

Von Neumann Algebras for Abstract Harmonic Analysis

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

This thesis develops the theory of operator algebras from the perspective of abstract harmonic analysis, and in particular, the theory of von Neumann algebras. Results from operator algebras are applied to the study of spaces of coefficient functions of unitary representations of locally compact groups, and in particular, the Fourier algebra of a locally compact group. The final result, which requires most of the material developed in earlier sections, is that the group von Neumann algebra of a locally compact group is in standard form.

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Chapter 1

Introduction

This thesis is intended to be an introduction to abstract harmonic analysis from the modern viewpoint of operator algebras. This begins with a treatment of the abstract theory of C^* -algebras and their connection to the representation theory of locally compact groups. It continues with the detailed development of relevant portions of the theory of von Neumann algebras. It concludes with an introduction to Fourier algebra of a general locally compact group, including Eymard's theorem on the spectrum, Herz's restriction theorem, and the fact that a group von Neumann algebra is always in standard form.

When I first began the study of abstract harmonic analysis, I soon realized that the literature is very scattered. In order to gain sufficiency in the subject to the point where one can read the recent research literature, one must learn from a number of different sources, spanning distinct historical periods and notational conventions. There is no one source that one can read for an introduction to the subject, and some important results can be difficult to locate in their most useful form. I set out to alleviate this problem and provide one source from which one can learn the fundamentals of the subject.

This thesis presumes a good working knowledge of functional analysis, particularly the duality theory of Banach spaces, a basic knowledge of Banach algebras, and a passing knowledge of integration theory on locally compact groups. However, besides these prerequisites, every argument is developed in full, often with more details than the proofs in the original papers or in other sources.

While this thesis is quite comprehensive in that it gives full proofs for all of the subjects that it covers, there are many relevant topics that are not covered. In particular, there are no serious examples of group representations, and there is no mention of weak containment of representations. The main reason for their exclusion is the lack of space, as this thesis is already well over the average length of a Master's thesis in mathematics. The type decomposition of von Neumann algebras was also excluded on the same grounds.

I consulted many sources in the preparation of this thesis. Besides the papers cited explicitly, I read most of the standard textbooks on the subject of operator algebras: Dixmier's two books [Dix69a] [Dix69b], Takesaki's introductory textbook [Tak02], Pedersen's book [Ped79], the book of Stratila and Zsido [SZ79], and Davidson's book [Dav96]. While all of the results in this thesis appear in one cited source or another, the proofs given here are often a synthesis of the different ideas that I have learned while studying the subject as a whole, rather than being taken directly from one particular place.

Since my goal was to give a synthetic treatment of the subject, looking back into the past rather than developing the subject genetically, I do not discuss the attribution of each theorem as it is proved. This is not meant to take credit away from those who toiled to prove it, rather it is meant to provide a more coherent narrative. Of course, there are exceptions made when a theorem is generally referred to by the name of the person who proved it. At the end of each section, I have written some notes on the history of the topic covered in that section, and these notes often provide additional information about the particular proofs that I chose to present.

Chapter 2

Banach $*$ -algebras

2.1 Definitions and Basic Properties

2.1.1 Definition. Let \mathcal{A} be an algebra over \mathbb{C} . An *involution* on \mathcal{A} is a map $*$: $\mathcal{A} \rightarrow \mathcal{A}$ such that for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$,

- (i) $(a^*)^* = a$,
- (ii) $(a + b)^* = a^* + b^*$,
- (iii) $(\lambda a)^* = \overline{\lambda} a^*$,
- (iv) $(ab)^* = b^* a^*$.

If \mathcal{A} is equipped with an involution, it is called a *$*$ -algebra*. By a *unital $*$ -algebra* we mean a $*$ -algebra with a multiplicative unit 1 such that $1 \neq 0$.

2.1.2 Examples.

- (i) The complex numbers form a $*$ -algebra, where the involution is conjugation.
- (ii) Let X be a locally compact Hausdorff space. Then the algebras $C_c(X)$, $C_0(X)$, $C_b(X)$, and $C(X)$ of continuous functions on X are $*$ -algebras, where the involution is pointwise conjugation.
- (iii) Let \mathcal{H} be a Hilbert space. Then the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on \mathcal{H} is a $*$ -algebra, where the involution is the adjoint operation.
- (iv) Let G be a locally compact group, and Δ its modular function. Then the algebras $C_c(G)$, $L^1(G)$, and $M(G)$, all equipped with the convolution product, are $*$ -algebras, where the measure μ^* is defined by

$$\int g(s) d\mu^*(s) = \int_G \overline{g(s^{-1})} d\mu(s)$$

for $f \in C_0(G)$. If $f \in L^1(G)$ and μ is the measure defined by

$$\int g(s) d\mu(s) = \int f(s)g(s) ds,$$

then

$$\int g(s) d\mu^*(s) = \int \overline{f(s)}g(s^{-1}) ds = \int \Delta(s^{-1})\overline{f(s^{-1})}g(s).$$

Hence $f^*(s) = \Delta(s^{-1})\overline{f(s^{-1})}$.

2.1.3 Definition. Let \mathcal{A} be a $*$ -algebra. If $a \in \mathcal{A}$ is such that $a = a^*$, we say that a is *self-adjoint*. We let \mathcal{A}_{sa} denote the set of self-adjoint elements of \mathcal{A} . We say that $a \in \mathcal{A}$ is *normal* if $aa^* = a^*a$, *unitary* if a is invertible and $a^{-1} = a^*$, and a *projection* if a is self-adjoint and $a^2 = a$.

2.1.4 Examples.

- (i) The self-adjoint elements of the \mathbb{C} are the real numbers. Every element of \mathbb{C} is normal. The unitary elements of \mathbb{C} are the elements of norm 1. The projections in \mathbb{C} are 0 and 1.
- (ii) Let X be a locally compact Hausdorff space. If $f \in C_0(X)$, then f is always normal (as $C_0(X)$ is commutative); f is unitary if and only if $\text{ran}(f) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, and f is a projection if and only if $\text{ran}(f) \subseteq \{0, 1\}$, in which case f is the characteristic function of a clopen set.
- (iii) Let \mathcal{H} be a Hilbert space. The self-adjoint elements of $\mathcal{B}(\mathcal{H})$ are the self-adjoint or Hermitian operators, i.e. those bounded operators T such that $\langle T\xi | \eta \rangle = \langle \xi | T\eta \rangle$ for all $\xi, \eta \in \mathcal{H}$. If $T \in \mathcal{B}(\mathcal{H})$, then T is normal, unitary, or a projection if and only if it is normal, unitary, or an orthogonal projection in the usual meaning of those words for operators. In particular, $U \in \mathcal{B}(\mathcal{H})$ is unitary if and only if $\langle U\xi | U\eta \rangle = \langle \xi | \eta \rangle$ for all $\xi, \eta \in \mathcal{H}$ and U has dense range.

It is important to note that if \mathcal{A} is a unital $*$ -algebra, then the unit 1 is a self-adjoint element of \mathcal{A} . This is because

$$1^* = 1^*1 = (1^*1)^* = 1^{**} = 1.$$

It is clear that \mathcal{A}_{sa} is a real vector subspace of \mathcal{A} . Generalizing the situation in the preceding two examples, every element $a \in \mathcal{A}$ can be expressed as $a = a_1 + ia_2$, where a_1 and a_2 are self-adjoint. Indeed, choose

$$a_1 = \frac{(x + x^*)}{2} \quad \text{and} \quad a_2 = \frac{(x - x^*)}{2i}.$$

These elements are referred to as the *real* and *imaginary parts* of a respectively. If \mathcal{A} is the complex numbers, these agree with the usual real and imaginary parts of a complex number, and if $\mathcal{A} = C(X)$, these are the pointwise real and imaginary parts of a function.

The natural maps between $*$ -algebras are the algebra homomorphisms that preserve the adjoint operation.

2.1.5 Definition. Let \mathcal{A} and \mathcal{B} be $*$ -algebras. We say that a homomorphism $f : \mathcal{A} \rightarrow \mathcal{B}$ is a $*$ -homomorphism if $f(a^*) = f(a)^*$ for every $a \in \mathcal{A}$.

If $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is a $*$ -homomorphism, then $\varphi(\mathcal{A})$ is a $*$ -subalgebra of \mathcal{B} , and $\ker(\varphi)$ is a self-adjoint ideal of \mathcal{A} .

If \mathcal{A} is a $*$ -algebra, then the linear functionals on \mathcal{A} inherit some of its involutive structure. If $f : \mathcal{A} \rightarrow \mathbb{C}$ is a linear functional, then we define the *adjoint* of f by

$$f^*(a) = \overline{f(a^*)}.$$

Clearly, f is also a linear functional on \mathcal{A} , and for linear functionals f and g on \mathcal{A} and $\lambda \in \mathbb{C}$,

- (i) $f^{**} = f$,
- (ii) $(f + g)^* = f^* + g^*$,
- (iii) $(\lambda f)^* = \overline{\lambda} f^*$.

We say that f is *self-adjoint* if $f = f^*$. If f is a linear functional on \mathcal{A} , then f can be expressed as $f = f_1 + if_2$, where f_1 and f_2 are linear functionals on \mathcal{A} . Indeed, choose

$$f_1 = \frac{(f + f^*)}{2} \quad \text{and} \quad f_2 = \frac{(f - f^*)}{2i}.$$

These elements are referred to as the *real* and *imaginary parts* of f respectively.

2.1.6 Definition. Let \mathcal{A} be a $*$ -algebra. If \mathcal{A} is also a Banach algebra that satisfies the additional condition

- (i) $\|a\| = \|a^*\|$,

we say that \mathcal{A} is a *Banach $*$ -algebra*. Note that some authors only require that involution on a Banach $*$ -algebra be continuous, and not necessarily an isometry. If, moreover, \mathcal{A} satisfies the condition

- (ii) $\|a^*a\| = \|a\|^2$,

we say that \mathcal{A} is a *C^* -algebra*.

If \mathcal{A} satisfies (ii), then it satisfies (i). Indeed, if $a \in \mathcal{A}$, then

$$\|a\|^2 = \|a^*a\| \leq \|a^*\| \|a\|,$$

so

$$\|a\| \leq \|a^*\| \leq \|a^{**}\| = \|a\|.$$

Also note that we do not require a unital Banach $*$ -algebra to satisfy $\|1\| = 1$. This assumption sometimes makes things easier to state, but there are some examples where the identity has norm greater than 1.

2.1.7 Examples.

- (i) The complex numbers form a C^* -algebra, where the involution is conjugation.
- (ii) Let X be a locally compact Hausdorff space. Then the algebras $C_b(X)$ and $C_0(X)$ of continuous functions are C^* -algebras.
- (iii) Let \mathcal{H} be a Hilbert space. Then the Banach algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on \mathcal{H} is a C^* -algebra, where the involution is the adjoint operation. The final condition of being a C^* -algebra holds because

$$\|A^*A\| = \sup_{\|x\|=\|y\|=1} \langle A^*Ax \mid y \rangle = \sup_{\|x\|=\|y\|=1} \langle Ax \mid Ay \rangle = \|A\|^2.$$

It follows that every closed self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ is also a C^* -algebra.

- (iv) Let G be a locally compact group. Then the algebras $L^1(G)$ and $M(G)$ are Banach $*$ -algebras.
- (v) Let \mathcal{A} be \mathbb{C}^2 equipped with the ∞ -norm and pointwise multiplication. Define an involution on \mathcal{A} by

$$(a, b)^* = (\bar{b}, \bar{a}).$$

Then \mathcal{A} is a commutative unital Banach $*$ -algebra, but \mathcal{A} is not a C^* -algebra. Indeed,

$$\|(1, 0)^*(1, 0)\| = \|(0, 1) \cdot (1, 0)\| = \|(0, 0)\| = 0.$$

This example may seem a bit contrived, but it illustrates what can happen in the absence of the C^* -algebra norm condition.

2.1.8 Proposition. *Let \mathcal{A} be a non-unital Banach algebra. Then there exists a unital Banach algebra \mathcal{A}_1 containing \mathcal{A} as a closed ideal of codimension 1. If \mathcal{A} is additionally a Banach $*$ -algebra, then we can take \mathcal{A}_1 to be a Banach $*$ -algebra containing \mathcal{A} as a closed self-adjoint ideal of codimension 1.*

PROOF. If \mathcal{A} is a Banach $*$ -algebra that is not unital (or even one that is unital, since it is sometimes useful to avoid a dichotomy in the middle of a proof), one can consider its canonical unitization \mathcal{A}_1 defined as follows. Let \mathcal{A}_1 be the vector space $\mathcal{A} \oplus \mathbb{C}$, with the $*$ -algebra structure given by

$$\begin{aligned} (a, \lambda) \cdot (b, \mu) &= (ab + \mu a + \lambda b, \lambda\mu); \\ (a, \lambda)^* &= (a^*, \bar{\lambda}). \end{aligned}$$

We will identify $a \in \mathcal{A}$ with its image $(a, 0)$ in \mathcal{A}_1 , which makes \mathcal{A} a self-adjoint maximal ideal of \mathcal{A}_1 of codimension 1. There are many possible ways to put a norm on \mathcal{A}_1 that will make it a Banach $*$ -algebra. The simplest is to take the ℓ^1 -direct sum of \mathcal{A} and \mathbb{C} , i.e. define

$$\|(a, \lambda)\| = \|a\| + |\lambda|.$$

Then, clearly, $\|\cdot\|$ is a norm on \mathcal{A}_I that makes it a Banach $*$ -algebra, and the embedding $a \mapsto (a, 0)$ of \mathcal{A} into \mathcal{A}_I is an isometry. Unfortunately, when \mathcal{A} is a C^* -algebra, this norm will not necessarily make \mathcal{A}_I a C^* -algebra. In this case, we will put a different norm on \mathcal{A}_I that does make it a C^* -algebra. \square

If \mathcal{A} is a non-unital Banach algebra or Banach $*$ -algebra, we will let \mathcal{A}_I denote the algebra constructed in the preceding proposition. If \mathcal{A} is a unital Banach algebra or Banach $*$ -algebra, we will simply let \mathcal{A}_I denote \mathcal{A} itself. This is sometimes useful to avoid an unnecessary dichotomy in the middle of a proof.

2.1.9 Proposition. *Let \mathcal{A} be a non-unital C^* -algebra. Then there exists a unital C^* -algebra \mathcal{A}_I containing \mathcal{A} a self-adjoint closed ideal of codimension 1.*

PROOF. Every $a \in \mathcal{A}$ acts on \mathcal{A} via the left multiplication operator $L_a : \mathcal{A} \rightarrow \mathcal{A}$, defined by $L_a b = ab$. Clearly, since \mathcal{A} is a Banach algebra, L_a is always a bounded linear operator on \mathcal{A} . The map $(a, \lambda) \mapsto L_a + \lambda I$ is an injective algebra homomorphism, and there is a natural involution on its range given by

$$(L_a + \lambda I)^* = L_{a^*} + \bar{\lambda} I$$

that makes this map a $*$ -algebra homomorphism. We define

$$\|(a, \lambda)\| = \|L_a + \lambda I\|.$$

Since $\mathcal{B}(\mathcal{A})$ is a Banach algebra, this defines an Banach algebra norm on \mathcal{A}_I . The embedding $a \mapsto (a, 0)$ of \mathcal{A} into \mathcal{A}_I is isometric because

$$\|a\| = \left\| a \frac{a^*}{\|a\|} \right\| \leq \|(a, 0)\| = \sup_{\|b\| \leq 1} \|ab\| \leq \|a\|.$$

All that we need to show is the C^* -algebra norm condition. We have

$$\begin{aligned} \|(a, \lambda)\|^2 &= \sup_{\|b\| \leq 1} \|ab + \lambda b\|^2 \\ &= \sup_{\|b\| \leq 1} \|(ab + \lambda b)^*(ab + \lambda b)\| \\ &= \sup_{\|b\| \leq 1} \|b^* a^* ab + \lambda b^* a^* b + \bar{\lambda} b^* ab + |\lambda|^2 b^* b\| \\ &\leq \sup_{\|b\| \leq 1} \|a^* ab + \lambda a^* b + \bar{\lambda} ab + |\lambda|^2 b\| \\ &= \|(a^* a + \lambda a^* + \bar{\lambda} a, |\lambda|^2)\| \\ &= \|(a, \lambda)^*(a, \lambda)\| \\ &\leq \|(a, \lambda)^*\| \|(a, \lambda)\|. \end{aligned}$$

Hence $\|(a, \lambda)\| \leq \|(a, \lambda)^*\|$. By symmetry, we have that $\|(a, \lambda)^*\| \leq \|(a, \lambda)\|$, so $\|(a, \lambda)\| = \|(a, \lambda)^*\|$. Therefore, the above inequality is an equality, and

the expression $\|(a, \lambda)^*(a, \lambda)\|$ in the middle of the inequality is equal to the first expression $\|(a, \lambda)\|^2$. This shows that \mathcal{A}_1 is a C^* -algebra. In the case of a generic Banach $*$ -algebra, we will use the first of the two norms, but whenever \mathcal{A} is assumed to be a C^* -algebra, we will use the second norm that makes \mathcal{A}_1 a C^* -algebra. It is easy to check that these two norms are always equivalent. If \mathcal{A} is already unital, we will let \mathcal{A}_1 denote \mathcal{A} itself. \square

If \mathcal{A} is a non-unital C^* -algebra, we will let \mathcal{A}_1 denote the algebra constructed in the preceding proposition. If \mathcal{A} is a unital C^* -algebra, we will simply let \mathcal{A}_1 denote \mathcal{A} . While this conflicts with the notation defined for more general Banach $*$ -algebra, it will always be clear from context whether we are using the C^* -algebra norm on \mathcal{A}_1 .

2.1.10 Example. Let X be a locally compact Hausdorff space that is not compact, so that $C_0(X)$ is not unital. Then $C_0(X)_1$ is $C(X \cup \{\infty\})$, where $X \cup \{\infty\}$ is the one-point compactification of X .

We have not assumed that the identity of a unital Banach $*$ -algebra has norm 1, but it is sometimes useful to assume that the identity has norm 1. A construction similar to the one used in the unitization of a C^* -algebra shows that a unital Banach algebra can always be renormed so that the identity has norm 1. In particular, if the identity of \mathcal{A} already has norm 1, this construction shows that \mathcal{A} is isometrically isomorphic to a subalgebra of $\mathcal{B}(X)$ for a Banach space X .

2.1.11 Proposition. *Let \mathcal{A} be a unital Banach algebra. For every $a \in \mathcal{A}$, define $L_a : \mathcal{A} \rightarrow \mathcal{A}$ by $L_a b = ab$. Then L_a is a bounded operator on \mathcal{A} . Define $\Phi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{A})$ by $\Phi(a) = L_a$. Then Φ is an injective Banach algebra homomorphism, and*

$$\frac{1}{\|1\|} \|a\| \leq \|\Phi(a)\| \leq \|\Phi\| \|a\|,$$

so the norm on \mathcal{A} is equivalent to the norm $\Phi(\mathcal{A})$ inherits from $\mathcal{B}(\mathcal{A})$. We have $\|\Phi(1)\| = 1$, and if $\|1\| = 1$ then Φ is an isometry.

PROOF. It is clear that each L_a is bounded and that Φ is a Banach algebra homomorphism. It is injective because $L_a 1 = a$. We have

$$\|a\| = \|\Phi(a)1\| \leq \|\Phi(a)\| \|1\|.$$

By dividing by $\|1\|$, it follows that

$$\frac{1}{\|1\|} \|a\| \leq \|\Phi(a)\| \leq \|\Phi\| \|a\|,$$

so the norm on \mathcal{A} is equivalent to the norm $\Phi(\mathcal{A})$ inherits from $\mathcal{B}(\mathcal{A})$. Since $\Phi(1)$ is the identity operator, $\|\Phi(1)\| = 1$. If $\|1\| = 1$, then $\|a\| \leq \|\Phi(a)\|$. Since

$$\|\Phi(a)b\| \leq \|ab\| \leq \|a\| \|b\|,$$

it follows that $\|\Phi(a)\| \leq \|a\|$ and $\|a\| = \|\Phi(a)\|$. \square

Unfortunately, it is not clear how to define an involution on $\Phi(\mathcal{A})$. Thus, in order to renorm a unital Banach $*$ -algebra we will introduce a new variant of this construction.

2.1.12 Definition. Let \mathcal{A} be a Banach algebra. A *double centralizer* of \mathcal{A} is a pair of bounded linear operators (L, R) on \mathcal{A} such that for all $a, b \in \mathcal{A}$,

- (i) $L(ab) = L(a)b$,
- (ii) $R(ab) = aR(b)$,
- (iii) $aL(b) = R(a)b$.

The *double centralizer algebra* $\mathcal{D}(\mathcal{A})$ of \mathcal{A} is the set of double centralizers of \mathcal{A} equipped with pointwise linear operations and the product

$$(L_1, R_1)(L_2, R_2) = (L_1L_2, R_1R_2).$$

If \mathcal{A} is also a Banach $*$ -algebra and $T : \mathcal{A} \rightarrow \mathcal{A}$ is a linear map, define $T^* : \mathcal{A} \rightarrow \mathcal{A}$ by

$$T^*(a) = T(a^*)^*.$$

Then $\mathcal{D}(\mathcal{A})$ becomes a $*$ -algebra when equipped with the involution

$$(L, R)^* = (R^*, L^*).$$

It is easy to check that $\mathcal{D}(\mathcal{A})$ is actual an algebra, and also a $*$ -algebra if \mathcal{A} is a Banach $*$ -algebra. If $I : \mathcal{A} \rightarrow \mathcal{A}$ is the identity map, then (I, I) is an identity for $\mathcal{D}(\mathcal{A})$. Every $a \in \mathcal{A}$ defines a double centralizer (L_a, R_a) of \mathcal{A} by

$$L_a b = ab \quad \text{and} \quad R_a b = ba.$$

The map $\Phi : \mathcal{A} \rightarrow \mathcal{D}(\mathcal{A})$ defined by $\Phi(a) = (L_a, R_a)$ is a homomorphism. If \mathcal{A} is a Banach $*$ -algebra, then this map is a $*$ -homomorphism.

We define a norm on $\mathcal{D}(\mathcal{A})$ by

$$\|(L, R)\| = \max(\|L\|, \|R\|),$$

where $\|L\|$ and $\|R\|$ are the operator norms of L and R , respectively. It is easy to check that $\mathcal{D}(\mathcal{A})$ is closed in $\mathcal{B}(\mathcal{A}) \times \mathcal{B}(\mathcal{A})$, and is thus a Banach algebra, and that $\Phi : \mathcal{A} \rightarrow \mathcal{D}(\mathcal{A})$ is a contractive Banach algebra homomorphism. If \mathcal{A} is a Banach $*$ -algebra, then the involution on $\mathcal{D}(\mathcal{A})$ is an isometry, making $\mathcal{D}(\mathcal{A})$ a Banach $*$ -algebra, and Φ a contractive Banach $*$ -algebra homomorphism. Also, note that

$$\|(1, 1)\| = \max(1, 1) = 1.$$

The double centralizer algebra is very interesting, particularly in the case where \mathcal{A} is not unital. However, we only introduced double centralizers to show that a unital Banach $*$ -algebra can be renormed so that the identity has norm 1, and in this case the double centralizers are precisely the pairs (L_a, R_a) , where $a \in \mathcal{A}$.

2.1.13 Proposition. *Let \mathcal{A} be a unital Banach algebra. Then the map $\Phi : \mathcal{A} \rightarrow \mathcal{D}(\mathcal{A})$ defined by $\Phi(a) = (L_a, R_a)$ is a Banach algebra isomorphism.*

PROOF. We need to show that Φ is injective and surjective. If $a \in \mathcal{A}$ and $\Phi(a) = 0$, then

$$a = L_a 1 = 0,$$

so $a = 0$. Therefore, Φ is injective. If $(L, R) \in \mathcal{D}(\mathcal{A})$ and $a \in \mathcal{A}$, then

$$L(a) = 1L(a) = R(1)a = L_{R(1)}(a)$$

and

$$R(a) = R(a)1 = aL(1) = R_{L(1)}(a).$$

Hence $(L, R) = (L_{R(1)}, R_{L(1)})$. However,

$$L(1) = 1L(1) = R(1)1 = R(1),$$

so $(L, R) = (L_a, R_a) = \Phi(a)$. Therefore, Φ is surjective. \square

Historical Notes

Particular examples of Banach algebras and Banach $*$ -algebras have been studied since the genesis of abstract analysis. von Neumann defined rings of operators in [vN29], which are now known as von Neumann algebras, and they were studied in detail by von Neumann and Murray in their “Rings of Operators” series of papers [MvN36], [MvN37], [vN40], and [MvN43]. There were some attempts to abstractly characterize rings of operators, but it was not clear at the time that the norm was of much importance, so the focus was mainly on the order-theoretic properties of von Neumann algebras and their closure in the various operator topologies on $\mathcal{B}(\mathcal{H})$.

General Banach algebras were first defined by Nagumo [Nag36], who called them linear metric rings, and Yosida [Yos36]. Most of the foundational work on Banach algebras and Banach $*$ -algebras was done from 1939 to 1944 by Gelfand and his collaborators in the Russian school, particularly Naimark and Raikov. In 1939, Gelfand published a series of announcements [Gel39a], [Gel39b], and [Gel39c], but due to the war the proofs of the announced theorems did not appear in the West until later. The Russian school produced many papers on this subject, so it is not possible to list them all, but some of the

more important ones are Gelfand's paper on the general theory of Banach algebras [Gel41], his paper with Naimark characterizing commutative C^* -algebras as $C_0(X)$ for a locally compact Hausdorff space X and general C^* -algebras as norm closed $*$ -subalgebras $\mathcal{B}(\mathcal{H})$ [GN43], and his paper with Raikov on the applications of this theory to group representations [GR43].

In the work of the Russian school, Banach algebras were called normed rings and Banach $*$ -algebras were called involutive normed rings. The terminology "Banach algebra" and "Banach $*$ -algebra" first appears in a paper of Ambrose [Amb45], and is apparently due to Max Zorn. This notation is somewhat strange, given that Banach essentially had nothing to do with the theory of Banach algebras. Several authors noted that the names "Gelfand algebra" and "Gelfand $*$ -algebra" would be more fitting, but nobody made the first leap to use them in their papers.

What is now called a C^* -algebra was originally called a B^* -algebra by Rickart [Ric46]. Gelfand and Naimark proved in [GN43] that a Banach $*$ -algebra \mathcal{A} is (algebraically) $*$ -isomorphic to a norm closed $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$ if and only if \mathcal{A} satisfies the conditions

$$\|a^*a\| = \|a\|^2 \quad \text{and} \quad 1 + a^*a \text{ is invertible}$$

for all $a \in \mathcal{A}$. These algebras were called C^* -algebras by Segal [Seg47], where the C originally meant that the algebra was closed in the norm topology on $\mathcal{B}(\mathcal{H})$, not that they were, as some might claim, a noncommutative analogue of the algebras $C_0(X)$. It turned out that the assumption of the invertibility of $1 + a^*a$ was unnecessary, which we will prove in Section 2.3. Ironically, after this development, the name " C^* -algebra" survived and the name " B^* -algebra" fell into abandon.

The double centralizer construction is originally due to Hochschild [Hoc47] in the purely algebraic case, and Johnson [Joh64] in the Banach algebraic case, although he was apparently unaware of Hochschild's work.

2.2 Commutative C^* -algebras

In this section we will assume familiarity with the basic Gelfand theory of commutative Banach algebras.

2.2.1 Theorem (Gelfand-Naimark). *Let \mathcal{A} be a commutative C^* -algebra. Then the Gelfand transform is an isometric $*$ -isomorphism of \mathcal{A} onto $C_0(\widehat{\mathcal{A}})$.*

PROOF. We will first prove the theorem under the assumption that \mathcal{A} is unital and then derive the general case. Let φ be a multiplicative linear functional on \mathcal{A} . We will show that $\varphi(a^*) = \overline{\varphi(a)}$. Suppose first that $a = a^*$ is self-adjoint.

Define a family of elements of \mathcal{A} by

$$u_t = e^{ita} = \sum_{n=0}^{\infty} \frac{(ita)^n}{n!},$$

where $t \in \mathbb{R}$. Each u_t is unitary, since

$$u_t^* = \sum_{n=0}^{\infty} \frac{(\overline{ita})^n}{n!} = \sum_{n=0}^{\infty} \frac{(-ita)^n}{n!} = e^{-ita} = u_t^{-1}.$$

Hence $\|u_t\|^2 = \|u_t^* u_t\| = \|1\| = 1$. Therefore,

$$1 \geq |\varphi(u_t)| = \left| \sum_{n \geq 0} \frac{it\varphi(a)^n}{n!} \right| = |e^{it\varphi(a)}| = e^{-t \operatorname{Im} \varphi(a)}.$$

Since this holds for all real t , we deduce that $\varphi(a)$ is real. If $x \in \mathcal{A}$ is not necessarily self-adjoint, let $x = a + bi$, where a and b are the real and imaginary parts of x respectively. Then $\varphi(a)$ and $\varphi(b)$ are real, so

$$\varphi(x^*) = \varphi(a - ib) = \varphi(a) - i\varphi(b) = \overline{\varphi(a) + i\varphi(b)} = \overline{\varphi(x)}.$$

Therefore, the Gelfand transform satisfies the property $\widehat{a^*} = \widehat{a}^*$ and is a $*$ -isomorphism.

It now remains to be shown that the Gelfand transform is an isometry. If $a \in \mathcal{A}$ is self-adjoint, $\|a^2\| = \|a^* a\| = \|a^2\|$, so by the Beurling spectral radius formula,

$$\|\widehat{a}\|_{\infty} = \operatorname{spr}(a) = \lim_{n \rightarrow \infty} \|a^{2^n}\|^{1/2^n} = \lim_{n \rightarrow \infty} (\|a\|^{2^n})^{1/2^n} = \|a\|.$$

If $a \in \mathcal{A}$ is not necessarily self-adjoint, $a^* a$ is self-adjoint, so we obtain

$$\|a\|^2 = \|a^* a\| = \|\widehat{a^* a}\|_{\infty} = \|\widehat{a^*} \widehat{a}\|_{\infty} = \|\widehat{a}\|_{\infty}^2.$$

Therefore, the Gelfand transform is an isometry.

The above implies that the image of \mathcal{A} under the Gelfand transform is a unital norm closed self-adjoint sub-algebra of $C(\widehat{\mathcal{A}})$ that separates points. Hence, by the Stone-Weierstrass Theorem, the Gelfand transform is surjective and thus a $*$ -isomorphism.

We now consider the non-unital case. If \mathcal{A} is not unital, then \mathcal{A} is a maximal ideal of codimension one in \mathcal{A}_1 , the unitization of \mathcal{A} . Clearly, \mathcal{A}_1 is also abelian, and the spectrum X of \mathcal{A}_1 is canonically homeomorphic to $\widehat{\mathcal{A}} \setminus \{\varphi_0\}$, where φ_0 is the multiplicative linear functional on \mathcal{A}_1 with kernel \mathcal{A} . By the argument above, the Gelfand transform is then a $*$ -isomorphism from \mathcal{A}_1 onto $C(X)$. The Gelfand transform then takes \mathcal{A} onto the ideal of functions vanishing at φ_0 , which can naturally be identified with $C_0(\widehat{\mathcal{A}})$. \square

If \mathcal{A} is a C^* -algebra and $S \subseteq \mathcal{A}$, the C^* -algebra generated by S is the least C^* -subalgebra of \mathcal{A} containing S , and is denoted $C^*(S)$. If \mathcal{A} is unital, then the convention is to include the identity in $C^*(S)$, although we may occasionally specify otherwise. In the case that $S = \{a\}$, $C^*(S)$ is denoted by $C^*(a)$ and is simply the norm closure of the set of polynomials in a and a^* ; if the identity is not included in $C^*(a)$, then it is the norm closure of the polynomials in a and a^* with zero constant term.

Note that if a is unitary or self-adjoint it is automatically normal, and that a is normal if and only if $C^*(a)$ is commutative. In this context, the preceding theorem has an important interpretation in terms of a continuous functional calculus for normal elements in arbitrary C^* -algebras.

2.2.2 Corollary (Continuous Functional Calculus). *Let \mathcal{A} be a C^* -algebra, and $a \in \mathcal{A}$ be normal. Then $C^*(a)$ is isometrically isomorphic to $C_0(\sigma(a))$ via the map that sends a to the identity function. The (not necessarily unital) C^* -algebra generated by a and a^* , without explicitly including the identity, is mapped onto $C_0(\sigma(a) \setminus \{0\})$.*

PROOF. Since a is normal, $C^*(a)$ is commutative. Therefore, we only need to determine $X = \widehat{C^*(a)}$. Note that a multiplicative linear functional φ in X is determined by $\varphi(a) = \lambda$, as then $\varphi(p(a, a^*)) = p(\lambda, \bar{\lambda})$ for every complex polynomial $p(x, y)$. Thus the map from X into \mathbb{C} given by $\varphi \mapsto \varphi(a)$ is a homeomorphism onto $\hat{a}(X)$. From the Gelfand theory, this implies that $\hat{a}(X) = \sigma(a)$. Hence the Gelfand transform identifies \hat{a} with the identity function, as desired. By Theorem 2.2.1, the Gelfand transform is an isometric $*$ -isomorphism. When a is not invertible, i.e. when $0 \in \sigma(a)$, the subalgebra generated by a and a^* , not including the identity, corresponds to the ideal of functions vanishing at 0, which is identified with $C_0(\sigma(a) \setminus \{0\})$. \square

We can use this isometric $*$ -isomorphism to construct a continuous functional calculus for a normal element of a C^* -algebra. If $a \in \mathcal{A}$ is normal, let $\Gamma : C^*(a) \rightarrow C_0(\sigma(a))$ denote the Gelfand transform. If $f \in C_0(\sigma(a))$, we let $f(a)$ denote $\Gamma^{-1}(f) \in C^*(a)$, so that $\widehat{f(a)} = f$. If $0 \in \sigma(a)$ and $f(0) = 0$, then $f(a)$ lies in the non-unital algebra generated by a and a^* . We will now state some of the more immediate properties of this functional calculus

2.2.3 Corollary. *Let \mathcal{A} be a C^* -algebra, and $a \in \mathcal{A}$ be normal. Then:*

- (i) *if $f \in C_0(\sigma(a))$ then $\sigma(f(a)) = f(\sigma(a))$;*
- (ii) *if $f \in C_0(\sigma(a))$ and $g \in C_0(\sigma(f(a))) = C_0(f(\sigma(a)))$, so that $g \circ f \in C_0(\sigma(a))$, then $(g \circ f)(a) = g(f(a))$;*

PROOF.

- (i) We have that

$$\sigma(f(a)) = \sigma(\widehat{f(a)}) = \sigma(f) = f(\sigma(a)).$$

- (ii) When $p(z, \bar{z})$ is a polynomial, it is immediate from the fact that the functional calculus is a $*$ -homomorphism that $p(f(a)) = (p \circ f)(a)$. Approximating g by polynomials using the Stone-Weierstrass Theorem establishes the general case. \square

2.2.4 Corollary. *Let \mathcal{A} be a C^* -algebra, and $a \in \mathcal{A}$ be normal. Then*

- (i) a is self-adjoint if and only if $\sigma(a) \subseteq \mathbb{R}$;
- (ii) a is unitary if and only if $\sigma(a) \subseteq \{\lambda \in \mathbb{C} : \|\lambda\| = 1\}$;
- (iii) a is a projection if and only if $\sigma(a) \subseteq \{0, 1\}$.

PROOF. These claims are immediate by the functional calculus and the fact that they all hold when $a \in C_0(X)$ for a locally compact Hausdorff space X . \square

It is often the case for a general element a of a Banach algebra that $\text{spr}(a)$ is strictly smaller than $\|a\|$. However, they are equal whenever a is a normal element of a C^* -algebra. We already showed this for self-adjoint a in the proof of the Gelfand-Naimark Theorem, but it can easily be recovered via the continuous functional calculus.

2.2.5 Corollary. *Let \mathcal{A} be a C^* -algebra, and $a \in \mathcal{A}$ be normal. Then $\|a\| = \text{spr}(a)$.*

PROOF. By Theorem 2.2.1 we have that

$$\|a\| = \|\hat{a}\|_{C_0(\sigma(a))} = \|z\|_{C_0(\sigma(a))} = \text{spr}(a). \quad \square$$

2.2.6 Corollary. *Let \mathcal{A} be a C^* -algebra. If $a \in \mathcal{A}$, then $\|a\| = \|a^*a\|^{1/2} = \text{spr}(a^*a)^{1/2}$. Therefore, the norm on a C^* -algebra is completely determined by the algebraic structure and is unique.* \square

2.2.7 Corollary. *Let \mathcal{A} be a C^* -algebra. Then \mathcal{A} is a semisimple Banach algebra.*

PROOF. Let R be the radical of \mathcal{A} . We would like to show that $R = \{0\}$. If $a \in R$, then $a^*a \in R$, so a^*a is nilpotent and $\text{spr}(a^*a) = 0$. However, by Corollary 2.2.5 this implies that $\|a\|^2 = \|a^*a\| = 0$, so $a = 0$ as desired. \square

Up to this point, we have been rather liberal with the notation $\sigma(a)$ when a is an element of a C^* -algebra by not specifying the particular C^* -algebra in which we are taking the spectrum. However, it turns out that the choice of C^* -algebra does not matter.

2.2.8 Proposition. *Let \mathcal{A} and \mathcal{B} be C^* -algebras. If \mathcal{B} is unital and \mathcal{A} is a unital subalgebra of \mathcal{B} , then $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a)$ for every $a \in \mathcal{A}$. If \mathcal{A} is a general C^* -subalgebra of \mathcal{B} , then $\sigma_{\mathcal{A}}(a) \cup \{0\} = \sigma_{\mathcal{B}}(a) \cup \{0\}$ for every $a \in \mathcal{A}$.*

PROOF. We can prove both claims at once by working in \mathcal{A}_1 and \mathcal{B}_1 . Since $\sigma_{\mathcal{B}}(a) \subseteq \sigma_{\mathcal{A}}(a)$ for every $a \in \mathcal{A}$, we need to show that if $a \in \mathcal{A}$ is invertible in \mathcal{B}_1 , then this inverse is contained in \mathcal{A}_1 . First, we will suppose that $a \in \mathcal{A}$ is self-adjoint. Then, by Theorem 2.2.1, the commutative subalgebra $C = C^*(\{a, a^{-1}\})$ of \mathcal{B}_1 is $*$ -isomorphic to $C(X)$ for some compact Hausdorff space X . Let \hat{a} denote the image of a under this isomorphism. Then \hat{a} is a nonzero function on X , so $0 \notin \sigma_C(a)$. Since a is self-adjoint, $\sigma_C(a)$ is necessarily a subset of the real line, so by the Stone-Weierstrass Theorem there are polynomials p_n such that $p_n(x)$ converges to x^{-1} on $\sigma_C(a)$. Then, since $\widehat{a^{-1}} = \lim_{n \rightarrow \infty} p_n(\hat{a})$, we see that $a^{-1} = \lim_{n \rightarrow \infty} p_n(a) \in C^*(a) \subseteq \mathcal{A}$.

If $a \in \mathcal{A}$ is not necessarily self-adjoint and $a^{-1} \in \mathcal{B}_1$, then $(a^*a)^{-1} = a^{-1}(a^{-1})^*$ lies in \mathcal{B} , and therefore by the above argument also lies in \mathcal{A} . Thus $a^{-1} = (a^*a)^{-1}a^*$ belongs to \mathcal{A} . \square

Historical Notes

The results of this section are due to Gelfand and Naimark [GN43].

2.3 Positivity and Order in C^* -algebras

2.3.1 Definition. Let \mathcal{A} be a C^* -algebra. An element $a \in \mathcal{A}$ is said to be *positive* if a is normal and $\sigma(a) \subseteq [0, \infty)$. The set of positive elements of \mathcal{A} is denoted by \mathcal{A}_+ . If $a \in \mathcal{A}_+$, we write $a \geq 0$.

By Corollary 2.2.4, if $a \in \mathcal{A}$ is positive then a is self-adjoint.

2.3.2 Example. Let X be a locally compact Hausdorff space. Then the positive elements of $C_0(X)$ are the functions that take only non-negative values.

This example is very important, because most of the basic facts about positive elements of a C^* -algebra are established by using the corresponding result for functions and applying the functional calculus.

2.3.3 Proposition. *Let \mathcal{A} be a C^* -algebra. If $a \in \mathcal{A}_+$, then there is a unique $b \in \mathcal{A}_+$ such that $b^2 = a$.*

PROOF. Let $f : [0, \|a\|] \rightarrow \mathbb{R}$ be the square root function. Since $\sigma(a) \subseteq [0, \|a\|]$ and $f(0) = 0$, we can use the continuous functional calculus to apply f to a . Let $b = f(a)$. Then b is normal and $\sigma(b) = f(\sigma(a)) \subseteq [0, \|a\|]$, so b is positive. Since $f^2 = \text{id}_{[0, \|a\|]}$, we have that $b^2 = f(a)f(a) = a$.

Suppose that c is another positive square root of a . Then by the functional calculus,

$$c = f(c^2) = f(a) = b. \quad \square$$

If $a \in \mathcal{A}_+$, we let $a^{1/2}$ denote the unique *square root* of a in Proposition 2.3.3.

2.3.4 Proposition. *Let \mathcal{A} be a C^* -algebra. If $a \in \mathcal{A}$ is self-adjoint then there exist $a_+, a_- \in \mathcal{A}_+$ such that $a = a_+ - a_-$ and $a_+a_- = 0$.*

PROOF. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = (x + |x|)/2$ and $g(x) = f(-x)$. Both f and g are positive, and $f(0) = g(0) = 0$, so we can use the functional calculus to apply them to a . Let $a_+ = f(a)$ and $a_- = a_+ - a = g(a)$. Then

$$a_+a_- = f(a)g(a) = (fg)(a) = 0. \quad \square$$

2.3.5 Proposition. *Let \mathcal{A} be a C^* -algebra. If $a \in \mathcal{A}_{\text{sa}}$, then the following are equivalent:*

- (i) $a \geq 0$,
- (ii) $a = b^2$ for some self-adjoint $b \in \mathcal{A}$,
- (iii) $\|C \cdot 1 - a\| \leq C$ for all $C \geq \|a\|$,
- (iv) $\|C \cdot 1 - a\| \leq C$ for some $C \geq \|a\|$.

PROOF. Throughout this proof, we will work in \mathcal{A}_1 rather than \mathcal{A} , as the latter two conditions require it. The implication (i) \implies (ii) is Proposition 2.3.3. Suppose (ii) holds, i.e. that there exists a self-adjoint $b \in \mathcal{A}$ such that $a = b^2$, and fix $C \geq \|a\|$. Then $a = f(b)$, where $f \in C(\sigma(b))$ is the function $f(x) = x^2$. Consequently, $\|f\|_{C(\sigma(b))} = \|a\|$, and thus $0 \leq f \leq \|a\| \leq C$. Therefore, $0 \leq C - f \leq C$. Hence

$$\|C \cdot 1 - a\| = \|(C - f)(b)\| = \|C - f\|_{C(\sigma(b))} \leq C,$$

so (iii) holds, which clearly implies (iv). Now, suppose that (iv) holds for some $C \geq \|a\|$. Then

$$C \geq \|C \cdot 1 - a\| = \|(C - z)(a)\| = \|c - z\|_{C(\sigma(b))}.$$

Therefore, the identity function is non-negative on $\sigma(a)$, i.e. $\sigma(a) \subseteq [0, \infty)$, showing that a is positive. \square

2.3.6 Corollary. *Let \mathcal{A} be a C^* -algebra. Then \mathcal{A}_+ defines a closed cone in \mathcal{A} , i.e. \mathcal{A}_+ is a closed subset of \mathcal{A} , and it is closed under addition and multiplication by positive scalars. Furthermore, $\mathcal{A}_+ \cap (-\mathcal{A}_+) = \{0\}$.*

PROOF. It is clear from the definition of positivity that \mathcal{A}_+ is closed under multiplication by positive scalars. We will first show that \mathcal{A}_+ is closed. Since the involution on \mathcal{A} defines an isometry, \mathcal{A}_{sa} is a closed subset of \mathcal{A} . By the equivalence of condition (iii) in Proposition 2.3.5, it is clear that \mathcal{A}_+ is a closed subset of \mathcal{A}_{sa} . To show that \mathcal{A} is closed under addition, suppose that $a, b \in \mathcal{A}_+$. By the equivalence of condition (iv) in Proposition 2.3.5, there exist $C_1, C_2 \in \mathbb{R}$ such that $C_1 \geq \|a\|$, $C_2 \geq \|b\|$, $\|C_1 \cdot 1 - a\| \leq C_1$, and $\|C_2 \cdot 1 - b\| \leq C_2$. Then

$$\|(C_1 + C_2) \cdot 1 - (a + b)\| \leq \|C_1 \cdot 1 - a\| + \|C_2 \cdot 1 - b\| \leq C_1 + C_2,$$

so by the equivalence of condition (iv) in Proposition 2.3.5, $a + b \in \mathcal{A}_+$. Finally, suppose that $a \in \mathcal{A}_+ \cap (-\mathcal{A}_+)$. Then $\sigma(a) \subseteq [0, \infty)$ and $\sigma(-a) \subseteq [0, \infty)$, so $\sigma(a) = \{0\}$ and $\text{spr}(a) = 0$. However, since a is positive it is self-adjoint, so $\|a\| = \text{spr}(a) = 0$ and $a = 0$. \square

2.3.7 Proposition. *Let \mathcal{A} be a C^* -algebra. If $a \in \mathcal{A}$, then $a^*a \in \mathcal{A}_+$.*

PROOF. Let $b = a^*a$. Since b is self-adjoint, by Proposition 2.3.4 there exist $b_+, b_- \in \mathcal{A}_+$ such that $b = b_+ - b_-$ and $b_+b_- = 0$. Let c be the positive square root of b_- , and let $d = ac$. Since c was defined by applying a continuous function to a self-adjoint element b_- via the functional calculus, the Stone-Weierstrass Theorem implies that c is a limit of polynomials in b_- , which can be taken to have zero constant coefficient, as the square root function maps 0 to 0. Hence $cb_+ = b_-b_+ = 0$. Then

$$-d^*d = -ca^*ac = -c(b_+ - b_-)c = cb_-c = b_-^2.$$

In particular, $-d^*d$ is positive.

Now, let $d = x + iy$, where x and y are the real and imaginary parts of d . Then

$$d^*d + dd^* = (x + iy)^*(x + iy) + (x + iy)(x + iy)^* = 2(x^2 + y^2).$$

This is a sum of positive elements, so by Corollary 2.3.6 it is positive. Therefore,

$$dd^* = (d^*d + dd^*) - d^*d = (d^*d + dd^*) + b_-^2$$

is a sum of positive elements, and hence is positive. It is a standard fact from Banach algebra theory that

$$\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$$

for any two elements a, b in a Banach algebra. Thus,

$$\sigma(d^*d) \cup \{0\} = \sigma(dd^*) \cup \{0\}.$$

Since dd^* is positive, its spectrum is contained in $[0, \infty)$, so the same holds for d^*d and d^*d is positive. But we showed above that $-d^*d$ is also positive. Therefore, $b_-^2 = -d^*d = 0$. Since b_- is positive it has a unique positive square root, so this implies that $b_- = 0$, i.e. that b is positive. \square

If $a \in \mathcal{A}$, we call $(a^*a)^{1/2}$ the *absolute value* of \mathcal{A} , and denote it by $|a|$.

The positive cone \mathcal{A}_+ allows us to put an ordering on \mathcal{A}_{sa} by defining $a \leq b$ to mean $b - a \in \mathcal{A}_+$.

2.3.8 Corollary. *Let \mathcal{A} be a C^* -algebra. If $a \leq b$ in \mathcal{A}_{sa} and $x \in \mathcal{A}$, then $xa^*x \leq x^*bx$.*

PROOF. Let c be the positive square root of $b - a$. Then

$$x^*bx - x^*ax = x^*(b - a)x = (cx)^*(cx) \geq 0. \quad \square$$

2.3.9 Corollary. *Let \mathcal{A} be a C^* -algebra. If $0 \leq a \leq b$ are invertible in \mathcal{A}_{sa} , then $b^{-1} \leq a^{-1}$.*

PROOF. By the preceding corollary,

$$1 - b^{-1/2}ab^{-1/2} = b^{-1/2}(b - a)b^{-1/2} \geq 0.$$

Thus $(a^{1/2}b^{-1/2})^*(a^{1/2}b^{-1/2}) \leq 1$, so $\|a^{1/2}b^{-1/2}\| \leq 1$. The adjoint has the same norm, so

$$1 \geq (a^{1/2}b^{-1/2})(a^{1/2}b^{-1/2})^* = a^{1/2}b^{-1}a^{-1/2}.$$

Multiplying on both sides by $a^{-1/2}$ and applying the preceding corollary yields

$$a^{-1} = a^{-1/2}1a^{-1/2} \geq a^{-1/2}(a^{1/2}b^{-1}a^{1/2})a^{-1/2} = b^{-1}. \quad \square$$

There is one important caveat with the interaction between the ordering on \mathcal{A}_{sa} and the continuous functional calculus. If $a, b \in \mathcal{A}_{\text{sa}}$, $a \leq b$, and $f : \sigma(a) \cup \sigma(b) \rightarrow \mathbb{R}$ is a monotone increasing continuous function, then it is not necessarily the case that $f(a) \leq f(b)$, even when f is a reasonably nice function.

2.3.10 Example. Let $\mathcal{A} = M_2(\mathbb{C})$, and let

$$a = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}.$$

It is easy to see that $a, b \in \mathcal{A}_{\text{sa}}$, and that

$$b - a = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \geq 0,$$

so that $a \leq b$. However, if $\alpha > 1$, then $a^\alpha \not\leq b^\alpha$. In particular, $a^2 \not\leq b^2$.

2.3.11 Definition. Let E be a subset of \mathbb{R} . If $f : S \rightarrow \mathbb{R}$, we say that f is *operator monotone (increasing)* if for every C^* -algebra \mathcal{A} and $a, b \in \mathcal{A}_{\text{sa}}$ such that $a \leq b$ and $\sigma(a) \cup \sigma(b) \subseteq E$ we have that $f(a) \leq f(b)$.

The assumption that $\alpha > 1$ in Example 2.3.10 is necessary, as is shown by the following proposition.

2.3.12 Proposition. *Suppose that $0 < \alpha \leq 1$. Then the function $f_\alpha : [0, \infty) \rightarrow \mathbb{R}$ given by $f(x) = x^\alpha$ is operator monotone.*

PROOF. Let

$$S = \{a \in (0, \infty) : f_\alpha \text{ is operator monotone}\}.$$

Then $1 \in S$, S is a closed subset of $(0, \infty)$, and it is closed under multiplication. By possibly replacing the ambient C^* -algebra with its unitization, without loss of generality we may assume that the ambient C^* -algebra is unital. If we wish to show that $\alpha \in S$ we need only show that whenever $0 \leq a \leq b$ and b is invertible we have that $a^\alpha \leq b^\alpha$. Indeed, if $0 \leq a \leq b$ and b is not necessarily invertible, then $a^\alpha \leq (b + \epsilon \cdot 1)^\alpha$ for all $\epsilon > 0$, and thus $a^\alpha \leq \lim_{\epsilon \rightarrow 0} (b + \epsilon \cdot 1)^\alpha = b^\alpha$. We will make another observation to further reduce the claim that needs to be proven. If $a, b \geq 0$ and b is invertible, then $a^\alpha \leq b^\alpha$ precisely when $b^{-\alpha/2} a^\alpha b^{-\alpha/2} \leq 1$, i.e. when

$$\|b^{-\alpha/2} a^\alpha b^{-\alpha/2}\| \leq 1.$$

Finally, we again recall the fact from elementary Banach algebra theory that whenever x and y are elements in a Banach algebra,

$$\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}.$$

In particular, $\text{spr}(xy) = \text{spr}(yx)$.

We will show that if $\alpha, \beta \in S$, then $\gamma = (\alpha + \beta)/2 \in S$. If $0 \leq x \leq y$ and y is invertible, we have that

$$\begin{aligned} \|b^{-\gamma/2} a^\gamma b^{-\gamma/2}\| &= \text{spr}(a^\gamma b^{-\gamma}) \\ &= \text{spr}(a^{\alpha/2} a^{\beta/2} b^{-\alpha/2} b^{-\beta/2}) \\ &= \text{spr}(b^{-\beta/2} a^\gamma b^{-\alpha/2}) \\ &\leq \|(b^{-\beta/2} a^{\beta/2})(a^{\alpha/2} b^{-\alpha/2})\| \\ &\leq \|(b^{-\beta/2} a^{\beta/2})\| \|(a^{\alpha/2} b^{-\alpha/2})\| \\ &\leq 1. \end{aligned}$$

From these properties about S , to show that $(0, 1] \subseteq S$ it suffices to show that $1/2 \in S$. If $0 \leq a \leq b$ and b is invertible, then $b^{-1/2}ab^{-1/2} \leq 1$, so $\|b^{-1/2}a^{1/2}\| \leq 1$. Hence

$$\text{spr}(b^{-1/4}a^{1/2}b^{-1/4}) = \text{spr}(b^{-1/2}a^{1/2}) \leq \|b^{-1/2}a^{1/2}\| \leq 1,$$

so $b^{-1/4}a^{1/2}b^{-1/4} \leq 1$, i.e. $a^{1/2} \leq b^{1/2}$. Therefore, $1/2 \in S$, showing that $(0, 1] \subseteq S$ as desired. \square

Historical Notes

The results of this section have an interesting history. Combining Proposition 2.3.5 and Proposition 2.3.7, we have that if \mathcal{A} is a C^* -algebra and $a \in \mathcal{A}_{\text{sa}}$, then the following are equivalent:

- (i) $a \geq 0$,
- (ii) $a = b^2$ for some self-adjoint $b \in \mathcal{A}$,
- (iii) $a = b^*b$ for some $b \in \mathcal{A}$,
- (iv) $\|C \cdot 1 - a\| \leq C$ for some $C \geq \|a\|$.

The implication (i) \implies (ii) is from the continuous functional calculus, due to Gelfand and Naimark [GN43]. The implications (ii) \implies (iii) is obvious, and the equivalence of (ii) \iff (iv) was proven by Kelley and Vaught [KV53]. The final implication (iii) \implies (i) follows from a remark of Kaplansky, mentioned in a review by Schatz [Sch54] of a paper of Fukiyama [Fuk52].

As mentioned in the historical remarks to Section 2.1, the definition of a C^* -algebra originally included an additional axiom, that $1 + a^*a$ is invertible. This statement follows from the equivalences mentioned above, so this axiom was no longer needed in the definition of a C^* -algebra.

Corollary 2.3.6, which states that $\mathcal{A}_+ \cap -\mathcal{A}_+ = \{0\}$ can be rewritten in the more concrete terms: if $a, b \in \mathcal{A}$ and $a^*a + b^*b = 0$, then $a = b = 0$. Fukiyama [Fuk52] showed that for a Banach $*$ -algebra, this property is equivalent to the C^* -norm condition that $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}$.

2.4 Bounded Approximate Identities

2.4.1 Definition. Let \mathcal{A} be a Banach algebra. A *left (resp. right) bounded approximate identity* for \mathcal{A} is a bounded net $(e_i)_{i \in I}$ such that $(e_i a)_{i \in I}$ (resp. $(a e_i)_{i \in I}$) converges to a for every $a \in \mathcal{A}$. We say that $(e_i)_{i \in I}$ is a *bounded approximate identity* if it is both a left and right bounded approximate identity. In any of the above cases, if $\|e_i\| \leq 1$ for all $i \in I$, we say that $(e_i)_{i \in I}$ is a *contractive*.

Some authors require a bounded approximate identity to be contractive, but we do not here, as there are many interesting Banach algebras with a bounded approximate identity but no contractive bounded approximate identity. Also, if \mathcal{A} is a C^* -algebra, some authors require that a bounded approximate identity be an increasing net of positive elements, but we do not.

2.4.2 Example. Let X be a locally compact Hausdorff space. Then a bounded approximate identity for $C_0(X)$ can be obtained by considering all positive functions of norm less than one, ordered by the usual ordering on $C_0(X)_{\text{sa}}$.

With some effort, we can use the continuous functional calculus to make this same method of constructing a bounded approximate identity work for an arbitrary C^* -algebra.

2.4.3 Proposition. *Let \mathcal{A} be a C^* -algebra. Then $\mathcal{A}_+ \cap \{a \in \mathcal{A} : \|a\| < 1\}$ is a bounded approximate identity of \mathcal{A} when it is given the ordering on self-adjoint elements of \mathcal{A} .*

PROOF. Let $\Lambda = \mathcal{A}_+ \cap \{a \in \mathcal{A} : \|a\| < 1\}$. It is not immediately clear that Λ is even a directed set. We will show this by establishing an order isomorphism of Λ with \mathcal{A}_+ , which is clearly a directed set. When \mathcal{A} is simply the complex numbers, $\mathcal{A}_+ = [0, \infty)$ and $\Lambda = [0, 1)$. In this case, the order isomorphism is established by the continuous functions $f : [0, 1) \rightarrow \mathbb{R}$ and $g : [0, \infty) \rightarrow \mathbb{R}$ given by

$$f(t) = (1 - t)^{-1} \quad \text{and} \quad g(t) = 1 - (1 + t)^{-1}.$$

For every $t \in [0, 1)$ we have

$$\begin{aligned} g(f(t)) &= 1 - (1 + f(t))^{-1} \\ &= 1 - (1 + (1 - t)^{-1} - 1)^{-1} \\ &= 1 - ((1 - t)^{-1})^{-1} \\ &= 1 - (1 - t) \\ &= t. \end{aligned}$$

Similarly, for every $t \in [0, \infty)$,

$$\begin{aligned} f(g(t)) &= (1 - g(t))^{-1} - 1 \\ &= (1 - (1 - (1 + t)^{-1}))^{-1} - 1 \\ &= ((1 + t)^{-1})^{-1} - 1 \\ &= 1 + t - 1 \\ &= t. \end{aligned}$$

Since $f(0) = g(0) = 0$, in \mathcal{A}_+ we have $f(\Lambda) \subseteq \mathcal{A}$ and $g(\mathcal{A}_+) \subseteq \mathcal{A}$. If $0 \leq s \leq t < 1$ then $f(s) \leq f(t)$, and if $0 \leq s \leq t < \infty$ then $g(s) \leq g(t)$. As we have

seen by example, this does not necessarily imply the same order properties when f and g are applied to members of \mathcal{A}_{sa} . However, by Corollary 2.3.9 and a few simple algebraic manipulations, if $0 \leq a \leq b < 1$ then $f(a) \leq f(b)$, and if $0 \leq a \leq b$ then $g(a) \leq g(b)$. Therefore, f and g establish an order isomorphism between Λ and \mathcal{A}_+ via the continuous functional calculus.

We now show the convergence criterion making Λ an approximate identity. If $a, b \in \Lambda$ are such that $a \leq b$ and $x \in \mathcal{A}$, then in \mathcal{A}_I ,

$$\|x - bx\|^2 = \|x^*(1 - b)^2x\| \leq \|x^*(1 - b)x\| \leq \|x^*(1 - a)x\|.$$

Similarly,

$$\|x - xb\|^2 \leq \|x(1 - a)x^*\|.$$

We will first prove the result for positive elements of \mathcal{A} and then derive it for arbitrary elements. If $x \in \mathcal{A}$ is positive, let $a_n = g(nx)$. We have that $a_n \in \mathcal{A}$ by the same reasoning as before. For $n \in \mathbb{N}$ define $h_n : [0, \infty) \rightarrow \mathbb{R}$ by

$$h(t) = t^2(1 - g(nt)) = t^2(1 + nt)^{-1}.$$

Clearly, $h_n(t) \leq t/n$, so

$$\|x(1 - a_n)x\| = \|h(x)\| \leq \|h\|_{C_0(\sigma(x))} \leq \|x\|/n.$$

Hence

$$\begin{aligned} \lim_{b \in \Lambda} \|x - bx\|^2 &\leq \lim_{n \rightarrow \infty} \sup_{\substack{b \in \Lambda \\ b \geq a_n}} \|x - bx\|^2 \\ &\leq \lim_{n \rightarrow \infty} \|x(1 - a_n)x\| \\ &= 0. \end{aligned}$$

A similar argument shows that

$$\lim_{b \in \Lambda} \|x - xb\|^2 = 0.$$

If $x \in \mathcal{A}$ is not necessarily positive, then we have

$$\|x - xb\|^2 = \|(1 - b)x^*x(1 - b)\| \leq \|x^*x - x^*xb\|,$$

allowing us to apply the above to the positive element x^*x , showing that

$$\lim_{b \in \Lambda} \|x - xb\|^2 \leq \lim_{b \in \Lambda} \|x^*x - x^*xb\|^2 = 0.$$

Similarly,

$$\lim_{b \in \Lambda} \|x - bx\|^2 = 0.$$

Therefore, Λ is a bounded approximate identity for \mathcal{A} . □

Occasionally, when \mathcal{A} is separable it is useful to have a sequence that is a bounded approximate identity. The following corollary is an easy consequence of the preceding theorem.

2.4.4 Corollary. *Let \mathcal{A} be a separable C^* -algebra. Then \mathcal{A} has a contractive bounded approximate identity that is an increasing sequence in \mathcal{A}_+ . \square*

Another important class of Banach algebras with bounded approximate identities is group algebras.

2.4.5 Proposition. *Let G be a locally compact group. Then $L^1(G)$ has a contractive approximate identity.*

PROOF. Let $(U_i)_{i \in I}$ be the directed system of all compact neighbourhoods of the identity, ordered by reverse inclusion. For every $i \in I$ let e_i be a positive function supported in U_i such that $e_i(s^{-1}) = e_i(s)$ and $\int e_i(s) ds = 1$. Fix $f \in C_c(G)$ supported on K and $\epsilon > 0$. Then there exists a compact neighbourhood U of the identity such that $|f(ts) - f(s)| < \epsilon$ whenever $t \in U$. Thus for all $i \in I$ such that $U_i \subseteq U$,

$$\begin{aligned} \|e_i * f - f\| &= \int |(e_i * f)(s) - f(s)| ds \\ &= \int \left| \int e_i(t)(f(ts) - f(s)) dt \right| ds \\ &\leq \iint_U e_i(t) |f(ts) - f(s)| dt ds \\ &= \int_U e_i(t) \int |f(ts) - f(s)| dx dt \\ &\leq \int e_i(t) \mu(U^{-1}K) \epsilon dy \\ &= \mu(U^{-1}K) \epsilon. \end{aligned}$$

Now, if $f \in L^1(G)$ choose $f_0 \in C_c(G)$ such that $\|f - f_0\| < \epsilon/3$. Then

$$\begin{aligned} \|e_i * f - f\| &\leq \|e_i\| \cdot \|f - f_0\| + \|e_i * f_0 - f_0\| + \|f_0 - f\| \\ &< \frac{2\epsilon}{3} + \|e_i * f_0 - f_0\| \\ &< \epsilon \end{aligned}$$

for sufficiently large $i \in I$, showing that $(e_i f)_{i \in I}$ converges to f . The argument showing that $(f e_i)_{i \in I}$ converges to f is similar. \square

If \mathcal{A} is a unital algebra, then every element of \mathcal{A} trivially factors as the product of two elements. By a remarkable theorem due to Cohen, this is also

true for all Banach algebras with a bounded left approximate identity. Cohen's theorem was soon recognized (in some part by Cohen himself, as well as Hewitt) to really be a theorem about Banach modules, which is the version we present here.

2.4.6 Definition. Let \mathcal{A} be a Banach algebra. A left normed \mathcal{A} -module is a normed space together with a map $\mathcal{A} \times X \rightarrow X$ (written $a \cdot x$) such that

- (i) $(a + b) \cdot x = a \cdot x + b \cdot x$ and $a \cdot (x + y) = a \cdot x + a \cdot y$;
- (ii) $(\alpha a) \cdot x = \alpha(a \cdot x) = a \cdot (\alpha x)$;
- (iii) $(ab) \cdot x = a \cdot (b \cdot x)$;
- (iv) $\|a \cdot x\| \leq M \cdot \|a\| \cdot \|x\|$ for some constant $M \geq 1$.

If X is complete then it is called a left Banach \mathcal{A} -module.

If \mathcal{A} is not unital, we can extend a left Banach \mathcal{A} -module to a left Banach \mathcal{A}_I -module by the action $(a + \alpha 1)x = ax + \alpha x$.

2.4.7 Definition. Let \mathcal{A} be a Banach algebra and let X be a left Banach \mathcal{A} -module. Then the closed linear subspace of X spanned by

$$\mathcal{A}X = \{a \cdot x : a \in \mathcal{A}, x \in X\}$$

is called the *essential part* of X and is denoted by X_e . If $X = X_e$, then X is said to be *essential*.

2.4.8 Proposition. Let \mathcal{A} be a Banach algebra with a bounded left approximate identity $(e_i)_{i \in I}$, and let X be a left Banach \mathcal{A} -module. Then

$$X_e = \overline{\mathcal{A}X} = \left\{ x \in X : \lim_{i \in I} e_i \cdot x = x \right\}.$$

PROOF. Fix $x \in X_e$. For every $\epsilon > 0$, there exist $a_1, \dots, a_n \in \mathcal{A}$, $x_1, \dots, x_n \in X$ such that

$$\left\| x - \sum_{k=1}^n a_k x_k \right\| < \epsilon.$$

Since

$$\lim_{i \in I} e_i \left(\sum_{k=1}^n a_k x_k \right) = \sum_{k=1}^n \lim_{i \in I} (e_i a_k) x_k = \sum_{k=1}^n a_k x_k$$

and $(e_i)_{i \in I}$ is bounded it follows that $\lim_{e_i x} = x$. □

2.4.9 Theorem (Cohen-Hewitt Factorization Theorem). Let \mathcal{A} be a Banach algebra with a left approximate identity bounded by $K \geq 1$, and let X be a left Banach \mathcal{A} -module. Then for every $z \in X_e$ and $\delta > 0$ there exist elements $a \in \mathcal{A}$ and $y \in X$ such that $z = ay$, $\|a\| \leq K$, $y \in \overline{\mathcal{A}z}$, and $\|y - z\| < \delta$.

PROOF. Let $C > 0$ be such that

$$0 < K < \frac{1-C}{C}.$$

For example, take $C = 1/(K+2)$. If $e \in \mathcal{A}$ and $\|e\| \leq K$ define

$$\varphi(e) = (1 - C + Ce)^{-1}.$$

Then $\varphi(e)$ is a well-defined element in \mathcal{A}_I since

$$1 - C + Ce = (1 - C)\left(1 + \frac{C}{1 - C}e\right)$$

and

$$\left\|1 - \left(1 + \frac{C}{1 - C}e\right)\right\| = \frac{C}{1 - C}\|e\| \leq \frac{C}{1 - C}K < 1.$$

The inequality

$$\begin{aligned} (1 - C - CK)\|\varphi(e)\| &\leq \|(1 - C)\varphi(e)\| - \|Ce\varphi(e)\| \\ &\leq \|(1 - C + Ce)\varphi(e)\| = 1 \end{aligned}$$

implies that

$$\|\varphi(e)\| \leq (1 - C - CK)^{-1}$$

Furthermore, since

$$\varphi(e)x - x = \varphi(e)(x - (1 - C + Ce)x) = C\varphi(e)(x - ex),$$

we have

$$\|\varphi(e)x - x\| \leq M(1 - C - CK)^{-1}\|ex - x\|$$

for every $x \in X$.

We shall inductively choose a sequence $(e_n)_{n \in \mathbb{N}}$ in \mathcal{A} such that $\|e_n\| \leq K$. Assuming we have such a sequence, let $a_1 = 1$ and

$$a_n = (1 - C)^n + \sum_{k=1}^n C(1 - C)^{k-1}e_k.$$

Then

$$\varphi(e_{n+1})a_{n+1} = (1 - C)^n + \sum_{k=1}^n C(1 - C)^{k-1}\varphi(e_{n+1})e_k,$$

$$\varphi(e_{n+1})a_{n+1} - a_n = \sum_{k=1}^n C(1 - C)^{k-1}(\varphi(e_{n+1})e_k - e_k),$$

and so

$$\|\varphi(e_{n+1})a_{n+1} - a_n\| \leq M(1 - C - CK)^{-1} \sum_{k=1}^n \|e_{n+1}e_k - e_k\|.$$

If a_n^{-1} exists and the right-hand side of this inequality is sufficiently small, then $\|\varphi(e_{n+1})a_{n+1}\|$ will be invertible, and thus a_{n+1} will be invertible. Let $y_n = a_n^{-1}z$. Then we have

$$\begin{aligned} \|\mathcal{Y}_{n+1} - \mathcal{Y}_n\| &\leq \|(\varphi(e_{n+1})a_{n+1})^{-1}\varphi(e_{n+1})z - a_n^{-1}\varphi(e_{n+1})z\| \\ &\quad + \|a_n^{-1}\varphi(e_{n+1})z - a_n^{-1}z\| \\ &\leq M\|(\varphi(e_{n+1})a_{n+1})^{-1} - a_n^{-1}\|\|\varphi(e_{n+1})z\| \|z\| \\ &\quad + M\|a_n^{-1}\|\|\varphi(e_{n+1})z - z\| \\ &\leq M(1 - C - CK)^{-1}\|z\|\|(\varphi(e_{n+1})a_{n+1})^{-1} - a_n^{-1}\| \\ &\quad + M^2(1 - C - CK)^{-1}\|a_n^{-1}\|\|e_{n+1}z - z\|. \end{aligned}$$

Thus, inductively choosing e_{n+1} in \mathcal{A} with $\|e_{n+1}\| \leq K$ such that $\|e_{n+1}e_k - e_k\|$ (for $1 \leq k \leq n$) and $\|e_{n+1}z - z\|$ are sufficiently small, we can conclude that a_{n+1}^{-1} exists in \mathcal{A}_I and

$$\|\mathcal{Y}_{n+1} - \mathcal{Y}_n\| < \frac{\delta}{2^n}.$$

Then

$$\sum_{n=1}^{\infty} \|\mathcal{Y}_{n+1} - \mathcal{Y}_n\| < \sum_{n=1}^{\infty} \frac{\delta}{2^n} = \delta.$$

It follows that (\mathcal{Y}_n) is a Cauchy sequence in X and thus has a limit y . We claim that $y \in \overline{\mathcal{A}z}$. Indeed, $y_n = a_n^{-1}z$ and $a_n^{-1} \in \mathcal{A}_I$, so $y_n \in \mathcal{A}_I z$ and hence $y \in \overline{\mathcal{A}_I z}$. But $\overline{\mathcal{A}_I z} = \overline{\mathcal{A}z} + Cz$ and $z \in \overline{\mathcal{A}z}$, so $y \in \overline{\mathcal{A}z}$.

