and $S_{\omega^*}S_{\tau^*} = S_{(\tau\omega)^*}$. Each τ and ω is either in the form $\alpha(\lambda)$ or $\beta(\mu)$. So we give a separate argument for each case.

First suppose that $\tau = \beta(\lambda)$. Since $s(\tau)T\Lambda = \beta(s(\lambda))$ and $s(\tau) = r(\omega)$, then $\omega = s(\beta(\lambda))$. Hence

$$\begin{aligned} S_{\beta(\lambda)}S_{\beta(s(\lambda))} &= \left(T_\lambda\left(T_{s(\lambda)} - F_{T,s(\lambda)}\right)\right)\left(T_{s(\lambda)} - F_{T,s(\lambda)}\right) \\ &= T_\lambda\left(T_{s(\lambda)} - F_{T,s(\lambda)}\right)^2 = T_\lambda\left(T_{s(\lambda)} - F_{T,s(\lambda)}\right) = S_{\beta(\lambda)}. \end{aligned} \tag{4.2.4}$$

Next suppose that $\tau = \alpha(\lambda)$ and $\omega = \beta(\mu)$. Then $\mu \in s(\lambda)\Lambda\backslash\{s(\lambda)\}$ since $s(\tau) = r(\omega)$, and Lemma 4.2.14(c) gives $F_{T,s(\lambda)}T_\mu = T_\mu$. Hence

$$\begin{aligned} S_{\alpha(\lambda)}S_{\beta(\mu)} &= \left(T_\lambda F_{T,s(\lambda)}\right)\left(T_\mu\left(T_{s(\mu)} - F_{T,s(\mu)}\right)\right) = T_\lambda T_\mu\left(T_{s(\lambda\mu)} - F_{T,s(\lambda\mu)}\right) \\ &= T_{\lambda\mu}\left(T_{s(\lambda\mu)} - F_{T,s(\lambda\mu)}\right) = S_{\beta(\lambda\mu)}. \end{aligned} \tag{4.2.5}$$

Finally suppose $\tau = \alpha(\lambda)$ and $\omega = \alpha(\mu)$. Then $S_{\beta(s(\lambda))}S_{\alpha(s(\lambda))} = 0$, so

$$S_{\beta(\lambda)}S_{\alpha(\mu)} = \left(S_{\beta(\lambda)}S_{\beta(s(\lambda))}\right)\left(S_{\alpha(s(\lambda))}S_{\alpha(\mu)}\right) = 0,$$

and

(4.2.6)
$$\begin{aligned} S_{\alpha(\lambda)}S_{\alpha(\mu)} &= \left(S_{\alpha(\lambda)} + S_{\beta(\lambda)}\right)\left(S_{\alpha(\mu)} + S_{\beta(\mu)}\right) - S_{\alpha(\lambda)}S_{\beta(\mu)} - S_{\beta(\lambda)}S_{\alpha(\mu)} - S_{\beta(\lambda)}S_{\beta(\mu)} \\ &= T_\lambda T_\mu - S_{\alpha(\lambda)}S_{\beta(\mu)} - S_{\beta(\lambda)}S_{\beta(\mu)}. \end{aligned}$$

If $\mu = s(\lambda)$, then $S_{\alpha(\lambda)}S_{\beta(\mu)} = \left(S_{\alpha(\lambda)}S_{\alpha(s(\lambda))}\right)S_{\beta(s(\lambda))} = 0$ because $S_{\alpha(s(\lambda))}S_{\beta(s(\lambda))} = 0$, and by (4.2.4), (4.2.6) becomes

$$S_{\alpha(\lambda)}S_{\alpha(s(\lambda))} = T_\lambda - S_{\beta(\lambda)} = S_{\alpha(\lambda)}.$$

On the other hand, if $\mu \neq s(\lambda)$, then $S_{\beta(\lambda)}S_{\beta(\mu)} = \left(S_{\beta(\lambda)}S_{\beta(s(\lambda))}\right)\left(S_{r(\beta(\mu))}S_{\beta(\mu)}\right) = 0$ (since $\beta(s(\lambda)) \neq r(\beta(\mu))$and $S_{\beta(s(\lambda))}S_{r(\beta(\mu))} = 0$) and by (4.2.5), (4.2.6) becomes

$$S_{\alpha(\lambda)}S_{\alpha(\mu)} = T_{\lambda\mu} - S_{\beta(\lambda\mu)} = S_{\alpha(\lambda\mu)}.$$

Therefore $S_\tau S_\omega = S_{\tau\omega}$ and a similar argument gives $S_{\omega^*}S_{\tau^*} = S_{(\tau\omega)^*}$. Thus the family $\{S_\tau, S_{\omega^*} : \tau, \omega \in T\Lambda\}$ satisfies (CP2).

We show that $\{S_\tau, S_{\omega^*} : \tau, \omega \in T\Lambda\}$ satisfies (CP3). Take $\tau, \omega \in T\Lambda$. We have to show $S_{\tau^*}S_\omega = \sum_{(\rho,\zeta)\in T\Lambda^{\min}(\tau,\omega)} S_\rho S_{\zeta^*}$. Note that each τ and ω is either in the form $\alpha(\lambda)$ or $\beta(\mu)$. We argue by cases.

