Stochastic Renewal Process Model for Condition-Based Maintenance

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Applied Science
in
Civil Engineering

Waterloo, Ontario, Canada, 2009

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This thesis deals with the reliability and maintenance of structures that are damaged by shocks arriving randomly in time. The degradation is modeled as a cumulative stochastic point process. Previous studies mostly adopted expected cost rate criterion for optimizing the maintenance policies, which ignores practical implications of discounting of maintenance cost over the life cycle of the system. Therefore, detailed analysis of expected discounted cost criterion has been done, which provides a more realistic basis for optimizing the maintenance. Examples of maintenance policies combining preventive maintenance with age-based replacement are analyzed. Derivaion for general cases involving preventive maintenance damage level have been discussed. Specific cases are also considered.

Acknowledgements

I am thankful to the Natural Sciences and Engineering Research Council of Canada (NSERC), University Network of Excellence in Nuclear Engineering (UNENE) and Ontario Research Fund for providing the financial support for this study.

I would like to thank Professor Mahesh D. Pandey for giving me the opportunity to study at the University of Waterloo. I am very grateful to former graduate secretary Marguarite Knechtel for help with additional scholarship.

I would like to thank Arun, Budhaditya and especially Anup from the department.

During the last three months of my eleven month stay in Waterloo, I have enjoyed playing poker and cricket with my new friends. I would like to thank Sharad, Karthik, Sarvagya, Darya, Bahar, Hiren, Prashant, Jalaj, Kammy, Avishek, Amod, Hrushikesh, Kush, Aniket, and Tapesh for the wonderful time I had with them.

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Chapter 1

Introduction

1.1 Introduction

One of the important area of interest in reliability theory is the study of various maintenance policies in order to reduce the occurrence of system failure. Over the years various probabilistic models have been introduced, each with its own intrinsic cost structure. One main problem is to compute an expression for the expected long run cost per unit time so that the optimization policy may be determined.

Critical engineering systems in nuclear power plants, such as reactor fuel core and piping systems experience degradation due to stresses and unfavorable environment produced by transients or shocks in the reactor. For example, unplanned shutdown and excursion to poor chemistry conditions cause degradation through corrosion, wear and fatigue processes. To control the risk due to failure of a critical engineering system in the plant, maintenance and replacements of degraded components are routinely performed. Because of uncertainty associated with the occurrence of shocks and damage produced by them, theory and stochastic processes play a key role in estimating reliability and developing maintenance strategies.

The failure of a system or structure occurs when its strength drops below a threshold that is necessary for resisting the applied stresses. Technically the total damage experienced by a system can be modeled as a sum of damage increments produced by individual shocks. To Incorporate uncertainties, shocks are modeled as a stochastic point processes and the damage produced by each shocks is modeled as a positive random variable. In essence, the cumulative damage is modeled as a compound point process (Smith 1958). The theory of stochastic processes and mathematical reliability analysis have been discussed in several monographs (Aven 1999, Barlow and Proschan [19], Cox [27], Feller [28] and Tijms 2003). Mercer [30] developed a stochastic model of degradation as a cumulative process in which shocks arrive as a poisson process. A more generalized formulation of damage using compound renewal process was presented by Morey [32], and he derived distribution of the first passage time or reliability function. Kahle and Wendt [29] modeled shocks as a double-stochastic Poisson process. Ebrahimi [31] proposed a cumulative damage model based on the Poisson shot noise process.

The cumulative damage models are popularly applied to the optimization of maintenance policies using the condition or age based criteria. Nakagawa [16] formulated a preventive maintenance policy was analyzed by Boland and Proschan [20]. Later several other policies were investigated by Nakagawa and co-workers (Nakagawa and Kijima [3], Nakagawa and Yasui 1993 and Qian et al. [2]). Aven [10] presented an efficient method for optimizing the cost rate. An in-depth discussion of inspection and maintenance optimization models is presented in a recent monograph ,Nakagawa [18].

Previous studies (like Qian et al [2]) mostly adopted asymptotic expected cost rate criterion for optimizing maintenance policies. However, in van der Weide et al [9] optimization of discounted cost, which is more pertinent to practical applications is considered.

1.2 Thesis Organization

In chapter (2), two of the most commonly used maintenance policies (Age and Block Replacement policies) are discussed. Relation between the two policies is also discussed. In chapter (3), various events and probabilities associated with stochastic process model are defined. Chapter (4) includes the derivation of asymptotic cost rate for various models in stochastic process model. Examples for the results derived in chapter (4) are given in chapter (5).

1.3 Research Objectives

The main objective of the thesis is to analyze the maintenance methods implemented to optimize the cost rate criteria. This thesis includes the study of common maintenance policies and stochastic process model. In particular, the thesis deals with following topics.

- To analyze the commonly used Age and Block Replacement policies for maintenance of engineering systems and relation between them.
- To analyze the stochastic process model used for optimizing cost rate for maintenance policies based on the assumptions that total damage and time of occurrence of shocks forms a stochastic process.
- To analyze the importance of discount factor in the stochastic process model since discount factor is more often applicable to empirical data.

Chapter 2

Common Maintenance Policies

2.1 Introduction

One of the important area of interest in reliability theory is the study of various maintenance policies in order to reduce the occurrence of system failure. Two policies of particular note are age replacement policy and block replacement policy. For an age replacement maintenance policy, scheduled replacement occur whenever an operating unit reaches age T. For block replacement maintenance policy, scheduled replacements occur every T units of time. Over the years various probabilistic models have been introduced, each with its own intrinsic cost structure. One main problem is to compute an expression for the expected long run cost per unit time so that the optimization policy may be determined. In the past, the calculations for the age replacement policy and the block replacement policy have been treated separately, we will show however that there is a simple and elegant cost relationship between theses two policies.

A general cost mechanism was introduced by Puri and Singh, [26]. However these authors were interested only in optimization results and considered only the age replacement policy. Thomas H. Savits, [1] gave a more general model. Section (2.2) and (2.3) delineate the notions of age and block replacement policies. The expected long run cost per unit of time is also calculated.

2.2 Age Replacement Policy

Failures of units are roughly classified into two failure modes: catastrophic failure in which a unit fails suddenly and completely, and degraded failure in which a unit fails gradually with time by its performance deterioration. In the former, failures during actual operation might sometimes be costly or dangerous. It is an important problem to determine when to replace or preventively maintain a unit before failure. In the latter, maintenance costs of a unit increase with its age, and inversely, its performance suffers some deterioration. In this case, it is also required to measure some performance parameters and to determine when to replace or preventively maintain a unit before it has been degraded into failure state.

In this section, we consider the replacement of a single unit with catastrophic failure mode, where its failure is very serious, and sometimes may incur a heavy loss. Some electronic and electric parts or equipment are typical examples. We introduce a high cost incurred for failure during operation and a low cost incurred for replacement before failure. The replacements after failure and before failure are called corrective replacement and preventive replacement, respectively. In this chapter we assume that the distribution of failure time of a unit is known a priori by investigating its life data, and the planning horizon is infinite. It is also assumed that an operating unit is supplied with unlimited spare units.

The most reasonable replacement policy for a unit is based on its age, which is called age replacement. A unit is always replaced at failure or time T if it has not failed up to time T, where T $(0 < T \leq \infty)$ is constant. In this case, it is appropriate to adopt the expected cost per unit of time as an objective function because the planning horizon is infinite.

2.2.1 Replacement Policy

Consider an age replacement policy in which a unit is replaced at constant time T after its installation or at failure, whichever occurs first. We call a specified time T the planned replacement time which ranges over $(0, \infty]$. The event $\{T = \infty\}$ represents that no replacement is made at all. It is assumed that failures are instantly detected and each failed unit is replaced with a new one, where its replacement time is negligible, and so, a new installed unit begins to operate instantly. Furthermore, suppose that the failure time $X_k, (k = 1, 2, ...)$ of each unit is independent and has an identical distribution $F(t) \equiv Pr\{X_k \leq t\}$ with finite mean μ, where $\bar{F} \equiv 1 - F$, $i.e.\mu \equiv \int_0^\infty \bar{F}(t)\,\mathrm{d}t < \infty$

A new unit is installed at time t = 0. Then, an age replacement procedure generates a renewal process as follows. Let $\{X_k\}_{k=1}^{\infty}$ be the failure times of the successive operating units. Define a new random variable $Z_k \equiv \min\{X_k, T\}(k = 1, 2, ...)$. Then, $\{Z_k\}_{k=1}^{\infty}$ represents the intervals between replacements caused by either failures or planned replacements such as shown in figure (2.1). A sequence of random variables $\{Z_k\}_{k=1}^{\infty}$ is independently and identically distributed, and forms a renewal process, and has an identical distribution

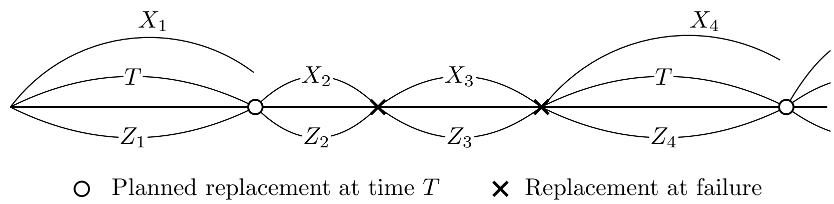


Figure 2.1: Age replacement policy

$$Pr\{Z_k \leq t\} = \begin{cases} F(t), & for \quad t < T \\ 1, & for \quad t \geq T \end{cases}$$

We consider the problem of minimizing the expected cost per unit of time for an infinite time span. Introduce the following costs. Cost c_1 is incurred for each failed unit that is replaced; this includes all costs resulting from a failure and its replacement. Cost c_2 ($< c_1$) is incurred for each non-failed unit that is replaced. Also, let $N_1(t)$ denote the number of failures during $(0, t]$ and $N_2(t)$ denote the number of replacements of non-failed units during $(0, t]$. Then, the expected cost during $(0, t]$ is given by

$$\widehat{C}(t) \equiv c_1 E\{N_1(t)\} + c_2 E\{N_2(t)\}$$

When the planning is infinite, it is appropriate to adopt the expected cost per unit of time $\lim_{t\to\infty} \widehat{C}(t)/t$ as an objective function.

We call the time interval from one replacement to the next replacement as one cycle. Then, the pairs of time and cost on each cycle are independently and identically distributed, and both have finite means. Thus, the expected cost per unit of time for an infinite time span is

$$C(T) = \lim_{t\to\infty} \frac{\widehat{C}(t)}{t} = \frac{\text{Expected cost of one cycle}}{\text{Mean time of one cycle}}$$

We call $C(T)$ the expected cost rate and generally adopt it as the objective function of an optimization problem.

When we set a planned replacement at time $T(0 < T \leq \infty)$ for a unit with failure time ζ, the expected cost of one cycle is

$$c_1 Pr\{X \leq T\} + c_2 Pr\{X > T\} = c_1 F(T) + c_2 \bar{F}(T) \tag{2.1}$$

and the mean time of one cycle is

$$\begin{aligned}
\int_0^T t \mathrm{d} Pr\{X \leq t\} + T\, Pr\{X > T\} &= \int_0^T t \mathrm{d}F(t) + T\bar{F}(T) \\
&= -\int_0^T t\, \mathrm{d}\bar{F}(t) + T\bar{F}(T) \\
&= -\left[t\bar{F}(T)\right]_0^T + \int_0^T \bar{F}(t)\, \mathrm{d}(t) + T\bar{F}(T) \\
&= \int_0^T \bar{F}(t)\mathrm{d}t
\end{aligned} \tag{2.2}$$

Thus, the expected cost rate is

$$C(T) = \frac{c_1 F(T) + c_2 \bar{F}(T)}{\int_0^T \bar{F}(t)\, \mathrm{d}t} \tag{2.3}$$

where c_1 = cost of replacement at failure and c_2 = cost of replacement at planned time T with $c_2 < c_1$.

If $T = \infty$ then the policy corresponds to replacement only at failure, and the expected cost rate is

$$C(\infty) \equiv \lim_{T \to \infty} C(T) = \frac{c_1}{\mu} \tag{2.4}$$

2.3 Block Replacement Policy

If a system consists of a block or group of units, their ages are not observed and only their failures are known, all units may be replaced periodically independently of their ages in use. The policy is called block replacement and is commonly used with complex electronic systems and many electrical parts.

2.3.1 Replacement Policy

A new unit begins to operate at time $t = 0$, and a failed unit is instantly detected and is replaced with a new one. Furthermore, a unit is replaced at periodic times $kT(k = 1, 2, 3...)$ independent of its age. Suppose that each unit has an identical failure distribution $F(t)$ with finite mean μ and $F^n(t)$ $(n = 1, 2...)$ is the n-fold Stieltjes convolution of $F(t)$ with itself ; *i.e.* $F^n(t) \equiv \int_0^t F^{n-1}(t-u)\mathrm{d}F(u)$ $(n = 1, 2, 3...)$ and $F^{(0)}(t) \equiv 1$ for $t > 0$.

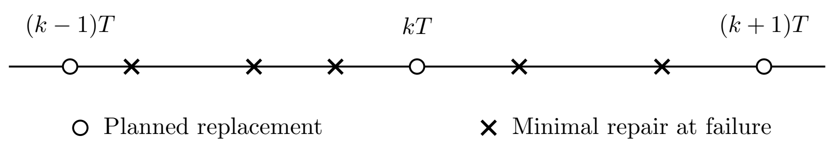


Figure 2.2: Block Replacement Policy

Consider a block consisting of one unit and one cycle with constant time T from the planned replacement to the next one. We consider the block of unit to calculate maintenance cost of one unit and its subsequent replacements. Let c_1 be the cost of replacement for a failed unit and c_2 be the cost of the planned replacement. Then, because the expected number of failed units during one cycle is $M(T), \equiv \sum_{n=1}^{\infty} F^{(n)}(T)$ (Sheldon M. Ross[33]), the expected cost in one cycle is,

$$c_1 E\{N_1(T)\} + c_2 E\{N_2(T)\} = c_1 M(T) + c_2$$

Therefore, the expected cost rate is

$$C(T) = \frac{[c_1 M(T) + c_2]}{T} \tag{2.5}$$

If a unit is replaced at failures, then, $T = \infty$. Taking into account the fact that $\lim_{T\to\infty} \frac{M(T)}{T} = \frac{1}{\mu}$, we conclude that expected cost rate in this case will be $\frac{c_1}{\mu}$

2.4 Cost Relationship between Age and Block Replacement Policies

This section analyzes the cost relationship between block replacement policy and age based replacement policy as proposed by Thomas H. Savits, 1988 [1]. The basic ingredient for this model is a stochastic process $\{R(t), 0 \leq t \leq X\}$. R(t) is interpreted as the operational cost of a unit on line during a time interval [0,t). Hence, we assume that $R(0) = 0$. The random variable X is assumed to be positive and designates an unscheduled replacement ; *i.e.* after ageing X units of time the item is to be replaced by a new identical unit. The cost for such an unplanned replacement is c_1.

The above mechanism does not preclude the possibility of unit breakdowns during the time interval $[0, X)$. In such a case, we assume that the unit is repaired each time and continues functioning at some level of operation. The cost for these repairs would be incorporated into the cost function $R(t)$. Thus X is distinguished as the time of a major breakdown which is not repairable; all other breakdowns are considered to be minor and repairable.

Note that $R(t)$ represents the unit operational cost over the time interval $[0, t)$ as opposed to the time interval $[0, t]$. This choice is just a matter of notational convenience.

It is customary to adopt some type of maintenance policy in order to reduce the occurrence of system failures. Two policies that we consider in this paper are age replacement and block replacement. In the case of an age replacement maintenance policy, a scheduled(or planned) replacement occurs whenever a functioning unit reaches age T. In the case of block replacement policy, a scheduled replacement occurs every T units of time, *i.e.* at the times $T, 2T, \cdots$. In either case, the cost of a planned replacement is c_2.

Thus there are three distinct types of costs built into our model: c_1, the cost for an unplanned replacement; c_2, the cost for a planned replacement due to some maintenance policy; and $R(t)$, the unit operational cost.

2.4.1 Operation Cost for Age Replacement Policy

We now want to analyze the total operational cost for each type of maintained system in more detail. In particular, we shall obtain expressions for the long-run cost per unit time, the so-called infinite-horizon case.

Let $\{R_i(t), 0 \le t \le X_i\}, i = 1, 2, \cdots$, be independent copies of $\{R(t), 0 \le t \le X\}$. $R_i(t)$ denote the operation cost of replaced unit in the i^{th} cycle. For convenience we shall ignore the cost of the initial unit at time $t = 0$.
We first introduce the following notation:

$$Z_k = \min(X_k, T) \tag{2.6}$$

$$X_k = \begin{cases} 0, & \text{if k = 0,} \\ Z_1 + \cdots + Z_k & \text{if k = 1,2,}\cdots \end{cases}$$

and

$$R_i^*(t) = \begin{cases} R_i(t+) & \text{if} \quad 0 \le t \le Z_i \\ R_i(Z_i) + c_1 I_{(X_i<T)} + c_2 I_{(X_i \ge T)} & \text{if} \quad t \ge Z_i. \end{cases}$$

for i=1,2,$\cdots$. Here I_A denotes the indicator function of the set A.

If $K_A(t)$ denotes the total operational cost over $[0, t]$, it is easy to see that for $X_k \le t \le X_{k+1}$, $k = 0, 1, \cdots$,

$$K_A(t) = \sum_{i=1}^{k} R_i^*(Z_i) + R_{k+1}^*(t - X_k) \tag{2.7}$$

Here we use the convention that an empty sum is 0. Since the expression (2.7) also depends on the maintenance parameter T, we sometimes use the more explicit notation $K_A(t;T)$ instead.

We now denote the expected total operational cost $E[K_A(t)]$ by $C_A(t)$ (or $C_A(t;T)$). Thus by the theory of renewal reward processes, we conclude that

$$J_A(T) = \lim_{t\to\infty} \frac{C_A(t)}{t} = \frac{E[R^*(t)]}{E[Z]} \tag{2.8}$$

(provided $E[|R^*(Z)|]$ is finite). Equation (2.8) says that the expected long run cost per unit time is the average cost per cycle divided by the average length of a cycle. It is now convenient to introduce the notation

$$A(T) = E[R^*(Z)] = E[R(T) + c_2; T \le X] + E[R(X) + c_1; X < T] \tag{2.9}$$

We also observe that $A(T) = E[K_A(Z;T)]$.