Define

$$a = \sum_{k=1}^{\infty} C(1 - C)^{k-1}e_k.$$

This series converges because $\|e_k\| \leq K$. Clearly, $a \in A$,

$$\|a\| \leq \sum_{k=1}^{\infty} C(1 - C)^{k-1}K = K,$$

and since $a = \lim a_n$,

$$z = \lim_{n \rightarrow \infty} a_n(a_n^{-1}z) = ay. \quad \square$$

In some applications of factorization, it is necessary to be able to factor multiple elements of the module as a product with the same element of the algebra. The following theorem was originally proven as a special case, but it follows readily from the general factorization theorem for Banach modules.

2.4.10 Corollary. *Let \mathcal{A} be a Banach algebra with a left approximate identity bounded by $K \geq 1$, and let X be a left Banach \mathcal{A} -module. Then for every sequence $(z_n)_{n=1}^{\infty} \in X_e$ converging to 0 and $\delta > 0$ there exists an $a \in \mathcal{A}$ and a sequence $(y_n)_{n \in \mathbb{N}} \in X$ converging to 0 such that $z_n = ay_n$, $\|a\| \leq K$, $y_n \in \overline{\mathcal{A}z_n}$, and $\|y_n - z_n\| < \delta$.*

PROOF. Define a new left Banach \mathcal{A} module

$$c_0(X) = \{(x_n)_{n=1}^\infty \subseteq X : \lim_{n \rightarrow \infty} x_n = 0\}$$

with norm

$$\|(x_n)_{n=1}^\infty\|_\infty = \sup_{n \in \mathbb{N}} \|x_n\|$$

and module action

$$a \cdot (x_n)_{n=1}^\infty = (a \cdot x_n)_{n=1}^\infty.$$

Then the desired conclusion follows from the Factorization Theorem Theorem 2.4.9 applied to $c_0(X)$. \square

Historical Notes

Theorem 2.4.9 was originally proven by Cohen [Coh59] in the case of a Banach algebra acting on itself. Various generalizations to Banach modules were achieved by many people independently, including Hewitt [Hew64] and Curtis and Figa-Talamanca [CFT66]. The factorization theorem for null sequences, Corollary 2.4.10, was proven by Varopoulos [Var64].

2.5 Homomorphisms, Ideals, and Quotients

One of the most important results about Banach $*$ -algebras is that any $*$ -homomorphism whose codomain is a C^* -algebra is automatically continuous, and in fact is a contraction. The proof is an easy application of the continuous functional calculus.

2.5.1 Proposition. *Let \mathcal{A} be a Banach $*$ -algebra and \mathcal{B} a C^* -algebra. If $\pi : \mathcal{A} \rightarrow \mathcal{B}$ is a $*$ -homomorphism, then $\|\pi(a)\| \leq \|a\|$ for all $a \in \mathcal{A}$.*

PROOF. By Corollary 2.2.5, $\|b\| = \text{spr}(b)$ for every normal $b \in \mathcal{B}$. For every $a \in \mathcal{A}$ we have that $\sigma_{\mathcal{B}}(\pi(a)) \subseteq \sigma_{\mathcal{A}}(a)$, so

$$\text{spr}(\pi(a)) \leq \text{spr}(a) \leq \|a\|.$$

This implies that

$$\begin{aligned} \|\pi(a)\|^2 &= \|\pi(a)^* \pi(a)\| \\ &= \|\pi(a^* a)\| \\ &= \text{spr}(\pi(a^* a)) \\ &\leq \|a^* a\| \\ &\leq \|a^*\| \|a\| \\ &= \|a\|^2. \end{aligned}$$

Therefore, $\|\pi(a)\| \leq \|a\|$ for all $a \in \mathcal{A}$. \square

2.5.2 Corollary. *Let \mathcal{A} and \mathcal{B} be C^* -algebras. If $\pi : \mathcal{A} \rightarrow \mathcal{B}$ is an injective $*$ -homomorphism then π is an isometry.*

PROOF. By the preceding proposition, π is a contraction. Suppose π is not isometric. Then there is an $a \in \mathcal{A}$ such that $\|\pi(a)\| < \|a\|$. Let $r = \|\pi(a^*a)\|$ and $s = \|a^*a\|$. Then

$$r = \|\pi(a^*a)\| = \|\pi(a)\|^2 < \|a\|^2 = \|a^*a\| = s.$$

Let $f \in C([0, s])$ be such that $f(t) = 0$ for $0 \leq t \leq r$ and $f(s) = 1$. Therefore, by the continuous functional calculus,

$$0 = f(\pi(a^*a)) = \pi(f(a^*a)).$$

Since f does not vanish on $\sigma(a^*a)$, $f(a^*a) \neq 0$. But this contradicts the injectivity of π . Therefore, our assumption that π is not isometric is false. \square

Whenever we speak of an ideal of a C^* -algebra, we mean a norm closed ideal, and we will assume that all ideals are two-sided unless explicitly stated otherwise.

2.5.3 Proposition. *Every ideal of a C^* -algebra is self-adjoint.*

PROOF. Let \mathcal{A} be a C^* -algebra, and \mathcal{J} be an ideal of \mathcal{A} . Define $\mathcal{B} = \mathcal{J} \cap \mathcal{J}^*$. Then \mathcal{B} is a C^* -subalgebra of \mathcal{A} , so by Proposition 2.4.3, \mathcal{B} contains a bounded approximate identity $(e_i)_{i \in I}$. If $a \in \mathcal{J}$ then $aa^* \in \mathcal{B}$, so

$$\lim_{i \in I} \|a^* - a^*e_i\|^2 = \lim_{i \in I} \|(aa^* - aa^*e_i) - e_i(aa^* - aa^*e_i)\| = 0.$$

Since $(e_i)_{i \in I}$ belongs to $\mathcal{B} \subseteq \mathcal{J}$, it follows that \mathcal{J}^* also belongs to \mathcal{J} . \square

If \mathcal{A} is a C^* -algebra and \mathcal{J} is an ideal of \mathcal{A} then \mathcal{A}/\mathcal{J} is easily seen to be a Banach $*$ -algebra, where the involution on \mathcal{A}/\mathcal{J} is defined by $(a + \mathcal{J})^* = a^* + \mathcal{J}$, but it is actually also a C^* -algebra.

2.5.4 Lemma. *Let \mathcal{A} be a C^* -algebra, \mathcal{J} be an ideal of \mathcal{A} , and $(e_i)_{i \in I}$ a bounded approximate identity for \mathcal{J} . Then*

$$\|a + \mathcal{J}\| = \lim_{i \in I} \|a - ae_i\|.$$

PROOF. Since $ae_i \in \mathcal{J}$, $\|a + \mathcal{J}\| \leq \|a - ae_i\|$. Towards the reverse inequality, if $\epsilon > 0$ there is a $b \in \mathcal{J}$ such that $\|a - b\| < \|a + \mathcal{J}\| + \epsilon$, so that in \mathcal{A}_1 ,

$$\begin{aligned} \lim_{i \in I} \|a - ae_i\| &\leq \lim_{i \in I} \|(a - b)(1 - e_i)\| + \|b - be_i\| \\ &\leq \|a - b\| \\ &< \|a + \mathcal{J}\| + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, the claim is established. \square

2.5.5 Proposition. *Let \mathcal{A} be a C^* -algebra and \mathcal{J} an ideal of \mathcal{A} . Then the quotient Banach algebra \mathcal{A}/\mathcal{J} is a C^* -algebra.*

PROOF. Let $(e_i)_{i \in I}$ be a bounded approximate identity for \mathcal{J} . We only need to verify the C^* -norm condition. If $a \in \mathcal{A}$, using the lemma and working in \mathcal{A}_I , we have that

$$\begin{aligned} \|(a + \mathcal{J})^*(a + \mathcal{J})\| &= \lim_{i \in I} \|a^* a (1 - e_i)\| \\ &\geq \lim_{i \in I} \|(1 - e_i) a^* a (1 - e_i)\| \\ &= \lim_{i \in I} \|a(1 - e_i)\|^2 \\ &= \|a + \mathcal{J}\|^2. \end{aligned}$$

Therefore, since the other direction of the equality holds in any Banach $*$ -algebra, \mathcal{A}/\mathcal{J} is a C^* -algebra. \square

The preceding proposition allows us to show that $*$ -homomorphisms between C^* -algebras have a particularly rigid structure. In particular, the image of a C^* -algebra via a $*$ -homomorphism is always closed, and thus a C^* -subalgebra of the codomain, and the $*$ -homomorphism factors as the composition of a quotient map and an isometry.

2.5.6 Proposition. *Let \mathcal{A} and \mathcal{B} be C^* -algebras, and let $\pi : \mathcal{A} \rightarrow \mathcal{B}$ be a nonzero $*$ -homomorphism. Then $\|\pi\| = 1$, $\pi(\mathcal{A})$ is a C^* -subalgebra of \mathcal{B} , and π factors as $\pi_0 \circ q$, where $q : \mathcal{A} \rightarrow \mathcal{A}/\ker(\pi)$ is the canonical quotient map and π_0 is the induced isometric $*$ -isomorphism of $\mathcal{A}/\ker(\pi)$ onto $\pi(\mathcal{A})$.*

PROOF. It is easy to verify that the described factorization exists and that π_0 is a $*$ -isomorphism. Since π_0 is injective, it is isometric by Corollary 2.5.2. It follows that $\pi(\mathcal{A})$ is closed, and thus is a C^* -subalgebra of \mathcal{B} . Since $\|\pi_0\| = 1$ and q is a quotient map of Banach spaces, it follows that $\|\pi\| = 1$. \square

Historical Notes

The results on the structure of $*$ -homomorphisms between C^* -algebras and the existence of quotients of C^* -algebras are due to Segal [Seg49].

2.6 Unitaries, Projections, and Partial Isometries

If \mathcal{A} is a unital C^* -algebra, the unitary elements of \mathcal{A} form a subgroup $\mathcal{U}(\mathcal{A})$ of \mathcal{A}^{-1} , which we will call the *unitary group* of \mathcal{A} .

2.6.1 Proposition. *Let \mathcal{A} be a unital C^* -algebra. Then every element of \mathcal{A} is the linear combination of four unitary elements.*

PROOF. Since every element of \mathcal{A} is the sum of its real and imaginary parts, it suffices to show that every self-adjoint element $a \in \mathcal{A}$ is the sum of two unitary elements. Without loss of generality, we may assume that $\|a\| \leq 1$. Using the continuous functional calculus, define

$$u = a + i(1 - a^2)^{\frac{1}{2}}.$$

It is easy to see that u is unitary and $a = \frac{1}{2}(u + u^*)$. □

Projections and unitaries on a Hilbert space \mathcal{H} both have the property that they define isometries when restricted to subspaces of \mathcal{H} .

2.6.2 Definition. Let \mathcal{A} be a unital C^* -algebra. We say that $v \in \mathcal{A}$ is a *partial isometry* if both v^*v and vv^* are projections, in which case we call v^*v the *initial* or *source projection* of v , and vv^* the *final* or *range projection* of v .

2.6.3 Example. Let \mathcal{H} be a Hilbert space and $v \in \mathcal{B}(\mathcal{H})$ a partial isometry. Then v^*v is the projection onto $\ker(v)^\perp$, and vv^* is the projection onto $v\mathcal{H}$.

2.6.4 Proposition. *Let \mathcal{A} be a unital C^* -algebra. If $v \in \mathcal{A}$, then the following are equivalent:*

- (i) v is a partial isometry;
- (ii) v^* is a partial isometry;
- (iii) v^*v is a projection;
- (iv) vv^* is a projection.

PROOF. Clearly, (i) and (ii) are equivalent, and (i) implies both (iii) and (iv). We will show that (iii) \implies (i). Then (iv) \implies (i) follows by exchanging v and v^* . Suppose that v^*v is a projection. Then we have

$$(vv^*)^2 = v(v^*v)v^* = v(v^*v)(v^*v)v^* = (vv^*)^3.$$

Since vv^* is self-adjoint, the continuous functional calculus implies that vv^* is a projection. □

Since the partial isometries in \mathbb{C} are simply the complex numbers of modulus 1, every complex number α can be written uniquely as $v|\alpha|$, where v is a partial isometry. There is a similar decomposition for operators, but in order to specify the correct uniqueness condition, we need to introduce the notion of the support projection of an operator.

If $a \in \mathcal{B}(\mathcal{H})$, we call the projection onto $\ker(a)^\perp$ the *right support projection* of a and the projection onto $\overline{a\mathcal{H}}$ the *left support projection* of a . We denote the right support projection by $s_r(a)$ and the left support projection by $s_l(a)$. It is easily checked that $s_r(a)$ is the least projection p in $\mathcal{B}(\mathcal{H})$ such that $ap = a$, and that $s_l(a)$ is the least projection p in $\mathcal{B}(\mathcal{H})$ such that $pa = p$. It follows that $s_r(a) = s_l(a^*)$, and that $s_r(a) = s_l(a)$ if a is self-adjoint. In this case, we simply call this projection the *support projection* of a , and denote it by $s(a)$. It is also easy to check that $s_r(a) = s(a^*a)$ and $s_l(a) = s(aa^*)$ for all $a \in \mathcal{B}(\mathcal{H})$.

2.6.5 Proposition (Polar Decomposition). *Let \mathcal{H} be a Hilbert space. If $a \in \mathcal{B}(\mathcal{H})$, then there exists a partial isometry $v \in \mathcal{B}(\mathcal{H})$ such that $a = v|a|$, $v^*v = s(|a|) = s_r(a)$, and $vv^* = s_l(a)$. If a is invertible, then we may take v to be unitary and $v \in C^*(a)$. Moreover, this decomposition is unique: if $w \in \mathcal{B}(\mathcal{H})$ is a partial isometry and $b \in \mathcal{B}(\mathcal{H})$ is a positive operator such that $a = wb$ and $v^*v = s(b)$, then $w = v$ and $b = |a|$.*

PROOF. Define an operator v_0 on the subspace $|a|\mathcal{H}$ of \mathcal{H} by

$$v_0(|a|\xi) = a\xi.$$

If $\xi \in \mathcal{H}$, then

$$\|v_0(|a|\xi)\|^2 = \|a\xi\|^2 = \langle a^*a\xi | \xi \rangle = \langle |a|^2\xi | \xi \rangle = \||a|\xi\|^2,$$

so v_0 can be extended, in a unique manner, to an isometry defined on the subspace $\overline{|a|\mathcal{H}} = s(|a|)\mathcal{H}$ of \mathcal{H} . It is then possible to define a partial isometry v on all of \mathcal{H} by

$$v\xi = \begin{cases} v_0\xi & \text{if } \xi \in s(|a|)\mathcal{H}, \\ 0 & \text{if } \xi \in (s(|a|)\mathcal{H})^\perp. \end{cases}$$

It is clear that $a = v|a|$,

$$v^*v = s_r(v) = s(|a|) = s_r(a) \quad \text{and} \quad vv^* = s_l(v) = s_l(a).$$

If a is invertible, then so is $|a| = (a^*a)^{1/2}$. Thus

$$v = a|a|^{-1} = a(a^*a)^{-1/2} \in C^*(a).$$

Since

$$v(|a|a^{-1}) = aa^{-1} = 1 \quad \text{and} \quad (|a|a^{-1})v = |a|a^{-1}a|a|^{-1} = 1,$$

v is invertible, and

$$v^*v = (a^*a)^{-1/2}(a^*a)(a^*a)^{-1/2} = 1,$$

so v is unitary.

Suppose $a = wb$, where w is a partial isometry such that $v^*v = s(b)$ and b is positive. Then

$$a^*a = b^*w^*wb = b^*b = b^2.$$

By the uniqueness of square roots, $|a| = (a^*a)^{1/2} = b$. If $\xi \in \mathcal{H}$, then

$$w(|a|\xi) = w(b\xi) = a\xi = v(|a|\xi),$$

so w agrees with v on the subspace $\overline{|a|\mathcal{H}} = s(|a|)\mathcal{H}$ of \mathcal{H} . Therefore,

$$w = ws_r(w) = ww^*w = ws(b) = ws(|a|) = vs(|a|) = vv^*v = vs_r(v) = v,$$

showing the uniqueness of the decomposition $a = v|a|$ of a . \square

We call the decomposition $a = v|a|$ in the preceding theorem the *polar decomposition* of a .

2.6.6 Examples.

- (i) It is necessary that we consider general partial isometries in the polar decomposition, not only unitaries. Indeed, let \mathcal{H} be a separable Hilbert space with orthonormal basis $(e_n)_{n=1}^\infty$. Let $a \in \mathcal{B}(\mathcal{H})$ be the right shift operator defined by $ae_n = e_{n+1}$. Then $(a^*a)^{1/2} = 1^{1/2} = 1$, so if $v \in \mathcal{B}(\mathcal{H})$ satisfies $a = v|a|$, we must have that $v = a$. However, a is not unitary, because $aa^* \neq 1$.
- (ii) If \mathcal{A} is a C^* -subalgebra of $\mathcal{B}(\mathcal{H})$ and $a \in \mathcal{A}$, then

$$|a| = (a^*a)^{1/2} \in \mathcal{A}$$

by the continuous functional calculus, but if $a = v|a|$ is the polar decomposition of a , it is not necessarily the case that $v \in \mathcal{A}$ as well. Indeed, consider $L^\infty([0, 1])$ acting on $L^2([0, 1])$ by left multiplication. If $f \in L^\infty([0, 1])$ is positive, then it is easily seen that $s(f)$ is the characteristic function of *essential support* of f , i.e. the complement of the union of all open subsets G of $[0, 1]$ such that f vanishes almost everywhere on G . It is easy to find continuous functions on $[0, 1]$ with an essential support that is neither \emptyset or $[0, 1]$, but $C([0, 1])$ does not contain any projections besides 0 and 1.

Partial isometries are intrinsically linked to the metric structure of a C^* -algebra. Recall that if X is a locally compact Hausdorff space, then the unit ball of $C_0(X)$ has an extreme point if and only if X is compact, i.e. if and only if $C_0(X)$ is unital, and that the extreme points are precisely the functions f such that $|f(x)| = 1$ for all $x \in X$, which are the partial isometries (or equivalently, unitaries) of $C_0(X)$. This characterization of when a C^* -algebra has extreme points is true for all C^* -algebras, and all of the extreme points

are partial isometries, but the exact description of the extreme points differs when the algebra is noncommutative.

One direction of our characterization of when the unit ball of a C^* -algebra is valid for all Banach algebras.

2.6.7 Proposition. *Let \mathcal{A} be a unital Banach algebra such that $\|1\| = 1$. Then 1 is an extreme point of the unit ball of \mathcal{A} .*

PROOF. By Proposition 2.1.11, there is a Banach space \mathcal{X} such that \mathcal{A} is isometrically isomorphic to a unital subalgebra of $\mathcal{B}(\mathcal{X})$. We will simply view \mathcal{A} as a subalgebra of $\mathcal{B}(\mathcal{X})$. To show that 1 is an extreme point of the unit ball of \mathcal{A} , it suffices to show that 1 is an extreme point of the unit ball of $\mathcal{B}(\mathcal{X})$. If $\|1 \pm a\| \leq 1$, then $\|1^* \pm a^*\| \leq 1$ by the properties of adjoints. If $f \in \mathcal{X}^*$, define

$$f_1 = (1^* + a^*)f \quad \text{and} \quad f_2 = (1^* - a^*)f,$$

so that $f = 1/2(f_1 + f_2)$. Since \mathcal{X}^* is a dual space, by the Krein-Milman Theorem its unit ball has an extreme point. If f is an extreme point of the unit ball of \mathcal{X}^* , then $f = f_1 = f_2$. Hence $a^*f = 0$, and so $a^* = 0$ and $a = 0$.

Now, suppose that $b, c \in \mathcal{A}$ are such that $\|b\| \leq 1$, $\|c\| \leq 1$, and $1 = 1/2(b + c)$. Let $a = 1 - b$. Then $1 - a = b$ and $1 + a = 2 \cdot 1 - b = c$. Therefore, by the above, $b = 1$ and $c = 1$, showing that 1 is an extreme point of the unit ball of \mathcal{X} . \square

While we will show that if the unit ball of a C^* -algebra has an extreme point then the algebra is unital, this is not true for Banach algebras in general.

2.6.8 Example. Let G be an infinite compact group and consider $L^2(G)$ with the convolution product. Since $L^2(G)$ is a dual space, by the Krein-Milman Theorem its unit ball has many extreme points. However, $L^2(G)$ is never unital when G is an infinite compact group.

2.6.9 Theorem. *Let \mathcal{A} be a C^* -algebra. Then the unit ball of \mathcal{A} has an extreme point if and only if \mathcal{A} is unital. Moreover, if \mathcal{A} is unital and e and f are projections in \mathcal{A} , the extreme points of $e\mathcal{A}f \cap \mathcal{A}^1$ are the elements v such that*

$$(e - vv^*)\mathcal{A}(f - v^*v) = \{0\},$$

and such a v is automatically a partial isometry.

PROOF. Suppose \mathcal{A} is unital. We showed that 1 is an extreme point of \mathcal{A}^1 in Proposition 2.6.7. In order to show the converse that \mathcal{A} is unital whenever \mathcal{A}^1 has an extreme point we will first need to prove the characterization of extreme points, working in the unitization of \mathcal{A} .

Suppose $v \in e\mathcal{A}f \cap \mathcal{A}^1$ is an extreme point but is not a partial isometry, i.e. v^*v is not a projection. Since $v \in e\mathcal{A}f$, we have

$$v^*v \in f^* \mathcal{A} e^* e \mathcal{A} f = f \mathcal{A} e e \mathcal{A} f = f \mathcal{A} f.$$

Let \mathcal{A}_0 be the (commutative) C^* -subalgebra of $f \mathcal{A} f$ generated by v^*v . By taking elements from a bounded approximate identity of \mathcal{A}_0 , there exists a positive sequence $(y_n)_{n=1}^\infty$ of elements in the unit ball of \mathcal{A}_0 such that

$$\lim_{n \rightarrow \infty} (v^*v)y_n = v^*v \quad \text{and} \quad \lim_{n \rightarrow \infty} (v^*v)y_n^2 = v^*v.$$

Since v^*v is not a projection, the Gelfand transform of v^*v takes a nonzero value less than 1 at some point $t \in \sigma(v^*v)$. Therefore, by Urysohn's lemma, there is a positive element $c \in \mathcal{A}_0$, whose Gelfand spectrum is nonzero at t , such that if $a_n = y_n + c$ and $b_n = y_n - c$, we have $\|v^*v a_n^2\| \leq 1$ and $\|v^*v b_n^2\| \leq 1$. Hence $v a_n$ and $v b_n$ are both in $e\mathcal{A}f \cap \mathcal{A}^1$. However,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \|v y_n - v\|^2 \\ &= \lim_{n \rightarrow \infty} \|(v y_n - v)^*(v y_n - v)\| \\ &= \lim_{n \rightarrow \infty} \|(v^*v)y_n^2 - (v^*v)y_n - (v^*v)y_n + v^*v\| \\ &= 0, \end{aligned}$$

so $v y_n \rightarrow v$. Thus, $v a_n \rightarrow v + v c$ and $v b_n \rightarrow v - v c$. Since

$$v = \frac{(v + v c) + (v - v c)}{2}$$

and v is an extreme point of $e\mathcal{A}f \cap \mathcal{A}^1$, $v = v + v c = v - v c$. Hence $v c = 0$, and

$$\|c v^* v c\| = \|v^* v c^2\| = 0,$$

contradicting the fact that the Gelfand transforms of both v^*v and c both take nonzero values at t . Therefore, our assumption that v is not a partial isometry is false.

We will now show that $(e - v v^*)\mathcal{A}(f - v^*v) = \{0\}$. Suppose otherwise that $(e - v v^*)\mathcal{A}(f - v^*v) \neq \{0\}$. Then there exists a nonzero a of the form $(e - v v^*)b(f - v^*v)$ with $\|b\| \leq 1$. If $p = v v^*$ and $q = v^*v$, we have

$$\begin{aligned} \|v \pm a\| &= \|p v q \pm (e - p)b(f - q)\| \\ &= \|(p v q \pm (e - p)b(f - q))^*(p v q \pm (e - p)b(f - q))\|^{1/2} \\ &= \|q v^* p v q + (f - q)b^*(e - p)b(f - q)\| \\ &\leq \max(\|v^* p v\|, \|b^*(e - p)b\|) \\ &\leq 1, \end{aligned}$$

so both $v + a$ and $v - a$ are in $e\mathcal{A}f \cap \mathcal{A}^1$. Since

$$v = \frac{(v + a) + (v - a)}{2},$$

this contradicts the fact that v is extreme. Therefore, our assumption that $(e - vv^*)\mathcal{A}(f - v^*v) \neq \{0\}$ is false. We can now show that if \mathcal{A}^1 has an extreme point, then \mathcal{A} is unital. Indeed, if $a \in \mathcal{A}$, then by the above,

$$(1 - v^*v)(1 - vv^*)a^*a(1 - vv^*)(1 - v^*v) = 0$$

Therefore, $a(1 - vv^*)(1 - v^*v) = 0$ for every $a \in \mathcal{A}$, which implies that

$$1 - v^*v - vv^* + (vv^*)(v^*v) = (1 - vv^*)(1 - v^*v) = 0,$$

or

$$1 = v^*v + vv^* - (vv^*)(v^*v) \in \mathcal{A},$$

showing that \mathcal{A} is unital.

Conversely, suppose $v \in e\mathcal{A}f$ is such that $(e - vv^*)\mathcal{A}(f - v^*v) = \{0\}$. Then, in particular,

$$0 = v^*(e - vv^*)v(f - v^*v) = v^*v(f - v^*v)^2.$$

It follows easily from the continuous functional calculus that v^*v is a projection, so by Proposition 2.6.4, v is a partial isometry. Let $p = vv^*$ and $q = v^*v$, and let $v = 1/2(a + b)$, with a and b in $e\mathcal{A}f \cap \mathcal{A}^1$. Then

$$v = vq = \frac{1}{2}(aq + bq),$$

so

$$q = v^*v \leq \frac{1}{2}(qa^*aq + qb^*bq) \leq q.$$

Since qa^*aq and qb^*bq belong to the unit ball of the C^* -algebra $q\mathcal{A}q$, for which q is the unit, it follows from the continuous functional calculus that

$$q = qa^*aq = qb^*bq.$$

However,

$$q = v^*v = \frac{1}{4}(qa^*aq + qa^*bq + qb^*aq + qb^*bq),$$

so that

$$q = \frac{1}{2}(qa^*bq + qb^*aq).$$

Since q is extreme in the unit ball of $q\mathcal{A}q$ by Proposition 2.6.7, we conclude that $q = qa^*bq = qb^*aq$. Therefore, $q(a - b)^*(a - b)q = 0$, and we have $aq = bq$. Similarly we can show that $pa = pb$, and thus by the assumption on v we have

$$a - b = (e - q)(a - b)(f - p) = 0,$$

so that b is an extreme point. \square

Historical Notes

The polar decomposition of bounded operators on a Hilbert space was developed by von Neumann [vN32]. The proof that the identity is an extreme point in the unit ball of a Banach algebra if it has norm 1 is due to Kakutani. The characterization of extreme points in a C^* -algebra was proven by Kadison [Kad51]. The Theorem 2.6.9 is more general, because it characterizes the extreme points of the unit ball of $p\mathcal{A}q$ for projections $p, q \in \mathcal{A}$, but there is no real change to the proof besides some bookkeeping.

2.7 Operator Topologies

There are many useful locally convex topologies on $\mathcal{B}(\mathcal{H})$ besides the norm, or uniform, topology. In this section we describe the basic operator topologies and their relationship to each other.

2.7.1 Definition. Let \mathcal{H} be a Hilbert space. The *weak operator topology* on $\mathcal{B}(\mathcal{H})$ is the locally convex topology defined by the seminorms

$$a \mapsto |\langle a\xi | \eta \rangle|,$$

where $\xi, \eta \in \mathcal{H}$. The *strong operator topology* on $\mathcal{B}(\mathcal{H})$ is the locally convex topology defined by the seminorms

$$a \mapsto \|a\xi\|,$$

where $\xi \in \mathcal{H}$. The *strong* operator topology* on $\mathcal{B}(\mathcal{H})$ is the locally convex topology defined by the seminorms

$$a \mapsto \|a\xi\| \quad \text{and} \quad a \mapsto \|a^*\xi\|,$$

or equivalently,

$$a \mapsto (\|a\xi\|^2 + \|a^*\xi\|^2)^{\frac{1}{2}},$$

where $\xi \in \mathcal{H}$.

By the polarization identity

$$\langle a\xi | \eta \rangle = \sum_{k=0}^3 i^k \langle a(\xi + i^k \eta) | \xi + i^k \eta \rangle,$$

it follows that the weak operator topology is also defined by the seminorms

$$a \mapsto |\langle a\xi | \xi \rangle|,$$

where $\xi \in \mathcal{H}$.

Often, for the sake of brevity, these topologies are referred to without the word ‘operator’ in their names. This is only potentially confusing for the weak operator topology, because the weak topology on a normed vector space is defined as the topology induced by the bounded linear functionals on that space. However, this topology is rarely used on $\mathcal{B}(\mathcal{H})$, so an unqualified reference to the weak topology on $\mathcal{B}(\mathcal{H})$ usually means the weak operator topology.

Convergence of nets in these three topologies is related. Indeed, for a net $(a_i)_{i \in I}$ in $\mathcal{B}(\mathcal{H})$, it is easy to see that $a_i \rightarrow 0$ in the strong operator topology if and only if $a_i^* a_i \rightarrow 0$ in the weak operator topology, and that $a_i \rightarrow 0$ in the strong* operator topology if and only if $a_i^* a_i + a_i a_i^* \rightarrow 0$ in the weak operator topology. If \supseteq indicates that the topology on the left is finer than the topology on the right, then it is easy to show the relations

$$\text{norm} \supseteq \text{strong}^* \supseteq \text{strong} \supseteq \text{weak}.$$

In fact, if \mathcal{H} is infinite-dimensional, each of these relations is strict, i.e. none of these topologies are equal.

2.7.2 Proposition. *Let \mathcal{H} be an infinite-dimensional Hilbert space. Then no two of the norm, strong*, strong, or weak operator topologies agree on $\mathcal{B}(\mathcal{H})$.*

PROOF. Since these topologies have the relations

$$\text{norm} \supseteq \text{strong}^* \supseteq \text{strong} \supseteq \text{weak},$$

we only need to show that these relations are strict. It suffices to prove only the case where \mathcal{H} is separable. Indeed, if \mathcal{H} is not separable, let \mathcal{K} be a separable subspace of \mathcal{H} . Then $\mathcal{B}(\mathcal{K})$ embeds isometrically in $\mathcal{B}(\mathcal{H})$ as $\mathcal{B}(\mathcal{K}) \oplus \{0_{\mathcal{K}^\perp}\}$, and the topologies under discussion are the same for $\mathcal{B}(\mathcal{K})$ and its isometric copy in $\mathcal{B}(\mathcal{H})$.

Let $(e_n)_{n=1}^\infty$ be an orthonormal basis for \mathcal{H} . We will first show that the weak operator topology on $\mathcal{B}(\mathcal{H})$ is strictly coarser than the strong operator topology. Let a be the right shift operator defined by $ae_n = e_{n+1}$. It is easy to verify that the sequence a^n converges to 0 in the weak operator topology. However, since each a^n is an isometry, the sequence does not converge at all in the strong operator topology.

To see that the strong operator topology is coarser than the strong* operator topology, define the sequence $(v_n)_{n=1}^\infty$ by letting $v_n e_n = e_1$ and $v_n e_k = 0$ for $k \neq n$. Then $v_n \rightarrow 0$ in the strong operator topology, but $(v_n)_{n=1}^\infty$ does not converge at all in the strong* operator topology.

Finally, to see that the norm topology is coarser than the strong* operator topology, consider the sequence of diagonal operators d_n defined by $d_n e_n = e_n$ and $d_n e_k = 0$ for $k \neq n$. It is easy to see that $d_n \rightarrow 0$ in the strong* operator topology, but $(d_n)_{n=1}^\infty$ does not converge at all in the norm topology. \square

If \mathcal{H} is finite-dimensional, then each of these topologies is the same as the norm topology, and all of the basic $*$ -algebraic operations are continuous with respect to them. Thus, for the rest of this discussion, we will assume that \mathcal{H} is infinite-dimensional. Addition and scalar multiplication are always continuous for locally convex topologies, so we need only concern ourselves with the algebra multiplication and the adjoint.

2.7.3 Proposition. *Let \mathcal{H} be an infinite-dimensional Hilbert space. Then the following table indicates which operations are continuous in the operator topologies:*

	<i>strong*</i>	<i>strong</i>	<i>weak</i>
<i>separate multiplication</i>	✓	✓	✓
<i>joint multiplication</i>	×	×	×
<i>joint multiplication (both sides bounded)</i>	✓	✓	×
<i>joint multiplication (one side bounded)</i>	✓	✓	×
<i>adjoint</i>	✓	×	✓

PROOF. For all of the counterexamples to continuity, we will work with a separable Hilbert space. The examples easily extend to general Hilbert spaces by considering a separable subspace. First, we deal with the weak operator topology. The separate continuity of multiplication is clear from the Cauchy-Schwarz inequality, and the adjoint is continuous because

$$|\langle a\xi | \eta \rangle| = |\langle a^* \eta | \xi \rangle|.$$

Now, suppose that \mathcal{H} is separable and let $(e_n)_{n=1}^\infty$ be an orthonormal basis for \mathcal{H} . Let $a \in \mathcal{B}(\mathcal{H})$ be the right shift operator defined by $ae_n = e_{n+1}$. Then a^* is the left shift operator defined by $ae_n = e_{n-1}$ for $n > 1$ and $ae_1 = 0$. Then $a^n \rightarrow 0$ and $(a^*)^n \rightarrow 0$ in the weak operator topology, but $a^n(a^*)^n = 1$, showing that multiplication is not jointly continuous, even when one of the sides is restricted to the unit ball.

Now, we consider the strong operator topology. The relation

$$ab\xi - a_0b_0\xi = a(b - b_0)\xi + (a - a_0)b_0\xi$$

establishes the joint continuity of multiplication when one of the sides is restricted to a bounded set, which includes the case of separate continuity of multiplication. Now, suppose that \mathcal{H} is separable and let $(e_n)_{n=1}^\infty$ be an orthonormal basis for \mathcal{H} . Define $a_n \in \mathcal{B}(\mathcal{H})$ by $a_n\xi = \langle e_n | \xi \rangle e_1$. Then $a_n \rightarrow 0$, but

$$\langle a_n^* e_1 | \xi \rangle = \langle e_1 | a_n \xi \rangle = \langle e_1 | e_1 \rangle \langle e_n | \xi \rangle,$$

showing that $a_n^* e_1 = e_n$, so $a_n^* e_1$ does not tend to zero. Hence the adjoint operation is not continuous. The discontinuity of joint multiplication is slightly more difficult. Let Λ denote the set of all (m, U) , where $m \in \mathbb{N}$ and U is

a strong operator topology neighbourhood of 0. Then Λ is a directed set equipped with the ordering saying that $(m, U) \leq (n, V)$ whenever $m \leq n$ and $U \supseteq V$. Let $a \in \mathcal{B}(\mathcal{H})$ be the right shift operator defined by $ae_n = e_{n+1}$. Then a^* is the left shift operator. If $\lambda = (m_\lambda, U_\lambda) \in \Lambda$ and $\xi \in \mathcal{H}$, then

$$\lim_{n \rightarrow \infty} m_\lambda \|(a^*)^n \xi\| = 0.$$

Therefore, $(m_\lambda (a^*)^n)_{n=1}^\infty$ converges to 0 in the strong operator topology. Therefore, there exists an N_λ such that $m_\lambda (a^*)^{N_\lambda} \in U_\lambda$. Define

$$b_\lambda = m_\lambda (a^*)^{N_\lambda} \quad \text{and} \quad c_\lambda = \frac{1}{m_\lambda} (a^*)^{N_\lambda}$$

Then

$$\lim_{\lambda \in \Lambda} \|c_\lambda\| = \lim_{\lambda \in \Lambda} \frac{1}{m_\lambda} = 0,$$

so $(c_\lambda)_{\lambda \in \Lambda}$ converges to 0 in the norm topology, and thus in the strong operator topology. Let U be a strong neighbourhood of 0, and define $\lambda_0 = (1, U)$. Then $b_{\lambda_0} \in U_{\lambda_0}$ and for every $\lambda \geq \lambda_0$, $b_\lambda \in U_\lambda \subseteq U_{\lambda_0}$. Therefore, $(b_\lambda)_{\lambda \in \Lambda}$ converges to 0 in the strong operator topology. However, $b_\lambda c_\lambda = 1$ for all $\lambda \in \Lambda$, so $(b_\lambda c_\lambda)_{\lambda \in \Lambda}$ converges to 1 in the strong operator topology. Therefore, multiplication is not jointly continuous in the strong operator topology.

Finally, we consider the strong* operator topology. The relation

$$ab\xi - a_0b_0\xi = a(b - b_0)\xi + (a - a_0)b_0\xi$$

again establishes the continuity of joint multiplication when one of the sides is restricted to a bounded set, which includes the case of one-sided multiplication. The adjoint operation is obviously continuous, because each of the defining seminorms of the strong* operator topology is invariant under the adjoint. The non-continuity of joint multiplication is similar to the example with the strong operator topology. \square

While the operator topologies are quite different, they all have the same continuous linear functionals, which implies that they have the same closed convex sets.

2.7.4 Proposition. *Let \mathcal{H} be a Hilbert space, and φ a bounded linear functional on $\mathcal{B}(\mathcal{H})$. Then the following are equivalent:*

- (i) φ is continuous in the weak operator topology;
- (ii) φ is continuous in the strong operator topology;
- (iii) φ is continuous in the strong* operator topology;
- (iv) there exist $\xi_1, \dots, \xi_n \in \mathcal{H}$ and $\eta_1, \dots, \eta_n \in \mathcal{H}$ such that

$$\varphi(a) = \sum_{i=1}^n \langle a\xi_i \mid \eta_i \rangle.$$

PROOF. The implications (i) \Rightarrow (ii), (ii) \Rightarrow (iii), and (iv) \Rightarrow (i) are clear, so all that needs to be shown is that (iii) \Rightarrow (iv). Suppose φ is continuous in the strong* operator topology, and let \mathbb{D} be the open unit disc of \mathbb{C} . Then $\varphi^{-1}(\mathbb{D})$ is an strong* open neighbourhood of zero, so it contains a basic open neighbourhood of zero, i.e. there exist $\xi_1, \dots, \xi_n \in \mathcal{H}$ such that

$$\begin{aligned} & \left\{ a \in \mathcal{B}(\mathcal{H}) : \sum_{i=1}^n (\|a\xi_i\|^2 + \|a^*\xi_i\|^2) \leq 1 \right\} \\ & \subseteq \{ a \in \mathcal{B}(\mathcal{H}) : (\|a\xi_i\|^2 + \|a^*\xi_i\|^2)^{\frac{1}{2}} \leq 1, 1 \leq i \leq n \} \\ & \subseteq \varphi^{-1}(\mathbb{D}). \end{aligned}$$

Therefore,

$$|\varphi(a)| \leq \left(\sum_{i=1}^n \|a\xi_i\|^2 + \|a^*\xi_i\|^2 \right)^{\frac{1}{2}}.$$

Define $\psi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{H}^{(n)}$ by

$$\psi(a) = (a\xi_1, \dots, a\xi_n).$$

We may define a bounded linear functional F on the range of ψ by $F(\psi(a)) = \varphi(a)$. By the Hahn-Banach Theorem, F extends to a bounded linear functional \tilde{F} on all of $\mathcal{H}^{(n)}$. By the Riesz-Fréchet Theorem, there exist vectors $\eta_1, \dots, \eta_n \in \mathcal{H}$ such that

$$\tilde{F}((x_1, \dots, x_n)) = \sum_{i=1}^n \langle x_i | \eta_i \rangle.$$

In particular,

$$\varphi(a) = \tilde{F}(\psi(a)) = \sum_{i=1}^n \langle a\xi_i | \eta_i \rangle. \quad \square$$

2.7.5 Corollary. *Let \mathcal{H} be a Hilbert space, and C a convex subset of $\mathcal{B}(\mathcal{H})$. Then the following are equivalent:*

- (i) C is closed in the weak operator topology;
- (ii) C is closed in the strong operator topology;
- (iii) C is closed in the strong* operator topology.

PROOF. The Hahn-Banach Theorem implies that locally convex topologies with the same continuous linear functionals have the same closed convex sets. Hence this corollary follows from Proposition 2.7.4. \square

If \mathcal{H} is a Hilbert space and α is a cardinal, we let $\mathcal{H}^{(\alpha)}$ denote the α -fold direct sum of \mathcal{H} . If $a \in \mathcal{B}(\mathcal{H})$, we let $a^{(\alpha)}$ denote the bounded operator on

$\mathcal{H}^{(\alpha)}$ that acts like a on each copy of \mathcal{H} . Clearly, $\|a^{(\alpha)}\| = \|a\|$. Similarly, if $S \subseteq \mathcal{B}(\mathcal{H})$, we let

$$S^{(\infty)} = \{a^{(\infty)} : a \in S\}.$$

We call $a^{(\infty)}$ and $S^{(\infty)}$ the α -fold amplification of a and S respectively. If $\alpha = \aleph_0$, we will often write ∞ instead of α .

With this in mind, there is a slight deficiency in the operator topologies we just presented. Let \mathcal{H} be a Hilbert space, and $\mathcal{H}^{(\infty)}$ the direct sum of countably many copies of \mathcal{H} . Then the amplification map $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}^{(\infty)})$ is not continuous with respect to the weak operator topologies on both spaces of operators. Indeed, let \mathcal{H} be a separable Hilbert space with an orthonormal basis $(e_n)_{n=1}^{\infty}$, and define a sequence $(\xi_n)_{n=1}^{\infty}$ by $\xi_n = (1/n^2)e_n$. Define a linear functional φ on $\mathcal{B}(\mathcal{H})$ by

$$\varphi(a) = \langle \Phi(a)(\xi_n)_{n=1}^{\infty} \mid (\xi_n)_{n=1}^{\infty} \rangle = \sum_{n=1}^{\infty} \langle a\xi_n \mid \xi_n \rangle = \frac{1}{n^4} \sum_{n=1}^{\infty} \langle ae_n \mid e_n \rangle.$$

It isn't too difficult to see that there is no $\xi \in \mathcal{H}$ such that

$$\varphi(a) = \langle a\xi \mid \xi \rangle,$$

so φ is not weak operator continuous. However, it is the restriction of weak operator continuous linear functional on $\mathcal{B}(\mathcal{H}^{(\infty)})$ to $\Phi(\mathcal{B}(\mathcal{H}))$, so Φ is not continuous with respect to the weak operator topology on $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H}^{(\infty)})$. The same is true for the strong and strong* operator topologies, because they have the same continuous linear functionals. We will now examine variants of the operator topologies that fix this defect.

2.7.6 Definition. Let \mathcal{H} be a Hilbert space. The σ -weak topology on $\mathcal{B}(\mathcal{H})$ is the locally convex topology defined by the seminorms

$$a \mapsto \left| \sum_{n=1}^{\infty} \langle a\xi_n \mid \eta_n \rangle \right|,$$

where $(\xi_n)_{n=1}^{\infty}, (\eta_n)_{n=1}^{\infty} \in \mathcal{H}^{(\infty)}$. The σ -strong topology on $\mathcal{B}(\mathcal{H})$ is the locally convex topology defined by the seminorms

$$a \mapsto \left(\sum_{n=1}^{\infty} \|a\xi_n\|^2 \right)^{\frac{1}{2}}$$

where $(\xi_n)_{n=1}^{\infty} \in \mathcal{H}^{(\infty)}$. The σ -strong* topology on $\mathcal{B}(\mathcal{H})$ is the locally convex topology defined by the seminorms

$$a \mapsto \sum_{n=1}^{\infty} \|a\xi_n\| \quad \text{and} \quad \sum_{n=1}^{\infty} \|a^*\xi_n\|,$$

or equivalently,

$$a \mapsto \left(\sum_{n=1}^{\infty} (\|a\xi_n\|^2 + \|a^*\xi_n\|^2) \right)^{\frac{1}{2}}$$

where $(\xi_n)_{n=1}^{\infty} \in \mathcal{H}^{(\infty)}$.

By the polarization identity

$$\langle a\xi | \eta \rangle = \sum_{k=0}^3 i^k \langle a(\xi + i^k \eta) | \xi + i^k \eta \rangle,$$

it follows that the σ -weak is also defined by the seminorms

$$a \mapsto \left| \sum_{n=1}^{\infty} \langle a\xi_n | \xi_n \rangle \right|,$$

where $(\xi_n)_{n=1}^{\infty} \in \mathcal{H}^{(\infty)}$.

If $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}^{\infty})$ be the amplification map, then the weak (resp. strong, strong*) operator topology inherited by $\Phi(\mathcal{B}(\mathcal{H}))$ is precisely the σ -weak (resp. σ -strong, σ -strong*) topology. The weak (resp. strong, strong*) operator topology is weaker than the σ -weak (resp. σ -strong, σ -strong*) operator topology.

Convergence of nets in these three topologies is related. Indeed, for a net $(a_i)_{i \in I}$ in $\mathcal{B}(\mathcal{H})$, it is easy to see that $a_i \rightarrow 0$ σ -strongly if and only if $a_i^* a_i \rightarrow 0$ σ -weakly, and that $a_i \rightarrow 0$ σ -strong* if and only if $a_i^* a_i + a_i a_i^* \rightarrow 0$ σ -weakly. If \supseteq indicates that the topology on the left is finer than the topology on the right, then it is easy to show the relations

$$\text{norm} \supseteq \sigma\text{-strong}^* \supseteq \sigma\text{-strong} \supseteq \sigma\text{-weak}.$$

If \mathcal{H} is infinite-dimensional then each of these relations is strict.

2.7.7 Proposition. *Let \mathcal{H} be an infinite-dimensional Hilbert space. Then no two of the norm, σ -strong*, σ -strong, or σ -weak topologies agree on $\mathcal{B}(\mathcal{H})$.*

PROOF. This is similar to Proposition 2.7.2, so we will omit the proof. \square

2.7.8 Proposition. *Let \mathcal{H} be an infinite-dimensional Hilbert space. Then the following table indicates which operations are continuous in the operator topologies:*

	σ -strong*	σ -strong	σ -weak
<i>separate multiplication</i>	✓	✓	✓
<i>joint multiplication</i>	×	×	×
<i>joint multiplication (both sides bounded)</i>	✓	✓	×
<i>joint multiplication (one side bounded)</i>	✓	✓	×
<i>adjoint</i>	✓	×	✓

PROOF. This is similar to Proposition 2.7.3, so we will omit the proof. \square

2.7.9 Proposition. *Let \mathcal{H} be a Hilbert space, and φ a bounded linear functional on $\mathcal{B}(\mathcal{H})$. Then the following are equivalent:*

- (i) φ is σ -weakly continuous;
- (ii) φ is σ -strongly continuous;
- (iii) φ is σ -strong* continuous;
- (iv) there exist sequences $(\xi_n)_{n=1}^{\infty}$ and $(\eta_n)_{n=1}^{\infty}$ in $\mathcal{H}^{(\infty)}$ such that

$$\varphi(a) = \sum_{n=1}^{\infty} \langle a\xi_n \mid \eta_n \rangle.$$

PROOF. The implications (i) \implies (ii), (ii) \implies (iii), and (iv) \implies (i) are clear, so all that needs to be shown is that (iii) \implies (iv). Suppose φ is strong* continuous, and let \mathbb{D} be the open unit disc of \mathbb{C} . Then $\varphi^{-1}(\mathbb{D})$ is an strong* open neighbourhood of zero, so it contains a basic open neighbourhood of zero, i.e. there exists a sequence $(\xi_n)_{n=1}^{\infty} \in \mathcal{H}^{(\infty)}$ such that

$$\left\{ a \in \mathcal{B}(\mathcal{H}) : \sum_{i=1}^{\infty} (\|a\xi_i\|^2 + \|a^*\xi_i\|^2) \leq 1 \right\} \subseteq \varphi^{-1}(\mathbb{D}).$$

Therefore,

$$|\varphi(a)| \leq \left(\sum_{i=1}^{\infty} \|a\xi_i\|^2 + \|a^*\xi_i\|^2 \right)^{\frac{1}{2}}.$$

Define $\psi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{H}^{(\infty)}$ by

$$\psi(a) = (a\xi_1, \dots, a\xi_n, \dots).$$

We may define a bounded linear functional F on the range of ψ by $F(\psi(a)) = \varphi(a)$. By the Hahn-Banach Theorem, F extends to a bounded linear functional \tilde{F} on all of $\mathcal{H}^{(n)}$. By the Riesz-Fréchet Theorem, there is a sequence $(\eta_n)_{n=1}^{\infty} \in \mathcal{H}^{(\infty)}$ such that

$$\tilde{F}((x_1, \dots, x_n, \dots)) = \sum_{i=1}^{\infty} \langle x_i \mid \eta_i \rangle.$$

In particular,

$$\varphi(a) = \tilde{F}(\psi(a)) = \sum_{i=1}^{\infty} \langle a\xi_i \mid \eta_i \rangle. \quad \square$$

2.7.10 Corollary. *Let \mathcal{H} be a Hilbert space, and C a convex subset of $\mathcal{B}(\mathcal{H})$. Then the following are equivalent:*

- (i) C is closed in the σ -weak topology;

- (ii) C is closed in the σ -strong topology;
- (iii) C is closed in the σ -strong* topology.

PROOF. The Hahn-Banach Theorem implies that locally convex topologies with the same continuous linear functionals have the same closed convex sets. Hence this corollary follows from Proposition 2.7.9. \square

We are now going to show that the σ -weak topology on $\mathcal{B}(\mathcal{H})$ is the weak* topology from a particular predual of $\mathcal{B}(\mathcal{H})$. Let \mathcal{H} be a Hilbert space and $\overline{\mathcal{H}}$ its conjugate space, so that $\mathcal{H}^* \cong \overline{\mathcal{H}}$ and $(\overline{\mathcal{H}})^* \cong \mathcal{H}$. Let $\mathcal{H} \otimes^y \overline{\mathcal{H}}$ be the Banach space projective tensor product of \mathcal{H} and $\overline{\mathcal{H}}$. We have

$$(\mathcal{H} \otimes^y \overline{\mathcal{H}})^* \cong \mathcal{B}(\mathcal{H}, (\overline{\mathcal{H}})^*) \cong \mathcal{B}(\mathcal{H})$$

via the pairing

$$\left\langle \sum_{n=1}^{\infty} \xi_n \otimes \overline{\eta_n}, a \right\rangle = \sum_{n=1}^{\infty} \langle a \xi_n \mid \eta_n \rangle.$$

2.7.11 Proposition. *Let \mathcal{H} be a Hilbert space and \mathcal{F} the collection of finite-dimensional subspaces of \mathcal{H} , ordered by inclusion. For every $F \in \mathcal{F}$, let p_F be the orthogonal projection of \mathcal{H} onto F . Then if $t \in \mathcal{H} \otimes^y \overline{\mathcal{H}}$,*

$$\lim_{F \in \mathcal{F}} (p_F \otimes p_F)t = t.$$

PROOF. We know from the basic theory of Hilbert spaces that

$$\lim_{F \in \mathcal{F}} p_F \xi = \xi$$

for all $\xi \in \mathcal{H}$. Fix $t \in \mathcal{H} \otimes^y \overline{\mathcal{H}}$ and $\epsilon > 0$. Let $(\xi_n)_{n=1}^{\infty}$ and $(\eta_n)_{n=1}^{\infty}$ be sequences in \mathcal{H} such that

$$t = \sum_{n=1}^{\infty} \xi_n \otimes \overline{\eta_n} \quad \text{and} \quad \sum_{n=1}^{\infty} \|\xi_n\| \|\eta_n\| < \|t\|_y + \frac{\epsilon}{4}.$$

Let $k \in \mathbb{N}$ be such that

$$\sum_{n=k+1}^{\infty} \|\xi_n\| \|\eta_n\| < \frac{\epsilon}{2}.$$

Then

$$\left\| t - \sum_{n=1}^k \xi_n \otimes \overline{\eta_n} \right\|_y < \frac{\epsilon}{2}.$$

Also, if F contains the span of $\{\xi_1, \dots, \xi_k, \eta_1, \dots, \eta_k\}$, then

$$\sum_{n=1}^k p_F \xi_n \otimes p_F \overline{\eta_n} - \sum_{n=1}^k \xi_n \otimes \overline{\eta_n} = 0,$$

so for a large enough $F \in \mathcal{F}$ we have

$$\begin{aligned}
\|(p_F \otimes p_F)t - t\|_y &\leq \|p_F \otimes p_F\| \cdot \left\| t - \sum_{n=1}^k \xi_n \otimes \bar{\eta}_n \right\|_y \\
&\quad + \left\| \sum_{n=1}^k p_F \xi_n \otimes p_F \bar{\eta}_n - \sum_{n=1}^k \xi_n \otimes \bar{\eta}_n \right\|_y \\
&\quad + \left\| \sum_{n=1}^k \xi_n \otimes \bar{\eta}_n - t \right\|_y \\
&\leq \frac{\epsilon}{2} + 0 + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Therefore,

$$\lim_{F \in \mathcal{F}} (p_F \otimes p_F)t = t. \quad \square$$

2.7.12 Corollary. *Let \mathcal{H} be a Hilbert space. The finite rank operators on \mathcal{H} are $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{H} \otimes^y \overline{\mathcal{H}})$ dense in $\mathcal{B}(\mathcal{H})$.*

PROOF. Let \mathcal{F} the family of finite-dimensional subspaces of \mathcal{H} , ordered by inclusion. For every $F \in \mathcal{F}$, let $p_F : \mathcal{H} \rightarrow \mathcal{H}$ be the orthogonal projection onto F . If $a \in \mathcal{B}(\mathcal{H})$ and $t \in \mathcal{H} \otimes^y \overline{\mathcal{H}}$, then by Proposition 2.7.11,

$$\lim_{F \in \mathcal{F}} \langle p_F a p_F, t \rangle = \lim_{F \in \mathcal{F}} \langle a, (p_F \otimes p_F)t \rangle = \langle a, t \rangle,$$

so a is the $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{H} \otimes^y \overline{\mathcal{H}})$ limit of finite rank operators. \square

Fix $t \in \mathcal{H} \otimes^y \overline{\mathcal{H}}$. Then there exist sequences $(\xi_n)_{n=1}^\infty$ and $(\eta_n)_{n=1}^\infty$ in \mathcal{H} such that

$$t = \sum_{n=1}^\infty \xi_n \otimes \bar{\eta}_n \quad \text{and} \quad \sum_{n=1}^\infty \|\xi_n\| \|\eta_n\| < \infty.$$

Define $\tilde{t} \in \mathcal{B}(\mathcal{H})$ by

$$\tilde{t}\xi = \left(\sum_{n=1}^\infty \xi_n \otimes \bar{\eta}_n \right) \xi = \sum_{n=1}^\infty \langle \xi | \eta_n \rangle \xi_n.$$

Note that \tilde{t} is a limit of finite rank operators, so $\tilde{t} \in \mathcal{K}(\mathcal{H})$.

2.7.13 Definition. Let \mathcal{H} be a Hilbert space. We say that $x \in \mathcal{B}(\mathcal{H})$ is a *trace-class operator* if $x = \tilde{t}$ for some $t \in \mathcal{H} \otimes^y \overline{\mathcal{H}}$. The set of trace-class operators on \mathcal{H} is denoted by $\mathcal{T}(\mathcal{H})$.

We will not distinguish between an element of $\mathcal{T}(\mathcal{H})$ and the operator in $\mathcal{B}(\mathcal{H})$ that it represents.

2.7.14 Proposition. *Let \mathcal{H} be a Hilbert space. Then*

- (i) *the map $t \mapsto \tilde{t}$ from $\mathcal{H} \otimes^y \overline{\mathcal{H}}$ to $\mathcal{T}(\mathcal{H})$ is injective;*
- (ii) *$\mathcal{T}(\mathcal{H})$ is a self-adjoint two-sided ideal in $\mathcal{B}(\mathcal{H})$;*
- (iii) *if $x \in \mathcal{T}(\mathcal{H})$, then $|x| \in \mathcal{T}(\mathcal{H})$;*
- (iv) *$\mathcal{T}(\mathcal{H})$ consists of the compact operators x on \mathcal{H} such that*

$$\sum_{n=1}^{\infty} \mu_n(x) < \infty,$$

where $(\mu_n(x))_{n=1}^{\infty}$ is the characteristic list of eigenvalues of $|x|$.

PROOF.

- (i) Suppose that $t \in \mathcal{H} \otimes^y \overline{\mathcal{H}}$ and $\tilde{t} = 0$. We want to show that $t = 0$. Suppose otherwise, that $t \neq 0$. Let $(\xi_n)_{n=1}^{\infty}$ and $(\eta_n)_{n=1}^{\infty}$ be sequences in \mathcal{H} such that

$$t = \sum_{n=1}^{\infty} \xi_n \otimes \overline{\eta_n} \quad \text{and} \quad \sum_{n=1}^{\infty} \|\xi_n\| \|\eta_n\| < \infty.$$

By Corollary 2.7.12, the finite rank operators are $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{H} \otimes^y \overline{\mathcal{H}})$ dense in $\mathcal{B}(\mathcal{H})$. Since the finite rank operators are the span of the rank one operators and $t \neq 0$, there exist $\xi, \eta \in \mathcal{H}$ such that

$$\langle t, \xi \otimes \overline{\eta} \rangle \neq 0.$$

However,

$$\begin{aligned} \langle \tilde{t}\xi | \eta \rangle &= \left\langle \left(\sum_{n=1}^{\infty} \xi_n \otimes \overline{\eta_n} \right) \xi \mid \eta \right\rangle \\ &= \sum_{n=1}^{\infty} \langle \xi | \eta_n \rangle \langle \xi_n | \eta \rangle \\ &= \sum_{n=1}^{\infty} \langle \eta_n \otimes \overline{\xi_n}, \xi \otimes \overline{\eta} \rangle \\ &= \left\langle \sum_{n=1}^{\infty} \xi_n \otimes \overline{\eta_n}, \xi \otimes \overline{\eta} \right\rangle \\ &= \langle t, \xi \otimes \overline{\eta} \rangle \\ &\neq 0, \end{aligned}$$

contradicting the fact that $\tilde{t} = 0$. Therefore, our assumption that $t \neq 0$ is false.

(ii) Fix $x \in \mathcal{T}(\mathcal{H})$ and let $(\xi_n)_{n=1}^\infty$ and $(\eta_n)_{n=1}^\infty$ be sequences in \mathcal{H} such that

$$x = \sum_{n=1}^{\infty} \xi_n \otimes \overline{\eta_n} \quad \text{and} \quad \sum_{n=1}^{\infty} \|\xi_n\| \|\eta_n\| < \infty.$$

Then, for every $a \in \mathcal{B}(\mathcal{H})$,

$$ax = \sum_{n=1}^{\infty} (a\xi_n) \otimes \overline{\eta_n} \in \mathcal{T}(\mathcal{H}) \quad \text{and} \quad xa = \sum_{n=1}^{\infty} \xi_n \otimes \overline{a\eta_n} \in \mathcal{T}(\mathcal{H}).$$

Therefore, $\mathcal{T}(\mathcal{H})$ is an ideal. Furthermore,

$$x^* = \sum_{n=1}^{\infty} \eta_n \otimes \overline{\xi_n} \in \mathcal{T}(\mathcal{H}),$$

so $\mathcal{T}(\mathcal{H})$ is self-adjoint.

(iii) Fix $x \in \mathcal{T}(\mathcal{H})$, and let $x = v|x|$ be the polar decomposition of x . Then

$$v^*x = v^*v|x| = s(|x|)|x| = |x|.$$

By (ii), $\mathcal{T}(\mathcal{H})$ is an ideal, so $|x| \in \mathcal{T}(\mathcal{H})$;

(iv) The operator norm on $\mathcal{H} \otimes \overline{\mathcal{H}}$, viewing tensors as finite rank operators, is clearly a cross norm. Therefore, the operator norm on $\mathcal{H} \otimes \overline{\mathcal{H}}$ is dominated by the projective norm, which is the norm on $\mathcal{T}(\mathcal{H})$. Since every operator in $\mathcal{T}(\mathcal{H})$ is the limit, in the projective norm, of finite rank operators, it is also the limit of finite rank operators in the operator norm. Therefore, every trace-class operator is compact. Now, suppose that $a \in \mathcal{K}(\mathcal{H})$. Then, by the Spectral Theorem for self-adjoint compact operators and the polar decomposition, there exist orthonormal sequences $(\xi_n)_{n=1}^\infty$ and $(\eta_n)_{n=1}^\infty$ in \mathcal{H} such that

$$a\xi = \sum_{n=1}^{\infty} \mu_n(a) \langle \xi | \eta_n \rangle \xi_n.$$

The desired conclusion is clear. □

We will now prove a useful criterion for determining whether an operator is compact or trace-class.

2.7.15 Proposition. *Let \mathcal{H} be a Hilbert space. If $a \in \mathcal{B}(\mathcal{H})$, then*

- (i) *a is compact if and only if for all orthonormal sequences $(e_n)_{n=1}^\infty$ and $(f_n)_{n=1}^\infty$ in \mathcal{H} , the sequence $(\langle ae_n | f_n \rangle)_{n=1}^\infty$ belongs to $c_0(\mathbb{N})$;*
- (ii) *a is trace-class if and only if for all orthonormal sequences $(e_n)_{n=1}^\infty$ and $(f_n)_{n=1}^\infty$ in \mathcal{H} , the sequence $(\langle ae_n | f_n \rangle)_{n=1}^\infty$ belongs to $\ell^1(\mathbb{N})$.*

PROOF.

- (i) Suppose that a is compact, and that $(e_n)_{n=1}^\infty$ and $(f_n)_{n=1}^\infty$ are orthonormal sequences in \mathcal{H} . Since $(e_n)_{n=1}^\infty$ is a weak null sequence and compact operators are completely continuous,

$$\lim_{n \rightarrow \infty} \|ae_n\| = 0.$$

By the Cauchy-Schwarz inequality, it follows that

$$0 \leq \lim_{n \rightarrow \infty} |\langle ae_n | f_n \rangle| \leq \lim_{n \rightarrow \infty} \|ae_n\|^{1/2} \|f_n\|^{1/2} = 0.$$

Therefore,

$$\lim_{n \rightarrow \infty} \langle ae_n | f_n \rangle = 0.$$

Conversely, suppose that for all orthonormal sequences $(e_n)_{n=1}^\infty$ and $(f_n)_{n=1}^\infty$ in \mathcal{H} , the sequence $(\langle ae_n | f_n \rangle)_{n=1}^\infty$ belongs to $c_0(\mathbb{N})$. We will show that a is approximable by finite rank operators, and is thus compact. Fix $\epsilon > 0$ such that $\epsilon < \|a\|$. Then there exist unit vectors $\xi, \eta \in \mathcal{H}$ such that $|\langle a\xi | \eta \rangle| > \epsilon$. Consequently, the collection \mathcal{F} of all pairs $((\xi_i)_{i \in I}, (\eta_i)_{i \in I})$ of orthonormal families $(\xi_i)_{i \in I}$ and $(\eta_i)_{i \in I}$ in \mathcal{H} such that $|\langle a\xi_i | \eta_i \rangle| > \epsilon$ for all $i \in I$ is non-empty. We can naturally put a partial ordering on \mathcal{F} by defining

$$((\xi_i)_{i \in I}, (\eta_i)_{i \in I}) \leq ((\xi'_j)_{j \in J}, (\eta'_j)_{j \in J})$$

whenever $I \subseteq J$ and $\xi_i = \xi'_i$ and $\eta_i = \eta'_i$ for all $i \in I$. It is easy to see that every chain in \mathcal{F} has an upper bound, so by Zorn's Lemma there exists a maximal element of \mathcal{F} , which we will denote by $((e_i)_{i \in I_\epsilon}, (f_i)_{i \in I_\epsilon})$. By our assumption on a , I_ϵ must be finite. Enumerate the elements of I_ϵ as $\{i_1, \dots, i_k\}$. Define projections p and q in $\mathcal{B}(\mathcal{H})$ by

$$p\xi = \sum_{n=1}^k \langle \xi | e_{i_n} \rangle e_{i_n} \quad \text{and} \quad q\xi = \sum_{n=1}^k \langle \xi | f_{i_n} \rangle f_{i_n}.$$

Both p and q have finite rank, so $b = ap + qa - qap$ is a finite rank operator in $\mathcal{B}(\mathcal{H})$. We claim that $\|a - b\| \leq \epsilon$.

Suppose to the contrary that $\|a - b\| > \epsilon$. Then there exist unit vectors $\xi, \eta \in \mathcal{H}$ such that $|\langle (a - b)\xi | \eta \rangle| > \epsilon$. Let $e_0 = \xi - p\xi$ and $f_0 = \eta - q\eta$, and observe that

$$e_0 \in \{e_i : i \in I_\epsilon\}^\perp \quad \text{and} \quad f_0 \in \{f_i : i \in I_\epsilon\}^\perp.$$

Note that $\|e_0\| \leq 1$ and $\|f_0\| \leq 1$. Since $(1 - q)a(1 - p) = a - b$,

$$\begin{aligned} |\langle ae_0 | f_0 \rangle| &= |\langle a(1 - p)\xi | (1 - q)\eta \rangle| \\ &= |\langle (1 - q)a(1 - p)\xi | \eta \rangle| \\ &= |\langle (a - b)\xi | \eta \rangle| \\ &> \epsilon \\ &\geq \epsilon \|e_0\| \|f_0\|. \end{aligned}$$

Therefore, e_0 and f_0 are non-zero, and

$$|\langle a(e_0/\|e_0\|) \mid f_0/\|f_0\| \rangle| > \epsilon,$$

contradicting the maximality of I_ϵ . Therefore, our assumption that $\|a - b\| > \epsilon$ is false, showing that a is compact.

- (ii) Suppose that a is trace-class. Then there exist orthonormal sequences $(\xi_n)_{n=1}^\infty$ and $(\eta_n)_{n=1}^\infty$ in \mathcal{H} such that

$$a\xi = \sum_{n=1}^{\infty} \mu_n(a) \langle \xi \mid \eta_n \rangle \xi_n,$$

where $(\mu_n(a))_{n=1}^\infty$ is the characteristic sequence of eigenvalues of a . By Hölder's inequality,

$$\begin{aligned} |\langle ae_n \mid f_n \rangle| &\leq \sum_{m=1}^{\infty} \mu_m(a)^{1/2} |\langle e_n \mid \xi_m \rangle| \mu_m(a)^{1/2} |\langle \eta_m \mid f_n \rangle| \\ &\leq \left(\sum_{m=1}^{\infty} \mu_m(a) |\langle e_n \mid \xi_m \rangle|^2 \right)^{1/2} \left(\sum_{m=1}^{\infty} \mu_m(a) |\langle \eta_m \mid f_n \rangle|^2 \right)^{1/2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} |\langle ae_n \mid f_n \rangle| &\leq \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \mu_m(a) |\langle e_n \mid \xi_m \rangle|^2 \right)^{1/2} \left(\sum_{m=1}^{\infty} \mu_m(a) |\langle \eta_m \mid f_n \rangle|^2 \right)^{1/2} \\ &\leq \left(\sum_{m,n=1}^{\infty} \mu_m(a) |\langle e_n \mid \xi_m \rangle|^2 \right)^{1/2} \left(\sum_{m,n=1}^{\infty} \mu_m(a) |\langle \eta_m \mid f_n \rangle|^2 \right)^{1/2} \\ &\leq \sum_{m=1}^{\infty} \mu_m(a) < \infty. \end{aligned}$$

Conversely, suppose that for all orthonormal sequences $(e_n)_{n=1}^\infty$ and $(f_n)_{n=1}^\infty$ in \mathcal{H} , the sequence $(\langle ae_n \mid f_n \rangle)_{n=1}^\infty$ belongs to $\ell^1(\mathbb{N})$. Since $\ell^1(\mathbb{N})$ is contained in $c_0(\mathbb{N})$, by (a) we have that a is compact. Then, by a combination of the Spectral Theorem for compact operators and the polar decomposition, there exist orthonormal sequences $(\xi_n)_{n=1}^\infty$ and $(\eta_n)_{n=1}^\infty$ in \mathcal{H} such that

$$a\xi = \sum_{n=1}^{\infty} \mu_n(a) \langle \xi \mid \eta_n \rangle \xi_n.$$

Since

$$\mu_n(a) = \langle a\xi_n \mid \eta_n \rangle,$$

our assumption implies that $(\mu_n)_{n=1}^\infty$ is summable, which by part (iii) of Proposition 2.7.14 implies that a is trace-class. \square

If $x \in \mathcal{T}(\mathcal{H})$, and $(\xi_n)_{n=1}^\infty$ and $(\eta_n)_{n=1}^\infty$ are sequences in \mathcal{H} such that

$$x = \sum_{n=1}^{\infty} \xi_n \otimes \bar{\eta}_n \quad \text{and} \quad \sum_{n=1}^{\infty} \|\xi_n\| \|\eta_n\| < \infty,$$

we define the *trace* of x to be the sum

$$\text{Tr}(x) = \sum_{n=1}^{\infty} \langle \xi_n | \eta_n \rangle.$$

By the Cauchy-Schwarz inequality,

$$\sum_{n=1}^{\infty} |\langle \xi_n | \eta_n \rangle|^2 \leq \sum_{n=1}^{\infty} \|\xi_n\| \|\eta_n\| < \infty,$$

so this sum is absolutely convergent. If $a \in \mathcal{B}(\mathcal{H})$, then

$$\text{Tr}(ax) = \sum_{n=1}^{\infty} \langle a\xi_n | \eta_n \rangle = \langle \xi_n | a^* \eta_n \rangle = \text{Tr}(xa),$$

and

$$\langle x, a \rangle = \sum_{n=1}^{\infty} \langle a\xi_n | \eta_n \rangle = \text{Tr}(ax).$$

Hence Tr implements the duality between $\mathcal{T}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$. If \mathcal{H} is finite dimensional, then $\mathcal{B}(\mathcal{H}) = \mathcal{K}(\mathcal{H})$ is reflexive, so $\mathcal{T}(\mathcal{H})$ is also isometrically isomorphic to $\mathcal{K}(\mathcal{H})^*$ via the same duality. It turns out that this result holds even if \mathcal{H} is not finite dimensional.

