Some functional equations connected with the utility of gains and losses

by

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I hereby declare that I am the sole author of this thesis.

This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

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Abstract

The behavioral properties shown by people when they make selections between different choices will be studied. Based on empirical and logical data a mathematical axiomatic model is built. D. Luce is a major contributor in this area. This thesis is based on his works and those of his many co-authors. Three approaches will be considered that lead to the rank-dependent utility representation of binary gambles composed only of gains (losses) relative to a status quo. The proofs involve the theory of functional equations which is a very powerful tool giving the precise numerical representations.

Contents

Chapter 1

Introduction

1.1 Alternatives and Gambles

A certain alternative or *certain consequence* is a result of some decision or some action, that can be positively or negatively valued and necessarily or certainly following from a set of conditions. For example, the result of the chess game between two players can be winning 20$ for one player if he wins, or losing 20$ if he loses (without any losing or winning sides in the case of draw). Thus, the chess game is an action, and the result of it is 20$, that is obviously valued positively (in the case of winning) or negatively (in the case of losing). Definitely the result follows from a set of conditions, that determines if the player wins or loses the game and there is no ambiguity about the benefit or cost if the game is placed. Together with the just defined terms we will use other equivalent terms such as "riskless" or "pure" alternatives or "riskless" or "pure" consequences. Here "certainty" is an ideal concept, but is a very close mathematical model in the modern world in some areas.

We define a domain of certain alternatives as a set $\mathcal{C}$. Since money is usually used to value different objects in the modern world, it is very useful to assume that $\mathcal{C}$ includes money as a special case , which is modeled as a subset $\mathcal{R}$ of the real numbers, R. So $\mathcal{R} = \mathcal{C} \cap R$.

Further we assume that the set of certain consequences includes a distinguished element e, which is very important in the utility theory, called *the status quo*. Intuitively, status quo is a neutral or a null consequence that does not alter the decision maker's current state. The status quo separates *gains* and *losses* in $\mathcal{C}$ in the obvious way, where the gains are consequences preferred to the status quo and the losses are consequences less preferred than the status quo. In real life people always compare consequences to the status quo, classifying consequences as losses or gains.

The next type of consequences arises when depending on the outcome of the *chance experiment* (for example, the toss of a coin or the draw of a ball from a urn) one of the finitely many pure consequences actually happens. This chance experiment carries out some process or act whose outcomes have a random aspect (for example, the chance experiment of tossing a dice has the face that comes up as the outcome with a random aspect). Some pure consequences (such as losing or winning money in the previous example of tossing a dice) may be attached to each outcome. The uncertainty about which consequence takes place is resolved only after the chance experiment is run. In this situation we speak about *uncertain alternatives*

If in an uncertain situation which a person (we will call him a decision maker) meets, there is some information, but no precise probability or certainty about likelihood of different outcomes, known to the decision maker, then the process of choosing among such outcomes is called *decision making under uncertainty.*

We distinguish *risky alternative* from uncertain alternative by defining the former as an uncertain alternative for which the probability is known or given for each chance outcome when the experiment is performed. The simplest risky alternatives, that are called *first-order* or *first-level* involve only consequences that are certain. An example is a bet on the toss of a dice, with consequences being the win of 20$ if the face with "1" arises (probability is $\frac{1}{6}$), and the loss of 10$ if the face with "5" or "6" arises (probability is $\frac{1}{3}$).
We can speak about *second-order* or *second-level risky alternatives* if one or more of consequences are first order risky alternatives and the probabilities are known at both levels. A common example is the playoff games when the winner of one pair goes to the next stage or level to play with the winner of another pair, and so on, assuming that probabilities are known.