2.4.2 Operation Cost for Block Replacement Policy

For block replacement policy we use the following notation:

$$\sigma_k = \begin{cases} 0 & \text{if} \quad k = 0 \\ X_1 + \cdots + X_k & \text{if} \quad k = 1, 2, \cdots \end{cases}$$

If $K_B(t)$ denotes the total operational cost over $[0, t]$, one can see that

$$K_B(t) = \begin{cases} R_1(T) + c_2 & \text{if} \quad 0 < T \leq \sigma_1, \\ \sum_{i=1}^{\infty} R_i(X_i) + kc_1 + R_{k+1}(T - \sigma_k) + c_2 & \text{if} \quad \sigma_k < T \leq \sigma_{k+1},\ k = 1, 2 \cdots \end{cases}$$

Now, since we have renewals every T units of time, it follows from the theory of renewal reward processes that the expected long run cost per unit time is the average cost per cycle divided by the average length of a cycle; *i.e.*.

$$J_B(T) = \lim_{t \to \infty} \frac{C_B(t)}{t} = \frac{E[K_B(T)]}{T} \tag{2.10}$$

2.4.3 Comparison between the two policies

Let $B(T) = E[K_B(T)]$

$$\begin{aligned} B(T) &= E[K_B(T)] \\ &= E[R_1(T) + c_2; 0 < T \leq \sigma_1] \\ &\quad + \sum_{k=1}^{\infty} E[\sum_{i=1}^{k} R_i(X_i) + kc_1 + R_{k+1}(T - \sigma_k) + c_2; \sigma_k < T \leq \sigma_{k+1}] \\ &= \sum_{k=1}^{\infty} \sum_{i=1}^{k} E[R_i(X_i) + c_1; \sigma_k < T \leq \sigma_{k+1}] \\ &\quad + \sum_{k=0}^{\infty} E[R_{k+1}(T - \sigma_k) + c_2; \sigma_k < T \leq \sigma_{k+1}] \\ &= \underbrace{\sum_{k=0}^{\infty} E[R_{k+1}(X_{k+1}) + c_1; \sigma_{k+1} < T]}_{\mathrm{I}} \\ &\quad + \underbrace{\sum_{k=0}^{\infty} E[R_{k+1}(T - \sigma_k) + c_2; \sigma_k < T \leq \sigma_{k+1}]}_{\mathrm{II}} \end{aligned} \tag{2.11}$$

$$\begin{aligned} \mathrm{I} &= \sum_{k=0}^{\infty} E[R_{k+1}(X_{k+1}) + c_1; \sigma_{k+1} < T] \\ &= \sum_{k=0}^{\infty} E[R_{k+1}(X_{k+1}) + c_1; \sigma_k + X_{k+1} < T] \\ &= \sum_{k=0}^{\infty} E[E[R_{k+1}(X_{k+1}) + c_1; \sigma_k + X_{k+1} < T]] \end{aligned}$$

The last step follows from the fact that $\{R_i(t), 0 \le t \le X_i\}$ are independent copies of $\{R(t), 0 \le t \le X\}$. Hence

$$\mathrm{I} = \int_{[0,T)} [E[R(X) + c_1; X < T - x] \,\mathrm{d}U(x). \tag{2.12}$$

$$\begin{aligned}
\mathrm{II} &= \sum_{k=0}^{\infty} E[R_{k+1}(T - \sigma_k) + c_2; \sigma_k < T \le \sigma_{k+1}] \\
&= \sum_{k=0}^{\infty} E[R_{k+1}(T - \sigma_k) + c_2; \sigma_k < T, X_{k+1} \ge T - \sigma_k] \\
&\qquad\qquad (\text{Noting that} \quad \sigma_{k+1} = \sigma_k + X_{k+1}) \\
&= \sum_{k=0}^{\infty} E[E[R_{k+1}(T - \sigma_k) + c_2; \sigma_k < T, X_{k+1} \ge T - \sigma_k]]
\end{aligned} \tag{2.13}$$

The last step follows from the fact that $\{R_i(t), 0 \le t \le X_i\}$ are independent copies of $\{R(t), 0 \le t \le \zeta\}$. Hence

$$\mathrm{II} = \int_{[0,T)} E[R(T - x) + c_2; X \ge T - x] \mathrm{d}\, U(x) \tag{2.14}$$

Where $U(x) = \sum_{k=0}^{\infty} P(\sigma_k \le x)$ is the renewal function generated by the independent and identically distributed sequence $X_1, X_2, \cdots$.

Hence, substituting (2.12) and (2.14) into equation (2.11) and recalling equation (2.9) we obtain the main result, which goes as follows

The average cost per cycle in the age replacement maintenance policy case, $A(T)$, and in the bloc replacement maintenance policy case, $B(T)$ are related by the following equation :

$$B(T) = \int_{[0,T)} A(T - x) \,\mathrm{d}U(x) \tag{2.15}$$

Hence if we know A, we can find B. conversely, if we know B, we can find A as follows,

$$\begin{aligned}
& A(T) + \int_0^T B(T-x)\,\mathrm{d}G(x) \qquad (\text{where } G(x) = P(X \le x)) && (2.16)\\
&= A(T) + \int_0^T \left(\int_{[0,T-x)} A(T-x-y)\,\mathrm{d}U(y) \right) \mathrm{d}G(x) \\
&= A(T) + \int_0^T \left(\int_{[0,T-x)} A(T-x-y)\,\mathrm{d}\left(\sum\nolimits_{k=0}^{\infty} P(\sigma_k \le y) \right) \right) \mathrm{d}G(x) \\
&= A(T) + \int_0^T \left(\int_{[0,T-x)} A(T-x-y)\,\mathrm{d}\left(\sum\nolimits_{k=0}^{\infty} G^k(y) \right) \right) \mathrm{d}G(x) \\
&= A(T) + \int_0^T A(T-x)\,\mathrm{d}\left(\sum\nolimits_{k=0}^{\infty} G^{k+1}(x) \right) && (2.17)
\end{aligned}$$

But

$$\begin{aligned}
\sum\nolimits_{k=0}^{\infty} G^k(x) &= 1 + \sum\nolimits_{k=0}^{\infty} G^{k+1}(x) \\
\therefore \qquad \sum\nolimits_{k=0}^{\infty} G^{k+1}(x) &= U(x) - 1 \\
\therefore \qquad \mathrm{d}\left(\sum\nolimits_{k=0}^{\infty} G^{k+1}(x) \right) &= \mathrm{d}U(x) - \delta\mathrm{d}(x) && (2.18)
\end{aligned}$$

From equations 2.17 and 2.18

$$\begin{aligned}
& A(T) + \int_0^T A(T-x)\ (\mathrm{d}U(x) - \delta\mathrm{d}(x)) \\
&= A(T) + \int_0^T A(T-x)\,\mathrm{d}U(x) - \int_0^T A(T-x)\,\delta\mathrm{d}(x) \\
&= A(T) + \int_0^T A(T-x)\,\mathrm{d}U(x) - A(T) \\
&= \int_0^T A(T-x)\,\mathrm{d}U(x) \\
&= B(T) && (2.19)
\end{aligned}$$

From equations 2.16 and 2.19

$$B(T) = A(T) + \int_0^T B(T-x)\,\mathrm{d}G(x) \tag{2.20}$$

Therefore, from Equations (2.15) and (2.20), we conclude that A can be found if B is known and conversely.

2.5 Discount Cost

In the coming chapters we will also include the discount factor in calculating expected cost rate. Discount cost describe the weights placed on costs that occur at different points in time. In economic literature, several mathematical functions give more weight to the present cost than to the future cost.

For better understanding of Discount factor, consider the following explanation. x units of money at time 0 does not have the same value as x units of money at time t. To find the relation between the value of money at two different times we introduce the notion of discount factor. **Discount Factor** is defined as the amount of money needed to be deposited at time 0 so that the value of money at time t is equal to 1.

It is quite natural for the discount factor to depend on the interest rate r. The most well-known discount function is the discount-utility model proposed by Samuelson [24]. According to this model, the discount factor at time t is given by

$$D(t) = e^{-rt}, r > 0 \tag{2.21}$$

This type of discounting is also called exponential discounting. Hence if we want to deposit x units at time t then we need to deposit xe^{-rt} units at time 0

As an alternative to exponential discounting, several hyperbolic functional forms for the discount function have been proposed: Herrnstein [21] and Mazur [22] suggested the function

$$D(t) = (1 + \beta t)^{-1}, \qquad \beta > 0, \tag{2.22}$$

and Loewenstein and Prelec [23] generalized this form to

$$D(t) = (1 + \beta t)^{-\frac{\beta}{\alpha}}, \qquad \alpha,\, \beta\, >\, 0. \tag{2.23}$$

Equations (2.22) and (2.23) are called hyperbolic discounting and generalized hyperbolic discounting, respectively. For hyperbolic discounting, future cost is attached more weight than for exponential discounting and a person's discount rate is declining over time rather than being a constant. A hyperbolic discount function often fits empirical data better than the exponential discount function. Exponential and hyperbolic discounting are special cases of generalized discounting: (2.23) converges to exponential discounting with rate β as $\alpha \to 0$ and (2.23) simplifies to hyperbolic discounting for $\alpha = \beta$.

There is a fundamental difference underlying hyperbolic and exponential discounting. To understand the difference consider the following example.

Suppose one deposits $v_1 = \$100$ in a bank which pays hyperbolic interest rate with $\beta = 0.3$. After 5 years, i.e. when $t_1 = 5$ we get $V_1 = \$250$. Now suppose at this point another deposit of $v_2 = \$200$ is made in the same bank. Now at this point of time we have $v_2 = V_2 = \$200$, $t_2 = 0$ and $V_1 = \$250$, so $V_2 < V_1$. After a gap of 10 years $t_1 = 15$, $t_2 = 10$, $v_1 = \$100$, $v_2 = \$200$. Finally, we get $V_1 = \$550$, and $V_2 = \$800$. So that $V_2 > V_1$. The conclusion is that the value reverses since duration (t) acts multiplicatively on the initial deposit (v). As the years pass, any differences in time of deposit are diminished in relative importance to difference in the amount of initial deposit.

On the other hand, suppose one deposits \$100 in bank which paid with exponential interest rate and after 5 years it grows to \$250. Now at this point, if one deposits \$200 in the same bank then there would be no time, no matter how long, where the values of two deposits reverse. Exponential growth describes compound interest. Interest is calculated based on the amount accumulated at the start of the period, not on the initial deposit.

Hence deviation from exponential discount functions are often considered by economists to be irrational (Strotz[25]). There is indirect evidence (Ainsle, 1974; Logue, 1988; Rachlin & Green, 1972) with non-human and human subjects that subjective delay discount functions do indeed deviate from exponential form. For instance, a pigeon might choose two pellets of food delayed by 14s over on pellet delayed by 10s(both alternatives fixed in time) but reverse its preference after 10s have passed, choosing one pellet immediately over two pellets delayed by 4s. This sort of preference reversal is predicted by hyperbolic discount functions but not by exponential ones.

On the basis of a series of experiments with pigeons as subjects, Mazur(1978) found direct evidence that pigeon's delay discount functions are not exponential but are hyperbolic.

2.6 Conclusion

The expected long run cost for the age based replacement policy and block based replacement policy is found to be $\frac{A(T)}{E[\min(X,T)]}$ and $\frac{B(T)}{T}$ respectively. The other important results gave the relationship between two maintenance policies in a way that the operation cost of one replacement policy could be calculated from the other. The results were as follows

$$\begin{aligned} B(T) &= \int_{[0,T)} A(T-x)\,\mathrm{d}U(x) \\ B(T) &= A(T) + \int_{[0,T)} B(T-x)\,\mathrm{d}G(x) \end{aligned}$$

These results help in calculating the operation cost due to age and block replacement policies, which are most commonly used. One can compare the operation cost due to these replacement policies with the replacement policies discussed in the coming chapters. And depending on the model one can use replacement policies which is cost effective.

Previous studies mostly adopted expected cost rate criterion for optimizing the maintenance policies, which ignores practical implication of discounting of maintenance cost over the life cycle of the system. Therefore, in the last section, concept of discounting is explained. Mainly the two discounting, hyperbolic and exponential discounting are discussed. In the coming chapters expected cost rate will be calculated considering exponential discounting.

Chapter 3

Stochastic Process Model

3.1 Introduction

Chapter (3), (4) and (5) gives in-depth discussion of optimization results derived by van der Weide et al [9]. In deriving the optimization results Qian et al [2] did not consider discounting, whereas discounted cost criterion is the main idea in van der Weide et al [9]. The primary objective of chapter (3) and chapter (4) is to present a conceptually clear derivation of expected discounted cost criterion for optimizing maintenance of systems subject to cumulative damage. The proposed derivation is general and it can be reduced to special cases of homogeneous or non-homogeneous Poisson processes or renewal process.

3.2 Background

Occurrences of shocks are random in time and the damage produced by each shock is also a random variable. The total damage at time t is a sum of damage increments produced by all j shocks occurred up to this time.

Random occurrences of shocks over time, $\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_j, ...$ are taken as points in a stochastic point process on $[0, \infty)$. The total number of shocks in the interval $(0, t]$ is denoted by $\mathcal{N}(t)$ and $\mathcal{N}(0) \equiv 0$.

Define the probability of occurrence of j shocks in $(0, t]$ as

$$H_j(t) = P(\mathcal{N}(t) = j), \tag{3.1}$$

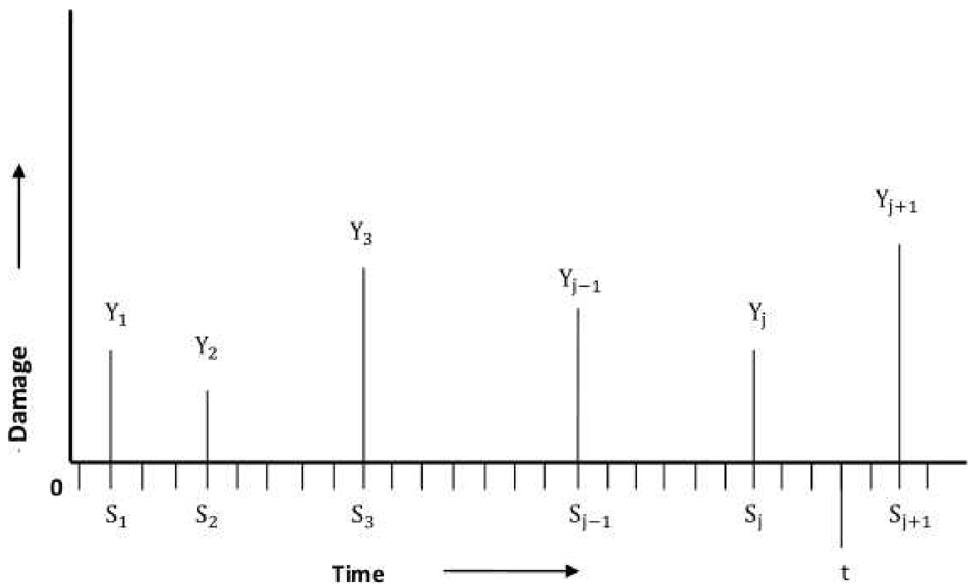


Figure 3.1: stochastic shock process causing random damage

and the expected number of shocks as

$$R(t) = E(\mathcal{N}(t)). \tag{3.2}$$

In a given time interval $(0, t]$ the probability associated with the number of shocks (j) is related with that of the time of occurrence of the j^{th} shock (S_j) as

$$F_j(t) = P(\mathcal{S}_j \leq t) = P(\mathcal{N}(t) \geq j) = \sum\nolimits_{i=j}^{\infty} H_i(t). \tag{3.3}$$

Using this, equation 3.1 can also be rewritten as

$$H_j(t) = P(\mathcal{N}(t) \geq j) - P(\mathcal{N}(t) \geq j+1) = F_j(t) - F_{j+1}(t). \tag{3.4}$$

Note that $F_j(t)$ depends on the distribution of the time between the shocks.

A shock produces a random amount of damage Y modeled by a cumulative distribution function

$$G(x) = P(Y \leq x) \tag{3.5}$$

The damage occurred at the j^{th} shock is denoted as Y_j. The evaluation of cumulative damage is based on two key assumptions:

1. damage increments, $Y_1, Y_2, ...$, are independent and identically distributed (*iid*).
2. The damage increments $(Y_j)_{j \geq 1}$ and the shock process $\mathcal{N}(t) : t \geq 0$ are independent.

The total damage caused by j shocks is given as

$$D_j = \sum_{i=1}^{j} Y_j, j \geq 1, \quad \text{and} \quad D_0 \equiv 0. \tag{3.6}$$

The distribution of D_j is obtained from the convolution of $G(x)$ as

$$P(D_j \leq x) = G^{(j)}(x), \tag{3.7}$$

where

$$G^{(j+1)}(x) = \int_0^x G^{(j)}(x-y)\, dG(y). \tag{3.8}$$

Note that

$$G^{(0)}(x) = 1,\ x \geq 0.$$

The total damage, $Z(t)$, at time t however depends on the number of shocks $\mathcal{N}(t)$ occurred in this interval, i.e.,

$$Z(t) = \sum_{j=1}^{N(t)} Y_i = D_{\mathcal{N}(t)}. \tag{3.9}$$

Using the total probability theorem and independence between the sequence $Y_1, Y_2, ...$ and $\mathcal{N}(t)$, we can write for $x > 0$

$$P(Z(t) > x) = \sum_{j=1}^{\infty} P(D_j > x, \mathcal{N}(t) = j) = \sum_{j=1}^{\infty} (1 - G^{(j)}(x)) H_j(t). \tag{3.10}$$

Using the facts that $\sum_{j=1}^{\infty} H_j(t) = 1$ and $G^{(0)}(x) = 1$, the distribution of the total damage can be written as

$$P(Z(t) \leq x) = H_0(t) + \sum_{j=1}^{\infty} G^{(j)}(x) H_j(t) = \sum_{j=0}^{\infty} G^{(j)}(x) H_j(t). \tag{3.11}$$

Substituting $H_0(t) = 1 - F_1(t)$ and $H_j(t) = F_j(t) - F_{(j+1)}(t)$ equation 3.11 can be written as

$$P(Z(t) \leq x) = 1 - \sum_{j=1}^{\infty} [G^{(j-1)}(x) - G^{(j)}(x)] F_j(t). \tag{3.12}$$

This is a fundamental expression that can be used to compute the system reliability. Suppose damage exceeding a limit x_{cr} causes the component failure, equation 3.12 provides $P(Z(t) \leq x_{cr})$ which is synonymous with the reliability function.