First suppose that $\tau = \beta(\lambda)$. Since $s(\tau)T\Lambda = \beta(s(\lambda))$, then $T\Lambda^{\min}(\tau,\omega) \neq \emptyset$ implies $\mathrm{MCE}(\tau,\omega) = \{\tau\}$. Hence if $T\Lambda^{\min}(\tau,\omega) \neq \emptyset$, then we have $\tau = \omega\beta(\nu)$ for some $\nu \in \Lambda$ and

$$S_{\tau^*}S_\omega = S_{\beta(\nu)^*}S_{\omega^*}S_\omega = S_{\beta(\nu)^*}S_{s(\omega)} = S_{\beta(\nu)^*} = \sum_{(\rho,\zeta)\in T\Lambda^{\min}(\tau,\omega)} S_\rho S_{\zeta^*}.$$

We suppose $T\Lambda^{\min}(\tau,\omega) = \emptyset$ and have to show that $S_{\tau^*}S_\omega = 0$. First note that regardless of whether ω is $\alpha(\mu)$ or $\beta(\mu)$, $S_{\tau^*}S_\omega$ has the form $\left(S_{\beta(\lambda)^*}T_\mu\right)b$. So it suffices to show that $S_{\beta(\lambda)^*}T_\mu = 0$. We have

$$S_{\beta(\lambda)^*}T_\mu = \left(T_{s(\lambda)} - F_{T,s(\lambda)}\right)T_{\lambda^*}T_\mu = \left(T_{s(\lambda)} - F_{T,s(\lambda)}\right)\sum_{(\nu,\gamma)\in\Lambda^{\min}(\lambda,\mu)} T_\nu T_{\gamma^*}. \tag{4.2.7}$$

If $\Lambda^{\min}(\lambda,\mu) = \emptyset$, then $S_{\beta(\lambda)^*}T_\mu = 0$, as required. So suppose $\Lambda^{\min}(\lambda,\mu) \neq \emptyset$. Since $T\Lambda^{\min}(\tau,\omega) = \emptyset$ and $\Lambda^{\min}(\lambda,\mu) \neq \emptyset$, then $\lambda \notin \mathrm{MCE}(\lambda,\mu)$. Hence for every $(\nu,\gamma) \in$

$\Lambda^{\min}(\lambda,\mu)$, we have $\nu \in s(\lambda)\Lambda\backslash\{s(\lambda)\}$ and by Lemma 4.2.14(c), $F_{T,s(\lambda)}T_\nu = T_\nu$. Hence we can rewrite (4.2.7) as

$$S_{\beta(\lambda)^*}T_\mu = \sum_{(\nu,\gamma)\in\Lambda^{\min}(\lambda,\mu)} (T_\nu - T_\nu)\,T_{\gamma^*} = 0.$$

Next suppose $\tau = \alpha(\lambda)$ and $\omega = \beta(\mu)$. A similar argument to that for the case $\tau = \beta(\lambda)$ gives $S_{\alpha(\lambda)^*}S_{\beta(\mu)} = \sum_{(\rho,\zeta)\in T\Lambda^{\min}(\alpha(\lambda),\beta(\mu))} S_\rho S_{\zeta^*}$.

Finally suppose $\tau = \alpha(\lambda)$ and $\omega = \alpha(\mu)$. We give a separate argument for whether $\alpha(\lambda)$ or $\alpha(\mu)$ belongs to $\mathrm{MCE}(\alpha(\lambda),\alpha(\mu))$. Suppose that at least one of $\alpha(\lambda)$ and $\alpha(\mu)$ belongs to $\mathrm{MCE}(\alpha(\lambda),\alpha(\mu))$. Without loss of generality, we suppose $\alpha(\lambda)\in\mathrm{MCE}(\alpha(\lambda),\alpha(\mu))$. (A similar argument applies when $\alpha(\mu)\in\mathrm{MCE}(\alpha(\lambda),\alpha(\mu))$.) Then $\alpha(\lambda) = \alpha(\mu\nu)$ for some $\nu\in\Lambda$ and

$$S_{\alpha(\lambda)^*}S_{\alpha(\mu)} = S_{\alpha(\nu)^*}S_{\alpha(\mu)^*}S_{\alpha(\mu)} = S_{\alpha(\nu)^*} = \sum_{(\rho,\zeta)\in T\Lambda^{\min}(\alpha(\lambda),\alpha(\mu))} S_\rho S_{\zeta^*},$$

as required.

So suppose that $\alpha(\lambda),\alpha(\mu)\notin\mathrm{MCE}(\alpha(\lambda),\alpha(\mu))$. Hence $\lambda,\mu\notin\mathrm{MCE}(\lambda,\mu)$. Then for every $(\nu,\gamma)\in\Lambda^{\min}(\lambda,\mu)$, we have $\nu\in s(\lambda)\Lambda\backslash\{s(\lambda)\}$ and $\gamma\in s(\mu)\Lambda\backslash\{s(\mu)\}$, and by Lemma 4.2.14(c), $F_{T,s(\lambda)}T_\nu = T_\nu$ and $T_{\gamma^*}F_{T,s(\mu)} = T_{\gamma^*}$. Therefore

$$\begin{aligned}(4.2.8)\qquad S_{\alpha(\lambda)^*}S_{\alpha(\mu)} &= \left(F_{T,s(\lambda)}T_{\lambda^*}\right)\left(T_\mu F_{T,s(\mu)}\right) = F_{T,s(\lambda)}\Big(\sum_{(\nu,\gamma)\in\Lambda^{\min}(\lambda,\mu)} T_\nu T_{\gamma^*}\Big)F_{T,s(\mu)}\\ &= \sum_{(\nu,\gamma)\in\Lambda^{\min}(\lambda,\mu)}\left(F_{T,s(\lambda)}T_\nu\right)\left(T_{\gamma^*}F_{T,s(\mu)}\right) = \sum_{(\nu,\gamma)\in\Lambda^{\min}(\lambda,\mu)} T_\nu T_{\gamma^*}.\end{aligned}$$