Although it is widely accepted that the term *gamble* means anything that involves risk or uncertainty or the act of risking anything that is valued on the result of something involving chance, here we will expand the definition of a gamble to include not only the risky and uncertain alternatives but also the certain ones. Indeed, the certain consequences can be considered as the degenerate gambles where the same consequences are attached to all outcomes of the chances experiment or where the consequence is assigned to a certain outcome of the chance experiment and this outcome occurs with probability 1. Let $\mathcal{G}$ denote the set of all gambles under consideration and let $\mathcal{E}$ denote the set of all chance experiments that describes the considered gambles. Then for each chance experiment $E \in \mathcal{E}$ we define Ω_E as the set of its possible outcomes. For example, in the experiment E of tossing a dice

$$\Omega_E = \{1, 2, 3, 4, 5, 6\},$$

where the digit from 1 to 6 means that the face showing it comes up.
Let $\mathcal{E}_E$ denote a family of subsets of Ω_E, for instance

$$\mathcal{E}_E = (\{1, 2\}, \{1, 5, 6\}, \{4, 2\})$$

in the previous example.

We assume that the following properties for $\mathcal{E}_E$ are satisfied:

(i) $\Omega_E \in \mathcal{E}_E$.

(ii) If $C \in \mathcal{E}_E$, then the complement of C relative to the set Ω_E denoted by $\bar{C}$ is also an element in $\mathcal{E}_E$.

(iii) If $C, D \in \mathcal{E}_E$, then $C \bigcup D \in \mathcal{E}_E$.

Therefore we have an algebra $\mathcal{E}_E$ and the elements of $\mathcal{E}_E$ are called *events*.

Our next aim is to define binary first-order and compound gambles. Let $\{E_1, ..., E_n\}$, $E_i \in \mathcal{E}_E$ denote a finite partition of Ω_E for some chance experiment $E \in \mathcal{E}$. We underline finite partition here since it is not realistic to think about the infinite number of outcomes for any experiment. In other words for $i, j \in \{1, 2, ..., n\}$ with $i \neq j$, $E_i \bigcap E_j = \emptyset$, and $\bigcup_{i=1}^{n} E_i = \Omega_E$. We assign to each event in the partition a pure consequence, and this assignment is called a *first-order* or *simple* gamble g, that is, the just defined gamble g is a function

$$g : \{E_1, ..., E_n\} \rightarrow \mathcal{C}. \tag{1.1}$$

Here we note that the result of running experiment E is exactly one element E_i in $\mathcal{E}_E$, that is, exactly one E_i in the partition occurs. Therefore if we denote $g(E_i) = g_i$, then the function g is the set

$$\{(g_1, E_1), (g_2, E_2), ..., (g_n, E_n)\}. \tag{1.2}$$

Even though the definition of the function g as a set is mathematically clear, in behavioral sciences the convention that

$$g_i \text{ is preferred to } g_j \Longleftrightarrow i < j$$

is commonly accepted. The notion of preference will be covered more fully in the next section. According to this convention we can rewrite (1.2) into the form of an ordered n-tuple

$$(g_1, E_1; g_2, E_2; ...; g_n, E_n), \tag{1.3}$$

which gives us another way of recording gamble g.

In this thesis we will be using n-tuple notation without assuming that the g's are ordered.

First-order binary gamble is the specific case of the just defined first-order gamble when $n = 2$. These are the most common gambles and we will deal with them in most cases. In this particular case it is convenient to introduce new notations : $x, y \in \mathcal{C}$, $g_1 = x, g_2 = y, E_1 = C$ and $E_2 = \bar{C} = \Omega_E \backslash C$. Then the first-order binary gamble becomes

$$(x, C; y, \Omega_E \backslash C) \equiv (x, C; y, \bar{C}) \equiv (x, C; y). \tag{1.4}$$

Thus, in the case of binary gambles we do not assume the convention that the first written outcome x is preferred to the second written outcome y, but still use the n-tuple notation.

Let $\mathcal{G}_0 = \mathcal{C}$ and let $\mathcal{G}_1$ denote the union of the set of first-order gambles and $\mathcal{G}_0$. Then by analogy with $\mathcal{G}_1$ we define $\mathcal{G}_2$ to be all assignments from a finite event partition into $\mathcal{G}_1$ together with $\mathcal{G}_1$. We can continue this process infinitely : $\mathcal{G}_k$ is the set of all assignments from a finite event partition into $\mathcal{G}_{k-1}$ together with $\mathcal{G}_{k-1}$. Or in other words, the sets $\mathcal{G}_k$ are defined recursively by

1. $\mathcal{G}_0 = \mathcal{C}$.