Formula for the mean value of the damage exceeding level B for the first time i.e. τ_B, is given by

$$\tau_B = \min\{t : Z(t) > B\} \tag{3.13}$$

So we have

$$\{\tau_B > t\} = \{Z(t) \leq B\} \qquad \text{and} \tag{3.14}$$

$$E(\tau_B) = \sum_{j=0}^{\infty} G^{(j)}(B) \int_0^{\infty} H_j(t)\mathrm{d}t. \tag{3.15}$$

3.3 Maintenance Model

In the proposed cumulative damage model, the failure can take place at time t when a shock occurring at t causes the total damage $Z(t)$ to exceed a critical threshold z_F. The failure prompts a corrective maintenance (CM) action involving the system renewal through replacement or complete overhaul (as good as new repair).

On the other hand, a preventive maintenance (PM) plan can be based on the following two strategies:

1. Condition-based strategy in which the system is renewed preventively as soon as $Z(t)$ exceeds a maintenance threshold value z_M, $z_m < z_F$. It is assumed here that system's degradation is continuously monitored.

2. Age-based strategy in which the system is replaced at certain age a, irrespective of its condition.

Reliability of safety-critical systems in a nuclear plant is maintained by implementing preventive maintenance and replacement programs.

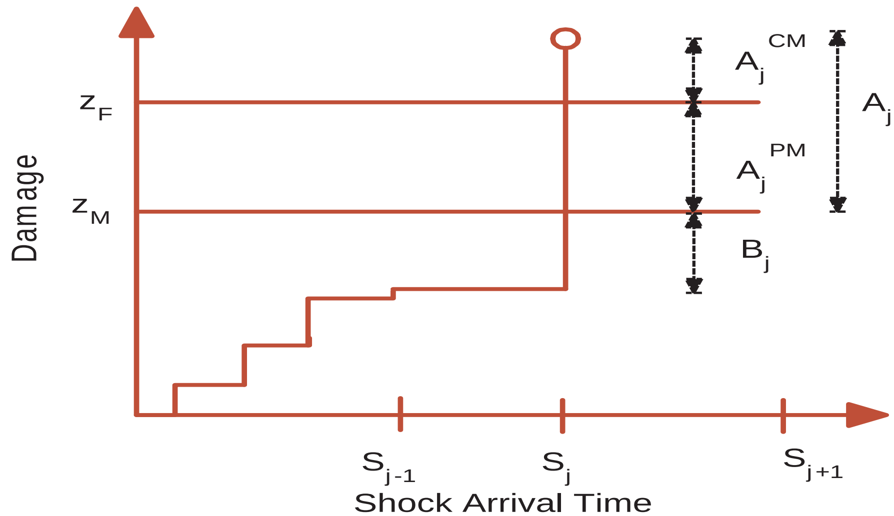


Figure 3.2: Shock Process

The probabilities of occurrence of any such maintenance actions at time t needs to be evaluated, considering the stochastic nature of the shock process and randomness associated with damage increments. this section describes derivation of these probability terms.

3.3.1 Formulation

First we investigate three basic disjoint events that can take place at the time of occurrence of any j^{th} shock. Define an event, A_j, that the total damage exceeds the PM threshold, z_M, at the j^{th} shock $(j = 1, 2, ...)$ as

$$A_j = \{D_{(j-1)} \leq z_M < D_j\} \tag{3.16}$$

The event A_j is the union of two disjoint events, namely, the occurrence of PM or CM action, and they are defined as follows

1. A_j^{PM} = Probability that the total damage exceeds z_M but is less than critical threshold = $\{A_j \cap \{D_j \leq z_F\}\}$

2. A_j^{CM} = Probability that the total damage exceeds z_M and $z_F = \{A_j \cap \{D_j > z_F\}\}$.

Finally, the event that no PM takes place by first j shocks is defined as

$$B_j = \{D_j \leq z_M\} = \bigcup_{i=j+1}^{\infty} A_i. \tag{3.17}$$

For future calculations, the following notations will be used

$$\begin{aligned} \alpha_j &= P(A_j) && (3.18) \\ \beta_j &= P(A_j^{PM}) && (3.19) \\ \gamma_j &= P(A_j^{CM}) && (3.20) \\ \pi_j &= P(B_j) && (3.21) \end{aligned}$$

Now these probabilities can be derived in terms of the distribution of damage increments $G(x)$ as follows

$$\begin{aligned} \alpha_j &= P(A_j) = P\{D_{j-1} \leq z_M < D_j\} \\ &= P(D_{j-1} \leq z_M, D_j > z_M) \\ &= P(D_{j-1} \leq z_M) - P(D_{j-1} \leq z_M, D_j \leq z_M) \end{aligned} \tag{3.22}$$

Since $\{D_j \leq z_M\}$ is contains the event $\{D_{j-1} \leq z_M\}$

$$\begin{aligned} \alpha_j &= P(D_{j-1} \leq z_M) - P(D_j \leq z_M) \\ &= G^{(j-1)}(z_M) - G^{(j)}(z_M) \end{aligned} \tag{3.23}$$

When the PM threshold is exceeded in shocks (j) to $(j-1)$, there are two possible mutually exclusive events: either the total damage is below z_F requiring a PM action, or the total damage exceeds level z_F leading to a CM action. It implies that $\alpha_j = \beta_j + \gamma_j$.

To calculate β_j, first note that

$$\begin{aligned} \beta_j &= P(A_j^{PM}) = P(D_{j-1} \leq z_M, z_M < D_j \leq z_F) \\ &= P(D_{j-1} \leq z_M, z_M < D_{j-1} + Y_j \leq z_F) \\ &= \int_0^{z_M} P(z_M < x + Y_j \leq z_F)\, dG^{(j-1)}(x) \\ &= \int_0^{z_M} [G(z_F - x) - G(z_M - x)]\, dG^{(j-1)}(x) \end{aligned} \tag{3.24}$$

In the same way we can evaluate γ_j as

$$\begin{aligned}
\gamma_j &= P(A_j^{CM}) = P(A_j \cap \{D_j > z_F\}). \\
&= P(D_{j-1} \leq z_M, D_j > z_F) \\
&= P(D_{j-1} \leq z_M, D_{j-1} + Y_j > z_F) \\
&= \int_0^{z_M} [1 - G(z_F - x)]\, dG^{(j-1)}(x).
\end{aligned} \tag{3.25}$$

Finally we calculate the probability $\pi_j = P(B_j)$

$$\begin{aligned}
\pi_j &= P(B_j) = P(D_j \leq z_M) \\
&= P(\cup_{i=j+1}^{\infty} A_i) \\
&= 1 - \sum\nolimits_{i=1}^{j} P(A_i) \\
&= 1 - \sum\nolimits_{i=1}^{j} \alpha_i. \\
&= 1 - \sum\nolimits_{i=1}^{j} [G^{(i-1)}(z_M) - G^{(i)}(z_M)] = G^{(j)}(z_M).
\end{aligned} \tag{3.26}$$

where we have used the fact that A_j are disjoint.

Now

$$\begin{aligned}
\{D_{(j-1)} \leq z_M < D_j\} &= \{A_j \cap \{D_j \leq z_F\}\} \bigcup \{A_j \cap \{D_j > z_F\}\} \\
P(\{D_{(j-1)} \leq z_M < D_j\}) &= P(\{A_j \cap \{D_j \leq z_F\}\}) \ + \ P(\{A_j \cap \{D_j > z_F\}\}) \\
\text{Hence} \qquad \alpha_j &= \beta_j + \gamma_j
\end{aligned} \tag{3.27}$$

Furthermore, for any $n \geq 1$, the collections $\{A_1^{PM}, A_1^{CM}, ..., A_n^{PM}, A_n^{CM}, B_n\}$ and $\{A_1, ..., A_n, B_n\}$ are both finite partitions of the sample space, so that we can write

$$\sum_{i=1}^{n} \alpha_i + \pi_n = \sum_{i=1}^{n} \beta_i + \sum_{i=1}^{n} \gamma_i + \pi_n = 1 \tag{3.28}$$

3.3.2 Probabilities Associated with Maintenance Actions

The proposed model involves an interaction of age-based and condition-based preventive maintenance criteria.

When we start at time 0, a new system is put into service. The system will be replaced at age a, should it survive up to this age. On the other hand, a corrective or preventive maintenance action before age a will be required if the cumulative damage at this time, $Z(a)$, exceeds the PM threshold z_M.

The probability associated with corrective or preventive maintenance action, P_{CP}, can be derived using equations (3.10) and (3.26) as

$$\begin{aligned} P_{CP} &= P(Z(a) > z_M) \\ &= 1 - \sum\nolimits_{j=1}^{\infty} G^{(j)}(z_M)\, H_j(a) \\ &= 1 - \sum\nolimits_{j=1}^{\infty} \pi_j\, H_j(a). \end{aligned} \tag{3.29}$$

Defining $\mathcal{A}_j = \sum_{i=1}^{j} \alpha_j$, it follows from Eqn. (3.26) that

$$\begin{aligned} P_{CP} &= 1 - \sum\nolimits_{j=1}^{\infty} \pi_j\, H_j(a) \\ &= 1 - \sum\nolimits_{j=1}^{\infty} \{1 - \sum\nolimits_{i=1}^{j} \alpha_i\}\, H_j(a) \\ &= \sum\nolimits_{j=1}^{\infty} \sum_{i=1}^{j} \alpha_i\, H_j(a) \\ &= \sum\nolimits_{j=1}^{\infty} \mathcal{A}_j\, H_j(a) \end{aligned} \tag{3.30}$$

P_{CP} can also be calculated using equations 3.12, 3.23 and 3.26 as follows.

$$\begin{aligned} P_{CP} &= P(Z(a) > z_M) = 1 - P(Z(a) \le z_M) \\ &= 1 - \left\{1 - \left\{ \sum\nolimits_{j=1}^{\infty} [G^{(j-1)}(z_M) - G^{(j)}(z_M)]\, F_j(a) \right\}\right\} \\ &= \sum\nolimits_{j=1}^{\infty} [G^{(j-1)}(z_M) - G^{(j)}(z_M)]\, F_j(a) \\ &= \sum\nolimits_{j=1}^{\infty} \alpha_j\, F_j(a). \end{aligned} \tag{3.31}$$

Let P_F be the probability that a corrective maintenance will be performed before age a. Now to calculate the value of P_F, note that the events A_j^{CM} and $\mathcal{S}_j$ are independent. Hence we get the following equation

$$\begin{aligned} P_F &= \sum\nolimits_{j=1}^{\infty} P(A_j^{CM}, \mathcal{S}_j < a) \\ &= \sum\nolimits_{j=1}^{\infty} \gamma_j \, F_j(a) = \sum\nolimits_{j=1}^{\infty} \gamma_j \sum\nolimits_{i=j}^{\infty} H_i(a) \qquad (3.32) \\ &= \sum\nolimits_{i=1}^{\infty} \sum\nolimits_{j=1}^{i} \gamma_j \, H_i(a) = \sum\nolimits_{i=1}^{\infty} \mathcal{C}_i \, H_i(a), \qquad (3.33) \end{aligned}$$

Where $\mathcal{C}_i = \sum_{j=1}^{i} \gamma_j$.

Similarly, the probability of preventive maintenance before a can be obtained as

$$\begin{aligned} P_M &= \sum\nolimits_{j=1}^{\infty} P(A_j^{PM}, \mathcal{S}_j < a) \\ &= \sum\nolimits_{j=1}^{\infty} \beta_j F_j(a) = \sum\nolimits_{j=1}^{\infty} \beta_j \sum\nolimits_{i=j}^{\infty} H_i(a) \qquad (3.34) \\ &= \sum\nolimits_{i=1}^{\infty} \sum\nolimits_{j=1}^{i} \beta_j \, H_i(a) = \sum\nolimits_{i=1}^{\infty} \mathcal{B}_i \, H_i(a) \qquad (3.35) \end{aligned}$$

Where $\mathcal{B}_i = \sum_{j=1}^{i} \beta_j$

The probability of age-based replacement at age a is as follows

$$\begin{aligned} P_A &= 1 - P_{CP} \\ &= P(Z(a) \le z_M) = 1 - P(Z(a) \ge z_M) \\ &= 1 - \sum\nolimits_{j=1}^{\infty} \alpha_j F_j(a) = 1 - \sum\nolimits_{j=1}^{\infty} \alpha_j \sum\nolimits_{i=j}^{\infty} H_i(a) \\ &= 1 - \sum\nolimits_{i=1}^{\infty} \sum\nolimits_{j=1}^{i} \alpha_j H_i(a) = 1 - \sum\nolimits_{i=1}^{\infty} \mathcal{A}_i \, H_i(a) \qquad (3.36) \end{aligned}$$

3.4 Summary

Probabilities associated with preventive and corrective maintenance are calculated in terms of time of occurrence of shocks, damage caused by shocks and age of replacement which will help in optimizing the cost rate defined in terms of these events.

Chapter 4

Maintenance Cost Analysis

4.1 Introduction

Let T denote the length of renewal cycles and C the cost associated with the renewal. After the first renewal at time T_1, a new cycle starts, and it survives the duration T_2 and so on. The renewal cost varies depending on the maintenance actions. We assume cost c_F for CM, c_A for replacement at age a and c_M for PM before age a.

For $\mathcal{N}(a) = n, n \geq 1$, a random vector of renewal cycles and associated costs, (T_m, C_m) is an *iid* sequence generated by random variables T and C with the following distribution.

$$\begin{aligned} P(T = \mathcal{S}_j) &= P(A_j^{PM} \cup A_j^{CM}) = P(A_j), \quad \text{for} \quad j \leq n, && (4.1) \\ P(T = a) &= P(\cup_{i=n+1}^{\infty} A_i) \quad \text{on} \quad B_n. && (4.2) \end{aligned}$$

$$\text{ie} \quad T = \begin{cases} \mathcal{S}_j & \text{on} \quad A_j, j \leq n \\ a & \text{on} \quad B_n \end{cases}$$

$$\begin{aligned} P(C = c_M) &= P(D_{j-1} \leq z_M < D_j \leq z_F) \quad \text{for} \quad A_j^{PM} \quad j \leq n && (4.3) \\ P(C = c_F) &= P(D_{j-1} \leq z_M < D_j, D_j > z_F) \quad \text{for} \quad A_j^{CM} \quad j \leq n && (4.4) \\ P(C = c_A) &= P(\cup_{i=n+1}^{\infty} A_i) \quad \text{on} \quad B_n. && (4.5) \end{aligned}$$

$$\text{ie}\quad C = \begin{cases} c_M & \text{on} \quad A_j^{PM}, j \leq n, \\ c_F & \text{on} \quad A_j^{CM}, j \leq n, \\ c_A & \text{on} \quad B_n \end{cases}$$

Putting the above facts together we get the following joint distribution for the sequence (T_m, C_m).

$$(T, C) = \begin{cases} (\mathcal{S}_j, c_M) & \text{on} \quad A_j^{PM}, j \leq n \\ (\mathcal{S}_j, c_F) & \text{on} \quad A_j^{CM}, j \leq n \\ (a, c_A) & \text{on} \quad B_n \end{cases}$$

If no shocks occurred up to time a, i.e. $\mathcal{N}(a) = 0$, then $(T, C) = (a, c_A)$. If $M(t)$ is the total number of completed renewal cycles, then the total cost over a time interval $(0, t]$ is a sum of costs incurred over these renewal cycles, which will be given as

$$K(t) = \sum_{j=1}^{M(t)} C_j. \tag{4.6}$$

Note that $K(t)$ is a random function involving random variables $M(t)$ and C, and its contribution is very difficult to evaluate. Therefore it is convenient to work with asymptotic formulas for long term expected cost.

So what we are looking for is simple form for the expression $Q = \lim_{t\to\infty} \frac{K(t)}{t}$. For that we need to know the relationships between $K(t)$, $M(t)$, C and T.