Since $s(\nu) = s(\gamma)$ for every $(\nu,\gamma)\in\Lambda^{\min}(\lambda,\mu)$, then by Lemma 4.2.14(a),

$$\begin{aligned}&S_{\alpha(\lambda)^*}S_{\alpha(\mu)}\\ &\quad= \sum_{(\nu,\gamma)\in\Lambda^{\min}(\lambda,\mu)}\left(\left(T_\nu F_{T,s(\nu)}\right)\left(F_{T,s(\gamma)}T_{\gamma^*}\right) + T_\nu\left(T_{s(\nu)} - F_{T,s(\nu)}\right)\left(T_{s(\gamma)} - F_{T,s(\gamma)}\right)T_{\gamma^*}\right)\\ &\quad= \sum_{(\nu,\gamma)\in\Lambda^{\min}(\lambda,\mu)}\left(S_{\alpha(\nu)}S_{\alpha(\gamma)^*} + S_{\beta(\nu)}S_{\beta(\gamma)^*}\right) = \sum_{(\rho,\zeta)\in T\Lambda^{\min}(\alpha(\lambda),\alpha(\mu))} S_\rho S_{\zeta^*},\end{aligned}$$

as required. Therefore $S_{\tau^*}S_\omega = \sum_{(\rho,\zeta)\in T\Lambda^{\min}(\tau,\omega)} S_\rho S_{\zeta^*}$ for all $\tau,\omega\in T\Lambda$. Thus the collection $\{S_\tau, S_{\omega^*} : \tau,\omega\in T\Lambda\}$ satisfies (CP3).

Lemma 4.2.13 says that to show that $\{S_\tau, S_{\omega^*} : \tau,\omega\in T\Lambda\}$ is a Kumjian-Pask $T\Lambda$-family, it suffices to show that $\prod_{g\in\alpha(v)T\Lambda^1}\left(S_{\alpha(v)} - S_g S_{g^*}\right) = 0$ for $\alpha(v)\in T\Lambda^0$. Take $\alpha(v)\in T\Lambda^0$. Then

$$(4.2.9)\qquad \prod_{g\in\alpha(v)T\Lambda^1}\left(S_{\alpha(v)} - S_g S_{g^*}\right)$$

$$= \prod_{e\in v\Lambda^1}\left(S_{\alpha(v)} - S_{\alpha(e)}S_{\alpha(e)^*}\right)\left(S_{\alpha(v)} - S_{\beta(e)}S_{\beta(e)^*}\right)$$

$$= \prod_{e\in v\Lambda^1}\left(T_vF_{T,v} - T_eF_{T,s(e)}^2T_{e^*}\right)\left(T_vF_{T,v} - T_e\left(T_{s(e)} - F_{T,s(e)}\right)^2T_{e^*}\right)$$

$$= \prod_{e\in v\Lambda^1}\left(T_vF_{T,v} - T_eF_{T,s(e)}T_{e^*}\right)\left(T_vF_{T,v} - T_e\left(T_{s(e)} - F_{T,s(e)}\right)T_{e^*}\right)$$

(by Lemma 4.2.14(a))

$$= \prod_{e\in v\Lambda^1}\left(F_{T,v} - T_eT_{e^*}\right)$$

by Lemma 4.2.14(a,c). So

$$\begin{aligned}\prod_{g\in\alpha(v)T\Lambda^1}\left(S_{\alpha(v)} - S_gS_{g^*}\right) &= \prod_{e\in v\Lambda^1}\left(F_{T,v} - T_eT_{e^*}\right)\\ &= F_{T,v}\prod_{e\in v\Lambda^1}\left(T_v - T_eT_{e^*}\right) \text{ (by Lemma 4.2.14(a,c))}\\ &= F_{T,v}\left(T_v - F_{T,v}\right) = F_{T,v}T_v - F_{T,v}^2\\ &= F_{T,v} - F_{T,v} \text{ (by Lemma 4.2.14(a,c))}\\ &= 0.\end{aligned}$$

Then $\{S_\tau, S_{\omega^*} : \tau,\omega\in T\Lambda\}$ is a Kumjian-Pask $T\Lambda$-family, as required.