2. for $k \geq 0$, $\mathcal{G}_{k+1} = \mathcal{G}_k \bigcup \{(g_1, E_1; ...; g_n, E_n) \mid g_i \in \mathcal{G}_k, E_i \in \mathcal{E}_E\ , n \geq\ 1\}$.

It makes sense to define k^{th}- *order gamble* or k^{th}-*level gamble* to be any element in $\mathcal{G}_k \backslash \mathcal{G}_{k-1}$. Any gamble which is not first-order and not pure consequence is called *compound*. It is obvious that the set $\mathcal{G}_n$ consists of all gambles of the level less than n including the gamble of the "zero" level i.e. pure consequences. We define the set of all gambles of any level by $\mathcal{G}_\infty = \bigcup\limits_{i=0}^{\infty} \mathcal{G}_i$, however, as it is shown in [8], in practice it is not realistic to suppose that a common person can perceive a compound gamble of a level higher than 2. Therefore in the rest of this thesis we will consider the case of gambles in $\mathcal{G}_2$ or in $\mathcal{G}_1$, and denote it by $\mathcal{G}$.

1.2 Choice, Preference and Utility Function

We recall that $\mathcal{G}$ is the set of all gambles under consideration and alternatives or gambles can be positively or negatively valued by decision maker, and therefore it is obvious to assume that some of the gambles can be more valuable or less valuable than others. Also we will assume the fact that if one gamble is seen to be more valuable than another, the more valuable one will be selected if we have to choose between them. We will call this situation *preference or revealed preference.*

If $g, h \in \mathcal{G}$, then we assume that just one of three situation must occur: g is seen to be preferred to h, which we denote by $g \succ h$; or if h is seen to be preferred to g, then we denote $h \succ g$; or if g and h are seen to be indifferent in preference, we denote $g \sim h$. Also we will meet situation when one gamble g is seen to be at least as preferred as another gamble h, that is , $g \succeq h$ or the union of $g \succ h$ and $g \sim h$. Assumption that this order is connected i.e. for all $g, h \in \mathcal{G}$ just one of three possibilities occurs either $g \prec h$, or $g \succ h$, or $g \sim h$, together with the transitivity which will be introduced later gives us the conclusion that $\preceq$ is a *weak preference* relation.

To evaluate a gamble we will establish its *certainty equivalence* or CE by connecting it to the equivalent amount of money, $CE(g) \in \mathcal{R} = \mathcal{C} \bigcap R$, such that

$$CE(g) \sim g. \tag{1.5}$$

Since we can expect that preference between money agrees with the numerical order, we have

$$g \preceq h \Longleftrightarrow CE(g) \leq CE(h).$$

More generally we introduce another numerical evaluation of a gamble which is called a *utility function*. For given behavioral properties in some general context, there exists a numerical function U over the domain of gambles which preserves preference between gambles, that is, such that for $g, h \in \mathcal{G}$,

$$g \preceq h \Longleftrightarrow U(g) \leq U(h).$$

Thus, we have built the preference structure $(\mathcal{C}, \mathcal{E}, \mathcal{G}, \preceq, e)$, where $\mathcal{C}$ is a set of pure or riskless consequences, $\mathcal{E}$ is a set of chance experiments, $\mathcal{G}$ is a set of gambles under consideration (often $\mathcal{G}_2$), $\preceq$ is a binary preference relation on $\mathcal{G}$, and $e \in \mathcal{G}$ is the status quo.

1.3 Basic Assumptions

In this section we will formulate few basic assumptions which we will assume throughout this thesis. There are many references and a lot of empirical information in the book of Luce [8] showing that some of these assumptions have been questioned empirically and proved right, but the rest of them was considered so obvious that they do not have to be and have not been tested empirically.