2.7.16 Theorem. *Let \mathcal{H} be a Hilbert space. Then $\mathcal{T}(\mathcal{H})$ is isometrically isomorphic to $\mathcal{K}(\mathcal{H})^*$ via the pairing $\langle a, x \rangle = \text{Tr}(ax)$.*

PROOF. Since $\mathcal{T}(\mathcal{H})^* \cong \mathcal{B}(\mathcal{H})$ isometrically via the same pairing and $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$, every $x \in \mathcal{T}(\mathcal{H})$ automatically defines an element $\mathcal{K}(\mathcal{H})^*$ with norm at most $\|x\|_y$. Fix $\varphi \in \mathcal{K}(\mathcal{H})^*$. If $\xi, \eta \in \mathcal{H}$, then the rank one operator $\xi \otimes \bar{\eta}$ has norm $\|\xi\| \|\eta\|$, so the sesquilinear form $[\cdot, \cdot]$ defined by

$$[\xi, \eta] = \varphi(\xi \otimes \bar{\eta})$$

is bounded. Therefore, by the Riesz-Fréchet Theorem, there exists an $x \in \mathcal{B}(\mathcal{H})$ such that

$$\varphi(\xi \otimes \bar{\eta}) = [\xi, \eta] = \langle x\xi | \eta \rangle$$

for all $\xi, \eta \in \mathcal{H}$. By linearity, $\varphi(a) = \text{Tr}(ax)$ for all finite rank $a \in \mathcal{K}(\mathcal{H})$.

We want to show that $x \in \mathcal{T}(\mathcal{H})$. Let $(e_n)_{n=1}^\infty$ and $(f_n)_{n=1}^\infty$ be orthonormal sequences in \mathcal{H} . Fix a sequence $(\alpha_n)_{n=1}^\infty$ in $c_0(\mathbb{N})$, and define a compact operator

$$b = \sum_{n=1}^{\infty} \alpha_n e_n \otimes f_n.$$

Then

$$\sum_{n=1}^{\infty} \alpha_n \langle x e_n | f_n \rangle = \sum_{n=1}^{\infty} \alpha_n \varphi(e_n \otimes \overline{f_n}) = \varphi(a).$$

Since $(\alpha_n)_{n=1}^{\infty}$ was arbitrary, the sequence $(\langle x e_n | f_n \rangle)_{n=1}^{\infty}$ belongs to $\ell^1(\mathbb{N})$. Therefore, by Proposition 2.7.15 (ii), $x \in \mathcal{T}(\mathcal{H})$. \square

2.7.17 Corollary. *Let \mathcal{H} be a Hilbert space. Then $\mathcal{K}(\mathcal{H})^{**}$ is isometrically isomorphic to $\mathcal{B}(\mathcal{H})$, and the natural embedding of $\mathcal{K}(\mathcal{H})$ into $\mathcal{K}(\mathcal{H})^{**}$ extends the inclusion of $\mathcal{K}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H})$.*

PROOF. Clear from Theorem 2.7.16 and duality between $\mathcal{T}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$. \square

2.7.18 Proposition. *Let \mathcal{H} be a Hilbert space. Then the σ -weak topology on $\mathcal{B}(\mathcal{H})$ coincides with the weak* topology from $\mathcal{T}(\mathcal{H})$.*

PROOF. Recall that the σ -weak topology is generated by linear functionals of the form

$$\varphi(a) = \sum_{n=1}^{\infty} \langle a \xi_n | \xi_n \rangle$$

for some sequence $(\xi_n)_{n=1}^{\infty}$ in $\mathcal{H}^{(\infty)}$. Consider such a functional. Then

$$\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty,$$

and

$$x = \sum_{n=1}^{\infty} \xi_n \otimes \overline{\xi_n}$$

defines an element of the projective tensor product $\mathcal{H} \otimes^y \overline{\mathcal{H}}$, i.e. an element of $\mathcal{T}(\mathcal{H})$. By the duality formulas for $\mathcal{T}(\mathcal{H})$, we have that

$$\langle a, x \rangle = \text{Tr}(ax) = \sum_{n=1}^{\infty} \langle a \xi_n | \xi_n \rangle = \varphi(a).$$

Therefore, the σ -weak topology is weaker than the weak* topology from $\mathcal{T}(\mathcal{H})$.

Conversely, suppose that $x \in \mathcal{T}(\mathcal{H})$. Let $x = v|x|$ be the polar decomposition of x . Then $|x|$ is a positive compact operator, so by the Spectral Theorem for compact operators there is an orthonormal sequence $(e_n)_{n=1}^{\infty}$ in \mathcal{H} such that

$$|x|\xi = \sum_{n=1}^{\infty} \mu_n(|x|) \langle \xi | e_n \rangle e_n,$$

where $(\mu_n(|x|))_{n=1}^{\infty}$ is the characteristic sequence of eigenvalues of $|x|$. Since $|v| \in \mathcal{T}(\mathcal{H})$, by Proposition 2.7.15 (ii), we have $\sum_{n=1}^{\infty} \mu_n(|x|) < \infty$. Define

$\xi_n = \mu_n(|x|)^{1/2} v e_n$ and $\eta_n = \mu_n(|x|)^{1/2} e_n$. Then both $(\xi_n)_{n=1}^\infty$ and $(\eta_n)_{n=1}^\infty$ are in $\mathcal{H}^{(\infty)}$, and

$$x = v|x| = \sum_{n=1}^{\infty} \xi_n \otimes \overline{\eta_n}.$$

If $a \in \mathcal{B}(\mathcal{H})$, we have

$$\langle a, x \rangle = \text{Tr}(ax) = \sum_{n=1}^{\infty} \langle a\xi_n | \eta_n \rangle.$$

Therefore, the weak* topology from $\mathcal{T}(\mathcal{H})$ is weaker than the σ -weak topology. \square

It is easy to see that the weak operator topology is generated by the rank one operators in $\mathcal{T}(\mathcal{H})$, which are dense in $\mathcal{T}(\mathcal{H})$. Hence the weak operator topology is weaker than the σ -weak topology. We will show that the weak operator topology agrees with the σ -weak topology on bounded sets, and derive similar facts for the other operator topologies.

2.7.19 Proposition. *Let \mathcal{H} be a Hilbert space. Then, if S is a bounded subset of $\mathcal{B}(\mathcal{H})$,*

- (i) *the weak operator topology and the σ -weak topology agree on S ;*
- (ii) *the strong operator topology and the σ -strong topology agree on S ;*
- (iii) *the strong* operator topology and the σ -strong* topology agree on S .*

PROOF.

- (i) Since the weak operator topology is weaker than the σ -weak topology, the identity map on S is σ -weak to weak operator continuous. Since the σ -weak topology is a weak*-topology, S is σ -weakly compact by the Banach-Alaoglu Theorem. Therefore, the identity map on S is a σ -weak to weak operator homeomorphism.
- (ii) This follows from (i) and the fact that for a net $(a_i)_{i \in I}$ in $\mathcal{B}(\mathcal{H})$,

$$\begin{aligned} a_i \rightarrow 0 \text{ (strong operator)} &\iff a_i^* a_i \rightarrow 0 \text{ (weak operator),} \\ a_i \rightarrow 0 \text{ (\sigma-strongly)} &\iff a_i^* a_i \rightarrow 0 \text{ (\sigma-weakly).} \end{aligned}$$

- (iii) This follows from (i) and the fact that for a net $(a_i)_{i \in I}$ in $\mathcal{B}(\mathcal{H})$,

$$\begin{aligned} a_i \rightarrow 0 \text{ (strong* operator)} &\iff a_i^* a_i + a_i a_i^* \rightarrow 0 \text{ (weak operator),} \\ a_i \rightarrow 0 \text{ (\sigma-strong*)} &\iff a_i^* a_i + a_i a_i^* \rightarrow 0 \text{ (\sigma-strong*)}. \quad \square \end{aligned}$$

2.7.20 Corollary. *Let \mathcal{H} be a Hilbert space. Then the closed unit ball of $\mathcal{B}(\mathcal{H})$ is weak operator compact.*

PROOF. By the Banach-Alaoglu Theorem, the unit ball of $\mathcal{B}(\mathcal{H})$ is σ -weakly compact. By Proposition 2.7.19, the weak operator topology agrees with the σ -weak topology on the unit ball. Therefore, the unit ball is weak operator compact. \square

2.7.21 Proposition. *Let \mathcal{H} be a Hilbert space, and $C \subseteq \mathcal{B}(\mathcal{H})$ a convex set. Let S be the unit ball of $\mathcal{B}(\text{cal}\mathcal{H})$. Then the following are equivalent:*

- (i) C is closed in the weak operator topology;
- (ii) C is closed in the strong operator topology;
- (iii) C is closed in the strong* operator topology;
- (iv) $C \cap rS$ is closed in the σ -weak topology for all $r > 0$;
- (v) $C \cap rS$ is closed in the σ -strong topology for all $r > 0$;
- (vi) $C \cap rS$ is closed in the σ -strong* topology for all $r > 0$.

If C is a linear subspace of $\mathcal{B}(\mathcal{H})$, then by scaling we need only consider $r = 1$ in the last three conditions.

PROOF. By Corollary 2.7.5, the weak operator, strong operator, and strong* operator topologies have the same closed convex sets, proving the equivalence of (i), (ii), and (iii). By Corollary 2.7.10, the σ -weak, σ -strong, and σ -strong* topologies have the same closed convex sets, proving the equivalence of (iv), (v), and (vi). Therefore, we need only prove the equivalence of (i) and (iv).

Recall the Krein-Šmulian Theorem, which states that a convex subset C of a dual space is weak* closed if and only if $C \cap rS$ is weak* closed for all $r > 0$, where S is the unit ball. Applied to the σ -weak topology, it implies that a convex subset C of $\mathcal{B}(\mathcal{H})$ is σ -weakly closed if and only if $C \cap rS$ is σ -weakly closed for all $r > 0$. By Proposition 2.7.19, the σ -weak topology and weak operator topology agree on bounded subsets of $\mathcal{B}(\mathcal{H})$. This implies the equivalence of (i) and (iv). \square

We will show in Corollary 2.8.18 that all of the topologies defined in this section have the same closed *-subalgebras of $\mathcal{B}(\mathcal{H})$.

The operator topologies rarely agree on subsets of $\mathcal{B}(\mathcal{H})$, but there are a few exceptions.

2.7.22 Proposition. *Let \mathcal{H} be a Hilbert space.*

- (i) *The adjoint operation is continuous in the strong operator topology when restricted to the set of normal operators, so the strong operator topology and the strong* operator topology agree on the set of normal operators.*
- (ii) *Similarly, the adjoint operation is continuous in the σ -strong topology when restricted to the set of normal operators, so the σ -strong topology and the σ -strong* topology agree on the set of normal operators.*

(iii) All of the operator topologies agree on the unitary group $\mathcal{U}(\mathcal{H})$.

PROOF. Clearly, (i) and (ii) follow from the following inequality for normal operators $a, b \in \mathcal{B}(\mathcal{H})$ and $\xi \in \mathcal{H}$:

$$\begin{aligned} \|(a^* - b^*)\xi\|^2 &= \langle aa^*\xi | \xi \rangle + \langle bb^*\xi | \xi \rangle - \langle ab^*\xi | \xi \rangle - \langle ba^*\xi | \xi \rangle \\ &= \|a\xi\|^2 - \|b\xi\|^2 + \langle (bb^* - ab^*)\xi | \xi \rangle + \langle (bb^* - ba^*)\xi | \xi \rangle \\ &= \|a\xi\|^2 - \|b\xi\|^2 + \langle (b - a)b^*\xi | \xi \rangle + \langle \xi | (b - a)b^*\xi \rangle \\ &\leq \|(a - b)\xi\|(\|a\xi\| + \|b\xi\|) + 2\|(b - a)b^*\xi\|\|\xi\|. \end{aligned}$$

By Proposition 2.7.19 and part (i), to show (iii) we need only show that the strong operator topology and the weak operator topology agree on $\mathcal{U}(\mathcal{H})$. Since the weak operator topology is weaker than the strong operator topology, we need only show that if a net of unitaries converges in the weak operator topology, then it converges in the strong operator topology. This follows from the following inequality for $u, v \in \mathcal{U}(\mathcal{H})$ and $\xi \in \mathcal{H}$:

$$\begin{aligned} \|(u - v)\xi\|^2 &= \langle (u - v)\xi | (u - v)\xi \rangle \\ &= \langle (u - v)\xi | u\xi \rangle - \langle (u - v)\xi | v\xi \rangle \\ &= \langle \xi | (u^* - v^*)u\xi \rangle - \langle (u - v)\xi | v\xi \rangle \\ &= \langle \xi | \xi - v^*u\xi \rangle - \langle (u - v)\xi | v\xi \rangle \\ &= \langle v\xi | v\xi - u\xi \rangle - \langle (u - v)\xi | v\xi \rangle \\ &= \overline{\langle u\xi - v\xi | v\xi \rangle} - \langle (u - v)\xi | v\xi \rangle \\ &= -2 \operatorname{Im} \langle (u - v)\xi | v\xi \rangle. \quad \square \end{aligned}$$

While none of the operator topologies on $\mathcal{B}(\mathcal{H})$ are metrizable when \mathcal{H} is infinite-dimensional, the situation is different on the unit ball, at least when \mathcal{H} is separable.

2.7.23 Proposition. *Let \mathcal{H} be a separable Hilbert space. Then the unit ball of $\mathcal{B}(\mathcal{H})$ is metrizable with respect to each of the weak operator, strong operator, and strong* operator topologies.*

PROOF. Let $(\xi_n)_{n=1}^\infty$ be a dense sequence in the unit ball of \mathcal{H} . Then the metric

$$d_w(a, b) = \sum_{n=1}^{\infty} \frac{1}{2^n} \langle (a - b)\xi_n | \xi_n \rangle$$

induces the weak operator topology on the unit ball of $\mathcal{B}(\mathcal{H})$, the metric

$$d_s(a, b) = \sum_{n=1}^{\infty} \frac{1}{2^n} \|(a - b)\xi_n\|$$

induces the strong operator topology, and the metric

$$d_{s^*}(a, b) = \sum_{n=1}^{\infty} \frac{1}{2^n} (\|(a - b)\xi_n\| + \|(a - b)^*\xi_n\|)$$

induces the strong* operator topology. \square

The self-adjoint part of $\mathcal{B}(\mathcal{H})$ is monotone order complete with respect to its usual ordering. Perhaps more importantly, the supremum of a bounded increasing net is also its strong operator limit.

2.7.24 Proposition. *Let \mathcal{H} be a Hilbert space, and $(a_i)_{i \in I}$ a bounded increasing net in $\mathcal{B}(\mathcal{H})_{\text{sa}}$. Then $(a_i)_{i \in I}$ has a supremum, which is also its strong operator limit.*

PROOF. Let M be a bound for the norms of the elements of the net $(a_i)_{i \in I}$. Then for every $\xi \in \mathcal{H}$, the net $(\langle a_i \xi | \xi \rangle)_{i \in I}$ is an increasing net of real numbers bounded above by $\langle M \xi | \xi \rangle = M \|\xi\|^2$. Thus we may define a quadratic form

$$\Lambda(a) = \lim_{i \in I} \langle a_i \xi | \xi \rangle = \sup_{i \in I} \langle a_i \xi | \xi \rangle.$$

Then, by the Riesz-Fréchet Theorem, there is an $a \in \mathcal{B}(\mathcal{H})$ defined by the polarization identity

$$\langle a \xi | \eta \rangle = \frac{1}{4} \sum_{k=1}^4 i^k \Lambda(\xi + i^k \eta).$$

As the polarization identity is valid for each a_i , it follows that

$$\lim_{i \in I} \langle a_i \xi | \eta \rangle = \langle a \xi | \eta \rangle$$

for every $\xi, \eta \in \mathcal{H}$. Therefore, a is the weak operator topology limit of $(a_i)_{i \in I}$. Since

$$\langle a \xi | \xi \rangle = \sup_{i \in I} \langle a_i \xi | \xi \rangle,$$

it is clear that $a_i \leq a$ for all $i \in I$, and that no lesser element of \mathcal{M}_{sa} has this property. Therefore, $a = \sup_{i \in I} a_i$.

Now we will show convergence in the strong operator topology. If $b \geq 0$, define the sesquilinear form $[\xi, \eta] = \langle b \xi | \eta \rangle$ on \mathcal{H} . Then, using the Cauchy-Schwarz inequality, we have

$$\|b \xi\|^2 = [\xi, b \xi] \leq [\xi, \xi]^{1/2} [b \xi, b \xi]^{1/2} = \langle b \xi | \xi \rangle^{1/2} \langle b^3 \xi | \xi \rangle^{1/2}.$$

Thus for $\xi \in \mathcal{H}$, $i_0 \in I$ and any $i \geq i_0$,

$$\begin{aligned} \|(a - a_i)\xi\|^2 &\leq \langle (a - a_i)\xi | \xi \rangle^{1/2} \langle (a - a_i)^3 \xi | \xi \rangle^{1/2} \\ &\leq \|a - a_{i_0}\|^3 \|\xi\|^2 \langle (a - a_i)\xi | \xi \rangle^{1/2}. \end{aligned}$$

The right side converges to 0, so $(a_i)_{i \in I}$ converges to a in the strong operator topology. \square

Historical Notes

The weak operator, strong operator, and σ -strong topologies were first studied by von Neumann [vN36]. The σ -weak topology was introduced by Dixmier [Dix50a], who realized that it was the weak* topology induced by the trace-class operators. In that article, Dixmier also characterized the continuous linear functionals of the weak operator and strong operator topologies. Dixmier was able to employ the general duality theory of Banach spaces, which was only in its infancy in the 1930s. The monotone order completeness of $\mathcal{B}(\mathcal{H})_{sa}$ shown in Proposition 2.7.24 is also due to Dixmier [Dix50b].

2.8 Density Theorems

2.8.1 Definition. Let \mathcal{H} be a Hilbert space. If $S \subseteq \mathcal{B}(\mathcal{H})$, the *commutant* of S is

$$S' = \{b \in \mathcal{B}(\mathcal{H}) : ab = ba \text{ for all } a \in S\}.$$

2.8.2 Proposition. Let \mathcal{H} be a Hilbert space.

- (i) If $S \subseteq \mathcal{B}(\mathcal{H})$ is closed under the adjoint, then S' is a unital $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed in the strong operator topology.
- (ii) If $a = \oplus_{i \in I} a_i$ is a bounded operator on $\mathcal{H} = \oplus_{i \in I} \mathcal{H}_i$ and $b = [b_{ij}] \in \mathcal{B}(\mathcal{H})$, then $b \in \{a\}'$ if and only if $b_{ij}a_j = a_i b_{ij}$ for all $i, j \in I$.
- (iii) If $a \in \mathcal{B}(\mathcal{H})$ and $b = [b_{ij}] \in \mathcal{B}(\mathcal{H}^{(n)})$, then $b \in \{a^{(n)}\}'$ if and only if $b_{ij}a = ab_{ij}$ for $i, j = 1, \dots, n$.
- (iv) If $S \subseteq \mathcal{B}(\mathcal{H})$, then $S^{(n)''} = S''^{(n)}$.

PROOF.

- (i) Since the identity commutes with all operators, $1 \in S'$. If $a, b \in S'$, then $a + b \in S'$ and $ab \in S'$. If $a \in S'$ and $\lambda \in \mathbb{C}$, then $\lambda a \in S'$. If $a \in S'$, then $ab = ba$ for all $b \in S$, so $b^*a^* = a^*b^*$ for all $b \in S$. Since S is closed under taking adjoints, this implies that $a^* \in S'$. Finally, pointwise limits of elements of S' are obviously also in S' , so S' is strong operator closed.
- (ii) This is a simple restatement of the identity $ab = ba$ for this particular setting.
- (iii) This is a special case of (ii).
- (iv) By applying (iii) to every $a \in S$, we have that $b = [b_{ij}] \in S^{(n)'}$ if and only if each matrix entry belongs to S' . Hence an operator $b = [b_{ij}]$ in $S^{(n)''}$ must commute with every matrix unit e_{ij} , which has 1 in the (i, j) entry and 0 elsewhere. This forces b to be diagonal and satisfy $b_{ii} = b_{jj}$ for $1 \leq i, j \leq n$. Thus, $b = b_{11}^{(n)}$. In addition, b commutes with $a^{(n)}$ for every $a \in S'$, so by (iii) $b_{11} \in S''$. Therefore, $S^{(n)''} = S''^{(n)}$. \square

A remarkable theorem of von Neumann shows that the double commutant of a non-degenerate self-adjoint algebra of operators is its closure in the strong operator topology. Note that it isn't even a priori clear that the closure of an algebra in any of the operator topologies is also an algebra.

2.8.3 Theorem (Double Commutant Theorem). *Let \mathcal{A} be a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ that acts nondegenerately on \mathcal{H} , i.e. such that $\overline{\mathcal{A}\mathcal{H}} = \mathcal{H}$. Then \mathcal{A}'' is the closure of \mathcal{A} in the strong operator topology.*

PROOF. Clearly, the strong operator topology closure of \mathcal{A} is contained in \mathcal{A}'' , because \mathcal{A}'' is closed in the strong operator topology and contains \mathcal{A} . Thus we want to show that \mathcal{A} is strong operator dense in \mathcal{A}'' . Fix an operator $b \in \mathcal{A}''$. By possibly scaling the vectors involved, an arbitrary basic strong operator neighbourhood of b is

$$\{a \in \mathcal{B}(\mathcal{H}) : \|(b - a)\xi_i\| < 1 \text{ for } i = 1, \dots, n\}$$

for some vectors $\xi_1, \dots, \xi_n \in \mathcal{H}$. It suffices to show that there exists an $a \in \mathcal{A}$ such that

$$\sum_{i=1}^n \|(b - a)\xi_i\|^2 < 1.$$

We will first consider the case where $n = 1$. Let p be the orthogonal projection onto the subspace $\overline{\mathcal{A}\xi_1}$, where the closure is taken with the norm topology. We claim that $p \in \mathcal{A}'$. Since $\mathcal{A}p\mathcal{H} \subseteq p\mathcal{H}$, we have that $pap = ap$ for every $a \in \mathcal{A}$. Therefore

$$pa = (a^*p)^* = (pa^*p)^* = pap = ap,$$

and $p \in \mathcal{A}'$. Since \mathcal{A}' is a unital algebra, $1 - p \in \mathcal{A}'$. We now want to show that there is an operator $a \in \mathcal{A}$ such that $\|(b - a)\xi_1\| < 1$. If $\eta = (1 - p)\xi_1$, then we have

$$\mathcal{A}\eta = \mathcal{A}(1 - p)\xi_1 = (1 - p)\mathcal{A}\xi_1 = p^\perp \mathcal{A}\xi_1 = 0.$$

Since \mathcal{A} acts non-degenerately on \mathcal{H} , $\eta = 0$, i.e. $\xi_1 \in \overline{\mathcal{A}\xi_1}$. Since $b \in \mathcal{A}''$, we have that $pb = bp$. Hence $b\xi_1 \in \overline{\mathcal{A}\xi_1}$. Therefore, there is an operator $a \in \mathcal{A}$ such that $\|(b - a)\xi_1\| < 1$.

Now, consider the case where $n \geq 2$. By Proposition 2.8.2 (iv), $\mathcal{A}^{(n)''} = (\mathcal{A}'')^{(n)}$. Applying the case where $n = 1$ to $b^{(n)} \in (\mathcal{A}'')^{(n)}$ and $\xi = (\xi_1, \dots, \xi_n)$ gives an operator $a \in \mathcal{A}$ such that

$$\|(b^{(n)} - a^{(n)})x\| < 1.$$

Thus

$$\|(b^{(n)} - a^{(n)})x\|^2 < 1,$$

and

$$\sum_{i=1}^n \|(b - a)x_i\|^2 = \|(b^{(n)} - a^{(n)})x\|^2 < 1,$$

so a lies in the given strong operator topology neighbourhood of T . \square

2.8.4 Definition. Let \mathcal{H} be a Hilbert space, and \mathcal{A} a self-adjoint unital subalgebra of $\mathcal{B}(\mathcal{H})$. We say that \mathcal{A} is a *von Neumann algebra* if \mathcal{A} is closed in the strong operator topology.

If $S \subseteq \mathcal{B}(\mathcal{H})$ is a self-adjoint set of operators, we call S'' the von Neumann algebra generated by S . In particular, if $a \in \mathcal{B}(\mathcal{H})$, we let $W^*(a)$ denote $\{a, a^*\}''$, the von Neumann algebra generated by a . Note that

$$\{a, a^*\} \subseteq C^*(a) \subseteq W^*(a),$$

so by the Double Commutant Theorem, $W^*(a)$ is the strong operator topology closure of $C^*(a)$. If a is normal, we can simplify this definition a bit.

2.8.5 Proposition (Fuglede-Putnam). *Let \mathcal{A} be a C^* -algebra and $m, n \in \mathcal{A}$ be normal. If $a \in \mathcal{A}$ satisfies $ma = an$, then $m^*a = an^*$.*

PROOF. It follows from the hypotheses that $m^k a = a n^k$ for all $k \in \mathbb{N}$. Hence $p(m)a = ap(n)$ for every univariate polynomial p with complex coefficients. For a fixed $\alpha \in \mathbb{C}$, $\exp(i\bar{\alpha}m)$ and $\exp(i\bar{\alpha}n)$ are limits of polynomials in m and n respectively, so $e^{i\bar{\alpha}m}a = ae^{i\bar{\alpha}n}$ for all $\alpha \in \mathbb{C}$, or equivalently, $a = e^{-i\bar{\alpha}m}ae^{i\bar{\alpha}n}$. Define $f : \mathbb{C} \rightarrow \mathcal{B}(\mathcal{H})$ by

$$f(\alpha) = e^{-i\alpha m^*} m^* a e^{i\alpha n^*}.$$

Since $e^{x+y} = e^x e^y$ when x and y commute, the normality of m and n implies that

$$f(\alpha) = e^{-i\alpha m^*} e^{-i\bar{\alpha}m} a e^{i\bar{\alpha}n} e^{i\alpha n^*} = e^{-i(\alpha m^* + \bar{\alpha}m)} a e^{i(\bar{\alpha}n + \alpha n^*)}.$$

For every $\alpha \in \mathbb{C}$, $\alpha m^* + \bar{\alpha}m$ and $\alpha n^* + \bar{\alpha}n$ are self-adjoint. Hence $e^{-i(\alpha m^* + \bar{\alpha}m)}$ and $e^{i(\bar{\alpha}n + \alpha n^*)}$ are unitary. Therefore, $\|f(\alpha)\| \leq \|a\|$. But f is a vector-valued entire function, so by Liouville's Theorem, f is constant. Therefore,

$$0 = f'(\alpha) = -im^* e^{-i\alpha n^*} a e^{i\alpha n^*} + ie^{-i\alpha m^*} a n^* e^{i\alpha n^*}.$$

Substituting 0 for α gives $-im^* a + iam^* = 0$, i.e. $m^* a = an^*$. \square

2.8.6 Corollary. *Let $a \in \mathcal{B}(\mathcal{H})$ be normal. Then $W^*(a) = \{a\}''$.*

PROOF. Clearly, $\{a, a^*\}' \subseteq \{a\}'$. If b commutes with a , then by the Fuglede-Putnam Theorem, b also commutes with a^* . Hence $\{a, a^*\}' = \{a\}'$, and $W^*(a) = \{a, a^*\}'' = \{a\}''$. \square

Another important example of a von Neumann algebra is $L^\infty(X, \mu)$, acting on $L^2(X, \mu)$ as multiplication operators, for any “reasonable” measure space (X, μ) .

2.8.7 Proposition. *Let (X, μ) be a localizable measure space. Then $L^\infty(X, \mu)$ is a maximal commutative von Neumann algebra, when viewed as multiplication operators on $L^2(X, \mu)$.*

PROOF. Since any maximal commutative subalgebra of $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra, we only need to show that $L^\infty(X, \mu)$ is maximal commutative, i.e. that if $a \in \mathcal{B}(L^2(X, \mu))$ is nonzero and commutes with every element of $L^\infty(X, \mu)$, then $a \in L^\infty(X, \mu)$.

We will first establish the case where (X, μ) is finite. The constant function 1 belongs to $L^2(X, \mu)$, so we can define a function $g \in L^2(X, \mu)$ by $g = a1$. We will show that $g \in L^\infty(X, \mu)$, $\|g\|_\infty \leq \|a\|$, and $a = M_g$. Note that for every $f \in L^\infty(X, \mu)$ we have

$$fg = M_f a 1 = a M_f 1 = a f.$$

Since $a \neq 0$, it follows that $g \neq 0$ and

$$\|fg\|_2 \leq \|a\| \|f\|_2.$$

Taking the case where $f = \chi_E$ for a Borel subset E of X , we have

$$\int_E |g|^2 d\mu = \|\chi_E g\|_2^2 \leq \|a\|^2 \|\chi_E\|_2^2 = \|a\|^2 \mu(E).$$

This inequality implies that $|g(x)| \leq \|a\|$ almost everywhere. Indeed, if $C \geq 0$ is any number such that

$$E = \{x \in X : |g(x)| > C\}$$

has positive measure, then the above inequality implies that

$$C^2 \mu(E) \leq \int_E |g|^2 \leq \|a\|^2 \mu(E),$$

and hence that $C \leq \|a\|$. Since $\|g\|_\infty$ is the supremum of all such C , it follows that $g \in L^\infty(X, \mu)$ and $\|g\|_\infty \leq \|a\|$.

Now, suppose that (X, μ) is a general localizable measure space. Then for the purposes of considering $L^\infty(X, \mu)$, we may assume that (X, μ) is strictly localizable, i.e. that there exists a partition $(E_i)_{i \in I}$ into measurable sets of finite measure such that $E \subseteq X$ is measurable if and only if $E \cap E_i$ is measurable for every $i \in I$, in which case

$$\mu(E) = \sum_{i \in I} \mu(E \cap E_i).$$

Let μ_i be the restriction of μ to E . Then $L^2(X, \mu)$ decomposes as a direct sum of Hilbert spaces

$$L^2(X, \mu) \cong \bigoplus_{i \in I} L^2(E_i, \mu_i).$$

For every $i \in I$, the projection from $L^2(X, \mu)$ onto $L^2(E_i, \mu_i)$ is given by multiplication by the characteristic function of E_i , and is thus contained in $L^\infty(X, \mu)$. In particular, it must commute with a , so a decomposes as a direct sum $a = \bigoplus_{i \in I} a_i$, where $a_i \in \mathcal{B}(L^2(E_i, \mu_i))$ for every $i \in I$. Since each a_i must commute with $L^\infty(E_i, \mu_i)$, it follows from the finite case done above that there exists an $f_i \in L^\infty(E_i, \mu_i)$ such that $a_i = M_{f_i}$ and

$$\|f_i\|_\infty \leq \|a_i\| \leq \|a\|.$$

Since the f_i are uniformly bounded, there is an $f \in L^\infty(X, \mu)$ such that $f|_{E_i} = f_i$ for every $i \in I$. It follows that $a = M_f$. \square

2.8.8 Remark. It is the case that every commutative von Neumann algebra \mathcal{M} is $*$ -isomorphic to $L^\infty(X, \mu)$ for some localizable measure space (X, μ) , although we will not prove this. If \mathcal{M} is a maximal commutative subalgebra of $\mathcal{B}(\mathcal{H})$, then this $*$ -isomorphism is implemented by a unitary operator between \mathcal{H} and $L^2(X, \mu)$. It is also true that while $L^\infty(X, \mu)$ is always a C^* -algebra, it is only a von Neumann algebra (acting as multiplication operators on $L^2(X, \mu)$) when (X, μ) is localizable.

2.8.9 Proposition. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then \mathcal{M} is a dual space.*

PROOF. Since \mathcal{M} is closed in the strong operator topology, it is closed in the σ -weak topology, which is the weak* topology from $\mathcal{T}(\mathcal{H})$. Let \mathcal{M}_\perp be the preannihilator of \mathcal{M} in $\mathcal{T}(\mathcal{H})$. Then, by the general theory of Banach space duality, $(\mathcal{T}(\mathcal{H})/\mathcal{M}_\perp)^*$ is isometrically isomorphic to \mathcal{M} . \square

The Double Commutant Theorem does not remain true as stated for subalgebras that act nondegenerately, because the commutant of any algebra contains the identity and there are many algebras closed in the strong operator topology that do not contain the identity, e.g. the natural embedding of $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ given by $A \mapsto A \oplus 0$. In general, the strong closure of a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ is not a von Neumann algebra, but it is a von Neumann algebra when it is restricted to the maximal subspace on which it acts nondegenerately. This fact turns out to be very useful when studying ideals in von Neumann algebras.

2.8.10 Corollary. *Let \mathcal{A} be a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ and \mathcal{M} the closure of \mathcal{A} in the strong operator topology. Define $\mathcal{H}_0 = \overline{\mathcal{A}\mathcal{H}}$. Then:*

- (i) \mathcal{M} is a unital C^* -algebra, where the identity for \mathcal{M} is the projection onto \mathcal{H}_0 ;
- (ii) the restriction algebra $\mathcal{M}_{|\mathcal{H}_0}$ of operators in \mathcal{M} restricted to \mathcal{H}_0 is a von Neumann algebra that is $*$ -isomorphic to \mathcal{M} ;
- (iii) if \mathcal{M}'' is the double commutant of \mathcal{M} , then

$$\mathcal{M}'' = \mathcal{M} \oplus \mathbb{C}(1 - e) = \{a + \alpha 1 : a \in \mathcal{M}, \alpha \in \mathbb{C}\}.$$

PROOF. Without the Double Commutant Theorem, which does not apply in this case due to the possibly degeneracy of the action of \mathcal{A} on \mathcal{H} , we do not know a priori that \mathcal{M} is an algebra.

Let $(e_i)_{i \in I}$ be a bounded approximate identity for \mathcal{A} , which we can take to be an increasing net of positive operators. Then $(e_i)_{i \in I}$ is a bounded increasing net in $\mathcal{B}(\mathcal{H})_{\text{sa}}$, so it strongly converges to its supremum $e \in \mathcal{M}$. Since $(e_i)_{i \in I}$ is a bounded approximate identity, $ae_i \rightarrow e_i$ in the norm topology for every $a \in \mathcal{A}$. By the separate continuity of multiplication in the strong operator topology, $ae_i \rightarrow ae$ in the strong operator topology. Since norm convergence implies strong convergence, this implies that $a = ae$ for every $a \in \mathcal{A}$. Similarly, $a = ea$ for every $a \in \mathcal{A}$. Again by the separate continuity of multiplication in the strong operator topology, it follows that $b = be$ and $b = eb$ for every $b \in \mathcal{M}$. In particular, $e = e^2$. Since e is self-adjoint by construction, e is a projection in $\mathcal{B}(\mathcal{H})$. Since e is the strong operator limit of a bounded approximate identity in \mathcal{A} , the range of e must be \mathcal{H}_0 . Since $b = be$ and $b = eb$ for every $b \in \mathcal{M}$, the range of every element of \mathcal{M} is contained in \mathcal{H}_0 . Since \mathcal{M} is strong operator topology closed it is also strong*-operator topology closed, so it is self-adjoint. The restriction map from \mathcal{M} to $\mathcal{M}_{|\mathcal{H}_0}$ is then a $*$ -isomorphism, in the sense that it is a adjoint-preserving vector space isomorphism that also preserves multiplication when it is defined. Since \mathcal{M} is the strong closure of \mathcal{A} , $\mathcal{M}_{|\mathcal{H}_0}$ is the strong closure of the restriction algebra $\mathcal{A}_{|\mathcal{H}_0}$ of operators in \mathcal{A} restricted to \mathcal{H}_0 . Since $\mathcal{A}_{|\mathcal{H}_0}$ acts nondegenerately on \mathcal{H}_0 , by the Double Commutant Theorem $\mathcal{M}_{|\mathcal{H}_0}$ is a von Neumann algebra. Hence \mathcal{M} is a unital C^* -algebra, where e is the identity element of \mathcal{M} . This shows (i) and (ii).

To show (iii), note that both \mathcal{H}_0 and \mathcal{H}_0^\perp are invariant under the action of \mathcal{A} . Since \mathcal{M} is the strong operator closure of \mathcal{A} , they are also both invariant under the action of \mathcal{M} , i.e. both e and $(1 - e)$ are in \mathcal{M}' . Since $\mathcal{M}' \subseteq (\mathcal{M}'')'$, both e and $(1 - e)$ are in $(\mathcal{M}'')'$, i.e. both \mathcal{H}_0 and \mathcal{H}_0^\perp are invariant under the action of \mathcal{M}'' . Clearly, the restriction algebra $\mathcal{M}_{|\mathcal{H}_0^\perp}$ is the trivial algebra $\{0\}$, so that $(\mathcal{M}_{|\mathcal{H}_0^\perp})' = \mathcal{B}(\mathcal{H}_0^\perp)$. If $a \in \mathcal{M}''$, then the restriction $a_{|\mathcal{H}_0^\perp}$ lies in $(\mathcal{M}_{|\mathcal{H}_0^\perp})'' = \mathbb{C} \cdot 1_{\mathcal{H}_0^\perp}$, so that $a - \lambda 1$ vanishes on \mathcal{H}_0^\perp and belongs to \mathcal{M}'' . Therefore, $(a - \lambda 1)_{|\mathcal{H}_0}$ lies in

$$(\mathcal{M}'')_{|\mathcal{H}_0} \subseteq (\mathcal{M}_{|\mathcal{H}_0})'' = \mathcal{M}_{|\mathcal{H}_0},$$

so there exists a $b \in \mathcal{M}$ such that $b_{|\mathcal{H}_0} = (a - \lambda 1)_{|\mathcal{H}_0}$. Both b and $a - \lambda 1$ vanish on \mathcal{H}_0^\perp , so that $b = a - \lambda 1$. Hence $a = b + \lambda 1$ as desired. \square

Since $L^\infty(X, \mu)$ is the prototypical example of a commutative von Neumann algebra, it is natural to expect normal operators in a von Neumann algebra to have a functional calculus for essentially bounded functions, analogous to the continuous functional calculus for C^* -algebras. Unfortunately, one can only speak of essentially bounded functions with respect to a particular Radon measure on the spectrum, and for operators on a non-separable Hilbert space, the question of which measure to choose becomes quite complicated. Therefore, we will describe a functional calculus for bounded Borel functions on the spectrum, which works without any additional complications for operators on non-separable Hilbert spaces as well.

If X is a locally compact Hausdorff space, let $B_b(X)$ be the Banach space of bounded complex-valued Borel functions on X , given the supremum norm. When equipped with pointwise operations, $B_b(X)$ is a C^* -algebra, where the identity is the constant function 1.

2.8.11 Theorem (Borel Functional Calculus). *Let $a \in \mathcal{B}(\mathcal{H})$ be a normal operator, and $\Gamma_0 : C(\sigma(a)) \rightarrow C^*(a)$ the inverse of the Gelfand transform. Then there exists an extension of Γ_0 to a $*$ -homomorphism $\Gamma : B_b(\sigma(a)) \rightarrow W^*(a)$. Moreover, if $(f_n)_{n=1}^\infty$ is a uniformly bounded sequence in $B_b(\sigma(a))$ converging pointwise to f , then $\Gamma(f_n) \rightarrow \Gamma(f)$ in the strong* operator topology.*

PROOF. For all $\xi, \eta \in \mathcal{H}$, define a linear functional $F_{\xi, \eta}$ on $C(\sigma(a))$ by

$$F_{\xi, \eta}(f) = \langle \Gamma(f)\xi \mid \eta \rangle.$$

Since Γ is a $*$ -homomorphism, it is a contraction, so by the Cauchy-Schwarz inequality we have $\|F_{\xi, \eta}\| \leq \|\xi\| \|\eta\|$. By the Riesz Representation Theorem, there exists a finite Radon measure $\mu_{\xi, \eta}$ on X such that

$$\langle \Gamma(f)\xi \mid \eta \rangle = F_{\xi, \eta}(f) = \int_{\sigma(a)} f \, d\mu_{\xi, \eta}$$

and $\|\mu_{\xi, \eta}\| \leq \|\xi\| \|\eta\|$. Now, for $f \in B_b(\sigma(a))$ and $\xi, \eta \in \mathcal{H}$, define a sesquilinear form B_f on \mathcal{H} by

$$B_f(\xi, \eta) = \int_{\sigma(a)} f \, d\mu_{\xi, \eta}.$$

It follows that $\|B_f\| \leq \|f\| \|\xi\| \|\eta\|$. Therefore, by the Riesz-Fréchet Theorem, there exists a bounded operator $\Gamma(f)$ on \mathcal{H} such that

$$\langle \Gamma(f)\xi \mid \eta \rangle = B_f(\xi, \eta) = \int_{\sigma(a)} f \, d\mu_{\xi, \eta}$$

and $\|\Gamma(f)\| \leq \|f\| \|\xi\| \|\eta\|$. This defines a linear map $\Gamma : B_b(\sigma(a)) \rightarrow \mathcal{B}(\mathcal{H})$ that extends Γ_0 .

We will first show that Γ is a $*$ -homomorphism. If $f \in C(\sigma(a))$ and $\xi, \eta \in \mathcal{H}$, then

$$\begin{aligned} \int_{\sigma(a)} f d\mu_{\xi, \eta} &= \langle \Gamma_0(f)\xi | \eta \rangle \\ &= \overline{\langle \Gamma_0(\bar{f})\eta | \xi \rangle} \\ &= \int_{\sigma(a)} \bar{f} d\mu_{\eta, \xi}. \end{aligned}$$

Hence $\overline{\mu_{\xi, \eta}} = \mu_{\eta, \xi}$. In particular, $\mu_{\xi, \xi}$ is a positive measure for every $\xi \in \mathcal{H}$. It follows by a similar computation that $\Gamma(\bar{f}) = \Gamma(f)^*$ for every $f \in B_b(\sigma(a))$. If $g \in C(\sigma(a))$ and $\xi, \eta \in \mathcal{H}$, we have $g \cdot \mu_{\xi, \eta} = \mu_{\Gamma(g)\xi, \eta}$. Indeed, if $f \in C(\sigma(a))$, then

$$\int_X fg d\mu_{\xi, \eta} = \langle \Gamma(fg)\xi | \eta \rangle = \langle \Gamma(f)\Gamma(g)\xi | \eta \rangle = \int_X f d\mu_{\Gamma(g)\xi, \eta}.$$

Similarly, if $f \in B_b(\sigma(a))$, then $f \cdot \mu_{\xi, \eta} = \mu_{\xi, \Gamma(f)^*\eta}$. Indeed, if $g \in C(\sigma(a))$, then

$$\begin{aligned} \int_X g d(f \cdot \mu_{\xi, \eta}) &= \int_X gf d\mu_{\xi, \eta} \\ &= \int_X f d\mu_{\Gamma(g)\xi, \eta} \\ &= \langle \Gamma(f)\Gamma(g)\xi | \eta \rangle \\ &= \langle \Gamma(g)\xi | \Gamma(f)^*\eta \rangle \\ &= \int_X g d\mu_{\xi, \Gamma(f)^*\eta}. \end{aligned}$$

Now, if $f, g \in B_b(\sigma(a))$, then

$$\begin{aligned} \langle \Gamma(fg)\xi | \eta \rangle &= \int_X fg d\mu_{\xi, \eta} \\ &= \int_X g d\mu_{\xi, \Gamma(f)^*\eta} \\ &= \langle \Gamma(g)\xi | \Gamma(f)^*\eta \rangle \\ &= \langle \Gamma(f)\Gamma(g)\xi | \eta \rangle. \end{aligned}$$

Therefore, Γ is a $*$ -homomorphism.

We will now show that Γ takes uniformly bounded sequences converging pointwise to sequences converging in the strong* operator topology. If $f \in B_b(\sigma(a))$, we have

$$\begin{aligned} \|\Gamma(f)\xi\|^2 &= \langle \Gamma(f)\xi | \Gamma(f)\xi \rangle \\ &= \langle \Gamma(f)^*\Gamma(f)\xi | \xi \rangle \\ &= \langle \Gamma(|f|^2)\xi | \xi \rangle \\ &= \int_{\sigma(a)} |f|^2 d\mu_{\xi, \xi}. \end{aligned}$$

Without loss of generality, we can let $(f_n)_{n=1}^\infty$ be a uniformly bounded sequence converging pointwise to 0. For every $\xi \in \mathcal{H}$ we have

$$\|\Gamma(f_n)\xi\|^2 = \int_{\sigma(a)} |f_n|^2 d\mu_{\xi,\xi}.$$

The right side converges to 0 as $n \rightarrow \infty$ by the Lebesgue Dominated Convergence Theorem, since $\mu_{\xi,\xi}$ is a positive finite measure on X and $|f_n|^2$ is a uniformly bounded sequences of positive functions converging pointwise to zero. Therefore, $\Gamma(f_n) \rightarrow 0$ in the strong operator topology. Since $(\overline{f_n})_{n=1}^\infty$ is also a uniformly bounded sequence in $B_b(\sigma(a))$ converging pointwise to 0, $\Gamma(f_n)^* \rightarrow 0$ in the strong operator topology, and thus $\Gamma(f_n) \rightarrow 0$ in the strong* operator topology.

Finally, we will show that the range of Γ is equal to $W^*(a)$. Note that $B_b(\sigma(a))$ naturally embeds into $C(\sigma(a))^{**} \cong M(\sigma(a))^*$ via the dual pairing

$$\langle \mu, f \rangle = \int_{\sigma(a)} f d\mu.$$

By the construction of Γ , it is clear that Γ is continuous with respect to the inherited weak* topology on $B_b(\sigma(a))$ and the weak operator topology on $\mathcal{B}(\mathcal{H})$. By Goldstine's Theorem, every element of $C(\sigma(a))^{**}$ is the weak* limit of a net in $C(\sigma(a))$. Therefore, every element in the range of Γ is the weak operator topology limit of a net in $C^*(a)$, which shows that the range of Γ is contained in $W^*(a)$. \square

Just like the continuous functional calculus, we will use $f(a)$ to denote the image of f in the *-homomorphism above. The main innovation of the Borel functional calculus compared to the continuous functional calculus is that we now have a functional calculus that includes characteristic functions of Borel subsets of the spectrum, which allows us to show that von Neumann algebras have many projections. In general, a C^* -algebra need not have any nontrivial projections.

2.8.12 Proposition. *Let \mathcal{M} be a von Neumann algebra. Then \mathcal{M} is the norm closed linear span of its projections.*

PROOF. Since \mathcal{M} is the linear span of its positive elements, we need only show that every positive $a \in \mathcal{M}$ is in the norm closed span of the projections of \mathcal{M} . By scaling, we may assume that $0 \leq a \leq 1$. Then, by the Borel functional calculus, we have

$$a = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \chi_{(k/n, 1]}(a).$$

\square

2.8.13 Proposition. *Let $a \in \mathcal{B}(\mathcal{H})$ be a positive operator and $\lambda > 0$ a positive number. Then there exists a projection $p \in W^*(a)$ such that*

$$ap \geq \lambda p \quad \text{and} \quad a(1 - p) \leq \lambda(1 - p).$$

PROOF. Let $p = \chi_{(\lambda, \infty)}(a)$. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(t) = \begin{cases} 0 & \text{if } t \leq \lambda \\ t - \lambda & \text{if } t > \lambda. \end{cases}$$

Then

$$\begin{aligned} ap - \lambda p &= a\chi_{(\lambda, \infty)}(a) - \lambda\chi_{(\lambda, \infty)}(a) \\ &= (\text{id}_{\mathbb{R}}\chi_{(\lambda, \infty)})(a) - (\lambda\chi_{(\lambda, \infty)})(a) \\ &= ((\text{id}_{\mathbb{R}} - \lambda)\chi_{(\lambda, \infty)})(a) \\ &= f(a) \\ &\geq 0, \end{aligned}$$

so $ap \geq \lambda p$. Similarly, define $g : \mathbb{R} \rightarrow \mathbb{R}$ by

$$g(t) = \begin{cases} \lambda - t & \text{if } t \leq \lambda \\ 0 & \text{if } t > \lambda. \end{cases}$$

Then

$$\begin{aligned} \lambda(1 - p) - a(1 - p) &= \lambda(1 - \chi_{(\lambda, \infty)}(a)) - a(1 - \chi_{(\lambda, \infty)}(a)) \\ &= (\lambda - \lambda\chi_{(\lambda, \infty)})(a) - (\text{id}_{\mathbb{R}} - \text{id}_{\mathbb{R}}\chi_{(\lambda, \infty)})(a) \\ &= ((\lambda - \text{id}_{\mathbb{R}})(1 - \chi_{(\lambda, \infty)}))(a) \\ &= g(a) \\ &\geq 0, \end{aligned}$$

so $a(1 - p) \leq \lambda(1 - p)$. □

If $\lambda \in \mathbb{R}$ is such that $0 \leq \lambda \leq 1$, then λ has a unique base-2 expansion. If X is a second countable locally compact Hausdorff space and $f : X \rightarrow \mathbb{R}$ is a Borel function such that $0 \leq f \leq 1$, then the construction of a base-2 expansion of a real number can be mimicked to give a “base-2 expansion” of f , in the sense that

$$f = \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{E_n}$$

for some sequence $(E_n)_{n=1}^{\infty}$ of Borel subsets of X . This should be intuitively clear, because the construction of a base-2 expansion of a real number only uses countable operations. Using the Borel functional calculus described above, we will now show a similar fact about positive operators, which includes, as a special case, the case of Borel functions.

2.8.14 Proposition. Let $a \in \mathcal{B}(\mathcal{H})$ be such that $0 \leq a \leq 1$. Then there exists a sequence of projections $(p_n)_{n \in \mathbb{N}}$ in $W^*(a)$ such that

$$a = \sum_{n=1}^{\infty} \frac{1}{2^n} p_n,$$

where convergence of infinite sums is taken in the norm topology.

PROOF. We will define the sequence $(p_n)_{n=1}^{\infty}$ by induction. By Proposition 2.8.13, there exists a projection $p_1 \in W^*(a)$ such that

$$ap_1 \geq \frac{1}{2}p_1 \quad \text{and} \quad a(1-p_1) \leq \frac{1}{2}(1-p_1).$$

Now, suppose that p_1, \dots, p_n have been defined so that

$$\sum_{k=1}^n \frac{1}{2^k} p_k \leq a.$$

Then, again by Proposition 2.8.13, there exists a projection $p_{n+1} \in W^*(a)$ such that

$$\left(a - \sum_{k=1}^n \frac{1}{2^k} p_k \right) p_{n+1} \geq \frac{1}{2^{n+1}} p_{n+1}$$

and

$$\left(a - \sum_{k=1}^n \frac{1}{2^k} p_k \right) (1 - p_{n+1}) \leq \frac{1}{2^{n+1}} (1 - p_{n+1}).$$

Since $0 \leq a \leq 1$, it follows by induction that

$$0 \leq a - \sum_{k=1}^n \frac{1}{2^k} p_k \leq \frac{1}{2^n}$$

for every $n \in \mathbb{N}$, so that

$$a = \sum_{n=1}^{\infty} \frac{1}{2^n} p_n. \quad \square$$

If (X, \mathcal{A}, μ) is a measure space and $\mathcal{A}_0(\mu)$ is the collection of all μ -null sets in \mathcal{A} , the μ -distance on \mathcal{A} is given by

$$d(E, F) = \min(\mu(E \triangle F), 1) = \min(\mu(E \setminus F) + \mu(F \setminus E), 1).$$

Passing to equivalence classes, it follows that $\mathcal{A}/\mathcal{A}_0(\mu)$ is a complete metric space and a Boolean σ -algebra, where the Boolean algebra operations are simply the usual operations on sets applied to equivalence classes. The quotient $\mathcal{A}/\mathcal{A}_0(\mu)$ is called the measure algebra of (X, \mathcal{A}, μ) , not to be confused with the measure algebra $M(G)$ of a locally compact group G . If (X, \mathcal{A}, μ) is a localizable measure space, then $\mathcal{A}/\mathcal{A}_0(\mu)$ is actually a complete Boolean algebra. The $\mathcal{A}/\mathcal{A}_0(\mu)$ can be identified with the set of projections in $L^\infty(X, \mathcal{A}, \mu)$, and a similar fact holds for the projections of an arbitrary von Neumann algebra.

2.8.15 Proposition. *Let \mathcal{M} be a von Neumann algebra. Then the projections in \mathcal{M} form a complete lattice, where the ordering is the usual ordering on \mathcal{M}_{sa} .*

PROOF. Suppose $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$, and let $(p_i)_{i \in I}$ be a family of projections in \mathcal{M} . Then

$$\mathcal{H}_0 = \bigcap_{i \in I} p_i \mathcal{H}$$

is a closed subspace of \mathcal{H} . Let p be the projection onto \mathcal{H}_0 . We want to show that $p \in \mathcal{M}$. By the Double Commutant Theorem, $\mathcal{M} = \mathcal{M}''$, so it suffices to show that p commutes with every $a \in \mathcal{M}'$. Fix $a \in \mathcal{M}'$. For every $i \in I$, $p_i \in \mathcal{M}$, so p_i commutes a , i.e. $p_i \mathcal{H}$ is an invariant subspace for a . Thus \mathcal{H}_0 , the intersection of invariant subspaces of a , is also an invariant subspace of a , i.e. p commutes with a . It is clear that p is the infimum of the family $(p_i)_{i \in I}$. If q is the infimum of the family $(1 - p_i)_{i \in I}$, then $1 - q$ is the supremum of $(p_i)_{i \in I}$. \square

The Double Commutant Theorem has a slight flaw in that a convergent net need not be bounded, so it does not imply that an element of the unit ball of \mathcal{A}'' can be approximated in any of the operator topologies by an element of the unit ball of \mathcal{A} . Since multiplication is not jointly continuous for any of the operator topologies, this can be problematic. The Double Commutant Theorem also does not imply that the positive elements of \mathcal{A}'' can be approximated by the positive elements of \mathcal{A} , or that the unitary elements of \mathcal{A}'' can be approximated by the unitary elements of \mathcal{A} , etc. We will now show that each of these statements is true.

2.8.16 Theorem (Kaplansky Density Theorem). *Let \mathcal{A} be a $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$ that acts nondegenerately on \mathcal{H} , and S the unit ball of $\mathcal{B}(\mathcal{H})$.*

- (i) *The unit ball $\mathcal{A} \cap S$ of \mathcal{A} is strong operator dense in $\mathcal{A}'' \cap S$.*
- (ii) *The self-adjoint part $\mathcal{A}_{\text{sa}} \cap S$ of the unit ball of \mathcal{A} is strong operator dense in $(\mathcal{A}'')_{\text{sa}} \cap S$.*
- (iii) *The positive part $\mathcal{A}_+ \cap S$ of the unit ball of \mathcal{A} is strong operator dense in $(\mathcal{A}'')_+ \cap S$.*
- (iv) *If \mathcal{A} is unital, then the unitary group $\mathcal{U}(\mathcal{A})$ of \mathcal{A} is strong operator dense in $\mathcal{U}(\mathcal{A}'')$.*

PROOF. We may assume that \mathcal{A} is norm closed, because each of these statements holds when \mathcal{A}'' is replaced with the norm closure of \mathcal{A} . Let $\mathcal{M} = \mathcal{A}''$. We will prove these assertions in multiple steps. The map defined on \mathcal{M} by

$$a \mapsto \frac{1}{2}(a + a^*)$$

is weak operator topology continuous and has range \mathcal{M}_{sa} . When restricted to \mathcal{A} , it has range \mathcal{A}_{sa} . Since \mathcal{A} is weak operator dense in \mathcal{M} , it follows that \mathcal{A}_{sa}

is weak operator dense in \mathcal{M}_{sa} . Thus, \mathcal{A}_{sa} is also strong operator dense in \mathcal{M}_{sa} , because the weak operator topology and strong operator topology have the same closed convex sets.

Fix $a \in \mathcal{M}_{\text{sa}} \cap S$. The function $f : [-1, 1] \rightarrow [-1, 1]$ defined by

$$f(t) = \frac{2t}{1+t^2}$$

is continuous, strictly increasing, and onto, so it has a continuous inverse. It follows that there exists a $b \in \mathcal{M}_{\text{sa}}$ such that

$$a = \frac{2b}{1+b^2}.$$

By the above, there exists a net $(b_i)_{i \in I}$ in \mathcal{A}_{sa} that converges to b in the strong operator topology. For every $i \in I$ define

$$a_i = \frac{2b_i}{1+b_i^2}.$$

Then $a_i \in \mathcal{A}_{\text{sa}} \cap S$. We have

$$\begin{aligned} a_i - a &= \frac{2b_i}{1+b_i^2} - \frac{2b}{1+b^2} \\ &= \frac{2b_i(1+b^2) - (1+b_i^2)2b}{(1+b_i^2)(1+b^2)} \\ &= 2 \frac{b_i - b}{(1+b_i^2)(1+b^2)} + 2 \frac{b - b_i}{(1+b_i^2)(1+b^2)}. \end{aligned}$$

Therefore, $(a_i)_{i \in I}$ converges to a in the strong operator topology, showing that $\mathcal{A}_{\text{sa}} \cap S$ is strong operator dense in $\mathcal{M}_{\text{sa}} \cap S$. This proves claim (ii) of the theorem.

Fix $a \in \mathcal{M}_+ \cap S$, and let $b = a^{1/2}$. By the above, there exists a net $(b_i)_{i \in I} \in \mathcal{A}_{\text{sa}} \cap S$ that converges to b in the strong operator topology. For every $i \in I$, define

$$a_i = b_i^* b_i \in \mathcal{A}_+ \cap S.$$

If $\xi, \eta \in \mathcal{H}$, we have

$$\begin{aligned} |\langle (a_i - a)\xi | \eta \rangle| &= |\langle (b_i^* b_i - b^* b)\xi | \eta \rangle| \\ &= |\langle b_i^* (b_i - b)\xi | \eta \rangle + \langle (b_i^* - b^*) b \xi | \eta \rangle| \\ &= |\langle (b_i - b)\xi | b_i \eta \rangle + \langle b \xi | (b_i - b)\eta \rangle| \\ &\leq \|(b_i - b)\xi\| \|\eta\| + \|(b_i - b)\eta\| \|\xi\|. \end{aligned}$$

Therefore, $(a_i)_{i \in I}$ converges to a in the weak operator topology, showing that $\mathcal{A}_+ \cap S$ is weak operator dense in $\mathcal{M}_+ \cap S$. Since the weak operator topology and

strong operator topology have the same closed convex sets, this implies that $\mathcal{A}_+ \cap S$ is strong operator dense in $\mathcal{M}_+ \cap S$, proving claim (iii) of the theorem.

Now, consider the Hilbert space $\mathcal{H}^{(2)}$ and the $*$ -algebras

$$M_2(\mathcal{A}) \subseteq M_2(\mathcal{B}(\mathcal{H})) \cong \mathcal{B}(\mathcal{H}^{(2)})$$

and

$$M_2(\mathcal{M}) \subseteq M_2(\mathcal{B}(\mathcal{H})) \cong \mathcal{B}(\mathcal{H}^{(2)}).$$

By Proposition 2.8.2 (iv) and the Double Commutant Theorem, $M_2(\mathcal{A})$ is strong operator dense in $M_2(\mathcal{M})$. Let S' be the unit ball of $\mathcal{B}(\mathcal{H}^{(2)})$. Fix $a \in \mathcal{M} \cap S$. Then

$$b = \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix}$$

defines an element of $M_2(\mathcal{M})_{\text{sa}} \cap S'$. By the above, it follows that there exists a net $(b_i)_{i \in I}$ in $M_2(\mathcal{A})_{\text{sa}} \cap S'$ that converges to b in the strong operator topology. In particular, the net of $(1, 2)$ matrix coefficients must also converge in the strong operator topology on $\mathcal{B}(\mathcal{H})$, showing that $\mathcal{A} \cap S$ is strong operator dense in $\mathcal{M} \cap S$. This proves claim (i) of the theorem.

Finally, if \mathcal{A} is unital, let $u \in \mathcal{M}$ be a unitary. We will use the Borel functional calculus to find a logarithm for u . Let \mathbb{T} be the unit circle in \mathbb{C} , and define $g : \mathbb{T} \rightarrow i\mathbb{R}$ by $g(e^{it}) = it$ when $-\pi < t \leq \pi$. Note that g is a Borel function such that $e^{g(z)} = z$ for every $z \in \mathbb{T}$. Let $a = -ig(u)$. Then a is a self-adjoint element of \mathcal{M} such that $u = e^{ia}$. By the above, there exists a net $(a_i)_{i \in I}$ in \mathcal{A}_{sa} that converges to a in the strong operator topology. Then $(e^{ia_i})_{i \in I}$ is a net of unitaries converging in the strong operator topology to u . This proves claim (iv) of the theorem. \square

The Kaplansky Density Theorem allows us to show that all of the operator topologies on $\mathcal{B}(\mathcal{H})$ have the same closed $*$ -subalgebras.

2.8.17 Corollary. *Let \mathcal{M} be a $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$ that acts nondegenerately on \mathcal{H} . Then the following are equivalent:*

- (i) \mathcal{M} is a von Neumann algebra;
- (ii) \mathcal{M} is σ -weakly closed;
- (iii) the unit ball of \mathcal{M} is σ -weakly closed.

PROOF. If \mathcal{M} is a von Neumann algebra, then \mathcal{M} is weak operator topology closed. Since the weak operator topology is weaker than the σ -weak topology, \mathcal{M} is σ -weakly closed as well. This shows that (i) \implies (ii).

If \mathcal{M} is σ -weakly closed, then the unit ball of \mathcal{M} is σ -weakly compact by the Banach-Alaoglu Theorem, showing that (ii) \implies (iii).

Finally, suppose that the unit ball of \mathcal{M} is weak operator topology compact. We want to show that \mathcal{M} is weak operator topology closed. Let a be a weak operator topology limit point of \mathcal{M} . By scaling, we may assume that $\|a\| \leq 1$. By Kaplansky's Density Theorem, there exists a net $(a_i)_{i \in I}$ in \mathcal{M} that converges to a such that $\|a_i\| \leq 1$ for every $a_i \in \mathcal{M}$. Since the σ -weak and weak operator topologies agree on bounded sets, the unit ball of \mathcal{M} is weak operator topology closed as well. Therefore, $a \in \mathcal{M}$, showing that \mathcal{M} is weak operator topology closed. This shows that (iii) \implies (i). \square

2.8.18 Corollary. *Let \mathcal{M} be a $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$. Then the following are equivalent:*

- (i) \mathcal{M} is closed in the weak operator topology;
- (ii) \mathcal{M} is closed in the strong operator topology;
- (iii) \mathcal{M} is closed in the strong* operator topology;
- (iv) \mathcal{M} is closed in the σ -weak topology;
- (v) \mathcal{M} is closed in the σ -strong topology;
- (vi) \mathcal{M} is closed in the strong* operator topology.

PROOF. This follows from Proposition 2.7.21 and Corollary 2.8.17. \square

In our proof of the Kaplansky Density Theorem, we heavily used the fact that the subspace of $\mathcal{B}(\mathcal{H})$ we are considering is a self-adjoint subalgebra. It turns out that the first statement of the theorem, that the unit ball of \mathcal{A} is strong operator dense in the strong operator topology closure of \mathcal{A} , is false for a general subspace of $\mathcal{B}(\mathcal{H})$. We will now give an example of such a subspace. In fact, we will also show how this example can be modified to give an example of a (necessarily non-self-adjoint) subalgebra of $\mathcal{B}(\mathcal{H})$ for which the Kaplansky Density Theorem fails.

2.8.19 Example. We claim that if the Kaplansky Density Theorem holds for all linear subspaces of $\mathcal{B}(\mathcal{H})$, i.e. if for every linear subspace \mathcal{X} of $\mathcal{B}(\mathcal{H})$ the unit ball of \mathcal{X} is strong operator dense in the strong operator topology closure of \mathcal{X} , then every σ -weakly closed subspace of $\mathcal{B}(\mathcal{H})$ is weak operator topology closed. Indeed, if \mathcal{X} is a σ -weakly closed subspace of $\mathcal{B}(\mathcal{H})$, then $\mathcal{X} \cap S$ is weak operator topology compact, where S is the unit ball of $\mathcal{B}(\mathcal{H})$. Let a be a weak operator topology limit point of \mathcal{X} . If the Kaplansky Density Theorem holds for subspaces of $\mathcal{B}(\mathcal{H})$, then clearly the same statement holds for the weak operator topology, i.e. for every linear subspace \mathcal{X} of $\mathcal{B}(\mathcal{H})$ the unit ball of \mathcal{X} is weak operator dense in the weak operator topology closure of \mathcal{X} , because the strong operator topology and weak operator topology have the same closed convex sets. Hence there exists a net $(a_i)_{i \in I}$ in $\mathcal{X} \cap S$ converging to a . Therefore, \mathcal{X} is weak operator closed. We will produce a linear subspace of $\mathcal{B}(\mathcal{H})$ that is σ -weakly closed but not weak operator topology.

Let \mathcal{H} be an infinite-dimensional Hilbert space, and x an infinite rank trace-class operator on \mathcal{H} . Define $E = \{\alpha x : \alpha \in \mathbb{C}\} \subseteq \mathcal{T}(\mathcal{H})$, and let $\mathcal{X} = E^\perp \subseteq \mathcal{B}(\mathcal{H})$. Then \mathcal{X} is σ -weakly closed. By the Bipolar Theorem applied to the $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{T}(\mathcal{H}))$ topology, i.e. the σ -weak topology, $\mathcal{X}_\perp = E$. Let $\mathcal{F}(\mathcal{H})$ be the subspace of $\mathcal{T}(\mathcal{H})$ consisting of the finite rank elements of $\mathcal{T}(\mathcal{H})$. It is easy to see that the $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{F}(\mathcal{H}))$ is the weak operator topology. Since $\mathcal{X}_\perp \cap \mathcal{F}(\mathcal{H}) = \{0\}$, the Bipolar Theorem applied to the $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{F}(\mathcal{H}))$ topology implies that the weak operator topology closure of \mathcal{X} is $\mathcal{B}(\mathcal{H})$. Therefore, \mathcal{X} is not weak operator topology closed.

If the Kaplansky Density Theorem holds for all subalgebras of $\mathcal{B}(\mathcal{H})$, then by the same reasoning as above, every σ -weakly closed subalgebra of $\mathcal{B}(\mathcal{H})$ is weak operator topology closed. Define $\mathcal{A} \subseteq M_2(\mathcal{B}(\mathcal{H})) \cong \mathcal{B}(\mathcal{H}^{(2)})$ by

$$\mathcal{A} = \left\{ \begin{bmatrix} \alpha & a \\ 0 & \beta \end{bmatrix} : \alpha, \beta \in \mathbb{C}, a \in \mathcal{X} \right\}.$$

Then \mathcal{A} is a σ -weakly closed subalgebra of $\mathcal{B}(\mathcal{H})$ that is not weak operator topology closed.

We will now show that many constructions in operator theory relativize to a general von Neumann algebra.

2.8.20 Proposition. *Let $a \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator, and $s(a)$ the support projection of a . Then $s(a) = \chi_{\mathbb{R} \setminus \{0\}}(a) \in W^*(a)$.*

PROOF. From the equality $t\lambda_{\mathbb{R} \setminus \{0\}}(t) = t$, it follows that $a\chi_{\mathbb{R} \setminus \{0\}} = a$, which implies that $s(a) \leq \chi_{\mathbb{R} \setminus \{0\}}(a)$. On the other hand, since $as(a) = a$, it follows that $f(a)s(a) = f(a)$ for every $f \in B_b(\sigma(a))$, first by considering polynomials with zero constant terms, and then by taking strong operator topology limits. In particular, $\chi_{\mathbb{R} \setminus \{0\}}(a)s(a) = \chi_{\mathbb{R} \setminus \{0\}}(a)$. Therefore, $\chi_{\mathbb{R} \setminus \{0\}} \leq s(a)$. \square

2.8.21 Proposition. *Let $a \in \mathcal{B}(\mathcal{H})$ be a positive operator, and $s(a)$ the support projection of a . Then there exists a sequence of projections $(e_n)_{n=1}^\infty$ in $W^*(a)$ such that*

$$ae_n \geq \frac{1}{n}e_n \quad \text{and} \quad e_n \nearrow s(a).$$

PROOF. By Proposition 2.8.20, it is clear that the sequence $e_n = \chi_{(1/n, \infty)}(a)$ is such a sequence. \square

In order to prove that the polar decomposition of a is contained in $W^*(a)$, we will first prove a simple criterion for whether an operator in $\mathcal{B}(\mathcal{H})$ belongs to \mathcal{M} .

2.8.22 Proposition. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then $\mathcal{M} = \mathcal{U}(\mathcal{M}')'$.*

PROOF. If $a \in \mathcal{M}$, then $a \in \mathcal{M}'' \subseteq \mathcal{U}(\mathcal{M}')'$. By Proposition 2.6.1, \mathcal{M}' is the linear span of $\mathcal{U}(\mathcal{M}')$. Thus, if $a \in \mathcal{U}(\mathcal{M}')'$ then $a \in \mathcal{M}'' = \mathcal{M}$. \square

2.8.23 Proposition. *Let $a \in \mathcal{B}(\mathcal{H})$ be an operator. If $a = v|a|$ is the polar decomposition of a , then v and $|a|$ are contained in $W^*(a)$.*

PROOF. Since $W^*(a)$ is a C^* -algebra, $|a| = (a^*a)^{1/2} \in W^*(a)$, so we need only show that $v \in W^*(a)$. By Proposition 2.8.22, it suffices to show that $v \in \mathcal{U}(W^*(a)')'$. If $u \in \mathcal{U}(W^*(a)')$, then

$$a = uu^*a = uau^* = uv|a|u^* = (uvu^*)|a|.$$

By Proposition 2.8.20, since $|a| \in W^*(a)$, $s(|a|) \in W^*(a)$, so

$$(uvu^*)^*(uvu) = uv^*u^*uvu = uv^*vu^* = us(|a|)u^* = uu^*s(|a|) = s(|a|),$$

and it follows from the uniqueness of the polar decomposition that $uvu^* = v$, i.e. v commutes with u . Therefore, $v \in \mathcal{U}(W^*(a)')' = W^*(a)$. \square

2.8.24 Corollary. *Let $a \in \mathcal{B}(\mathcal{H})$ be an operator. Then the right support projection $s_r(a)$ of a , left support projection $s_l(a)$ of a , and the projection onto the kernel of a are all contained in $W^*(a)$.*

PROOF. This follows from Proposition 2.8.23, because if $a = v|a|$ is the polar decomposition of a , then $s_r(a) = v^*v$, $s_l(a) = vv^*$, and the projection onto the kernel of a is $1 - s_r(a)$. \square

If \mathcal{A} is a nondegenerate C^* -subalgebra of $\mathcal{B}(\mathcal{H})$, then $\overline{\mathcal{A}\mathcal{H}} = \mathcal{H}$. In fact, \mathcal{A} has a bounded approximate identity, so applying the Cohen-Hewitt Factorization Theorem to the Banach \mathcal{A} -module \mathcal{H} gives that every $\xi \in \mathcal{H}$ can be written as $a\eta$ for some $\eta \in \mathcal{H}$. Sometimes it may be more useful to consider $\overline{\mathcal{A}S}$ for a particular subset S of \mathcal{H} , and in particular, the case where $S = \{\xi\}$ for some $\xi \in \mathcal{H}$.