To start with, observe that

$$E(T) < \infty \Rightarrow P(T < \infty) = 1. \tag{4.7}$$

Because, if we had $P(T = \infty) = c > 0$ then we would have $E(T) = \infty$.
Hence we have

$$P(T_1 + T_2 + ... + T_n < \infty) = 1 \tag{4.8}$$

This in turn implies that

$$P(\lim_{t\to\infty} M(t) = \infty) = 1 \tag{4.9}$$

because if we had

$$P(\lim_{t\to\infty} M(t) = \tau < \infty) = c > 0, \quad \text{(say)} \tag{4.10}$$

then

$$P(T_1 + T_2 + ...T_\tau = \infty) = c \tag{4.11}$$

which would be a contradiction to (4.8). Hence we have

$$\lim_{t\to\infty} M(t) = \infty \quad \text{with probability 1,}$$
$$T_1 + T_2 + ... + T_{M(t)} \le t < T_1 + T_2 + ... + T_{M(t)+1}$$

the above inequality gives

$$\frac{T_1 + T_2 + ... + T_{M(t)}}{M(t)} \le \frac{t}{M(t)} < \frac{T_1 + T_2 + ... + T_{M(t)+1}}{M(t)+1} \frac{M(t)+1}{M(t)}$$

By the strong law of large numbers for a sequence of independent and identically distributed random variables, we have

$$\lim_{n\to\infty} \frac{T_1 + T_2 + ... + T_n}{n} = E[T] \quad \text{with probability 1}$$

Hence letting $t \to \infty$ we get the following equation

$$\begin{aligned} \lim_{t\to\infty} \frac{t}{M(t)} &= E[T] \\ \text{ie} \quad \lim_{t\to\infty} \frac{M(t)}{t} &= \frac{1}{E[T]} \end{aligned}$$

Finally we can get the expression for long-term expected average cost rate Q per unit time given as

$$Q = \lim_{t\to\infty} \frac{K(t)}{t}$$
$$\sum\nolimits_{i=1}^{M(t)} C_i = K(t) \le \sum\nolimits_{i=1}^{M(t)+1} C_i.$$
$$\frac{\sum_{i=1}^{M(t)} C_i}{t} = \frac{K(t)}{t} \le \frac{\sum_{i=1}^{M(t)+1} C_i}{t}.$$
$$\frac{\sum_{i=1}^{M(t)} C_i}{M(t)} \times \frac{M(t)}{t} = \frac{K(t)}{t} \le \frac{\sum_{i=1}^{M(t)+1} C_i}{M(t)+1} \times \frac{M(t)+1}{t}.$$

Taking the limit as $t \to \infty$, we get

$$\lim_{t\to\infty} \frac{\sum_{i=1}^{M(t)} C_i}{M(t)} \times \frac{M(t)}{t} = \lim_{t\to\infty} \frac{K(t)}{t} \le \lim_{t\to\infty} \frac{\sum_{i=1}^{M(t)+1} C_i}{M(t)+1} \times \frac{M(t)+1}{t}$$

By the use of strong law of large number for the sequence $\{C_n\}$ we get

$$\frac{E(C)}{E(T)} = \lim_{t\to\infty} \frac{K(t)}{t} \le \frac{E(C)}{E(T)}$$

So we have

$$\lim_{t\to\infty} \frac{k(t)}{t} = \frac{E(C)}{E(T)}$$

Now the discounted cost evaluated over a time horizon (0,t] considering an exponential discounting factor with interest rate r per unit time is given as

$$K(t,r) = \sum_{j=1}^{M(t)} e^{-rU_j} C_j$$

Where $U_j = T_1 + \cdots + T_j$ denotes the time of j^{th} renewal.

The long-term expected equivalent average cost per unit time is given as (van der Weide [6]):

$$Q(r) = \lim_{t\to\infty} rE(K(t,r)) = \frac{rE(Ce^{-rT})}{1 - E(e^{-rT})} \tag{4.12}$$

Note that

$$\lim_{r\to 0} Q(r) = \lim_{r\to 0} \frac{rE(Ce^{-rT})}{1 - E(e^{-rT})} = \frac{E(C)}{E(T)} = Q \tag{4.13}$$

4.2 General Expression for Cost Rate (With Discounting)

The expected discounted cost expended is given by

$$E[Ce^{-rT}] = \sum_{n=0}^{\infty} E[Ce^{-rT}; \mathcal{N}(a) = n]$$

For n=0 we have a simple form

$$E[Ce^{-rT};\mathcal{N}(a)=0] = C_A e^{-ra} H_0(a), \tag{4.14}$$

For $n \geq 1$ we split the expectations over the disjoint events $A_1^{PM}, A_1^{CM}, \cdots, A_n^{PM}$, A_n^{CM}, B_n.

$$\begin{aligned} E[Ce^{-rT}\,;\mathcal{N}(a)=n] &= \sum\nolimits_{j=1}^{n} E[Ce^{-rT}\,;\mathcal{N}(a)=n\,|\,A_j^{PM}]\,P(A_j^{PM}) \\ &\quad + \sum\nolimits_{j=1}^{n} E[Ce^{-rT}\,;\mathcal{N}(a)=n\,|\,A_j^{CM}]\,P(A_j^{CM}) \\ &\quad + E[Ce^{-rT}\,;\mathcal{N}(a)=n\,|\,B_n]\,P(B_n). \\ E[Ce^{-rT};\mathcal{N}(a)=n] &= C_M \sum\nolimits_{j=1}^{n} \beta_j E[e^{-rS_j};\mathcal{N}(a)=n] \\ &\quad + C_F \sum\nolimits_{j=1}^{n} \gamma_j E[e^{-rS_j};\mathcal{N}(a)=n] \\ &\quad + C_A e^{-ra}\pi_n H_n(a). \end{aligned} \tag{4.15}$$

Substituting Eq (4.15) into Eq (4.14) we get the following equation involving double summations.

$$\begin{aligned} E[Ce^{-rT}] &= C_M \sum\nolimits_{n=1}^{\infty} \sum\nolimits_{j=1}^{n} \beta_j E[e^{-r\mathcal{S}_j};\mathcal{N}(a)=n] \\ &\quad + C_F \sum\nolimits_{n=1}^{\infty} \sum\nolimits_{j=1}^{n} \gamma_j E[e^{-r\mathcal{S}_j};\mathcal{N}(a)=n] \\ &\quad + C_A \sum\nolimits_{n=0}^{\infty} e^{-ra}\pi_n H_n(a). \\ &= C_M \sum\nolimits_{j=1}^{\infty} \beta_j \sum\nolimits_{n=1}^{j} E[e^{-r\mathcal{S}_j};\mathcal{N}(a)=n] \\ &\quad + C_F \sum\nolimits_{j=1}^{\infty} \gamma_j \sum\nolimits_{n=1}^{j} E[e^{-r\mathcal{S}_j};\mathcal{N}(a)=n] \\ &\quad + C_A \sum\nolimits_{n=0}^{\infty} e^{-ra}\pi_n H_n(a). \qquad (4.16) \\ &= C_M \sum\nolimits_{j=1}^{\infty} \beta_j E[e^{-r\mathcal{S}_j};\mathcal{N}(a)\geq j] \\ &\quad + C_F \sum\nolimits_{j=1}^{\infty} \gamma_j E[e^{-r\mathcal{S}_j};\mathcal{N}(a)\geq j] \\ &\quad + C_A \sum\nolimits_{n=0}^{\infty} e^{-ra}\pi_n H_n(a). \qquad (4.17) \\ E[Ce^{-rT}] &= C_M \sum\nolimits_{j=1}^{\infty} \beta_j E[e^{-rS_j};\mathcal{S}_j \leq a] \\ &\quad + C_F \sum\nolimits_{j=1}^{\infty} \gamma_j E[e^{-rS_j};\mathcal{S}_j \leq a] \\ &\quad + C_A \sum\nolimits_{n=0}^{\infty} e^{-ra}\pi_n H_n(a). \qquad (4.18) \end{aligned}$$

4.2.1 Cost Rate in Terms of $\mathcal{S}_j$

The third term in equation (4.18) can be simplified in terms of $\mathcal{S}_j$ as follows

$$\begin{aligned}
\sum_{n=1}^{\infty} \pi_n H_n(a) &= \sum_{n=1}^{\infty}(1-\sum_{j=1}^{n} \alpha_j) P(\mathcal{N}(a)=n) \\
&= \sum_{n=1}^{\infty} P(\mathcal{N}(a)=n) - \sum_{j=1}^{\infty}\sum_{n=j}^{\infty} \alpha_j P(\mathcal{N}(a)=n) \\
&= P(\mathcal{S}_1 \le a) - \sum_{j=1}^{\infty} \alpha_j P(\mathcal{S}_j \le a). \qquad (4.19)
\end{aligned}$$

Finally we obtain,

$$\begin{aligned}
E[Ce^{-rT}] &= C_M \sum_{j=1}^{\infty} \beta_j E[e^{-r\mathcal{S}_j}; \mathcal{S}_j \le a] \\
&\quad + C_F \sum_{j=1}^{\infty} \gamma_j E[e^{-r\mathcal{S}_j}; \mathcal{S}_j \le a] \\
&\quad + C_A e^{-ra}\left(P(\mathcal{S}_1 \le a) - \sum_{j=1}^{\infty} \alpha_j P(\mathcal{S}_j \le a)\right). \qquad (4.20) \\
&= \sum_{j=1}^{\infty} \big(C_M\beta_j + C_F\gamma_j\big) E[e^{-r\mathcal{S}_j}; \mathcal{S}_j \le a] \\
&+ C_A e^{-ra}\left(P(\mathcal{S}_1 \le a) - \sum_{j=1}^{\infty} \alpha_j P(\mathcal{S}_j \le a)\right). \qquad (4.21)
\end{aligned}$$

Taking in this formula $C_A = C_M = C_F = 1$ and $\beta_j + \gamma_j = \alpha_j$, we get after simplification

$$E[e^{-rT}] = e^{-ra}\left(P(\mathcal{S}_1 \le a) - \sum_{j=1}^{\infty} \alpha_j P(\mathcal{S}_j \le a)\right) + \sum_{j=1}^{\infty} \alpha_j E[e^{-r\mathcal{S}_j}; \mathcal{S}_j \le a] \qquad (4.22)$$

Using equation 4.12, from the above calculations it follows that the long-term expected equivalent average discounted cost is given as

$$Q(r) = r\frac{\sum_{j=1}^{\infty} \big(C_M\beta_j + C_F\gamma_j\big) E[e^{-rS_j}; \mathcal{S}_j \le a] + C_A e^{-ra}\left(P(\mathcal{S}_1 \le a) - \sum_{j=1}^{\infty} \alpha_j P(\mathcal{S}_j \le a)\right)}{1 - e^{-ra}\left(P(\mathcal{S}_1 \le a) - \sum_{j=1}^{\infty} \alpha_j P(\mathcal{S}_j \le a)\right) - \sum_{j=1}^{\infty} \alpha_j E[e^{-r\mathcal{S}_j}; \mathcal{S}_j \le a]} \qquad (4.23)$$

4.2.2 Cost Rate in Terms of H_n

The term $E[e^{-r\mathcal{S}_j}; \mathcal{S}_j \le a]$ can be evaluated in terms of its distribution F_j of $\mathcal{S}_j$ as

$$\begin{aligned} E[e^{-r\mathcal{S}_j}; \mathcal{S}_j \le a] &= \int_0^a e^{-rx}\mathrm{d}F_j(x) = e^{-ra}F_j(a) + \int_0^a F_j(x) r e^{-rx}\mathrm{d}x \qquad (4.24)\\ &= e^{-ra}(\textstyle\sum_{i=j}^{\infty} H_i(a)) + \displaystyle\int_0^a (\textstyle\sum_{i=j}^{\infty} H_i(x)) r e^{-rx}\mathrm{d}x \\ &= \textstyle\sum_{i=j}^{\infty} \displaystyle\left\{ e^{-ra}H_i(t) + \int_0^a H_i(t) r e^{-rx}\mathrm{d}x \right\} \qquad (4.25) \end{aligned}$$

$$\begin{aligned} E[Ce^{-rT}] &= C_A\, e^{-ra} \textstyle\sum_{i=0}^{\infty} \pi_i\, H_i(a) \\ &\quad + \textstyle\sum_{j=1}^{\infty} (C_M\beta_j + C_F\gamma_j) \sum_{i=j}^{\infty} \displaystyle\left\{ e^{-ra}H_i(a) + \int_0^a H_i(x) r e^{-rx}\mathrm{d}x \right\} \\ &= C_A\, e^{-ra} \textstyle\sum_{i=0}^{\infty} \pi_i\, H_i(a) \\ &\quad + \textstyle\sum_{i=1}^{\infty} \sum_{j=1}^{i} (C_M\beta_j + C_F\gamma_j) \displaystyle\left\{ e^{-ra}H_i(a) + \int_0^a H_i(x) r e^{-rx}\mathrm{d}x \right\} \\ &= C_A\, e^{-ra} \textstyle\sum_{i=0}^{\infty} \pi_i\, H_i(a) \\ &\quad + \textstyle\sum_{i=1}^{\infty} (C_M\mathcal{B}_i + C_F\mathcal{C}_i) \displaystyle\left\{ e^{-ra}H_i(a) + \int_0^a H_i(x) r e^{-rx}\mathrm{d}x \right\} \end{aligned}$$

Rearranging the terms we can also write

$$\begin{aligned} E[Ce^{-rT}] = C_A e^{-ra} H_0(a) &+ e^{-ra} \textstyle\sum_{n=1}^{\infty} (C_A\pi_n + C_M\mathcal{B}_n + C_F\mathcal{C}_n) H_n(a) \\ &+ \textstyle\sum_{n=1}^{\infty} (C_M\mathcal{B}_n + C_F\mathcal{C}_n) \displaystyle\int_0^a H_n(x) r e^{-rx}\mathrm{d}x. \end{aligned} \qquad (4.26)$$

This formula can be simplified by taking $C_0 = C_A = C_M$ and $C_F = C_0 + \delta_F$,

$$\begin{aligned} E[Ce^{-rT}] &= e^{-ra}(C_0 + \delta_F \textstyle\sum_{n=1}^{\infty} \mathcal{C}_n H_n(a)) \\ &\quad + \textstyle\sum_{n=1}^{\infty} (C_0\mathcal{B}_n + C_F\mathcal{C}_n) \displaystyle\int_0^a H_n(x) r e^{-rx}\mathrm{d}x. \end{aligned} \qquad (4.27)$$

Substituting $C_0 = C_F = 1$ in equation 4.27 we get

$$\begin{aligned} E[e^{-rT}] &= e^{-ra} + \textstyle\sum_{n=1}^{\infty} (1 - \pi_n) \displaystyle\int_0^a H_n(x) r e^{-rx}\mathrm{d}x \\ &= 1 - \textstyle\sum_{n=0}^{\infty} \pi_n \displaystyle\int_0^a H_n(x) r e^{-rx}\mathrm{d}x. \end{aligned} \qquad (4.28)$$

From the above calculations it follows that the long-term expected equivalent average discounted cost in terms of H_n is given as

$$Q(r) \;=\; r\frac{C_A\, e^{-ra}\sum_{i=0}^{\infty}\pi_i\, H_i(a)+\sum_{i=1}^{\infty}(C_M\mathcal{B}_n+C_F\mathcal{C}_n)\Big\{e^{-ra}H_i(t)+\int_0^a H_i(t)re^{-rx}\mathrm{d}x\Big\}}{\sum_{n=0}^{\infty}\pi_n\int_0^a H_n(x)re^{-rx}\mathrm{d}x.} \tag{4.29}$$

The expected length of the renewal cycle can be derived by differentiating equation 4.28 with respect to r and then substituting $r = 0$:

$$E[T] = \sum_{n=0}^{\infty}\pi_n\int_0^a H_n(x)\mathrm{d}x. \tag{4.30}$$

4.3 General Expression for Cost Rate (Without Discounting)

As discussed before, there are three possible ways for the renewal of a structure at time T: PM when $z_M \le Z(T) < z_F$, replacement when $T = a$, and CM when $Z(T) \ge z_F$. The expected length of the renewal cycle is given as

$$E(T) = \sum_{j=0}^{\infty}\pi_j\int_0^a H_j(x)\,\mathrm{d}x.$$

The expected cost incurred in a renewal cycle is simply a product of probabilities of the three possible renewal actions with the associated costs:

$$\begin{aligned} E(C) &= C_A\,P_A + C_M\,P_M + C_F\,P_F &(4.31)\\ &= C_A\Big\{1-\sum_{j=1}^{\infty}\alpha_j\,F_j(a)\Big\}+C_M\Big\{\sum_{j=1}^{\infty}\beta_j\,F_j(a)\Big\}+C_F\Big\{\sum_{j=1}^{\infty}\gamma_j\,F_j(a)\Big\} &\\ &= C_A+\sum_{j=1}^{\infty}[C_M\beta_j+C_F\gamma_j-C_A\alpha_j]\,F_j(a). &(4.32)\end{aligned}$$

Assuming that $C_0 = C_M = C_A$ and $C_F = C_0 + \delta_F$, we get

$$\begin{aligned}
E(C) &= C_0 + \sum_{j=1}^{\infty} [C_0\,\beta_j + C_0\,\gamma_j - (C_0 + \delta_F)\alpha_j]\, F_j(a). \\
&= C_0 + \sum_{j=1}^{\infty} [C_0(\beta_j + \gamma_j - \alpha_j) + \delta_F\,\gamma_j\, F_j(a). \\
&= C_0 + \delta_F \sum_{j=1}^{\infty} \gamma_j\, F_j(a) \\
&= C_0 + \delta_F \sum_{j=1}^{\infty} \gamma_j \sum_{i=j}^{\infty} H_i(a) \\
&= C_0 + \delta_F \sum_{i=1}^{\infty} \sum_{j=1}^{i} \gamma_j\, H_i(a) \\
&= C_0 + \delta_F \sum_{i=1}^{\infty} \mathcal{C}_i\, H_i(a). \qquad (4.33)
\end{aligned}$$

The asymptotic cost rate Q can now be obtained by substituting the values of $E(C)$ and $E(T)$.

$$Q = \frac{C_0 + \delta_F \sum_{i=1}^{\infty} \mathcal{C}_i\, H_i(a)}{\sum_{j=0}^{\infty} \pi_j \int_0^a H_j(x)\,\mathrm{d}x} \qquad (4.34)$$

It is also possible to express Q in terms of the distribution functions F_j of the occurrence times S_j of the shocks.

$$\begin{aligned}
Q &= \frac{C_0 + \delta_F \sum_{i=1}^{\infty} \gamma_i\, F_i(a)}{\int_0^a H_0(x)\,\mathrm{d}x + \sum_{j=1}^{\infty} \left(1 - \sum_{i=1}^{j} \alpha_i\right) \int_0^a H_j(x)\,\mathrm{d}x} \\
&= \frac{C_0 + \delta_F \sum_{i=1}^{\infty} \gamma_i\, F_i(a)}{\int_0^a H_0(x)\,\mathrm{d}x + \sum_{j=1}^{\infty} \int_0^a H_j(x)\,\mathrm{d}x - \sum_{j=1}^{\infty} \sum_{i=1}^{j} \alpha_i \int_0^a H_j(x)\,\mathrm{d}x} \\
&= \frac{C_0 + \delta_F \sum_{i=1}^{\infty} \gamma_i\, F_i(a)}{a - \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \alpha_i \int_0^a H_j(x)\,\mathrm{d}x} \\
&= \frac{C_0 + \delta_F \sum_{i=1}^{\infty} \gamma_i\, F_i(a)}{a - \sum_{i=1}^{\infty} \alpha_i \int_0^a \sum_{j=i}^{\infty} H_j(x)\,\mathrm{d}x} \\
&= \frac{C_0 + \delta_F \sum_{i=1}^{\infty} \gamma_i\, F_i(a)}{a - \sum_{i=1}^{\infty} \alpha_i \int_0^a F_i(x)\,\mathrm{d}x} \qquad (4.35)
\end{aligned}$$