Next we show (b). Suppose that $rT_v \neq 0$ and $r\prod_{e\in v\Lambda^1}\left(T_v - T_eT_e^*\right) \neq 0$ for all $v\in\Lambda^0$, and $\pi_S : \mathrm{KP}_R(T\Lambda) \to A$ is the R-algebra homomorphism such that $\pi_S(s_\tau) = S_\tau$ and $\pi_S(s_{\omega^*}) = S_{\omega^*}$ for $\tau,\omega\in T\Lambda$. We have to show π_S is injective. Since $rT_v \neq 0$ for all $r\in R\setminus\{0\}$ and $v\in\Lambda^0$, then by Lemma 4.2.14(d), $rF_{T,v}\neq 0$ for all $r\in R\setminus\{0\}$ and $v\in\Lambda^0$. Therefore for all $r\in R\setminus\{0\}$ and $v\in\Lambda^0$,

$$rS_{\alpha(v)} = rT_vF_{T,v} = rF_{T,v} \neq 0$$

and

$$rS_{\beta(v)} = rT_v\left(T_v - F_{T,v}\right) = r\left(T_v - F_{T,v}\right) = r\prod_{e\in v\Lambda^1}\left(T_v - T_eT_{e^*}\right) \neq 0.$$

Hence $rS_x \neq 0$ for all $r\in R\setminus\{0\}$ and $x\in T\Lambda^0$. Since $T\Lambda$ is aperiodic, then by Theorem 3.7.1, π_S is injective. □

One immediate application of Lemma 4.2.15 is:

Theorem 4.2.16 ([19, Theorem 4.13]). *Suppose that Λ is a row-finite k-graph with no sources and that R is a commutative ring with 1. Suppose that $\{t_\lambda, t_{\mu^*} : \lambda,\mu\in\Lambda\}$ is* *the universal Cohn Λ-family and $\{s_\tau, s_{\omega^*} : \tau, \omega \in T\Lambda\}$ is the universal Kumjian-Pask $T\Lambda$-family. For $\tau, \omega \in T\Lambda$, define*

$$S_\tau := \begin{cases} t_\lambda F_{t,s(\lambda)} & \text{if} \quad \tau = \alpha(\lambda) \\ t_\lambda \left(t_{s(\lambda)} - F_{t,s(\lambda)}\right) & \text{if} \quad \tau = \beta(\lambda), \end{cases}$$

$$S_{\omega^*} := \begin{cases} F_{t,s(\mu)} t_{\mu^*} & \text{if} \quad \omega = \alpha(\mu) \\ \left(t_{s(\mu)} - F_{t,s(\mu)}\right) t_{\mu^*} & \text{if} \quad \omega = \beta(\mu). \end{cases}$$

Then

(a) *There exists an R-algebra homomorphism $\pi : \mathrm{KP}_R(T\Lambda) \to \mathrm{C}_R(\Lambda)$ such that $\pi(s_\tau) = S_\tau$ and $\pi(s_{\omega^*}) = S_{\omega^*}$ for $\tau, \omega \in T\Lambda$. Furthermore, π is an isomorphism.*

(b) *The subsets*

$$\mathrm{C}_R(\Lambda)_n := \mathrm{span}_R \{t_\lambda t_{\mu^*} : \lambda, \mu \in \Lambda, d(\lambda) - d(\mu) = n\}$$

form a $\mathbb{Z}^k$-grading of $\mathrm{C}_R(\Lambda)$.

Proof. First we show part (a). By Lemma 4.2.15(a), $\{S_\tau, S_{\omega^*} : \tau, \omega \in T\Lambda\}$ is a Kumjian-Pask $T\Lambda$-family and by the universal property of Kumjian-Pask $T\Lambda$-family [18, Theorem 3.7(a)], there exists an R-algebra homomorphism $\pi : \mathrm{KP}_R(T\Lambda) \to \mathrm{C}_R(\Lambda)$ such that $\pi(s_\tau) = S_\tau$ and $\pi(s_{\omega^*}) = S_{\omega^*}$ for $\tau, \omega \in T\Lambda$. On the other hand, Theorem 4.1.5(b) says that $rt_v \neq 0$ and $r \prod_{e \in v\Lambda^1} (t_v - t_e t_e^*) \neq 0$ for all $r \in R \setminus \{0\}$ and $v \in \Lambda^0$. Hence by Lemma 4.2.15(b), π is injective.

Now we show the surjectivity of π. Since

$$\mathrm{C}_R(\Lambda) = \mathrm{span}_R \{t_\lambda t_{\mu^*} : \lambda, u \in \Lambda, s(\lambda) = s(\mu)\}$$

(Proposition 4.1.3(b)), it suffices to show that for $\lambda, \mu \in \Lambda$, both t_λ and t_{μ^*} belong to the image of π. Take $\lambda, \mu \in \Lambda$. Then

$$\begin{aligned} t_\lambda &= t_\lambda t_{s(\lambda)} = t_\lambda F_{t,s(\lambda)} + t_\lambda t_{s(\lambda)} - t_\lambda F_{t,s(\lambda)} \\ &= t_\lambda F_{t,s(\lambda)} + t_\lambda \left(t_{s(\lambda)} - F_{t,s(\lambda)}\right) = \pi\left(s_{\alpha(\lambda)}\right) + \pi\left(s_{\beta(\lambda)}\right), \end{aligned} \tag{4.2.10}$$

and

$$\begin{aligned} t_{\mu^*} &= t_{s(\mu)} t_{\mu^*} = F_{t,s(\mu)} t_{\mu^*} + t_{s(\mu)} t_{\mu^*} - F_{t,s(\mu)} t_{\mu^*} \\ &= F_{t,s(\mu)} t_{\mu^*} + \left(t_{s(\mu)} - F_{t,s(\mu)}\right) t_{\mu^*} = \pi\left(s_{\alpha(\mu)^*}\right) + \pi\left(s_{\beta(\mu)^*}\right). \end{aligned} \tag{4.2.11}$$

Therefore π is an isomorphism.