2.8.25 Definition. Let \mathcal{A} be a $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$. We say that a subset S is *cyclic* (for \mathcal{A}) if $\overline{\mathcal{A}S} = \mathcal{H}$, and we say that S is *separating* (for \mathcal{A}) if $aS = 0$ implies that $a = 0$. In the particular case of $S = \{\xi\}$, we say that ξ is *cyclic* (for \mathcal{A}) or that ξ is *separating* (for \mathcal{A}).

Intuitively, if \mathcal{A} has a cyclic vector then \mathcal{A} is “large”, and if \mathcal{A} has a separating vector, then \mathcal{A} is “small”.

2.8.26 Proposition. *Let \mathcal{A} be a $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$. If S is a subset of \mathcal{H} , then S is cyclic for \mathcal{A} if and only if S is separating for \mathcal{A}' .*

PROOF. Suppose S is cyclic for \mathcal{A} , and that $aS = \{0\}$ for some $a \in \mathcal{M}'$. Let p be the projection onto the kernel of a . By Proposition 2.8.23, $p \in \mathcal{A}'$, so $S \subseteq p\mathcal{H}$. Since $p \in \mathcal{A}'$ and S is cyclic for \mathcal{A} , it follows that $\mathcal{H} = \overline{\mathcal{A}S} \subseteq p\mathcal{H}$, so $p = 1$ and $a = 0$.

Conversely, suppose that S is separating for \mathcal{A}' . Let p denote the projection of \mathcal{H} onto $\overline{\mathcal{A}S}$. Since $\overline{\mathcal{A}S}$ is an invariant subspace for \mathcal{A} , $p \in \mathcal{A}$. Clearly, $(1 - p)S = \{0\}$, so since S is separating for \mathcal{A}' it follows that $1 - p = 0$. Hence $\overline{\mathcal{A}S} = \mathcal{H}$ and S is cyclic for \mathcal{A} . \square

Historical Notes

The Double Commutant Theorem is due to von Neumann [vN29]. Von Neumann algebras were called “rings of operators” by Murray and von Neumann; Dieudonné suggested the current terminology in the French tradition of naming mathematical objects after people, and it was used heavily by Dixmier. The name “ W^* -algebra” was introduced by Segal [Seg50], where the W was meant to stand for “weakly closed”, just as C^* -algebras are $*$ -subalgebras of $\mathcal{B}(\mathcal{H})$ closed in the norm topology. Nowadays, the name W^* -algebra is used by some to refer to an abstract C^* -algebra that is $*$ -isomorphic to some von Neumann algebra. Ironically, Kaplansky [Kap51b] defined the notion of an AW^* -algebra, intending to characterize those abstract C^* -algebras that are , but there are commutative AW^* -algebras that are not W^* -algebras.

The realization that every von Neumann algebra is a dual space is due to Dixmier [Dix53].

Kaplansky proved the density theorem that bears his name in [Kap51c]. The original proof proceeds by proving the strong operator continuity of the continuous functional calculus for self-adjoint elements for functions that decay no faster than $|\cdot|$. The proven given here is a bit simpler, but it is still essentially the same argument.

2.9 Representations

2.9.1 Definition. Let \mathcal{A} be a $*$ -algebra. A $*$ -homomorphism $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$, where \mathcal{H} is a Hilbert space, is said to be a $*$ -representation of \mathcal{A} .

We will often refer to a $*$ -representation simply as a *representation* when the context is clear, and we will often refer to a representation π and its representation Hilbert space \mathcal{H} as a pair (π, \mathcal{H}) . If not otherwise named, we will

denote the representation space of π as \mathcal{H}_π . We will generally only consider representations of Banach $*$ -algebras, but we will introduce the basic notions for arbitrary $*$ -algebras. Before we continue, we will introduce some helpful notation. If \mathcal{H} is a Hilbert space and S is a subset of \mathcal{H} , we will let $[S]$ denote the norm closed span of S in \mathcal{H} .

2.9.2 Examples.

- (i) Let X be a locally compact Hausdorff space, and let μ be a finite Radon measure on X . The inclusion of $C_0(X)$ into $L^\infty(X, \mu)$ gives a representation of $C_0(X)$, because $L^\infty(X, \mu)$ acts on $L^2(X, \mu)$ by left multiplication.
- (ii) It may very well happen that a Banach $*$ -algebra \mathcal{A} has no nontrivial representations. Let \mathcal{A} be \mathbb{C}^2 equipped with the ∞ -norm and pointwise multiplication. Define an involution on \mathcal{A} by

$$(a, b)^* = (\bar{b}, \bar{a}).$$

Then \mathcal{A} is a commutative unital Banach $*$ -algebra, but it has no nontrivial representations, because

$$(1, 0)^*(1, 0) = (0, 0) \quad \text{and} \quad (0, 1)^*(0, 1) = (0, 0).$$

and $\{(1, 0), (0, 1)\}$ is a basis of \mathcal{A} . Thus, if $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ is a representation,

$$\|\pi((1, 0))\|^2 = \|\pi((1, 0)^*(1, 0))\| = 0$$

and

$$\|\pi((0, 1))\|^2 = \|\pi((0, 1)^*(0, 1))\| = 0,$$

which implies that π is the zero map. Note that for this example we took a Banach $*$ -algebra with a faithful representation and modified the involution so that it has no nontrivial representations.

2.9.3 Definition. Let \mathcal{A} be a $*$ -algebra, and $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ a representation of \mathcal{A} .

- (i) We say that π is *nondegenerate* if $\overline{\pi(\mathcal{A})\mathcal{H}} = \mathcal{H}$.
- (ii) We say that $\xi \in \mathcal{H}$ is a *cyclic vector* for π if ξ is cyclic for $\pi(\mathcal{A})$, i.e. if $\overline{\pi(\mathcal{A})\xi} = \mathcal{H}$. If π has a cyclic vector, then it is said to be *cyclic*.
- (iii) We say that π is *topologically irreducible*, or simply *irreducible*, if the only closed invariant subspaces of $\pi(\mathcal{A})$ are $\{0\}$ and \mathcal{H} .

2.9.4 Proposition (Schur's Lemma). Let \mathcal{A} be a $*$ -algebra, and $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ a nondegenerate representation of \mathcal{A} . Then the following are equivalent:

- (i) π is *irreducible*.
- (ii) $\pi(\mathcal{A})' = \mathbb{C}$.
- (iii) Either $\dim(\mathcal{H}) = 1$ or every nonzero $\xi \in \mathcal{H}$ is a cyclic vector for π .

PROOF. We will show that (i) and (ii) are equivalent, and then that (i) and (iii) are equivalent.

Suppose that π is irreducible and $T \in \pi(\mathcal{A})'$. Since $\pi(\mathcal{A})'$ is a von Neumann algebra, it is the norm closure of linear combinations of its projections Proposition 2.8.12. The projections in $\pi(\mathcal{A})'$ are the projections onto closed invariant subspaces of $\pi(\mathcal{A})$, but since π is irreducible, these projections are only 0 and 1. Therefore, T is a scalar.

Conversely, suppose that $\pi(\mathcal{A})' = \mathbb{C}$. Then the only projections commuting with $\pi(\mathcal{A})$ are 0 and 1, so π has no nontrivial closed invariant subspaces and is irreducible.

We will now show that (i) and (iii) are equivalent. Suppose that π is irreducible, $\dim(\mathcal{H}) > 1$, and $\xi \in \mathcal{H}$ is nonzero. Then $\overline{\pi(\mathcal{A})\xi}$ is a closed invariant subspace of \mathcal{A} . However, it may possibly be zero. If it is zero, i.e. if $\pi(a)\xi = 0$ for all $a \in \mathcal{A}$, then $\{\lambda\xi : \lambda \in \mathbb{C}\}$ is a closed invariant subspace of π . Since this subspace is nonzero by definition, by our assumption that π is irreducible it must be all of \mathcal{H} , contradicting our assumption that $\dim(\mathcal{H}) > 1$. Hence $\overline{\pi(\mathcal{A})\xi}$ is a nonzero closed invariant subspace of \mathcal{A} , so it must be equal to all of \mathcal{H} , showing that ξ is a cyclic vector for π .

Conversely, suppose that either $\dim(\mathcal{H}) = 1$ or every nonzero $\xi \in \mathcal{H}$ is a cyclic vector for π . Let X be a nonzero closed invariant subspace of π . If $\dim(\mathcal{H}) = 1$, then clearly $X = \mathcal{H}$. Otherwise, let $\xi \in X$ be nonzero, so that $\overline{\pi(\mathcal{A})\xi} \subseteq X$. Since ξ is cyclic, $\overline{\pi(\mathcal{A})\xi} = \mathcal{H}$, so $X = \mathcal{H}$. Therefore, π is irreducible. \square

2.9.5 Corollary. *Let \mathcal{A} be a commutative $*$ -algebra. Then every irreducible representation of \mathcal{A} is one-dimensional.*

PROOF. If $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ is irreducible, then $\pi(\mathcal{A})' = \mathbb{C}$. However, since \mathcal{A} is commutative, $\pi(\mathcal{A})' \subseteq \pi(\mathcal{A})$. Therefore, $\pi(\mathcal{A}) = \mathbb{C}$. \square

Let \mathcal{A} be a Banach $*$ -algebra. If $(\pi_i)_{i \in I}$ is a family of representations of \mathcal{A} , then the *direct sum* of this family is the representation

$$\bigoplus_{i \in I} \pi_i : \mathcal{A} \rightarrow \mathcal{B} \left(\bigoplus_{i \in I} \mathcal{H}_{\pi_i} \right)$$

defined by

$$\left(\bigoplus_{i \in I} \pi_i \right) (a) = \bigoplus_{i \in I} \pi_i(a).$$

Since every representation is a contraction, $\|\pi_i(a)\| \leq \|a\|$ for every $i \in I$, showing that every operator defined above is actually bounded.

2.9.6 Proposition. *Let \mathcal{A} be a Banach $*$ -algebra. Then every nondegenerate representation of \mathcal{A} is a direct sum of cyclic representations.*

PROOF. Let $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ be a representation of \mathcal{A} , and let \mathcal{F} denote the family of all subsets F of \mathcal{H} such that $[\pi(\mathcal{A})\xi]$ and $[\pi(\mathcal{A})\eta]$ are orthogonal for every distinct pair $\xi, \eta \in F$. When ordered by inclusion, it is obvious that every chain in \mathcal{F} has an upper bound. Therefore, by Zorn's Lemma, there exists a maximal element $F = \{\xi_i\}_{i \in I}$ of \mathcal{F} . For each $i \in I$, the subspace $\mathcal{H}_i = [\pi(\mathcal{A})\xi_i]$ is invariant for π , so $(\pi|_{\mathcal{H}_i}, \mathcal{H}_i)$ is a subrepresentation of π . By definition, ξ_i is a cyclic vector for $\pi|_{\mathcal{H}_i}$. The maximality of F implies that $\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i$, so $\pi = \bigoplus_{i \in I} \pi|_{\mathcal{H}_i}$. \square

A useful fact about nondegenerate $*$ -representations of Banach $*$ -algebras is that they take bounded approximate identities to nets converging to 1 in the strong operator topology.

2.9.7 Proposition. *Let \mathcal{A} be a Banach $*$ -algebra, and $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ a $*$ -representation of \mathcal{A} . If $(e_i)_{i \in I}$ is a bounded approximate identity of \mathcal{A} , then $\pi(e_i)$ converges to 1 in the strong operator topology.*

PROOF. Since $\pi(e_i) \leq \|e_i\|$ and $(e_i)_{i \in I}$ is bounded, $(\pi(e_i))_{i \in I}$ is a bounded net in $\mathcal{B}(\mathcal{H})$. If $a \in \mathcal{A}$ and $\xi \in \mathcal{H}$, then

$$\pi(e_i)\pi(a)\xi = \pi(e_i a)\xi$$

tends strongly to $\pi(a)\xi$ because

$$\|\pi(e_i a) - \pi(a)\xi\| \leq \|e_i a - a\|$$

tends to 0. Since $\pi(\mathcal{A})\mathcal{H}$ is dense in \mathcal{H} and $(\pi(e_i))_{i \in I}$ is a bounded net in $\mathcal{B}(\mathcal{H})$, this implies that $(\pi(e_i))_{i \in I}$ converges to the identity in the strong operator topology. \square

2.9.8 Definition. Let \mathcal{A} be a Banach $*$ -algebra, and let (π_1, \mathcal{H}_1) and (π_2, \mathcal{H}_2) be nondegenerate representations of \mathcal{A} . We say that $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is an *intertwining operator* from π_1 to π_2 if

$$T\pi_1(a) = \pi_2(a)T$$

for every $a \in \mathcal{A}$. The set of all intertwining operators from π_1 to π_2 is denoted $\mathcal{R}(\pi_1, \pi_2)$.

It is clear that if $T \in \mathcal{R}(\pi_1, \pi_2)$ and $S \in \mathcal{R}(\pi_2, \pi_3)$, then $ST \in \mathcal{R}(\pi_1, \pi_3)$, and that the identity is always in $\mathcal{R}(\pi_1, \pi_1)$. Hence the class of representations

of \mathcal{A} forms a category where the morphisms from π_1 to π_2 are the intertwining operators from π_1 to π_2 .

If π is a representation of \mathcal{A} , then the intertwining operators from π to itself are precisely the bounded operators on \mathcal{H}_π that commute with π . Thus $\mathcal{R}(\pi, \pi) = \pi(\mathcal{A})'$, and the commutant of a representation can be viewed as the algebra of endomorphisms of that representation.

The most important intertwining operators are the unitaries, i.e. the isomorphisms of the underlying Hilbert spaces that convert the action of one representation into another. If there is a unitary intertwining operator between two representations, then we say that they are *unitarily equivalent*. For all of our purposes, unitarily equivalent representations are essentially considered to be the same representation.

If G is a locally compact group, the representations of $L^1(G)$ are of particular interest, because they correspond to the unitary representations of G .

2.9.9 Definition. Let G be a locally compact group. A homomorphism $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ is said to be a *unitary representation* of G , or simply a *representation*, if it is continuous with respect to the strong operator topology.

By Proposition 2.7.22 (iii), all of the operator topologies on agree on $\mathcal{U}(\mathcal{H})$, so the strong operator topology may be replaced with any of the other operator topologies.

We can extend unitary representations of G to $*$ -representations of $L^1(G)$ and $M(G)$ by integration.

2.9.10 Proposition. Let G be a locally compact group, and $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ be a unitary representation of G . Define $\pi_{M(G)} : M(G) \rightarrow \mathcal{B}(\mathcal{H})$ by

$$\langle \pi_{M(G)}(\mu)\xi \mid \eta \rangle = \int \langle \pi(s)\xi \mid \eta \rangle d\mu(s).$$

Then $\pi_{M(G)}$ is a $*$ -representation of $M(G)$, and its restriction to $L^1(G)$ is nondegenerate and is given by

$$\langle \pi_{L^1(G)}(f)\xi \mid \eta \rangle = \int f(s)\langle \pi(s)\xi \mid \eta \rangle ds.$$

PROOF. First, note that $\|\pi(\mu)\| \leq \|\mu\|$, so $\pi(\mu) \in \mathcal{B}(\mathcal{H})$ for all $\mu \in M(G)$. Clearly, π is linear, so we need only show that it is multiplicative and $*$ -

preserving. If $\mu, \nu \in M(G)$ and $\xi, \eta \in \mathcal{H}$, then

$$\begin{aligned}
\langle \pi_{M(G)}(\mu * \nu)\xi | \eta \rangle &= \int \langle \pi(s)\xi | \eta \rangle d(\mu * \nu)(s) \\
&= \iint \langle \pi(st)\xi | \eta \rangle d\mu(s) d\nu(t) \\
&= \iint \langle \pi(t)\xi | \pi(s^{-1})\eta \rangle d\mu(s) d\nu(t) \\
&= \int \langle \pi_{M(G)}(\nu)\xi | \pi(s^{-1})\eta \rangle d\mu(s) \\
&= \int \langle \pi(s)\pi_{M(G)}(\nu)\xi | \eta \rangle d\mu(s) \\
&= \langle \pi_{M(G)}(\mu)\pi_{M(G)}(\nu)\eta | \xi \rangle,
\end{aligned}$$

showing that $\pi_{M(G)}$ is multiplicative. If $\mu \in M(G)$ and $\xi, \eta \in \mathcal{H}$, then

$$\begin{aligned}
\langle \pi_{M(G)}(\mu^*)\xi | \eta \rangle &= \int \langle \pi(s^{-1})\xi | \eta \rangle d\mu(s) \\
&= \int \overline{\langle \pi(s)\eta | \xi \rangle} d\mu(s) \\
&= \overline{\langle \pi_{M(G)}(\mu)\eta | \xi \rangle} \\
&= \langle \pi_{M(G)}(\mu)\xi | \eta \rangle,
\end{aligned}$$

showing that $\pi_{M(G)}$ is $*$ -preserving.

It only remains to be shown that $\pi_{L^1(G)}$ is nondegenerate. Let $(u_i)_{i \in I}$ be a bounded approximate identity for $L^1(G)$ such that each u_i is positive real-valued and $\text{supp}(u_j) \subseteq \text{supp}(u_i)$ whenever $i \leq j$, which exists by Proposition 2.4.5. Then $(\pi_{L^1(G)}(u_i))_{i \in I}$ converges to 1 in the weak operator topology. Indeed if $\xi, \eta \in \mathcal{H}$, then

$$\begin{aligned}
|\langle (\pi_{L^1(G)}(u_i) - 1)\xi | \eta \rangle| &= \left| \int \langle u_i(s)\pi(s)\xi | \eta \rangle ds - \langle \xi | \eta \rangle \right| \\
&= \int |\langle u_i(s)\pi(s)\xi - \xi | \eta \rangle| ds \\
&\leq \int u_i(s) |\langle \pi(s)\xi - \xi | \eta \rangle| ds \\
&\leq \sup_{s \in \text{supp}(u_i)} |\langle \pi(s)\xi - \xi | \eta \rangle|.
\end{aligned}$$

As i increases, the last expression tends to 0 by the continuity of π . Therefore, 1 is in the weak operator topology closure of the range of $\pi_{L^1(G)}$. Since the weak operator topology and the strong operator topology have the same closed convex sets, 1 is in the strong operator topology closure of the range of $\pi_{L^1(G)}$, which show that $\pi_{L^1(G)}$ is nondegenerate. \square

If $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ is a unitary representation of G , we can easily recover π from $\pi_{M(G)}$ using the embedding $s \mapsto \delta_s$ of G into $M(G)$ as the point

masses, and that this would yield a unitary representation of G for every $*$ -representation ρ of $M(G)$. Unfortunately, if $\pi_{M(G)}$ is a $*$ -representation of $M(G)$, then the representation π of G obtained by restricting to the point masses is not necessarily continuous. However, if we assume that $\pi_{M(G)}$ is nondegenerate when restricted to $L^1(G)$, then we can show that π is continuous.

2.9.11 Proposition. *Let G be a locally compact group, and $\pi_{M(G)} : M(G) \rightarrow \mathcal{B}(\mathcal{H})$ a $*$ -representation of $M(G)$ whose restriction to $L^1(G)$ is nondegenerate. Define $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ by $\pi(s) = \pi(\delta_s)$. Then π is a continuous representation of G .*

PROOF. Clearly, π is a homomorphism whose range is contained in $\mathcal{U}(\mathcal{H})$, so we need only show that π is strong operator topology continuous. Suppose $(s_n)_{n=1}^\infty$ is a sequence in G converging to e . If $f \in C_c(G)$ and $\xi \in \mathcal{H}$, we claim that $\pi(s_n)\pi_{M(G)}(f)\xi \rightarrow \pi(\delta_{s_n} * f)\xi$ in the strong operator topology and $\delta_{s_n} * f(t) = \Delta(s_n^{-1})f(s_n^{-1}t)$. By the Lebesgue Dominated Convergence Theorem, $\delta_{s_n} * f$ converges in $L^1(G)$ to f . Hence

$$\|\pi(s_n)\pi_{M(G)}(f)\xi - \pi_{M(G)}(f)\xi\| \leq \|\delta_{s_n} * f - f\|_1 \|\xi\|.$$

The right-hand side tends to 0 as $n \rightarrow \infty$. Hence, by the nondegeneracy of the restriction of $\pi_{M(G)}$ to $L^1(G)$ (and the fact that $C_c(G)$ is dense in $L^1(G)$), $\pi(s_n)\xi \rightarrow \xi$ on a dense subspace of \mathcal{H} . Fix $\eta \in \mathcal{H}$, $\epsilon > 0$, and $\xi \in \mathcal{H}$ chosen so that $\|\xi - \eta\| < \epsilon/3$. Since each $\pi(s_n)$ is unitary, for sufficiently large n we have

$$\begin{aligned} \|\pi(s_n)\eta - \eta\| &\leq \|\pi(s_n)\eta - \pi(s_n)\xi\| + \|\pi(s_n)\xi - \xi\| + \|\xi - \eta\| \\ &< \frac{2\epsilon}{3} + \|\pi(s_n)\xi - \xi\| \\ &< \epsilon. \end{aligned}$$

Therefore, π is strong operator topology continuous. \square

Given a $*$ -representation of $M(G)$, we can restrict it to $L^1(G)$, but how do we know that every nondegenerate $*$ -representation of $L^1(G)$ corresponds to a unitary representation of G , or a $*$ -representation of $M(G)$? We would like this to be the case, because in many respects the algebra $L^1(G)$ is much more tractable than the algebra $M(G)$. However, since $L^1(G)$ is an ideal in $M(G)$ with a bounded approximate identity, we are able to uniquely extend $*$ -representations of $L^1(G)$ to those of $M(G)$. We will carry out this construction, and then derive the correspondence between representations of G , $L^1(G)$, and $M(G)$.

2.9.12 Lemma. *Let \mathcal{A} be a Banach $*$ -algebra, \mathcal{J} a closed self-adjoint ideal of \mathcal{A} with a bounded approximate identity, and $\pi : \mathcal{J} \rightarrow \mathcal{B}(\mathcal{H})$ a cyclic representation*

of \mathcal{B} . Then π has a unique extension to a representation $\tilde{\pi}$ of \mathcal{A} . Every cyclic vector for π is a cyclic vector for $\tilde{\pi}$ and vice-versa.

PROOF. Let $(u_i)_{i \in I}$ be a bounded approximate identity for \mathcal{A} , and let $\xi \in \mathcal{H}$ be a cyclic vector for π . Fix $a \in \mathcal{A}$. We will define $\tilde{\pi}(a)$ first as a bounded linear operator on the dense subspace $\pi(\mathcal{J})\xi$ of \mathcal{H} and then extend it to all of \mathcal{H} by continuity. Define $\tilde{\pi}(a)$ on $\pi(\mathcal{J})\xi$ by

$$\tilde{\pi}(a)\pi(x)\eta = \pi(ax)\eta.$$

This operator is bounded, because if M is a bound for $(e_i)_{i \in I}$,

$$\begin{aligned} \|\tilde{\pi}(a)\pi(x)\xi\| &= \|\pi(ax)\xi\| \\ &= \lim_{i \in I} \|\pi(au_i x)\xi\| \\ &= \lim_{i \in I} \|\pi(au_i)\pi(x)\xi\| \\ &\leq \sup_{i \in I} \|au_i\| \cdot \|\pi(x)\xi\| \\ &\leq \sup_{i \in I} M \cdot \|a\| \cdot \|\pi(x)\| \cdot \|\xi\|. \end{aligned}$$

Therefore, $\tilde{\pi}(a)$ extends by continuity to a bounded operator on all of \mathcal{H} . If $x, y \in \mathcal{J}$, then

$$\tilde{\pi}(a)\pi(x)\pi(y)\xi = \tilde{\pi}(a)\pi(xy)\xi = \pi(axy) = \pi(ax)\pi(y)\xi.$$

Since ξ is cyclic for π , this shows that $\hat{\pi}(a)\pi(x) = \pi(ax)$ for every $x \in \mathcal{J}$.

The map $\tilde{\pi}(a) : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ is clearly linear, so we only have to show that it is multiplicative and self-adjoint. If $a, b \in \mathcal{A}$ and $x \in \mathcal{J}$ then

$$\tilde{\pi}(ab)\pi(x) = \pi(abx) = \tilde{\pi}(a)\pi(bx) = \tilde{\pi}(a)\tilde{\pi}(b).$$

If $a \in \mathcal{A}$ and $x, y \in \mathcal{J}$ then

$$\begin{aligned} \langle \tilde{\pi}(a)\pi(x)\xi | \eta \rangle &= \langle \pi(ax)\xi | \pi(y)\xi \rangle \\ &= \langle \xi | \pi(ax)^* \pi(y)\xi \rangle \\ &= \langle \xi | \pi(x^* a^* y)\xi \rangle \\ &= \langle \xi | \pi(x^*) \pi(a^* y)\xi \rangle \\ &= \langle \xi | \pi(x)^* \tilde{\pi}(a^*) \pi(y)\xi \rangle \\ &= \langle \pi(x)\xi | \tilde{\pi}(a^*)\eta \rangle. \end{aligned}$$

Since ξ is cyclic for π , this shows that $\tilde{\pi}(a^*) = \tilde{\pi}(a)^*$ for every $a \in \mathcal{A}$. Therefore, $\tilde{\pi}$ is a representation. Since $\mathcal{H} = [\pi(\mathcal{J})\xi] \subseteq [\tilde{\pi}(\mathcal{A})\xi] \subseteq \mathcal{H}$, ξ is a cyclic vector for $\tilde{\pi}$. Conversely, let $\eta \in \mathcal{H}$ be a cyclic vector for $\hat{\pi}$. To show that η is cyclic for π we need only show that $\hat{\pi}(\mathcal{A})\eta \subseteq [\pi(\mathcal{J})\eta]$. Fix $a \in \mathcal{A}$, and let $(x_n)_{n=1}^{\infty}$ be a sequence in \mathcal{J} such that $\pi(x_n)\xi \rightarrow \eta$. Since

$\pi(ax_n) = \tilde{\pi}(a)\pi(x_n)$, $\pi(ax_n)\eta \rightarrow \hat{\pi}(a)\eta$, showing that $\tilde{\pi}(a) \in [\pi(\mathcal{J})\eta]$. Therefore, η is cyclic for π .

Finally, we will show the uniqueness of this extension. If $a \in \mathcal{A}$ and $T \in \mathcal{B}(\mathcal{H})$ is such that $T\pi(x) = \pi(ax)$ for every $x \in \mathcal{J}$, then

$$T\pi(x)\eta = \tilde{\pi}(a)\pi(x)\eta$$

for every $x \in \mathcal{J}$ and $\eta \in \mathcal{H}$. Since π is nondegenerate, $T = \tilde{\pi}(a)$. \square

2.9.13 Proposition. *Let \mathcal{A} be a Banach $*$ -algebra, \mathcal{J} a closed self-adjoint ideal of \mathcal{A} with a bounded approximate identity, and $\pi : \mathcal{J} \rightarrow \mathcal{B}(\mathcal{H})$ a representation of \mathcal{B} . Then π has an extension to a representation $\tilde{\pi}$ of \mathcal{A} . If π is nondegenerate then this extension is unique.*

PROOF. We will first assume that π is nondegenerate and then derive the general case from this. By Proposition 2.9.6, π is the direct sum of cyclic representations. Hence the claim follows from Lemma 2.9.12, because a direct sum of representations has an extension if and only if every summand in the direct sum has an extension.

Now, suppose that π is not necessarily nondegenerate. Let $\mathcal{H}_0 = [\pi(\mathcal{J})\mathcal{H}]$, and let $\pi_0 : \mathcal{J} \rightarrow \mathcal{B}(\mathcal{H}_0)$ be the subrepresentation $\pi_0(x) = \pi(x)|_{\mathcal{H}_0}$. Then π_0 is nondegenerate and has an extension $\tilde{\pi}_0$ to all of \mathcal{A} . Let $\rho : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_0^\perp)$ be any representation, such as the zero representation. Then $\tilde{\pi}_0 \oplus \rho$ is an extension of π . \square

2.9.14 Corollary. *Let \mathcal{A} be a Banach $*$ -algebra, and \mathcal{J} a closed self-adjoint ideal of \mathcal{A} with a bounded approximate identity. If $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ and $\rho : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ are representations of \mathcal{A} such that $\pi|_{\mathcal{J}}$ is nondegenerate and $\pi|_{\mathcal{J}} = \rho|_{\mathcal{J}}$ then $\pi = \rho$.*

PROOF. Since π and ρ are both extensions of $\pi|_{\mathcal{J}} = \rho|_{\mathcal{J}}$, the claim follows from the uniqueness in Proposition 2.9.13. \square

2.9.15 Proposition. *Let \mathcal{A} be a Banach $*$ -algebra, \mathcal{J} a closed self-adjoint ideal of \mathcal{A} with a bounded approximate identity, and (π_1, \mathcal{H}_1) and (π_2, \mathcal{H}_2) nondegenerate representations of \mathcal{J} . Then $\mathcal{R}(\tilde{\pi}_1, \tilde{\pi}_2) = \mathcal{R}(\pi_1, \pi_2)$.*

PROOF. Clearly,

$$\mathcal{R}(\tilde{\pi}_1, \tilde{\pi}_2) \subseteq \mathcal{R}(\pi_1, \pi_2).$$

To show the converse, fix $T \in \mathcal{R}(\pi_1, \pi_2)$. Then for every $a \in \mathcal{A}$, $x \in \mathcal{J}$, and $\xi \in \mathcal{H}_1$,

$$T\tilde{\pi}_1(a)\pi_1(x)\xi = T\pi_1(ax)\xi = \pi_2(ax)T\xi = \tilde{\pi}_2(a)\pi_2(x)T\xi = \tilde{\pi}_2(a)T\pi_1(x)\xi.$$

Since π_1 is nondegenerate, we have

$$T\widetilde{\pi}_1(a)\xi = \widetilde{\pi}_2(a)T\xi$$

for every $a \in \mathcal{A}$ and $\xi \in \mathcal{H}_1$, showing that $T \in \mathcal{R}(\widetilde{\pi}_1, \widetilde{\pi}_2)$. \square

2.9.16 Proposition. *Let G be a locally compact group. Then the integration of a unitary representation of G to a $*$ -representation of $M(G)$ and restriction of a $*$ -representation of $M(G)$ to $L^1(G)$ give a bijective correspondence between*

- (i) *continuous unitary representations of G ,*
- (ii) *$*$ -representations of $L^1(G)$,*
- (iii) *$*$ -representations of $M(G)$ whose restrictions to $L^1(G)$ are nondegenerate.*

Moreover, this correspondence preserves intertwining operators and cyclic vectors.

PROOF. The correspondence holds by Proposition 2.9.10, Proposition 2.9.11, and Proposition 2.9.13, so we need only show that it preserves intertwining operators and cyclic vectors. Let $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ and $\rho : G \rightarrow \mathcal{U}(\mathcal{K})$ be continuous representations of G . Then, if $T : \mathcal{H} \rightarrow \mathcal{K}$ is a bounded linear operator, using the nondegeneracy of $\pi_{L^1(G)}$ in the appropriate places gives

$$\begin{aligned} & T \text{ is an intertwiner of } \pi \text{ and } \rho \\ \Leftrightarrow & T\pi(s)\xi = \rho(s)T\xi \quad \text{for } s \in G \text{ and } \xi \in \mathcal{H} \\ \Leftrightarrow & \langle T\pi(s)\xi | \eta \rangle = \langle \rho(s)T\xi | \eta \rangle \quad \text{for } s \in G, \xi \in \mathcal{H}, \eta \in \mathcal{K} \\ \Leftrightarrow & \int f(s)\langle T\pi(s)\xi | \eta \rangle ds = \int f(s)\langle \rho(s)T\xi | \eta \rangle ds \\ & \text{for } f \in L^1(G), \xi \in \mathcal{H}, \eta \in \mathcal{K} \\ \Leftrightarrow & \langle T\pi_{L^1(G)}(f)\xi | \eta \rangle = \langle \rho_{L^1(G)}(f)T\xi | \eta \rangle \quad \text{for } f \in L^1(G), \xi \in \mathcal{H}, \eta \in \mathcal{K} \\ \Leftrightarrow & T\pi_{L^1(G)}(f)\xi = \rho_{L^1(G)}(f)T\xi \quad \text{for } f \in L^1(G), \xi \in \mathcal{H}, \eta \in \mathcal{K} \\ \Leftrightarrow & T \text{ is an intertwiner of } \pi_{L^1(G)} \text{ and } \rho_{L^1(G)}. \end{aligned}$$

The case of cyclic vectors is similar, so we will leave it as an exercise.

The passage between representations of $L^1(G)$ and $M(G)$ preserves intertwining operators and cyclic vectors by Lemma 2.9.12 and Proposition 2.9.15. \square

Since this correspondence preserves intertwining operators between two representations, it also preserves the commutant of a single representation. If $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ is a representation, then $\pi(G) \subseteq \mathcal{B}(\mathcal{H})$ is a self-adjoint set of operators, so $\pi(G)'$ is a von Neumann algebra that agrees with $\pi_{L^1(G)}(L^1(G))'$ and $\pi_{M(G)}(M(G))$. Thus $\pi(G)''$ is also a von Neumann algebra, which we will call the *von Neumann algebra generated by π* .

The two most important representations of G are the *left regular representation* and *right regular representation* on $L^2(G)$, defined by

$$(\lambda(g)\xi)(s) = \xi(g^{-1}s)$$

and

$$(\rho(g)\xi)(s) = \Delta(g)^{1/2}\xi(sg)$$

respectively. It is easy to check that these define continuous unitary representations of G . The integrated forms of these representations on $L^1(G)$ are given by

$$(\lambda_{L^1(G)}(f)\xi)(t) = (f * \xi)(t) = \int f(s)\xi(s^{-1}t) ds$$

and

$$(\rho_{L^1(G)}(f)\xi)(t) = (\xi * f)(t) = \int \Delta(s)^{1/2}f(s)\xi(ts) ds.$$

2.9.17 Proposition. *Let G be a locally compact group. Then the left and right regular representations are faithful unitary representations of G that give faithful representations of $L^1(G)$ and $M(G)$.*

PROOF. We will only give proofs for the left regular representation. The case of the right regular representation is similar.

First, we will show that λ gives a faithful representation of G . Fix $s \in G$ such that $s \neq e$. Choose a relatively compact symmetric neighbourhood V of e so that $s \notin V^2$. Then V has a finite non-zero Haar measure, so that $\chi_V \in L^2(G)$. We have

$$\lambda(s)\chi_V = \chi_{sV} \neq \chi_V = \lambda(e)\chi_V$$

Therefore, λ is faithful when viewed as a representation of G .

Now, we will show that λ gives a faithful representation of $M(G)$, and thus also of $L^1(G)$. Fix a nonzero $\mu \in M(G)$. Since $M(G)$ is the dual of $C_0(G)$, which contains $C_c(G)$ as a dense subspace, there exists an $f \in C_c(G)$ such that $\int f d\mu \neq 0$. The function $\mu * f$ on G defined by

$$(\mu * f)(t) = \int f(s^{-1}t) d\mu(s)$$

is continuous, since f is uniformly continuous, and we have

$$|(\mu * f)(s) - (\mu * f)(t)| \leq \|f_s - f_t\|_\infty \|\mu\|.$$

Since $(\mu * f)(e) = \int f d\mu$ is nonzero,

$$\|\lambda(\mu)f\|_2^2 = \int |(\mu * f)(s)|^2 ds > 0,$$

which implies that $\lambda(\mu) \neq 0$. Therefore, λ gives a faithful representation of $M(G)$. \square

The von Neumann algebras $\lambda(G)''$ and $\rho(G)''$ are called the *left group von Neumann algebra* and *right group von Neumann algebra* of G respectively. Usually, the left algebra is preferred and is simply called the *group von Neumann algebra* of G .

Unitary representations of groups have additional structure that is not shared by representations of $*$ -algebras. If $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ and $\rho : G \rightarrow \mathcal{U}(\mathcal{K})$ are representations of G , then the *tensor product* of π and ρ is the representation $\pi \otimes \rho : G \rightarrow \mathcal{U}(\mathcal{H} \otimes \mathcal{K})$ defined by

$$(\pi \otimes \rho)(s) = \pi(s) \otimes \rho(s).$$

Let $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ be a representation of G and $\overline{\mathcal{H}}$ the conjugate Hilbert space of \mathcal{H} . The *contragredient* of π is the representation $\overline{\pi} : G \rightarrow \mathcal{U}(\overline{\mathcal{H}})$ defined by

$$\langle \overline{\pi}(s)\overline{\xi} | \overline{\eta} \rangle = \overline{\langle \pi(s)\xi | \eta \rangle}.$$

If operators on \mathcal{H} and $\overline{\mathcal{H}}$ are represented as (possibly infinite) matrices with respect to some orthonormal basis, then the matrix representing $\overline{\pi}(s)$ is simply the transpose of the matrix representing $\pi(s^{-1}) = \pi(s)^*$.

If G and H are both locally compact groups, then there is another notion of tensor product of a representation of G and a representation of H . If $G = H$, then this notion does not agree with the notion of tensor product. Hence this tensor product is often called the *Kronecker product* of representations, reserving the use of the phrase “tensor product” for the product defined above.

Since representations of a Banach $*$ -algebra (or a locally compact group) always generate C^* -subalgebras of $\mathcal{B}(\mathcal{H})$, it would be more convenient for the purposes of representation theory to replace a Banach $*$ -algebra or locally compact group with a C^* -algebra that is a universal object with respect to the representation theory of the algebra or group. Then the study of representations of these objects could be replaced entirely by the study of representations of C^* -algebras. We will carry out this construction for all Banach $*$ -algebras with a separating family of $*$ -representations.

2.9.18 Definition. Let \mathcal{A} be a Banach $*$ -algebra with a separating family of $*$ -representations. Define $\|\cdot\|_{\text{rep}} : \mathcal{A} \rightarrow [0, \infty)$ by

$$\|a\|_{\text{rep}} = \sup\{\|\pi(a)\| : \pi \text{ is a } * \text{-representation of } \mathcal{A}\}.$$

We refer to $\|\cdot\|_{\text{rep}}$ as the *representation norm* on \mathcal{A} .

Of course, we must justify our choice of notation and actually show that $\|\cdot\|_{\text{rep}}$ is a norm on \mathcal{A} .

2.9.19 Proposition. Let \mathcal{A} be a Banach $*$ -algebra with a separating family of $*$ -representations. Then

- (i) $\|\cdot\|_{\text{rep}}$ is a norm on \mathcal{A} ;
- (ii) $\|ab\|_{\text{rep}} \leq \|a\|_{\text{rep}}\|b\|_{\text{rep}}$ for all $a, b \in \mathcal{A}$;
- (iii) $\|a^*\|_{\text{rep}} = \|a\|_{\text{rep}}$ for every $a \in \mathcal{A}$;
- (iv) $\|a^*a\|_{\text{rep}} = \|a\|_{\text{rep}}^2$ for every $a \in \mathcal{A}$.

Therefore, the completion of \mathcal{A} with respect to $\|\cdot\|_{\text{rep}}$ is a C^* -algebra.

PROOF.

- (i) If $a, b \in \mathcal{A}$, then $\|\pi(a+b)\| \leq \|\pi(a)\| + \|\pi(b)\|$ for every representation π of \mathcal{A} , so $\|a+b\|_{\text{rep}} \leq \|a\|_{\text{rep}} + \|b\|_{\text{rep}}$. Similarly, if $a \in \mathcal{A}$ and $\alpha \in \mathbb{C}$, then $\|\pi(\alpha a)\| = |\alpha| \cdot \|\pi(a)\|$ for every representation π of \mathcal{A} , so $\|\alpha a\|_{\text{rep}} = |\alpha| \|a\|_{\text{rep}}$. Nondegeneracy follows from the assumption that \mathcal{A} has a separating family of $*$ -representations.
- (ii) If $a, b \in \mathcal{A}$, then $\|\pi(ab)\| \leq \|\pi(a)\| \cdot \|\pi(b)\|$ for every representation π of \mathcal{A} , so $\|ab\|_{\text{rep}} \leq \|a\|_{\text{rep}}\|b\|_{\text{rep}}$.
- (iii) If $a \in \mathcal{A}$, then $\|\pi(a^*)\| = \|\pi(a)\|$ for every representation π of \mathcal{A} , so $\|a^*\|_{\text{rep}} = \|a\|_{\text{rep}}$.
- (iv) If $a \in \mathcal{A}$, then $\|\pi(a^*a)\| = \|\pi(a)\|^2$ for every representation π of \mathcal{A} , so $\|a^*a\|_{\text{rep}} = \|a\|_{\text{rep}}^2$. \square

2.9.20 Definition. Let \mathcal{A} be a Banach $*$ -algebra with a separating family of $*$ -representations. The completion of \mathcal{A} with respect to the representation norm $\|\cdot\|_{\text{rep}}$ is called the *enveloping C^* -algebra* of \mathcal{A} , and is denoted by $C^*(\mathcal{A})$.

The enveloping C^* -algebra gets its name because every $*$ -homomorphism from \mathcal{A} to a C^* -algebra \mathcal{B} factors uniquely through $C^*(\mathcal{A})$. However, we are not yet able to show this, because we have not proven that every C^* -algebra has a faithful $*$ -representation.

2.9.21 Example. Let G be a locally compact group. Then, by Proposition 2.9.17, the left regular representation of $L^1(G)$ is faithful, so $L^1(G)$ has a separating family of $*$ -representations. The enveloping C^* -algebra of $L^1(G)$ is called the *group C^* -algebra* of G and is denoted by $C^*(G)$. If π is a representation of G , it extends uniquely to a representation $\pi_{C^*(G)}$ of $C^*(G)$, so we can define $C_\pi^*(G)$ as the quotient $C^*(G)/\ker(\pi_{C^*(G)})$. In particular, if λ is the left regular representation of G , $C_\lambda^*(G)$ is denoted by $C_r^*(G)$, and is called the *reduced group C^* -algebra* of G . Even though λ integrates to a faithful representation of $L^1(G)$, it is not necessarily faithful when extended to $C^*(G)$. In fact, it is a theorem of Hulanicki [Hul64] that λ gives a faithful representation of $C^*(G)$ if and only if G is amenable.

Historical Notes

Representations of $*$ -algebras were considered in the earliest papers on C^* -algebras by Gelfand and Naimark [GN43] and Segal [Seg47], as they both characterized C^* -algebras as those Banach $*$ -algebras that are isometrically isomorphic to a $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$.

Schur's Lemma (Proposition 2.9.4) is due to Schur in the case of finite-dimensional representation of a finite group. The original version actually concerns intertwining operators of two irreducible representations, where it is shown that irreducible representations have intertwining operators if and only if they are equivalent, in which case all of the intertwining operators are scalar multiples of the identity. It was used as a lemma in the proof of the famous Schur orthogonality relations of characters. Schur made the important realization that the developing theory of representations and characters of finite groups could be recast in terms of linear algebra, which paved the way for the generalization of representation theory to infinite groups in terms of operators on infinite-dimensional spaces.

The connection between representations of locally compact groups and $*$ -representations of C^* -algebras has been known since the beginning of operator algebras. However, the first person to explicitly state that there exists a C^* -algebra $C^*(G)$ whose representation theory is equivalent to that of G is Kaplansky [Kap51a]. The reduced group C^* -algebra was already introduced by Segal [Seg47].

2.10 Positive Linear Functionals

2.10.1 Definition. Let \mathcal{A} be a $*$ -algebra. If $\omega : \mathcal{A} \rightarrow \mathbb{C}$ is a linear functional, we say that \mathcal{A} is *positive* if $\omega(a^*a) \geq 0$ for every $a \in \mathcal{A}$.

2.10.2 Examples.

- (i) Let X be a locally compact Hausdorff space, and let μ be a positive finite Radon measure on X . Then the map $\omega : C_0(X) \rightarrow \mathbb{C}$ defined by

$$\omega(f) = \int f(x) d\mu(x)$$

is a continuous positive linear functional on \mathcal{A} . By the Riesz Representation Theorem, every continuous positive linear functional on \mathcal{A} is of this form.

- (ii) Let $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ be a representation of \mathcal{A} , and fix $\xi \in \mathcal{H}$. Define $\omega : \mathcal{A} \rightarrow \mathbb{C}$ by

$$\omega(a) = \langle \pi(a)\xi | \xi \rangle.$$

Then ω is obviously linear, and

$$\omega(a^*a) = \langle \pi(a^*a)\xi | \xi \rangle = \langle \pi(a)^*\pi(a)\xi | \xi \rangle = \langle \pi(a)\xi | \pi(a)\xi \rangle \geq 0,$$

so ω is positive. Since π is continuous, ω is also continuous. A positive linear functional expressed in this way is said to be in *vector form*.

- (iii) The sum of any two positive linear functionals on \mathcal{A} is a positive linear functional, and if ω is a positive linear functional on \mathcal{A} , then so is $\lambda\omega$ for every $\lambda \geq 0$.

The main objective of this section is to show that this second example is the typical example of a positive linear functional. However, there are a few immediate complications:

- (i) Since a $*$ -homomorphism whose range is a C^* -algebra is always continuous, any positive functional ω on \mathcal{A} of the form

$$\omega(a) = \langle \pi(a)\xi | \xi \rangle,$$

where π is a representation of \mathcal{A} and $\xi \in \mathcal{H}$, must also be continuous. However, it is not very difficult to produce non-continuous positive linear functionals on a Banach $*$ -algebra. The easiest examples are related to the failure of factorization results for \mathcal{A} . Let \mathcal{A}^2 be the span of all products of two elements in \mathcal{A} . If \mathcal{A} has a bounded approximate identity, then the simplest form of the Cohen Factorization Theorem implies that $\mathcal{A}^2 = \mathcal{A}$. However, if \mathcal{A} does not have a bounded approximate identity, it is very well possible that \mathcal{A}^2 is strictly smaller than \mathcal{A} . In such a case, any linear functional ω of \mathcal{A} that vanishes on \mathcal{A}^2 is positive, but there are many such functionals that are not continuous. We will show in Theorem 2.10.9 that the existence of a bounded approximate identity in a Banach $*$ -algebra implies that every positive linear functional on the algebra is continuous.

- (ii) Even when φ is assumed to be continuous, there are other simple restrictions that φ must satisfy in order to have a representation in our desired form. Suppose that $\omega(a) = \langle \pi(a)\xi | \xi \rangle$, where π is a representation of \mathcal{A} and $\xi \in \mathcal{H}$. It is easy to extend π to \mathcal{A}_1 , the unitization of \mathcal{A} , by simply defining

$$\tilde{\pi}(a, \lambda) = \pi(a) + \lambda \cdot 1.$$

Then ω also extends to \mathcal{A}_1 as

$$\tilde{\omega}(a, \lambda) = \langle \tilde{\pi}(a, \lambda)\xi | \xi \rangle.$$

Just as a bounded approximate identity in a closed self-adjoint ideal J of a Banach $*$ -algebra \mathcal{A} allows one to extend representations of J to representations of \mathcal{A} Proposition 2.9.13, it is reasonable to expect that a

bounded approximate identity in \mathcal{A} would allow one to extend positive functionals on \mathcal{A} to \mathcal{A}_1 , since \mathcal{A} is a closed self-adjoint ideal of \mathcal{A}_1 . We could show this directly, but we will get it as a consequence of the construction of a representation from ω .

We will show that if \mathcal{A} has a bounded approximate identity then every positive linear functional on \mathcal{A} arises from a nondegenerate (and even cyclic) representation in this fashion. The key idea behind the construction of such representations is that for any positive linear functional ω on \mathcal{A} we can define a sesquilinear form $[\cdot, \cdot]$ on \mathcal{A} by

$$[a, b] = \omega(b^*a).$$

With a small amount of work, we can show that $[a, b] = \overline{[b, a]}$, as we do in the next proposition, where we also reprove the Cauchy-Schwarz inequality for ω for the sake of completeness. If $[\cdot, \cdot]$ were an inner product, we could represent \mathcal{A} by left multiplication on itself, equipped with this inner product, and then try completing the inner product to get a representation of \mathcal{A} on a Hilbert space. However, there is one thing preventing $[\cdot, \cdot]$ from being an inner product: it is quite possible that $[a, a] = 0$, i.e. $\omega(a^*a) = 0$, when $a \neq 0$. To correct this problem, we will define a subset \mathcal{N}_ω of \mathcal{A} associated to ω that is essentially the kernel of $[\cdot, \cdot]$ as a bilinear map. We will then show that \mathcal{N}_ω is a left ideal of \mathcal{A} , allowing us to redefine $[\cdot, \cdot]$ as an inner product on $\mathcal{A}/\mathcal{N}_\omega$.

2.10.3 Proposition. *Let \mathcal{A} be a $*$ -algebra, and ω a positive linear functional on \mathcal{A} . Then:*

- (i) $\omega(b^*a) = \overline{\omega(a^*b)}$ for all $a, b \in \mathcal{A}$;
- (ii) $|\omega(b^*a)|^2 \leq \omega(a^*a)\omega(b^*b)$ for all $a, b \in \mathcal{A}$;

PROOF.

(i) We have

$$\omega(a^*a) + \omega(b^*b) + \omega(a^*b) + \omega(b^*a) = \omega((a+b)^*(a+b)) \geq 0,$$

so $\text{Im}(\omega(b^*a) + \omega(a^*b)) = 0$. Hence $\text{Im}(\omega(b^*a)) = -\text{Im}(\omega(a^*b))$. By replacing a by ia ,

$$\begin{aligned} \text{Re}(\omega(b^*a)) &= \text{Im}(i\omega(b^*a)) \\ &= \text{Im}(\omega(b^*(ia))) \\ &= -\text{Im}(\omega((ia)^*b)) \\ &= -\text{Im}(-i\omega(a^*b)) \\ &= \text{Im}(i\omega(a^*b)) \\ &= \text{Re}(\omega(a^*b)), \end{aligned}$$

showing that $\omega(b^*a) = \overline{\omega(a^*b)}$.

(ii) If $\omega(b^*b) = 0$, the inequality is trivial. Thus we will assume that $\omega(b^*b) \neq 0$. For every $\alpha \in \mathbb{C}$,

$$\omega(a^*a) - \bar{\alpha}\omega(b^*) - \alpha\overline{\omega(b^*b)} + |\alpha|^2\omega(b^*b) = \omega((a - \alpha b)^*(a - \alpha b)) \geq 0.$$

In particular, when $\alpha = \omega(b^*a)/\omega(b^*b)$, we get that

$$\omega(a^*a) - \frac{|\omega(b^*a)|^2}{\omega(b^*b)} \geq 0,$$

establishing the desired inequality. \square

2.10.4 Definition. Let \mathcal{A} be a $*$ -algebra, and ω a positive linear functional on \mathcal{A} . The *left kernel* of ω is the set

$$\mathcal{N}_\omega = \{a \in \mathcal{A} : \omega(a^*a) = 0\}.$$

2.10.5 Proposition. Let \mathcal{A} be a $*$ -algebra, and ω a positive linear functional on \mathcal{A} . Then \mathcal{N}_ω is a left ideal of \mathcal{A} . Furthermore, the sesquilinear form $\langle \cdot | \cdot \rangle$ on $\mathcal{A}/\mathcal{N}_\omega$ defined by

$$\langle a + \mathcal{N} | b + \mathcal{N} \rangle = \omega(b^*a)$$

is a well-defined inner product on $\mathcal{A}/\mathcal{N}_\omega$.

PROOF. If $a, b \in \mathcal{N}_\omega$,

$$\begin{aligned} \omega((a+b)^*(a+b)) &= \omega(a^*a + a^*b + b^*a + b^*b) \\ &= 2\operatorname{Re}(\omega(a^*b)) \\ &\leq 2\omega(a^*a)^{\frac{1}{2}}\omega(b^*b)^{\frac{1}{2}} \\ &= 0, \end{aligned}$$

so $a+b \in \mathcal{N}_\omega$. If either $a \in \mathcal{A}$ or $a \in \mathbb{C}$ and $b \in \mathcal{N}_\omega$ then

$$\begin{aligned} \omega((ab)^*(ab)) &= \omega(b^*a^*ab) \\ &\leq \omega(b^*b)^{\frac{1}{2}}\omega((a^*ab)^*(a^*ab))^{\frac{1}{2}} \\ &= 0, \end{aligned}$$

so $ab \in \mathcal{N}_\omega$. Hence \mathcal{N}_ω is a left ideal.

The bilinear form $\langle \cdot | \cdot \rangle$ is well-defined because if $n_1, n_2 \in \mathcal{N}_\omega$,

$$\omega((b+n_2)^*(a+n_1)) = \omega(b^*a) + \omega((b+n_2)^*n_1) + \overline{\omega(a^*n_2)} = \omega(b^*a).$$

By part (ii) of Proposition 2.10.3,

$$\langle a + \mathcal{N}_\omega | b + \mathcal{N}_\omega \rangle = \omega(b^*a) = \overline{\omega(a^*b)} = \overline{\langle b + \mathcal{N}_\omega | a + \mathcal{N}_\omega \rangle}.$$

If $\langle a + \mathcal{N}_\omega | a + \mathcal{N}_\omega \rangle = 0$, then $\omega(a^*a) = 0$ and $a \in \mathcal{N}_\omega$, i.e. $a + \mathcal{N}_\omega = 0 + \mathcal{N}_\omega$. Therefore $\langle \cdot | \cdot \rangle$ is an inner product. \square

We noted earlier in this section that continuity of a positive linear functional is necessary for it to be derived from a representation. We will first prove the continuity of a positive linear functional ω on \mathcal{A} whenever \mathcal{A} has a bounded approximate identity, as the continuity of ω will be essential in showing that the representation of \mathcal{A} on $\mathcal{A}/\mathcal{N}_\omega$ is bounded.

One of the most important corollaries of the continuous functional calculus for C^* -algebras is the existence of square roots of positive elements. In many cases, one can also use the holomorphic functional calculus to obtain a square root for an element in a general Banach $*$ -algebra. Obviously, this is not always possible, due to the existence of multiple branches of the square root function for negative real numbers. However, the natural argument works by simply restricting the spectrum of an element so that it lies in the remainder of the unit disc.

2.10.6 Proposition. *Let \mathcal{A} be a unital Banach $*$ -algebra. If $a \in \mathcal{A}$ is self-adjoint and such that $\text{spr}(1 - a) < 1$ then there exists a self-adjoint $b \in \mathcal{A}$ such that $b^2 = a$.*

PROOF. Let $D = \{\lambda \in \mathbb{C} : \|\lambda - 1\| < 1\}$. Since $\text{spr}(1 - a) < 1$, we have that $\sigma(a) \subseteq D$. Let $f : D \rightarrow \mathbb{C}$ be the analytic continuation of the square root function from the open interval $(0, 1)$ to D . Let $b = f(a)$. Then $b^2 = a$ by the holomorphic functional calculus. Since the Taylor series expansion of f around 1 has real coefficients converging in D , b is self-adjoint. \square

2.10.7 Lemma. *Let \mathcal{A} be a unital Banach $*$ -algebra, and ω a positive linear functional on \mathcal{A} . Then ω is continuous, and if $\|1\| = 1$, we have $\|\omega\| = \omega(1)$.*

PROOF. By Proposition 2.1.13, we may renorm \mathcal{A} so that $\|1\| = 1$. If $a \in \mathcal{A}$ is self-adjoint and $\|a\| < 1$, then $\text{spr}(a) < 1$, so $\text{spr}(1 - a) < 1$. Thus, by Proposition 2.10.6, there is a self-adjoint $b \in \mathcal{A}$ such that $b^*b = 1 - a$, so

$$\omega(1) - \omega(a) = \omega(1 - a) = \omega(b^*b) \geq 0.$$

If $\|a\| < 1$, then $\|a^*a\| \leq \|a\|^2 < 1$. Since a^*a is self-adjoint, the above gives us that $\omega(a^*a) \leq \omega(1)$. Hence by Proposition 2.10.3,

$$|\omega(a)|^2 = |\omega(1 \cdot a)|^2 \leq \omega(a^*a)\omega(1) \leq \omega(1)^2.$$

Therefore, $\|\omega\| \leq \omega(1)$, and ω is continuous. If $\|1\| = 1$, then $\|\omega\| \geq \omega(1)$, so $\|\omega\| = \omega(1)$. \square

2.10.8 Proposition. *Let \mathcal{A} be a Banach $*$ -algebra, and ω a positive linear functional on \mathcal{A} . Fix $b \in \mathcal{A}$, and define $\rho : \mathcal{A} \rightarrow \mathbb{C}$ by $\rho(x) = \omega(b^*ab)$. Then ρ is a continuous positive linear functional and $\|\rho\| \leq \omega(b^*b)$. In particular, $|\omega(b^*ab)| \leq \|a\|\omega(b^*b)$ for all $a, b \in \mathcal{A}$.*

PROOF. Let \mathcal{A}_1 be the unitization of \mathcal{A} . Define $\tilde{\rho} : \mathcal{A}_1 \rightarrow \mathbb{C}$ by $\tilde{\rho}(a) = \omega(b^*ab)$, which makes sense because \mathcal{A} is an ideal of \mathcal{A}_1 and thus $b^*ab \in \mathcal{A}$. Obviously, $\tilde{\rho}$ is a linear functional, and

$$\tilde{\rho}(a^*a) = \omega(b^*a^*ab) = \omega((ab)^*(ab)) \geq 0,$$

$\tilde{\rho}$ is positive. Therefore, by Lemma 2.10.7, $\tilde{\rho}$ is continuous, and $\|\tilde{\rho}\| = \tilde{\rho}(1) = \omega(b^*b)$. Restricting to \mathcal{A} , we have that ρ is a continuous positive linear functional and $\|\rho\| \leq \|\tilde{\rho}\| = \omega(b^*b)$. The final inequality holds because

$$|\omega(b^*ab)| = |\rho(a)| \leq \|a\| \cdot \|\rho\| \leq \|a\|\omega(b^*b). \quad \square$$

2.10.9 Theorem. *Let \mathcal{A} be a Banach $*$ -algebra with a bounded approximate identity. Then every positive linear functional on \mathcal{A} is continuous.*

PROOF. Let ω be a positive linear functional on \mathcal{A} , and let $(x_n)_{n=1}^\infty$ be any sequence in \mathcal{A} such that $x_n \rightarrow 0$. Then by Corollary 2.4.10 there is an $a \in \mathcal{A}$ and a sequence $(y_n)_{n=1}^\infty$ in \mathcal{A} such that $x_n = ay_n$ for all $n \geq 1$ and $y_n \rightarrow 0$. Applying the same theorem to \mathcal{A}^{op} , there is a $b \in \mathcal{A}$ and a sequence $(z_n)_{n=1}^\infty$ in \mathcal{A} such that $y_n = z_nb$ for all $n \geq 1$ and $z_n \rightarrow 0$. Define $\rho : \mathcal{A} \rightarrow \mathbb{C}$ by $\rho(x) = \omega(axb)$. From the polarization identity

$$4axb = \sum_{k=0}^3 i^k (a^* + i^k b)^* x (a^* + i^k b)$$

we have

$$\rho(x) = \omega(axb) = \frac{1}{4} \left(\sum_{k=0}^3 i^k \omega((a^* + i^k b)^* x (a^* + i^k b)) \right),$$

so ρ is a linear combination of functionals of the form $x \mapsto \omega(c^*xc)$ for some $c \in \mathcal{A}$, which are all continuous by Proposition 2.10.8. Therefore, ρ is also continuous, and

$$\lim_{n \rightarrow \infty} \omega(x_n) = \lim_{n \rightarrow \infty} \omega(az_nb) = \lim_{n \rightarrow \infty} \rho(z_n) = 0,$$

showing that ω is continuous. □

The automatic continuity of positive linear functionals in the presence of a bounded approximate identity allows us to generalize some inequalities that are easy to prove in the unital case.

2.10.10 Proposition. *Let \mathcal{A} be a Banach $*$ -algebra with a bounded approximate identity, and ω a positive linear functional on \mathcal{A} . Then:*

- (i) $\omega(a^*) = \overline{\omega(a)}$ for all $a \in \mathcal{A}$;
- (ii) if K is a bound for a bounded approximate identity of \mathcal{A} , then

$$|\omega(a)|^2 \leq K^2 \|\omega\| \omega(a^*a)$$

for all $a \in \mathcal{A}$;

PROOF.

- (i) Let $(e_i)_{i \in I}$ be a bounded approximate identity of \mathcal{A} . For every $a \in \mathcal{A}$,

$$\begin{aligned} \omega(a^*) &= \lim_{i \in I} \omega(a^* e_i) \\ &= \lim_{i \in I} \overline{\omega(e_i^* a)} \\ &= \lim_{i \in I} \overline{\omega((a^* e_i)^*)} \\ &= \overline{\omega(a^{**})} \\ &= \overline{\omega(a)}. \end{aligned}$$

- (ii) Let $(e_i)_{i \in I}$ be a bounded approximate identity of \mathcal{A} with bound K . For every $a \in \mathcal{A}$,

$$\begin{aligned} |\omega(a)|^2 &= \lim_{i \in I} |\omega(e_i^* a)|^2 \\ &\leq \lim_{i \in I} \sup \omega(e_i^* e_i) \omega(a^* a) \\ &\leq \lim_{i \in I} \sup \|e_i^* e_i\| \|\omega\| \omega(a^* a) \\ &\leq \lim_{i \in I} \sup \|e_i\|^2 \|\omega\| \omega(a^* a) \\ &\leq K^2 \|\omega\| \omega(a^* a). \end{aligned}$$

□

If \mathcal{A} is an algebra with a contractive approximate identity, such as a C^* -algebra or $L^1(G)$ for a locally compact group G , then the positive linear functionals on \mathcal{A} are more well-behaved than in the general case of a bounded approximate identity.

2.10.11 Proposition. *Let \mathcal{A} be a Banach $*$ -algebra with a contractive approximate identity, and ω a positive linear functional on \mathcal{A} . Then:*

- (i) ω has a unique extension $\tilde{\omega}$ to \mathcal{A}_1 such that $\|\tilde{\omega}\| = \|\omega\|$;
- (ii) $\|\omega\| = \sup_{a \in \mathcal{A}, \|a\| \leq 1} \omega(a^* a)$;
- (iii) if $(a_i)_{i \in I}$ is a net in \mathcal{A} such that $\omega(a_i) \rightarrow \|\omega\|$, then $\omega(a_i^* a_i) \rightarrow \|\omega\|$.

Let $\mathcal{N} = \mathcal{N}_{\tilde{\omega}}$, so that $\mathcal{A}_1/\mathcal{N}$ is the inner product space associated with $\tilde{\omega}$. Then:

- (iv) if $(a_i)_{i \in I}$ is a net in \mathcal{A} such that $\omega(a_i) \rightarrow \|\omega\|$, then $a_i + \mathcal{N} \rightarrow 1 + \mathcal{N}$;

- (v) the image of \mathcal{A} is dense in $\mathcal{A}_I/\mathcal{N}$;
- (vi) if $(e_i)_{i \in I}$ is a contractive approximate identity of \mathcal{A} , then $\omega(e_i) \rightarrow \|\omega\|$;
- (vii) if ψ is another positive linear functional on \mathcal{A} , then

$$\|\omega + \psi\| = \|\omega\| + \|\psi\| \quad \text{and} \quad \widetilde{\omega + \psi} = \widetilde{\omega} + \widetilde{\psi}.$$

PROOF.

- (i) By Lemma 2.10.7, the norm of a positive linear functional on \mathcal{A}_I is determined by its value at the identity. Since \mathcal{A} has codimension 1 in \mathcal{A}_I , the extension of any positive linear functional to \mathcal{A}_I , if it exists, is unique. Define a linear functional $\widetilde{\omega}$ on \mathcal{A}_I by

$$\widetilde{\omega}(a + \lambda 1) = \omega(a) + \lambda \|\omega\|.$$

Then, using part (ii) of Proposition 2.10.10, we have

$$\begin{aligned} \widetilde{\omega}((a + \lambda 1)^*(a + \lambda 1)) &= \omega(a^*a + \bar{\lambda}a + \lambda a^*) + |\lambda|^2 \|\omega\| \\ &= \omega(a^*a) + 2 \operatorname{Re} \bar{\lambda} \omega(a) + |\lambda|^2 \|\omega\| \\ &\geq \omega(a^*a) - 2|\lambda| \|\omega\|^{1/2} \omega(a^*a) + |\lambda|^2 \|\omega\| \\ &= (\omega(a^*a)^{1/2} - |\lambda| \|\omega\|^{1/2})^2 \\ &\geq 0, \end{aligned}$$

so $\widetilde{\omega}$ is positive. By Lemma 2.10.7, $\|\widetilde{\omega}\| = \widetilde{\omega}(1) = \|\omega\|$.

- (ii) By part (ii) of Proposition 2.10.10, we have

$$\|\omega\|^2 = \sup_{a \in \mathcal{A}, \|a\| \leq 1} |\omega(a)|^2 \leq \sup_{a \in \mathcal{A}, \|a\| \leq 1} \|\omega\| \omega(a^*a) \leq \|\omega\|^2,$$

so dividing by $\|\omega\|$ gives

$$\|\omega\| = \sup_{a \in \mathcal{A}, \|a\| \leq 1} \omega(a^*a).$$

- (iii) By part (ii) of Proposition 2.10.10, we have

$$\omega(a_i)^2 \leq \|\omega\| \omega(a_i^* a_i) \leq \|\omega\|^2.$$

Hence $\omega(a_i^* a_i) \rightarrow \|\omega\|$.

- (iv) By Lemma 2.10.7 and part (i) of Proposition 2.10.10, we have

$$\begin{aligned} \langle a_i - 1 + \mathcal{N} \mid a_i - 1 + \mathcal{N} \rangle &= \widetilde{\omega}((a_i - 1)^*(a_i - 1)) \\ &= \widetilde{\omega}(a_i^* a_i - a_i^* - a_i + 1) \\ &= \omega(a_i^* a_i) - \overline{\omega(a_i)} - \omega(a_i) + \|\omega\|. \end{aligned}$$

By part (iii), the first term tends to $\|\omega\|$, and the middle two terms each tend to $-\|\omega\|$, so

$$\lim_{i \in I} \langle a_i - 1 + \mathcal{N} \mid a_i - 1 + \mathcal{N} \rangle = 0,$$

and $a_i + \mathcal{N} \rightarrow 1 + \mathcal{N}$.

(v) Let $(a_n)_{n=1}^{\infty}$ be a sequence in the unit ball of \mathcal{A} such that $|\omega(a_n)| \rightarrow \|\omega\|$. Define

$$b_n = \overline{\operatorname{sgn}(\omega(a_n))} a_n,$$

so that $\omega(b_n) \rightarrow \|\omega\|$. By part (iv), $b_n + \mathcal{N} \rightarrow 1 + \mathcal{N}$. Fix $c \in \mathcal{A}_I$, and define $c_n = cb_n$. Then $(c_n)_{n=1}^{\infty}$ is a sequence in \mathcal{A} , and by Proposition 2.10.8,

$$\begin{aligned} \|c_n - c + \mathcal{N}\|^2 &= \|(c(b_n - 1) + \mathcal{N})\| \\ &= \widetilde{\omega}((c(b_n - 1))^* c(b_n - 1)) \\ &= \widetilde{\omega}((b_n - 1)^* c^* c(b_n - 1)) \\ &\leq \|c^* c\| \widetilde{\omega}((b_n - 1)^* (b_n - 1)) \\ &= \|c^* c\| \cdot \|b_n - 1 + \mathcal{N}\|. \end{aligned}$$

Therefore, $c_n + \mathcal{N} \rightarrow c + \mathcal{N}$.

(vi) If $a \in \mathcal{A}$,

$$\lim_{i \in I} \langle e_i + \mathcal{N} \mid a + \mathcal{N} \rangle = \lim_{i \in I} \omega(a^* e_i) = \omega(a^*) = \langle 1 + \mathcal{N} \mid a + \mathcal{N} \rangle,$$

By part (v) the image of \mathcal{A} is dense in $\mathcal{A}_I/\mathcal{N}$, so $e_i + \mathcal{N} \rightarrow 1 + \mathcal{N}$ weakly. Therefore,

$$\lim_{i \in I} \omega(e_i) = \lim_{i \in I} \langle e_i + \mathcal{N} \mid 1 + \mathcal{N} \rangle = \langle 1 + \mathcal{N} \mid 1 + \mathcal{N} \rangle = \widetilde{\omega}(1) = \|\omega\|.$$

(vii) The first claim follows from part (vi), and the second follows from the first. \square

Before we finally show that every positive linear functional on a Banach $*$ -algebra with a bounded approximate identity can be expressed in vector form with respect to a representation of the algebra, we will show that such a representation is unique.

2.10.12 Proposition. *Let \mathcal{A} be a $*$ -algebra, $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ a representation of \mathcal{A} , and ξ a cyclic vector for π . If $\rho : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{K})$ is another representation of \mathcal{A} and $\eta \in \mathcal{K}$ is a cyclic vector for ρ such that*

$$\langle \pi(a)\xi \mid \xi \rangle = \langle \rho(a)\eta \mid \eta \rangle$$

for all $a \in \mathcal{A}$, then there exists a unique unitary $U : \mathcal{H} \rightarrow \mathcal{K}$ establishing a unitary equivalence of π and ρ and taking ξ to η .

PROOF. Define $U_0 : \pi(\mathcal{A})\xi \rightarrow \rho(\mathcal{A})\eta$ by

$$U_0 \pi(a)\xi = \rho(a)\eta.$$

We have

$$\begin{aligned}
\langle U_0\pi(a)\xi | U_0\pi(b)\xi \rangle &= \langle \rho(a)\eta | \rho(b)\eta \rangle \\
&= \langle \rho(b)^*\rho(a)\eta | \eta \rangle \\
&= \langle \rho(b^*a)\eta | \eta \rangle \\
&= \omega(b^*a) \\
&= \langle \pi(b^*a)\xi | \xi \rangle \\
&= \langle \pi(a)\xi | \pi(b)\xi \rangle.
\end{aligned}$$

Hence U_0 is well-defined and an isometry from $\pi(\mathcal{A})\xi$ onto $\rho(\mathcal{A})\eta$, so it extends by uniform continuity to a unitary U from \mathcal{H} onto \mathcal{K} , as ξ and η are cyclic vectors for π and ρ respectively. Then for all $a, b \in \mathcal{A}$,

$$\begin{aligned}
\rho(a)U_0\pi(b)\xi &= \rho(a)\rho(b)\eta \\
&= \rho(ab)\eta \\
&= U_0\pi(ab)\xi \\
&= U_0\pi(a)\pi(b)\xi.
\end{aligned}$$

Therefore $\rho(a)U = U\pi(a)$ for every $a \in \mathcal{A}$, and U establishes the unitary equivalence of π and π' . The uniqueness is clear because U is uniquely determined by U_0 , and U_0 is the only way that we could establish a unitary equivalence between π and ρ exchanging the cyclic vectors. \square

2.10.13 Theorem (Gelfand-Naimark-Segal). *Let \mathcal{A} be a Banach $*$ -algebra with a bounded approximate identity. If ω is a positive linear functional on \mathcal{A} there is a representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ and a cyclic vector ξ for π such that*

$$\omega(a) = \langle \pi(a)\xi | \xi \rangle.$$

Moreover, if \mathcal{A} has a contractive approximate identity, then $\|\omega\| = \|\xi\|^2$.