These expressions are useful if the shock process is a renewal process, i.e. inter-occurrence times of shocks are *iid*

4.4 Without Age Replacement

Consider the simple condition based maintenance strategy in which the system is renewed preventively as soon as $Z(t)$ exceeds a maintenance threshold value $z_M, z_M < z_F$, without age replacement. In this case a renewal cycle ends if the total damage exceeds the PM level z_M. We have

$$(T, C) = \begin{cases} (\mathcal{S}_j, c_M) & \text{on} \quad A_j^{PM} \\ (\mathcal{S}_j, c_F) & \text{on} \quad A_j^{CM} \end{cases}$$

In order to derive the asymptotic cost rate Q (i.e cost per unit time) and the long term expected equivalent average cost per unit time, a number of expectations have to be evaluated:

4.4.1 Cost Rate in Terms of $\mathcal{S}_j$

$$\begin{aligned} E(Ce^{-rT}) &= E[E(Ce^{-rT}|C)] \\ &= \sum\nolimits_{j=1}^{\infty} E[C_M e^{-r\mathcal{S}_j}]\beta_j + \sum\nolimits_{j=1}^{\infty} E[C_F e^{-r\mathcal{S}_j}]\gamma_j \\ &= \sum\nolimits_{j=1}^{\infty} (C_M\beta_j + C_F\gamma_j)E(e^{-r\mathcal{S}_j}) \end{aligned} \tag{4.36}$$

and substituting $C \equiv 1$, we get

$$E(e^{-rT}) = \sum_{j=1}^{\infty} \alpha_j E(e^{-r\mathcal{S}_j}) \tag{4.37}$$

and substituting $r = 0$ yields

$$E(C) = \sum_{j=1}^{\infty} (C_M\beta_j + C_F\gamma_j) \tag{4.38}$$

The expected length of the renewal cycle can be derived by differentiating Eq (4.37) with respect to r and then substituting $r = 0$

$$E(T) = \sum_{j=1}^{\infty} \alpha_j E(\mathcal{S}_j). \tag{4.39}$$

It follows that asymptotic cost rate per unit time is given as

$$Q = \frac{\sum_{j=1}^{\infty}(C_M\beta_j + C_F\gamma_j)}{\sum_{j=1}^{\infty}\alpha_j E(\mathcal{S}_j)} \tag{4.40}$$

And the asymptotic cost rate per unit time considering the discount factor is given as

$$Q(r) = \frac{r\sum_{j=1}^{\infty}(C_M\beta_j + C_F\gamma_j)E(e^{-r\mathcal{S}_j})}{1 - \sum_{j=1}^{\infty}\alpha_j E(e^{-r\mathcal{S}_j})}. \tag{4.41}$$

These formulas for Q and $Q(r)$ are calculated in terms of the occurrence times of the shocks. When the shock process is a renewal process we can simplify formula (4.40) and (4.41). In this case $\mathcal{S}_j = X_1 + \cdots + X_j$ where (X_j) is the *iid* sequence of inter-occurrence times and we can substitute $E(\mathcal{S}_j) = jm$ and $E(e^{-r\mathcal{S}_j}) = \omega^j$, where $m = E(X_1)$ and $\omega = E(e^{-rX_1})$

4.4.2 Cost Rate in Terms of H_n

We now continue with expressions for Q and $Q(r)$ in terms of the probability distribution of the total number of shocks. Using integration by parts, we get

$$\begin{aligned} E(e^{-r\mathcal{S}_j}) = \int_0^{\infty} e^{-rx}\,\mathrm{d}F_j(x) &= [e^{-rx}F_j(x)]_0^{\infty} + r\int_0^{\infty} e^{-rx}F_j(x)\,\mathrm{d}x \\ &= r\sum_{j=1}^{\infty}\int_0^{\infty} H_i(x)e^{-rx}\mathrm{d}x \end{aligned} \tag{4.42}$$

and differentiating with respect to r and substituting $r = 0$ we get

$$E(\mathcal{S}_j) = \sum_{i=j}^{\infty}\int_0^{\infty} H_i(x)\,\mathrm{d}x \tag{4.43}$$

$$\begin{aligned} \sum\nolimits_{j=1}^{\infty}\alpha_j E(e^{-r\mathcal{S}_j}) &= \sum\nolimits_{j=1}^{\infty}\alpha_j r\sum\nolimits_{i=j}^{\infty}\int_0^{\infty} H_i(x)e^{-rx}\,\mathrm{d}x \\ &= r\sum\nolimits_{i=1}^{\infty}\sum\nolimits_{j=1}^{i}\alpha_i\int_0^{\infty} H_i(x)e^{-rx}\,\mathrm{d}x \\ &= r\sum\nolimits_{i=1}^{\infty}\mathcal{A}_i\int_0^{\infty} H_i(x)e^{-rx}\,\mathrm{d}x. \end{aligned} \tag{4.44}$$

It follows that

$$Q = \frac{\sum_{j=1}^{\infty}(C_M\beta_j + C_F\gamma_j)}{\sum_{i=1}^{\infty}\mathcal{A}_i\int_0^{\infty}H_i(x)\,\mathrm{d}x}, \tag{4.45}$$

and

$$Q(r) = \frac{r^2\sum_{i=1}^{\infty}(C_M\mathcal{B}_i + C_F\mathcal{C}_i)\int_0^{\infty}H_i(x)e^{-rx}\,\mathrm{d}x}{1 - r\sum_{i=1}^{\infty}\mathcal{A}_i\int_0^{\infty}H_i(x)e^{-rx}\,\mathrm{d}x} \tag{4.46}$$

4.5 Specific Models

In this section we analyze specific cases of PM policy as analyzed in Quian et al [2] and van der Weide et al [9] Following are the two proposed models

1. **Model 1 :** The system undergoes PM at age a or at the occurrence of first shock $\mathcal{S}_1$ producing damage Y_1, whichever occurs first. This means that the PM level, z_M=0. This type of model could be used in nuclear power plants where damage beyond certain level could be catastrophic.
2. **Model 2 :** The system undergoes PM at time a or CM if the total damage exceeds the failure level z_F, whichever occurs first. This means that the PM level $z_M = z_F$. This type of model can be used where failures do not cause much difference in production or the failures are not catastrophic in nature.

4.5.1 Model 1

In this case PM occurs at the first shock or at the age a. This means that $z_M = 0$ implies that $P(B_n) = \pi_n = P(D_n \leq z_M) = 0$ for all $n \geq 1$ and $\pi_0 = 1$, since each shock produces a finite amount of damage. There can be either no shock or one shock before the age a is reached. Hence the possible values for n are 0 and 1. Note that $H_0(x) + H_1(x) = 1$.
From equation 4.27,

$$E(Ce^{-rT}) = e^{-ra}(C_0 + \delta_F\sum_{n=1}^{\infty}\mathcal{C}_nH_n(a)) + \sum_{n=1}^{\infty}(C_0\mathcal{B}_n + C_F\mathcal{C}_n)\int_0^a H_n(x)re^{-rx}\mathrm{d}x. \tag{4.47}$$

To Simplify this expression, we need to evaluate the value of α_j, β_j, and γ_j
Recall

$$\begin{cases} \alpha_j &= P(A_j) &= P(D_{j-1} \leq z_M < D_j); \\ \beta_j &= P(A_j^{PM}) &= P(A_j \bigcap \{D_j \leq z_F\}); \\ \gamma_j &= P(A_j^{CM}) &= P(A_j \bigcap \{D_j > z_F\}). \end{cases}$$

Hence we get $\alpha_1 = 1$, since $z_M = 0$, $\beta_1 = G(z_F)$ and $\gamma_1 = 1 - G(z_F)$. Rest of the terms will be zero.
Now,

$$\begin{aligned} E(Ce^{-rT}) &= e^{-ra}(C_0 + \delta_F \sum\nolimits_{n=1}^{\infty} \mathcal{C}_n H_n(a)) \\ &\quad + \sum\nolimits_{n=1}^{\infty} (C_0 \mathcal{B}_n + C_F \mathcal{C}_n) \int_0^a H_n(x) r e^{-rx} \mathrm{d}x. \\ &= e^{-ra}(C_0 + \delta_F \mathcal{C}_0 H_0(a) + \delta_F \mathcal{C}_1 H_1(a)) \\ &\quad + (C_0 \mathcal{B}_1 + C_F \mathcal{C}_1) \int_0^a H_1(x) r e^{-rx} \mathrm{d}x. \\ &= e^{-ra}(C_0 + \delta_F \bar{G}(z_F)(1 - H_0(a))) \\ &\quad + (C_0 G(z_F) + (C_0 + \delta_F)\bar{G}(z_F)) \int_0^a (1 - H_0(x)) r e^{-rx} \mathrm{d}x. \\ &= C_0(1 - \int_0^a H_0(x) r e^{-rx} \, \mathrm{d}x) \\ &\quad + \delta_F \bar{G}(z_F)[1 - e^{-ra} H_0(a) - \int_0^a H_0(x) r e^{-rx} \, \mathrm{d}x] \end{aligned} \tag{4.48}$$

From Eq (4.28), We have

$$\begin{aligned} E(e^{-rT}) &= 1 - \sum\nolimits_{n=0}^{\infty} \pi_n \int_0^a H_n(x) r e^{-rx} \, \mathrm{d}x. \\ &= 1 - \int_0^a H_0(x) r e^{-rx} \, \mathrm{d}x \end{aligned} \tag{4.49}$$

Hence the expected long term discounted value of the expected equivalent average cost rate is given as

$$Q(a, r) = \frac{rE(Ce^{-rT})}{1 - E(e^{-rT})}$$

where $E(Ce^{-rT})$ and $E(e^{-rT})$ are given by equations 4.48 and 4.49.

4.5.2 Model 2

In this case the system undergoes maintenance at age a or when the damage exceeds z_F. That means $z_M = z_F$. Hence in this case we have

- $\beta_j = P(A_j^{PM}) = P(A_j \bigcap \{D_j \leq z_F\}) = 0$, since it is not possible to have $\{z_M < D_j\}$ and $\{D_j \leq z_M\}$ simultaneously.
- $\alpha_j = P(A_j) = P(A_j^{PM}) + P(A_j^{CM}) = \beta_j + \gamma_j = \gamma_j$
- $\pi_j = G^j(z_M) = G^j(z_F)$

Now,

$$\begin{aligned}
E(Ce^{-rT}) &= e^{-ra}[C_0 + \delta_F \sum\nolimits_{n=0}^{\infty} \mathcal{C}_n H_n(a)] \\
&\quad + \sum\nolimits_{n=1}^{\infty} (C_0 \mathcal{B}_n + C_F \mathcal{C}_n) \int_0^a H_n(x) r e^{-rx} \, \mathrm{d}x \\
&= e^{-ra}[C_0 + \delta_F \sum\nolimits_{n=0}^{\infty} (\sum\nolimits_{i=1}^{n} \gamma_i) H_n(a)] \\
&\quad + \sum\nolimits_{n=1}^{\infty} (C_F (\sum\nolimits_{i=1}^{n} \gamma_i)) \int_0^a H_n(x) r e^{-rx} \, \mathrm{d}x \\
&= e^{-ra}[C_0 + \delta_F \sum\nolimits_{n=0}^{\infty} (1 - \pi_n) H_n(a)] \\
&\quad + \sum\nolimits_{n=1}^{\infty} (C_F (1 - \pi_n)) \int_0^a H_n(x) r e^{-rx} \, \mathrm{d}x \\
&= e^{-ra}[C_0 + \delta_F] - e^{-ra} \delta_F \sum\nolimits_{n=0}^{\infty} \pi_n H_n(a) + C_F \int_0^a r e^{-rx} \, \mathrm{d}x \\
&\quad - C_F \sum\nolimits_{n=1}^{\infty} \pi_n \int_0^a H_n(x) r e^{-rx} \, \mathrm{d}x \\
&= C_F \left(1 - \sum\nolimits_{n=0}^{\infty} \pi_n \int_0^a H_n(x) r e^{-rx} \, \mathrm{d}x \right) \\
&\quad - \delta_F e^{-ra} \sum\nolimits_{n=0}^{\infty} \pi_n H_n(a) \qquad (4.50)
\end{aligned}$$

And from equation 4.28, we get

$$E(e^{-rT}) = 1 - \sum\nolimits_{n=0}^{\infty} G^j(z_F) \int_0^a H_j(x) r e^{-rx} \, \mathrm{d}x \qquad (4.51)$$

The expected equivalent average cost rate can be found by substituting the value of $E(e^{-rT})$ and $E(Ce^{-rT})$ in equation (4.12)

4.6 Summary

General expression for asymptotic cost rate for maintenance is derived with and without considering discount factor. These cost rate's are calculated in terms of occurrence time of shocks as well as in terms of number of shocks. Expression for cost rate is function of damage caused by shocks, preventive level, failure level, and replacement age. Expression are also derived for case of replacement without considering age replacement and couple of other models.

Chapter 5

Applications

5.1 Introduction

For the analytical results derived in chapter 4, numerical examples of maintenance cost optimization are presented to illustrate the proposed model. van der Weide et al, [9] and Qian et al, [2] modelled shock process as Poisson process and exponential damage distribution. However, van der Weide et al, [9] has given more examples involving discount factor which have been discussed in this chapter. The optimization variables are the PM damage threshold (z_M) and the replacement age (a). Also in the case where age replacement is not considered, shock process is taken with Weibull inter-occurrence time. In all the derivations cost due to preventive maintenance and cost due to age replacement is taken to be the same. Sections 5.3 and 5.4 give the expression for most general cases. Section 5.5 describes the few cases where age replacement is not considered. In section 5.6, couple of specific models are considered.

5.2 Exponential Damage Distribution

This section is about the analysis of damage process when the shock follows exponential distribution is given as

$$G(x) = 1 - e^{-\lambda x},\ x \geq 0$$

First we calculate the distribution of the sum D_k of k damage increments. The general expression for the distribution D_k is given by

$$P(D_k \leq x) = \int_0^x G^{(k-1)}(x-y)\,\mathrm{d}G(y) \qquad for \qquad k \geq 1$$

First observe that

$$\begin{aligned} P(D_1 \leq x) = G(x) &= 1 - e^{-\lambda x} \\ P(D_2 \leq x) = G^{(2)}(x) &= 1 - \sum_{i=0}^{1} \frac{(\lambda x)^i}{i!} e^{-\lambda x} \qquad (5.1) \\ P(D_{k+1} \leq x) = G^{(k+1)}(x) &= 1 - \sum_{i=0}^{k} \frac{(\lambda x)^i}{i!} e^{-\lambda x} \qquad (5.2) \end{aligned}$$

We proceed with calculation of α_j, β_j, γ_j and π_j, which we need for the analysis of damage process.

$$\begin{aligned} \beta_j &= \int_0^{z_M} \Big[G(z_F - x) - G(z_M - x)\Big]\mathrm{d}G^{(j-1)}(x) \\ &= \int_0^{z_M} (1 - G(z_M - x))\mathrm{d}G^{j-1}(x) - \int_0^{z_M} (1 - G(z_F - x))\mathrm{d}G^{(j-1)}(x) \qquad (5.3) \end{aligned}$$

In order to proceed further we evaluate the integral $I = \int_0^{z_M} \Big[1 - G(\xi - x)\Big]\,\mathrm{d}G^{j-1}(x)$.

$$\begin{aligned} I &= \Big[e^{-\lambda(\xi - x)} G^{(j-1)}(x)\Big]_0^{z_M} - \int_0^{z_M} \frac{\mathrm{d}e^{-\lambda(\xi - x)}}{\mathrm{d}x} G^{(j-1)}(x)\mathrm{d}x. \\ &= e^{-\lambda(\xi - z_M)} - \sum_{i=0}^{j-2} \frac{(\lambda z_M)^i}{i!} e^{-\lambda \xi} - \lambda e^{-\lambda \xi} \Bigg[\int_0^{z_M} e^{\lambda x} \Bigg(1 - \sum_{i=0}^{j-2} \frac{(\lambda x)^i}{i!} e^{-\lambda x}\,\mathrm{d}x\Bigg)\Bigg] \\ &= \frac{(\lambda z_M)^{j-1}}{(j-1)!} \qquad (5.4) \end{aligned}$$

On substituting this value we get

$$\beta_j = \frac{(\lambda z_M)^{j-1}}{(j-1)!}\left(e^{-\lambda z_M} - e^{-\lambda z_F}\right) \tag{5.5}$$

$$\begin{aligned}
\alpha_j = G^{j-1}(z_M) - G^j(z_M) &= \left[1 - \sum_{i=0}^{j-2}\frac{(\lambda z_M)^i}{i!}e^{-\lambda z_M}\right] - \left[1 - \sum_{i=0}^{j-1}\frac{(\lambda z_M)^i}{i!}e^{-\lambda z_M}\right] \\
&= \frac{(\lambda z_M)^{(j-1)}}{(j-1)!}e^{-\lambda z_M} \qquad (5.6)
\end{aligned}$$

$$\begin{aligned}
\gamma_j = \alpha_j - \beta_j &= \frac{(\lambda z_M)^{j-1}}{(j-1)!}e^{-\lambda z_M} - \left[\frac{(\lambda z_M)^{(j-1)}}{(j-1)!}\left(e^{-\lambda z_M} - e^{-\lambda z_F}\right)\right] \\
&= \frac{(\lambda z_M)^{(j-1)}}{(j-1)!}e^{-\lambda z_F} \qquad (5.7)
\end{aligned}$$

$$\begin{aligned}
\pi_j = 1 - \sum_{i=1}^{j}\alpha_i &= G^j(z_M) \\
&= 1 - \sum_{i=0}^{j-1}\frac{(\lambda z_M)^i}{i!}e^{-\lambda z_M} \qquad (5.8) \\
&= 1 - \sum_{i=0}^{n}\alpha_i = 1 - \mathcal{A}_n \qquad (5.9) \\
\mathcal{B}_n = \sum_{j=1}^{n}\beta_j &= \sum_{j=1}^{n}\frac{(\lambda z_M)^{j-1}}{(j-1)!}\left(e^{-\lambda z_M} - e^{-\lambda z_F}\right) \\
&= \sum_{i=1}^{n}\frac{(\lambda z_M)^{j-1}}{(j-1)!}e^{-\lambda z_M}\left(1 - e^{-\lambda(z_F - z_M)}\right) \\
&= \left(1 - e^{-\lambda(z_F - z_M)}\right)\mathcal{A}_n \qquad (5.10) \\
\mathcal{C}_n = \sum_{i=1}^{n}\gamma_i &= \sum_{i=1}^{n}\frac{(\lambda z_M)^{i-1}}{(i-1)!}e^{-\lambda z_F} \\
&= e^{-\lambda(z_F - z_M)}\sum_{i=1}^{n}\frac{(\lambda z_M)^{i-1}}{(i-1)!}e^{-\lambda z_M} \qquad (5.11) \\
&= e^{-\lambda(z_F - z_M)}\mathcal{A}_n \qquad (5.12)
\end{aligned}$$