There are three fundamental principles of rationality that are true for all gambles and do not depend on any utility of gambling:

- Two distinct descriptions of an alternative should be seen as the same and therefore be indifferent.

- Consider a set S of alternatives with one gamble g being at least as preferred as all other in S. If g' is created by replacing some aspect of g by something at least as preferred, all else fixed, then no alternative in S is preferred to g'.
- If the chance of receiving the result that is more valued is made more likely at the expense of the result that is less valued, the modified alternative is preferred to the original one.

According to above stated basic behavioral properties we will form the axiomatic mathematical system, which consists of nine axioms: four of them are pure indifference and the rest are preference axioms.

1.3.1 Elementary Indifferences

Axiom 1. For every $E \in \mathcal{E}, C \in \mathcal{E}_E$, and $g \in \mathcal{G}_1$,

$$(g, C; g) \sim g, \tag{1.6}$$

where it is understood that the partition is $\{C, \Omega_E \backslash C\}$.

If axiom 1 holds, we say that the preference substructure of binary gambles from $\langle \mathcal{C}, \mathcal{E}, \mathcal{G}_2, \succeq \rangle$ is *idempotent*. In words, here we have a gamble $g' = (g, C; g)$ and they can not be equal since $g \in \mathcal{G}_1$, whereas $g' \in \mathcal{G}_2 \backslash \mathcal{G}_1$. As the result of running an experiment E the gamble g happens if C occurs and, equally, if C fails to occur, the result is also the gamble g. This axiom asserts that for the decision maker it is indifferent to choose between g and g' as the preferable one. The decision maker might still prefer one gamble to another, if, for example, there is a reason why the experiment should run or why time should be wasted for running this experiment. But we ignore these behavioral reasons, and based on the nature of each gamble, consider them equivalent. The first axiom is the direct consequence of the first fundamental principle because running or not running experiment C does not influence on the result of two gambles $(g, C; g)$ and g and therefore these gambles are seen indifferent.

Axiom 2. For every $E \in \mathcal{E}$, and $g, h \in \mathcal{G}_1$,

$$(g, \Omega_E; h) \sim g. \tag{1.7}$$

If axiom 2 holds than we say that the preference substructure of binary gambles from $\langle \mathcal{C}, \mathcal{E}, \mathcal{G}_2, \succeq \rangle$ is *certain*.

As well as Axiom 1 this axiom is based on the first fundamental assumption. Here the gamble is very primitive in the sense that it gives only one result g whatever outcome from Ω_E occurs. Thus, for the decision maker it does not make any difference whether one runs such an experiment or not since the result is obvious for him.

Axiom 3. For every $E \in \mathcal{E}, C \in \mathcal{E}_E$, and $g, h \in \mathcal{G}_1$,

$$(g, C; h) \sim (h, \bar{C}; g), \tag{1.8}$$

where $\bar{C} = \Omega_E \backslash C$.

If axiom 3 holds than we say that the preference substructure of binary gambles from $\langle \mathcal{C}, \mathcal{E}, \mathcal{G}_2, \succeq \rangle$ is *complemental*. Here the two sides say exactly the same thing except for the order of writing. If it is assumed that $g \succeq e \succeq h$ i.e. a gamble between gains and losses, then this axiom shows the connection between losses and gains, that is, we have the gain h, if C occurs, and otherwise we have the loss g. This assumption also gives us the right to consider gambles as the class equivalence with respect to orders of writing events and consequences, and therefore this is another interpretation of the first principle.

Axiom 4. For all $f, g, h \in \mathcal{G}_1$,

$$\text{If} \quad f \preceq g \quad \text{and} \quad g \preceq h, \quad \text{then} \quad f \preceq h. \tag{1.9}$$

If axiom 4 holds then we say that the preference substructure of binary gambles from $\langle \mathcal{C}, \mathcal{E}, \mathcal{G}_2, \succeq \rangle$ is *transitive*. Transitivity is a special case of the second rationality principle. Here for $S = \{f, g\}$ we have that no element from S is preferred to g, and if we create another gamble h such that it is at least as preferred as g, then no gamble from S is preferred to g.