PROOF. Let $\mathcal{N} = \mathcal{N}_\omega = \{a \in \mathcal{A} : \omega(a^*a) = 0\}$, and $\langle \cdot | \cdot \rangle$ the bilinear form on \mathcal{A}/\mathcal{N} defined by

$$\langle a + \mathcal{N} | b + \mathcal{N} \rangle = \omega(b^*a).$$

By Proposition 2.10.5, $\langle \cdot | \cdot \rangle$ is actually a well-defined inner product on \mathcal{A}/\mathcal{N} . Let \mathcal{H} denote the completion of \mathcal{A}/\mathcal{N} as an inner product space. Let $\pi_0 : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{N}$ denote the left regular representation of \mathcal{A} on \mathcal{A}/\mathcal{N} defined by

$$\pi_0(a)(x + \mathcal{N}) = a(x + \mathcal{N}),$$

which is well-defined because \mathcal{N} is a left ideal. It is obviously linear, but it also preserves the involution because

$$\begin{aligned}
\langle \pi_0(a)(x + \mathcal{N}) | y + \mathcal{N} \rangle &= \omega(y^*ax) \\
&= \omega((a^*y)^*x) \\
&= \langle x + \mathcal{N} | \pi_0(a^*)(y + \mathcal{N}) \rangle \\
&= \langle \pi_0(a^*)^*(x + \mathcal{N}) | y + \mathcal{N} \rangle,
\end{aligned}$$

showing that $\pi_0(a^*) = \pi_0(a)^*$. Moreover, $\pi_0(a)$ is bounded for every $a \in \mathcal{A}$ as Proposition 2.10.10 implies that for all $a, x \in \mathcal{A}$,

$$\begin{aligned} \|\pi_0(a)(x + \mathcal{N})\|^2 &= \|a(x + \mathcal{N})\|^2 \\ &= \|ax + \mathcal{N}\|^2 \\ &= \omega(x^*a^*ax) \\ &\leq \|a^*a\|\omega(x^*x) \\ &= \|a^*a\| \cdot \|x + \mathcal{N}\|^2. \end{aligned}$$

Thus π_0 extends by continuity to a $*$ -representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$.

Now, we want to get the cyclic vector. If \mathcal{A} is unital, then it is easy to see that $1 + \mathcal{N}$ is a cyclic vector for π . If \mathcal{A} has a contractive approximate identity $(e_i)_{i \in I}$, then we can use the fact that $\omega(e_i) \rightarrow \|\omega\|$ to show that $e_i + \mathcal{N}$ defines a Cauchy net in \mathcal{H} , whose limit will be a cyclic vector. However, in the case of a general bounded approximate identity, this approach does not necessarily work. Instead, we show that ω descends to a bounded linear functional on the quotient \mathcal{A}/\mathcal{N} , and thus defines a bounded linear functional on \mathcal{H} , finally using the Riesz-Fréchet Theorem to get the cyclic vector. Define a linear functional ψ_0 on \mathcal{A}/\mathcal{N} by $\psi_0(a + \mathcal{N}) = \omega(a)$. To see that this is well-defined, let $(e_i)_{i \in I}$ be a bounded approximate identity for \mathcal{A} with bound K . Then, by part (ii) of Proposition 2.10.3, for all $n \in \mathcal{N}$,

$$|\omega(n)|^2 = \lim_{i \in I} |\omega(e_i^*n)|^2 \leq \lim_{i \in I} K^2 \omega(n^*n)\omega(e_i^*e_i) = 0.$$

By Proposition 2.10.10,

$$|\psi_0(a + \mathcal{N})| = |\omega(a)| \leq \|\omega\|^{\frac{1}{2}} \omega(a^*a)^{\frac{1}{2}} = \|\omega\|^{\frac{1}{2}} \|a + \mathcal{N}\|.$$

Therefore, ψ_0 is bounded and extends by continuity to a bounded linear functional ψ on \mathcal{H} . By the Riesz-Fréchet Theorem, there exists a unique vector $\xi \in \mathcal{H}$ such that $\psi(\eta) = \langle \eta | \xi \rangle$ for every $\eta \in \mathcal{H}$, and thus $\omega(a) = \langle a + \mathcal{N} | \xi \rangle$ for every $a \in \mathcal{A}$. We then have, for all $a, b \in \mathcal{A}$,

$$\begin{aligned} \langle a + \mathcal{N} | b + \mathcal{N} \rangle &= \omega(b^*a) \\ &= \langle b^*a + \mathcal{N} | \xi \rangle \\ &= \langle \pi(b^* + \mathcal{N})(a + \mathcal{N}) | \xi \rangle \\ &= \langle a + \mathcal{N} | \pi(b + \mathcal{N})\xi \rangle. \end{aligned}$$

Hence $a + \mathcal{N} = \pi(a)\xi$ and

$$\omega(a) = \langle a + \mathcal{N} | \xi \rangle = \langle \pi(a)\xi | \xi \rangle$$

for every $a \in \mathcal{A}$. Since $\pi(a)\xi = a + \mathcal{N}$, $\overline{\pi(\mathcal{A})\xi} = \overline{\mathcal{A}/\mathcal{N}} = \mathcal{H}$, so ξ is a cyclic vector for π .

Now, assume that \mathcal{A} has a contractive bounded approximate identity $(e_i)_{i \in I}$. Since π is nondegenerate, $\pi(e_i) \rightarrow 1$ in the strong operator topology by Proposition 2.9.7. Hence, by

$$\|\omega\| = \lim_{i \in I} |\omega(e_i)| = \lim_{i \in I} \langle \pi(e_i)\xi \mid \xi \rangle = \|\xi\|^2. \quad \square$$

The construction in the preceding theorem is called the *Gelfand-Naimark-Segal construction*, or simply the *GNS construction*. We will often denote the triple of the representation, Hilbert space, and cyclic vector associated with a positive linear functional ω by $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$, and call it the *GNS triple* associated with ω .

If G is a locally compact group, then the positive functionals on $L^1(G)$ are all continuous, because $L^1(G)$ has a bounded approximate identity. Hence they are elements of $L^\infty(G)$. We would like a characterization of the positive functionals on $L^1(G)$ in terms of their concrete properties as (equivalence classes of) functions on G .

2.10.14 Definition. Let G be a locally compact group. If φ is a continuous function on G , we say that φ is of *positive type*, or is *positive definite*, if for every finite sequence $s_1, \dots, s_n \in G$, the matrix $[\varphi(s_i^{-1}s_j)]_{1 \leq i, j \leq n}$ is positive. We let $P(G)$ denote the set of all functions on G of positive type.

2.10.15 Examples.

- (i) If $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ is a representation of G and $\xi \in \mathcal{H}$, then $\varphi : G \rightarrow \mathbb{C}$ defined by

$$\varphi(s) = \langle \pi(s)\xi \mid \xi \rangle$$

is of positive type. Indeed, if $s_1, \dots, s_n \in G$ and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$, then

$$\begin{aligned} \sum_{1 \leq i, j \leq n} \overline{\alpha_i} \alpha_j \langle \pi(s_i^{-1}s_j)\xi \mid \xi \rangle &= \sum_{1 \leq i, j \leq n} \langle \overline{\alpha_i} \alpha_j \pi(s_i^{-1}s_j)\xi \mid \xi \rangle \\ &= \sum_{1 \leq i, j \leq n} \langle \pi(s_j)\alpha_i \xi \mid \pi(s_i)\alpha_j \xi \rangle \\ &= \left\langle \sum_{i=1}^n \pi(s_i)\alpha_i \xi \mid \sum_{i=1}^n \pi(s_i)\alpha_i \xi \right\rangle \\ &= \left\| \sum_{i=1}^n \pi(s_i)\alpha_i \xi \right\|^2 \\ &\geq 0. \end{aligned}$$

- (ii) The special case of (i) where π is the left regular representation is particularly interesting. Recall that if f is a function on G , then f^\vee is the function on G defined by $f^\vee(s) = f(s^{-1})$. If $\xi, \eta \in L^2(G)$, then $\overline{\xi} * \eta^\vee \in C_0(G)$ and

$$\langle \lambda(s)\xi \mid \eta \rangle = (\overline{\eta} * \xi^\vee)(s).$$

Suppose, first, that $\xi, \eta \in C_c(G)$. Then

$$\langle \lambda(s)\xi | \eta \rangle = \int \xi(s^{-1}t)\overline{\eta(t)} dt = \int \overline{\eta(t)}\xi^\vee(t^{-1}s) dt = \overline{\eta} * \xi^\vee(s).$$

Since $\xi, \eta \in C_c(G)$, it is clear that $\overline{\eta} * \xi^\vee(s)$ also has compact support and is in $C_0(G)$. If $\xi, \eta \in L^2(G)$, then there exist sequences $(\xi_n)_{n=1}^\infty$ and $(\eta_n)_{n=1}^\infty$ in $C_c(G)$ converging to ξ and η respectively. If $s \in G$ we have

$$\begin{aligned} |\langle \lambda(s)\xi | \eta \rangle - \langle \lambda(s)\xi_n | \eta_n \rangle| &\leq \|\lambda(s)(\xi - \xi_n)\| \|\eta\| + \|\lambda(s)\xi_n\| \|\eta - \eta_n\| \\ &= \|\xi - \xi_n\| \|\eta\| + \|\xi_n\| \|\eta - \eta_n\|. \end{aligned}$$

Therefore, the coefficient function $\langle \lambda(\cdot)\xi | \eta \rangle$ is approximated in the uniform norm by compactly supported functions, so it must be in $C_0(G)$. Since

$$\langle \lambda(s)\xi_n | \eta_n \rangle = (\overline{\eta_n} * \xi_n^\vee)(s)$$

for every $n \in \mathbb{N}$, we have

$$\langle \lambda(s)\xi | \eta \rangle = (\overline{\eta} * \xi^\vee)(s).$$

In particular, if $\xi \in L^2(G)$, then $\overline{\xi} * \xi^\vee$ is a function of positive type.

- (iii) The sum of any two functions of positive type on G is of positive type, and if $\varphi : G \rightarrow \mathbb{C}$ is of positive type, then so is $\lambda\varphi$ for every $\lambda \geq 0$.
- (iv) A function of positive type is not necessarily positive valued. For example, the function $\varphi : \mathbb{R} \rightarrow \mathbb{C}$ defined by $\varphi(t) = e^{it}$ is a one-dimensional representation of \mathbb{R} and thus a function of positive type.

2.10.16 Proposition. *Let G be a locally compact group, and $\varphi : G \rightarrow \mathbb{C}$ be a function of positive type. Then*

- (i) $\varphi(s^{-1}) = \overline{\varphi(s)}$ for every $s \in G$;
- (ii) $|\varphi(s)| \leq \varphi(e)$ for every $s \in G$;
- (iii) $\|\varphi\|_\infty = \varphi(e)$.

PROOF.

- (i) Since the matrix

$$\begin{pmatrix} \varphi(e) & \varphi(s) \\ \varphi(s^{-1}) & \varphi(e) \end{pmatrix}$$

is positive it is also self-adjoint, showing that $\varphi(s^{-1}) = \overline{\varphi(s)}$.

- (ii) Since the matrix $[\varphi(e)]$ is positive, $\varphi(e) \geq 0$. The matrix

$$\begin{pmatrix} \varphi(e) & \varphi(s) \\ \varphi(s^{-1}) & \varphi(e) \end{pmatrix}$$

is also positive, so it has positive determinant, i.e.

$$\varphi(e)^2 - \varphi(s)\varphi(s^{-1}) \geq 0.$$

By (i), this implies that

$$\varphi(e)^2 - |\varphi(s)|^2 = \varphi(e)^2 - \varphi(s)\overline{\varphi(s)} = \varphi(e)^2 - \varphi(s)\varphi(s^{-1}) \geq 0.$$

Hence $|\varphi(s)| \leq \varphi(e)$.

(iii) This is immediate from (ii). □

Following part (iii), we let $P_1(G)$ denote the set of functions of positive type of norm 1 on G .

We will see that there is a natural correspondence between functions of positive type on G and positive linear functionals on $L^1(G)$, where a positive linear functional on $L^1(G)$ is constructed by integrating against a function of positive type, and we will show that every function of positive type arises in this fashion. Thus the functions of positive type on G can be viewed as concrete realizations of the positive linear functionals on $C^*(G)$ as actual continuous functions on G . We first need a lemma about the extreme points of the unit ball of $M(G)$

2.10.17 Proposition. *Let X be a locally compact Hausdorff space. Then the set of extreme points of the unit ball of $M(X)$ is*

$$\{\alpha\delta_x : \alpha \in \mathbb{C}, |\alpha| = 1 \text{ and } x \in X\}.$$

The set of extreme points of the set of probability measures on X is

$$\{\delta_x : x \in X\}.$$

PROOF. We leave it as an easy exercise to show that $\alpha\delta_x$ is an extreme point of the unit ball of $M(X)$ when $|\alpha| = 1$ and $x \in X$.

Now, let μ be an extreme point of the unit ball of $M(X)$. Suppose that the support of μ is $\{x\}$ for some $x \in X$. Then it is easy to see that $\mu = \alpha\delta_x$ for some $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$.

Suppose that the support of μ contains two distinct points x_1 and x_2 . Let U_1 and U_2 be neighbourhoods of x_1 and x_2 respectively with disjoint closures. By Urysohn's Lemma, there is an $f \in C_0(X)$ such that $0 \leq f(x) \leq 1$ for all $x \in X$, $f(x) = 1$ for all x in the closure of U_1 and $f(x) = 0$ for all x in the closure of U_2 . Consider the measures $f \cdot \mu$ and $(1 - f) \cdot \mu$. Let

$$\alpha = \|f \cdot \mu\| = \int |f| d|\mu| = \int f d|\mu|.$$

Then

$$\alpha = \int f d|\mu| \leq \|\mu\| = 1 \quad \text{and} \quad \alpha = \int f d|\mu| \geq |\mu|(U_1) > 0,$$

because U_1 is open and $U_1 \cap \text{supp}(\mu) \neq \emptyset$. Also,

$$1 - \alpha = 1 - \int f d|\mu| = \int (1 - f) d|\mu| = \|(1 - f) \cdot \mu\|,$$

and thus

$$1 - \alpha \geq \int_{U_2} d|\mu| = |\mu|(U_2) > 0,$$

because $U_2 \cap \text{supp}(\mu) \neq \emptyset$. Hence $0 < \alpha < 1$. However,

$$\left\| \frac{1}{\alpha} f \cdot \mu \right\| \leq 1 \quad \text{and} \quad \left\| \frac{1}{1 - \alpha} (1 - f) \cdot \mu \right\| \leq 1,$$

and

$$\mu = \alpha \left(\frac{f \cdot \mu}{\alpha} \right) + (1 - \alpha) \left(\frac{(1 - f) \cdot \mu}{1 - \alpha} \right).$$

Since μ is an extreme point of the unit ball of $M(X)$ and $\alpha \neq 0$, this implies that $\mu = (1/\alpha)f \cdot \mu$. This implies that f is equal to $\alpha < 1$ μ -almost everywhere on the support of μ . However, f is 1 on U_1 and $|\mu|(U_1) > 0$, a contradiction. Therefore, our assumption that the support of μ is more than a single point is false, and $\mu = \alpha \delta_x$ for some $\alpha \in \mathbb{C}$.

In order to prove the characterization of the extreme points of the probability measures on X it suffices to show that any extreme point of the set of probability measures is also an extreme point of $M(X)$. Suppose μ is an extreme point of the probability measures, and let $\mu_1, \mu_2 \in M(X)$ be such that $\|\mu_1\| \leq 1$, $\|\mu_2\| \leq 1$, and

$$\mu = \frac{1}{2}(\mu_1 + \mu_2).$$

Then

$$1 = \|\mu\| \leq \frac{1}{2}(\|\mu_1\| + \|\mu_2\|) \leq 1.$$

Hence $\|\mu_1\| + \|\mu_2\| = 2$, so $\|\mu_1\| = \|\mu_2\| = 1$. Also,

$$1 = \mu(X) = \frac{1}{2}(\mu_1(X) + \mu_2(X)).$$

Therefore,

$$1 = \|\mu\| \leq \frac{1}{2}(\|\mu_1\| + \|\mu_2\|) \leq 1.$$

which implies that $\|\mu_1\| = \|\mu_2\| = 1$. We want to show that μ_1 and μ_2 are both probability measures. We have

$$1 = \mu(X) = \frac{1}{2}(\mu_1(X) + \mu_2(X))$$

Since $|\mu_1(X)| \leq 1$, $|\mu_2(X)| \leq 1$, and 1 is an extreme point of the unit ball of \mathbb{C} , we have

$$\|\mu_1\| = \mu_1(X) = 1 \quad \text{and} \quad \|\mu_2\| = \mu_2(X) = 1.$$

By a well-known fact from measure theory, this equality implies that μ_1 and μ_2 are probability measures. We reprove this fact for bounded linear functionals on possibly noncommutative C^* -algebras in Proposition 2.12.1. Since μ is an extreme point of the set of probability measures, this implies that μ is an extreme point of the unit ball. \square

2.10.18 Theorem. *Let G be a locally compact group, and suppose $\varphi \in C_b(G)$. Then the following are equivalent:*

- (i) φ is a function of positive type;
- (ii) φ represents a positive linear functional on $L^1(G)$, i.e.

$$\langle f^* * f, \varphi \rangle = \iint \varphi(s^{-1}t) \overline{f(s)} f(t) ds dt \geq 0$$

for all $f \in L^1(G)$;

- (iii) φ represents a positive linear functional on $M(G)$, i.e.

$$\langle \mu^* * \mu, \varphi \rangle = \iint \varphi(s^{-1}t) \overline{d\mu(s)} d\mu(t) \geq 0$$

for all $\mu \in M(G)$.

PROOF. Suppose φ is of positive type. We only need to show that the condition in (ii) holds for functions in $C_c(G)$, since it is a dense subset of $L^1(G)$. Fix $f \in C_c(G)$. The function

$$(s, t) \mapsto \varphi(s^{-1}t) \overline{f(s)} f(t)$$

on $G \times G$ is continuous and of compact support. Let S denote the support of this function. Then there is a compact subset K of G such that $S \subseteq K \times K$. By the Krein-Milman Theorem and Proposition 2.10.17, the restriction of the Haar measure to K is the weak* limit of a bounded net of positive measures $(\nu_i)_{i \in I}$ of finite support, so the restriction of the Haar measure of $G \times G$ to $K \times K$ is the weak* limit of the net of product measures $(\nu_i)_{i \in I}$. Let ν be a measure of finite support defined by the positive masses $\alpha_1, \dots, \alpha_n$ at the points s_1, \dots, s_n . Then we have

$$\iint \varphi(s^{-1}t) \overline{f(s)} f(t) d\nu(s) d\nu(t) = \sum_{1 \leq i, j \leq n} \alpha_i \alpha_j \varphi(s_i^{-1} s_j) \overline{f(s_i)} f(s_j) \geq 0.$$

Therefore, taking the weak* limit,

$$\iint \varphi(s^{-1}t) \overline{f(s)} f(t) ds dt \geq 0.$$

Suppose (ii) holds. If $\mu \in M(G)$ has compact support K and $f \in C_c(G)$, then

$$\langle f^* * \mu^* * \mu * f, \varphi \rangle = \iint \left(\iint \varphi(xyzs) f^*(x) f(s) dx ds \right) d\mu^*(y) d\mu(z) \geq 0.$$

Fix $\epsilon > 0$. Using the compactness of $K^{-1} \times K$, there is a relatively compact symmetric neighbourhood V of e such that if $x, s \in V$, then

$$|\varphi(xyzs) - \varphi(yz)| < \epsilon$$

for all $(y, z) \in K^{-1} \times K$. Hence if f_V is a positive function supported on V such that $\int f_V(x) dx = 1$, then

$$\begin{aligned} & |\langle \mu^* * \mu, \varphi \rangle - \langle f_V^* * \mu^* * \mu * f_V, \varphi \rangle| \\ &= \left| \iint ((\varphi(yz) - \varphi(xyzs)) f_V^*(x) f_V(s) dx ds) d\mu^*(y) d\mu(z) \right| \\ &\leq \iint \left(\iint |\varphi(yz) - \varphi(xyzs)| f_V^*(x) f_V(s) dx ds \right) d|\mu^*|(y) d|\mu|(z) \\ &\leq \epsilon |\mu^*|(K^{-1}) |\mu|(K). \end{aligned}$$

Taking ϵ sufficiently small shows that $\langle \mu^* * \mu, \varphi \rangle \geq 0$ if $\mu \in M(G)$ is of compact support. Since the measures of compact support are dense in $M(G)$,

$$\langle \mu^* * \mu, \varphi \rangle \geq 0$$

for all $\mu \in M(G)$.

Finally, suppose (iii) holds. Fix $s_1, \dots, s_n \in G$ and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$, and let $\mu = \sum_{i=1}^n \alpha_i s_i \in M(G)$. Since

$$\mu^* * \mu = \sum_{i,j=1}^n \overline{\alpha_i} \alpha_j \varphi(s_i^{-1} s_j),$$

we have

$$\sum_{i,j=1}^n \overline{\alpha_i} \alpha_j \varphi(s_i^{-1} s_j) = \langle \mu^* * \mu, \varphi \rangle \geq 0.$$

Therefore, φ is of positive type. □

2.10.19 Theorem. *Let G be a locally compact group, and suppose $\varphi \in L^\infty(G)$. Then the following are equivalent:*

- (i) φ is in the equivalence class of a (necessarily unique) continuous function of positive type;
- (ii) φ represents a positive linear functional on $L^1(G)$, i.e.

$$\langle f^* * f, \varphi \rangle = \iint \varphi(s^{-1}t) \overline{f(s)} f(t) ds dt \geq 0$$

for all $f \in L^1(G)$;

- (iii) there exists a unitary representation $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ and a $\xi \in \mathcal{H}$ such that

$$\varphi(s) = \langle \pi(s)\xi | \xi \rangle.$$

Moreover, ξ may be taken to be cyclic for π .

PROOF. The equivalence of (i) and (ii) follows from Theorem 2.10.18. Suppose (ii) holds. Then, by the GNS Theorem, there exists a representation $\pi_{M(G)} : M(G) \rightarrow \mathcal{B}(\mathcal{H})$ and a cyclic vector $\xi \in \mathcal{H}$ such that

$$\langle \mu, \varphi \rangle = \langle \pi(\mu)\xi \mid \xi \rangle.$$

Let $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ be the associated representation of G defined by

$$\pi(g) = \pi_{M(G)}(\delta_g).$$

Then

$$\varphi(s) = \langle \delta_s, \varphi \rangle = \langle \pi_{M(G)}(\delta_s)\xi \mid \xi \rangle = \langle \pi(s)\xi \mid \xi \rangle$$

for all $s \in G$, proving (iv). Finally, the implication (iii) \implies (i) is part (i) of Examples 2.10.15. \square

Since unitary representations of groups have the same intertwiners and cyclic vectors as their integrated forms on $L^1(G)$, the same uniqueness that holds for the GNS construction of $L^1(G)$ holds for the corresponding representation of a function of positive type: if $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ and $\rho : G \rightarrow \mathcal{U}(\mathcal{K})$ are unitary representations of G with cyclic vectors $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$ such that

$$\langle \pi(s)\xi \mid \xi \rangle = \langle \rho(s)\eta \mid \eta \rangle,$$

then there exists a unique unitary $U : \mathcal{H} \rightarrow \mathcal{K}$ establishing a unitary equivalence of π and ρ and taking ξ to η . If φ is a function of positive type on G , we denote the triple of the representation, Hilbert space, and cyclic vector associated with φ by $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$, and call it the *GNS triple* associated with φ .

One consequence of this characterization of functions of positive type is that the functions of positive type are closed under pointwise multiplication, and that the product is easily computed using the tensor product of representations. The closure of functions of positive type under multiplication also follows directly from the definition and the fact that the Schur (or coordinate-wise) product of positive matrices is positive, but the concrete form is more useful. Similarly, complex conjugates of functions of positive type are also functions of positive type, which is given by the contragredient of a unitary representation.

2.10.20 Corollary. *Let G be a locally compact group.*

- (i) *If φ and ψ are functions of positive type on G , then $\varphi \cdot \psi$ is a function of positive type.*
- (ii) *If φ is a function of positive type on G , then $\overline{\varphi}$ is a function of positive type.*

PROOF.

- (i) Let $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ be $(\pi_\psi, \mathcal{H}_\psi, \xi_\psi)$ be the GNS triples associated with φ and ψ . Then

$$\begin{aligned} (\varphi \cdot \psi)(s) &= \langle \pi_\varphi(s)\xi_\varphi \mid \xi_\varphi \rangle \langle \pi_\psi(s)\xi_\psi \mid \xi_\psi \rangle \\ &= \langle (\pi_\varphi(s)\xi_\varphi) \otimes (\pi_\psi(s)\xi_\psi) \mid \xi_\varphi \otimes \xi_\psi \rangle \\ &= \langle (\pi_\varphi \otimes \pi_\psi)(s)\xi_\varphi \otimes \xi_\psi \mid \xi_\varphi \otimes \xi_\psi \rangle, \end{aligned}$$

which shows that $\varphi \cdot \psi$ is of positive type.

- (ii) Let $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ be the GNS triple associated with φ . Then

$$\begin{aligned} \overline{\varphi}(s) &= \overline{\varphi(s)} \\ &= \overline{\langle \pi_\varphi(s)\xi_\varphi \mid \xi_\varphi \rangle} = \langle \overline{\pi_\varphi \xi_\varphi} \mid \overline{\xi_\varphi} \rangle, \end{aligned}$$

which shows that $\overline{\varphi}$ is of positive type. □

The natural topology for the positive functionals on a Banach $*$ -algebra is the weak* topology, but in the case of $L^1(G)$ for a locally compact group, we would really like a concrete realization of this topology as a topology of functions on the group. We will show that the weak* topology on $L^\infty(G)$ is the topology of uniform convergence on compact sets when restricted to the set of states on $L^1(G)$. Unfortunately, the proof is of a somewhat more technical flavour than the other results about functions of positive type, and there does not really seem to be any way to reduce the amount of technicality. First, we will show an inequality for functions of positive type that shows they are continuous.

2.10.21 Proposition. *Let G be a locally compact group, and let $\varphi : G \rightarrow \mathbb{C}$ be a function of positive type. Then*

$$|\varphi(x) - \varphi(y)|^2 \leq 2\varphi(e)[\varphi(e) - \operatorname{Re} \varphi(x^{-1}y)].$$

PROOF. Let (π, \mathcal{H}, ξ) be the GNS representation associated with φ . Then

$$\begin{aligned} |\varphi(x) - \varphi(y)|^2 &= |\langle (\pi(x) - \pi(y))\xi \mid \xi \rangle|^2 \\ &\leq \|\xi\|^2 \cdot \|\pi(x)\xi - \pi(y)\xi\|^2 \\ &= \varphi(e)[\|\pi(x)\xi\|^2 + \|\pi(y)\xi\|^2 - 2\operatorname{Re}\langle \pi(x)\xi \mid \pi(y)\xi \rangle] \\ &= \varphi(e)[2\|\xi\|^2 - 2\operatorname{Re}\langle \pi(x^{-1}y)\xi \mid \xi \rangle] \\ &= 2\varphi(e)[\varphi(e) - \operatorname{Re} \varphi(x^{-1}y)]. \end{aligned} \quad \square$$

2.10.22 Theorem (Raikov). *Let G be a locally compact group. Then the weak* topology on $P_1(G)$ agrees with the topology of uniform convergence on compact sets.*

PROOF. Let $(\varphi_i)_{i \in I}$ be a net in $P_1(G)$ converging to some $\varphi \in P_1(G)$ uniformly on every compact subset of G . Fix $f \in L^1(G)$ and $\epsilon > 0$. Choose a compact subset K of G and an $i_0 \in I$ such that

$$\|(1 - \chi_K)f\|_1 < \frac{\epsilon}{4} \quad \text{and} \quad \|\chi_K(\varphi_i - \varphi_{i_0})\|_\infty < \frac{\epsilon}{2\|f\|_1}$$

for every $i \geq i_0$. Then, for every $i \geq i_0$ we have

$$\begin{aligned} |\langle f, \varphi_i - \varphi \rangle| &= |\langle f, \chi_K(\varphi_i - \varphi) \rangle| + |\langle f, (1 - \chi_K)(\varphi_i - \varphi) \rangle| \\ &\leq \|f\|_1 \|\chi_K(\varphi_i - \varphi)\|_\infty + \|(1 - \chi_K)f\|_1 \|\varphi_i - \varphi\|_\infty \\ &\leq \frac{\epsilon}{2} + 2\|(1 - \chi_K)f\|_1 \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore, $(\varphi)_{i \in I}$ converges in the weak* topology on $L^\infty(G)$.

Conversely, suppose that $(\varphi_i)_{i \in I}$ is a net in $P_1(G)$ converging to some φ in the weak* topology on $L^\infty(G)$. Fix a compact set K of G . Let V be a relatively compact neighbourhood of the identity $e \in G$ such that

$$V \subseteq \{s \in G : |1 - \varphi(s)| \leq \epsilon\}.$$

Define $f_V = \chi_V/m(V) \in L^1(G)$. Fix $\epsilon > 0$, and choose $i_1 \in I$ such that $|\langle f_V, \varphi_i - \varphi \rangle| < \epsilon$ for all $i \geq i_1$, i.e.

$$\left| \int_V \varphi_i(s) - \varphi(s) ds \right| < \epsilon m(V).$$

For every $i \geq i_1$, we have

$$\left| \int_V (1 - \varphi_i(s)) ds \right| \leq \int_V |1 - \varphi(s)| ds + \left| \int_V \varphi_i(s) - \varphi(s) ds \right| < 2\epsilon m(V).$$

By Proposition 2.10.21, if $i \in I$ and $s, t \in G$,

$$\begin{aligned} |\varphi_i(t^{-1}s) - \varphi_i(s)| &\leq \sqrt{2\varphi_i(e)(\varphi_i(e) - \operatorname{Re} \varphi_i((t^{-1}s)^{-1}s))} \\ &= \sqrt{2 - 2 \operatorname{Re} \varphi_i(t)}. \end{aligned}$$

Therefore, for every $i \geq i_1$ and $s \in G$,

$$\begin{aligned}
|f_V * \varphi_i(s) - \varphi_i(s)| &= \left| \int_V f_V(t) (\varphi_i(t^{-1}s) - \varphi_i(s)) dt \right| \\
&\leq \frac{1}{m(V)} \int_V |\varphi_i(t^{-1}s) - \varphi_i(s)| dt \\
&\leq \frac{1}{m(V)} \int_V \sqrt{2 - 2 \operatorname{Re} \varphi_i(t)} dt \\
&\leq \frac{1}{m(V)} \left(\int_V 2 - 2 \operatorname{Re} \varphi_i(t) dt \right)^{\frac{1}{2}} m(V)^{\frac{1}{2}} \\
&\leq \frac{1}{m(V)} \left| 2 \int_V 1 - \varphi_i(t) dt \right|^{\frac{1}{2}} m(V)^{\frac{1}{2}} \\
&\leq \sqrt{2\epsilon}.
\end{aligned}$$

Hence, for every $i \geq i_1$,

$$\|f_V * \varphi_i - \varphi_i\|_\infty \leq \sqrt{2\epsilon}$$

Similarly, since $|1 - \varphi(s)| < \epsilon$ for all $s \in V$, we could repeat the above argument to get

$$\|f_V * \varphi - \varphi\|_\infty \leq \sqrt{2\epsilon}.$$

If $f \in L^1(G)$ and $s \in G$, define ${}_s f \in L^1(G)$ by ${}_s f(t) = f(s^{-1}t)$. For a fixed $s \in G$, the map $f \mapsto {}_s f$ on $L^1(G)$ is continuous. Since K is compact, so is K^{-1} , and thus the subset $\{{}_s(f_V) : s \in K^{-1}\}$ of $L^1(G)$ is compact in the norm topology. If $\psi \in L^\infty(G)$, define $\psi^\vee \in L^\infty(G)$ by $\psi^\vee(s) = \psi(s^{-1})$. It is easily seen that the mapping given by \vee is weak*-continuous. Therefore, the net $(\varphi_i^\vee)_{i \in I}$ converges to φ^\vee in the weak* topology, and there exists some $i_2 \in I$ such that

$$\sup_{s \in K} |\langle {}_{s^{-1}} f_V, \varphi_i^\vee - \varphi^\vee \rangle| < \epsilon$$

for every $i \geq i_2$. If $f \in L^1(G)$ and $\psi \in L^\infty(G)$, then

$$\begin{aligned}
\langle {}_{s^{-1}} f, \psi^\vee \rangle &= \int \psi^\vee(t) f(st) dt \\
&= \int \psi^\vee(s^{-1}t) f(t) dt \\
&= \int f(t) \psi(t^{-1}s) dt \\
&= f * \psi(s).
\end{aligned}$$

Hence, for every $i \geq i_2$,

$$\sup_{s \in K} |f_V * \varphi_i(s) - f_V * \varphi(s)| \leq \epsilon.$$

Let i_0 be an element of I that dominates i_1 and i_2 . Then for every $i \geq i_0$,

$$\begin{aligned} & \sup_{s \in K} |\varphi_i(s) - \varphi(s)| \\ & \leq \|\varphi_i - f_V * \varphi_i\|_\infty + \sup_{s \in K} |f_V * \varphi_i(s) - f_V * \varphi(s)| + \|f_V * \varphi - \varphi\|_\infty \\ & \leq 2\sqrt{2\epsilon} + \epsilon + 2\sqrt{2\epsilon}. \end{aligned}$$

Therefore, the net $(\varphi_i)_{i \in I}$ converges to φ uniformly on K . \square

If \mathcal{A} is a Banach $*$ -algebra with a bounded approximate identity and a separating family of $*$ -representations, then the positive linear functionals on \mathcal{A} correspond to the positive linear functionals on $C^*(\mathcal{A})$, and this identification is isometric. If we restrict our attention to the positive linear functionals of norm 1, then this identification is a homeomorphism with respect to the weak $*$ topologies on \mathcal{A}^* and $C^*(\mathcal{A})^*$.

2.10.23 Proposition. *Let \mathcal{A} be a Banach $*$ -algebra with a bounded approximate identity and a separating family of $*$ -representations. Then*

- (i) *every positive linear functional ω on \mathcal{A} has a unique extension to a positive linear functional $\tilde{\omega}$ on $C^*(\mathcal{A})$;*
- (ii) *the map $\omega \mapsto \tilde{\omega}$ is a norm-preserving bijection from \mathcal{A}_+^* onto $C^*(\mathcal{A})_+^*$;*
- (iii) *the map $\omega \mapsto \tilde{\omega}$ is a homeomorphism onto its range when restricted to any bounded subset of \mathcal{A}_+^* , where both \mathcal{A}_+^* and $C^*(\mathcal{A})_+^*$ are equipped with the weak $*$ topology.*

PROOF.

- (i) Without loss of generality, we can assume that $\|\omega\| \leq 1$. By the GNS Theorem, there exists a representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ and a vector $\xi \in \mathcal{H}$ such that $\|\xi\| \leq 1$ and $\omega(a) = \langle \pi(a)\xi | \xi \rangle$. Then for every $a \in \mathcal{A}$,

$$\begin{aligned} |\omega(a)| &= |\langle \pi(a)\xi | \xi \rangle| \\ &\leq \|\pi(a)\xi\| \|\xi\| \\ &\leq \|\pi(a)\| \|\xi\| \|\xi\| \\ &\leq \|\pi(a)\| \\ &\leq \|a\|_{\text{rep}}, \end{aligned}$$

showing that ω is continuous with respect to the representation norm. Therefore, it extends to a unique linear functional $\tilde{\omega}$ on $C^*(\mathcal{A})$. To see that $\tilde{\omega}$ is also positive, fix $b \in C^*(\mathcal{A})$. Then there exists a sequence $(a_n)_{n=1}^\infty$ in \mathcal{A} such that converges to b . Since ω is positive, $\omega(a_n^*a_n) \geq 0$ for every $n \geq 1$. Thus

$$\tilde{\omega}(b^*b) = \lim_{n \rightarrow \infty} \tilde{\omega}(a_n^*a_n) = \lim_{n \rightarrow \infty} \omega(a_n^*a_n) \geq 0,$$

showing that $\tilde{\omega}$ is positive.

- (ii) Let ω be a positive linear functional on \mathcal{A} . The map $\omega \mapsto \tilde{\omega}$ is obviously injective. Since the restriction of every positive linear functional on $C^*(\mathcal{A})$ to a positive linear functional on \mathcal{A} , this map is also bijective. Since the representation norm on \mathcal{A} is dominated by its usual norm, for every $a \in \mathcal{A}$ such that $\|a\| \leq 1$,

$$|\omega(a)| = |\tilde{\omega}(a)| \leq \|\tilde{\omega}\| \|a\| \leq \|\tilde{\omega}\|.$$

Hence $\|\omega\| \leq \|\tilde{\omega}\|$. For the reverse inequality, note that by Proposition 2.10.10, for every $a \in \mathcal{A}$ such that $\|a\|_{\text{rep}} \leq 1$ we have

$$\begin{aligned} |\tilde{\omega}(a)| &= |\omega(a)| \\ &\leq \|\omega\|^{\frac{1}{2}} \omega(a^*a)^{\frac{1}{2}} \\ &= \|\omega\|^{\frac{1}{2}} \tilde{\omega}(a^*a)^{\frac{1}{2}} \\ &\leq \|\omega\|^{\frac{1}{2}} \|\tilde{\omega}\|^{\frac{1}{2}} \|a^*a\|^{\frac{1}{2}} \\ &\leq \|\tilde{\omega}\|^{\frac{1}{2}} \|\tilde{\omega}\|^{\frac{1}{2}} (\|a\|^2)^{\frac{1}{2}} \\ &= \|\tilde{\omega}\| \|a\|_{\text{rep}} \\ &\leq \|\tilde{\omega}\|. \end{aligned}$$

Since the unit ball of \mathcal{A} equipped with the $\|\cdot\|_{\text{rep}}$ norm is dense in the unit ball of $C^*(\mathcal{A})$, this establishes that $\|\tilde{\omega}\| \leq \|\omega\|$.

- (iii) Let M be a bounded subset of \mathcal{A}_+^* , and let $N = \{\hat{\omega} : \omega \in M\}$. By (ii), N is also bounded. The map from N onto M defined by $\hat{\omega} \mapsto \omega$ is obviously uniformly continuous with respect to the weak* topologies on M and N , because \mathcal{A} is a dense subset of $C^*(\mathcal{A})$. Since N is bounded, it is precompact, and thus the map $\hat{\omega} \mapsto \omega$ is a homeomorphism when restricted to N , i.e. the map $\omega \mapsto \hat{\omega}$ is a homeomorphism when restricted to M . \square

The preceding result is very useful when applied to $L^1(G)$ for a locally compact group G , because it implies that the functions of positive type on G can be identified with the positive linear functionals on $C^*(G)$.

Historical Notes

The GNS construction is independently due to Gelfand and Naimark [GN43] and Segal [Seg47], who only considered C^* -algebras. The generalization to Banach $*$ -algebras with a bounded approximate identity is clear, as long as the positive linear functional in question is assumed to be continuous. This assumption was shown to be unnecessary by Varopoulos, who proved in [Var64] that all positive linear functionals on a Banach $*$ -algebra with a bounded approximate identity are automatically continuous.

Functions of positive type are a part of classical Fourier analysis on the real line. The theory of functions of positive type was generalized to general locally

compact groups by Gelfand and Raikov [GR43] and Godement [God48]. Raikov proved the equivalence of the weak* topology on $P_1(G)$ with the topology of the uniform convergence compact sets (Theorem 2.10.22) in [Rai47].

2.11 Pure Positive Functionals and Irreducible Representations

The Gelfand-Naimark-Segal construction gives an association of representations to positive functionals. We would like to know when this results in an irreducible representation.

The key tool in resolving this question is an operator-theoretic analogue of the Radon-Nikodym Theorem in measure theory. In the case of a commutative C^* -algebra, where positive linear functionals correspond to positive finite Radon measures, the order structure on the positive functionals has an interpretation in terms of a familiar property from measure theory. Let X be a locally compact Hausdorff space, and μ and ν positive finite Radon measures on X . Then $\mu \leq C\nu$ for some $C > 0$, when μ and ν are viewed as positive functionals on $C_0(X)$, precisely when μ is absolutely continuous with respect to ν , i.e. every ν -null set is also a μ -null set. The Radon-Nikodym Theorem states that $\mu \leq C\nu$ for some $C > 0$ if and only if there exists a $g \in L^1(X, \nu)$ (which, due to the positivity of the measures μ and ν , is necessarily positive) such that

$$\mu(E) = \int_E g \, d\nu,$$

in which case

$$\int f \, d\mu = \int fg \, d\nu.$$

We will present an operator-theoretic analogue of the Radon-Nikodym Theorem here, although it is not strictly a generalization, because it is impossible to recover the classical Radon-Nikodym Theorem from the version we give here.

2.11.1 Proposition. *Let \mathcal{A} be a $*$ -algebra, $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ a representation of \mathcal{A} , and $\xi \in \mathcal{H}$. Define a positive linear functional ω on \mathcal{A} by*

$$\omega(a) = \langle \pi(a)\xi \mid \xi \rangle.$$

If $T \in \pi(\mathcal{A})'$ is such that $0 \leq T \leq 1$, let ω_T be the linear functional on \mathcal{A} defined by

$$\omega_T(a) = \langle T\pi(a)\xi \mid \xi \rangle.$$

Then ω_T is a positive functional on \mathcal{A} such that $\omega_T \leq \omega$. Moreover, if ξ is cyclic then the map $T \mapsto \omega_T$ is injective.

PROOF. Since the product of commuting positive operators is positive, ω_T is positive. If $a \in \mathcal{A}$, we have

$$\begin{aligned}\omega_T(a^*a) &= \langle T\pi(a^*a)\xi | \xi \rangle \\ &= \langle T^{1/2}\pi(a)\xi | T^{1/2}\pi(a)\xi \rangle \\ &= \|T\pi(a)\xi\|^2 \\ &\leq \|\pi(a)\xi\|^2 \\ &= \omega(a^*a),\end{aligned}$$

so $\omega_T \leq \omega$. Now, suppose that ξ is cyclic, and that $\omega_T = \omega_{T'}$. Then

$$\langle T\pi(a)\xi | \xi \rangle = \langle T'\pi(a)\xi | \xi \rangle.$$

Since ξ is cyclic, this implies that $T\xi = T'\xi$, i.e. that $(T - T')\xi = 0$. Since ξ is cyclic for $\pi(\mathcal{A})$, it is separating for $\pi(\mathcal{A})'$, so $(T - T') = 0$, i.e. $T = T'$. \square

2.11.2 Proposition. *Let \mathcal{A} be a Banach $*$ -algebra with a bounded approximate identity, and ω and ψ be positive linear functionals on \mathcal{A} . Furthermore, let (π, \mathcal{H}, ξ) denote the GNS representation given by ω . If $\psi \leq \omega$, then there is a $T \in \pi(\mathcal{A})'$ such that $0 \leq T \leq 1$ and*

$$\psi(a) = \langle T\pi(a)\xi | \xi \rangle$$

for all $a \in \mathcal{A}$.

PROOF. Using that ξ is cyclic for π , define a sesquilinear form $[\cdot, \cdot]$ on \mathcal{H} by

$$[\pi(a)\xi, \pi(b)\xi] = \psi(b^*a).$$

As in the proof of the GNS Theorem, $[\cdot, \cdot]$ is a bounded sesquilinear form. Therefore, by the Riesz-Fréchet Theorem there is a $T \in \mathcal{B}(\mathcal{H})$ such that $[\eta, \zeta] = \langle T\eta | \zeta \rangle$ for all $\eta, \zeta \in \mathcal{H}$. We will now show that $T \in \pi(\mathcal{A})'$. If $a, b, c \in \mathcal{A}$, then

$$\begin{aligned}\langle T\pi(a)(\pi(c)\xi) | \pi(b)\xi \rangle &= \psi(b^*(ac)) \\ &= \psi((a^*b)^*c) \\ &= \langle T\pi(c)\xi | \pi(a^*)(\pi(b)\xi) \rangle \\ &= \langle \pi(a)T(\pi(c)\xi) | \pi(b)\xi \rangle.\end{aligned}$$

Therefore, fixing a and letting b and c range over \mathcal{A} , we conclude that $T\pi(a) = \pi(a)T$, i.e. $T \in \pi(\mathcal{A})'$. Finally, we will show that $0 \leq T \leq 1$. We have

$$\langle T\pi(a)\xi | \pi(a)\xi \rangle = \psi(a^*a)$$

for every $a \in \mathcal{A}$, so T is positive. Since

$$\langle T\pi(a)\xi | \pi(a)\xi \rangle = \psi(a^*a) \leq \omega(a^*a) = \langle \pi(a)\xi | \pi(a)\xi \rangle$$

for every $a \in \mathcal{A}$, the norm of T is at most 1. \square

2.11.3 Definition. Let \mathcal{A} be a $*$ -algebra, and ω a positive linear functional on \mathcal{A} . We say that ω is *pure* if whenever ψ is a positive linear functional such that $\psi \leq \omega$, there exists a $\lambda \geq 0$ such that $\psi = \lambda\omega$.

2.11.4 Proposition. *Let \mathcal{A} be a Banach $*$ -algebra with a bounded approximate identity, and ω a positive linear functional on \mathcal{A} . Then ω is pure if and only if the GNS representation associated with ω is irreducible.*

PROOF. Let (π, \mathcal{H}, ξ) be the GNS representation associated to ω . Suppose that ω is pure but π is not irreducible. By Proposition 2.9.4, there is a projection $p \in \pi(\mathcal{A})'$ such that $p \neq 0$ and $p \neq 1$. Then $p\xi \neq 0$, as otherwise $p(\pi(a)\xi) = \pi(a)p\xi = 0$ for all $a \in \mathcal{A}$ and $p = 0$. Similarly, $(1-p)\xi \neq 0$. For $a \in \mathcal{A}$, define

$$\omega_1(a) = \langle \pi(a)p\xi | p\xi \rangle = \omega(p\pi(a))$$

and

$$\omega_2(a) = \langle \pi(a)(1-p)\xi | (1-p)\xi \rangle = \omega((1-p)\pi(a)).$$

Then ω_1 and ω_2 are positive linear functionals on \mathcal{A} and $\omega = \omega_1 + \omega_2$. Since ω is pure, $\omega_1 = \lambda\omega$ and $\omega_2 = (1-\lambda)\omega$ for some λ satisfying $0 < \lambda < 1$. Fix $\epsilon > 0$. Since ξ is cyclic for π , there is an $a \in \mathcal{A}$ such that $\|\pi(a)\xi - p\xi\| < \epsilon$. We have $\|\pi(a)\xi\|^2 = \omega(a^*a) = \|p\xi\|^2$, so

$$\|(1-p)\pi(a)\xi\| = \|(1-p)(\pi(a)\xi - p\xi)\| < \epsilon$$

and

$$(1-\lambda)\|p\xi\|^2 = (1-\lambda)\omega(a^*a) = \omega_2(a^*a) = \|(1-p)\pi(a)\xi\|^2 < \epsilon^2.$$

Since this holds for all $\epsilon > 0$, $\lambda = 1$, a contradiction. Therefore, our assumption that ω is not irreducible is false.

Conversely, suppose π is irreducible but ω is not pure. Then there exists a positive linear functional ψ on \mathcal{A} such that $\psi \leq \omega$ but $\psi \neq \lambda\omega$ for any $\lambda \in [0, 1]$. In particular, $\psi \neq \omega$. Therefore, by Proposition 2.11.2 there exists a $T \in \pi(\mathcal{A})'$ such that $0 \leq T \leq 1$

$$\psi(a) = \langle T\pi(a)\xi | \xi \rangle$$

for all $a \in \mathcal{A}$. Since $\|\psi\| = 1$ and $\psi \neq \omega$, $0 < T < 1$. Therefore, $\pi(\mathcal{A})' \neq \mathbb{C}$, which contradicts the irreducibility of π by Proposition 2.9.4. Therefore, ω is pure. \square

2.11.5 Proposition. *Let \mathcal{A} be a normed $*$ -algebra. If $M \geq 0$, then the set of positive linear functionals on \mathcal{A} of norm at most M is a weak* compact convex subset of \mathcal{A}^* .*

PROOF. By the Banach-Alaoglu Theorem, the unit ball of \mathcal{A}^* is weak* compact. The set of positive linear functionals on \mathcal{A} is a weak* closed subset of \mathcal{A}^* , so its intersection with the unit ball is weak* compact. \square

2.11.6 Proposition. *Let \mathcal{A} be a Banach $*$ -algebra with a contractive bounded approximate identity, and let B be the set of positive linear functionals on \mathcal{A} with norm at most 1. Then $\omega \in B$ is an extreme point if and only if ω is pure and either $\omega = 0$ or $\|\omega\| = 1$.*

PROOF. Suppose ω is a nonzero extreme point of B . Clearly, $\|\omega\| = 1$. Let ω_1 be a positive linear functional on \mathcal{A} such that $\omega_1 \leq \omega$. If $\omega_1 = \omega$, we are done, so suppose $\omega_1 < \omega$. Then if $\omega_2 = \omega - \omega_1$, $\omega = \omega_1 + \omega_2$, where both ω_1 and ω_2 are nonzero positive linear functionals. Let $\lambda = \|\omega\|$, so that $\|\omega_2\| = 1 - \lambda$ (since the norms of positive functionals are additive), and let $\psi_1 = \lambda^{-1}\omega_1$, $\psi_2 = (1 - \lambda)^{-1}\omega_2$. Then $\psi_1, \psi_2 \in B$ and

$$\omega = \lambda\psi_1 + (1 - \lambda)\psi_2.$$

Since ω is an extreme point of B , $\omega = \omega_1 = \omega_2$. Therefore, $\omega_1 = \lambda\omega$, which shows that ω is pure.

Conversely, suppose that ω is pure and either $\omega = 0$ or $\|\omega\| = 1$. To show that 0 is an extreme point of B , it suffices to show that if ω is a linear functional on \mathcal{A} such that $\omega \in B$ and $-\omega \in B$, then $\omega = 0$. Since both ω and $-\omega$ are positive, $\omega(a^*a) = 0$ for all $a \in \mathcal{A}$. By part (ii) of Proposition 2.10.10, we have the inequality $|\omega(a)|^2 \leq \|\omega\|\omega(a^*a) = 0$ for all $a \in \mathcal{A}$, which shows that $\omega = 0$. Suppose that ω is pure and nonzero, $\|\omega\| = 1$, and that $\omega = \lambda\omega_1 + (1 - \lambda)\omega_2$ for some positive linear functionals ω_1 and ω_2 on \mathcal{A} of norm at most $\|\omega\|$ and λ such that $0 < \lambda < 1$. Then $\lambda\omega_1 \leq \omega$, so there exists an $\alpha \geq 0$ such that $\lambda\omega_1 = \alpha\omega$. Since $0 < \lambda < 1$, we have $0 \leq \alpha \leq 1$. Since

$$\|\omega\| \leq \lambda\|\omega_1\| + (1 - \lambda)\|\omega_2\|$$

and $\|\omega_1\|, \|\omega_2\| \leq \|\omega\|$, we must have $\|\omega_1\| = \|\omega_2\| = \|\omega\|$. Therefore, $\lambda = \alpha$ and $\omega = \omega_1 = \omega_2$, which shows that ω is an extreme point of B . \square

2.11.7 Corollary. *Let \mathcal{A} be a Banach $*$ -algebra with a contractive bounded approximate identity, and let B be the set of positive linear functionals on \mathcal{A} with norm at most 1. Then B is the weak* closed convex hull of the pure positive functionals of norm 1 and 0.*

PROOF. This is clear from Proposition 2.11.6 and the Krein-Milman Theorem. \square

Historical Notes

The results of this section are due to Gelfand and Raikov [GR43] and Segal [Seg47].

2.12 Positive Linear Functionals on C^* -algebras

By Lemma 2.10.7, positive functionals on unital Banach $*$ -algebras achieve their norm at the identity. For C^* -algebras, there is a converse: every bounded linear functional that achieves its norm at any positive element in the unit ball is positive.

2.12.1 Proposition. *Let \mathcal{A} be a C^* -algebra, and ω a bounded linear functional on \mathcal{A} . If there is an $a \in \mathcal{A}_+$ such that $\|a\| \leq 1$ and $\omega(a) = \|\omega\|$, then ω is positive.*

PROOF. We will first prove the proposition in the case where \mathcal{A} is unital, and then derive the general case. Since $0 \leq a \leq 1$, we have $\|a + e^{i\theta}(1 - a)\| \leq 1$ for every $\theta \in \mathbb{R}$. Choosing a θ such that $e^{i\theta}\omega(1 - a)$, we have

$$\|\omega\| \leq \omega(a) + e^{i\theta}\omega(1 - a) = \omega(a + e^{i\theta}(1 - a)) \leq \|\omega\|,$$

so $\omega(1 - a) = 0$, and $\omega(1) = \omega(a) = \|\omega\|$. Without loss of generality, we may assume that $\|\omega\| = 1$ by possibly replacing ω with $\omega/\|\omega\|$. We claim that $\omega(\mathcal{A}_{sa}) \subseteq \mathbb{R}$. Fix $b \in \mathcal{A}_{sa}$ and suppose $\omega(b) = \alpha + \beta i$ for $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$. Without loss of generality, we may assume that $\beta > 0$ by possibly replacing b with $-b$. Furthermore, since $\omega(b - \alpha \cdot 1) = \beta i$, we may assume that $\alpha = 0$. Then, for $\lambda \geq 0$,

$$|\omega(b + \lambda i \cdot 1)| = \beta + \lambda \leq \|\omega(b + \lambda i \cdot 1)\| \leq (\|b\|^2 + \lambda^2)^{\frac{1}{2}},$$

which is impossible for sufficiently large λ . Therefore, β must be zero. Hence $\omega(\mathcal{A}_{sa}) \subseteq \mathbb{R}$, so that $\omega(b^*) = \overline{\omega(b)}$ for all $b \in \mathcal{A}$.

Now, for an arbitrary $b \in \mathcal{A}_+$, we have $\| \|b\| \cdot 1 - b \| \leq \|b\|$, so that

$$\|b\| - \omega(b) = \omega(\|b\| \cdot 1 - b) \leq \|b\|.$$

Therefore, $\omega(b) \geq 0$, i.e. ω is positive.

If \mathcal{A} is not unital, consider \mathcal{A}_I and a Hahn-Banach extension $\widetilde{\omega}$ of ω to \mathcal{A}_I . Then $\widetilde{\omega}$ satisfies the conditions of the proposition, so by the unital case shown above, $\widetilde{\omega}$ is positive. The restriction of a positive functional to a subalgebra is also positive, so ω is positive. \square

An important consequence of the Gelfand-Naimark-Segal Theorem is that in order to produce sufficiently many representations of a Banach $*$ -algebra with a bounded approximate identity, it suffices to construct sufficiently many positive linear functionals on \mathcal{A} . However, this is not always possible, even when \mathcal{A} is commutative, unital, and finite-dimensional.

2.12.2 Example. Let \mathcal{A} be \mathbb{C}^2 equipped with the ∞ -norm, pointwise multiplication, and the involution $(a, b)^* = (\bar{b}, \bar{a})$. Since \mathcal{A} is unital, the GNS Theorem holds, and hence to show that \mathcal{A} has no non-zero positive linear functionals it suffices to show that \mathcal{A} has no non-zero representations. Let $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ be a representation of \mathcal{A} , and let $a = (1, 0)$. Since $\pi(a)^*\pi(a) = \pi(a^*a) = \pi(0) = 0$, we must have that $\pi(a) = \pi(a^*) = 0$. But a and a^* are a basis for \mathcal{A} , so π must be identically zero.

Therefore, in order to show the existence of nontrivial positive linear functionals on a Banach $*$ -algebra it will be necessary to either assume a very strong condition on the algebra, such as being a C^* -algebra, or to work with a specific class of examples, such as $L^1(G)$ for a locally compact group G . We will deal with the case of a C^* -algebra here.

2.12.3 Proposition. *Let \mathcal{A} be a C^* -algebra, and $a \in \mathcal{A}$ self-adjoint. Then there exists a pure state ω on \mathcal{A} such that $|\omega(a)| = \|a\|$.*

PROOF. If \mathcal{A} is not unital, then we will work in \mathcal{A}_1 . Then $C^*(a)$ is a subalgebra of \mathcal{A}_1 . Since $\|a\| = \text{spr}(a)$, there is a multiplicative linear functional ω on $C^*(a)$ such that $|\omega(a)| = \|a\|$. Because ω is multiplicative, we have $\|\omega\| = 1 = \omega(1)$. Let ρ be any Hahn-Banach extension of ω to a functional on \mathcal{A}_1 . Then $\|\rho\| = 1 = \rho(1)$, so by Proposition 2.12.1 ρ is a state on \mathcal{A}_1 , and thus $\rho|_{\mathcal{A}}$ is a state on \mathcal{A} .

Let \mathcal{F} be the set consisting of all states φ on \mathcal{A} such that $\varphi(a) = \rho(a)$. By the above, this set is a non-empty weak* closed bounded convex subset of \mathcal{A}^* . Thus, by the Banach-Alaoglu Theorem, \mathcal{F} is weak* compact. By the Krein-Milman Theorem, it has an extreme point φ_0 . We want to show that φ_0 is an extreme point of $S(\mathcal{A})$ and is thus pure. To do this, it suffices to show that \mathcal{F} is a face of $S(\mathcal{A})$.

Suppose $\varphi \in \mathcal{F}$ is such that $\varphi = (\psi_1 + \psi_2)/2$ for $\psi_1, \psi_2 \in S(\mathcal{A})$. Then

$$\|a\| = |\varphi(a)| \leq \frac{|\psi_1(a)| + |\psi_2(a)|}{2} \leq \|a\|.$$

This can only occur if $\psi_1(a) = \psi_2(a) = \varphi(a) = \rho(a)$. Thus $\psi_1, \psi_2 \in \mathcal{F}$, showing that \mathcal{F} is a face. \square

2.12.4 Corollary. *Let \mathcal{A} be a C^* -algebra. Then for every $a \in \mathcal{A}$ there exists an irreducible representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ of \mathcal{A} and a unit vector $\xi \in \mathcal{H}$ such that $\|\pi(a)\xi\| = \|a\|$.*

PROOF. By the preceding proposition, there exists a pure state φ on \mathcal{A} such that $\varphi(a^*a) = \|a\|^2$. Let π be a representation of \mathcal{A} and $\xi \in \mathcal{H}_\pi$ the cyclic

vector obtained from applying the GNS construction to φ . Since φ is pure, by Proposition 2.11.4, π is irreducible. Finally,

$$\|\pi(a)\xi\|^2 = \langle \pi(a^*a)\xi | \xi \rangle = \varphi(a^*a) = \|a\|^2. \quad \square$$

2.12.5 Theorem (Gelfand-Naimark). *Let \mathcal{A} be a C^* -algebra. Then the representation*

$$\pi = \bigoplus_{\omega \in \mathcal{A}_+^*} (\pi_\omega, \mathcal{H}_\omega)$$

is faithful. If \mathcal{A} is separable, there is a faithful representation of \mathcal{A} on a separable Hilbert space.

PROOF. It follows from the preceding corollary that $\|\pi(a)\| = \|a\|$ for every $a \in \mathcal{A}$. To show the statement about separability, pick a countable subset of \mathcal{A}_+^* that separates the points of \mathcal{A} . \square

The representation mentioned in the statement of the Gelfand-Naimark Theorem is called the *universal representation* of \mathcal{A} .

If G is a locally compact group, then the left regular representation of $L^1(G)$ is faithful, so many of the preceding results also have consequences for group representations as well.

2.12.6 Proposition. *Let G be a locally compact group, and φ a function of positive type on G . Then φ is the limit, in the topology of uniform convergence on compact sets, of convex combinations of pure functions of positive type of norm $\|\varphi\|$ and 0.*

PROOF. By Corollary 2.11.7, the weak* closed convex hull of the pure positive functionals of norm $\|\varphi\|$ and 0 is the set of functions of positive type of norm at most $\|\varphi\|$. By Theorem 2.10.22, the weak* topology on functions of positive type is the topology of uniform convergence on compact sets. \square

2.12.7 Corollary. *Let G be a locally compact group. Then every function in $C_0(G)$ is the limit, in the topology of uniform convergence of compact sets, of linear combinations of pure functions of positive type.*

PROOF. Since $C_c(G)$ is dense in $C_0(G)$ in the topology of uniform convergence on compact sets, it suffices to prove the claim for functions in $C_c(G)$. Fix $f \in C_c(G)$, and let $(e_i)_{i \in I}$ be a bounded approximate identity of $C_0(G)$ contained

in $C_c(G)$. Then f is the uniform limit of the net $(f * e_i)_{i \in I}$ in $C_c(G)$. We have the polarization identity

$$\begin{aligned} f * e_i(s) &= \langle \lambda(s) e_i^\vee | \bar{f} \rangle \\ &= \frac{1}{4} \sum_{k=0}^3 i^k \langle \lambda(s) (e_i^\vee + i^k \bar{f}) | e_i^\vee + i^k \bar{f} \rangle \\ &= \frac{1}{4} \sum_{k=0}^3 i^k \overline{(e_i + i^k \bar{f})} * (e_i^\vee + i^k \bar{f})^\vee. \end{aligned}$$

Hence $f * e_i$ is the linear combination of functions of the form $\bar{h} * h^\vee$, each of which is a function of positive type. The result then follows by Proposition 2.12.6. \square

2.12.8 Corollary (Gelfand-Raikov). *Let G be a locally compact group. If $s \in G$ is not equal to the identity, then there exists an irreducible representation π of G such that $\pi(s) \neq 1$.*

PROOF. There exists a function in $C_0(G)$ that takes different values at s and e , so by Proposition 2.12.6 there exists a pure function of positive type φ on G that takes different values at s and e . If (π, \mathcal{H}, ξ) is the GNS triple associated to φ , then π is irreducible by Proposition 2.11.4 and

$$\langle \pi(s)\xi | \xi \rangle = \varphi(s) \neq \varphi(e) = \langle \xi | \xi \rangle.$$

Therefore, $\pi(s) \neq 1$. \square

If \mathcal{A} is a commutative C^* -algebra, then positive linear functionals on \mathcal{A} correspond to positive finite regular Borel measures on the spectrum of \mathcal{A} . The Jordan decomposition from measure theory implies that any signed finite regular Borel measure on a locally compact Hausdorff space μ is the difference of two positive finite regular Borel measures μ_+ and μ_- such that $|\mu| = |\mu_+| + |\mu_-|$. The decomposition is found by partitioning the measure space into two sets and restricting the measure to each of them, and it is unique up to any two such partitions differing by null sets.

There is a generalization of this result to all C^* -algebras, and it can actually be proven by using a careful application of the Jordan decomposition for ordinary measures. First, we will prove a proposition that will be useful in establishing the appropriate uniqueness claim for the decomposition.

2.12.9 Proposition. *Let \mathcal{A} be a unital C^* -algebra, and let φ_+ and φ_- be positive linear functionals on \mathcal{A} . Then the following are equivalent:*

(i) $\|\varphi_+ - \varphi_-\| = \|\varphi_+\| + \|\varphi_-\|.$

(ii) For every $\epsilon > 0$ there is a $z \in \mathcal{A}_+^1$ such that $\varphi_+(1-z) < \epsilon$ and $\varphi_-(z) < \epsilon$.

PROOF. Suppose that $\|\varphi_+ - \varphi_-\| = \|\varphi_+\| + \|\varphi_-\|$. Since $\varphi_+ - \varphi_- \in \mathcal{A}_{\text{sa}}^*$ there is an $a \in \mathcal{A}_{\text{sa}}^1$ such that

$$\varphi_+(a) - \varphi_-(a) + \epsilon \geq \|\varphi_+ - \varphi_-\|.$$

But then

$$\varphi_+(a) - \varphi_-(a) + \epsilon \geq \|\varphi_+\| + \|\varphi_-\| = \varphi_+(1) + \varphi_-(1),$$

so $\varphi(1-a) + \psi(1+a) < \epsilon$. Since $0 \leq 1-a \leq 2$ and $0 \leq 1+a \leq 2$ we can choose $z = 1/2(1+a)$ so that $1-z = 1/2(1-a)$.

Conversely, suppose that for every $\epsilon > 0$ there is a $z \in \mathcal{A}_+^1$ such that $\varphi_+(1-z) < \epsilon$ and $\varphi_-(z) < \epsilon$. Clearly, $\|\varphi_+ - \varphi_-\| \leq \|\varphi_+\| + \|\varphi_-\|$. But if $\varphi_+(1-z) < \epsilon$ and $\varphi_-(z) < \epsilon$,

$$\begin{aligned} \|\varphi_+\| + \|\varphi_-\| &= \varphi_+(1) + \varphi_-(1) \\ &\leq \varphi_+(2z-1) + \varphi_-(1-2z) + 4\epsilon \\ &= (\varphi_+ - \varphi_-)(2z-1) + 4\epsilon \\ &\leq \|\varphi_+ - \varphi_-\| + 4\epsilon, \end{aligned}$$

since $\|2z-1\| \leq 1$. Since $\epsilon > 0$ is arbitrary, $\|\varphi_+\| + \|\varphi_-\| \leq \|\varphi_+ - \varphi_-\|$. \square

2.12.10 Theorem (Jordan Decomposition). *Let \mathcal{A} be a C^* -algebra. Then if φ is a self-adjoint bounded linear functional on \mathcal{A} , there exist unique positive linear functionals φ_+ and φ_- on \mathcal{A} such that $\varphi = \varphi_+ - \varphi_-$ and $\|\varphi\| = \|\varphi_+\| + \|\varphi_-\|$.*

PROOF. We will first show existence. If $a \in \mathcal{A}$, then $a = a_1 + ia_2$ for self-adjoint elements $a_1, a_2 \in \mathcal{A}$. Hence φ is determined by its values on \mathcal{A}_{sa} , where it only takes real values.

Let $X = \{\omega \in \mathcal{A}_+^* : \|\omega\| \leq 1\}$, and equip X with the weak* topology. By the Banach-Alaoglu Theorem, X is a compact Hausdorff space. Every $a \in \mathcal{A}_{\text{sa}}$ defines a real-valued continuous function f_a on X by $f_a(\omega) = \omega(a)$. Also, by Corollary 2.12.12,

$$\begin{aligned} \|f_a\|_\infty &= \sup_{\omega \in X} |f_a(\omega)| \\ &= \sup_{\omega \in X} |\omega(a)| \\ &= \sup\{|\langle \pi(a)\xi | \xi \rangle| : \pi \text{ a representation of } \mathcal{A}, \xi \in \mathcal{H}_\pi, \|\xi\| \leq 1\}. \end{aligned}$$

Let $V = \{f_a : a \in \mathcal{A}_{\text{sa}}\}$ be the vector subspace of $C(X)$ formed by the image of \mathcal{A}_{sa} . Define a linear functional F on this subspace by $F(f_a) = \varphi(a)$. Then

$$\begin{aligned} \|F\| &= \sup_{\|f_a\|_\infty \leq 1} |\varphi(a)| \\ &= \sup_{\|a\| \in \mathcal{A}_{\text{sa}}, \|a\| \leq 1} |\varphi(a)| \\ &\leq \|\varphi\|, \end{aligned}$$

so F is bounded. Hence, by the Hahn-Banach Theorem, F extends to a bounded linear functional defined on all of $C(X)$, which we will also denote by F , with the same norm. The Riesz Representation Theorem implies that there is a real-valued finite regular Borel measure μ on X such that $|\mu| = \|F\| \leq \|\varphi\|$ such that

$$F(f) = \int_X f(\omega) d\mu(\omega).$$

In particular, if $a \in \mathcal{A}_{\text{sa}}$,

$$\varphi(a) = F(f_a) = \int_X f_a(\omega) d\mu(\omega).$$

Let $\mu = \mu_+ - \mu_-$ be the Jordan decomposition of μ as the difference of positive finite regular Borel measures μ_+ and μ_- on X . For $a \in \mathcal{A}_{\text{sa}}$, define

$$\varphi_+(a) = \int_X f_a(\omega) d\mu_-(\omega) \quad \text{and} \quad \varphi_-(a) = \int_X f_a(\omega) d\mu_+(\omega).$$

Extend φ_+ and φ_- to all of \mathcal{A} by defining

$$\varphi_\pm(a_1 + a_2) = \varphi_\pm(a_1) + i\varphi_\pm(a_2)$$

for $a_1, a_2 \in \mathcal{A}_{\text{sa}}$. Clearly, φ_+ and φ_- define positive linear functionals on \mathcal{A} , and $\varphi = \varphi_+ - \varphi_-$. Note that

$$|\varphi_\pm(a)| \leq \|f_a\|_\infty |\mu_\pm| = \|a\| |\mu_\pm|,$$

so $\|\varphi_\pm\| \leq |\mu_\pm|$. Therefore,

$$\begin{aligned} \|\varphi\| &= \|\varphi_+ - \varphi_-\| \\ &\leq \|\varphi_+\| + \|\varphi_-\| \\ &\leq |\mu_+| + |\mu_-| \\ &= |\mu| \\ &= \|F\| \\ &\leq \|\varphi\|, \end{aligned}$$

showing that $\|\varphi\| = \|\varphi_+\| + \|\varphi_-\|$.