Define the sums

$$\textstyle\sum_1 = \sum_{n=1}^{\infty} \mathcal{A}_n H_n(a) \tag{5.13}$$

$$\textstyle\sum_2 = \sum_{n=1}^{\infty} \mathcal{A}_n \int_0^a H_n(x) \mathrm{d}x \tag{5.14}$$

$$\textstyle\sum_3 = \sum_{n=1}^{\infty} \mathcal{A}_n \int_0^a H_n(x)\, e^{-rx} \mathrm{d}x \tag{5.15}$$

5.3 General Expression for Cost Rate (Without Discounting)

From equation (4.34) we get the following expression for asymptotic cost rate

$$Q = \frac{C_0 + \delta_F \sum_{i=1}^{\infty} \mathcal{C}_i\, H_i(a)}{\sum_{j=0}^{\infty} \pi_j \int_0^a H_j(x)\, \mathrm{d}x} \tag{5.16}$$

It follows from the above calculations that

$$\begin{aligned} E(C) &= C_0 + \delta_F \sum_{i=1}^{\infty} \mathcal{C}_i\, H_i(a) \\ &= C_0 + \delta_F \sum_{i=1}^{\infty} e^{-\lambda(z_F - z_M)}\, \mathcal{A}_i\, H_i(a) \\ &= C_0 + \delta_F e^{-\lambda(z_F - z_M)} \textstyle\sum_1 \end{aligned} \tag{5.17}$$

$$\begin{aligned} E(T) &= \sum_{j=0}^{\infty} \pi_j \int_0^a H_j(x)\, \mathrm{d}x \\ &= \sum_{j=0}^{\infty} (1 - \mathcal{A}_j) \int_0^a H_j(x)\, \mathrm{d}x \\ &= a - \textstyle\sum_2 \end{aligned} \tag{5.18}$$

Hence the asymptotic non-discounted cost-rate can be given as

$$Q = \frac{C_0 + \delta_F e^{-\lambda(z_F - z_M)} \sum_1}{a - \sum_2} \tag{5.19}$$

5.3.1 Homogenous Poisson process

When the shock follows homogenous Poisson process

$$\begin{aligned}
H_i(a) &= P(N(a)=i) = \frac{(\mu a)^i e^{-\mu a}}{i!} \\
\mathcal{A}_i = \sum_{j=1}^{i} \alpha_j &= \sum_{j=1}^{i} \frac{(\lambda z_M)^{(j-1)}}{(j-1)!} e^{-\lambda z_M} \\
&= \sum_{j=0}^{i-1} \frac{(\lambda z_M)^j}{j!} e^{-\lambda z_M} \\
\therefore \qquad Q(a) &= \frac{C_0 + \delta_F e^{-\lambda(z_F - z_M)} \sum_{i=1}^{\infty} \left(\sum_{j=0}^{i-1} \frac{(\lambda z_M)^j}{j!} e^{-\lambda z_M} \right) H_i(a)}{a - \sum_{i=1}^{\infty} \left(\sum_{j=0}^{i-1} \frac{(\lambda z_M)^j}{j!} e^{-\lambda z_M} \right) \int_0^a H_i(x) \mathrm{d}x.} \qquad (5.20)
\end{aligned}$$

Now in order to proceed further we need to evaluate the integral

$$\begin{aligned}
I_i(a) = \int_0^a H_i(x)\,\mathrm{d}x &= \int_0^a \frac{(\mu x)^i e^{-\mu x}}{i!} \mathrm{d}x = \frac{\mu^i}{i!} \int_0^a x^i e^{-\mu x} \mathrm{d}x \\
&= \frac{\mu^i}{i!} \left\{ \left[\frac{-x^i e^{-\mu x}}{\mu} \right]_0^a + \int_0^a \frac{i x^{i-1} e^{-\mu x}}{\mu}\,\mathrm{d}x \right\} \\
&= -\frac{a}{i} H_{(i-1)}(a) + I_{(i-1)}(a) \qquad (5.21) \\
I_0(a) &= \int_0^a H_0(x)\,\mathrm{d}x = \int_0^a e^{-\mu x}\,\mathrm{d}x \\
&= -\frac{1}{\mu} [e^{-\mu x}]_0^a = \frac{1 - e^{-\mu a}}{\mu} \qquad (5.22)
\end{aligned}$$

Considering Q as a function of a we want to see how the cost rate varies with age.

$$Q(a) = \frac{C_0 + \delta_F e^{-\lambda(z_F - z_M)} \sum_{i=1}^{\infty} \left(\sum_{j=0}^{i-1} \frac{(\lambda z_M)^j}{j!} e^{-\lambda z_M} \right) H_i(a)}{a - \sum_{i=1}^{\infty} \left(\sum_{j=0}^{i-1} \frac{(\lambda z_M)^j}{j!} e^{-\lambda z_M} \right) \int_0^a H_i(x) \mathrm{d}x.} \qquad (5.23)$$

To begin with consider $C_0 = 20, z_M = 28, z_F = 30, \lambda = 0.5$.

$$Q(a) = \frac{20 + \delta_F e^{-1} \sum_{i=1}^{\infty} \left(\sum_{j=0}^{i-1} \frac{(14)^j}{j!} e^{-14} \right) H_i(a)}{a - \sum_{i=1}^{\infty} \left(\sum_{j=0}^{i-1} \frac{14^j}{j!} e^{-14} \right) \int_0^a H_i(x) \mathrm{d}x.} \qquad (5.24)$$

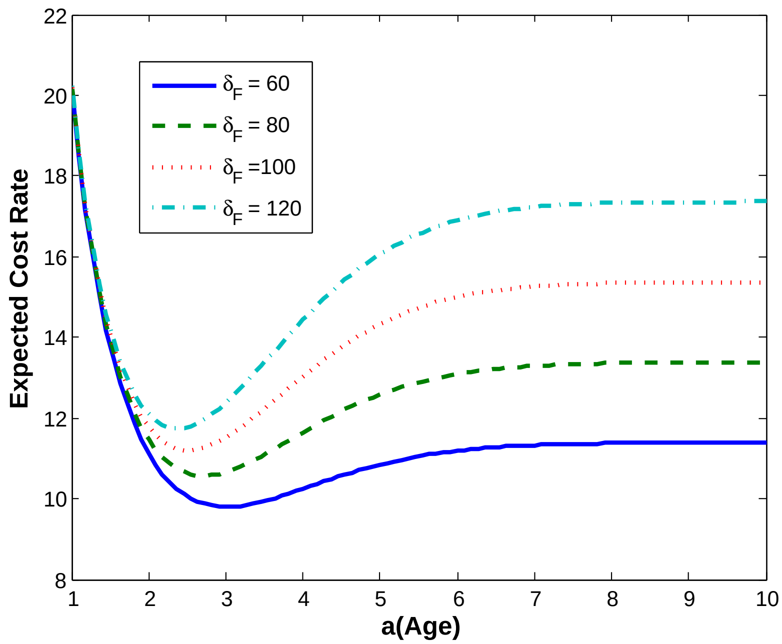


Figure 5.1: General Cost Rate for HPP without Discounting

When the cost due to corrective maintenance as compared to preventive maintenance is high (i.e. δ_F is high), as it happens in the case of nuclear power plants, it would be reasonable to decrease preventive damage level as the following diagrams show, Consider $C_0 = 20, \delta_F = 120, z_F = 30, \lambda = 0.5$.

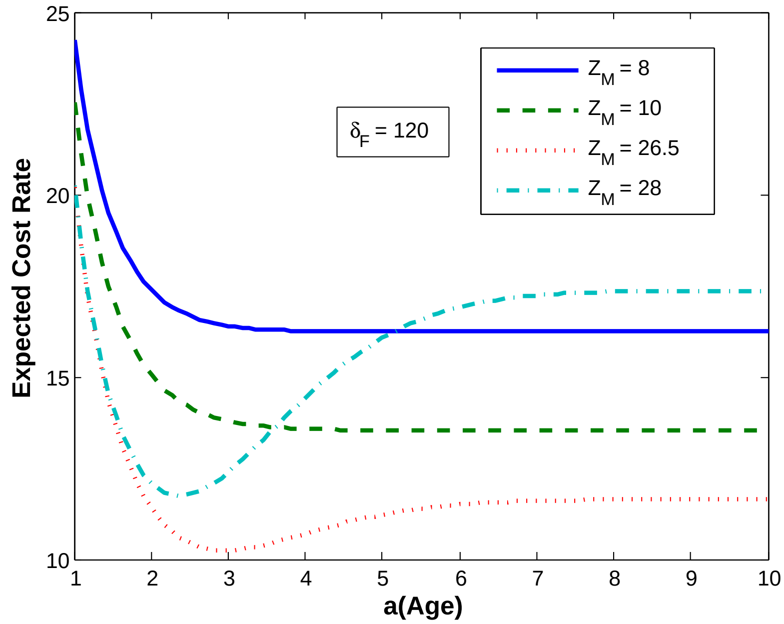


Figure 5.2: General Cost Rate for HPP without Discounting

When the maintenance cost due to failure level is very high, it would be natural to decrease the PM level, as the above graph shows. Decreasing z_M from 28 to 26.5 decreases the cost rate. However further decrease in z_M increases the cost rate as decrease in z_M implies frequent repair.

5.3.2 Non-Homogeneous Poisson process

When the shock follows non-homogeneous Poisson process with $\mu(t) = 2t$

$$H_i(a) = P(\mathcal{N}(a) = i) = \frac{a^{2i}e^{-a^2}}{i!} \tag{5.25}$$

$$Q(a) = \frac{C_0 + \delta_F e^{-\lambda(z_F - z_M)} \sum_{i=1}^{\infty} \left(\sum_{j=0}^{i-1} \frac{(\lambda z_M)^j}{j!} e^{-\lambda z_M} \right) H_i(a)}{a - \sum_{i=1}^{\infty} \left(\sum_{j=0}^{i-1} \frac{(\lambda z_M)^j}{j!} e^{-\lambda z_M} \right) \int_0^a H_i(x) \mathrm{d}x.} \tag{5.26}$$

To get recursive relation for $I_i = \int_0^a H_i(x)\,\mathrm{d}x$ consider following equation

$$\frac{d}{dx}(x^{2i+1}e^{-x^2}) = (2i+1)x^{2i}e^{-x^2} + x^{2i+1}e^{-x^2}(-2x)$$

$$
\begin{aligned}
\therefore \quad 2\int_0^a x^{2(i+1)} e^{-x^2}\,\mathrm{d}x &= (2i+1)\int_0^a x^{2i} e^{-x^2}\,\mathrm{d}x - a^{2i+1}e^{-a^2} \\
\therefore \quad I_{i+1}(a) &= \frac{2i+1}{2(i+1)} I_i(a) - \frac{a}{2(i+1)} H_i(a) \\
\text{or} \quad I_i(a) &= \frac{2i-1}{2i} I_{i-1}(a) - \frac{a}{2i} H_{(i-1)}(a) \\
\text{and} \quad I_0(a) &= \int_0^a e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}\,\mathrm{erf}(a) \qquad (5.27)
\end{aligned}
$$

As discussed in the homogeneous case we want to see how the cost rate varies with age in non-homogeneous case. Consider $C_0 = 20, z_M = 28, z_F = 30, \lambda = 0.5$
On substituting these values we get

$$
Q(a) = \frac{20 + \delta_F e^{-1} \sum_{i=1}^{\infty} \left(\sum_{j=0}^{i-1} \frac{(14)^j}{j!} e^{-14} \right) H_i(a)}{a - \sum_{i=1}^{\infty} \left(\sum_{j=0}^{i-1} \frac{14^j}{j!} e^{-14} \right) \left(\frac{2i-1}{2i} I_{i-1}(a) - \frac{a}{2i} H_{(i-1)}(a) \right)} \qquad (5.28)
$$

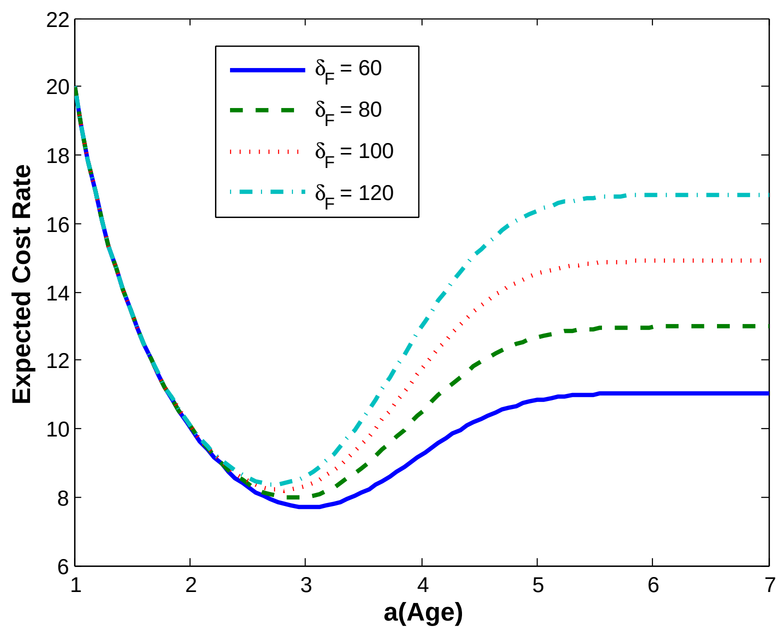


Figure 5.3: General Cost Rate for NHPP without Discounting

Hence for $\mu(t) = 2t$ it is seen that minimum expected cost rate remains almost the same for different values of δ_F. However, the difference increase with increase in the value of a.

5.4 General Expression for Cost Rate (With Discounting)

From equation (4.27) and (4.28)

$$
\begin{aligned}
E(Ce^{-rT}) &= e^{-ra}\left(C_0 + \delta_F \sum_{n=1}^{\infty} \mathcal{C}_n H_n(a)\right) + \sum_{n=1}^{\infty} (C_0\mathcal{A}_n + \delta_F \mathcal{C}_n) \int_0^a H_n(x) r e^{-rx}\, \mathrm{d}x \\
&= e^{-ra}\left(C_0 + \delta_F \sum_{n=1}^{\infty} e^{-\lambda(z_F - z_M)} \mathcal{A}_n H_n(a)\right) \\
&\quad + r\left[C_0 \sum_{n=0}^{\infty} \mathcal{A}_n \int_0^a H_n(x) e^{-rx}\, \mathrm{d}x + \delta_F \sum_{n=1}^{\infty} \mathcal{C}_n \int_0^a H_n(x)\, e^{-rx}\, \mathrm{d}x\right] \\
&= e^{-ra}\left(C_0 + \delta_F e^{\lambda(z_F - z_M)} \sum_{n=1}^{\infty} \mathcal{A}_n H_n(a)\right) \\
&\quad + r\left[C_0 {\textstyle\sum_3} + \delta_F \sum_{n=1}^{\infty} e^{-\lambda(z_F - z_M)} \mathcal{A}_n \int_0^a H_n(x) e^{-rx} \mathrm{d}x\right] \\
&= e^{-ra}\left(C_0 + \delta_F e^{-\lambda(z_F - z_M)} {\textstyle\sum_1}\right) + r\left[C_0 + \delta_F e^{-\lambda(z_F - z_M)}\right] {\textstyle\sum_3} \qquad (5.29) \\
E[e^{-rT}] &= 1 - \sum_{n=0}^{\infty} \pi_n \int_0^a H_n(x) r e^{-rx} \mathrm{d}x \\
&= 1 - \sum_{n=0}^{\infty} (1 - \mathcal{A}_n) \int_0^a H_n(x) e^{-rx} \mathrm{d}x \\
&= 1 - \left[\int_0^a e^{-rx}\, \mathrm{d}x - {\textstyle\sum_3}\right] \\
&= 1 - \left[\frac{1 - e^{-ra}}{r} - {\textstyle\sum_3}\right] \qquad (5.30)
\end{aligned}
$$

Expected equivalent average discounted cost rate per unit time $Q(r)$, is given by

$$\begin{aligned}
Q(r) &= \frac{r\,E(Ce^{-rT})}{1-E(e^{-rT})} \\
&= \frac{r\left[e^{-ra}\left(C_0+\delta_F e^{-\lambda(z_F-z_M)}\sum_1\right)+r\left(C_0+\delta_F e^{-\lambda(z_F-z_M)}\right)\sum_3\right]}{1-\left[1-\left\{\frac{1-e^{-ra}}{r}-\sum_3\right\}\right]} \\
&= \frac{r^2\left[e^{-ra}\left(C_0+\delta_F e^{-\lambda(z_F-z_M)}\sum_1\right)+r\left(C_0+\delta_F e^{-\lambda(z_F-z_M)}\right)\sum_3\right]}{\left\{1-e^{-ra}-r\sum_3\right\}} \qquad (5.31)
\end{aligned}$$

One gets relationship between age and expected cost rate similar to the one calculated without considering discount factor using proper recursive relation for calculating $\sum_3$. Now in order to calculate the value of $Q(r)$, we need to simplify the expression for $\sum_3$