As a consequence of transitivity for the weak preference $\preceq$, the indifference relation $\sim$ must also be transitive:

$$\text{If } f \sim g \text{ and } g \sim h, \text{ then } f \sim h.$$

It is not necessary to check and it is obviously true that money preference is transitive as well.

1.3.2 Preference Consequences

The following five axioms involve the full preference relation $\succeq$ over $\mathcal{G}$, not just the indifference relation $\sim$ as in the first four. These are all subject to some descriptive doubt, so they have been explored empirically reasonably carefully by many authors (see the second chapter of [8]).

Axiom 5. If money is modeled as R, then for all $\alpha, \beta \in R$

$$\alpha \succeq \beta \Longleftrightarrow \alpha \geq \beta. \tag{1.10}$$

This axiom is believed to be truthful for all people and seems to be useless to check empirically, since every decision maker prefers more money to less money if everything else holds fixed.

Axiom 6. For all $E \in \mathcal{E}, g_i, g_i' \in \mathcal{G}_1$, $i \in \{1, 2, ..., n\}$, and partitions $\{E_1, E_2, ..., E_n\}$ of Ω_E, where $E_i \in \mathcal{E}_E$,

$$g_i' \succeq g_i \quad \text{if and only if}$$

$$(g_1, E_1; ...; g_i', E_i; ...; g_n, E_n) \succeq (g_1, E_1; ...; g_i, E_i; ...; g_n, E_n). \tag{1.11}$$

If axiom 6 holds then we say that the preference substructure of gambles from $\langle \mathcal{C}, \mathcal{E}, \mathcal{G}_2, \succeq \rangle$ is *consequence monotonic.* In the binary case this axiom simplifies to

$$g' \succeq g \Longleftrightarrow (g', C; h) \succeq (g, C; h)$$

and

$$h' \succeq h \Longleftrightarrow (g, C; h') \succeq (g, C; h).$$

Again we see that consequence monotonicity is another interpretation of the second rationality principle in which $S = \{f\}$, where

$$f = (g_1, E_1; ...; g_i, E_i; ...; g_n, E_n),$$

and if we replace g_i by g'_i which is at least as preferred as g_i, then the transformed gamble $(g_1, E_1; ...; g'_i, E_i; ...; g_n, E_n)$ is seen at least as preferable as the original one. Another interpretation of this axiom is that it shows the "separable" form of the gambles in the sense that the relation between g_i and g'_i is not influenced by the rest of the gamble structure.

When we speak about monotonicity of the gambles, there are two quite different forms to be considered. One involves consequences and is very close to the previous axiom when improving one part of the gambles improves the whole gamble. Another (which is considered below) arises when we have a binary operation and it is analogous to the numerical monotonicity of addition, which says that $\alpha \geq \beta$ if and only if $\alpha + \gamma \geq \beta + \gamma$. This property is a mathematically strong condition and it can be seen from [8] that there are some doubts about its validity, therefore we will not assume it as an axiom. However this condition is important so we will introduce it here.
The preference substructure of binary gambles from $\langle \mathcal{C}, \mathcal{E}, \mathcal{G}_2, \succeq \rangle$ is said to be *event monotonic* if and only if for all $E \in \mathcal{E}, A, B, C \in \mathcal{E}_E$ with $A \cap C = B \cap C = \emptyset$ and $g, h \in \mathcal{G}_1$, with $g \succ h$

$$(g, A; h) \succeq (g, B; h) \quad \text{if and only if} \quad (g, A \cup C; h) \succeq (g, B \cup C; h). \tag{1.12}$$

This condition can be established from the third fundamental principle. Since the decision maker prefers the gamble which makes g more likely to occur than h, and since a gamble $(g, A; h)$ is at least as preferred as a gamble $(g, B; h)$, the decision maker sees from rationality principle that an event A happens at least as likely as an event B. Therefore adjoining the disjoint event C to both sides, the order of preference is preserved. This principle can be easily transformed to the converse when the gamble makes g less likely to occur than h, in this case by removing the common part we still have the gamble that is at most as preferred as the first.