Now, we will show uniqueness. We will first prove it in the case where \mathcal{A} is unital and then derive the general case. Suppose that ψ_+ and ψ_- are positive

linear functionals on \mathcal{A} such that $\varphi = \psi_+ - \psi_-$ and $\|\varphi\| = \|\psi_+\| + \|\psi_-\|$. Fix $\epsilon > 0$. By Proposition 2.12.9, there is a $z \in \mathcal{A}_+^1$ such that $\varphi_+(1-z) < \epsilon$ and $\varphi_-(z) < \epsilon$. Then

$$\psi_+(z) \geq \psi_+(z) - \psi_-(z) = \varphi_+(z) - \varphi_-(z) > \varphi_+(1) - 2\epsilon.$$

Likewise, $\psi_-(1-z) > \varphi_-(1) - 2\epsilon$, so that

$$\psi_+(z) + \psi_-(1-z) > \|\varphi_+\| + \|\varphi_-\| - 4\epsilon = \|\psi_+\| + \|\psi_-\| - 4\epsilon.$$

It follows that $\psi_+(1-z) + \psi_-(z) < 4\epsilon$. Since $\varphi_+ - \psi_+ = \varphi_- - \psi_-$ we have for every $a \in \mathcal{A}$,

$$\begin{aligned} \varphi_+(a) - \psi_+(a) &= \varphi_+(az) - \psi_+(az) + \varphi_+(a(1-z)) - \psi_+(a(1-z)) \\ &= \varphi_-(az) - \psi_-(az) + \varphi_+(a(1-z)) - \psi_+(a(1-z)). \end{aligned}$$

Then

$$\begin{aligned} |\varphi_-(az)|^2 &\leq \varphi_-(aa^*)\varphi_-(z^2) \\ &\leq \|a\|^2\|\varphi_-\|\varphi_-(z) \\ &\leq \|a\|^2\|\varphi_+ - \varphi_-\|\varphi_-(z). \end{aligned}$$

Similarly, we have the inequalities

$$\begin{aligned} |\psi_-(az)|^2 &\leq \|a\|^2\|\varphi_+ - \varphi_-\|\psi_-(z) \\ |\varphi_+(a(1-z))|^2 &\leq \|a\|^2\|\varphi_+ - \varphi_-\|\varphi_+(1-z) \\ |\psi_+(a(1-z))|^2 &\leq \|a\|^2\|\varphi_+ - \varphi_-\|\psi_+(1-z) \end{aligned}$$

Combining all of the above work, we have

$$\begin{aligned} &|\varphi_+(a) - \psi_+(a)| \\ &= \|\varphi_-(az) - \psi_-(az) + \varphi_+(a(1-z)) - \psi_+(a(1-z))\| \\ &\leq \|\varphi_-(az)\| + \|\psi_-(az)\| + \|\varphi_+(a(1-z))\| + \|\psi_+(a(1-z))\| \\ &\leq \|a\|\|\varphi_+ - \varphi_-\|^{\frac{1}{2}}(\varphi_-(z^2)^{\frac{1}{2}} + \psi_-(z)^{\frac{1}{2}} + \varphi_+(1-z)^{\frac{1}{2}} + \psi_+(1-z)^{\frac{1}{2}}) \\ &\leq \|a\|\|\varphi_+ - \varphi_-\|^{\frac{1}{2}}(\epsilon^{\frac{1}{2}} + (4\epsilon)^{\frac{1}{2}} + \epsilon^{\frac{1}{2}} + (4\epsilon)^{\frac{1}{2}}) \\ &\leq \|a\|\|\varphi_+ - \varphi_-\|^{\frac{1}{2}}6\epsilon^{\frac{1}{2}} \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\varphi_+ = \psi_+$ and thus $\varphi_- = \psi_-$.

Suppose that \mathcal{A} is not unital. All of the positive linear functionals φ_+ , φ_- , ψ_+ , and ψ_- extend uniquely to positive linear functionals on \mathcal{A}_I of the same norm. Since

$$\mathcal{A}_I = \mathcal{A} \oplus \mathbb{C} \cdot 1,$$

the extension of every positive linear functional on \mathcal{A} to \mathcal{A}_I takes the value of its norm at 1, and

$$\|\varphi_+\| + \|\varphi_-\| = \|\psi_+\| + \|\psi_-\|,$$

the algebraic relation $\varphi_+ - \varphi_- = \psi_+ - \psi_-$ is preserved by extending the positive functionals to \mathcal{A}_I . \square

2.12.11 Corollary. *Let \mathcal{A} be a C^* -algebra. Then if φ is a bounded linear functional on \mathcal{A} , there exist positive linear functionals $\omega_1, \dots, \omega_4$ on \mathcal{A} such that*

$$\varphi = \omega_1 - \omega_2 + i(\omega_3 - \omega_4).$$

PROOF. Apply the Jordan decomposition to the real and imaginary parts of φ . □

Coupled with the Gelfand-Naimark-Segal Theorem, the Jordan decomposition provides a description of all bounded linear functionals on a C^* -algebra.

2.12.12 Corollary. *Let \mathcal{A} be a C^* -algebra. If φ is a bounded linear functional on \mathcal{A} , then there exists a cyclic representation π of \mathcal{A} and vectors $\xi, \eta \in \mathcal{H}_\pi$ such that*

$$\varphi(a) = \langle \pi(a)\xi | \eta \rangle.$$

PROOF. Let $\omega_1, \dots, \omega_4$ be positive linear functionals on \mathcal{A} such that

$$\varphi = \omega_1 - \omega_2 + i(\omega_3 - \omega_4).$$

By the GNS Theorem, there exist representations π_i of \mathcal{A} and vectors $\xi_i \in \mathcal{H}_{\pi_i}$ such that $\omega_i(a) = \langle \pi_i(a)\xi_i | \xi_i \rangle$. Let

$$\omega = \omega_1 + \omega_2 + \omega_3 + \omega_4,$$

and let (π, \mathcal{H}, ξ) be the GNS triple associated with ω . Since $\omega_i \leq \omega$, each ω_i induces a bounded linear functional ψ_i on the subspace $\pi(\mathcal{A})\xi$ of \mathcal{H} by

$$\psi_i(\pi(a)\xi) = \omega_i(a).$$

Since ξ is cyclic for π , ψ_i extends uniquely to a bounded linear functional on $\mathcal{H} = \overline{\pi(\mathcal{A})\xi}$. Therefore, by the Riesz-Fréchet Theorem, there exists an $\eta_i \in \mathcal{H}$ such that

$$\omega_i(a) = \psi_i(\pi(a)\xi) = \langle \pi(a)\xi | \eta_i \rangle.$$

Letting $\eta = \eta_1 - \eta_2 + i(\eta_3 - \eta_4)$ gives the decomposition

$$\varphi(a) = \langle \pi(a)\xi | \eta \rangle. \quad \square$$

We will show in Corollary 3.6.4 that the norm of φ may be achieved by a particular choice of π , ξ , and η , but this result is by no means trivial.

The Jordan Decomposition of bounded linear functionals on a C^* -algebra has particularly important implications to the representation theory of locally compact groups. In Section 2.10, we saw the correspondence between positive linear functionals on $C^*(G)$, $L^1(G)$, and $M(G)$ and functions of positive type on G . The method of proof used there does not allow any similar correspondence between arbitrary bounded linear functionals on $C^*(G)$ and a class of continuous functions on G , because in general the dual spaces of $C^*(G)$, $L^1(G)$, and $M(G)$ are all distinct.

2.12.13 Definition. Let G be a locally compact group. The *Fourier-Stieltjes Algebra* of G is the linear span of the functions of positive type in $L^\infty(G)$, and is denoted by $B(G)$.

Since every function of positive type is in the unique equivalence class of some continuous function, we may also view $B(G)$ as a subspace of $C_b(G)$. By the Jordan Decomposition of functionals on $C^*(G)$, we may identify $B(G)$ with the dual space of $C^*(G)$. Indeed, if $\varphi \in B(G)$ and $f \in L^1(G)$, we have

$$\langle f, \varphi \rangle = \int \varphi(s) f(s) ds.$$

2.12.14 Proposition. Let G be a locally compact group. If $\varphi \in L^\infty(G)$, then $\varphi \in B(G)$ if and only if there exists a representation $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ and vectors $\xi, \eta \in \mathcal{H}$ such that

$$\varphi(s) = \langle \pi(s)\xi, \eta \rangle,$$

in which case we may choose ξ to be a cyclic vector for π . Moreover, $B(G)$ is a $*$ -subalgebra of $C_b(G)$.

PROOF. The first claim follows from Corollary 2.12.12, and the second claim follows from Corollary 2.10.20 and the definition of $B(G)$ as the linear span of the functions of positive type. \square

We would like to show that $B(G)$ is a Banach $*$ -algebra when equipped with the norm from $C^*(G)^*$, but this result will have to wait until Corollary 3.6.5.

2.12.15 Proposition. Let G be a locally compact group. Then $B(G)$ is invariant under translation by elements of G .

PROOF. Since translation by an element of G is linear on $L^\infty(G)$, we need only show that if $\varphi \in B(G)$ is a function of positive type, then any translation of φ is the linear combination of functions of positive type. Let φ be a function of positive type on G , and let (π, \mathcal{H}, ξ) be the GNS triple associated with φ . Fix $g, h \in G$ and let $\eta = \pi(g)\xi$ and $\zeta = \pi(h)^*\xi$. Then, using a standard polarization identity,

$$\begin{aligned} 4\varphi(hsg) &= 4\langle \pi(hsg)\xi | \xi \rangle \\ &= 4\langle \pi(s)\eta | \zeta \rangle \\ &= \sum_{k=0}^3 i^k \langle \pi(s)(\eta + i^k\zeta) | (\eta + i^k\zeta) \rangle. \end{aligned} \quad \square$$

Historical Notes

The Jordan decomposition of self-adjoint bounded linear functionals on a C^* -algebra is due to Takeda [Tak54], but the norm formula $\|\varphi\| = \|\varphi_+\| + \|\varphi_-\|$ and the uniqueness are due to Grothendieck [Gro57]. Grothendieck's original proof of uniqueness relied on the polar decomposition of normal linear functionals on a von Neumann algebra, which we prove in Theorem 3.6.2. The proof given here is due to Pedersen [Ped69]. The Fourier-Stieltjes algebra of a generally locally compact group was introduced by Eymard in his thesis [Eym64].

Chapter 3

Von Neumann Algebras

3.1 Basic Properties

Recall that a von Neumann algebra is a strong operator closed, or equivalently, σ -weakly closed, unital $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$. In this section, we will develop some of the basic properties and definitions of the theory of von Neumann algebras. First, we will examine the structure of σ -weakly closed ideals.

3.1.1 Proposition. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and \mathcal{I} a σ -weakly closed left (resp. right) ideal of \mathcal{M} . Then there exists a unique projection $p \in \mathcal{M}$ such that $\mathcal{I} = \mathcal{M}p$ (resp. $\mathcal{I} = p\mathcal{M}$). If \mathcal{I} is a two-sided ideal then p is central.*

PROOF. Suppose \mathcal{I} is a σ -weakly closed left ideal of \mathcal{M} . Then $\mathcal{J} = \mathcal{I} \cap \mathcal{I}^*$ is a σ -weakly closed $*$ -algebra of operators on \mathcal{H} , so there is a largest projection $p \in \mathcal{J}$. Then we have $\mathcal{M}p \subseteq \mathcal{I}$ because $p \in \mathcal{I}$ and \mathcal{I} is a left ideal. Conversely, fix $a \in \mathcal{I}$. Since a^*a is in \mathcal{I} and self-adjoint, $a^*a \in \mathcal{J}$. Then $|a| = (a^*a)^{\frac{1}{2}}$ is in \mathcal{J} because \mathcal{J} is a C^* -algebra, so $|a|p = |a|$. Considering the polar decomposition $a = u|a|$ of a shows $a = ap$, so $a \in \mathcal{M}p$. Therefore, $\mathcal{I} \subseteq \mathcal{M}p$. The proof for the case where \mathcal{I} is a right ideal is similar.

If \mathcal{I} is a two-sided ideal, then we for every $a \in \mathcal{A}$ we have $pa \in \mathcal{I} = \mathcal{M}p$, and thus $pa = (pa)a = pap$. Considering a^* , we have $pa^* = pa^*p$, so

$$pa = pap = (pa^*p)^* = (pa^*)^* = ap. \quad \square$$

If \mathcal{H} has finite dimension n , then it is a well-known fact that $\mathcal{B}(\mathcal{H}) \cong M_n(\mathbb{C})$ is a simple algebra, i.e. it has no nontrivial ideals. This is no longer the case if \mathcal{H} is infinite-dimensional, because the compact operators $\mathcal{K}(\mathcal{H})$ form a nontrivial self-adjoint closed ideal of $\mathcal{B}(\mathcal{H})$. However, if we restrict our attention to σ -weakly closed ideals, then the situation is different.

3.1.2 Definition. Let \mathcal{M} be von Neumann algebra. We say that \mathcal{M} is a factor if $\mathcal{Z}(\mathcal{M}) = \mathbb{C}1$.

3.1.3 Corollary. Let \mathcal{M} be a factor. Then \mathcal{M} has no nontrivial σ -weakly closed two-sided ideals.

PROOF. Let J be a two-sided ideal of \mathcal{M} . By Proposition 3.1.1, there exists a central projection $p \in \mathcal{Z}(\mathcal{M})$ such that $J = p\mathcal{M}$. Since \mathcal{M} is a factor, $\mathcal{Z}(\mathcal{M}) = \mathbb{C}1$, so $p = 0$ or $p = 1$. \square

If \mathcal{M} is a von Neumann algebra, then by the Double Commutant Theorem $\mathcal{Z}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$ is also a von Neumann algebra. In particular, the central projections of \mathcal{M} form a complete lattice.

3.1.4 Definition. Let \mathcal{M} be a von Neumann algebra. If $p \in \mathcal{M}$ is a projection, then $c(p)$, the *central cover* or *central support* of p , is the least central projection z in \mathcal{M} such that $zp = p$.

If p is a projection, then it is easily seen that $c(p)$ is the least central projection z such that $zp = p$. If \mathcal{M} is represented on a Hilbert space, we can concretely describe $c(p)$.

3.1.5 Proposition. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If $p \in \mathcal{M}$ is a projection, then $c(p)$ is the projection onto the norm closure of $(\mathcal{M}p)\mathcal{H}$.

PROOF. Let z be the projection onto the norm closure of $(\mathcal{M}p)\mathcal{H}$. Since $(\mathcal{M}p)\mathcal{H}$ is an invariant subspace for both \mathcal{M} and \mathcal{M}' , it follows that $z \in \mathcal{M} \cap \mathcal{M}' = \mathcal{Z}(\mathcal{M})$. Since $(\mathcal{M}p)\mathcal{H} \subseteq p\mathcal{H}$, it follows that $zp = p$. Hence $z \geq c(p)$. Conversely, $c(p)\mathcal{H}$ is obviously an invariant subspace for \mathcal{M} and $c(p)\mathcal{H} \supseteq p\mathcal{H}$. Hence, we have that $c(p)\mathcal{H} \supseteq (\mathcal{M}p)\mathcal{H}$, i.e. $c(p) \geq z$. Therefore, $c(p) = z$. \square

There is another interpretation of the central cover of p in terms of unitary conjugates of p .

3.1.6 Proposition. Let \mathcal{M} be a von Neumann algebra. If $p \in \mathcal{M}$ is a projection, then

$$c(p) = \sup\{u^*pu : u \in \mathcal{U}(\mathcal{M})\}.$$

PROOF. If $z \in \mathcal{Z}(\mathcal{M})$ is a projection, and $z \geq p$, then

$$z = u^*zu \geq u^*pu$$

for all $u \in \mathcal{U}(\mathcal{M})$. Therefore, $c(p)$ dominates

$$z = \sup\{u^*pu : u \in \mathcal{U}(\mathcal{M})\}.$$

The definition of z is as an element of \mathcal{M} , but in order to show that $z \in \mathcal{Z}(\mathcal{M})$ we need to show that $z \in \mathcal{M}$. From the definition of z , we have $u^*zu = z$ for all $u \in \mathcal{U}(\mathcal{M})$, so $zu = uz$. Thus, z commutes with every unitary in \mathcal{M} , which by Proposition 2.8.22 implies that $z \in \mathcal{M}'$, and thus $z \in \mathcal{Z}(\mathcal{M})$. By taking $u = 1$ in the definition of z , we get that $z \geq p$. Therefore, $c(p) \geq z \geq p$, and $z = c(p)$. \square

There are two basic constructions for “cutting down” a von Neumann algebra with respect to a projection in either the algebra or its commutant. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and $p \in \mathcal{M}$ a projection. Every operator in $p\mathcal{M}p$ leaves $p\mathcal{H}$ invariant, so that the set

$$\{a|_{p\mathcal{H}} : a \in p\mathcal{M}p\}$$

is a nondegenerate $*$ -subalgebra of $\mathcal{B}(p\mathcal{H})$. We denote this algebra by $p\mathcal{M}p$, and call it the *reduced von Neumann algebra* of \mathcal{M} with respect to p . The notations \mathcal{M}_p and $\mathcal{M}_{p\mathcal{H}}$ are also common, to specify that this algebra acts on a potentially smaller Hilbert space, but there is no risk of confusion in anything that we will do, so we will simply use the notation $p\mathcal{M}p$.

Similarly, every operator in $\mathcal{M}'p$ also leaves $p\mathcal{H}$ invariant, so that the set

$$\{a|_{p\mathcal{H}} : a \in \mathcal{M}'p\}$$

is a nondegenerate $*$ -subalgebra of $\mathcal{B}(p\mathcal{H})$. We denote this algebra by $\mathcal{M}'p$, and call it the *induced von Neumann algebra* of \mathcal{M}' with respect to p . Since p commutes with \mathcal{M}' , this is equal to $p\mathcal{M}'p$ as a set, but we will always use distinct notation to distinguish the induced von Neumann algebra from the reduced von Neumann algebra.

First, we will prove that these $*$ -algebras are actually von Neumann algebras, and that they are commutants of each other.

3.1.7 Proposition. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $p \in \mathcal{M}$ a projection. Then $p\mathcal{M}p$ and $\mathcal{M}'p$ are both von Neumann algebras on $p\mathcal{H}$ and $(p\mathcal{M}p)' = \mathcal{M}'p$.*

PROOF. Let $\rho : \mathcal{M} \rightarrow p\mathcal{M}p$ be the restriction map given by $\rho(a) = a|_{p\mathcal{H}}$. It is easily checked that ρ is a surjective σ -weakly continuous $*$ -homomorphism, and that $p\mathcal{M}p$ is norm closed. A surjective $*$ -homomorphism between C^* -algebras is a quotient map, so ρ takes the unit ball of \mathcal{M} onto the unit ball of $p\mathcal{M}p$. Since ρ is σ -weakly continuous and the unit ball of \mathcal{M} is σ -weakly compact, the unit ball of $p\mathcal{M}p$ is σ -weakly compact. By Corollary 2.8.17, this

implies that $p\mathcal{M}p$ is a von Neumann algebra. An identical argument shows that $\mathcal{M}'p$ is a von Neumann algebra.

We will prove that $p\mathcal{M}p = (\mathcal{M}'p)'$, which is equivalent to $(p\mathcal{M}p)' = \mathcal{M}'p$ by the Double Commutant Theorem. It is obvious that $p\mathcal{M}p \subseteq (\mathcal{M}'p)'$. Conversely, fix $a_{p\mathcal{H}} \in (\mathcal{M}'p)'$ so that $a = pap$. For every $b \in \mathcal{M}'$, we have $a_{p\mathcal{H}}b_{p\mathcal{H}} = b_{p\mathcal{H}}a_{p\mathcal{H}}$, so $ab = ba$. Therefore, $a \in \mathcal{M}'' = \mathcal{M}$. \square

If $p' \in \mathcal{M}'$ is a projection, the central cover of p' determines whether the restriction map from \mathcal{M} to $\mathcal{M}p'$ is a $*$ -isomorphism.

3.1.8 Proposition. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and $p' \in \mathcal{M}'$ a projection. The restriction map from \mathcal{M} onto $\mathcal{M}p'$ is a $*$ -isomorphism if and only if $c(p') = 1$.*

PROOF. Let $\rho : \mathcal{M} \rightarrow \mathcal{M}p'$ be the restriction map, and suppose that ρ is a $*$ -isomorphism. Since $p' \leq c(p')$,

$$\rho(1 - c(p')) = (1 - c(p'))_{p'\mathcal{H}} = 0.$$

Since ρ is a $*$ -isomorphism, $1 - c(p') = 0$, i.e. $c(p') = 1$.

Conversely, suppose that $c(p') = 1$. By Proposition 3.1.5, we have that the closure of $(\mathcal{M}'p')\mathcal{H}$ is \mathcal{H} . If $a \in \mathcal{M}$ and $\rho(a) = a_{p'\mathcal{H}} = 0$, then $ap'\mathcal{H} = \{0\}$, and

$$a(\mathcal{M}'p'\mathcal{H}) = \mathcal{M}'(ap'\mathcal{H}) = \{0\}.$$

Therefore, $a = 0$, showing that ρ is a $*$ -isomorphism. \square

Historical Notes

The results in this section are due to Murray and von Neumann [MvN36] [MvN37]. It is often assumed that the name “factor” is due to the fact that, at least on a separable Hilbert space, every von Neumann algebra may be written as a generalized direct sum, or “direct integral” of factors, but this is not the case. Murray and von Neumann were interested in von Neumann algebras $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ such that $\mathcal{M} \bar{\otimes} \mathcal{M}' \cong \mathcal{B}(\mathcal{H})$, where $\bar{\otimes}$ represents the σ -weak closure of the algebraic tensor product $\mathcal{M} \otimes \mathcal{M}'$. Hence factors were originally supposed to be factors in a tensor product decomposition of $\mathcal{B}(\mathcal{H})$. However, such a decomposition can not exist for all factors, only the so-called Type I factors, which are all $*$ -isomorphic to $\mathcal{B}(\mathcal{K})$ for some Hilbert space \mathcal{K} . Thus, in retrospect, the name “factor” is somewhat poorly chosen.

3.2 Comparison of Projections

If p and q are projections in $\mathcal{B}(\mathcal{H})$, then the most natural way to compare p and q is by comparing the dimensions of their subspaces as cardinal numbers. However, if p and q are projections in a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$, then this isn't necessarily a very good way to compare p and q , because it doesn't involve \mathcal{M} in any way, and it is dependent on the representation of \mathcal{M} on a Hilbert space. What is actually important in the case of $\mathcal{B}(\mathcal{H})$ is that if p and q are two projections whose ranges have the same dimension, then there exists a partial isometry v such that $v^*v = p$ and $vv^* = q$. This definition is entirely algebraic and is easily relativized to a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$, so it will be the one we use.

3.2.1 Definition. Let \mathcal{M} be a von Neumann algebra. If p and q are projections in \mathcal{M} , we say that p and q are (*Murray-von Neumann*) *equivalent*, denoted $p \sim q$, if there exists an element $u \in \mathcal{M}$ such that $uu^* = p$ and $uu^* = q$. If p is equivalent to a projection $q_1 \leq q$, we denote this by $p \preceq q$.

3.2.2 Proposition. *Let \mathcal{M} be a von Neumann algebra. Then \sim is an equivalence relation on the projections of \mathcal{M} .*

PROOF. Clearly, \sim is reflexive and symmetric, so we need only show that it is transitive. If $p, q, r \in \mathcal{M}$ are projections and $v, w \in \mathcal{M}$ are such that $v^*v = p$, $vv^* = q$ and $w^*w = q$, $ww^* = r$, then

$$(wv)^*wv = v^*w^*wv = v^*qv = v^*v = p$$

and

$$wv(wv)^* = wv v^*w^* = wqw^* = ww^* = r.$$

Therefore, $p \sim q$. □

3.2.3 Examples.

- (i) If $p, q \in \mathcal{B}(\mathcal{H})$ are projections, then $p \sim q$ if and only if the ranges of p and q have the same dimension, and $p \preceq q$ if and only if the dimension of the range of p is at most the dimension of the range of q .
- (ii) If $p, q \in \mathcal{M}$ are central projections, then $p \sim q$ if and only if $p = q$, from which it follows that $p \preceq q$ if and only if $p \leq q$. Indeed, let $v \in \mathcal{M}$ be such that $v^*v = p$ and $vv^* = q$. Since $1 - p$ is also a central projection,

$$((1 - p)v)^*(1 - p)v = (1 - p)v^*v = (1 - p)p = 0,$$

which implies that

$$(1 - p)q = (1 - p)vv^* = (1 - p)v((1 - p)v)^* = 0.$$

Similarly, $(1 - q)p = 0$. Therefore, $p = q$.

- (iii) If $a \in \mathcal{M}$, consider the polar decomposition $a = v|a|$ of a . Then the left and right support projections of a , given by vv^* and v^*v respectively, are equivalent.
- (iv) Similarly, if $\varphi \in \mathcal{M}_*$, consider the polar decomposition $\varphi = v \cdot |\varphi|$ of φ . Then the left and right support projections of a , given by vv^* and v^*v respectively, are equivalent.

3.2.4 Proposition. *Let \mathcal{M} be a von Neumann algebra.*

- (i) $p \sim 0$ if and only if $p = 0$;
- (ii) if $p \sim q$ and z is a central projection, then $zp \sim zq$;
- (iii) if $p \preceq q$, then $c(p) \leq c(q)$;
- (iv) if $(p_i)_{i \in I}$ and $(q_i)_{i \in I}$ are two families of mutually orthogonal projections in \mathcal{M} such that $p_i \sim q_i$ for all $i \in I$, then $\sum_{i \in I} p_i \sim \sum_{i \in I} q_i$, where the sums are taken in the strong operator topology.

PROOF.

- (i) If $v^*v = 0$, then $v = 0$ and $vv^* = 0$.
- (ii) Let $v \in \mathcal{M}$ be such that $v^*v = p$ and $vv^* = q$. Then

$$(zv)^*zv = v^*zzv = zv^*v = zp \quad \text{and} \quad zv(zv)^* = zvv^*z = zq.$$

- (iii) We need only show that if $p \sim q$, then $c(p) = c(q)$. If z is a central projection annihilating p then $zp = 0$ and $zp \sim zq$, which implies that $zq = 0$. By symmetry, p and q are annihilated by the same central projections, and thus have the same central cover.
- (iv) For every $i \in I$, let $v_i \in \mathcal{M}$ be such that $v_i^*v_i = p_i$ and $v_iv_i^* = q_i$. Define

$$v = \sum_{i \in I} v_i,$$

where the sum is taken in the strong operator topology. We need to actually show that this sum converges. Let J be a finite subset of I , and define

$$v_J = \sum_{i \in J} v_i.$$

Let \mathcal{F} be the directed set of all finite subsets of I , ordered by reverse inclusion. We want to show that the net $(v_J)_{J \in \mathcal{F}}$ converges in the strong operator topology. By the Banach-Steinhaus Theorem, we only need to show that $(v_J\xi)_{J \in \mathcal{F}}$ converges in norm for every $\xi \in \mathcal{H}$. Since the vectors $v_i\mathcal{H}$ are mutually orthogonal for all $i \in I$, this is equivalent to the condition

$$\sum_{i \in I} \|v_i\|^2 = \lim_{J \in \mathcal{F}} \sum_{i \in J} \|v_i\xi\|^2 < \infty.$$

We have

$$\sum_{i \in I} \|v_i \xi\|^2 = \sum_{i \in I} \langle v_i^* v_i \xi | \xi \rangle = \sum_{i \in I} \langle p_i \xi | \xi \rangle = \sum_{i \in I} \langle (\bigvee_{i \in I} p_i) \xi | \xi \rangle < \infty.$$

Therefore, the net $(v_i)_{i \in I}$ converges in the strong operator topology to some $v \in \mathcal{M}$, as desired. Similarly, the net $(v_i^*)_{i \in I}$ converges in the strong operator topology to some $w \in \mathcal{M}$. Since $v = \sum_{i \in I} v_i$ in the strong operator topology, $v = \sum_{i \in I} v_i$ in the weak operator topology as well, so $v^* = \sum_{i \in I} v_i^*$ in the weak operator topology. This implies that $v^* = w$. From the equalities $v_i^* v_j = \delta_{ij} p_i$ and $v_i v_j^* = \delta_{ij} p_j$, it follows that

$$v^* v = \sum_{i \in I} p_i \quad \text{and} \quad v v^* = \sum_{i \in I} p_i. \quad \square$$

3.2.5 Proposition. *Let \mathcal{M} be a von Neumann algebra. If p and q are projections in \mathcal{M} , then the following are equivalent:*

- (i) $c(p)$ and $c(q)$ are not orthogonal;
- (ii) $p\mathcal{M}q \neq \{0\}$;
- (iii) there exist nonzero projections $p_1 \leq p$ and $q_1 \leq q$ in \mathcal{M} such that $p_1 \sim q_1$.

PROOF. We will first show that (i) \implies (ii). Suppose $p\mathcal{M}q = \{0\}$, and let $\mathcal{J} = \{a \in \mathcal{M} : p\mathcal{M}a = \{0\}\}$. Then \mathcal{J} is a σ -weakly closed two-sided ideal of \mathcal{M} , so there exists a central projection $z \in \mathcal{M}$ such that $\mathcal{J} = z\mathcal{M}$. Since $q \in \mathcal{J}$, we have $q \leq z$, so $c(q) \leq z$. By definition, $pz = 0$, so $c(q)p = 0$. Hence $p \leq 1 - c(q)$. Therefore, $c(p)$ and $c(q)$ are orthogonal.

Next, we show that (ii) \implies (iii). Fix a nonzero $a \in p\mathcal{M}q$. Then $paq = a$, so letting p_1 be the left support projection of a and q_1 be the right support projection of a we have that $p_1 \leq a$, $q_1 \leq q$, and $p_1 \sim q_1$.

Finally, we show (iii) \implies (i). If there exist nonzero projections $p_1 \leq p$ and $q_1 \leq q$ in \mathcal{M} such that $p_1 \sim q_1$, then $c(p_1) = c(q_1) \neq 0$, $c(p_1) \leq c(p)$, and $c(q_1) \leq c(q)$, so $c(p)$ and $c(q)$ are not orthogonal. \square

If p and q are projections in $\mathcal{B}(\mathcal{H})$, then $p \leq q$ if and only if the dimension of the range of p is less than the dimension of the range of q . Therefore, if $p \leq q$ and $q \leq p$, $p \sim q$ by the Schroeder-Bernstein Theorem. The same fact holds for an arbitrary von Neumann algebra, with essentially the same proof.

3.2.6 Theorem (Schroeder-Bernstein Theorem). *Let \mathcal{M} be a von Neumann algebra. If p and q are projections in \mathcal{M} such that $p \leq q$ and $q \leq p$, then $p \sim q$.*

PROOF. Let q_1 be a projection in \mathcal{M} and v a partial isometry in \mathcal{M} such that

$$p = v^* v, \quad v v^* = q_1 \leq q.$$

Similarly, let p_1 and w be such that

$$q = w^*w, \quad ww^* = p_1 \leq p.$$

By induction, we construct two decreasing sequences of projections by defining

$$p_{n+1} = wp_nw^*, \quad q_{n+1} = vp_nv^*.$$

Let p_∞ and q_∞ be the infima of these two sequences of projections. Then

$$p = \sum_{n=0}^{\infty} (p_n - p_{n+1}) + p_\infty, \quad q = \sum_{n=0}^{\infty} (q_n - q_{n+1}) + q_\infty,$$

where $p_0 = p$ and $q_0 = q$. Clearly, we have

$$v(p_n - p_{n+1})v^* = q_{n+1} - q_{n+2}, \quad w(q_n - q_{n+1})w^* = p_{n+1} - p_{n+2},$$

and

$$vp_\infty v^* = q_\infty, \quad wq_\infty w^* = p_\infty.$$

Hence

$$p_{2n} - p_{2n+1} \sim q_{2n+1} - q_{2n+2}, \quad p_{2n+1} - p_{2n+2} \sim q_{2n} - q_{2n+1}.$$

Therefore,

$$\begin{aligned} p &= \sum_{n=0}^{\infty} (p_{2n} - p_{2n+1}) + \sum_{n=0}^{\infty} (p_{2n+1} - p_{2n+2}) + p_\infty \\ &\sim \sum_{n=0}^{\infty} (q_{2n+1} - q_{2n+2}) + \sum_{n=0}^{\infty} (q_{2n} - q_{2n+1}) + q_\infty \\ &= q. \end{aligned} \quad \square$$

If p and q are projections in $\mathcal{B}(\mathcal{H})$, then either $p \leq q$ or $q \leq p$. This does not hold for more general von Neumann algebras, because by part (iii) of Proposition 3.2.4 Murray-von Neumann comparability of projections implies comparability of their central covers in the usual ordering on self-adjoint elements, and this is not always the case. For example, in any nontrivial direct product $\mathcal{M}_1 \times \mathcal{M}_2$, the central cover of a nonzero projection $(p, 0)$ is orthogonal to the central cover of a nonzero projection $(0, q)$. However, this is essentially the only impediment to comparison of two projections, in the sense that there always exists a central projection breaking the von Neumann algebra into two pieces where comparison is possible.

3.2.7 Theorem (Comparison Theorem). *Let \mathcal{M} be a von Neumann algebra. If p and q are projections in \mathcal{M} then there exists a central projection $z \in \mathcal{M}$ such that*

$$zp \leq zq \quad \text{and} \quad (1-z)p \geq (1-z)q.$$

PROOF. Let $(p_i)_{i \in I}$ and $(q_i)_{i \in I}$ be a maximal pair of families of mutually orthogonal projections such that

$$p_i \leq p, \quad q_i \leq q, \quad p_i \sim q_i$$

for every $i \in I$. By the additivity of equivalence, it follows that

$$p_1 = \bigvee_{i \in I} p_i \sim \bigvee_{i \in I} q_i = q_1.$$

If $p_2 = p - p_1$ and $q_2 = q - q_1$, then, due to the maximality of the chosen families and Proposition 3.2.5, it follows that

$$c(p_2)c(q_2) = 0.$$

Let $z = c(q_2)$. Then we have

$$zp = zp_1 + zp_2 = zp_1 + c(q_2)c(q_2)p_2 = zp_1 \sim zq_1 \leq zq,$$

and similarly,

$$(1 - z)q = (1 - z)q_1 + (1 - z)q_2 = (1 - z)q_1 \sim (1 - z)p_1 \leq (1 - z)p. \quad \square$$

Historical Notes

The comparison theory of projections is originally due to Murray and von Neumann [MvN36] [MvN37]. However, they proved many of their results only in the case of factors, or in the case of a separable Hilbert space. These results were extended to general von Neumann algebras by Dixmier [Dix49] and Kaplansky [Kap51b].

3.3 Normal Linear Functionals

Since the self-adjoint part of a von Neumann algebra is monotone complete, it makes sense to consider maps between von Neumann algebras that respect this additional structure. We will first consider linear functionals, and then use the results for linear functionals to derive similar results for more general linear maps between von Neumann algebras.

3.3.1 Definition. Let \mathcal{M} be a von Neumann algebra, and let φ be a bounded linear functional on \mathcal{M} . We say that φ is *normal* if whenever $(a_i)_{i \in I}$ is a bounded increasing net in \mathcal{M}_{sa} ,

$$\varphi \left(\sup_{i \in I} a_i \right) = \lim_{i \in I} \varphi(a_i).$$

Equivalently, φ is normal if whenever $(a_i)_{i \in I}$ is a bounded decreasing net in \mathcal{M}_+ whose greatest lower bound is 0,

$$\lim_{i \in I} \varphi(a_i) = 0.$$

We let \mathcal{M}_* denote the set of all normal linear functionals on \mathcal{M} .

3.3.2 Examples.

- (i) Since the strong operator topology limit of a bounded increasing net in \mathcal{M}_{sa} is its supremum, any linear functional on \mathcal{M} that is strong operator topology continuous on bounded parts is normal. By Proposition 2.7.9, these are precisely the σ -weakly continuous linear functionals on \mathcal{M} .
- (ii) As a particular case of (i), every linear functional in vector form is normal, i.e. if $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ and $\xi, \eta \in \mathcal{H}$, then the linear functional φ defined by $\varphi(a) = \langle a\xi | \eta \rangle$.

Note that the definition of normality only depends on the structure of \mathcal{M} as a C^* -algebra, and thus is independent of the particular representation of \mathcal{M} on a Hilbert space. The notation \mathcal{M}_* suggests that \mathcal{M}_* is a Banach space predual of \mathcal{M} . Our first goal will be to show that this is indeed the case, i.e. that \mathcal{M}_* is precisely the space of linear functionals on \mathcal{M} continuous with respect to the σ -weak topology on \mathcal{M} .

Recall that if \mathcal{A} is a Banach algebra, then \mathcal{A}^* has a canonical Banach \mathcal{A} -bimodule structure given by

$$(b \cdot \varphi)(a) = \varphi(ab) \quad \text{and} \quad (\varphi \cdot b)(a) = \varphi(ba).$$

We first prove that \mathcal{M}_* is a closed subspace of \mathcal{M}^* that is invariant under all of the basic operations.

3.3.3 Proposition. *Let \mathcal{M} be a von Neumann algebra. Then*

- (i) \mathcal{M}_* is a closed subspace of \mathcal{M}^* ;
- (ii) if $\varphi \in \mathcal{M}_*$, then $\varphi^* \in \mathcal{M}_*$;
- (iii) if $\varphi \in \mathcal{M}_*$ and $b \in \mathcal{M}$, then $b \cdot \varphi \in \mathcal{M}_*$ and $\varphi \cdot b \in \mathcal{M}_*$;
- (iv) if $\varphi \in \mathcal{M}_*$ is self-adjoint, then $\varphi_+ \in \mathcal{M}_*$ and $\varphi_- \in \mathcal{M}_*$.

PROOF.

- (i) It is clear that \mathcal{M}_* is a linear subspace of \mathcal{M}^* . Let $(\varphi_n)_{n=1}^\infty$ be a sequence in \mathcal{M}_* converging to $\varphi \in \mathcal{M}^*$, and let $(a_i)_{i \in I}$ be a bounded monotone increasing net with supremum a . Obviously,

$$\varphi(a) \geq \sup_{i \in I} \varphi(a_i).$$

Fix $\epsilon > 0$. Then there exists an index $i \in I$ such that $\varphi_n(a) - \epsilon < \varphi_n(a_i)$, since φ_n is normal. We have

$$\begin{aligned}\varphi(a_i) &\geq \varphi_n(a_i) - \epsilon \|a_i\| \\ &\geq \varphi_n(a) - \epsilon \|a\| - \epsilon - \epsilon \|a_i\| \\ &\geq \varphi(a) - \epsilon \|a\| - \epsilon - \epsilon \|a_i\| \\ &\geq \varphi(a) - (1 + 2\|a\|)\epsilon.\end{aligned}$$

Therefore, we have

$$\varphi(a) \leq \sup_{i \in I} \varphi(a_i),$$

showing that φ is normal.

- (ii) The definition of normality only mentions self-adjoint elements of \mathcal{M} , so if $\varphi \in \mathcal{M}_*$ then $\varphi^* \in \mathcal{M}_*$.
- (iii) If $a, b \in \mathcal{M}$ and $a \geq 0$, then $b^*ab^* \geq 0$, so it follows that $b \cdot \varphi b^* \in \mathcal{M}_*$ for every $\varphi \in \mathcal{M}_*$. Hence, by the polarization identity

$$4ba = \sum_{n=0}^3 i^n (a + i^n)^* b (a + i^n),$$

the functional $b \cdot \varphi$ is a linear combination of normal functionals, so $b \cdot \varphi \in \mathcal{M}_*$. Since \mathcal{M}_* is $*$ -invariant, we have $\varphi \cdot b = (b^* \cdot \varphi)^* \in \mathcal{M}_*$.

- (iv) Fix $\epsilon > 0$. By Proposition 2.12.9 we can find $z \in \mathcal{M}_+^1$ such that $\varphi_+(1-z) < \epsilon$ and $\varphi_-(z) < \epsilon$. Then for every $a \in \mathcal{M}$,

$$\begin{aligned}|\varphi_+(a) - \varphi_+(za)| &\leq |\varphi_+((1-z)a)| + |\varphi_-(za)| \\ &\leq \varphi_+(1-z)^{\frac{1}{2}} \|\varphi_+\|^{\frac{1}{2}} \|a\| + \varphi_-(z)^{\frac{1}{2}} \|\varphi_-\|^{\frac{1}{2}} \|a\| \\ &< 2\epsilon^{\frac{1}{2}} \|\varphi\|^{\frac{1}{2}} \|a\|.\end{aligned}$$

Since $\varphi \cdot z \in \mathcal{M}_*$ and \mathcal{M}_* is norm-closed it follows that φ_+ and thus φ_- belong to \mathcal{M}_* . \square

3.3.4 Lemma. *Let \mathcal{M} be a von Neumann algebra and ω a normal state on \mathcal{M} . Then there is a family $(p_i)_{i \in I}$ of mutually orthogonal nonzero projections in \mathcal{M} with $\sum_{i \in I} p_i = 1$ such that each functional $p_i \cdot \omega$ is weak operator continuous.*

PROOF. Using Zorn's Lemma, let $(p_i)_{i \in I}$ be a maximal family of mutually orthogonal projections in \mathcal{M} such that $p_i \cdot \omega$ is weak operator continuous for every $i \in I$. We claim that $p_0 = \sum_{i \in I} p_i = 1$. Suppose otherwise. Choose a unit vector $\xi \in (1 - p_0)\mathcal{H}$ and define a linear functional ψ on \mathcal{M} by $\psi(a) = 2\langle a\xi | \xi \rangle$. Again using Zorn's Lemma, let $(q_i)_{i \in I}$ be a maximal family of mutually orthogonal projections in $(1 - p_0)\mathcal{M}(1 - p_0)$ such that $\omega(q_i) \geq \psi(q_i)$ for every $i \in I$, and let $q_0 = \sum_{i \in I} q_i$. Since ω and ψ are both normal, $\omega(q_0) \geq$

$\psi(q_0)$. Therefore, $q_0 \neq 1 - p_0$. Let $p_1 = 1 - p_0 - q_0$. Then $p_1 \neq 0$ and for each projection $p \leq p_1$ we have $\omega(p) < \psi(p)$ by the maximality of q_0 . Since each element in \mathcal{M}_+ can be approximated in norm by positive linear combinations of projections this shows that $\omega \leq \psi$ on $p_1\mathcal{M}p_1$. But then

$$|\omega(ap_1)|^2 \leq \omega(p_1a^*ap_1) \leq \psi(p_1a^*ap_1) = 2\|ap_1\xi\|^2,$$

which implies that the functional $\omega \cdot p_1$ is strong operator continuous, and thus also weak operator continuous by Proposition 2.7.4. This contradicts the maximality of the family $(p_i)_{i \in I}$. \square

3.3.5 Theorem. *Let \mathcal{M} be a von Neumann algebra and φ be a bounded linear functional on \mathcal{M} . Then the following are equivalent:*

- (i) φ is normal;
- (ii) φ is σ -weakly continuous;
- (iii) φ is σ -strongly continuous;
- (iv) φ is σ -strong* continuous;
- (v) φ is weak-operator continuous on the unit ball of \mathcal{M} ;
- (vi) φ is strong-operator continuous on the unit ball of \mathcal{M} ;
- (vii) φ is strong*-operator continuous on the unit ball of \mathcal{M} ;
- (viii) there exist sequences $(\xi_n)_{n=1}^\infty$ and $(\eta_n)_{n=1}^\infty$ in $\mathcal{H}^{(\infty)}$ such that

$$\varphi(a) = \sum_{n=1}^{\infty} \langle a\xi_n \mid \eta_n \rangle.$$

PROOF. To show that (i) \implies (v), by Proposition 3.3.3 we may assume that φ is a state. By Lemma 3.3.4, there exists a $(p_i)_{i \in I}$ of mutually orthogonal nonzero projections in \mathcal{M} with $\sum_{i \in I} p_i = 1$ such that each functional $p_i \cdot \omega$ is weak-operator continuous. Fix $\epsilon > 0$. It is easy to see that φ must vanish on all but finitely many p_i , by adding a large enough finite number of the p_i , we can find a projection $p \in \mathcal{M}$ such that $p \cdot \varphi$ is weak-operator continuous and $\varphi(1 - p) < \epsilon$. If $(a_j)_{j \in J}$ is a bounded net in \mathcal{M} converging weakly to zero then

$$|\varphi(a_j)| \leq |\varphi(a_j p)| + \|a_j\| \|\varphi\|^{\frac{1}{2}} \epsilon^{\frac{1}{2}},$$

which shows that $(\varphi(a_j))_{j \in J}$ converges to zero. The conditions (ii) - (viii) are all equivalent by Proposition 2.7.9 and Proposition 2.7.19.

To finish the proof, we will show that (iii) \implies (i). Suppose that φ is σ -strongly continuous. The supremum of a bounded monotone net in \mathcal{M}_{sa} agrees with its σ -strong limit, so φ is normal. \square

3.3.6 Theorem (Sakai). *Let \mathcal{M} be a von Neumann algebra. Then the Banach space dual of \mathcal{M}_* is isometrically isomorphic to \mathcal{M} via the natural map. Moreover, if X a Banach space such that X^* is isometrically isomorphic to \mathcal{M} , then X is isometrically isomorphic to \mathcal{M}_* .*

PROOF. Let S be the closed unit ball of \mathcal{M} . Since \mathcal{M}_* is the space of σ -weakly continuous linear functionals on \mathcal{M} and the σ -weak topology is a weak* topology for \mathcal{M} , the Banach space dual of \mathcal{M}_* is \mathcal{M} . Let X be another Banach space such that X^* is isometrically isomorphic to \mathcal{M} . We claim that \mathcal{M}_{sa} is $\sigma(\mathcal{M}, X)$ -closed. Since \mathcal{M}_{sa} is convex, by the Krein-Šmulian Theorem it suffices to show that $\mathcal{M}_{\text{sa}} \cap S$ is $\sigma(\mathcal{M}, X)$ -closed.

Let $(a_j)_{j \in J}$ be a net in $\mathcal{M}_{\text{sa}} \cap S$ converging in the $\sigma(\mathcal{M}, X)$ -topology to a . Let $a = b + ic$ be the decomposition of a into real and imaginary parts, so that $b, c \in \mathcal{M}_{\text{sa}}$. By the Banach-Alaoglu Theorem, S is $\sigma(\mathcal{M}, X)$ -compact and thus $\sigma(\mathcal{M}, X)$ -closed, so we only need to show that $a \in \mathcal{M}_{\text{sa}}$. If $c = 0$, then $a \in \mathcal{M}_{\text{sa}}$ and we are done. Otherwise, if $c \neq 0$, for a sufficiently large positive or negative $n \in \mathbb{Z}$ we have

$$\|b + i(c + n1)\| \geq \|c\| + |n| > \sqrt{1 + n^2} \geq \|a_j + in1\|.$$

By the Banach-Alaoglu Theorem, $B(\mathcal{M}, \sqrt{1 + n^2})$ is $\sigma(\mathcal{M}, X)$ -closed, so $(a_j + in1)_{j \in J}$ can not converge to $a + i(b + n1)$, contradicting the assumption that $c \neq 0$. Therefore, \mathcal{M}_{sa} is $\sigma(\mathcal{M}, X)$ -closed. Since

$$\mathcal{M}_+ \cap S = (\mathcal{M}_{\text{sa}} \cap S) \cap (1 - \mathcal{M}_{\text{sa}} \cap S)$$

is $\sigma(\mathcal{M}, X)$ -closed and \mathcal{M}_+ is convex, \mathcal{M}_+ is $\sigma(\mathcal{M}, X)$ -closed by the Krein-Šmulian Theorem.

Let X_+ denote the set of positive functionals in X . We claim that X_+ separates the points of \mathcal{M} . Every C*-algebra is the linear span of its positive cone, so it suffices to show that X separates the points of \mathcal{M}_+ . Since \mathcal{M}_+ is a $\sigma(\mathcal{M}, X)$ -closed cone, if $a \in \mathcal{M}_+$ then by the Hahn-Banach Theorem there exists $\varphi \in X$ such that $\varphi(\mathcal{M}_+) \geq 0$ and $\varphi(a) > 0$. Finally, we claim that every functional in X_+ is normal. Fix an increasing net $(a_i)_{i \in I}$ in \mathcal{M}_{sa} bounded in norm by $K > 0$. Since $\mathcal{M}_{\text{sa}} \cap B(\mathcal{M}, K)$ is $\sigma(\mathcal{M}, X)$ -compact, there exists a subsequence $(a_j)_{j \in J}$ of $(a_i)_{i \in I}$ that converges to some $a \in \mathcal{M}_{\text{sa}}$. We want to show that a is actually the least upper bound of $(a_i)_{i \in I}$. For every a_i we eventually have $a_j \geq a_i$, i.e. $a_j - a_i \in \mathcal{M}_+$, so $a \geq a_i$ for all $i \in I$ because \mathcal{M}_+ is $\sigma(\mathcal{M}, X)$ -closed. If $\omega \in X_+$, then

$$\lim_{i \in I} \omega(a_i) \leq \omega(a) = \lim_{j \in J} \omega(a_j) \leq \lim_{i \in I} \omega(a_i),$$

so ω is normal. The above shows that X is a subspace of \mathcal{M}_* that separates the points of \mathcal{M} . Therefore, $X = \mathcal{M}_*$ by the Hahn-Banach Theorem. \square

There is a converse to the preceding corollary due to Sakai [Sak56], namely that if \mathcal{A} is a C*-algebra and X is a Banach space such that X^* is isometrically isomorphic to a von Neumann algebra, then there exists a von Neumann algebra \mathcal{M} and a weak* homeomorphic *-isomorphism between \mathcal{A} and \mathcal{M} that

identifies X with \mathcal{M}_* . Tomiyama [Tom57] found a simplified proof of this result.

Given a dual Banach space X , it is not always possible to extend a weak* continuous linear functional on a subspace X_0 of X to a weak* continuous linear functional on X with the same norm. Similarly, if a dual Banach space is equipped with an order structure, an order-preserving weak* continuous linear functional on a subspace of X_0 need not have an order-preserving weak* continuous extension to all of X .

However, the situation for normal linear functionals on a von Neumann algebra is much different. If $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra and φ is a normal linear functional on \mathcal{M} , then there exist sequences $(\xi_n)_{n=1}^\infty$ and $(\eta_n)_{n=1}^\infty$ in $\mathcal{H}^{(\infty)}$ such that

$$\varphi(a) = \sum_{n=1}^{\infty} \langle a\xi_n | \eta_n \rangle$$

and

$$\|\varphi\| = \|(\xi_n)_{n=1}^\infty\| \|(\eta_n)_{n=1}^\infty\|.$$

We will first prove this in the positive case, and we will derive the general case from the positive case in Corollary 3.6.3.

3.3.7 Proposition. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and ω a normal positive functional on \mathcal{M} . Then there exists a sequence $(\xi_n)_{n=1}^\infty$ in $\mathcal{H}^{(\infty)}$ such that*

$$\omega(a) = \sum_{n=1}^{\infty} \langle a\xi_n | \xi_n \rangle.$$

In particular,

$$\|\omega\| = \|(\xi_n)_{n=1}^\infty\|^2 = \sum_{n=1}^{\infty} \|\xi_n\|^2,$$

and ω extends to a normal positive functional on $\mathcal{B}(\mathcal{H})$ with the same norm.

PROOF. Since ω is a normal linear functional on \mathcal{M} , there exist sequences $(\xi_n)_{n=1}^\infty$ and $(\eta_n)_{n=1}^\infty$ in $\mathcal{H}^{(\infty)}$ such that

$$\omega(a) = \sum_{n=1}^{\infty} \langle a\xi_n | \eta_n \rangle.$$

Let $\rho : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}^{(\infty)})$ be the amplification map. If $\xi = (\xi_n)_{n=1}^\infty$ and $\eta = (\eta_n)_{n=1}^\infty$, then we have

$$\omega(a) = \langle \rho(a)\xi | \eta \rangle.$$

If $a \in \mathcal{M}$ is positive, then

$$\begin{aligned}
4\omega(a) &= 2(\omega(a) + \omega^*(a)) \\
&= 2(\langle a\xi | \eta \rangle + \langle a\eta | \xi \rangle) \\
&= \langle a(\xi + \eta) | \xi + \eta \rangle - \langle a(\xi - \eta) | \xi - \eta \rangle \\
&\leq \langle a(\xi + \eta) | \xi + \eta \rangle.
\end{aligned}$$

Therefore, by Proposition 3.3.7, there exists a $T \in \rho(\mathcal{M})'$ such that

$$\omega(a) = \langle \rho(a)T^{1/2}(\xi + \eta) | T^{1/2}(\xi + \eta) \rangle.$$

Define $\zeta = (\zeta_n)_{n=1}^\infty$ in $\mathcal{H}^{(\infty)}$ by $\zeta = T^{1/2}(\xi + \eta)$. Then

$$\omega(a) = \langle \rho(a)\zeta | \zeta \rangle = \sum_{n=1}^{\infty} \langle a\zeta_n | \zeta_n \rangle. \quad \square$$

3.3.8 Definition. A von Neumann algebra is said to be *countably decomposable*, or σ -finite, if it admits at most countably many orthogonal projections.

3.3.9 Proposition. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then the following three statements are equivalent:

- (i) \mathcal{M} is countably decomposable;
- (ii) there exists a countable separating subset of \mathcal{H} for \mathcal{M} ;
- (iii) there exists a faithful positive functional in \mathcal{M}_* .

PROOF. Suppose that \mathcal{M} is countably decomposable, and let $(\xi_n)_{n=1}^\infty$ be a maximal family of vectors in \mathcal{H} such that for distinct m, n , $[\mathcal{M}'\xi_m]$ and $[\mathcal{M}'\xi_n]$ are orthogonal, where the index must be countable because the projections e_n of \mathcal{H} onto $[\mathcal{M}'\xi_i]$ are in \mathcal{M} and are orthogonal. By the maximality of this decomposition, we have

$$\sum_{n=1}^{\infty} e_i = 1.$$

Hence $(\xi_i)_{n=1}^\infty$ is cyclic for \mathcal{M}' , so it is separating for \mathcal{M} .

Suppose that $(\xi_n)_{n=1}^\infty$ is a countable separating family of vectors in \mathcal{H} for \mathcal{M} . Let φ be the normal positive functional defined by

$$\varphi(a) = \sum_{n=1}^{\infty} \frac{1}{2^n \|\xi_n\|^2} \langle a\xi_n | \xi_n \rangle.$$

Since $\varphi(a^*a) = 0$ if and only if $a\xi_n = 0$ for all n , φ is faithful.

Let φ be a faithful normal positive functional on \mathcal{M} , and let $(e_i)_{i \in I}$ be an orthogonal family of projections. Let I_n be the set of indices i with $\varphi(e_i) \geq 1/n$. Since φ is faithful, we have that $I = \bigcup_{n=1}^{\infty} I_n$. But the inequality

$$\varphi(1) \geq \varphi\left(\sum_{i \in I_n} e_i\right) \geq \sum_{i \in I_n} \varphi(e_i) \geq \frac{1}{n}|I_n|$$

shows that each I_n must be finite, so I is countable. \square

Historical Notes

The equivalence between the normality and σ -weak continuity of a linear functional is due to Dixmier [Dix53]. Originally he only considered positive linear functionals, but the generalization to general linear functionals is easy given Pedersen's version of the Jordan decomposition for self-adjoint bounded linear functionals on a C^* -algebra [Ped69]. Similarly, the generalization to topologies other than the σ -weak topology is easy given the results of Section 2.7.

Proposition 3.3.7, which states that every normal positive functional φ on a von Neumann algebra on \mathcal{H} can be written in the form

$$\varphi(a) = \sum_{n=1}^{\infty} \langle a\xi_n \mid \xi_n \rangle$$

for some sequence of vectors $(\xi_n)_{n=1}^{\infty}$ in $\mathcal{H}^{(\infty)}$ is due to Dixmier [Dix54].

3.4 Normal maps

Just as in the case of linear functionals we can define the notion of a normal linear map between two von Neumann algebras. Unfortunately, the theory is unsatisfactory in the general case, because a bounded linear map need not be the linear combination of four positive maps. Therefore, we will restrict our attention only to positive maps.

3.4.1 Definition. Let \mathcal{M} and \mathcal{N} be von Neumann algebras, and let $\rho : \mathcal{M} \rightarrow \mathcal{N}$ be a positive map. We say that ρ is normal if, whenever $(a_i)_{i \in I}$ is a bounded increasing net in \mathcal{M}_{sa} , then

$$\rho\left(\sup_{i \in I} a_i\right) = \sup_{i \in I} \rho(a_i).$$

3.4.2 Examples.

- (i) Since the definition of normality of a positive map between von Neumann algebras only depends on the properties of the von Neumann algebras as abstract C^* -algebras, every $*$ -isomorphism between von Neumann algebras is automatically normal.

- (ii) Since the supremum of a bounded increasing net in \mathcal{M}_{sa} is also its strong operator topology limit, every positive map between von Neumann algebras that is strong operator topology continuous on bounded parts is normal.
- (iii) As a particular case of (ii), if α is a cardinal, then the α -fold amplification map $\rho : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}^{(\alpha)})$ is a normal $*$ -homomorphism.
- (iv) As another particular case of (ii), if $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra and $p \in \mathcal{M}'$ is a projection, then the induction map from \mathcal{M} to \mathcal{M}_p is a normal $*$ -homomorphism. If p has central cover 1, then the induction map is a normal $*$ -isomorphism by Proposition 3.1.8.

The three concrete examples of normal maps above are the prototypical examples, and we will show that any normal $*$ -isomorphism between von Neumann algebras is the composition of such maps in Theorem 3.4.8.

We will now show the equivalence of normality of a $*$ -homomorphism with continuity in the various operator topologies. Our argument is actually valid for a more general class of positive maps, because it depends only on the fact that a generalized Cauchy-Schwarz inequality holds for $*$ -homomorphisms. Such an inequality holds for other positive maps that are not necessarily $*$ -homomorphisms. The reader who is familiar with operator spaces should note that it holds for all completely positive maps.

3.4.3 Proposition. *Let \mathcal{M} and \mathcal{N} be von Neumann algebras, and $\rho : \mathcal{M} \rightarrow \mathcal{N}$ a positive map that satisfies a generalized Cauchy-Schwarz inequality*

$$\rho(a)^* \rho(a) \leq K \rho(a^* a),$$

for some $K > 0$. Then the following are equivalent:

- (i) ρ is normal.
- (ii) ρ is continuous with respect to the σ -weak topologies on \mathcal{M} and \mathcal{N} ;
- (iii) ρ is continuous with respect to the σ -strong topologies on \mathcal{M} and \mathcal{N} ;
- (iv) ρ is continuous with respect to the σ -strong* topologies on \mathcal{M} and \mathcal{N} ;
- (v) ρ is continuous with respect to the weak-operator topologies on bounded parts of \mathcal{M} and \mathcal{N} ;
- (vi) ρ is continuous with respect to the strong-operator topologies on bounded parts of \mathcal{M} and \mathcal{N} ;
- (vii) ρ is continuous with respect to the strong*-operator topologies on bounded parts of \mathcal{M} and \mathcal{N} .

PROOF. The equivalence (i) \iff (ii) is clear, because the σ -weak topology is generated by the predual, so ρ is σ -weakly continuous if and only if $\varphi \circ \rho \in \mathcal{M}_*$ for every $\varphi \in \mathcal{N}_*$, and ρ is normal if and only if $\varphi \circ \rho$ is normal for every $\varphi \in \mathcal{N}_*$.

The implications (ii) \implies (v) holds because the σ -weak topology agrees with the weak operator topology on bounded sets, and similarly for (iii) \implies (vi) and (iv) \implies (vii).

We will now prove that (ii) \implies (iii). Suppose that ρ is σ -weakly continuous and $a_i \rightarrow 0$ σ -strongly. Then $a_i^* a_i \rightarrow 0$ σ -weakly, so $\rho(a_i^* a_i) \rightarrow 0$ σ -weakly. Then

$$\rho(a_i)^* \rho(a_i) \leq K \rho(a_i^* a_i),$$

so $\rho(a_i)^* \rho(a_i) \rightarrow 0$ σ -weakly, i.e. $\rho(a_i) \rightarrow 0$ σ -strongly. The proofs of (ii) \implies (iv), (v) \implies (vi), and (v) \implies (vii) are similar.

Finally, (vi) \implies (i) and (vii) \implies (i) hold because the supremum of a bounded increasing net in \mathcal{M}_{sa} is its strong operator and strong* operator limit. \square

3.4.4 Remark. Kadison [Kad52] proved that if \mathcal{A} and \mathcal{B} are C*-algebras and $\rho : \mathcal{A} \rightarrow \mathcal{B}$ is a positive map, then $\rho(a)^2 \leq \rho(a^2)$ for all $a \in \mathcal{A}$. Since the adjoint is continuous in the σ -strong* topology, the decomposition of a general element into its real and imaginary parts is also continuous with respect to the σ -strong* topology. Therefore, using Kadison's inequality, the proof of the previous theorem can be adopted to show that ρ is σ -weakly continuous if and only if ρ is σ -strong* continuous, without the assumption of any generalized Cauchy-Schwarz inequality.

3.4.5 Proposition. *Let \mathcal{M} be a von Neumann algebra, and let ω be a positive functional on \mathcal{M} . Then ω is normal if and only if the GNS representation π_ω is normal.*

PROOF. Let (π, \mathcal{H}, ξ) be the GNS representation associated with ω . Suppose that ω is normal, and let $(a_i)_{i \in I}$ be a bounded increasing net in \mathcal{M}_{sa} with supremum a . Then $(\pi(a_i))_{i \in I}$ is a bounded increasing net in $\mathcal{B}(\mathcal{H})_{\text{sa}}$, so it has a supremum b . For every $x \in \mathcal{M}$ we have

$$\begin{aligned} \langle \pi(a) \pi(x) \xi \mid \pi(x) \xi \rangle &= \omega(x^* a x) \\ &= \lim_{i \in I} \omega(x^* a_i x) \\ &= \lim_{i \in I} \langle \pi(a_i) \pi(x) \xi \mid \pi(x) \xi \rangle \\ &= \langle \pi(b) \pi(x) \xi \mid \pi(x) \xi \rangle, \end{aligned}$$

where the limits are taken in the strong* topology. Since ξ is as a cyclic vector for π , this implies that $\pi(a) = b$. Therefore, π is normal.

Conversely, suppose that π_ω is normal. Then

$$\omega(a) = \langle \pi(a) \xi \mid \xi \rangle,$$

so ω is the composition of π and the normal functional $b \mapsto \langle b \xi \mid \xi \rangle$ on $\pi(\mathcal{M})$. Therefore, ω is normal. \square

3.4.6 Remark. The reader who is familiar with the theory of operator spaces and completely bounded maps should note that it is possible to define the notion of a normal completely bounded map, by modifying the definition of a normal positive map so that a completely bounded map $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ is normal if, whenever $(a_i)_{i \in I}$ is a bounded increasing net in \mathcal{M}_{sa} , then

$$\varphi \left(\sup_{i \in I} a_i \right) = \lim_{i \in I} \varphi(a_i).$$

There are versions of the Stinespring and Wittstock theorems for normal completely positive and completely bounded maps, and the proofs of these theorems are similar to that of the GNS Theorem for normal maps. In particular, the Wittstock Structure Theorem for normal maps allows one to prove the analogue of Proposition 3.4.3 for all completely bounded maps.

3.4.7 Proposition. *Let \mathcal{M} and \mathcal{N} be von Neumann algebras, and $\rho : \mathcal{M} \rightarrow \mathcal{N}$ a normal $*$ -homomorphism. Then $\ker(\rho)$ is a σ -weakly closed ideal and $\rho(\mathcal{M})$ is a σ -weakly closed subalgebra of \mathcal{N} , which is a von Neumann subalgebra if ρ is unital.*

PROOF. The first claim is clear. Since a $*$ -homomorphism between C^* -algebras is a quotient map onto its range, ρ maps the closed unit ball of \mathcal{M} onto the closed unit ball of $\rho(\mathcal{M})$. Since ρ is σ -weakly continuous and the unit ball of \mathcal{M} is σ -weakly compact, this implies that the unit ball of $\rho(\mathcal{M})$ is σ -weakly compact. Therefore, by Corollary 2.8.17, $\rho(\mathcal{M})$ is a von Neumann algebra. \square

We will now prove two structure theorems for $*$ -isomorphisms.

3.4.8 Theorem. *Let \mathcal{M} and \mathcal{N} be von Neumann algebras, and let $\rho : \mathcal{M} \rightarrow \mathcal{N}$ be a surjective normal $*$ -homomorphism. Then ρ is the composition $\rho_3 \circ \rho_2 \circ \rho_1$ of an amplification $\rho_1 : \mathcal{M} \rightarrow \mathcal{M}^{(\alpha)}$, an induction $\rho_2 : \mathcal{M}^{(\alpha)} \rightarrow p\mathcal{M}^{(\alpha)}$ defined by a projection $p \in \mathcal{M}^{(\alpha)'}$, and a spatial isomorphism $\rho_3 : p\mathcal{M}^{(\alpha)} \rightarrow \mathcal{N}$. Moreover, if ρ is cyclic, then p must have central cover 1.*

PROOF. Suppose $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{N} \subseteq \mathcal{B}(\mathcal{K})$. We will first prove the claim in the case where ρ is a normal cyclic representation. Let $\xi \in \mathcal{K}$ be a cyclic vector for ρ , and define a normal positive functional on \mathcal{M} by

$$\omega(a) = \langle \rho(a)\xi \mid \xi \rangle.$$

Since ω is normal, there exists a sequence $\eta = (\xi_n)_{n=1}^{\infty}$ in $\mathcal{H}^{(\infty)}$ such that

$$\omega(a) = \langle \rho_1(a)\eta \mid \eta \rangle,$$

where $\rho_1 : \mathcal{M} \rightarrow \mathcal{M}^{(\infty)}$ is the amplification map. Let $p \in \mathcal{M}^{(\infty)'}$ be the projection onto $\mathcal{M}^{(\infty)}\eta$, and $\rho_2 : \mathcal{M}^{(\infty)} \rightarrow p\mathcal{M}^{(\infty)}$ the corresponding induction map. Then η is a cyclic vector for the homomorphism $\rho_2 \circ \rho_1$ and

$$\langle \rho(a)\xi \mid \xi \rangle = \omega(a) = \langle \rho_1(a)\eta \mid \eta \rangle = \langle p\rho_1(a)\eta \mid \eta \rangle = \langle (\rho_2 \circ \rho_1)(a)\eta \mid \eta \rangle.$$

By the uniqueness of the GNS construction (see Proposition 2.10.12), there exists a spatial isomorphism $\rho_3 : p\mathcal{M}^{(\infty)} \rightarrow \mathcal{N}$ such that $\rho_3 \circ \rho_2 \circ \rho_1 = \rho$.

We will now derive the general case from the cyclic case. Let $(\xi_i)_{i \in I}$ be a maximal family of nonzero vectors in \mathcal{K} such that the subspaces $\mathcal{K}_i = \mathcal{N}\xi_i$ of \mathcal{K} are mutually orthogonal. Since the family is maximal, $\mathcal{K} = \bigoplus_{i \in I} \mathcal{K}_i$. For every $i \in I$, let $q_i \in \mathcal{M}'$ be the projection onto \mathcal{K}_i and $\rho_i : \mathcal{M} \rightarrow q_i\mathcal{N}$ the surjective normal $*$ -homomorphism defined by $\rho_i(a) = q_i\rho(a)$. If $i \in I$, then ξ_i is a cyclic vector for ρ_i , so exists a an amplification $\rho_{i,1} : \mathcal{M} \rightarrow \mathcal{M}^{(\infty)}$, an induction $\rho_{i,2} : \mathcal{M}^{(\infty)} \rightarrow p_i\mathcal{M}^{(\infty)}$ defined by a projection $p_i \in \mathcal{M}^{(\infty)'}$, and a spatial isomorphism $\rho_{i,3} : p_i\mathcal{M}^{(\infty)} \rightarrow q_i\mathcal{N}$ such that $\rho^i = \rho_{i,3} \circ \rho_{i,2} \circ \rho_{i,1}$. If $\alpha = \aleph_0 \cdot \text{card}(I)$, then the amplification map $\rho_1 : \mathcal{M} \rightarrow \mathcal{M}^{(\alpha)}$ is simply the direct sum of the amplifications $\rho_{i,1}$. Similarly, if $p = \bigoplus_{i \in I} p_i$, then the induction map $\rho_2 : \mathcal{M}^{(\alpha)} \rightarrow p\mathcal{M}^{(\alpha)}$ is the coordinate-wise sum of each of the $\rho_{i,2}$, and the spatial equivalences $\rho_{i,3}$ combine to give a spatial equivalence $\rho_3 : p\mathcal{M}^{(\alpha)} \rightarrow \mathcal{N}$ such that $\rho_1 \circ \rho_2 \circ \rho_3 = \rho$.

Finally, if ρ is a $*$ -isomorphism, then ρ_2 must be a $*$ -isomorphism. It follows from Proposition 3.1.8 that p has central cover 1. \square

3.4.9 Theorem. *Let \mathcal{M} and \mathcal{N} be von Neumann algebras, and let $\rho : \mathcal{M} \rightarrow \mathcal{N}$ be a $*$ -isomorphism. Then there exist projections $p \in \mathcal{M}'$ and $q \in \mathcal{N}'$, both with central cover 1, and a spatial isomorphism $\rho_0 : p\mathcal{M} \rightarrow q\mathcal{N}$ such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\rho} & \mathcal{N} \\ \downarrow & & \downarrow \\ p\mathcal{M} & \xrightarrow{\rho_0} & q\mathcal{N} \end{array}$$

PROOF. By Theorem 3.4.8, we only need to prove the theorem under the assumption that $\mathcal{M} = e\mathcal{R}$ and $\mathcal{N} = f\mathcal{R}$, where $\mathcal{R} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra, and e and f are projections in \mathcal{R}' with central cover 1.

Since the unit ball of the subspace $f\mathcal{M}'e$ of \mathcal{M}' is convex and σ -weakly compact, by the Krein-Milman Theorem it has an extreme point v . By Theorem 2.6.9, v is a partial isometry, and if $p = v^*v$ and $q = vv^*$ are the source and range projections of v , we have

$$(f - q)\mathcal{M}'(e - p) = \{0\}.$$

Therefore, if $u \in \mathcal{M}'$ is unitary, $u^*(f - q)u$ is orthogonal to $e - p$. By Proposition 3.1.6, $c(f - q)$ is the supremum of the projections of the form $u^*(f - q)u$,

where $u \in \mathcal{M}'$ is unitary, so $c(f - q)$ is orthogonal to $e - p$. Therefore, $c(f - q)$ and $c(e - p)$ are orthogonal. We have

$$1 = c(e) \leq c(e - p) + c(p) \quad \text{and} \quad 1 = c(f) \leq c(f - q) + c(q).$$

Since $p \sim q$, $c(p) = c(q)$, and it follows that

$$1 \leq c(f - q) + c(q) \leq 1 - c(e - p) + c(q) \leq c(p) + c(q) = 2c(p).$$

Therefore, $c(p) = c(q) = 1$. □

Historical Notes

The continuity result Proposition 3.4.3 is due to Dixmier [Dix53]. The structure theorems for normal $*$ -homomorphisms, Theorem 3.4.8 and Theorem 3.4.9, are also due to Dixmier [Dix54].

3.5 The Enveloping von Neumann Algebra of a C^* -algebra

Every representation of a C^* -algebra \mathcal{A} generates a von Neumann algebra, so some aspects of C^* -algebra theory are invariably von Neumann algebraic in nature. Analogous to the enveloping C^* -algebra of a Banach $*$ -algebra, it would be useful to construct a universal enveloping von Neumann algebra of \mathcal{A} that acts as a “stand-in” for \mathcal{A} in the category of von Neumann algebras. Such a von Neumann algebra always exists, and it may be identified with the bidual \mathcal{A}^{**} of \mathcal{A} .

3.5.1 Proposition. *Let \mathcal{A} be a C^* -algebra, π a representation of \mathcal{A} , and define $\mathcal{M} = \pi(\mathcal{A})''$. Then the adjoint of the restriction of the adjoint of π to \mathcal{M}_* is the unique surjective linear map of $\tilde{\pi} : \mathcal{A}^{**} \rightarrow \mathcal{M}$ such that $\pi = \tilde{\pi} \circ \iota$, where $\iota : \mathcal{A} \rightarrow \mathcal{A}^{**}$ is the canonical embedding. Moreover:*

- (i) $\tilde{\pi}$ is continuous with respect to the weak* topology on \mathcal{A}^{**} and the σ -weak topology on \mathcal{M} ;
- (ii) $\tilde{\pi}$ maps the closed unit ball of \mathcal{A}^{**} onto the closed unit ball of \mathcal{M} .