For part (b), we recall from 3.1.8(c) that the subsets

$$\mathrm{KP}_R\left(T\Lambda\right)_n := \mathrm{span}_R\left\{s_\tau s_{\omega^*} : \tau, \omega \in T\Lambda, d\left(\tau\right) - d\left(\omega\right) = n\right\}$$

forms a $\mathbb{Z}^k$-grading of $\mathrm{KP}_R\left(\Lambda\right)$. For every $v \in \Lambda^0$, $d\left(t_{s(\lambda)} - F_{t,s(\lambda)}\right) = 0 = d\left(F_{t,s(\lambda)}\right)$. Hence regardless of whether τ and ω are in the form $\alpha\left(\lambda\right)$ or $\beta\left(\mu\right)$, we have $d\left(\tau\right) - d\left(\omega\right) = d\left(\lambda\right) - d\left(\mu\right)$ and $s_\tau s_{\omega^*} \in \mathrm{C}_R\left(\Lambda\right)_n$, which implies $\pi\left(s_\tau s_{\omega^*}\right) \in \mathrm{KP}_R\left(T\Lambda\right)_n$. Since π is an isomorphism, the $\mathrm{C}_R\left(\Lambda\right)_n$ form a grading for $\mathrm{C}_R\left(\Lambda\right)$, as required. □

Remark 4.2.17. Our Theorem 4.2.16 generalises results about Cohn path algebras associated to 1-graphs. In particular, Theorem 4.2.16(a) generalises [5, Theorem 5] (which is also stated in [2, Theorem 1.5.18]), and Theorem 4.2.16(b) generalises [2, Corollary 2.1.5(ii)].

Proof of Theorem 4.2.1. Since the family $\{t_\lambda, t_{\mu^*} : \lambda, \mu \in \Lambda\}$ is a Cohn Λ-family, then so is $\{\phi\left(t_\lambda\right), \phi\left(t_{\mu^*}\right) : \lambda, \mu \in \Lambda\}$. For $\tau, \omega \in T\Lambda$, define

$$S_\tau := \begin{cases} \phi\left(t_\lambda\right) F_{\phi(t),s(\lambda)} & \text{if} \quad \tau = \alpha\left(\lambda\right) \\ \phi\left(t_\lambda\right)\left(\phi\left(t_{s(\lambda)}\right) - F_{\phi(t),s(\lambda)}\right) & \text{if} \quad \tau = \beta\left(\lambda\right), \end{cases}$$

$$S_{\omega^*} := \begin{cases} F_{\phi(t),s(u)}\phi\left(t_{\mu^*}\right) & \text{if} \quad \omega = \alpha\left(\mu\right) \\ \left(\phi\left(t_{s(u)}\right) - F_{\phi(t),s(u)}\right)\phi\left(t_{\mu^*}\right) & \text{if} \quad \omega = \beta\left(\mu\right). \end{cases}$$

Lemma 4.2.15(a) says that $\{S_\tau, S_{\omega^*} : \tau, \omega \in T\Lambda\}$ is a Kumjian-Pask $T\Lambda$-family, and by the universal property of Kumjian-Pask $T\Lambda$-family, there exists an R-algebra homomorphism $\pi_S : \mathrm{KP}_R\left(\Lambda\right) \to A$ such that $\pi_S\left(s_\tau\right) = S_\tau$ and $\pi_S\left(s_{\omega^*}\right) = S_{\omega^*}$ for $\tau, \omega \in T\Lambda$. On the other hand, since $\phi\left(rt_v\right) \neq 0$ and $\phi\left(r\prod_{e\in v\Lambda^1}\left(t_v - t_e t_e^*\right)\right) \neq 0$ for all $r \in R\backslash\{0\}$ and $v \in \Lambda^0$, Lemma 4.2.15(b) implies that π_S is injective.

Theorem 4.2.16(a) says that $\pi : \mathrm{KP}_R\left(T\Lambda\right) \to \mathrm{C}_R\left(\Lambda\right)$ is an isomorphism with

$$\pi\left(s_\tau\right) = \begin{cases} t_\lambda F_{t,s(\lambda)} & \text{if} \quad \tau = \alpha\left(\lambda\right); \\ t_\lambda\left(t_{s(\lambda)} - F_{t,s(\lambda)}\right) & \text{if} \quad \tau = \beta\left(\lambda\right), \end{cases}$$

$$\pi\left(s_{\omega^*}\right) = \begin{cases} F_{t,s(\mu)}t_{\mu^*} & \text{if} \quad \omega = \alpha\left(\mu\right); \\ \left(t_{s(\mu)} - F_{t,s(\mu)}\right)t_{\mu^*} & \text{if} \quad \omega = \beta\left(\mu\right). \end{cases}$$