The next assumption is a special case of event monotonicity which seems to be empirically right, and therefore we accept it as an axiom.

Axiom 7. For all $E \in \mathcal{E}, C, D \in \mathcal{E}_E$ and $g, h \in \mathcal{G}_1$,

$$\text{if} \quad g \succ h \quad \text{and} \quad C \supset D, \quad \text{then} \quad (g, C; h) \succ (g, D; h). \tag{1.13}$$

If axiom 7 holds then we say that the preference substructure of binary gambles from $\langle \mathcal{C}, \mathcal{E}, \mathcal{G}_2, \succeq \rangle$ is *monotonicity of event inclusion.* This is the first part of event monotonicity which arises from the third basic principle. Here since a gamble g is preferred to a gamble h, and an event C is more likely to occur than an event D, the decision maker sees the gamble $(g, C; h)$ preferable to the gamble $(g, D; h)$. If we switch the order of preference and the inclusion of the events, then this axiom is also true .
The next proposition shows the relation between event monotonicity and monotonicity of event inclusion, and since the converse of this proposition is not generally true, it shows that event monotonicity is a mathematically stronger property.

Proposition 1 *Suppose idempotence,certainty, consequence monotonicity, and transitivity hold in the structure $\langle \mathcal{C}, \mathcal{E}, \mathcal{G}_2, \succeq \rangle$. Then event monotonicity implies monotonicity of event inclusion.*

Proof. Suppose $C \supset D$ and $g \succ h$, and we have to prove that $(g, C; h) \succ (g, D; h)$. Since D is a proper subset of C, $A = C \backslash D$ is nonempty. By complementarity

$(g, \emptyset; h) \sim (h, E; g)$, by certainty we have $(h, E; g) \sim h$, further by idempotence $h \sim (h, A; h)$, and finally the consequence monotonicity axiom gives $(h, A; h) \prec (g, A; h)$. Thus, $(g, \emptyset; h) \prec (g, A; h)$ and by event monotonicity adjoining D to both sides of this relation , we conclude

$$(g, D; h) \prec (g, A \cup D; h) \sim (g, C; h)$$

which proves this proposition.

□

We note again that the converse of this theorem is not true. Intuitively, it is clear since monotonicity of event inclusion depends on the order of inclusion , on the other hand event monotonicity does not imply any event inclusion. The example known as Ellsberg paradox that can be found in the book [8], page 55, proves this assertion.

Axiom 8. For all $E \in \mathcal{E}, C, D \in \mathcal{E}_E$ and $g, h \in \mathcal{G}_1$ with $g, h \succ e$

$$(g, C; e) \succeq (g, D; e) \Longleftrightarrow (h, C; e) \succeq (h, D; e). \tag{1.14}$$

If axiom 8 holds we say that the preference substructure of binary gambles from $\langle \mathcal{C}, \mathcal{E}, \mathcal{G}_2, \succeq, e \rangle$, where e is the status quo shows *order-independence of events*.
It is possible to phrase this axiom in a different way using arbitrary z instead of e with the same property $g, h \succeq z$. This axiom also allows us to introduce an induced order $\succeq_{\mathcal{E}_E}$ over $\mathcal{E}_E$ in the following way: we say that $C \succeq_{\mathcal{E}_E} D$ if $(g, C; e) \succeq (g, D; e)$ holds for all $g \in \mathcal{G}_1$.
Order-independence of events for the first-order gambles is intuitively understandable, but for the second-order gamble it is not so clear. Indeed, in the

case of a first-order gamble since g is a gain, from the first inequality it follows that the event C should occur more likely than the event D. Therefore, since h is also the gain, the second inequality should hold. But it is not known, based on [8], page 58 , whether order-independence has been checked empirically.

It is now appropriate to put these axioms in one definition:

> **Definition.** A preference structure of binary gambles from $\langle \mathcal{C}, \mathcal{E}, \mathcal{G}_2, \succeq \rangle$ is said to be *elementary rational* if and only if it satisfies the following conditions: $\succeq$ is a weak order (transitive and connected), idempotence, certainty, complementarity, consequence monotonicity, monotonicity of event inclusion , and order-independence of events.