PROOF. Let $\pi^* : \mathcal{M}^* \rightarrow \mathcal{A}^*$ be the adjoint of π , and let $\tilde{\pi} : \mathcal{A}^{**} \rightarrow \mathcal{M}_*$ be the adjoint of the restriction of π^* to \mathcal{M}_* . Then for every $a \in \mathcal{A}$ and $\varphi \in \mathcal{M}_*$,

$$\langle \pi(a), \varphi \rangle = \langle \iota(a), \pi^*(\varphi) \rangle = \langle \tilde{\pi}(\iota(a)), \varphi \rangle,$$

so $\pi = \tilde{\pi} \circ \iota$. By Goldstine's Theorem, \mathcal{A} is weak* dense in \mathcal{A}^{**} , so $\tilde{\pi}$ is uniquely determined by this property. Since $\tilde{\pi}$ is the adjoint of a map from \mathcal{M}_* to \mathcal{A}^* , it is continuous with respect to the weak* topology on \mathcal{A}^{**} .

By Goldstine's Theorem, the unit ball of $\iota(\mathcal{A})$ is weak* dense in \mathcal{A}^{**} , so $\tilde{\pi}$ maps the closed unit ball of \mathcal{A}^{**} onto the closed unit ball of \mathcal{M} . Since $\tilde{\pi}$ is the adjoint of the restriction of an adjoint of a contraction, it is a contraction itself. Since $\pi = \tilde{\pi} \circ \iota$, $\tilde{\pi}(\mathcal{A}^{**})$ contains $\pi(\mathcal{A})$, so by the Kaplansky Density Theorem, $\tilde{\pi}$ maps the unit ball of \mathcal{A}^{**} onto the closed unit ball of \mathcal{M} . \square

3.5.2 Theorem. *Let \mathcal{A} be a C^* -algebra and $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ the universal representation of \mathcal{A} . Define $\mathcal{M} = \pi(\mathcal{A})''$, and recall the definition of the map $\tilde{\pi} : \mathcal{A}^{**} \rightarrow \mathcal{M}$ of Proposition 3.5.1. Then:*

- (i) *if $\varphi \in \mathcal{A}^*$, then there exist $\xi, \eta \in \mathcal{H}$ such that $\varphi(a) = \langle \pi(a)\xi | \eta \rangle$, and if φ is positive we can choose ξ and η such that $\xi = \eta$;*
- (ii) *$\tilde{\pi}$ is an isometric isomorphism and a homeomorphism with respect to the weak* topology on \mathcal{A}^{**} and the σ -weak topology on \mathcal{M} ;*
- (iii) *for every von Neumann algebra \mathcal{N} and $*$ -homomorphism $\rho_0 : \mathcal{A} \rightarrow \mathcal{N}$ there exists a normal $*$ -homomorphism $\rho : \mathcal{M} \rightarrow \mathcal{N}$ such that $\rho_0 = \rho \circ \pi$;*

PROOF.

- (i) By Corollary 2.12.12, there exists a cyclic representation $\rho : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_\rho)$ and $\xi, \eta \in \mathcal{H}_\rho$ such that $\varphi(a) = \langle \rho(a)\xi | \eta \rangle$. Since ρ is a direct summand of π , we may regard ξ and η as the projection onto \mathcal{H}_ρ of elements ξ' and η' of \mathcal{H} , so that $\varphi(a) = \langle \pi(a)\xi' | \eta' \rangle$. Similarly, the assertion about positivity follows directly from the GNS Theorem.
- (ii) Since π is an isometry, π^* is a quotient map. Clearly, (i) implies that the restriction $(\pi^*)|_{\mathcal{M}_*} : \mathcal{M}_* \rightarrow \mathcal{A}^*$ is surjective, so it is a quotient map, and its adjoint $\tilde{\pi}$ is an isometry. By Proposition 3.5.1, $\tilde{\pi}$ is surjective, so it is an isometric isomorphism. Also, $\tilde{\pi}$ is continuous with respect to the weak* topology on \mathcal{A}^{**} and the σ -weak topology on \mathcal{M} . Therefore, by the Krein-Šmulian Theorem, $\tilde{\pi}$ is a homeomorphism with respect to the weak* topology on \mathcal{A}^{**} and the σ -weak topology on \mathcal{M} .
- (iii) Let $\rho_0 : \mathcal{A} \rightarrow \mathcal{N}$ be a $*$ -homomorphism, and let $\tilde{\rho}_0 : \mathcal{A}^{**} \rightarrow \rho_0(\mathcal{A})''$ be the map defined in Proposition 3.5.1. Define $\rho = \rho_0 \circ \tilde{\pi}^{-1}$. Then ρ is normal and $\rho_0 = \rho \circ \pi$. \square

3.5.3 Examples.

- (i) The bidual of $c_0(\mathbb{N})$ is isometrically isomorphic to $\ell^\infty(\mathbb{N})$, and the usual product on $\ell^\infty(\mathbb{N})$ extends the product on $c_0(\mathbb{N})$, so $\ell^\infty(\mathbb{N})$ is the enveloping von Neumann algebra of $c_0(\mathbb{N})$.
- (ii) Consider the algebra $\mathcal{K}(\mathcal{H})$ of compact operators on some Hilbert space \mathcal{H} . By Corollary 2.7.17, the bidual of $\mathcal{K}(\mathcal{H})$ is $\mathcal{B}(\mathcal{H})$, and the usual product on $\mathcal{B}(\mathcal{H})$ extends the product on $\mathcal{K}(\mathcal{H})$, so $\mathcal{B}(\mathcal{H})$ is the enveloping von Neumann algebra of $\mathcal{K}(\mathcal{H})$. However, since $\mathcal{B}(\mathcal{H})' = \mathbb{C}$, $\mathcal{B}(\mathcal{H})$ can not be identified spatially with a direct sum of representations, whereas

the enveloping von Neumann algebra of $\mathcal{K}(\mathcal{H})$ is defined as a direct sum of cyclic representations. Therefore, $\mathcal{B}(\mathcal{H})$ is not spatially isomorphic to $\mathcal{K}(\mathcal{H})^{**}$.

- (iii) Let G be a locally compact group. Then the enveloping von Neumann algebra of $C^*(G)$ is denoted by $W^*(G)$, and is called the *enveloping von Neumann algebra* of G . Since the Fourier-Stieltjes algebra $B(G)$ is the dual of $C^*(G)$, it is the predual of $W^*(G)$.

Given a von Neumann algebra \mathcal{M} and the bimodule action of \mathcal{M} on \mathcal{M}_* , it is natural to examine the subsets of \mathcal{M}_* that are invariant under this action.

3.5.4 Definition. Let \mathcal{M} be a von Neumann algebra. A subset V of \mathcal{M}_* is said to be *left-invariant* (resp. *right-invariant*) if $a \cdot V \subseteq V$ (resp. $V \cdot a \subseteq V$) for every $a \in \mathcal{M}$. If V is both left-invariant and right-invariant, it is simply called *invariant*.

3.5.5 Examples.

- (i) Let \mathcal{M} be a von Neumann algebra and φ a normal linear function on \mathcal{M} . Define

$$V_\varphi = [\mathcal{M}\varphi] \quad \text{and} \quad W_\varphi = [\varphi\mathcal{M}].$$

Then V_φ and W_φ are respectively left-invariant and right-invariant norm closed subspaces of \mathcal{M}_* .

- (ii) Let \mathcal{A} be a C^* -algebra and π a representation of \mathcal{A} . The *coefficient space* of π is the subspace $\pi^*(\mathcal{M}_*(\pi))$ of \mathcal{A}^* , and is denoted by A_π . It is called the coefficient space because the prototypical element of A_π is the elementary coefficient function

$$\pi_{\xi,\eta}(a) = \langle \pi(a)\xi | \eta \rangle$$

for some $\xi, \eta \in \mathcal{H}$. However, A_π is only the norm closed span of these coefficient functions, and it is not necessarily the case that every element of A_π is a coefficient function. Since π is a quotient map, π^* is an isometry, so A_π is a norm closed subspace of \mathcal{A}^* . It is also invariant under the bimodule action of \mathcal{A}^{**} , because if $a, b, x \in \mathcal{A}$ and $\varphi \in \mathcal{M}_*(\pi)$ then

$$\begin{aligned} \langle x, a \cdot \pi^*(\varphi) \cdot b \rangle &= \langle bxa, \pi^*(\varphi) \rangle \\ &= \langle \pi(bxa), \varphi \rangle \\ &= \langle \pi(b)\pi(x)\pi(a), \varphi \rangle \\ &= \langle \pi(x), \pi(a) \cdot \varphi \cdot \pi(b) \rangle \\ &= \langle x, \pi^*(\pi(a) \cdot \varphi \cdot \pi(b)) \rangle. \end{aligned}$$

In the case where \mathcal{A} is $C^*(G)$ for a locally compact group G , the dual \mathcal{A}^* can be identified with the Fourier-Stieltjes algebra $B(G)$, and thus A_π

may be regarded as the subspace of $B(G)$ generated by the elementary coefficient functions of the associated representation of G . We will often slightly conflate notation and refer to A_π in the case where π is a unitary representation of G when we really mean the coefficient space of the integrated form of π on $C^*(G)$.

- (iii) Let G be a locally compact group, and $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ the universal representation of G . Then the enveloping von Neumann algebra $W^*(G)$ is generated by π , and the Fourier-Stieltjes algebra $B(G)$ is the predual of $W^*(G)$. The left and right actions of $\pi(G)$ on $B(G)$ are given by right and left translation respectively. Indeed, if $\varphi \in B(G)$, $\xi, \eta \in \mathcal{H}$ are such that $\varphi(s) = \langle \pi(s)\xi | \eta \rangle$, and $g \in G$, then

$$(\pi(g) \cdot \varphi)(s) = \langle \pi(sg)\xi | \eta \rangle = \varphi(sg)$$

and

$$(\varphi \cdot \pi(g))(s) = \langle \pi(gs)\xi | \eta \rangle = \varphi(gs).$$

Therefore, the left-invariant subsets of $B(G)$ are the subsets invariant under right translations, and the right-invariant subsets of $B(G)$ are the subsets invariant under left translations.

3.5.6 Theorem. *Let \mathcal{M} be a von Neumann algebra. Then the map $p \mapsto \mathcal{M}_* \cdot p$ (resp. $p \mapsto p \cdot \mathcal{M}_*$) establishes a bijective correspondence between*

- (i) *projections in \mathcal{M} , and*
- (ii) *norm closed left-invariant (resp. right-invariant) subspaces of \mathcal{M}_* .*

This correspondence induces a bijective correspondence between

- (i) *central projections in \mathcal{M}*
- (ii) *norm closed invariant subspaces of \mathcal{M}_* ,*

PROOF. Suppose $p \in \mathcal{M}$ is a projection. We want to show that

$$\mathcal{M}_* \cdot p = ((1 - p)\mathcal{M})^\perp.$$

If $\varphi \in \mathcal{M}_*$ and $a \in (1 - p)\mathcal{M}$, then $a = (1 - p)b$ for some $b \in \mathcal{M}$, so

$$(\varphi \cdot p)(a) = \varphi(pa) = \varphi(p(1 - p)b) = \varphi(0b) = 0,$$

showing that $\mathcal{M}_* \cdot p \subseteq ((1 - p)\mathcal{M})^\perp$. If $\varphi \in ((1 - p)\mathcal{M})^\perp$ and $a \in \mathcal{M}$, then

$$\begin{aligned} \varphi(a) &= \varphi(pa + (1 - p)a) \\ &= \varphi(pa) + \varphi((1 - p)a) \\ &= \varphi(pa) + 0 \\ &= (\varphi \cdot p)(a), \end{aligned}$$

so $\varphi = \varphi \cdot p$ and $\varphi \in \mathcal{M}_* \cdot p$. Therefore,

$$\mathcal{M}_* \cdot p = ((1 - p)\mathcal{M})^\perp,$$

which is clearly a norm closed left-invariant subspace of \mathcal{M}_* . Similarly,

$$p \cdot \mathcal{M}_* = (\mathcal{M}(1 - p))^\perp,$$

which is clearly a norm closed right-invariant subspace of \mathcal{M} . The map $p \mapsto \mathcal{M}_* \cdot p$ (resp. $p \mapsto p \cdot \mathcal{M}_*$) is injective because $(1 - p)$ is the largest projection in $(1 - p)\mathcal{M}$ (resp. in $\mathcal{M}(1 - p)$).

We now want to show that the map $p \mapsto \mathcal{M}_* \cdot p$ is surjective. Suppose V is a norm closed left-invariant subspace of \mathcal{M}_* . Then V^\perp is a σ -weakly closed subspace of \mathcal{M} . If $x \in V^\perp$, $a \in \mathcal{A}$, and $\varphi \in V$, then $a \cdot \varphi \in V$ and

$$\varphi(xa) = (a \cdot \varphi)(x) = 0,$$

showing that $xa \in V^\perp$. Hence V^\perp is a right ideal of \mathcal{M} . By Proposition 3.1.1, there exists a projection $p \in \mathcal{M}$ such that $V^\perp = p\mathcal{M}$. Thus

$$V = V^{\perp\perp} = ((1 - (1 - p))\mathcal{M})^\perp = \mathcal{M}_* \cdot (1 - p).$$

Similarly, if V is a norm closed right-invariant subspace of \mathcal{M}_* , then V^\perp is a σ -weakly closed left ideal of \mathcal{M} , and by Proposition 3.1.1 there exists a projection $p \in \mathcal{M}$ such that $V^\perp = \mathcal{M}p$. Thus

$$V = V^{\perp\perp} = (\mathcal{M}(1 - (1 - p)))^\perp = (1 - p) \cdot \mathcal{M}_*.$$

If V is invariant on both sides then V^\perp is a σ -weakly closed two-sided ideal of \mathcal{M} , so by Proposition 3.1.1 there is a central projection $p \in \mathcal{M}$ such that

$$V^\perp = p\mathcal{M} = \mathcal{M}p.$$

Therefore,

$$V = \mathcal{M}_* \cdot (1 - p) = (1 - p) \cdot \mathcal{M}_*. \quad \square$$

The correspondence between projections and invariant subspaces of the predual in the preceding theorem will be of considerable interest to us throughout this work, so we will give it its own nomenclature.

3.5.7 Definition. Let \mathcal{M} be a von Neumann algebra. If V is a norm closed left-invariant (resp. right-invariant) subspace of \mathcal{M}_* , then the *left support* (resp. *right support*) of V is the unique projection $p \in \mathcal{M}$ such that $V = \mathcal{M}_* \cdot p$ (resp. $V = p \cdot \mathcal{M}_*$). If, in addition, V is invariant on both sides, p is simply called the *support* of V .

3.5.8 Examples.

- (i) Let \mathcal{M} be a von Neumann algebra and φ a normal linear function on \mathcal{M} . Define

$$V_\varphi = [\mathcal{M}\varphi] \quad \text{and} \quad W_\varphi = [\varphi\mathcal{M}].$$

Then V_φ and W_φ are respectively left-invariant and right-invariant norm closed subspaces of \mathcal{M}_* , so by Theorem 3.5.6 there exist projections $p, q \in \mathcal{M}$ such that

$$V_\varphi = \mathcal{M}_* \cdot p \quad \text{and} \quad W_\varphi = q \cdot \mathcal{M}_*.$$

The projections p and q are, respectively, the smallest projections in \mathcal{M} such that $\varphi = \varphi \cdot p$ and $\varphi = q \cdot \varphi$. Thus the projections p and q are called, respectively, the *right support projection* and *left support projection* of φ and are denoted by $s_l(\varphi)$ and $s_r(\varphi)$. We have $s_r(\varphi) = s_l(\varphi^*)$, so if φ is self-adjoint, then $V_\varphi = W_\varphi$ and $s_r(\varphi) = s_l(\varphi)$, in which case we denote both by $s(\varphi)$, the *support projection* of φ .

- (ii) Let \mathcal{A} be a C^* -algebra and π a representation of \mathcal{A} . The coefficient space A_π of π is a norm closed invariant subspace of \mathcal{A}^* , the predual of \mathcal{A}^{**} , so there exists a unique central projection $z \in \mathcal{A}^{**}$ such that $A_\pi = z \cdot \mathcal{A}^*$. This central projection is called the *central support* of π and is denoted by $c(\pi)$. Moreover, we can associate a representation of \mathcal{A} to any central projection $z \in \mathcal{A}^{**}$. Let $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ be the universal representation of \mathcal{A} and $z \in \mathcal{A}^{**}$ a central projection. Define a representation $\rho : \mathcal{A} \rightarrow \mathcal{B}(z\mathcal{H})$ by $\rho(a) = z\pi(a)$. Then the central support of ρ is clearly z .
- (iii) Let G be a locally compact group. Then the Fourier-Stieltjes algebra $B(G)$ is the predual of the enveloping von Neumann algebra $W^*(G)$, and the invariant subspaces of $B(G)$ under the bimodule action of $W^*(G)$ are simply the translation-invariant subspaces.

If \mathcal{A} is a C^* -algebra, then unitarily equivalent representations of \mathcal{A} clearly have equal central support in \mathcal{A}^{**} , but it is impossible for the central support of representations to distinguish them up to unitary equivalence, simply because there are not “set many” equivalence classes of unitary representations of \mathcal{A} . Since the central support of a representation π depends only on its coefficient space A_π , which is a copy of $\mathcal{M}_*(\pi)$ sitting inside of \mathcal{A}^* , it is natural to consider the notion of equivalence of representations by considering normal $*$ -isomorphisms of the respective von Neumann algebras that preserve the action of \mathcal{A} .

3.5.9 Definition. Let \mathcal{A} be a C^* -algebra, and let π_1 and π_2 be nondegenerate representations of \mathcal{A} . We say that π_1 and π_2 are *quasiequivalent* if there exists a normal $*$ -isomorphism $\rho : \mathcal{M}(\pi_1) \rightarrow \mathcal{M}(\pi_2)$ such that $\rho \circ \pi_1 = \pi_2$.

3.5.10 Examples.

- (i) Unitarily equivalent representations are quasiequivalent.
- (ii) Let π a representation of \mathcal{A} and α a nonzero cardinal number. Then the amplification map $\rho : \mathcal{M}(\pi) \rightarrow \mathcal{M}(\pi^{(\alpha)})$ is a normal $*$ -isomorphism such that $\rho \circ \pi = \pi^{(\alpha)}$. Therefore, π and $\pi^{(\alpha)}$ are quasiequivalent.
- (iii) Let π be a representation of \mathcal{A} , and $p \in \mathcal{M}(\pi)'$ a projection with central cover 1. Define a representation σ of \mathcal{A} by $\sigma(a) = p\pi(a)$. Then $\mathcal{M}(\sigma) = p\mathcal{M}(\pi)$, and if $\rho : \mathcal{M}(\pi) \rightarrow \mathcal{M}(\sigma)$ is the induction map with respect to p , we have $\rho \circ \pi = \sigma$. Therefore, π and σ are quasiequivalent.

3.5.11 Theorem. *Let \mathcal{A} be a C^* -algebra. and let π_1 and π_2 be nondegenerate representations of \mathcal{A} . Then the following are equivalent:*

- (i) π_1 and π_2 are quasiequivalent;
- (ii) $c(\pi_1) = c(\pi_2)$;
- (iii) $A_{\pi_1} = A_{\pi_2}$.

PROOF. We will first show the equivalence (i) \iff (iii), because the equivalence (ii) \iff (iii) follows from Theorem 3.5.6. Suppose that π_1 and π_2 are quasiequivalent, and let $\rho : \mathcal{M}(\pi_1) \rightarrow \mathcal{M}(\pi_2)$ be a normal $*$ -isomorphism such that $\rho \circ \pi_1 = \pi_2$. Then, since ρ is a σ -weakly bicontinuous isometric isomorphism, the preadjoint ρ_* of ρ is an isometric isomorphism from $\mathcal{M}_*(\pi_2)$ onto $\mathcal{M}_*(\pi_1)$, and $\pi_1^* \circ \rho_* = \pi_2^*$. Therefore, we have

$$A_{\pi_2} = \pi_2^*(\mathcal{M}_*(\pi_2)) = \pi_1^* \circ \rho_*(\mathcal{M}_*(\pi_2)) = \pi_1^*(\mathcal{M}_*(\pi_1)) = A_{\pi_1}.$$

Conversely, suppose that $A_{\pi_1} = A_{\pi_2}$. The restriction of π_i^* to $\mathcal{M}_*(\pi_i)$ establishes an isometric isomorphism between $\mathcal{M}_*(\pi_i)$ and A_{π_i} . Hence

$$\sigma = ((\pi_1^*)|_{\mathcal{M}_*(\pi_1)})^{-1} \circ (\pi_2^*)|_{\mathcal{M}_*(\pi_2)}$$

is an isometric isomorphism from $\mathcal{M}_*(\pi_2)$ onto $\mathcal{M}_*(\pi_1)$. Let $\rho = \sigma^*$ be the adjoint of σ . Then ρ is a σ -weakly bicontinuous isometric isomorphism of $\mathcal{M}(\pi_1)$ onto $\mathcal{M}(\pi_2)$. If $a, b \in \mathcal{A}$ and $\varphi \in \mathcal{M}_*(\pi_2)$, we have

$$\begin{aligned} \langle \varphi, \rho(\pi_1(a)^* \pi_2(b)) \rangle &= \langle \sigma(\varphi), \pi_1(a)^* \pi_2(b) \rangle \\ &= \langle \sigma(\varphi), \pi_1(a^* b) \rangle \\ &= \langle \pi_2^*(\varphi), a^* b \rangle \\ &= \langle \varphi, \pi_2(a^* b) \rangle \\ &= \langle \varphi, \pi_2(a)^* \pi_2(b) \rangle. \end{aligned}$$

Therefore, ρ is a normal $*$ -isomorphism and $\rho \circ \pi_1 = \pi_2$. □

Since quasiequivalence of von Neumann algebras is specified by a normal $*$ -isomorphism, the two structure theorems we proved for normal $*$ -isomorphisms of von Neumann algebras have implications about quasiequivalent representations.

3.5.12 Proposition. *Let \mathcal{A} be a C^* -algebra, and let π_1 and π_2 be nondegenerate representations of \mathcal{A} . If π_1 and π_2 are quasiequivalent, then there is a representation π , quasiequivalent to π_1 and π_2 , such that both π_1 and π_2 are unitarily equivalent to subrepresentations of π .*

PROOF. Clear from Theorem 3.4.8 and the fact that amplifications, inductions, and spatial equivalences all preserve quasiequivalence. \square

3.5.13 Proposition. *Let \mathcal{A} be a C^* -algebra, and let π_1 and π_2 be nondegenerate representations of \mathcal{A} . If π_1 and π_2 are quasiequivalent, then there are unitarily equivalent subrepresentations σ_1 and σ_2 of π_1 and π_2 respectively that are both quasiequivalent to π_1 and π_2 .*

PROOF. Clear from Theorem 3.4.9 and the fact that amplifications, inductions, and spatial equivalences all preserve quasiequivalence. \square

Another property of representations closely related to quasiequivalence is disjointness.

3.5.14 Definition. Let \mathcal{A} be a C^* -algebra, and let π_1 and π_2 be nondegenerate representations of \mathcal{A} . We say that π_1 and π_2 are *disjoint* if π_1 and π_2 have no nonzero intertwining operators.

3.5.15 Theorem. *Let \mathcal{A} be a C^* -algebra, and let π_1 and π_2 be nondegenerate representations of \mathcal{A} . Then the following are equivalent:*

- (i) π_1 and π_2 are disjoint;
- (ii) π_1 and π_2 have no nonzero unitarily equivalent subrepresentations;
- (iii) $c(\pi_1)$ and $c(\pi_2)$ are orthogonal;
- (iv) $A_{\pi_1} \cap A_{\pi_2} = \{0\}$;
- (v) $(\pi_1 \oplus \pi_2)(\mathcal{A})'' = \pi_1(\mathcal{A})'' \oplus \pi_2(\mathcal{A})''$;
- (vi) $(\pi_1 \oplus \pi_2)(\mathcal{A})' = \pi_1(\mathcal{A})' \oplus \pi_2(\mathcal{A})'$.

PROOF. We will first show (i) \implies (ii). Suppose that π_1 and π_2 are disjoint but there exist nonzero subrepresentations ρ_1 and ρ_2 of π_1 and π_2 that are unitarily equivalent by some unitary $U : \mathcal{H}_{\rho_1} \rightarrow \mathcal{H}_{\rho_2}$. Define an operator $V : \mathcal{H}_{\pi_1} \rightarrow \mathcal{H}_{\pi_2}$ by extending U to vanish on $\mathcal{H}_{\rho_1}^\perp$. Then V is a partial isometry from \mathcal{H}_{π_1} to \mathcal{H}_{π_2} that intertwines π_1 and π_2 , contradicting the disjunction of π_1 and π_2 .

Conversely, suppose that (ii) holds, and let $T : \mathcal{H}_{\pi_1} \rightarrow \mathcal{H}_{\pi_2}$ be a nonzero intertwiner for π_1 and π_2 , i.e.

$$T\pi_1(a) = \pi_2(a)T$$

for all $a \in \mathcal{A}$. Taking adjoints, we have

$$\pi_1(a^*)T^* = T^*\pi_2(a^*).$$

Hence T^* is an intertwiner of π_2 and π_1 . This implies that the positive operator T^*T commutes with $\pi_1(\mathcal{A})$, and thus $|T| = (T^*T)^{1/2} \in \pi_1(\mathcal{A})'$. Let $T = V|T|$ be the polar decomposition of T . Then V implements a unitary equivalence between the subrepresentations of π_1 and π_2 defined by the source and range spaces of V respectively. Since T is nonzero, V is also nonzero.

We will now show (ii) \implies (iii). Suppose that (ii) holds but $c(\pi_1)$ and $c(\pi_2)$ are not orthogonal. Let $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ denote the canonical extensions of π_1 and π_2 to \mathcal{A}^{**} . Let ρ_1 and ρ_2 be the subrepresentations of π_1 and π_2 respectively determined by the projections $\widetilde{\pi}_1(c(\pi_2))$ and $\widetilde{\pi}_2(c(\pi_1))$ in $\pi_1(\mathcal{A})'$ and $\pi_2(\mathcal{A})'$ respectively. Both of these representations have central cover $c(\pi_1)c(\pi_2)$, and thus are quasiequivalent by Theorem 3.5.11. If $c(\pi_1)$ and $c(\pi_2)$ are not orthogonal, then these subrepresentations are nonzero, and thus by Proposition 3.5.13, π_1 and π_2 have nonzero unitarily equivalent subrepresentations, a contradiction.

The equivalence (iii) \iff (iv) is clear, because $A_{\pi_1} = c(\pi_1) \cdot \mathcal{A}^*$ and similarly for A_{π_2} . We will show that (iii) \implies (v). Let $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ denote the canonical extensions of π_1 and π_2 to \mathcal{A}^{**} . If $a \in \mathcal{A}^{**}$, then $(\widetilde{\pi}_1 \oplus \widetilde{\pi}_2)(a) = 0$ if and only if $\widetilde{\pi}_1(a) = 0$ and $\widetilde{\pi}_2(a) = 0$. If π is a representation of \mathcal{A} and $\tilde{\pi}$ is the canonical extension of π to \mathcal{A}^{**} , then the kernel of $\tilde{\pi}$ is $\mathcal{A}^{**}(1 - c(\pi))$. Therefore,

$$\begin{aligned} \ker(\widetilde{\pi}_1 \oplus \widetilde{\pi}_2) &= \ker(\widetilde{\pi}_1) \cap \ker(\widetilde{\pi}_2) \\ &= \mathcal{A}^{**}(1 - c(\pi_1))(1 - c(\pi_2)) \\ &= \mathcal{A}^{**}(1 - (c(\pi_1) + c(\pi_2))). \end{aligned}$$

It follows that $(\widetilde{\pi}_1 \oplus \widetilde{\pi}_2)(\mathcal{A}^{**})$ is $*$ -isomorphic to

$$\mathcal{A}^{**}(c(\pi_1) + c(\pi_2)) = \mathcal{A}^{**}c(\pi_1) \oplus \mathcal{A}^{**}c(\pi_2).$$

Therefore,

$$(\pi_1 \oplus \pi_2)(\mathcal{A})'' = \pi_1(\mathcal{A})'' \oplus \pi_2(\mathcal{A})''.$$

The equivalence (v) \iff (vi) holds by the Double Commutant Theorem, so all that remains to be shown is that (vi) \implies (ii). Suppose that (vi) holds but there exist nonzero subrepresentations ρ_1 and ρ_2 of π_1 and π_2 that are unitarily equivalent by some unitary $U : \mathcal{H}_{\rho_1} \rightarrow \mathcal{H}_{\rho_2}$. Define an operator $V : \mathcal{H}_{\pi_1} \rightarrow \mathcal{H}_{\pi_2}$ by extending U to vanish on $\mathcal{H}_{\rho_1}^\perp$. Then V is a partial isometry

from \mathcal{H}_{π_1} to \mathcal{H}_{π_2} that intertwines π_1 and π_2 . It is easy to verify that $V^*V \in \pi_1(\mathcal{A})'$, $VV^* \in \pi_2(\mathcal{A})'$, and

$$V^*(\pi_2(a)VV^*)V = \pi_1(a)V^*V$$

for all $a \in \mathcal{A}$. Consider V as a bounded operator on $\mathcal{H}_{\pi_1} \oplus \mathcal{H}_{\pi_2}$. Then

$$V = \begin{pmatrix} 0 & 0 \\ V_{21} & 0 \end{pmatrix}$$

for some $V_{21} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. We have

$$\begin{aligned} (\pi_1 \oplus \pi_2)(a)V &= (\pi_1(a) + \pi_2(a))V \\ &= \pi_2(a)V \\ &= V\pi_1(a) \\ &= V(\pi_1(a) + \pi_2(a)) \\ &= V(\pi_1 \oplus \pi_2)(a). \end{aligned}$$

Hence $V \in (\pi_1 \oplus \pi_2)(\mathcal{A})'$. By our assumption of (vi),

$$(\pi_1 \oplus \pi_2)(\mathcal{A})' = \pi_1(\mathcal{A})' \oplus \pi_2(\mathcal{A})' \subseteq \mathcal{B}(\mathcal{H}_{\pi_1}) \oplus \mathcal{B}(\mathcal{H}_{\pi_2}),$$

Therefore, $V = 0$. □

We will now discuss some applications of the theory developed above to direct sums of representations.

3.5.16 Proposition. *Let \mathcal{A} be a C^* -algebra. If $(\pi_i)_{i \in I}$ is a family of mutually disjoint nondegenerate representations of \mathcal{A} and $\pi = \sum_{i \in I} \pi_i$, then*

$$A_\pi = \left\{ u \in \mathcal{B}(G) : u = \sum_{i \in I} u_i \text{ where } u_i \in A_{\pi_i} \text{ and } \sum_{i \in I} \|u_i\| < \infty \right\}.$$

Moreover, if $u \in A_\pi$, then

$$\|u\| = \inf \left\{ \sum_{i \in I} \|u_i\| : u = \sum_{i \in I} u_i \text{ and } u_i \in A_{\pi_i} \right\}.$$

PROOF. Fix $u \in A_\pi$, and sequences $(\xi_n)_{n=1}^\infty$ and $(\eta_n)_{n=1}^\infty$ such that

$$u = \sum_{n=1}^\infty \pi_{\xi_n, \eta_n} \quad \text{and} \quad \sum_{n=1}^\infty \|\xi_n\| \|\eta_n\| < \infty.$$

Since $\mathcal{H}_\pi = \oplus_{i \in I} \mathcal{H}_{\pi_i}$, we can write $\xi_n = (\xi_{n,i})_{i \in I}$ and $\eta_n = (\eta_{n,i})_{i \in I}$ for $i \in I$ such that

$$\|\xi_n\| = \left(\sum_{i \in I} \|\xi_{n,i}\|^2 \right)^{\frac{1}{2}}, \quad \|\eta_n\| = \left(\sum_{i \in I} \|\eta_{n,i}\|^2 \right)^{\frac{1}{2}},$$

and

$$\pi_{\xi_n, \eta_n} = \sum_{i \in I} \pi_{\xi_{n,i}, \eta_{n,i}}.$$

For every $i \in I$, define

$$u_i = \sum_{n=1}^{\infty} \pi_{\xi_{n,i}, \eta_{n,i}}.$$

Then $u_i \in A_\pi$, because

$$\sum_{n=1}^{\infty} \|\xi_{n,i}\| \|\eta_{n,i}\| \leq \sum_{n=1}^{\infty} \|\xi_n\| \|\eta_n\| < \infty.$$

We have

$$\begin{aligned} \sum_{i \in I} \|u_i\| &\leq \sum_{i \in I} \sum_{n=1}^{\infty} \|\xi_{n,i}\| \|\eta_{n,i}\| \\ &= \sum_{n=1}^{\infty} \sum_{i \in I} \|\xi_{n,i}\| \|\eta_{n,i}\| \\ &\leq \sum_{n=1}^{\infty} \left(\sum_{i \in I} \|\xi_{n,i}\|^2 \right)^{\frac{1}{2}} \left(\sum_{i \in I} \|\eta_{n,i}\|^2 \right)^{\frac{1}{2}} \\ &= \sum_{n=1}^{\infty} \|\xi_n\| \|\eta_n\|. \end{aligned}$$

Hence $\sum_{i \in I} u_i \in \mathcal{A}^*$. Since the sums above converge absolutely, we may rearrange them to obtain

$$\sum_{i \in I} u_i = \sum_{i \in I} \sum_{n=1}^{\infty} (\pi_i)_{\xi_{n,i}, \eta_{n,i}} = \sum_{n=1}^{\infty} \sum_{i \in I} (\pi_i)_{\xi_{n,i}, \eta_{n,i}} = \sum_{n=1}^{\infty} \pi_{\xi_n, \eta_n} = u.$$

Conversely, it is clear that any sum of the form $\sum_{i \in I} u_i$, where $u_i \in A_{\pi_i}$ and $\sum_{i \in I} \|u_i\| < \infty$, is also in A_π .

The norm formula follows from the above calculations and the fact that A_π is the quotient of $\mathcal{H} \otimes^y \overline{\mathcal{H}}$, where $\mathcal{H} = \oplus_{i \in I} \mathcal{H}_{\pi_i}$. \square

3.5.17 Proposition. *Let \mathcal{A} be a C^* -algebra. If π_1 and π_2 are disjoint nondegenerate representations of \mathcal{A} , then $A_{\pi_1 \oplus \pi_2} = A_{\pi_2} \oplus_1 A_{\pi_1}$, where \oplus_1 means a direct sum in the ℓ^1 sense.*

PROOF. Since π_1 and π_2 are disjoint, by Theorem 3.5.15 we have that $A_{\pi_1} \cap A_{\pi_2} = \emptyset$. Since A_{π_1} and A_{π_2} are both closed subspaces of \mathcal{A}^{**} , the conclusion follows from Proposition 3.5.16. \square

3.5.18 Proposition. *Let \mathcal{A} be a C^* -algebra. If $(\pi_i)_{i \in I}$ is a family of mutually disjoint nondegenerate representations of \mathcal{A} and $\pi = \sum_{i \in I} \pi_i$, then $A_\pi = \sum_{i \in I} A_{\pi_i}$, where the direct sum is a direct sum in the ℓ^1 sense.*

PROOF. We claim that for every every $j \in I$, π_j and $\oplus_{i \in I, i \neq j} \pi_i$ are disjoint. Indeed, by Theorem 3.5.15, if they are not disjoint, there are nonzero subrepresentations ρ and σ of π_j and $\oplus_{i \in I, i \neq j} \pi_i$ respectively that are unitarily equivalent. Let \mathcal{K} be the subspace of $\oplus_{i \in I, i \neq j} \mathcal{H}_{\pi_i}$ corresponding to σ . Since σ is nonzero, there is an $i \in I$ such that $\sigma \cap \mathcal{H}_{\pi_i} \neq 0$. This defines a nonzero subrepresentation of π_i that is unitarily equivalent to some subrepresentation of ρ , and thus π_j , contradicting our assumption.

Now, by Proposition 3.5.17, we have that $A_\pi = A_{\pi_j} \oplus_1 A_\rho$, where $\rho = \oplus_{i \in I, i \neq j} \pi_i$. The conclusion of the theorem follows by (possibly transfinite) induction. \square

3.5.19 Proposition. *Let \mathcal{A} be a C^* -algebra. If π_1 and π_2 are nondegenerate representations of \mathcal{A} , then $A_{\pi_1} \subseteq A_{\pi_2}$ if and only if π_1 is quasiequivalent to a subrepresentation of π_2 .*

PROOF. Suppose that $A_{\pi_1} \subseteq A_{\pi_2}$. Then $c(\pi_1) \leq c(\pi_2)$. Let $\tilde{\pi}_1$ and $\tilde{\pi}_2$ be the canonical extensions of π_1 and π_2 to \mathcal{A}^{**} . Then $\tilde{\pi}_1(c(\pi_1)) \in \pi_1(\mathcal{A})'$ and $\tilde{\pi}_2(c(\pi_2)) \in \pi_2(\mathcal{A})'$ define quasiequivalent subrepresentations of π_1 and π_2 respectively. However, $\tilde{\pi}_1(c(\pi_1)) = 1$, which implies that π_1 is quasiequivalent to a subrepresentation of π_2 .

Conversely, suppose that π_1 is quasiequivalent to a subrepresentation ρ of π_2 . Then by Theorem 3.5.11, $A_{\pi_1} = A_\rho \subseteq A_{\pi_2}$. \square

Historical Notes

The result establishing the existence of an enveloping von Neumann algebra was first announced by Sherman [She50], but a proof never appeared. The first published proof was given by Takeda [Tak54].

The correspondence between projections in a von Neumann algebra and invariant subspaces of its predual is due to Effros [Eff63].

The concepts of quasiequivalence and disjointness of representations are due to Mackey [Mac53] [Mac76]. The theory of coefficient spaces of representations of C^* -algebras and groups is part of the folklore of the subject, so it is difficult to pin down the exact origin of most of the results given here. Our treatment closely follows Arzac's PhD thesis [Ars76]. However, we use the enveloping C^* -algebra in some places where Arzac quotes results about subrepresentations of an arbitrary representation.

3.6 The Polar Decomposition of Normal Functionals

If φ is a normal linear functional on $\mathcal{B}(\mathcal{H})$, then there exists a trace-class operator $x \in \mathcal{T}(\mathcal{H})$ such that

$$\varphi(a) = \text{Tr}(ax).$$

If $x = v|x|$ is the polar decomposition of x , then $|x| \in \mathcal{T}(\mathcal{H})$, and

$$\varphi(a) = \text{Tr}(av|x|) = v \cdot \text{Tr}(a|x|),$$

so every normal linear functional on $\mathcal{B}(\mathcal{H})$ is the product of a partial isometry and a normal positive functional. If $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra and φ is a normal linear functional on \mathcal{M} , then we can extend φ to $\mathcal{B}(\mathcal{H})$ and get the same decomposition, but there is no reason to believe that the partial isometry v lies in \mathcal{M} . In this section, we develop a similar decomposition relative to an arbitrary von Neumann algebra.

3.6.1 Proposition. *Let \mathcal{M} be a von Neumann algebra, and φ a normal linear functional on \mathcal{M} . If a projection $p \in \mathcal{M}$ satisfies $\|p \cdot \varphi\| = \|\varphi\|$ then $p \cdot \varphi = \varphi$.*

PROOF. We may assume that $\|\varphi\| = 1$. Let $q = 1 - p$. We will show that $q \cdot \varphi = 0$, proving the result. If $q \cdot \varphi \neq 0$ then there exists a $b \in \mathcal{M}$ with $\|b\| = 1$ such that $(q \cdot \varphi)(b) = \delta > 0$. By the Banach-Alaoglu Theorem, there exists an $a \in \mathcal{M}$ with $\|a\| = 1$ such that $(p \cdot \varphi)(a) = \|a\| = 1$. Since

$$\begin{aligned} \|ap + \delta bq\|^2 &= \|(ap + \delta bq)(ap + \delta bq)^*\| \\ &= \|apa^* + \delta^2 bqb^*\| \\ &\leq 1 + \delta^2, \end{aligned}$$

it follows that

$$\|ap + \delta bq\| \leq (1 + \delta^2)^{\frac{1}{2}} < 1 + \delta^2.$$

On the other hand,

$$\varphi(ap + \delta bq) = (p \cdot \varphi)(a) + \delta(q \cdot \varphi)(b) = 1 + \delta^2,$$

which contradicts the fact that $\|\varphi\| = 1$. Therefore, $q \cdot \varphi = 0$. \square

3.6.2 Theorem. *Let \mathcal{M} be a von Neumann algebra, and φ a normal linear functional on \mathcal{M} . Then there exists a unique normal positive linear functional $|\varphi| \in \mathcal{M}_*$ and partial isometry $v \in \mathcal{M}$ such that*

$$\varphi = v \cdot |\varphi|$$

and

$$v^*v = s(|\varphi|).$$

Moreover, $\| |\varphi| \| = \|\varphi\|$, $s_r(\varphi) = s(|\varphi|)$, and $s_l(\varphi) = vv^*$.

PROOF. By the σ -weak compactness of the unit sphere of \mathcal{M} , there is an $a \in \mathcal{M}$ with $\|a\| = 1$ such that $\varphi(a) = \|\varphi\|$. Let $a^* = u|a^*|$ be the polar decomposition of a^* . Then

$$a = |a^*|u^*$$

and we have

$$\|\varphi\| = \varphi(a) = \varphi(|a^*|u^*) = (u^* \cdot \varphi)(|a^*|).$$

Let $|\varphi| = u^* \cdot \varphi$. Then by Proposition 2.12.1, $|\varphi|$ is positive because $0 \leq |a^*| \leq 1$. Let $p = uu^*$. Then we have

$$u \cdot |\varphi| = uu^* \cdot \varphi = p \cdot \varphi$$

and

$$\|\varphi\| = \varphi(a) = \varphi(ap) = (p \cdot \varphi)(a),$$

so $\|p \cdot \varphi\| = \|\varphi\|$. Hence by Proposition 3.6.1, $p \cdot \varphi = \varphi$, i.e. $\varphi = u \cdot |\varphi|$. By the equality

$$u^*u \cdot |\varphi| = u^* \cdot \varphi = |\varphi|$$

we have $u^*u \geq s(|\varphi|)$. Hence $v = us(|\varphi|)$ is a partial isometry in \mathcal{M} with $\varphi = v \cdot |\varphi|$ and $v^*v = s(|\varphi|)$. Since

$$|\varphi| = u^* \cdot \varphi \quad \text{and} \quad \varphi = u \cdot |\varphi|,$$

we have

$$\| |\varphi| \| \leq \|\varphi\| \quad \text{and} \quad \|\varphi\| \leq \| |\varphi| \|.$$

Therefore, $\| |\varphi| \| = \|\varphi\|$, obtaining the desired decomposition of φ .

Now, we will show uniqueness. Let ω be a normal positive functional on \mathcal{M} and $w \in \mathcal{M}$ a partial isometry such that $\varphi = w \cdot \omega$ and $w^*w = s(\omega)$. Then we have

$$\begin{aligned} |\varphi|(1) &= |\varphi|(v^*v) \\ &= \varphi(v^*) \\ &= \omega(v^*w) \\ &= \omega(w^*wv^*w) \\ &= \varphi(w^*wv^*) \\ &= |\varphi|(w^*wv^*v) \\ &= |\varphi|(w^*w), \end{aligned}$$

so $w^*w \geq s(|\varphi|) = v^*v$. Similarly, $v^*v \geq w^*w$, so $v^*v = w^*w$. Let $p = v^*v = w^*w$. Since

$$w^*v = w^*ww^*vv^*v = pw^*vp \in p\mathcal{M}p,$$

there exist $a, b \in (p\mathcal{M}p)_{\text{sa}}$ such that $w^*v = a + ib$. We have

$$|\varphi|(a) + i|\varphi|(b) = |\varphi|(w^*v) = \varphi(w^*) = \omega(w^*w) = \|\omega\| = \| |\varphi| \|.$$

Hence $|\varphi|(a) = \|\varphi\| = |\varphi|(1) = |\varphi|(p)$, so $|\varphi|(p-a) = 0$. Since $p = s(|\varphi|)$, $|\varphi|$ is faithful when restricted to $p\mathcal{M}p$. Therefore, because a is self-adjoint, $p-a = 0$, i.e. $a = p$. Since $\|w^*v\| \leq 1$, $w^*v = p + ib$, and $\|p + ib\| \leq \|p\| + \|ib\| = 1 + \|b\|$, we have $\|b\| = 0$, so $b = 0$. Hence $w^*v = p$, and $v^*w = (w^*v)^* = p$ as well. We have

$$v = vp = vv^*w \quad \text{and} \quad w = wp = ww^*v$$

so that

$$vv^* = vv^*ww^*vv^* \leq ww^* \quad \text{and} \quad ww^* = ww^*vv^*ww^* \leq vv^*,$$

which implies that $s_1(\varphi) = vv^* = ww^*$. Finally, we have

$$v = (vv^*)v = (ww^*)v = wp = ww^*v = w.$$

Therefore,

$$|\varphi| = v^* \cdot \varphi = w^* \cdot \varphi = \omega. \quad \square$$

The expression of φ in Theorem 3.6.2 is called the *polar decomposition* of φ and $|\varphi|$ is called the *absolute value* of φ .

3.6.3 Corollary. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and φ a normal linear functional on \mathcal{M} . Let $\varphi = v \cdot |\varphi|$ be the polar decomposition of φ and $(\eta_n)_{n=1}^{\infty}$ a sequence in $\mathcal{H}^{(\infty)}$ such that*

$$|\varphi|(a) = \sum_{n=1}^{\infty} \langle a\eta_n | \eta_n \rangle.$$

If $\xi_n = v\eta_n$, then

$$\varphi(a) = \sum_{n=1}^{\infty} \langle a\xi_n | \eta_n \rangle \quad \text{and} \quad \|\varphi\| = \|(\xi_n)_{n=1}^{\infty}\| \|(\eta_n)_{n=1}^{\infty}\|.$$

PROOF. We have

$$\varphi(a) = (v \cdot |\varphi|)(a) = |\varphi|(av) = \sum_{n=1}^{\infty} \langle av\eta_n | \eta_n \rangle = \sum_{n=1}^{\infty} \langle a\xi_n | \eta_n \rangle.$$

Clearly, $\|\varphi\| \leq \|(\xi_n)_{n=1}^{\infty}\| \|(\eta_n)_{n=1}^{\infty}\|$, and

$$\|\varphi\| = \|\varphi\| = |\varphi|(1) = \|(\eta_n)_{n=1}^{\infty}\|^2 \geq \|(\xi_n)_{n=1}^{\infty}\| \|(\eta_n)_{n=1}^{\infty}\|,$$

so $\|\varphi\| = \|(\xi_n)_{n=1}^{\infty}\| \|(\eta_n)_{n=1}^{\infty}\|$. □

3.6.4 Corollary. *Let \mathcal{A} be a C^* -algebra. If $\varphi \in \mathcal{A}^*$, then there exists a representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ and $\xi, \eta \in \mathcal{H}$ such that*

$$\varphi(a) = \langle \pi(a)\xi | \eta \rangle \quad \text{and} \quad \|\varphi\| = \|\xi\|\|\eta\|.$$

Moreover, if φ is positive, we may choose $\xi = \eta$.

PROOF. This follows from Corollary 3.6.3 and the fact that if $\varphi = \omega_\xi$, then

$$\|\varphi\| = \varphi(1) = \langle \xi | \xi \rangle = \|\xi\|^2. \quad \square$$

3.6.5 Corollary. *Let G be a locally compact group. Then $B(G)$ is a Banach algebra with respect to pointwise operations and the norm from $C^*(G)^*$. Moreover, $\|u\|_\infty \leq \|u\|_{B(G)}$ for every $u \in B(G)$.*

PROOF. We only need to show that the norm in $B(G)$ is submultiplicative with respect to pointwise multiplication and that the adjoint is an isometry. Fix $\varphi, \psi \in B(G)$. Since $B(G)$ is the predual of $W^*(G)$, by Corollary 3.6.4 there exist representations π and ρ of G , $\xi_1, \eta_1 \in \mathcal{H}_\pi$, and $\xi_2, \eta_2 \in \mathcal{H}_\rho$ such that $\varphi = \pi_{\xi_1, \eta_1}$, $\rho = \pi_{\xi_2, \eta_2}$, $\|\varphi\| = \|\xi_1\|\|\eta_1\|$, and $\|\rho\| = \|\xi_2\|\|\eta_2\|$. Then

$$(\varphi\psi)(s) = \langle (\pi \otimes \rho)(s)\xi_1 \otimes \xi_2 | \eta_1 \otimes \eta_2 \rangle,$$

so

$$\|\varphi\psi\| \leq \|\xi_1\|\|\eta_1\|\|\xi_2\|\|\eta_2\| = \|\varphi\|\|\psi\|.$$

We also have

$$\overline{\varphi}(s) = \overline{\varphi(s)} = \overline{\langle \pi(s)\xi_1 | \eta_1 \rangle} = \langle \overline{\pi}(s)\overline{\xi_1} | \overline{\eta_1} \rangle,$$

so

$$\|\overline{\varphi}\| \leq \|\overline{\xi_1}\|\|\overline{\eta_1}\| = \|\xi_1\|\|\eta_1\| = \|\varphi\|.$$

Since complex conjugation is an involution, this implies that $\|\varphi\| = \|\overline{\varphi}\|$. The inclusion map $\iota : B(G) \rightarrow C_b(G)$ is a $*$ -homomorphism and $C_b(G)$ is a C^* -algebra, so $\|u\|_\infty = \|\iota(u)\|_\infty \leq \|u\|_{B(G)}$. \square

Historical Notes

The polar decomposition of normal linear functionals is due to Sakai [Sak58]. The corollary that $B(G)$ is a Banach algebra is due to Eymard [Eym64].

3.7 The Radon-Nikodym Theorem for Normal Functionals

In this section, we will prove a Radon-Nikodym theorem for normal positive functionals on a von Neumann algebra. We will frequently use vector functionals defined on both an algebra and its commutant. To avoid confusion, we will introduce some notation that allows us to distinguish between the two cases. If $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra and $\xi \in \mathcal{H}$, then let ω_ξ denote the restriction of the vector functional ω_ξ on $\mathcal{B}(\mathcal{H})$ to \mathcal{M} and ω'_ξ denote the restriction of ω_ξ to \mathcal{M}' .

We will first prove a lemma of independent interest.

3.7.1 Proposition. *Let \mathcal{M} be a von Neumann algebra and φ a normal positive functional on \mathcal{M} . If $x \in \mathcal{M}$, then $|x \cdot \varphi| \leq \|x\|\varphi$.*

PROOF. Let $x \cdot \varphi = v \cdot |x \cdot \varphi|$ be the polar decomposition of $x \cdot \varphi$. Then

$$|x \cdot \varphi|(a) = (x \cdot \varphi)(av^*) = \varphi(av^*x)$$

and

$$\varphi(ax) = (x \cdot \varphi)(a) = |x \cdot \varphi|(av)$$

for all $a \in \mathcal{M}$. Let $y = v^*x$. Then $y \cdot \varphi = |x \cdot \varphi|$ is positive, and for all $a \in \mathcal{M}$,

$$\varphi(ay) = (y \cdot \varphi)(a) = (y\varphi)^*(a) = \overline{\varphi(a^*y)} = \overline{\varphi((y^*a)^*)} = \varphi(y^*a).$$

Hence $\varphi(ay^2) = \varphi(y^*ay)$ for all $a \in \mathcal{M}$, and

$$\varphi(a^*ay^2) = \varphi(y^*a^*ay) = \varphi((ay)^*(ay)) \geq 0$$

for all $a \in \mathcal{M}$, so $y^2 \cdot \varphi$ is positive. Similarly,

$$(y^{2n+1} \cdot \varphi)(a) = \varphi(ay^{2n+1}) = \varphi(y^{2n}ay^{2n})$$

for all $a \in \mathcal{M}$ and $n \geq 0$, so $y^{2n+1} \cdot \varphi$ is positive. Therefore, if $a \in \mathcal{M}_+$,

$$\begin{aligned} |x \cdot \varphi|(a) &= \varphi(ay) \\ &= \varphi(a^{1/2}(a^{1/2}y)) \\ &\leq \varphi(a)^{1/2} \varphi(y^*ay)^{1/2} \\ &= \varphi(a)^{1/2} \varphi(ay^2)^{1/2} \\ &\leq \varphi(a)^{1/2} \varphi(a)^{1/4} \varphi(ay^4)^{1/4} \\ &\quad \vdots \\ &\leq \varphi(a)^{1-2^{-n}} \varphi(ay^{2n})^{2^{-n}} \\ &\leq \varphi(a)^{1-2^{-n}} (\|\varphi\| \|a\| \|y\|^{2n})^{2^{-n}}. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ gives

$$|x \cdot \varphi|(a) \leq \|y\| \varphi(a) = \|v^*x\| \varphi(a) \leq \|x\| \varphi(a). \quad \square$$

3.7.2 Theorem. *Let \mathcal{M} be a von Neumann algebra and ψ a normal positive functional on \mathcal{M} . If φ is another normal positive functional on \mathcal{M} such that $\varphi \leq \psi$, then there exists a unique $x \in \mathcal{M}$ such that $0 \leq x \leq 1$, $s(x) \leq s(\psi)$, and $\varphi = x \cdot \psi \cdot x$. Moreover, $s(x) = s(\varphi)$.*

PROOF. By considering the reduced algebra $s(\psi)\mathcal{M}s(\psi)$, we may assume that ψ is faithful. Then, by the normal GNS construction Proposition 3.4.5, we may assume that $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ and $\psi = \omega_\xi$ for some $\xi \in \mathcal{H}$. Since ψ is faithful, ξ is a separating vector for \mathcal{M} . Since $\varphi \leq \psi$, by Proposition 2.11.2 there exists a $t' \in \mathcal{M}'$, $0 \leq t' \leq 1$, such that

$$\varphi(a) = \langle at'\xi | \xi \rangle = \langle as'\xi | s'\xi \rangle,$$

where $s' = (t')^{1/2}$. Let φ' denote the normal linear functional on \mathcal{M}' defined by

$$\varphi'(a) = (s' \cdot \omega'_\xi)(a) = \langle as\xi | \xi \rangle.$$

By Proposition 3.7.1, $|\varphi'| \leq \omega'_\xi$, so there is an $x \in \mathcal{M}$, $0 \leq x \leq 1$, such that $|\varphi'| = x \cdot \omega'_\xi$. Let $\varphi' = v'|\varphi'|$ denote the polar decomposition of φ' . Then, for all $b \in \mathcal{M}'$,

$$\begin{aligned} \langle x\xi | b\xi \rangle &= \langle b^*x\xi | \xi \rangle \\ &= |\varphi'| (b^*) \\ &= \varphi'(b^*(v')^*) \\ &= \langle b^*(v')^*s'\xi | \xi \rangle \\ &= \langle (v')^*s'\xi | b\xi \rangle. \end{aligned}$$

Since ξ is separating for \mathcal{M} , it is cyclic for \mathcal{M}' , so the above implies that $x\xi = (v')^*s'\xi$. Similarly, for all $b \in \mathcal{M}'$,

$$\begin{aligned} \langle v'(v')^*s\xi | b\xi \rangle &= \langle b^*v'(v')^*s'\xi | \xi \rangle \\ &= |\varphi'| (b^*v') \\ &= \varphi'(b^*) \\ &= \langle b^*s'\xi | \xi \rangle \\ &= \langle s'\xi | b\xi \rangle, \end{aligned}$$

so $v'x\xi = v'(v')^*s'\xi = s'\xi$. Then, for every $a \in \mathcal{M}$,

$$\begin{aligned} \varphi(a) &= \langle as'\xi | s'\xi \rangle \\ &= \langle as'\xi | vx\xi \rangle \\ &= \langle a(v')^*s'\xi | x\xi \rangle \\ &= \langle ax\xi | x\xi \rangle \\ &= \langle xax\xi | \xi \rangle \\ &= \psi(xax). \end{aligned}$$

The claim that $s(x) = s(\varphi)$ is clear.

We will now show uniqueness. Suppose $y \in \mathcal{M}$ is such that $0 \leq y \leq 1$, $s(y) \leq s(\psi)$ and

$$\varphi(a) = \psi(yay)$$

for all $a \in \mathcal{A}$. Since $s(y) \leq s(\psi)$, we may continue our convention of only considering the reduced algebra $s(\psi)\mathcal{M}s(\psi)$ represented on the GNS Hilbert space associated with ψ . Since $\psi = \omega_\xi$, for all $a \in \mathcal{A}$ we have

$$\omega_\xi(xax) = \omega_\xi(yay),$$

which implies

$$\|ax\xi\|^2 = \omega_\xi(xa^*ax) = \omega_\xi(ya^*ay) = \|ay\xi\|^2.$$

Hence we can define a partial isometry $u' \in \mathcal{M}'$ by

$$u'(ax\xi) = ay\xi.$$

Consider the normal functionals φ'_x and φ'_y on \mathcal{M}' given by

$$\varphi'_x(b) = \langle bx\xi | \xi \rangle \quad \text{and} \quad \varphi'_y(b) = \langle by\xi | \xi \rangle.$$

These functionals are positive, because the product of two commuting positive elements in a C*-algebra is positive. If $b \in \mathcal{M}'$, then

$$\varphi'_y(b) = \langle bv'y\xi | \xi \rangle = (v' \cdot \varphi'_x)(b),$$

so $\varphi'_y = v' \cdot \varphi'_x$. It is also clear from the definitions of u' and φ'_x that $(u')^*u' = s(\varphi'_x)$. Since both φ'_x and φ'_y are positive, the uniqueness of polar decomposition implies that u' is the identity on the range of $s(\varphi'_x)$, which is clearly $[\mathcal{M}'x\xi]$. In particular,

$$x\xi = u'x\xi = y\xi.$$

Since ξ is a separating vector for \mathcal{M} , this implies that $x = y$. □

We will show a number of results about the vector implementation of normal linear functionals and the relationship between a von Neumann algebra and its commutant. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If $\xi \in \mathcal{H}$, let p_ξ and p'_ξ be the projections onto $[\mathcal{M}'\xi]$ and $[\mathcal{M}\xi]$ respectively. Then $p_\xi \in \mathcal{M}$ and $p'_\xi \in \mathcal{M}'$. If ω_ξ is the restriction of the vector functional ω_ξ on $\mathcal{B}(\mathcal{H})$ to \mathcal{M} and ω'_ξ is the restriction of ω_ξ to \mathcal{M}' , it is easy to see that

$$s(\omega_\xi) = s(\omega_{\xi|\mathcal{M}}) = p_\xi \quad \text{and} \quad s(\omega'_\xi) = s(\omega_{\xi|\mathcal{M}'}) = p'_\xi.$$

Note that ξ is cyclic if and only if $p'_\xi = 1$, and separating if and only if $p_\xi = 1$. For this reason p_ξ and p'_ξ are called *cyclic projections* of \mathcal{M} and \mathcal{M}' respectively. This notation can be slightly confusing, because $p_\xi = 1$ if and only if ξ is separating, not cyclic, but it is standard.

3.7.3 Proposition. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and φ a normal positive functional on \mathcal{M} . If $\xi \in \mathcal{H}$ is such that

$$\varphi \geq \omega_\xi \quad \text{and} \quad s(\varphi) = p_\xi,$$

then there exists an $\eta \in \mathcal{H}$ such that

$$\varphi = \omega_\eta \quad \text{and} \quad p'_\eta = p'_\xi.$$

PROOF. Since $\varphi \geq \omega_\xi$, by Theorem 3.7.2 there exists an $x \in \mathcal{M}$, $0 \leq x \leq 1$, such that

$$\omega_\xi = x \cdot \varphi \cdot x \quad \text{and} \quad s(x) = s(\varphi).$$

Since $s(\omega_\xi) = s(\varphi)$, it follows that $s(x) = s(\varphi) = p_\xi$. By Proposition 2.8.21, there exists an increasing sequence $(e_n)_{n=1}^\infty$ of projections in \mathcal{M} , such that e_n and x commute,

$$xe_n \geq \frac{1}{n}e_n \quad \text{and} \quad e_n \nearrow s(x).$$

Then xe_n is invertible in $e_n\mathcal{M}e_n$, so there exists a positive $a_n \in e_n\mathcal{M}e_n$ such that $xa_n = e_n$. Let $\eta_n = a_n\xi$. Then, for $n \geq m$, we have

$$\begin{aligned} \|\eta_m - \eta_n\|^2 &= \langle (a_m - a_n)\xi \mid (a_m - a_n)\xi \rangle \\ &= \omega_\xi((a_m - a_n)^2) \\ &= \varphi(x(a_m - a_n)^2x) \\ &= \varphi(e_m - e_n). \end{aligned}$$

Since φ is normal and $e_n \rightarrow s(x)$ in the strong operator topology, we have $\varphi(e_n) \rightarrow \varphi(s(x))$. It follows that $(\eta_n)_{n=1}^\infty$ is a Cauchy sequence. Let η be its limit. Then, for every $a \in \mathcal{M}$,

$$\begin{aligned} \varphi(a) &= \varphi(s(x)as(a)) \\ &= \lim_{n \rightarrow \infty} \varphi(e_nae_n) \\ &= \lim_{n \rightarrow \infty} \varphi(xa_naa_n) \\ &= \lim_{n \rightarrow \infty} \langle aa_n\xi \mid a_n\xi \rangle \\ &= \langle a\eta \mid \eta \rangle \\ &= \omega_\eta(a), \end{aligned}$$

so $\varphi = \omega_\eta$. We have $\eta_n \in \mathcal{M}\xi$ for every $n \in \mathbb{N}$, so $\eta \in [\mathcal{M}\xi]$ and $p'_\eta \leq p'_\xi$. Conversely, we have

$$x\eta = \lim_{n \rightarrow \infty} xa_n\xi = \lim_{n \rightarrow \infty} e_n\xi = s(a)\xi = p_\xi(\xi)\xi.$$

Therefore, $\xi \in \mathcal{M}\eta$ and $p'_\xi \leq p'_\eta$. □

3.7.4 Theorem. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and φ a normal positive functional on \mathcal{M} . If $\xi \in \mathcal{H}$ has the property that $s(\varphi) \leq p_\xi$, then there exists an $\eta \in [\mathcal{M}\xi] \cap [\mathcal{M}'\xi]$ such that $\varphi = \omega_\eta$. Moreover, if $s(\varphi) = p_\xi$, then there exists an $\eta \in \mathcal{H}$ such that $\varphi = \omega_\eta$ and $p'_\eta = p'_\xi$.

PROOF. Let $\psi = \varphi + \omega_\xi$. By Proposition 3.7.3, there exists an $\eta_0 \in \mathcal{H}$ such that

$$\psi = \omega_{\eta_0} \quad \text{and} \quad p'_{\eta_0} = p'_\xi.$$

Since $\varphi \leq \psi$, it follows from Theorem 3.7.2 that there exists an $x \in \mathcal{M}$, $0 \leq x \leq 1$, such that

$$\varphi = x \cdot \psi \cdot x \quad \text{and} \quad s(x) \leq s(\psi) = s(\xi).$$

Let $\eta = x\eta_0$. Then for any $a \in \mathcal{M}$, we have

$$\varphi(a) = \omega_{\eta_0}(xax) = \omega_\eta(a),$$

so $\varphi = \omega_\eta$. Since $\eta = x\eta_0 \in \mathcal{M}\eta_0$, it follows that $\eta \in [\mathcal{M}\xi]$. On the other hand, we have

$$\eta = x\eta_0 \in s(x)\mathcal{H} \subseteq p_\xi\mathcal{H} = [\mathcal{M}'\xi].$$

Let us now assume that $s(\varphi) = p_\xi$. Then, it follows that $s(a) = p_{\eta_0}$. By Proposition 2.8.21, there exists a sequence $(e_n)_{n=1}^\infty$ of projections in \mathcal{M} and an increasing sequence $(a_n)_{n=1}^\infty$ of positive elements of \mathcal{M} such that $a_n \in e_n\mathcal{M}e_n$,

$$xa_n = a_nx = e_n \quad \text{and} \quad e_n \nearrow s(a).$$

Let $\eta_n = a_n\eta = a_nx\eta_0 = e_n\eta_0$. Since $\eta_n = a_n\eta \in \mathcal{M}\eta$, we have

$$\eta_0 = p_{\eta_0}(\eta_0) = s(a)(\eta_0) = \lim_{n \rightarrow \infty} e_n\eta_0 \in [\mathcal{M}\eta].$$

Thus, $p'_{\eta_0} \leq p'_\eta$. But $p'_{\eta_0} = p'_\xi$, so this implies that $p'_\xi \leq p'_\eta$. On the other hand, it is obvious that $p'_\eta \leq p'_\xi$. Therefore, $p'_\eta = p'_\xi$. \square

The following corollary is one of the most important results in the theory of von Neumann algebras.

3.7.5 Corollary. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra with a separating vector. If φ is a normal linear functional on \mathcal{M} , then there exist $\xi, \eta \in \mathcal{H}$ such that

$$\varphi = \omega_{\xi, \eta} \quad \text{and} \quad \|\varphi\| = \|\xi\|\|\eta\|.$$

Moreover, if φ is positive, then we may choose $\xi = \eta$, and if φ is positive and faithful and ζ is a separating vector of \mathcal{M} , then we may choose ξ such that $p'_\xi = p'_\zeta$.

PROOF. If ζ is a separating vector for \mathcal{M} , then $p_\zeta = 1$. Thus, the result follows from Theorem 3.7.4, Corollary 3.6.3, and the fact that if $\varphi = \omega_\xi$, then

$$\|\varphi\| = \varphi(1) = \langle \xi | \xi \rangle = \|\xi\|^2. \quad \square$$

If φ is a faithful positive functional on $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$, then the GNS representation of φ is a $*$ -homomorphism and thus normal. By Proposition 3.4.5, this implies that φ is normal as well. If we assume that \mathcal{M} is in the GNS representation with respect to φ , and $\xi \in \mathcal{H}$ is a cyclic vector such that $\varphi(a) = \langle a\xi | \xi \rangle$, then ξ is also separating, because φ is faithful. Conversely, if $\eta \in \mathcal{H}$ is both cyclic and separating, then the linear functional ψ on \mathcal{M} defined by

$$\psi(a) = \langle a\xi | \xi \rangle$$

is a faithful normal positive functional. The following theorem shows that all $*$ -isomorphisms between von Neumann algebras with a cyclic and separating vector are spatial isomorphisms.

3.7.6 Theorem. *Let $\mathcal{M}_1 \subseteq \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{M}_2 \subseteq \mathcal{B}(\mathcal{H}_2)$ be von Neumann algebras, and let $\xi_1 \in \mathcal{H}_1$ and $\xi_2 \in \mathcal{H}_2$ be vectors that are cyclic and separating for \mathcal{M}_1 and \mathcal{M}_2 respectively. If $\rho : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is a $*$ -isomorphism, then there exists a unitary $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that $\rho(a) = UaU^*$.*

PROOF. Define a normal positive functional φ on \mathcal{M}_1 by

$$\varphi(a) = \omega_{\xi_2}(\rho(a)).$$

Then φ is faithful, because ξ_2 is separating for \mathcal{M}_2 and ρ is a $*$ -isomorphism. By Corollary 3.7.5, there exists a vector $\eta_1 \in \mathcal{H}_1$ such that $\varphi = \omega_{\eta_1}$ and $p'_{\eta_1} = p'_{\xi_1}$. Since ξ_1 is cyclic, $p'_{\eta_1} = p'_{\xi_1}$, and η_1 is cyclic as well.

Define a linear map $U_0 : \mathcal{M}_1\eta_1 \rightarrow \mathcal{M}_2\xi_2$ by

$$U_0(a\eta_1) = \rho(a)\xi_2.$$

If $a \in \mathcal{M}_1$, then

$$\|a\eta_1\|^2 = \omega_{\eta_1}(a^*a) = \varphi(a^*a) = \omega_{\xi_2}(\rho(a^*a)) = \|\rho(a)\xi_2\|^2.$$

Hence U_0 is an isometry. Since the vectors η_1 and ξ_2 are cyclic for \mathcal{M}_1 and \mathcal{M}_2 respectively, it follows that U_0 can be extended by uniform continuity to a unitary $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$. If $a, b \in \mathcal{M}_1$, we have

$$\rho(a)\rho(b)\xi_2 = \rho(ab)\xi_2 = U_0ab\eta_1 = (U_0a)(b\eta_1) = U_0aU_0^{-1}\rho(b)\xi_2.$$

Therefore, $\rho(a) = uau^*$ for all $a \in \mathcal{M}_1$. \square

3.7.7 Theorem. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If $\xi, \eta \in \mathcal{H}$, then $p_\xi \preceq p_\eta$ if and only if $p'_\xi \preceq p'_\eta$.*

PROOF. By symmetry, we need only prove the forwards direction. Moreover, if q is a subprojection of p_η , then

$$\overline{\mathcal{M}'q\xi} = \overline{q\mathcal{M}'\xi} = qp\mathcal{H} = q\mathcal{H},$$

so $q = p_{q\xi}$. Similarly, if q' is a subprojection of p'_η , then $q' = p'_{q'\xi}$. Therefore, we need only prove that $p_\xi \sim p_\eta$ implies $p'_\xi \sim p'_\eta$.