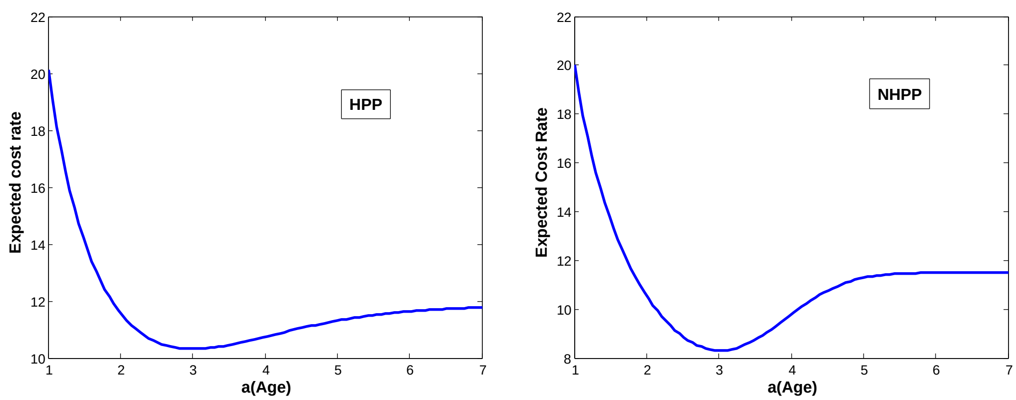
When $H_n(x)$ is a Homogeneous Poisson Process with parameter μ.

$$\begin{aligned}
\sum\nolimits_3 &= \sum_{n=1}^{\infty}\mathcal{A}_n\int_0^a H_n(x)\,e^{-rx}\mathrm{d}x = \sum_{n=1}^{\infty}\mathcal{A}_n\mathcal{I}_n(a) \qquad (5.32)\\
\mathcal{I}_n(a) &= \int_0^a H_n(x)\,e^{-rx}\mathrm{d}x = \int_0^a \frac{(\mu x)^n}{n!}e^{-\mu x}\,e^{-rx}\mathrm{d}x \\
&= -\frac{\mu a}{n}\frac{e^{-ra}}{\mu+r}H_{n-1}(a)+\frac{\mu}{\mu+r}\mathcal{I}_{n-1}(a) \qquad (5.33)\\
\mathcal{I}_0(a) &= \int_0^a H_0(x)e^{-rx}\,\mathrm{d}x = \int_0^a e^{-\mu x-rx}\,\mathrm{d}x \\
&= \left[-\frac{e^{-(\mu+r)x}}{\mu+r}\right]_0^a = \frac{1-e^{-(\mu+r)a}}{\mu+r} \qquad (5.34)
\end{aligned}$$

When $H_n(x)$ is Non-Homogeneous Poisson Process with parameter $2x$.

$$\begin{aligned}
\sum\nolimits_3 &= \sum_{n=1}^{\infty}\mathcal{A}_n\int_0^a H_n(x)\,e^{-rx}\mathrm{d}x = \sum_{n=1}^{\infty}\mathcal{A}_n\mathcal{I}_n(a) \\
\mathcal{I}_n(a) &= \int_0^a H_n(x)e^{-rx}\,\mathrm{d}x = \int_0^a \frac{e^{-x^2}x^{2n}}{n!}e^{-rx}\,\mathrm{d}x \\
&= \frac{1}{n!2}\left[-a^{2n-1}e^{-ra}e^{-a^2}-r\mathcal{I}(2n-1)+(2n-1)\mathcal{I}(2n-2)\right] \qquad (5.35)
\end{aligned}$$

$$
\begin{aligned}
\mathcal{I}_0(a) &= \int_0^a e^{-ry-y^2}\,\mathrm{d}y \\
&= \frac{\sqrt{\pi}}{2} e^{(\frac{r}{2})^2} \left[\operatorname{erf}(a+\frac{r}{2}) - \operatorname{erf}(\frac{r}{2})\right] \qquad (5.36) \\
\mathcal{I}_1(a) &= \int_0^a y e^{-ry-ry^2}\,\mathrm{d}y \\
&= \frac{1}{2}\left[1 - e^{-ra-a^2}\right] - \frac{r}{2}\mathcal{I}(0) \qquad (5.37)
\end{aligned}
$$



(a) HPP

(b) NHPP

Figure 5.4: General Cost Rate with Discounting

The parameter used for investigating the variation of cost rate with age in the above case are $C_0 = 20, z_M = 28, z_F = 30, \lambda = 0.5, r = 0.04$

5.5 Without Age Replacement

As Discussed in the previous chapter, in this case the system is renewed preventively as soon as $Z(t)$ exceeds a maintenance threshold value z_M, without age replacement.

5.5.1 Cost Rate in Terms of H_n

The expression for Q and $Q(r)$ in terms of the probability distribution of the number of shocks, using equation (4.45) and (4.46) is given by

$$Q = \frac{\sum_{j=1}^{\infty}(C_M\beta_j + C_F\gamma_j)}{\sum_{i=1}^{\infty}\mathcal{A}_i\int_0^{\infty}H_i(x)\,\mathrm{d}x}, \tag{5.38}$$

$$Q(r) = \frac{r^2\sum_{i=1}^{\infty}(C_M\mathcal{B}_i + C_F\mathcal{C}_i)\int_0^{\infty}H_i(x)e^{-rx}\,\mathrm{d}x}{1 - r\sum_{i=1}^{\infty}\mathcal{A}_i\int_0^{\infty}H_i(x)e^{-rx}\,\mathrm{d}x} \tag{5.39}$$

For HPP, using recursive relations similar to equations (5.21) and (5.33) we calculate the integrals involved in equations (5.38) and (5.39).

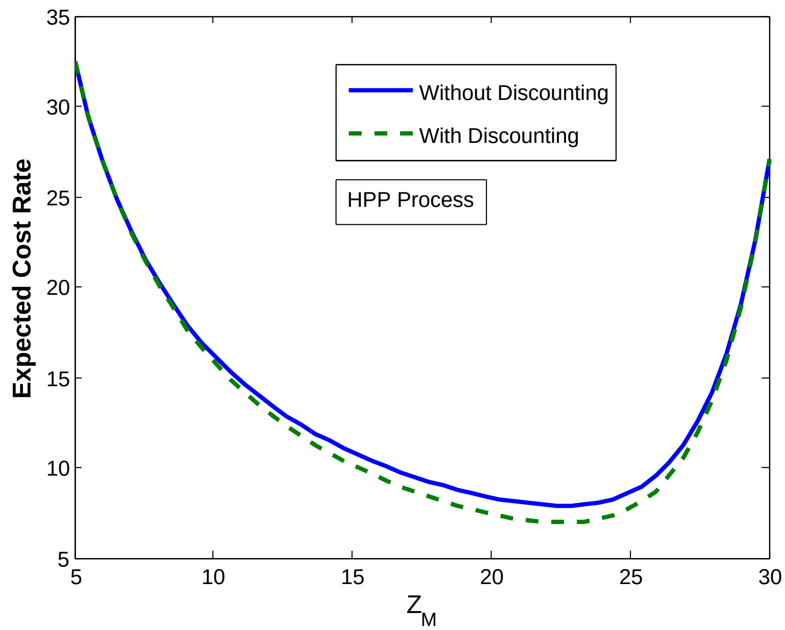


Figure 5.5: HPP Shock Process

For NHPP, using recursive relations similar to equations (5.27) and (5.35) integrals involved in equations (5.38) and (5.39) can be simplified to obtain results numerically. Parameters for Figures 5.5 and 5.6 are $C_0 = 20, C_F = 100, \delta_F = 80, z_F = 30, r = 0.04, \mu = 4.06$

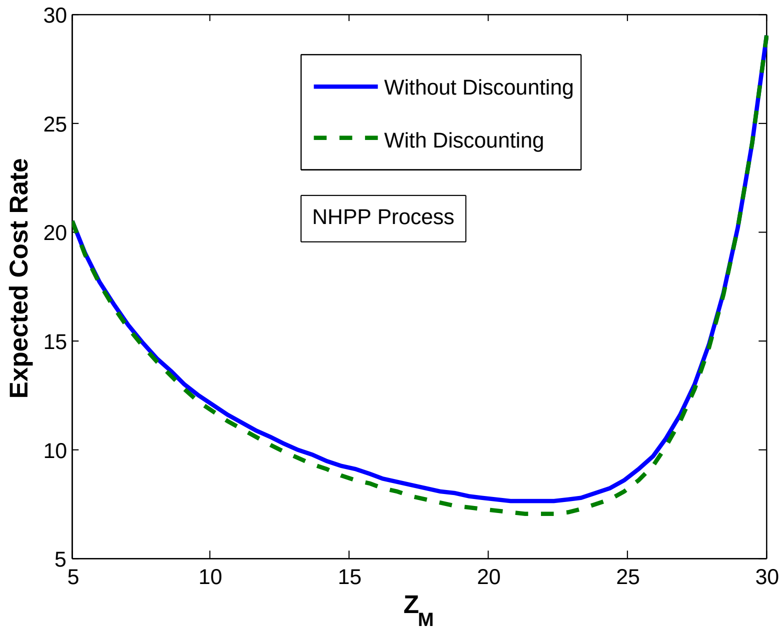


Figure 5.6: NHPP Shock Process

5.5.2 Cost Rate in Terms of $\mathcal{S}_j$

Asymptotic non-discounted cost rate using equation. (4.40) is given by

$$Q = \frac{\sum_{j=1}^{\infty} (C_M\beta_j + C_F\gamma_j)}{\sum_{j=1}^{\infty} \alpha_j E(\mathcal{S}_j)} \tag{5.40}$$

Asymptotic discounted rate using equation. (4.41) is given by

$$\begin{aligned} Q(r) &= \frac{r\sum_{j=1}^{\infty} (C_M\beta_j + C_F\gamma_j)\, E(e^{-r\mathcal{S}_j})}{1-\sum_{j=1}^{\infty} \alpha_j E(e^{-r\mathcal{S}_j})} \\ &= \frac{r\sum_{j=1}^{\infty} (C_0(\beta_j + \gamma_j) + \delta_F\gamma_j)\, E(e^{-r\mathcal{S}_j})}{1-\sum_{j=1}^{\infty} \alpha_j E(e^{-r\mathcal{S}_j})} \\ &= \frac{r\left[C_0\sum_{j=1}^{\infty} \alpha_j E(e^{-r\mathcal{S}_j}) + \delta_F\sum_{j=1}^{\infty} \gamma_j E(e^{-r\mathcal{S}_j})\right]}{1-\sum_{j=1}^{\infty} \alpha_j E(e^{-r\mathcal{S}_j})} \end{aligned} \tag{5.41}$$

When $\mathcal{S}_j$ is a renewal process with inter-arrival times X_i, the expressions for equations 5.40 and 5.41 can be further simplified.

For asymptotic **non-discounted** cost rate

$$\begin{aligned} \text{Numerator} &= \sum_{j=1}^{\infty} [C_M\beta_j + C_F\gamma_j] = \sum_{j=1}^{\infty} C_0(\beta_j+\gamma_j) + \delta_F\gamma_j \\ &= C_0 + \delta_F e^{-\lambda(z_F - z_M)} \end{aligned} \tag{5.42}$$

$$\begin{aligned} \text{Denominator} &= \sum_{j=1}^{\infty} \alpha_j E(\mathcal{S}_j) = \sum_{j=1}^{\infty} \alpha_j E(X_1 + \ldots + X_j) \\ &= \sum_{j=1}^{\infty} \alpha_j (jm) = \frac{m}{\lambda z_M}\sum_{j=1}^{\infty} j^2 \frac{(\lambda z_M)^j}{j!} e^{-\lambda z_M} \\ &= m(\lambda z_M + 1) \end{aligned} \tag{5.43}$$

Where $m = E(X)$
Substituting equations 5.42 and 5.43 in 5.40,

$$\therefore \qquad Q = \frac{C_0 + \delta_F e^{-\lambda(z_F - z_M)}}{m(\lambda z_M + 1)} \tag{5.44}$$

For asymptotic **discounted** cost rate

$$\begin{aligned}
\text{Numerator} &= r\left[C_0 \sum_{j=1}^{\infty} \alpha_j \left(E(e^{-rX_1})\right)^j + \delta_F \sum_{j=1}^{\infty} \gamma_j \left(E(e^{-rX_1})\right)^j\right] \\
&= r\left[C_0 + \delta_F \omega e^{-\lambda(z_F - z_M)}\right] \omega e^{-\lambda z_M (1-\omega)} \qquad (5.45) \\
\text{Denominator} &= 1 - \sum_{j=1}^{\infty} \alpha_j E(e^{-r\mathcal{S}_j}) \\
&= 1 - \omega e^{-\lambda z_M (1-\omega)} \qquad (5.46)
\end{aligned}$$

Where $\omega = E(e^{-rX})$

Substituting equations (5.45) and (5.46) in (5.41),

$$\therefore \qquad Q(r) = \frac{r\left[C_0 + \delta_F \omega e^{-\lambda(z_F - z_M)}\right] \omega e^{-\lambda z_M (1-\omega)}}{1 - \omega e^{-\lambda z_M (1-\omega)}} \qquad (5.47)$$

Example (1)

Here we consider the example where shock process is renewal with Weibull inter-occurrence times. Probability density of Weibull distribution with scale parameter θ, and shape parameter α is given by

$$f(x;\theta,\alpha) = \frac{\alpha}{\theta}(\frac{x}{\theta})^{\alpha-1} e^{-(\frac{x}{\theta})^{\alpha}}$$

For $\theta = 3$ and $\alpha = 2$

$$f(x;3,2) = \frac{2}{3}(\frac{x}{3}) e^{-(\frac{x}{3})^2}$$

$$\begin{aligned}
m = E[X] =& \theta\,\Gamma(1 + \frac{1}{\alpha}) = 3\,\Gamma(1 + \frac{1}{2}) = 2.6587 \\
\omega = E&[e^{-rX}] = 0.8776
\end{aligned}$$

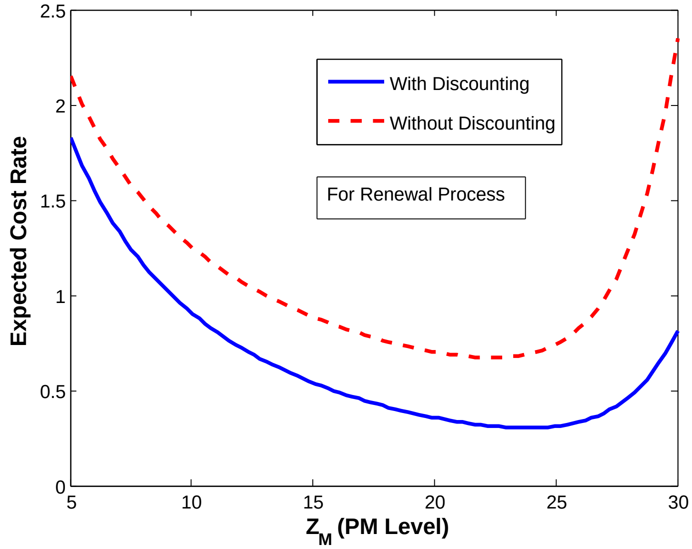


Figure 5.7: Variation of **cost rate** with $\boldsymbol{z_M}$

The parameters used in the above example are $C_0 = 20, \delta_F = 80, \lambda = 0.5, z_F = 30, r = 0.05$

Example (2)
Now consider the example where shock process is renewal whose inter-occurrence times are Chi-square distributed. Probability density of Chi-square distribution with k degrees of freedom is given by

$$f(x;k) = \begin{cases} \frac{1}{2^{k/2}\Gamma(k/2)} x^{(k/2)-1} e^{-x/2} & \text{for} \quad x > 0, \\ 0 & \text{for} \quad x \leq 0 \end{cases}$$

For $k = 1$

$$f(x;k) = \begin{cases} \frac{1}{2^{1/2}\Gamma(1/2)} x^{-(1/2)} e^{-x/2} & \text{for} \quad x > 0, \\ 0 & \text{for} \quad x \leq 0 \end{cases}$$

$$m = E[X] = k = 1$$
$$\omega = E[e^{-rX}] = (1-2r)^{-1/2}$$

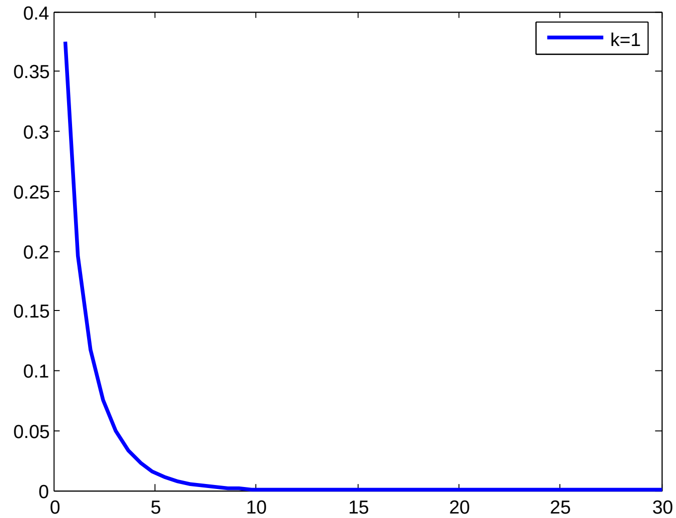


Figure 5.8: PDF of χ^2 distribution with $k = 1$

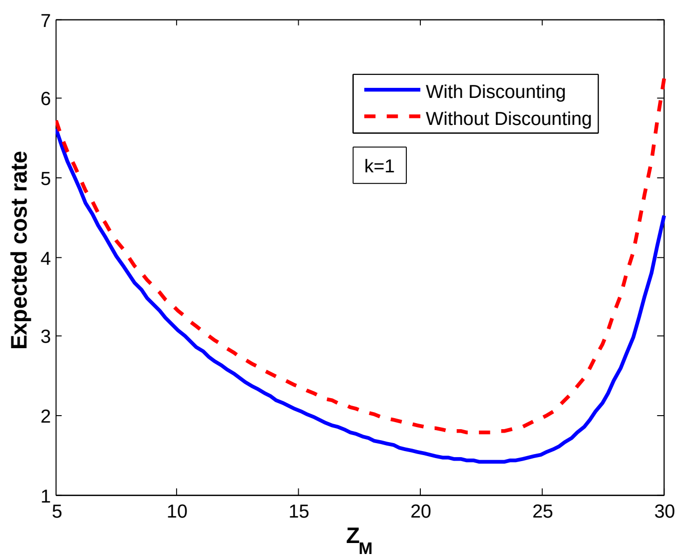


Figure 5.9: Variation of **cost rate** with $\boldsymbol{z_M}$

Again,the parameters used in the above example are $C_0 = 20, \delta_F = 80, \lambda = 0.5, z_F = 30, r = 0.05$. Most often, inter-arrival times of renewal shock process is not so less initially. So it is not practical to consider χ^2 distribution with 1 degree of freedom. Figure (5.8) shows probability density of χ^2 distributed inter-occurrences times with one degree of freedom. Following is χ^2 probability distribution with five, ten and fifteen degrees of freedom, which is more reasonable assumption.