For $\lambda, \mu \in \Lambda$, we have $t_\lambda = \pi\left(s_{\alpha(\lambda)}\right) + \pi\left(s_{\beta(\lambda)}\right)$ and $t_{\mu^*} = \pi\left(s_{\alpha(\mu)^*}\right) + \pi\left(s_{\beta(\mu)^*}\right)$ (see (4.2.10) and (4.2.11)). Hence

$$\begin{aligned}\left(\pi_S \circ \pi^{-1}\right)\left(t_\lambda\right) &= \left(\pi_S \circ \pi^{-1}\right)\left(\pi\left(s_{\alpha(\lambda)}\right) + \pi\left(s_{\beta(\lambda)}\right)\right) = \pi_S\left(s_{\alpha(\lambda)}\right) + \pi_S\left(s_{\alpha(\lambda)}\right) \\ &= S_{\alpha(\lambda)} + S_{\alpha(\lambda)} = \phi\left(t_\lambda\right) F_{\phi(t),s(\lambda)} + \phi\left(t_\lambda\right)\left(\phi\left(t_{s(\lambda)}\right) - F_{\phi(t),s(\lambda)}\right)\end{aligned}$$

$$= \phi\left(t_{\lambda}\right)$$

and

$$\begin{aligned}\left(\pi_S \circ \pi^{-1}\right)\left(t_{\mu^*}\right) &= \left(\pi_S \circ \pi^{-1}\right)\left(\pi\left(s_{\alpha(\mu)^*}\right) + \pi\left(s_{\beta(\mu)^*}\right)\right) = \pi_S\left(s_{\alpha(\mu)^*}\right) + \pi_S\left(s_{\beta(\mu)^*}\right) \\ &= S_{\alpha(\mu)^*} + S_{\alpha(\mu)^*} = F_{\phi(t),s(\mu)}\phi\left(t_{\mu^*}\right) + \left(\phi\left(t_{s(\mu)}\right) - F_{\phi(t),s(\mu)}\right)\phi\left(t_{\mu^*}\right) \\ &= \phi\left(t_{\mu^*}\right).\end{aligned}$$

These imply $\phi = \pi_S \circ \pi^{-1}$ since $\mathrm{C}_R(\Lambda) = \operatorname{span}_R\{t_\lambda t_{\mu^*} : \lambda, u \in \Lambda, s(\lambda) = s(\mu)\}$ (Proposition 4.1.3(b)). The injectivity of π^{-1} and π_S imply that ϕ is injective. $\square$

Remark 4.2.18. The Cohn Λ-family $\{T_\lambda, T_{\mu^*} : \lambda, u \in \Lambda\}$ as constructed in Proposition 4.1.4 satisfies $rT_v \neq 0$ and $r\prod_{e\in v\Lambda^1}\left(T_v - T_eT_e^*\right) \neq 0$ for all $r \in R\backslash\{0\}$ and $v \in \Lambda^0$. Hence Theorem 4.2.1 tells that the R-algebra homomorphism $\phi_T : \mathrm{C}_R(\Lambda) \to \operatorname{End}\left(\mathbb{F}_R\left(W_\Lambda\right)\right)$ such that $\phi_T\left(t_\lambda\right) = T_\lambda$ and $\phi_T\left(t_{\mu^*}\right) = T_{\mu^*}$ for $\lambda, \mu \in \Lambda$, is injective.

Remark 4.2.19. When Λ is a 1-graph, that is, when $k = 1$, then Λ is the path category of a directed graph E. One consequence of Theorem 4.1.5 and Theorem 3.7.1 is that the universal Cohn algebra $\mathrm{C}_R(\Lambda)$ that we have constructed is isomorphic to the Cohn path algebra associated to E as defined in [2, Definition 1.5.1]. Since [2, Definition 1.5.1] only considers the situation where R is a field, our construction gives a generalisation of the Cohn path algebra to the setting where R is an arbitrary commutative ring with 1.

4.3 Examples and Applications

4.3.1 Higher-rank graph Toeplitz algebras.

As mentioned in the introduction to Section 4.2, the uniqueness theorem for Cohn path algebras (Theorem 4.2.1) is an analogue of the uniqueness theorem for Toeplitz algebras (Theorem 1.2.3). We show that if Λ is a row-finite k-graph with no sources, then its Cohn path algebra over the complex numbers is isomorphic to a dense *-subalgebra of the Toeplitz algebra associated to Λ (Proposition 4.3.1).

Recall from page 7 and Footnote 1.2 that a *Toeplitz-Cuntz-Krieger Λ-family* is a collection of partial isometries $\{Q_\lambda : \lambda \in \Lambda\}$ in a C^*-algebra B satisfying (TCK1-3). For a row-finite k-graph Λ, the *Toeplitz algebra* of Λ is the C^*-algebra generated by a universal Toeplitz-Cuntz-Krieger Λ-family $\{q_\lambda : \lambda \in \Lambda\}$. Furthermore, for $v \in \Lambda^0$, we have $q_v \neq 0$ and $\prod_{e\in v\Lambda^1}\left(q_v - q_eq_e^*\right) \neq 0$ [49, Corollary 3.7.7].

Proposition 4.3.1 ([19, Proposition 5.2]). *Suppose that Λ is a row-finite k-graph with no sources. Suppose that $\{q_\lambda : \lambda \in \Lambda\}$ is the universal Toeplitz-Cuntz-Krieger Λ-family and $\{t_\lambda, t_{\mu^*} : \lambda, u \in \Lambda\}$ is the universal (complex) Cohn Λ-family. Then there is an isomorphism*

$$\phi_q : \mathrm{C}_{\mathbb{C}}(\Lambda) \to \operatorname{span}\{q_\lambda q_\mu^* : \lambda, \mu \in \Lambda\}$$

such that $\phi_q(t_\lambda) = q_\lambda$ and $\phi_q(t_{\mu^*}) = q_\mu^*$ for $\lambda, u \in \Lambda$. In particular, $\mathrm{C}_{\mathbb{C}}(\Lambda)$ is isomorphic to a dense subalgebra of $TC^*(\Lambda)$.