Suppose $p_\xi \sim p_\eta$, and let $v \in \mathcal{M}$ be such that $v^*v = p_\xi$ and $vv^* = p_\eta$. Let $\xi_0 = v\xi$. Then

$$\overline{\mathcal{M}\xi_0} = \overline{\mathcal{M}v\xi} \subseteq \overline{\mathcal{M}\xi} = \overline{\mathcal{M}v^*v\xi} \subseteq \overline{\mathcal{M}v\xi} = \overline{\mathcal{M}\xi_0},$$

so $\overline{\mathcal{M}\xi_0} = \overline{\mathcal{M}\xi}$. Thus, $p'_\xi = p'_{\xi_0}$, and we need only prove that $p'_{\xi_0} \sim p'_\eta$. We have

$$\overline{\mathcal{M}'\xi_0} = \overline{\mathcal{M}'v\xi} = \overline{v\mathcal{M}'\xi} = \overline{\mathcal{M}'\eta}.$$

Hence $p_{\xi_0} = p_\eta$. Since p_η is the support projection of ω_η , by Theorem 3.7.4 there exists a $\eta_0 \in \mathcal{H}$ such that $\omega_\eta = \omega_{\eta_0}$ and $p'_{\eta_0} = p'_{\xi_0}$. Define $w_0 : \mathcal{M}\eta_0 \rightarrow \mathcal{M}\eta$ by $w_0a\eta_0 = a\eta$. If $a \in \mathcal{M}$, then

$$\|w_0a\eta_0\|^2 = \|a\eta\|^2 = \omega_\eta(a) = \omega_{\eta_0}(a) = \|a\eta_0\|^2,$$

so w_0 is an isometry. Extend w_0 by uniform continuity to an isometry $w_0 : \overline{\mathcal{M}\eta_0} \rightarrow \overline{\mathcal{M}\eta}$. Extend w_0 to a partial isometry $w : \mathcal{H} \rightarrow \mathcal{H}$ by defining w to be zero on $\overline{\mathcal{M}\eta_0}^\perp$. Clearly, $w \in \mathcal{M}'$,

$$w^*w = p'_{\eta_0} = p'_{\xi_0} = p'_\xi,$$

and $ww^* = p'_\eta$. Therefore, $p'_\xi \sim p'_\eta$. \square

3.7.8 Corollary. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If \mathcal{M} has both a cyclic vector and a separating vector, then \mathcal{M} has a vector that is both cyclic and separating.*

PROOF. Let ξ be a cyclic vector for \mathcal{M} and η a separating vector for \mathcal{M} . Then $p'_\xi = 1$ and $p_\eta = 1$. Thus $p_\xi \preceq p_\eta$ and $p'_\eta \preceq p'_\xi$. By Theorem 3.7.7, $p'_\xi \preceq p'_\eta$, so $p'_\xi \sim p'_\eta$. Let $v \in \mathcal{M}'$ be such that $v^*v = p'_\eta$ and $vv^* = p'_\xi$. Since $p'_\xi = 1$, the range of v must be \mathcal{H} . Let $\xi_0 = v\xi$. Then

$$\overline{\mathcal{M}\xi_0} = \overline{\mathcal{M}v\xi} = \overline{v\mathcal{M}\xi} = \mathcal{H}.$$

Thus, ξ_0 is cyclic for \mathcal{M} . Since $a\xi_0 = av\xi = v a\xi$ for all $a \in \mathcal{M}$, we have that $a\xi_0 = 0$ if and only if $a\xi = 0$. Since ξ is separating, this implies that ξ_0 is separating as well. \square

Historical Notes

The Radon-Nikodym Theorem for normal positive functionals is due to Sakai [Sak65]. The remaining results on implementation of normal positive functionals as vector functionals are essentially due to Murray and von Neumann [MvN37], but their original proofs are much more complicated than the ones given here, relying on nontrivial results from unbounded operator theory. Murray and von Neumann first proved Theorem 3.7.7, the equivalence of $p_\xi \preceq p_\eta$ and $p'_\xi \preceq p'_\eta$, and derived the other results as corollaries of this theorem. In his survey article [Kad58], Kadison sketched proofs of the other results assuming Theorem 3.7.4, the theorem on vector implementation of normal functionals. These proofs are much simpler than the original proofs and are the ones given here. Vowden [Vow69] was the one to finally prove Theorem 3.7.4 directly from Sakai's Radon-Nikodym Theorem.

Chapter 4

The Fourier Algebra

4.1 Definition and Basic Properties

In Section 3.5, we discussed coefficient spaces of representations of Banach $*$ -algebras with a bounded approximate identity. In this section, we will discuss the properties of coefficient spaces of representations of locally compact groups, and in particular, the coefficient space of the left regular representation. First, we will prove some functorial properties of coefficient spaces with respect to continuous homomorphisms of locally compact groups.

If G and H are locally compact groups, $\sigma : H \rightarrow G$ is a continuous homomorphism, and π is a representation of G , then $\pi \circ \sigma$ is a representation of H . If $u = \pi_{\xi, \eta} \in B(G)$, then $u \circ \sigma = (\pi \circ \sigma)_{\xi, \eta} \in B(H)$. The map $j_\sigma : B(G) \rightarrow B(H)$ given by $j_\sigma(u) = u \circ \sigma$ is clearly a linear contraction.

4.1.1 Proposition. *Let G and H be locally compact groups and $\sigma : H \rightarrow G$ a continuous homomorphism. If $j_\sigma : B(G) \rightarrow B(H)$ is the map induced by σ and π is a representation of G , then:*

- (i) $j_\sigma(A_\pi) = A_{\pi \circ \sigma}$;
- (ii) *for all $v \in A_{\pi \circ \sigma}$, there is a $u \in A_\pi$ such that $j_\sigma(u) = v$ and $\|u\|_{B(G)} = \|v\|_{B(H)}$.*

PROOF.

- (i) Clearly, $j_\sigma(F_\pi) \subseteq F_{\pi \circ \sigma}$, so $j_\sigma(A_\pi) \subseteq A_{\pi \circ \sigma}$. Conversely,

$$A_{\pi \circ \sigma} = \left\{ v \in B(H) : v = \sum_{n=1}^{\infty} (\pi \circ \sigma)_{\xi_n, \eta_n} \text{ where } \sum_{n=1}^{\infty} \|\xi_n\| \|\eta_n\| < \infty \right\},$$

which is clearly contained in $j_\sigma(A_\pi)$.

(ii) Fix $v \in A_{\pi \circ \sigma}$. By Corollary 3.6.3 applied to $(\pi \circ \sigma)(G)''$, there exist sequences $(\xi_n)_{n=1}^\infty$ and $(\eta_n)_{n=1}^\infty$ in $\mathcal{H}^{(\infty)}$ such that

$$v = \sum_{n=1}^{\infty} (\pi \circ \sigma)_{\xi_n, \eta_n} \quad \text{and} \quad \|v\|_{B(H)} = \sum_{n=1}^{\infty} \|\xi_n\| \|\eta_n\|.$$

Let

$$u = \sum_{n=1}^{\infty} \pi_{\xi_n, \eta_n}.$$

Then $j(u) = v$, so $\|v\|_{B(H)} \leq \|u\|_{B(G)}$, and

$$\|u\|_{B(G)} \leq \sum_{n=1}^{\infty} \|\xi_n\| \|\eta_n\| = \|v\|_{B(H)}.$$

Therefore, $\|u\|_{B(G)} = \|v\|_{B(H)}$. □

4.1.2 Corollary. *Let G be a locally compact group, π a representation of G , and H a closed subgroup of G . Then $A_{\pi|_H}$ is a closed subspace of $B(H)$, and the restriction map $u \mapsto u|_H$ is a quotient map from A_π onto $A_{\pi|_H}$.*

PROOF. Let $\sigma : H \rightarrow G$ be the inclusion map. Then the restriction map $u \mapsto u|_H$ is simply j_σ . Hence $A_{\pi|_H} = j_\sigma(A_\pi) = A_{\pi \circ \sigma}$ is closed in $B(H)$. The fact that j_σ is a quotient map follows from part (ii) of Proposition 4.1.1. □

Since $B(G)$ has the additional structure of a Banach $*$ -algebra, it is natural to ask which coefficient spaces are subalgebras and ideals of $B(G)$. The following characterization is quite natural.

4.1.3 Proposition. *Let G be a locally compact group and π a representation of G . Then:*

- (i) A_π is a subalgebra of $B(G)$ if and only if $\pi \otimes \pi$ is quasiequivalent to a subrepresentation of π ;
- (ii) A_π is an ideal of $B(G)$ if and only if $\rho \otimes \pi$ is quasiequivalent to a subrepresentation of π for every representation ρ of G .

PROOF.

- (i) We have the following equivalences, using Proposition 3.5.19 for the last one:

$$\begin{aligned} A_\pi \text{ is an algebra} &\iff A_\pi A_\pi \subseteq A_\pi \\ &\iff F_\pi F_\pi = F_{\pi \otimes \pi} \subseteq A_\pi \\ &\iff A_{\pi \otimes \pi} \subseteq A_\pi \\ &\iff \pi \otimes \pi \text{ is quasiequivalent to} \\ &\quad \text{a subrepresentation of } \pi. \end{aligned}$$

(ii) We have the following equivalences, using Proposition 3.5.19 for the last one:

$$\begin{aligned}
A_\pi \text{ is an ideal} &\iff A_\rho A_\pi \subseteq A_\pi \text{ for all } \rho \\
&\iff F_\rho F_\pi = F_{\rho \otimes \pi} \subseteq A_\pi \text{ for all } \rho \\
&\iff A_{\rho \otimes \pi} \subseteq A_\pi \text{ for all } \rho \\
&\iff \rho \otimes \pi \text{ is quasiequivalent to a} \\
&\quad \text{subrepresentation of } \pi \text{ for all } \rho. \quad \square
\end{aligned}$$

We will show that A_λ is an ideal of $B(G)$ by using an even stronger statement, that $\pi \otimes \lambda$ is quasiequivalent to λ itself in a very natural way for every representation π of G .

4.1.4 Proposition (Fell's Absorption Principle). *Let G be a locally compact group and $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ a representation of G . Let α be the dimension of \mathcal{H} if the dimension is infinite, and \aleph_0 otherwise. Then $\pi \otimes \lambda$ is unitarily equivalent to the representation $\lambda^{(\alpha)}$. In particular, $\pi \otimes \lambda$ is quasiequivalent to λ .*

PROOF. Define $U_1 : \mathcal{H} \otimes L^2(G) \rightarrow L^2(G, \mathcal{H})$ by

$$U_1(\xi \otimes f)(s) = f(s)\pi(s^{-1})\xi.$$

Let $(e_i)_{i \in I}$ be an orthonormal basis for \mathcal{H} , and define $U_2 : L^2(G, \mathcal{H}) \rightarrow L^2(G)^{(I)}$ by

$$U_2 f = (\langle f(\cdot) | e_i \rangle)_{i \in I}.$$

Clearly, both U_1 and U_2 are unitaries, so $U = U_2 U_1 : \mathcal{H} \otimes L^2(G) \rightarrow L^2(G)^{(I)}$ is a unitary. For all s and almost all t in G , we have for an elementary tensor $\xi \otimes f$ in $\mathcal{H} \otimes L^2(G)$,

$$\begin{aligned}
U(\pi \otimes \lambda(s))(\xi \otimes f(t)) &= U_2 U_1(\pi(s)\xi \otimes \lambda(s)f(t)) \\
&= U_2 f(s^{-1}t)\pi(t^{-1}s)\xi \\
&= (f(s^{-1}t)\langle \xi | \pi(s^{-1}t)e_i \rangle)_{i \in I} \\
&= \lambda^{(I)}(s)(f(t)\langle \xi | \pi(t)e_i \rangle)_{i \in I} \\
&= \lambda^{(I)}(s)U(\xi \otimes f(t)).
\end{aligned}$$

Hence $\pi \otimes \lambda$ is unitarily equivalent to $\lambda^{(I)}$, and thus quasiequivalent to λ . \square

4.1.5 Corollary. *Let G be a locally compact group. Then A_λ is an ideal of $B(G)$.*

PROOF. Follows from part (ii) of Proposition 4.1.3 and Proposition 4.1.4. \square

4.1.6 Definition. Let G be a locally compact group. The *Fourier algebra* of G is the coefficient space A_λ of the left regular representation.

Recall part (ii) of Examples 2.10.15, which states that if $\xi, \eta \in L^2(G)$, then $\bar{\xi} * \eta^\vee \in A(G)$ and

$$\langle \lambda(s)\xi \mid \eta \rangle = (\bar{\eta} * \xi^\vee)(s).$$

Indeed, the span of such functions is dense in $A(G)$. We will show in Corollary 4.3.17 that every function in $A(G)$ is of this form, but this result is highly nontrivial.

An important basic fact about the Fourier algebra is that it satisfies a property similar to Urysohn's lemma.

4.1.7 Definition. Let \mathcal{A} be a Banach algebra of functions in $C_0(X)$ on a locally compact Hausdorff space X . We say that \mathcal{A} is regular if for every compact subset K of X and open subset U of X such that $K \subseteq U$, there is a $u \in \mathcal{A}$ such that $u(s) = 1$ for all $s \in K$ and $u(s) = 0$ for all $s \in X \setminus U$.

4.1.8 Proposition. *Let G be a locally compact group. Then $A(G)$ is a regular Banach algebra of functions in $C_0(G)$.*

PROOF. Fix a compact subset K of G . Let V be a relatively compact symmetric neighbourhood of the identity such that $KV^2 \subseteq U$, and define

$$u = \frac{1}{m(V)} \chi_{KV} * \chi_V \in A(G).$$

Then, if $s \in G$, we have

$$u(s) = \int \frac{1}{m(V)} \chi_{KV}(t) \chi_V(t^{-1}s) dt = \frac{1}{m(V)} \int \chi_{KV \cap sV}(t) dt = \frac{m(KV \cap sV)}{m(V)},$$

so u clearly has the desired properties. \square

Using this Urysohn property and the fact that $A(G)$ is an ideal of $B(G)$, it is not difficult to show that $A(G)$ is the closure of compactly supported functions in $B(G)$.

4.1.9 Definition. Let \mathcal{A} be a Banach algebra of functions on a locally compact Hausdorff space X . We say that \mathcal{A} is Tauberian if the compactly supported elements of \mathcal{A} are dense in \mathcal{A} .

4.1.10 Theorem. *Let G be a locally compact group. Then $A(G)$ is the closure of $B(G) \cap C_c(G)$ in $B(G)$. In particular, $A(G)$ is Tauberian.*

PROOF. We will first show that $B(G) \cap C_c(G) \subseteq A(G)$. Fix $v \in B(G) \subseteq C_c(G)$ and let K be the support of v . By Proposition 4.1.8, there exists some $u \in A(G)$ such that $u(s) = 1$ for all $s \in K$. Therefore, $v = vu \in A(G)$, as $A(G)$ is an ideal in $B(G)$.

Now we will show that $B(G) \cap C_c(G)$ is dense in $A(G)$. To show this, it suffices to show that it is dense in F_λ . Fix $u \in F_\lambda$, and let $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n \in \mathcal{H}$ be such that

$$u = \sum_{i=1}^k \lambda_{\xi_i, \eta_i} = \sum_{i=1}^k \overline{\eta_i} * \xi_i^\vee.$$

Let $(f_{i,n})_{n=1}^\infty$ and $(g_{i,n})_{n=1}^\infty$ be sequences in $C_c(G)$ such that

$$\lim_{n \rightarrow \infty} \|f_{i,n} - \xi_i\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|g_{i,n} - \eta_i\| = 0.$$

Then it is easy to check that

$$u = \lim_{n \rightarrow \infty} \sum_{i=1}^k \overline{g_{i,n}} * f_{i,n}^\vee$$

and that

$$\sum_{i=1}^k \overline{g_{i,n}} * f_{i,n}^\vee \in C_c(G)$$

for every $n \in \mathbb{N}$. □

4.1.11 Corollary. *Let G be a locally compact group. Then $A(G)$ is a dense subalgebra of $C_0(G)$.*

PROOF. Recall that the $B(G)$ norm dominates the uniform norm. Since $A(G) \cap C_c(G)$ is dense in $A(G)$ by Theorem 4.1.10, it follows that $A(G) \subseteq C_0(G)$. By Proposition 4.1.8, $A(G)$ is regular, so it is a point-separating self-adjoint subalgebra of $C_0(G)$, so it is dense by the Stone-Weirstrass Theorem. □

Since $A(G)$ is a Banach algebra of continuous functions on G that separates the points of G , the spectrum of G at least contains the point evaluation functional for each element of G . We will now show that this is the entirety of the spectrum of $A(G)$.

4.1.12 Definition. Let X be a locally compact Hausdorff space, and \mathcal{A} a Banach algebra of functions in $C_0(X)$. We let \mathcal{A}_c denote the compactly supported elements of \mathcal{A} . If $x \in X$, let

$$I(x) = \{f \in \mathcal{A} : f(x) = 0\} \quad \text{and} \quad J(x) = \{f \in \mathcal{A}_c : x \notin \text{supp}(f)\},$$

We say that x is a *point of local synthesis* if $\mathcal{A}_c \cap I(x)$ is in the closure of $J(x)$.

4.1.13 Proposition. *Let G be a locally compact group. Then every element of G is a point of local synthesis for $A(G)$.*

PROOF. Fix $x \in G$ and $f \in A(G)_c \cap I(x)$. Fix ϵ such that $0 < \epsilon < \|f\|_\infty$. Let

$$W = \{s \in G : \|f_s - f\| \leq \epsilon\}.$$

Then W is a compact subset of G that contains an open neighbourhood V of the identity. Let v be the function on G defined by

$$v(s) = \begin{cases} f(s) & \text{if } s \in xV, \\ 0 & \text{otherwise.} \end{cases}$$

Define $u : G \rightarrow \mathbb{C}$ by

$$u(s) = \frac{1}{m(V)} \chi_V,$$

where $m(V)$ is the Haar measure of V . Then $u \in L^2(G)$ and $\int u(s) ds = 1$. We also have $f - v \in L^2(G)$. Thus,

$$\varphi = (f - v) * u^\vee \in A(G).$$

If $s \in G$, we have

$$\varphi(s) = \int (f - v)(st)u(t) dt.$$

Hence $\varphi(s) = 0$ if and only if $s\text{supp}(u) \subseteq xV$. It follows that $\text{supp}(\varphi)$ is a compact set not containing x . We have

$$f - \varphi = (f - f * u^\vee) + (v * u^\vee).$$

Since $\text{supp}(u) \subseteq W$,

$$\|f - f * u^\vee\| \leq \int \|f - f_s\| u(s) ds \leq \epsilon.$$

Also,

$$\|v * u^\vee\| \leq \|u\|_2 \|v\|_2,$$

but the particular choice of u implies that $\|u\|_2 = m(V)^{-1/2}$. Therefore,

$$\text{dist}(f, J(x)) \leq \epsilon + m(V)^{-\frac{1}{2}} \left(\int_{xV} |f(s)|^2 ds \right)^{\frac{1}{2}}.$$

Since $\epsilon > 0$ is arbitrary, we have $\text{dist}(f, J(x)) = 0$. Therefore, x is a point of local synthesis for $A(G)$. \square

We now define the notion of support for a bounded linear functional on a regular Banach algebra. Note that this notion agrees with the support of a measure in the case of $C_0(X)$.

4.1.14 Definition. Let X be a locally compact Hausdorff space and \mathcal{A} a Banach algebra of functions in $C_0(X)$. If $\varphi \in \mathcal{A}^*$, we define $\text{supp}(\varphi)$, the *support* of φ , by saying that $x \in \text{supp}(\varphi)$ if and only if there exists a neighbourhood U of x such that $\langle u, \varphi \rangle = 0$ for $u \in \mathcal{A}$ such that $\text{supp}(u) \subseteq U$.

It follows from the regularity of \mathcal{A} that $\text{supp}(\varphi)$ is the smallest closed subset E of X such that $\langle u, \varphi \rangle = 0$ for every $u \in \mathcal{A}$ with compact support disjoint from E . Note that if $x \in X$, then $x \in \text{supp}(\varphi)$ if and only if $\langle u, \varphi \rangle = 0$ for every $u \in J(x)$. Also, the statement that $\text{supp}(\varphi) = \emptyset$ if and only if $\varphi = 0$ is equivalent to the Tauberianness of \mathcal{A} .

4.1.15 Proposition. *Let X be a locally compact Hausdorff space and \mathcal{A} a regular Tauberian Banach algebra of functions in $C_0(X)$. If $\varphi \in \sigma(\mathcal{A})$ is nonzero, then $\text{supp}(\varphi) = \{x\}$ for some $x \in X$.*

PROOF. Since \mathcal{A} is Tauberian and φ is nonzero, $\text{supp}(\varphi) \neq \emptyset$. Fix $x \in \text{supp}(\varphi)$. If U is an open neighbourhood of x , by the definition of $\text{supp}(\varphi)$ there exists a $u \in \mathcal{A}$ with compact support such that $\langle u, \varphi \rangle \neq 0$ and $\text{supp}(u) \subseteq U$. By the regularity of \mathcal{A} , there exists a $v \in \mathcal{A}$ such that $v = 1$ on $\text{supp}(\varphi)$ and $v = 0$ off of U . Since φ is multiplicative, we have

$$\langle u, \varphi \rangle = \langle uv, \varphi \rangle = \langle u, \varphi \rangle \langle v, \varphi \rangle,$$

so $\langle v, \varphi \rangle = 1$ and hence $v \cdot \varphi = \varphi$. Therefore, $\text{supp}(\varphi) \subseteq U$. Since U is an arbitrary open neighbourhood of x , it follows that $\text{supp}(\varphi) = \{x\}$. \square

If $x \in X$, we let δ_x denote the linear functional on $C_0(X)$ defined by evaluation at x . We will also use δ_x to denote the restriction of δ_x to any Banach algebra of functions in $C_0(X)$.

4.1.16 Proposition. *Let X be a locally compact Hausdorff space and \mathcal{A} a regular Banach algebra of functions in $C_0(X)$. If x is a point of local synthesis for \mathcal{A} and the support $\varphi \in \mathcal{A}^*$ is $\{x\}$, then $\varphi = \alpha \delta_x$ for some $\alpha \in \mathbb{C}$. Thus, if φ is multiplicative, $\varphi = \delta_x$.*

PROOF. We want to show that $\langle u, \varphi \rangle$ depends only on $u(x)$, in which case it follows that $\varphi = \alpha \delta_x$ for some $\alpha \in \mathbb{C}$. Fix $u, v \in \mathcal{A}_c$ such that $u(x) = v(x)$. Then $u - v \in \mathcal{A}_c \cap I(X)$. Since x is a point of local synthesis for \mathcal{A} , $u - v$ is in the closure of $J(x)$. Since $\text{supp}(\varphi) = \{x\}$, we have $\langle u - v, \varphi \rangle = 0$. Thus, $\langle u, \varphi \rangle = \langle v, \varphi \rangle$, and there exists an $\alpha \in \mathbb{C}$ such that $\varphi = \alpha \delta_x$. If φ is multiplicative, it follows that $\alpha = 1$ and $\varphi = \delta_x$. \square

If \mathcal{A} is a Banach algebra of functions in $C_0(X)$, then the map $x \mapsto \delta_x$ is a continuous embedding of X into the spectrum of \mathcal{A} . Thus, when we say that the spectrum of \mathcal{A} is X , we mean that the topology on the spectrum is the same as the usual topology on X .

4.1.17 Corollary. *Let X be a locally compact Hausdorff space and \mathcal{A} a regular Tauberian Banach algebra of functions in $C_0(X)$ such that every point of X is a point of local synthesis for \mathcal{A} . Then the spectrum of \mathcal{A} is X .*

PROOF. This follows immediately from the combination of Proposition 4.1.15 and Proposition 4.1.16. \square

4.1.18 Corollary (Eymard). *Let G be a locally compact group. Then the spectrum of $A(G)$ is G .*

PROOF. By Proposition 4.1.8 and Theorem 4.1.10, $A(G)$ is a regular Tauberian Banach algebra of functions on $C_0(G)$. By Proposition 4.1.13, every point of G is a point of local synthesis for $A(G)$. Therefore, by Corollary 4.1.17, the spectrum of $A(G)$ is G . \square

Historical Notes

The general results on the coefficient spaces of representations at the beginning of this section are taken from [Ars76], but they are likely folklore. The Fourier algebra was first defined in general by Eymard [Eym64], who proved it is an algebra and an ideal in $B(G)$. Stinespring already defined it and proved it is an algebra in the unimodular case [Sti59], but his methods do not generalize to general locally compact groups without the deep modular theory of von Neumann algebras due to Tomita and Takesaki [Tak70].

Eymard's original proof that $A(G)$ is an ideal in $B(G)$ relies on defining it as the closure of the compactly supported functions in $B(G)$, whereas we defined it as the coefficient space of the left regular representation and used Fell's Absorption Principle. Fell proved the principle that bears his name in [Fel62], but it was known much earlier. It was a well-known fact in the development of the representation theory of finite groups, and in the case of a general locally compact group G , it is an easy consequence of Mackey's subgroup theorem [Mac52].

Eymard proved his theorem on the spectrum of $A(G)$ in [Eym64]. The simplified proof given here is due to Herz [Her73].

4.2 Herz's Restriction Theorem

Let G be a locally compact group and H a closed subgroup of G . If $u \in B(G)$, then there exists a representation $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ of G and $\xi, \eta \in \mathcal{H}$ such that

$$u(s) = \langle \pi(s)\xi \mid \eta \rangle \quad \text{and} \quad \|u\| = \|\xi\|\|\eta\|$$

By restricting π to H , we have

$$u|_H(s) = \langle \pi|_H(s)\xi \mid \eta \rangle,$$

which shows that $u|_H \in B(H)$ and $\|u|_H\| \leq \|\xi\|\|\eta\| = \|u\|$. Since $A(G)$ is the closure of $B(G) \cap C_c(G)$ in $B(G)$ and $A(H)$ is the closure of $B(H) \cap C_c(H)$ in $B(H)$, it follows from the contractivity of the restriction map that if $u \in A(G)$, then $u|_H \in A(H)$.

It is natural to ask whether the restriction of functions in $A(G)$ is *all* of $A(H)$. This is indeed the case. We will give a clever proof of this fact based on an analogue of the Stone-Weierstrass Theorem for $A(H)$ that characterizes $A(H)$ amongst its closed subalgebras.

Let \mathcal{E} be the span of all products $f * g$, where $f, g \in C_c(G)$. Then \mathcal{E} is dense in $A(G)$, and \mathcal{E} is closed under complex conjugation and the \vee -operation.

Recall that $\rho : G \rightarrow \mathcal{U}(L^2(G))$ is the right regular representation of G given by

$$(\rho(g)\xi)(s) = \Delta(g)^{1/2}\xi(sg),$$

or for $f \in L^1(G)$,

$$(\rho_{L^1(G)}(f)\xi)(t) = (\xi * f)(t) = \int \Delta(s)^{1/2}f(s)\xi(ts) ds.$$

4.2.1 Proposition. *Let G be a locally compact group. If $f \in L^1(G)$, then $\rho(f) \in \text{VN}(G)'$.*

PROOF. If $\xi \in L^2(G)$ and $f, g \in L^1(G)$, then by the associativity of convolution, we have

$$\begin{aligned} \lambda(f)\rho(g)\xi &= \lambda(f)(\xi * g) \\ &= f * (\xi * g)(f * \xi) * g \\ &= \rho(g)(f * \xi) \\ &= \rho(g)\lambda(f)\xi. \end{aligned} \quad \square$$

To avoid confusion in this section, we will temporarily suspend our convention of using lowercase letters to refer to elements of von Neumann algebras, and we will generally use T to refer to an element of $\text{VN}(G)$. Also, if $h \in \mathcal{E}$, we will be very careful to distinguish between $Th \in L^2(G)$ and $T \cdot h \in A(G)$.

4.2.2 Proposition. *Let G be a locally compact group. If $T \in \text{VN}(G)$ and $h \in \mathcal{E}$, then $Th \in A(G)$ and $\langle h, T \rangle = (Th^\vee)(e)$.*

PROOF. Consider $h^\vee = f * g$, where $f, g \in C_c(G)$. Then, since ρ commutes with $\text{VN}(G)$,

$$Th^\vee = T(f * g) = T\rho(g)f = \rho(g)Tf = (Tf) * g \in A(G).$$

By linearity, $Th \in A(G)$ for all $h \in \mathcal{E}$. If $h \in \mathcal{E}$ is such that $h^\vee = f * g$, we have $h = \lambda_{f, \overline{g^\vee}}$, and

$$\begin{aligned}
\langle h, T \rangle &= \langle Tf | \overline{g^\vee} \rangle \\
&= \int (Tf)(s) \overline{g(s^{-1})} ds \\
&= ((Tf) * g)(e) \\
&= (T(f * g))(e) \\
&= (Th^\vee)(e). \quad \square
\end{aligned}$$

We require one more simple computation before we prove our theorem.

4.2.3 Proposition. *Let G be a locally compact group. If $f \in L^1(G)$ and $u \in A(G)$, then $\lambda(f) \cdot u = (f * u^\vee)^\vee$.*

PROOF. If $s \in G$, we have

$$\begin{aligned}
(\lambda(f) \cdot u)(s) &= \langle \lambda(f) \cdot u, \lambda(s) \rangle \\
&= \langle u, \lambda(s) \lambda(f) \rangle \\
&= \langle u, \lambda(s f) \rangle \\
&= \int u(t) f(s^{-1}t) dt \\
&= \int f(t) u(st) dt \\
&= \int f(t) u^\vee(t^{-1}s^{-1}) dt \\
&= (f * u^\vee)(s^{-1}) \\
&= (f * u^\vee)^\vee(s). \quad \square
\end{aligned}$$

4.2.4 Theorem. *Let G be a locally compact group. If \mathcal{B} is a point-separating translation-invariant self-adjoint closed subalgebra of $A(G)$ such that for every $s \in G$ there is a $u \in \mathcal{B} \cap C_c(G)$ such that $u(s) \neq 0$, then $\mathcal{B} = A(G)$.*

PROOF. Since \mathcal{B} is closed under translations, it follows that \mathcal{B} is a $VN(G)$ -subbimodule of $A(G)$. (In fact, by Theorem 3.5.6, it follows that \mathcal{B} is the coefficient space of some representation of G .) Suppose that $\mathcal{B} \neq A(G)$. Then, by the Hahn-Banach Theorem, there exists a nonzero $T \in VN(G)$ such that $\langle u, T \rangle = 0$ for all $u \in \mathcal{B}$. Moreover, since \mathcal{B} is a $VN(G)$ -subbimodule of $A(G)$ it follows that

$$\langle u, T \lambda(f) \rangle = \langle \lambda(f) \cdot u, T \rangle = 0$$

for every $u \in \mathcal{B}$ and $f \in L^1(G)$. Therefore, by Proposition 4.2.2 and Proposition 4.2.3, if $w \in A(G) \cap C_c(G)$ and $h \in \mathcal{E} \subseteq L^1(G)$,

$$\begin{aligned} \langle w, T\lambda(h) \rangle &= \langle \lambda(h) \cdot w, T \rangle \\ &= \langle (h * w^\vee)^\vee, T \rangle \\ &= (T(h * w^\vee))(e) \\ &= ((Th) * w^\vee)(e) \\ &= \int (Th)(s)w(s) ds. \end{aligned}$$

In particular, by our choice of T ,

$$\int (Th)(s)w(s) ds = \langle w, T\lambda(h) \rangle = 0.$$

Since $T \neq 0$ and \mathcal{E} is dense in $L^2(G)$, there is $h \in \mathcal{E}$ such that $Th \neq 0$. In particular, there exists some $s \in G$ such that $(Th)(s) \neq 0$. By Proposition 4.2.2, $Th \in A(G)$. Thus, applying the hypotheses in the statement of this theorem, there exists some $u \in \mathcal{B} \cap C_c(G)$ such that $u(s) \neq 0$. If we let uTh denote the pointwise product of u and Th , then $(uTh)(s) \neq 0$, and so $uTh \neq 0$. If $v \in \mathcal{B}$, then $uv \in \mathcal{B} \cap C_c(G)$, so

$$\int u(s)(Th)(s)v(s) ds = \int (Th)(s)u(s)v(s) ds = \langle uv, T\lambda(h) \rangle = 0.$$

However, by the Stone-Weierstrass Theorem, the closure of \mathcal{B} in the uniform topology must be $C_0(G)$, so it follows that

$$\int u(s)(Th)(s)v(s) ds = 0$$

for all $v \in C_0(G)$, which contradicts the fact that $uTh \neq 0$. Therefore, our assumption that $\mathcal{B} \neq A(G)$ is false. \square

4.2.5 Corollary (Herz's Restriction Theorem). *Let G be a locally compact group and H a closed subgroup of G . Then the restriction map from $A(G)$ to $A(H)$ is a contractive surjective homomorphism. Moreover, if $v \in A(H)$, there exists a $u \in A(G)$ such that $u|_H = v$ and $\|u\|_{A(G)} = \|v\|_{A(H)}$.*

PROOF. Let $\mathcal{B} = A(G)|_H \subseteq A(H)$. Clearly, \mathcal{B} is a translation-invariant closed subalgebra of $A(H)$, and the restriction map is a quotient map by Corollary 4.1.2. Hence, we need only show that the restriction map is surjective. If $u \in A(G) \cap C_c(G)$, then $u|_H \in A(G) \cap C_c(G)$, so surjectivity follows from Theorem 4.2.4. \square

Herz's Restriction Theorem has an important corollary that allows us to represent $VN(H)$ naturally as a subalgebra of $VN(G)$.

4.2.6 Corollary. *Let G be a locally compact group and H a closed subgroup of G . Let $\text{VN}_H(G)$ be the σ -weakly closed span of $\lambda_G(H)$. Then there is a $*$ -isomorphism $\Phi : \text{VN}(H) \rightarrow \text{VN}_H(G)$ such that $\Phi(\lambda_G(s)) = \lambda_H(s)$ for all $s \in H$.*

PROOF. By Corollary 4.2.5,

$$A_{\lambda_{G|H}} = A(G)|_H = A(H) = A_{\lambda_H},$$

so by Theorem 3.5.11, $\lambda_{G|H}$ and λ_H are quasiequivalent. Therefore, there exists an intertwining $*$ -isomorphism $\Phi : \mathcal{M}(\lambda_H) \rightarrow \mathcal{M}(\lambda_{G|H})$. Our conclusion follows immediately, because $\mathcal{M}(\lambda_H) = \text{VN}(H)$ and $\mathcal{M}(\lambda_{G|H}) = \text{VN}_H(G)$. \square

Historical Notes

Herz proved his restriction theorem in [Her70]. The clever proof we give here is due to Arzac [Ars76].

4.3 Standard Form and the Fourier Algebra

The prototypical element of $A(G)$ is of the form $\lambda_{\xi,\eta}$, for $\xi, \eta \in L^2(G)$. We would like to show that every element of $A(G)$ is of this form. Since $A(G)$ is the space of normal linear functionals on $\text{VN}(G)$, the natural way to approach this problem is to examine when a normal linear functional on a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is of the form

$$\omega_{\xi,\eta}(a) = \langle a\xi | \eta \rangle$$

for some $\xi, \eta \in \mathcal{H}$. Unfortunately, this doesn't always happen, even in simple cases like $\mathcal{B}(\mathcal{H})$. If every normal state on \mathcal{M} is a vector functional, then the σ -weak topology, which is generated by the normal linear functionals, is equal to the weak operator topology, which is generated by the vector functionals. This is not true for many von Neumann algebras, including $\mathcal{B}(\mathcal{H})$ for an infinite-dimensional Hilbert space \mathcal{H} .

It might seem strange that $\text{VN}(G)$ always has the nice property that normal linear functionals can be put into vector form, when even a simple and relatively well-behaved von Neumann algebra like $\mathcal{B}(\mathcal{H})$ fails to have it. When \mathcal{M} is $L^\infty(X, \mu)$ for some localizable measure space (X, μ) , every normal linear functional on \mathcal{M} can be put into vector form. Indeed, if φ is a normal positive functional on \mathcal{M} , then by the duality of $L^1(X, \mu)$ and $L^\infty(X, \mu)$ (which is essentially the Radon-Nikodym Theorem) there exists a positive $g \in L^1(X, \mu)$ such that

$$\varphi(f) = \int f(x)g(s) ds.$$

Then $g^{1/2} \in L^2(X, \mu)$, and

$$\varphi(f) = \langle M_f g^{1/2} \mid g^{1/2} \rangle.$$

The general case follows by considering the polar decomposition of φ .

It turns out that there is a general “noncommutative integration theory” for von Neumann algebras that gives a canonical representation of an arbitrary von Neumann algebra for which many analogues of facts from measure theory hold. In this representation, one can show that every normal linear functional is a vector functional, essentially using the same argument as in the case of $L^\infty(X, \mu)$. However, this theory is unfortunately quite involved, so we will not develop it here. Instead, we will use a simpler method that suffices for many cases, including $VN(G)$ for any locally compact group G .

In order to axiomatize this situation, it makes sense to consider a $*$ -algebra \mathcal{A} that is simultaneously a inner product space such that the action of \mathcal{A} on itself by left multiplication defines a nondegenerate bounded representation of \mathcal{A} consisting of bounded operators. The first difficulties arise when describing the interaction between the involution and the inner product. The simplest condition to place on the involution is that it defines a conjugate linear isometry on the Hilbert space, so we will consider that case first.

4.3.1 Definition. A *Hilbert algebra* is a $*$ -algebra \mathcal{A} also equipped with an inner product such that

- (i) if $a \in \mathcal{A}$, then the left multiplication operator $\lambda(a) : \mathcal{A} \rightarrow \mathcal{A}$ defined by $\lambda(a)a = ab$ is bounded;
- (ii) $\langle ab \mid c \rangle = \langle b \mid a^*c \rangle$ for all $a, b, c \in \mathcal{A}$;
- (iii) $\langle a \mid b \rangle = \langle b^* \mid a^* \rangle$ for all $a, b \in \mathcal{A}$;
- (iv) the linear span \mathcal{A}^2 of products of elements in \mathcal{A} is dense in \mathcal{A} ;

Let \mathcal{H} be the completion of \mathcal{A} as an inner product space. Since $\lambda(a)$ is bounded for every $a \in \mathcal{A}$, it extends uniquely to a bounded linear operator on \mathcal{H} , which we will also denote by $\lambda(a)$. The map λ is then a $*$ -homomorphism, and is called the *left regular representation* of \mathcal{A} . By the last axiom of a Hilbert algebra, λ is a nondegenerate representation. The von Neumann algebra $\lambda(\mathcal{A})''$ is called the *left von Neumann algebra* of \mathcal{A} . If $a \in \mathcal{A}$, define a linear map $\rho(a) : \mathcal{A} \rightarrow \mathcal{A}$ by $\rho(a)b = ba$. If $\xi, \eta \in \mathcal{A}$, then

$$\langle \rho(a)\xi \mid \rho(a)\eta \rangle = \langle \xi a \mid \eta a \rangle = \langle a^* \eta^* \mid a^* \xi^* \rangle = \langle \lambda(a^*)\eta^* \mid \lambda(a)^* \xi^* \rangle,$$

so $\|\rho(a)\| = \|\lambda(a^*)\| = \|\lambda(a)\| < \infty$. Hence $\rho(a)$ is bounded and extends uniquely to a bounded linear operator on \mathcal{H} , which we will also denote by $\rho(a)$. The map ρ is then a $*$ -anti-homomorphism, and is called the *right regular representation* of \mathcal{A} . Again using the last axiom of a Hilbert algebra, $\rho(\mathcal{A})$ is a nondegenerate $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$. The von Neumann algebra $\rho(\mathcal{A})''$ is called the *right von Neumann algebra* of \mathcal{A} .

4.3.2 Examples.

- (i) Let (X, μ) be a decomposable measure space. Then $L^2(X, \mu) \cap L^\infty(X, \mu)$ is a Hilbert algebra under pointwise multiplication and conjugation.
- (ii) Let G be a unimodular locally compact group. Then $C_c(G)$ is a Hilbert algebra with the usual convolution product and inner product, and the involution $f^*(s) = \Delta(s^{-1})\overline{f(s^{-1})} = \overline{f(s^{-1})}$.

Unfortunately, if G is not unimodular then $C_c(G)$ need not be a Hilbert algebra. In fact, the standard involution on $C_c(G)$ may not even be bounded. There is a more general notion of a *left Hilbert algebra* that suffices to construct the standard form of an arbitrary von Neumann algebra. Unfortunately, the theory of left Hilbert algebras requires some very delicate arguments involving unbounded operators. Since we are only immediately interested in the case of $VN(G)$ for a locally compact group G , we will develop a simpler generalization of Hilbert algebras, known as quasi-Hilbert algebras.

The basic idea behind the definition of a quasi-Hilbert algebra is best described in the case of the $C_c(G)$. Since the problem with non-unimodular groups arises due to the presence of the modular function in the involution on $C_c(G)$, we simply remove the modular function. The missing modular function has to be incorporated somewhere, so we redefine the inner product on $C_c(G)$ to incorporate the modular function, and define a new involution on $C_c(G)$, which gets introduced during multiplication, to also incorporate the modular function. Unfortunately, this requires redefining a small amount of existing notation. This new notation will only be used in this section, and the reader should be warned that this contradicts our notation elsewhere as well as accepted notation in the literature when dealing with $C_c(G)$ as a left Hilbert algebra rather than a quasi-Hilbert algebra.

Let G be a locally compact group, and define a new inner product on $C_c(G)$ by

$$\langle f | g \rangle = \int \Delta(s)^{-\frac{1}{2}} f(s) \overline{g(s)} ds.$$

Breaking usual conventions, define

$$f^*(s) = \overline{f(s^{-1})} \quad \text{and} \quad f^\#(s) = \Delta(s)^{-\frac{1}{2}} f(s).$$

Let b be the inverse of ${}^\#$, so that $f^b(s) = \Delta(s)^{\frac{1}{2}} f(s)$. We will now axiomatize this situation.

4.3.3 Definition. A *quasi-Hilbert algebra* is a $*$ -algebra \mathcal{A} also equipped with an inner product and a bijective linear mapping $x \mapsto x^\#$, whose inverse will be denoted x^b , such that

- (i) if $a \in \mathcal{A}$, then the left multiplication operator $\lambda(a) : \mathcal{A} \rightarrow \mathcal{A}$ defined by $\lambda(a)b = ab$ is bounded;

- (ii) $\langle a | b \rangle = \langle b^* | a^* \rangle$ for all $a, b \in \mathcal{A}$;
- (iii) $\langle ab | c \rangle = \langle b | a^* \# c \rangle$ for $a, b, c \in \mathcal{A}$;
- (iv) the linear span \mathcal{A}^2 of products of elements in \mathcal{A} is dense in \mathcal{A} ;
- (v) if \mathcal{H} is the completion of \mathcal{A} and $\xi, \eta \in \mathcal{H}$ are such that

$$\langle \xi | ab \rangle = \langle \eta | a^* b^* \rangle$$

for all $a, b \in \mathcal{A}$, then there exists a sequence $(x_n)_{n=1}^{\infty}$ in \mathcal{A} such that $x_n \rightarrow \eta$ and $x_n^{\#} \rightarrow \xi$.

Note that every Hilbert algebra becomes a quasi-Hilbert algebra by defining $\#$ to be the identity function on \mathcal{A} .

4.3.4 Proposition. *Let G be a locally compact group. Then $C_c(G)$ is a quasi-Hilbert algebra.*

PROOF. Fix $f \in C_c(G)$. We want to show that the map $\lambda(f)$ is bounded. If $g, h \in C_c(G)$, then

$$\begin{aligned} |\langle f * g | h \rangle| &= \left| \int \Delta(t)^{-\frac{1}{2}} (f * g)(t) \overline{h(t)} dt \right| \\ &= \left| \int \Delta(t)^{-\frac{1}{2}} \int f(s) g(s^{-1}t) ds \overline{h(t)} dt \right| \\ &\leq \iint |\Delta(t)^{-\frac{1}{2}} f(s) g(s^{-1}t) \overline{h(t)}| ds dt \\ &= \int |f(s)| \int \Delta(t)^{-\frac{1}{2}} |g(s^{-1}t) h(t)| dt ds \\ &= \int |f(s)| \int |\Delta(t)^{-\frac{1}{4}} g(s^{-1}t)| |\Delta(t)^{-\frac{1}{4}} h(t)| dt ds \end{aligned}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} &|\langle f * g | h \rangle| \\ &\leq \int |f(s)| \left(\int \Delta(t)^{-\frac{1}{2}} |g(s^{-1}t)|^2 dt \right)^{\frac{1}{2}} \left(\int \Delta(t)^{-\frac{1}{2}} |h(t)|^2 dt \right)^{\frac{1}{2}} ds \\ &= \int |f(s)| \left(\Delta(s)^{-\frac{1}{2}} \int \Delta(t)^{-\frac{1}{2}} |g(t)|^2 dt \right)^{\frac{1}{2}} \left(\int \Delta(t)^{-\frac{1}{2}} |h(t)|^2 dt \right)^{\frac{1}{2}} ds \\ &= \int \Delta(s)^{-\frac{1}{4}} |f(s)| \left(\int \Delta(t)^{-\frac{1}{2}} |g(t)|^2 dt \right)^{\frac{1}{2}} \left(\int \Delta(t)^{-\frac{1}{2}} |h(t)|^2 dt \right)^{\frac{1}{2}} ds \\ &\leq \int \|g\| \|h\| \Delta(s)^{-\frac{1}{4}} |f(s)| ds \end{aligned}$$

Thus, if C is the maximum value of $|f(t)|$ for $t \in G$, then

$$\|f * g\| \leq \|g\| \int \Delta(s)^{-\frac{1}{4}} |f(s)| dt,$$

where the integral is finite. If $f, g \in C_c(G)$, then

$$\begin{aligned}
\langle f | g \rangle &= \int \Delta(s)^{-\frac{1}{2}} f(s) \overline{g(s)} ds \\
&= \int \Delta(s)^{-\frac{1}{2}} g^*(s^{-1}) \overline{f^*(s^{-1})} ds \\
&= \int \Delta(s^{-1}) \Delta(s^{-1})^{-\frac{1}{2}} g^*(s) \overline{f^*(s)} ds \\
&= \int \Delta(s)^{-1} \Delta(s)^{\frac{1}{2}} g^*(s) \overline{f^*(s)} ds \\
&= \int \Delta(s)^{-\frac{1}{2}} g^*(s) \overline{f^*(s)} ds \\
&= \langle g^* | f^* \rangle.
\end{aligned}$$

We leave the axiom $\langle fg | h \rangle = \langle g | f^{\#} h \rangle$ as an exercise. It is proven in a similar fashion to the previous axioms.

To show that $C_c(G)^2$ is dense in $C_c(G)$, simply take the standard bounded approximate identity for $C_c(G)$.

To prove the last axiom of a Hilbert algebra, fix $\xi, \eta \in \mathcal{H}$ such that

$$\langle \xi | fg \rangle = \langle \eta | f^{\#} g^{\#} \rangle$$

for all $f, g \in C_0(G)$. Clearly, this implies the equality of the linear functionals on \mathcal{H} defined by integration against $\xi(s)$ and $\Delta(s)^{-\frac{1}{2}} \eta(s)$, so there exists a sequence $(h_n)_{n=1}^{\infty}$ in $C_c(G)$ such that $h_n \rightarrow \eta$ and $h_n^{\#} \rightarrow \xi$. \square

Let \mathcal{H} denote the completion of \mathcal{A} with respect to its inner product. Since $\lambda(a)$ is bounded for every $a \in \mathcal{A}$, it extends uniquely to a bounded linear operator on \mathcal{H} , which we will also denote by $\lambda(a)$. It may not be wise to call λ the left regular representation of \mathcal{A} , because it is not actually a $*$ -representation. If $a \in \mathcal{A}$, define a linear map $\rho(a) : \mathcal{A} \rightarrow \mathcal{A}$ by $\rho(a)b = ba$. If $\xi, \eta \in \mathcal{A}$, then

$$\langle \rho(a)\xi | \rho(a)\eta \rangle = \langle \xi a | \eta a \rangle = \langle a^* \eta^* | a^* \xi^* \rangle = \langle \lambda(a^*) \eta^* | \lambda(a^*) \xi^* \rangle,$$

so $\|\rho(a)\| = \|\lambda(a^*)\| = \|\lambda(a)\| < \infty$. Hence $\rho(a)$ is bounded and extends uniquely to a bounded linear operator on \mathcal{H} , which we will also denote by $\rho(a)$.

Since $\langle a | b \rangle = \langle a^* | b^* \rangle$ for all $a, b \in \mathcal{A}$, the involution on \mathcal{A} extends uniquely to a conjugate linear isometry $J : \mathcal{H} \rightarrow \mathcal{H}$. There are a number of useful formulas that can easily be verified:

$$\begin{aligned}
\lambda(\alpha a + \beta b) &= \alpha \lambda(a) + \beta \lambda(b), & \rho(\alpha a + \beta b) &= \alpha \rho(a) + \beta \rho(b), \\
\lambda(a^{\#}) &= \lambda(a)^*, & \rho(a^{\#\#}) &= \rho(a)^*, \\
\lambda(a)\rho(b) &= \rho(b)\lambda(a),
\end{aligned}$$

$$J\lambda(a)J = \rho(a^*) \quad J\rho(a)J = \lambda(a^*).$$

Thus, $\lambda(\mathcal{A})$ and $\rho(\mathcal{A})$ are $*$ -subalgebras of $\mathcal{B}(\mathcal{H})$, which are nondegenerate because of the assumption that \mathcal{A}^2 is dense in \mathcal{A} . We call $\lambda(\mathcal{A})''$ and $\rho(\mathcal{A})''$ the left and right von Neumann algebras of \mathcal{A} respectively. From the above formulas, it is easy to see that $\lambda(\mathcal{A})''$ and $\rho(\mathcal{A})''$ commute. Our goal is to show that $\lambda(\mathcal{A})' = \rho(\mathcal{A})''$ and $\rho(\mathcal{A})' = \lambda(\mathcal{A})''$. However, we will first make a brief aside about the relationship between the left von Neumann algebra of $C_c(G)$ as a quasi-Hilbert algebra and $\text{VN}(G)$.

We have changed the inner product on $C_c(G)$, so we should note that the resulting von Neumann algebra is unitarily equivalent to $\text{VN}(G)$. Indeed, define $U_0 : C_c(G) \rightarrow C_c(G)$ by

$$(U_0 f)(s) = \Delta(s)^{\frac{1}{4}} f(s).$$

Clearly, U_0 is a linear bijection. If $f, g \in C_c(G)$, then

$$\begin{aligned} \langle U_0(f) | U_0(g) \rangle &= \int \Delta(s)^{-\frac{1}{2}} (U_0 f)(s) \overline{(U_0 g)(s)} ds \\ &= \int \Delta(s)^{-\frac{1}{2}} \Delta(s)^{\frac{1}{4}} f(s) \overline{\Delta(s)^{\frac{1}{4}} g(s)} ds \\ &= \int f(s) \overline{g(s)} \\ &= \langle f | g \rangle. \end{aligned}$$

Therefore, U_0 extends by uniform continuity to a unitary $U : L^2(G) \rightarrow \mathcal{H}$, where $L^2(G)$ denotes $C_c(G)$ completed with the usual inner product and \mathcal{H} denotes $C_c(G)$ completed with the inner product given here. There is a subtlety here that limits the use of quasi-Hilbert algebra techniques. While U is the “natural” unitary between $C_c(G)$ with the usual inner product and this new inner product, it is not necessarily an intertwiner for the usual representation of $C_c(G)$ on itself by left multiplication and the representation λ . Let $L : C_c(G) \rightarrow \mathcal{B}(L^2(G))$ denote the usual left regular and right regular representations, given by

$$(L(f)\xi)(t) = (f * \xi)(t) = \int f(s)\xi(s^{-1}t) ds$$

Let $\lambda : C_c(G) \rightarrow \mathcal{B}(\mathcal{H})$ denote the left regular representations given by viewing $C_c(G)$ as a quasi-Hilbert algebra. Then, for every $f \in C_c(G)$ and $\xi \in L^2(G)$, we have

$$\begin{aligned} (\lambda(f)U\xi)(t) &= (f * (U\xi))(t) \\ &= \int f(s)\Delta(s^{-1}t)^{\frac{1}{4}}\xi(s^{-1}t) ds \\ &= \int f(s)\Delta(s)^{-\frac{1}{4}}\Delta(t)^{\frac{1}{4}}\xi(s^{-1}t) ds \\ &= \Delta(t)^{\frac{1}{4}} \int \Delta(s)^{-\frac{1}{4}} f(s)\xi(s^{-1}t) ds \end{aligned}$$

If G is not unimodular, then it is not necessarily the case that

$$f(s) = \Delta(s)^{-1/4} f(s).$$

However, if we let $h(s) = \Delta(s)^{-1/4} f(s) \in C_c(G)$, then we have $\lambda(f)U\xi = UL(h)\xi$. This shows that U takes $L(C_c(G))$ to $\lambda(C_c(G))$ setwise, which is enough to guarantee that the von Neumann algebras $\lambda(C_c(G))''$ and $L(C_c(G))''$ are spatially equivalent. Thus, $\lambda(C_c(G))''$ is spatially equivalent to $\text{VN}(G)$.

4.3.5 Definition. An element $a \in \mathcal{H}$ is said to be left bounded (resp. right bounded) if there exists an operator $\lambda(a)$ (resp. $\rho(a)$) such that $\lambda(a)x = \rho(x)a$ (resp. $\rho(a)x = \lambda(x)a$) for all $x \in \mathcal{A}$. We let \mathfrak{m} denote the set of $\lambda(a)$, where a is a left bounded element of \mathcal{H} , and let \mathfrak{n} denote the $\rho(a)$, where a is a right bounded elements of \mathcal{H} .

The elements of \mathcal{A} are both left and right bounded, and the notation $\lambda(a)$ and $\rho(a)$ is consistent with the above when $a \in \mathcal{A}$. Moreover, the equality $\lambda(a)x = \rho(x)a$ shows that if we let $\rho(x)$ converge weakly to the identity, then $a \in \overline{\lambda(a)\mathcal{H}}$, and similarly with λ and ρ exchanged. This implies that the maps $a \rightarrow \lambda(a)$ and $a \rightarrow \rho(a)$ are injective.

4.3.6 Proposition. *If a is left bounded and $T \in \rho(\mathcal{A})'$, then Ta is left-bounded, $T\lambda(a) = \lambda(Ta)$. Hence the set \mathfrak{m} forms a right ideal of $\lambda(\mathcal{A})''$. Similarly, if a is right bounded and $T \in \lambda(\mathcal{A})'$, then Ta is right bounded and $T\rho(a) = \rho(Ta)$, so the set \mathfrak{n} forms a left ideal of $\rho(\mathcal{A})''$.*

PROOF. Let a be left bounded and fix $x, y \in \mathcal{A}$. We have that

$$\lambda(a)\rho(x)y = \lambda(a)(yx) = \rho(yx)a = \rho(x)\rho(y)a = \rho(x)\lambda(a)y,$$

so that $\lambda(a)$ commutes with the $\rho(x)$, showing that $\lambda(a) \in \rho(\mathcal{A})'$. If $T \in \rho(\mathcal{A})'$, we have

$$T\lambda(a)x = T\rho(x)a = \rho(x)Ta,$$

and so that Ta is left-bounded and $T\lambda(a) = \lambda(Ta)$. A similar argument applies for the case of right bounded elements. \square

4.3.7 Proposition. *Let $\mathfrak{m}_1 = \mathfrak{m} \cap \mathfrak{m}^*$ and $\mathfrak{n}_1 = \mathfrak{n} \cap \mathfrak{n}^*$. We have*

$$\mathfrak{m}_1'' = \rho(\mathcal{A})' \quad \text{and} \quad \mathfrak{n}_1'' = \lambda(\mathcal{A})'.$$

PROOF. We will only prove the first claim, because the the proof of the second claim is similar. By the previous proposition, Proposition 4.3.6, we have that $\mathfrak{m}_1'' \subseteq \rho(\mathcal{A})'$, so we only need to show that $\rho(\mathcal{A})' \subseteq \mathfrak{m}_1''$. Fix $T \in \rho(\mathcal{A})'$ and $T_1 \in \mathfrak{m}_1'$, and let us show that $TT_1 = T_1T$. The previous proposition, Proposition 4.3.6, implies immediately that for $x, x' \in \mathcal{A}$, we have that $\lambda(x')^*T\lambda(x) \in \mathfrak{m}_1$. Hence

$$\lambda(x')^*T\lambda(x)T_1 = T_1\lambda(x')^*T\lambda(x),$$

and it suffices to let $\lambda(x)$ and $\lambda(x')$ converge weakly to the identity. \square

4.3.8 Proposition. m_1 and n_1 commute.

PROOF. Fix $\lambda(a) \in m_1, \rho(c) \in n_1$. We then have $\lambda(a)^* = \lambda(b)$ and $\rho(c)^* = \rho(d)$, with a and b left-bounded, and c and d right-bounded. For all $x, y \in \mathcal{A}$, we have

$$\begin{aligned}
 \langle a | xy \rangle &= \langle a | \rho(y)x \rangle \\
 &= \langle \rho(y^{\#*})a | x \rangle \\
 &= \langle \lambda(a)y^{\#*} | x \rangle \\
 &= \langle y^{\#*} | \lambda(b)x \rangle \\
 &= \langle y^{\#*} | \rho(x)b \rangle \\
 &= \langle \rho(x^{\#*}y^{\#*} | b \rangle \\
 &= \langle (x^{\#}y^{\#})^* | b \rangle \\
 &= \langle Jb | x^{\#}y^{\#} \rangle.
 \end{aligned}$$

By the last axiom of a quasi-Hilbert algebra, there exists a sequence $(x_n)_{n=1}^{\infty}$ in \mathcal{A} such that $x_n^* \rightarrow b$ and $x_n^{\#} \rightarrow a$. A similar calculation yields the existence of a sequence $(y_n)_{n=1}^{\infty}$ in \mathcal{A} such that $y_n^* \rightarrow c$, $y_n^{\#} \rightarrow d$. Then

$$\begin{aligned}
 \langle \lambda(a)\rho(c)x | y \rangle &= \langle \rho(c)x | \lambda(b)y \rangle \\
 &= \langle \lambda(x)c | \rho(y)b \rangle \\
 &= \lim \langle \lambda(x)y_n^* | \rho(y)x_n^* \rangle \\
 &= \lim \langle xy_n^* | x_n^*y \rangle \\
 &= \lim \langle x_n^{\#}x | yy_n^{\#} \rangle \\
 &= \lim \langle \rho(x)x_n^{\#} | \lambda(y)y_n^{\#} \rangle \\
 &= \langle \rho(x)a | \lambda(y)d \rangle \\
 &= \langle \lambda(a)x | \rho(d)y \rangle \\
 &= \langle \rho(c)\lambda(a)x | y \rangle,
 \end{aligned}$$

and hence $\lambda(a)\rho(c) = \rho(c)\lambda(a)$. □

4.3.9 Theorem. $\lambda(\mathcal{A})' = \rho(\mathcal{A})$ and $\rho(\mathcal{A})' = \lambda(\mathcal{A})$.

PROOF. We already know that $\lambda(\mathcal{A}) \subseteq \rho(\mathcal{A})'$. Moreover,

$$\rho(\mathcal{A})' = m_1'' \subseteq n_1''' = \lambda(\mathcal{A})'' = \lambda(\mathcal{A}),$$

and hence $\lambda(\mathcal{A}) = \rho(\mathcal{A})'$. By the Double Commutant Theorem, it follows that $\rho(\mathcal{A})' = \lambda(\mathcal{A})$. □

The J operator defined above is a conjugate linear isometry of \mathcal{H} that takes $\lambda(\mathcal{A})''$ onto its commutant $\rho(\mathcal{A})''$. We will axiomatize its properties, because of the importance of such operators in von Neumann algebra theory.

4.3.10 Definition. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. A *unitary involution* of \mathcal{M} is a conjugate linear operator $J : \mathcal{H} \rightarrow \mathcal{H}$ with the following properties:

- (i) $\langle J\xi | J\eta \rangle = \langle \eta | \xi \rangle$,
- (ii) $J^2 = 1$,
- (iii) $J\mathcal{M}J = \mathcal{M}'$,
- (iv) $JaJ = a^*$ for all $a \in \mathcal{Z}(\mathcal{M})$.

If \mathcal{M} is equipped with a unitary involution, it is said to be in *standard form*.

Theorem 4.3.9 shows that if \mathcal{A} is a quasi-Hilbert algebra, then $\lambda(\mathcal{A})''$ is in standard form. In particular, since being in standard form is preserved by spatial equivalence, $\text{VN}(G)$ is in standard form.

4.3.11 Proposition. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $z \in \mathcal{M}$ a central projection. Then \mathcal{M} is in standard form if and only if $\mathcal{M}z$ and $\mathcal{M}(1 - z)$ are in standard form.*

PROOF. Suppose that J is a unitary involution of \mathcal{M} . Define $J_1 : z\mathcal{H} \rightarrow z\mathcal{H}$ and $J_2 : (1 - z)\mathcal{H} \rightarrow (1 - z)\mathcal{H}$ by

$$J_1\xi = Jz\xi \quad \text{and} \quad J_2\xi = J(1 - z)\xi.$$

We will show that J_1 is a unitary involution for $\mathcal{M}z$, in which case it follows that J_2 is also a unitary involution for $\mathcal{M}(1 - z)$ by symmetry. We have

$$J_1\xi = Jz\xi = JJzJ\xi = zJ\xi,$$

so the range of J_1 is actually in $z\mathcal{H}$. If $\xi, \eta \in z\mathcal{H}$, then

$$\langle J_1\xi | J_1\eta \rangle = \langle J\xi | J\eta \rangle = \langle \eta | \xi \rangle.$$

If $\xi \in z\mathcal{H}$, then

$$J_1J_1\xi = JzJz\xi = z^*z\xi = z\xi = \xi,$$

so $J_1^2 = 1$. If $a \in \mathcal{M}'$, then $J_1azJ_1 = (JzazJ)|_{z\mathcal{H}}$. Since $(\mathcal{M}z)' = \mathcal{M}'z$, this shows that $J\mathcal{M}J = \mathcal{M}'$. Finally, if $az \in \mathcal{Z}(\mathcal{M}z) = \mathcal{M}z \cap \mathcal{M}'z$, then

$$J_1azJ_1 = JzazJz = JazJz = z^*a^*z = (az)^*.$$

Conversely, suppose that $J_1 : z\mathcal{H} \rightarrow z\mathcal{H}$ and $J_2 : (1 - z)\mathcal{H} \rightarrow (1 - z)\mathcal{H}$ are unitary involutions for $\mathcal{M}z$ and $\mathcal{M}(1 - z)$ respectively. Define $J : \mathcal{H} \rightarrow \mathcal{H}$ by

$$J\xi = J_1z\xi + J_2(1 - z)\xi.$$

Arguments similar to those employed in the forward direction show that J is a unitary involution of \mathcal{M} . □

4.3.12 Proposition. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a countably decomposable von Neumann algebra. Then there exists a central projection $z \in \mathcal{M}$ such that $z\mathcal{M}z$ has a cyclic vector and $(1 - z)\mathcal{M}(1 - z)$ has a separating vector.*

PROOF. Using Zorn's Lemma, Let $(\xi_i)_{i \in I}$ be a maximal family of nonzero vectors in \mathcal{H} such that $(p_{\xi_i})_{i \in I}$ and $(p'_{\xi_i})_{i \in I}$ are both orthogonal families of cyclic projections, in \mathcal{M} and \mathcal{M}' respectively. Define projections $p \in \mathcal{M}$ and $p' \in \mathcal{M}'$ by

$$p = \sum_{i \in I} p_{\xi_i} \quad \text{and} \quad p' = \sum_{i \in I} p'_{\xi_i}.$$

Let $q = 1 - p$ and $q' = 1 - p'$. If $qq' \neq 0$, then $q\mathcal{H} \cap q'\mathcal{H}$ contains a nonzero vector η . In this case, $\overline{\mathcal{M}'\eta}$ is orthogonal to each $\overline{\mathcal{M}'\xi_i}$ and $\overline{\mathcal{M}\eta}$ is orthogonal to each $\overline{\mathcal{M}\xi_i}$, contradicting our assumption of the maximality of $(\xi_i)_{i \in I}$. Thus, $qq' = 0$, which implies that $c(q)q' = 0$ and $c(q)c(q') = 0$. This implies that

$$c(q) \leq 1 - q' = p' \quad \text{and} \quad 1 - c(q) \leq p.$$

Since \mathcal{M} is countably decomposable, I must be countable, so we may assume that

$$\sum_{i \in I} \|\xi_i\|^2 < \infty$$

by possibly replacing each ξ_i by some scalar multiple. Let $\xi = \sum_{i \in I} \xi_i$. Then we have

$$\xi_i = p_{\xi_i} \xi = p'_{\xi_i} \xi.$$

Hence both $\overline{\mathcal{M}\xi}$ and $\overline{\mathcal{M}'\xi}$ contain each ξ_i , so that

$$p_{\xi} \geq \sum_{i \in I} p_{\xi_i} = p \quad \text{and} \quad p'_{\xi} \geq \sum_{i \in I} p'_{\xi_i} = p'.$$

But $\xi = p\xi = p'\xi$, so $p_{\xi} \leq p$ and $p'_{\xi} \leq p'$. Therefore, $p = p_{\xi}$ and $p' = p'_{\xi}$. Let $z = c(q)$. Then

$$z = zp' = zp_{\xi} = p'_{z\xi},$$

so $z\xi$ is cyclic for $\mathcal{M}z$. Similarly,

$$1 - z = (1 - z)p = (1 - z)p_{\xi} = p_{(1-z)\xi}.$$

Thus, $(1 - z)\xi$ is cyclic for $\mathcal{M}'(1 - z)$, so it is separating for $\mathcal{M}(1 - z)$. \square

4.3.13 Corollary. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a countably decomposable von Neumann algebra. If \mathcal{M} is in standard form, then \mathcal{M} has a vector that is both cyclic and separating. Hence the identity representation of \mathcal{M} is unitarily equivalent to the GNS representation from a faithful normal state.*

PROOF. By Proposition 4.3.12, there exists a central projection $z \in \mathcal{M}$ such that $z\mathcal{M}z$ has a cyclic vector and $(1-z)\mathcal{M}(1-z)$ has a separating vector. By Proposition 4.3.11, both $z\mathcal{M}z$ and $(1-z)\mathcal{M}(1-z)$ are in standard form. However, if $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra and J is a unitary involution of \mathcal{N} , then $\xi \in \mathcal{H}$ is cyclic if and only if $J\xi$ is separating and $\xi \in \mathcal{K}$ is separating if and only if $J\xi$ is cyclic. Since $z\mathcal{M}z$ and $(1-z)\mathcal{M}(1-z)$ each have both cyclic vectors and separating vectors, by Corollary 3.7.8 they each have a vector that is both cyclic and separating. Taking the sum of the two vectors yields a vector that is both cyclic and separating for \mathcal{M} . \square

In order to reduce to the countably decomposable case, we need to consider induction by the product of a projection in \mathcal{M} and a projection in \mathcal{M}' .

4.3.14 Proposition. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and let p and p' be projections in \mathcal{M} and \mathcal{M}' respectively. Let $q = pp'$, and consider $q\mathcal{M}q$ as an algebra of operators on $q\mathcal{H}$. Then*

- (i) $q\mathcal{M}q$ is a von Neumann algebra;
- (ii) $(q\mathcal{M}q)' = q\mathcal{M}'q$;
- (iii) $\mathcal{Z}(q\mathcal{M}q) = q\mathcal{Z}(\mathcal{M})q$;
- (iv) the map $pap \mapsto qaq$ is a $*$ -isomorphism of $p\mathcal{M}p$ onto $q\mathcal{M}q$ if and only if $c(p) \leq c(p')$.

PROOF. The map $a \mapsto qaq$ is the composition of a reduction $a \mapsto pap$ onto $p\mathcal{M}p$ followed by an induction $pap \mapsto qaq$ of $p\mathcal{M}p$ onto $q\mathcal{M}q$, where q is regarded as an element of $(p\mathcal{M}p)' = p\mathcal{M}'p$. Thus, the claims of the theorem follow from the corresponding claims in Proposition 3.1.7 and Proposition 3.1.8. \square

4.3.15 Corollary. *Let \mathcal{M} be a von Neumann algebra in standard form, J a unitary involution for \mathcal{M} , and $p \in \mathcal{M}$ a projection. Let $p' = JpJ$ and $q = pp'$. Then the induction map $pap \mapsto qaq$ is a $*$ -isomorphism of $p\mathcal{M}p$ onto $q\mathcal{M}q$.*

PROOF. Since J commutes with central projections, we have

$$c(p') = Jc(p)J \geq Jp'J = p.$$

Hence $c(p) \leq c(p')$, so by part (iv) of Proposition 4.3.14, the induction map $pap \mapsto qaq$ is a $*$ -isomorphism. \square

4.3.16 Theorem. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra in standard form. If φ is a normal linear functional on \mathcal{M} , then there exist $\xi, \eta \in \mathcal{H}$ such that*

$$\varphi = \omega_{\xi, \eta} \quad \text{and} \quad \|\varphi\| = \|\xi\| \|\eta\|.$$

Moreover, if φ is positive, then we may choose $\xi = \eta$.

PROOF. We will only prove the case where φ is positive, because the general case follows from Theorem 3.6.2 and Corollary 3.6.3. Let φ be a normal positive functional on \mathcal{M} , and let $p = s(\varphi)$, $p' = JpJ$, and $q = pp'$. By Corollary 4.3.15, the induction map from $p\mathcal{M}p$ to $q\mathcal{M}q$ is an isomorphism, there exists a normal positive functional ψ on $(q\mathcal{M}q)_*$ such that

$$\varphi(a) = \psi(qaq)$$

for all $a \in \mathcal{M}$. Since $q\mathcal{M}q$ is countably decomposable and $J|_{q\mathcal{H}}$ is a unitary involution for $q\mathcal{M}q$, it follows that there exists a $\xi \in q\mathcal{H}$ such that

$$\varphi(a) = \psi(qaq) = \langle qa q \xi \mid \xi \rangle = \langle a q \xi \mid q \xi \rangle = \langle a \xi \mid \xi \rangle$$

for all $a \in \mathcal{A}$. □

4.3.17 Corollary. *Let G be a locally compact group. If $\varphi \in A(G)$, then there exist $\xi, \eta \in L^2(G)$ such that*

$$\varphi = \lambda_{\xi, \eta} \quad \text{and} \quad \|\varphi\| = \|\xi\| \|\eta\|.$$

Moreover, if φ is positive, then we may choose $\xi = \eta$.

PROOF. Follows from Theorem 4.3.9 and Theorem 4.3.16. □

Historical Notes

Hilbert algebras were first introduced by Nakano [Nak50] as a generalization of the H^* -algebras studied by Ambrose [Amb45]. Quasi-Hilbert algebras were introduced by Dixmier [Dix52] for the purpose of extending the commutation theorem for the left and right regular representations to general locally compact groups. The original definition of a quasi-Hilbert algebra is somewhat messier than the one given here, which is taken from [Dix69a]. In some sense, the theory of quasi-Hilbert algebras is just a trick to prove the commutation theorem for general locally compact groups. The theory of quasi-Hilbert algebras doesn't immediately generalize all of the results derived from the Hilbert algebra machinery for unimodular groups, and it is not the case that every von Neumann algebra arises from a quasi-Hilbert algebra.

The problem of developing a true generalization of Hilbert algebras remained open for quite some time, until Tomita produced some preprints in the late 1960s. Unfortunately, these preprints contained serious errors, which led to their rejection by most of the operator algebras community. Takesaki thought that the crucial ideas were recoverable, and after a great deal of work he produced a rigorous account of Tomita's ideas in [Tak70]. Today, this theory is known as the Tomita-Takesaki or modular theory of von Neumann algebras, and it is a cornerstone of the theory of operator algebras.

The proof that every normal linear functional on $VN(G)$ is a vector functional ultimately requires a reduction to the countably decomposable case. Eymard uses structure theory of locally compact groups to make the reduction [Eym64], whereas we use a nice observation of Haagerup [Haa75].

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