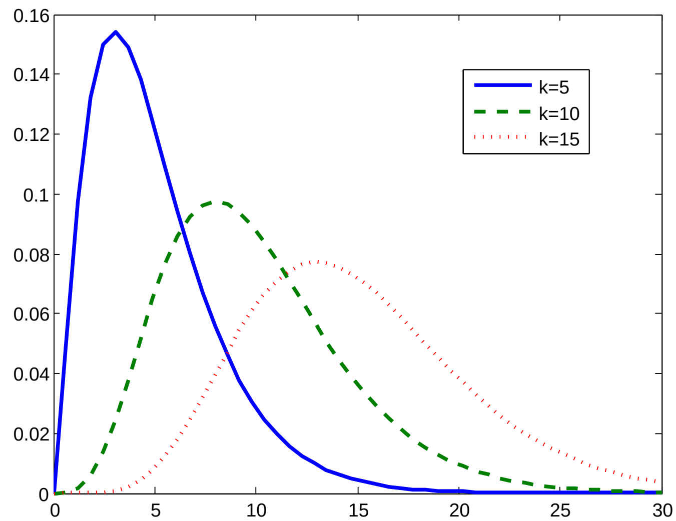


Figure 5.10: PDF of χ^2 distribution

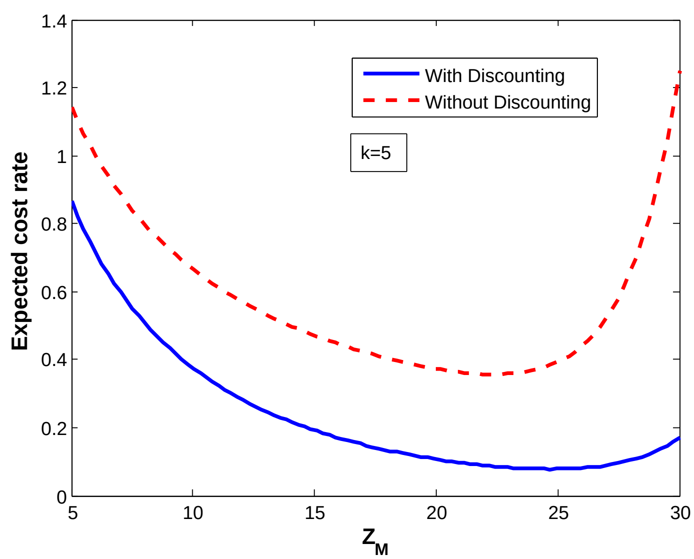


Figure 5.11: Variation of **cost rate** with $\boldsymbol{z_M}$

Figure (5.11) displays the variation of cost rate with z_M when inter-occurrence times are χ^2 distributed with five degrees of freedom. For this case there is significant difference between discounted and non-discounted cost rate.

Example (3)
Consider the example where shock process is renewal whose inter-occurrence times follow Erlang distribution. Probability density of Erlang distribution with shape parameter k and scale parameter θ is given by

$$f(x,k,\theta) = \frac{x^{k-1}e^{-\frac{x}{\theta}}}{\theta^k(k-1)!} x \geq 0, \theta > 0$$

As in the case with χ^2 distribution, it is more practical to consider scale parameter 2 and shape parameter 3 for the Erlang distribution, as it often fits practical data better than other parameters.

$$m = E[X] = k\theta = 6$$
$$\omega = E[e^{-rX}] = (1 - r\theta)^{-k} = (1 - 0.05 \times 2)^{-2} = 1.2346$$

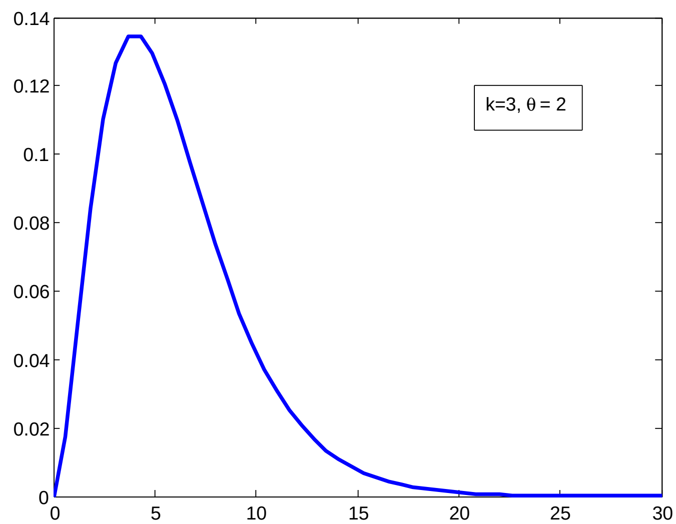


Figure 5.12: PDF of Erlang distribution

Figure (5.12) indicates the probability density of inter occurrence times. Figure (5.13) indicates the cost rate, when the inter arrival times follow Erlang distribution.

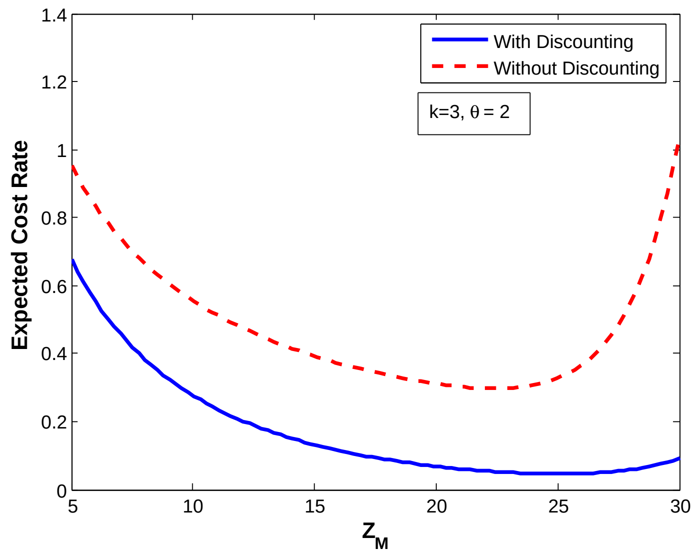


Figure 5.13: Variation of **cost rate** with $\boldsymbol{z_M}$

5.6 Specific Maintenance Models

In this section we analyze the three maintenance models introduced in the previous chapter. We make the following assumptions

1. The shock process $\mathcal{N}(t)$ is a non-homogeneous poisson process with intensity function $\mu(t) = 2t$, and
2. The damage size per shock is exponentially distributed with parameter $\lambda > 0$.

5.6.1 Model 1

Expected long term discounted value of the expected equivalent average cost rate when PM occurs at the first shock or at the age a is given by (from equation (4.48) and (4.49))

$$\begin{aligned} Q(a,r) &= r\frac{C_0(1-\int_0^a H_0(x)re^{-rx}\,\mathrm{d}x) + \delta_F\bar{G}(z_F)[1-e^{-ra}H_0(a)-\int_0^a H_0(x)re^{-rx}\,\mathrm{d}x]}{\int_0^a H_0(x)re^{-rx}\,\mathrm{d}x} \\ &= \frac{C_0(1-r\int_0^a H_0(x)e^{-rx}\,\mathrm{d}x) + \delta_F\bar{G}(z_F)[1-e^{-ra}H_0(a)-r\int_0^a H_0(x)e^{-rx}\,\mathrm{d}x]}{\int_0^a H_0(x)e^{-rx}\,\mathrm{d}x} \end{aligned} \tag{5.48}$$

Where

$$\begin{aligned}\int_0^a H_0(y)e^{-ry}\,\mathrm{d}y &= \int_0^a e^{-ry-y^2}\,\mathrm{d}y \\ &= \frac{\sqrt{\pi}}{2}e^{\left(\frac{r}{2}\right)^2}\left[\operatorname{erf}(a+\frac{r}{2})-\operatorname{erf}(\frac{r}{2})\right]\end{aligned}$$

In this model we have only the age a as a decision variable. We choose a such that the long term discounted value of the expected equivalent average cost rate is minimized. Following figure illustrates the dependence of the long-term discounted value of the expected equivalent average cost rate on the age a. We took in this example $\lambda = 0.56$, $z_F = 3$ and $C_0 = 20$.

Following figures demonstrate the change in expected equivalent average cost rate with age a. Figure (5.14) illustrates change with different values of r (interest rate), while Figure (5.15) illustrates change with different values of δ_F.

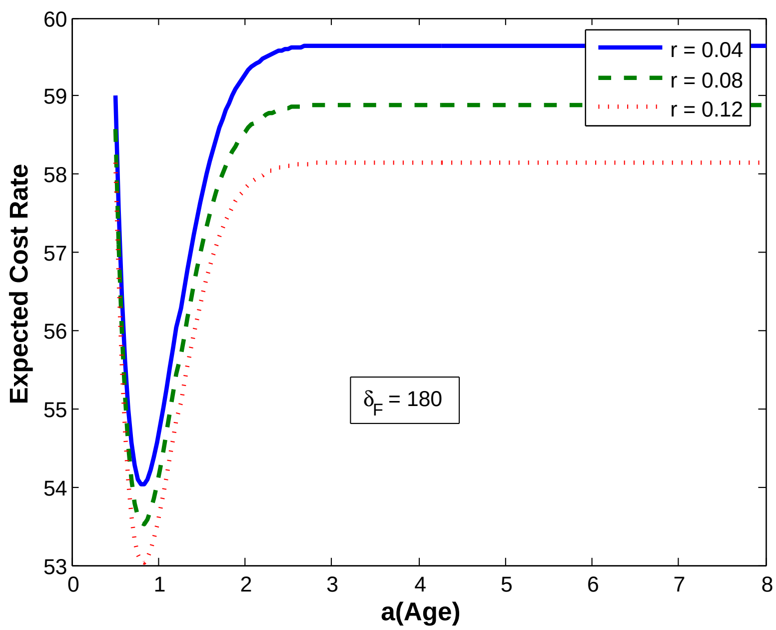


Figure 5.14: PM occurs at the first shock or age a

For all the values of r, minimum is observed at $a = 0.875$. The table below summarizes the observations.

	$r = 0.04$	r=0.08	r=0.12
Minimum value of $Q(r)$	54.0319	53.5317	53.0358
Mean cycle length for $a = 0.875$. $(E(T))$	0.8120	0.8120	0.8120
Probability that no shock occurs during $(0, a]$, $(H_0(a))$	0.465	0.465	0.465

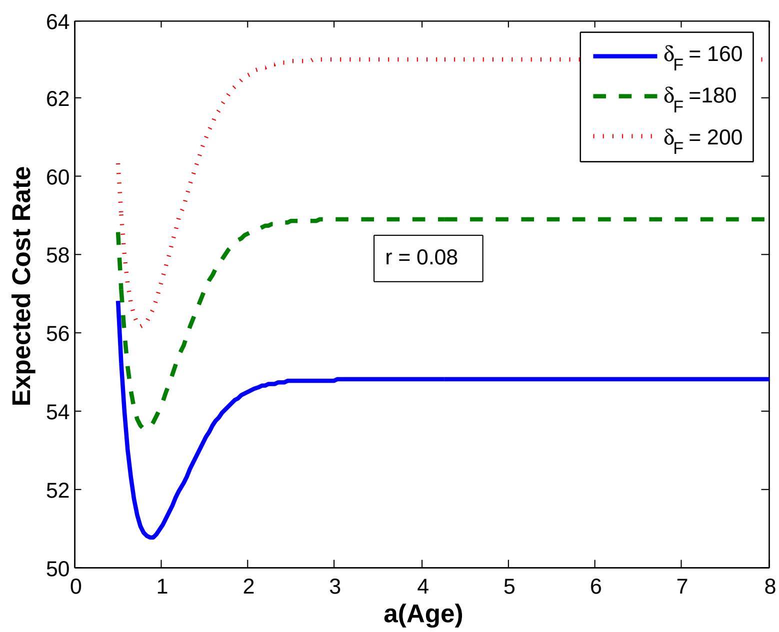


Figure 5.15: PM occurs at the first shock or age a

5.6.2 Model 2

From equations (4.50) and (4.51), expected long term discounted value of the expected equivalent average cost rate when the maintenance occurs at failure or at age a is given by

$$Q(a,r) = r\frac{C_F\left(1 - \sum_{n=0}^{\infty}\pi_n \int_0^a H_n(x)re^{-rx}\,\mathrm{d}x\right) - \delta_F e^{-ra}\sum_{n=0}^{\infty}\pi_n H_n(a)}{\sum_{n=0}^{\infty} G^j(z_F)\int_0^a H_j(x)re^{-rx}\,\mathrm{d}x} \tag{5.49}$$

The integral, $\int_0^a H_j(x)re^{-rx}\,\mathrm{d}x$ can be calculated as in Eq (5.35).
Since the maintenance occurs at failure, $z_M = z_F$ and hence

$$\pi_n = 1 - \sum_{i=0}^{n-1}\frac{(\lambda z_M)^i}{i!}e^{-\lambda z_M} = 1 - \sum_{i=0}^{n-1}\frac{(\lambda z_F)^i}{i!}e^{-\lambda z_F} \tag{5.50}$$

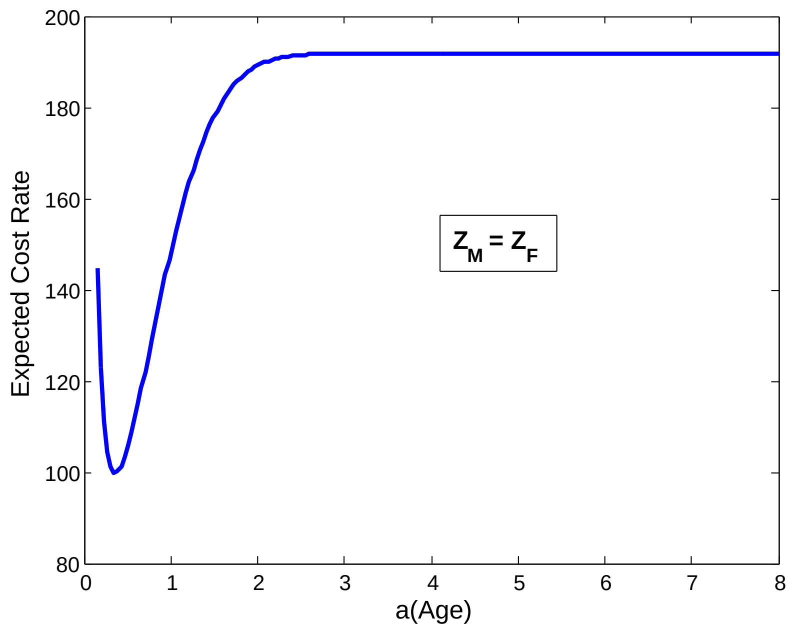


Figure 5.16: PM occurs at failure or age a

The parameters used in the performing the cost analysis of Model 2 are $C_0 = 20, \lambda = 0.56, z_M = z_F = 30, r = 0.04$

5.7 Conclusion

The maintenance cost rate can be optimized with respect to both z_M and a. For a specified z_M, an optimal a can be found that would minimize the cost rate. The global minimum of the cost rate is insensitive to the age replacement(a), and it is the same as the minimum cost for only condition-based strategy. This suggests that for minimizing the cost rate, a condition-based strategy will always supersede age-based replacement. The proposed formulation can incorporate a general point process as a shock process. For example, renewal process can be included in the model. The derivation of discounted cost rate provides more practical solutions to the optimization of maintenance policies than those based on non-discounted cost rate criterion.

Chapter 6

Conclusions

6.1 Conclusion & Contributions

Stochastic process model remains the critical component in preventive maintenance of engineering systems. The main purpose of this thesis is to study the maintenance models of engineering systems. Detailed analysis of work done by Savits, T. H. [1] and van der Weide et al [9] is done. In addition, detailed analysis has been done for the case when the inter occurrence times follow Chi-Square and Erlang distributions and the cost rate has been derived by considering the parameters which give distribution that can fit empirical data. In chapter (2) common existing maintenance policies and relation between them have been discussed. These policies can be used where failure is not catastrophic and preventive maintenance is not required. However, the best among these common maintenance policies could be implemented by calculating the cost rate as derived in detail. Very often it is not possible to calculate cost rate for one maintenance policy. In that case, the cost relationship as indicated by equations 2.15 and 2.20 could be implemented. Chapters 3, 4, & 5 show how maintenance policies can be optimized by suitable combination of preventive maintenance and age replacement. General expressions and expressions without age replacement of cost rate considering preventive maintenance, age based replacement and cost replacement are discussed. Some commonly observed specific cases have also been considered.

6.2 Future Work

For the models discussed in the thesis, asymptotic discount cost is calculated at the beginning of the cycle. Instead of considering the cost at the beginning of cycle one can consider terminating stream of fixed payments over a specified period of time. One can also work with several cases when cost is constant over time. Because of mathematical convenience, most of the literature available today is on exponential discounting. Some literature, as discussed in section (2.5) suggests that other types of discounting may be appropriate to work with in several cases. Damage distribution is take to be exponential in van der Weide et al, [9] and Qian et al, [2]. However, one fails to evaluate integrals of the type (5.3) even for gamma distribution. Different methods could be implemented for other distributions as well.

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