Proof. Since $\{q_\lambda : \lambda \in \Lambda\}$ satisfies (TCK1-3), then with $q_{\mu^*} := q_\mu^*$, $\{q_\lambda, q_{\mu^*} : \lambda, \mu \in \Lambda\}$ is a Cohn Λ-family in $TC^*(\Lambda)$. Thus the universal property of $\mathrm{C}_{\mathbb{C}}(\Lambda)$ gives a homomorphism ϕ_q from $\mathrm{C}_{\mathbb{C}}(\Lambda)$ onto the dense subalgebra

$$A := \operatorname{span}_{\mathbb{C}}\{q_\lambda q_\mu^* : \lambda, \mu \in \Lambda\} \subseteq TC^*(\Lambda).$$

On the other hand, for all $r \in \mathbb{C}\setminus\{0\}$ and $v \in \Lambda^0$, we have $\frac{1}{r}\phi_q(rt_v) = q_v \neq 0$ and

$$\frac{1}{r}\phi_q\Big(r\prod_{e \in v\Lambda^1}(t_v - t_e t_{e^*})\Big) = \prod_{e \in v\Lambda^1}(q_v - q_e q_e^*) \neq 0.$$

So $\phi_q(rt_v) \neq 0$ and $\phi_q\Big(r\prod_{e \in v\Lambda^1}(t_v - t_e t_{e^*})\Big) \neq 0$ for all $r \in \mathbb{C}\setminus\{0\}$ and $v \in \Lambda^0$. Then Theorem 4.2.1 implies that ϕ_q is injective. $\square$

Remark 4.3.2. For $k = 1$, Proposition 4.3.1 tells that the Cohn path algebra of 1-graph E is isomorphic to a dense subalgebra of $TC^*(E)$.

4.3.2 Groupoids and Steinberg algebras.

In Proposition 3.4.1, we show that each Kumjian-Pask algebra is isomorphic to a Steinberg algebra. Thus Theorem 4.2.16 implies that the Cohn path algebra of Λ is also isomorphic to a Steinberg algebra associated to $T\Lambda$. However, this is somewhat obscure because one has to go through $T\Lambda$. We improve this result by showing that there exists a groupoid associated to Λ such that its Steinberg algebra is isomorphic to the Cohn path algebra of Λ (Proposition 4.3.4).

Recall the path groupoid $\mathcal{TG}_\Lambda$ of a row-finite k-graph Λ with no sources from Example 2.8.1.

Proposition 4.3.3 ([19, Proposition 5.6]). *Suppose that Λ is a row-finite k-graph with no sources. Then the path groupoid $\mathcal{TG}_\Lambda$ is* effective*, in the sense that the interior of*

$$\operatorname{Iso}(\mathcal{TG}_\Lambda) := \{a \in \mathcal{TG}_\Lambda : s(a) = r(a)\}$$

is $\mathcal{TG}_\Lambda^{(0)}$.

Proof. For $x \in \mathcal{TG}_\Lambda^{(0)}$, we have $(x,0,x) \in \operatorname{Iso}(\mathcal{TG}_\Lambda)$ and then $\mathcal{TG}_\Lambda^{(0)}$ belongs to the interior of $\operatorname{Iso}(\mathcal{TG}_\Lambda)$. Now we show the reverse inclusion. Take an interior point a of $\operatorname{Iso}(\mathcal{TG}_\Lambda)$. Then there exists $TZ_\Lambda(\lambda *_s \mu \backslash G)$ such that $a \in TZ_\Lambda(\lambda *_s \mu \backslash G) \subseteq \operatorname{Iso}(\mathcal{TG}_\Lambda)$. We have to show $\lambda = \mu$. Since $a \in TZ_\Lambda(\lambda *_s \mu \backslash G)$, then $TZ_\Lambda(\lambda *_s \mu \backslash G)$ is not empty. Thus $s(\lambda) \notin G$. Hence $(\lambda, d(\lambda) - d(\mu), \mu) \in TZ_\Lambda(\lambda *_s \mu \backslash G)$, and since $TZ_\Lambda(\lambda *_s \mu \backslash G) \subseteq \operatorname{Iso}(\mathcal{TG}_\Lambda)$, this implies $\lambda = \mu$. Therefore $\mathcal{TG}_\Lambda$ is effective. □

Proposition 4.3.4 ([19, Proposition 5.8]). *Suppose that Λ is a row-finite k-graph with no sources, that $\mathcal{TG}_\Lambda$ is its path groupoid and that R is a commutative ring with 1. Then there is an isomorphism $\phi_Q : \mathrm{C}_R(\Lambda) \to A_R(\mathcal{G}_\Lambda)$ such that $\phi_Q(t_\lambda) = 1_{TZ_\Lambda(\lambda *_s s(\lambda))}$ and $\phi_Q(t_{\mu^*}) = 1_{TZ_\Lambda(s(\mu) *_s \mu)}$ for $\lambda, u \in \Lambda$.*

Before proving Proposition 4.3.4, we first note that the argument of Lemma 3.4.3 also applies to the path groupoid $\mathcal{TG}_\Lambda$ and we get the following result.