Denote a second-order gamble with at least one gain and at least one loss by

$$g = (g_1, E_1; ...; g_n, E_n), \quad E_i \cap E_j = \emptyset, \quad i \neq j, \quad \text{and} \bigcup_{i=1}^{n} E_i = E,$$

where the indices are chosen such that the consequences are ordered by preference, i.e.

$$g_1 \succeq g_2 \succeq ... \succeq g_k \succeq e \succeq g_{k+1} \succeq ... \succeq g_n.$$

Thus, the first k consequences are gains and those from $k+1$ to n are losses. The status quo e is included in the first group.

The last axiom is based on the fact that people formulate a general gamble as a binary one composed of two subgambles. One subgamble consists of those events with only gains including the status quo as consequences; the other only of events with only losses as consequences . Formally, define

$$E_+ = \bigcup_{i=1}^{k} E_i, \quad \text{and} \quad E_- = \bigcup_{i=k+1}^{n} E_i$$

and

$$g_+ = (g_1, E_1; ...; g_k, E_k), \quad \text{and} \quad g_- = (g_{k+1}, E_{k+1}; ...; g_n, E_n).$$

By this definition the subgamble g_+ is conditioned on the subevent E_+ of E and g_- is conditional on E_-.

Axiom 9. For all $E \in \mathcal{E}, g \in \mathcal{G}_1$ with at least one gain and at least one loss consequence

$$g \sim (g_+, E_+; g_-, E_-). \tag{1.15}$$

These two gambles cannot be equal since $g \in \mathcal{G}_1$ whereas $(g_+, E_+; g_-, E_-) \in \mathcal{G}_2/\mathcal{G}_1$.

If axiom 9 holds then we say that the preference substructure of binary gambles from $\langle \mathcal{C}, \mathcal{E}, \mathcal{G}_2, \succeq, e \rangle$, where e is the status quo shows *gain-loss decomposition.*
It seems to be reasonable to assume this gain-loss decomposition since many people including sophisticated decision makers, think about any gambles with at least one gain and at least one loss in terms of such a decomposition. For example, most of us deal with potential losses, such as losing job or accidents, as very distinct from the gains associated with buying new car or house.
Mathematically this axiom simplifies a complicated gamble by reducing it to three simpler ones: one deals only with losses, another with only gains and the third is the binary mixed gamble. Then we can consider the case for losses to be the same as for the gain case since we always can assume that a loss is a negative gain. But again it is not known according to Luce [8], page 60, whether or not there is empirical data for checking this axiom.

1.4 Binary Rank-Dependent Utility (RDU)

Having built the axiomatical model we are about to define the notion of rank-dependent utility representation. The main problem of this thesis is: under what assumptions and how can we get the rank-dependent representation for the binary gambles? From here on, according to the ninth axiom, we consider gambles involving only gains as consequences. We will make slight changes in the notations and we will also give the definition of RDU representation and state its three properties.

At first we recall that $\mathcal{G}_0 = \mathcal{C}$ is the set of certain alternatives, $\mathcal{G}_1$ is the first-order gambles together with $\mathcal{G}_0$, $\mathcal{G}_2$ is the second-order gambles together with $\mathcal{G}_1$, and $\succeq$ is a weak preference order. We introduce more specific notations:

- $\mathcal{G}_0^+ = \mathcal{C}^+ = \{x : x \in \mathcal{C} \text{ and } x \succeq e\}$ is the set of certain gains including the status quo e.
- $\mathcal{G}_i^+$ is the subset of $\mathcal{G}_i$ for all gambles that are generated inductively from $\mathcal{C}^+$ and $\mathcal{E}$.
- $\mathbf{G}_i^+ = \langle \mathcal{C}^+, \mathcal{E}, \mathcal{G}_i^+, \succeq, e\rangle$ i.e. the substructure of the gain gambles.