Lemma 4.3.5 ([19, Lemma 5.9]). *Suppose that Λ is a row-finite k-graph with no sources. Suppose that $\{TZ_\Lambda(\lambda_i *_s \mu_i \backslash G_i)\}_{i=1}^n$ is a finite collection of compact open bisection sets and that*

$$U := \bigcup_{i=1}^n TZ_\Lambda(\lambda_i *_s \mu_i \backslash G_i).$$

Then

$$1_U \in \operatorname{span}_R \left\{ 1_{TZ_\Lambda(\lambda *_s \mu \backslash G)} : (\lambda, \mu) \in \Lambda *_s \Lambda, G \subseteq s(\lambda)\Lambda \right\}.$$

Proof of Proposition 4.3.4. Define $Q_\lambda := 1_{TZ_\Lambda(\lambda *_s s(\lambda))}$ and $Q_{\mu^*} := 1_{TZ_\Lambda(s(\mu) *_s \mu)}$ for $\lambda, \mu \in \Lambda$. By [24, Theorem 6.9] and [58, Example 7.1], $\{Q_\lambda : \lambda \in \Lambda\}$ is a Toeplitz-Cuntz-Krieger Λ-family. Then $\{Q_\lambda, Q_{\mu^*} : \lambda, u \in \Lambda\}$ is a Cohn Λ-family in $A(\mathcal{TG}_\Lambda)$. Hence there exists a homomorphism $\phi_Q : \mathrm{C}_R(\Lambda) \to A_R(\mathcal{TG}_\Lambda)$ such that $\phi_Q(t_\lambda) = Q_\lambda$ and $\phi_Q(t_{\mu^*}) = Q_{\mu^*}$ for $\lambda, \mu \in \Lambda$.

We show that ϕ_Q is injective. For all $r \in R \backslash \{0\}$ and $v \in \Lambda^0$, we have

$$\phi_Q(rt_v) = rQ_v = r1_{TZ_\Lambda(v *_s v)} \neq 0$$

and

$$\begin{aligned}
\phi_Q\Big(r \prod_{e \in v\Lambda^1} (t_v - t_e t_{e^*})\Big) &= r \prod_{e \in v\Lambda^1} (Q_v - Q_e Q_{e^*}) = r \prod_{e \in v\Lambda^1} \left(1_{TZ_\Lambda(v *_s v)} - 1_{TZ_\Lambda(e *_s e)}\right) \\
&= r \prod_{e \in v\Lambda^1} 1_{TZ_\Lambda(v *_s v \backslash \{e\})} = r1_{\prod_{e \in v\Lambda^1} TZ_\Lambda(v *_s v \backslash \{e\})} \\
&= r1_{TZ_\Lambda(v *_s v \backslash v\Lambda^1)} \neq 0.
\end{aligned}$$

Hence Theorem 4.2.1 implies that ϕ_Q is injective, as required.

To see the surjectivity of ϕ_Q, we take $f \in A_R(\mathcal{TG}_\Lambda)$. By Proposition 2.9.1, we write f as $\sum_{U \in F} a_U 1_U$ where $a_U \in R$, each U is in the form $\bigcup_{i=1}^n TZ_\Lambda(\lambda_i *_s \mu_i \setminus G_i)$ for some $n \in \mathbb{N}$, and F is finite set of mutually disjoint elements. Hence to show that $f \in \operatorname{im}(\phi_Q)$, it suffices to show that each $1_U \in \operatorname{im}(\phi_Q)$ where

$$U := \bigcup_{i=1}^n TZ_\Lambda(\lambda_i *_s \mu_i \setminus G_i)$$

for some positive integer n and collection $\{TZ_\Lambda(\lambda_i *_s \mu_i \setminus G_i)\}_{i=1}^n$. By Lemma 4.3.5, 1_U can be written as the sum of elements in the form $1_{TZ_\Lambda(\lambda *_s \mu \setminus G)}$. On the other hand, by following the argument of [18, Equation 5.5], we have

$$1_{TZ_\Lambda(\lambda *_s \mu \setminus G)} = Q_\lambda \Big(\prod_{\nu \in G} (Q_{s(\lambda)} - Q_\nu Q_{\nu^*}) \Big) Q_\mu$$

for all $(\lambda, \mu) \in \Lambda *_s \Lambda$ and finite $G \subseteq s(\lambda)\Lambda$. Hence every $1_{TZ_\Lambda(\lambda *_s \mu \setminus G)}$ belongs to $\operatorname{im}(\phi_Q)$ and so does 1_U, as required. Hence ϕ_Q is surjective, and is an isomorphism. $\square$

Remark 4.3.6. Proposition 3.4.1 shows that the Kumjian-Pask algebra of $T\Lambda$ is isomorphic to the Steinberg algebra associated to the boundary-path groupoid $\mathcal{G}_{T\Lambda}$ of [58]. We could have shown that the path groupoid $\mathcal{TG}_\Lambda$ of Example 2.8.1 is topologically isomorphic to the boundary-path groupoid $\mathcal{G}_{T\Lambda}$, and deduced Proposition 4.3.4. However, the direct argument above takes about the same amount of effort.

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