We note here that when we deal with $\mathcal{G}_i^+$ and its ordering $\succeq$, we simply mean the restriction of the original ordering to the subsystem.

Definition. $\mathbf{G}_2^+$ is said to have a binary *rank-dependent utility (RDU)* representation if and only if there exists a mapping $U : \mathcal{G}_2^+ \to R_+^* = \{\alpha : \alpha \in R$

and $\alpha \geq 0\}$ and , for each $E \in \mathcal{E}$, a mapping $W_E : \mathcal{E}_E \to [0,1]$ with

$$W_E(\emptyset) = 0, \qquad W_E(\Omega_E) = 1,$$

such that U is order preserving in the sense that for $g', h' \in \mathcal{G}_2^+$,

$$g' \succeq h' \Longleftrightarrow U(g') \geq U(h'),$$

$$U(e) = 0,$$

and for $g, h \in \mathcal{G}_1^+$,

$$U(g, C; h, \bar{C}) = \begin{cases} U(g)W_E(C) + U(h)[1 - W_E(C)], & g \succ h \\ U(g), & g \sim h \\ U(g)[1 - W_E(\bar{C})] + U(h)W_E(\bar{C}), & g \prec h. \end{cases} \tag{1.16}$$

This model is called "rank-dependent" since we used different forms depending on whether $g \succ h$, or $g \sim h$, or $g \prec h$. We will call U a utility function and W_E a weighting function which is defined for some particular experiment E and which maps events into the closed interval $[0,1]$. In the general case, when we consider both gains and losses we distinguish weighting functions for gains and losses. W_E need not in general be an additive function, that is, we need not have $W_E(C) + W_E(\bar{C}) = 1$ since otherwise we would not need rank dependency. We also notice that in equation (1.16) the second line follows directly from idempotence, and the third line follows from the first by complementarity.

Definition. The representation is said to be *dense in intervals* if the images of U and W_E are each dense in an interval, i.e. for r, s in the interval with $r > s$, there exists a t in the image of the function such that $r > t > s$.

The representation is said to be *onto intervals* if U is onto a real interval including 0 and there is at least one experiment $K \in \mathcal{E}$ with W_K onto $[0, 1]$.

Having one experiment (later defined as canonical) that has a rank - dependent representation onto an interval gives us the chance to even further simplify the consideration of all binary gambles to the binary gambles based on this experiment.

Definition. In a structure $\mathbf{G}_2^+$ of binary gambles of gains, an experiment $K \in \mathcal{E}$ is said to be *canonical* if and only if for any $E \in \mathcal{E}$ and any $C \subseteq \Omega_E$, there exists $D \subseteq \Omega_K$ depending on C and Ω_E such that for all $g, h \in \mathcal{G}_1^+$ with $g \succeq h$,

$$(g, C; h, \Omega_E \backslash C) \sim (g, D; h, \Omega_K \backslash D). \tag{1.17}$$

Theorem 1 *If $\mathbf{G}_2^+$ has an RDU representation and there is an experiment K for which W_K is onto $[0, 1]$, then K is canonical.*

Proof. Because W_E maps into the interval $[0, 1]$ and W_K is surjective onto $[0, 1]$, for any $C \subseteq \Omega_E$ there exists $D \subseteq \Omega_K$ such that $W_K(D) = W_E(C)$. So by the RDU representation for $g \succeq h$

$$U(g, C; h, \Omega_E \backslash C) = U(g)W_E(C) + U(h)[1 - W_E(C)]$$

$$= U(g)W_K(D) + U(h)[1 - W_K(D)] = U(g, D; h, \Omega_K \backslash D),$$

and therefore (1.17) holds.

The existence of the canonical experiment is a strong property. But to have this property it is not necessary to have RDU onto an interval, because it is enough to have the specific property of the function W_K for some experiment K.

We now explore three necessary properties of the RDU representation which will play the main role in axiomatizing it:

1. Separability.

For each $E \in \mathcal{E}$ the substructure $\langle \mathcal{C}^+, \mathcal{E}_E, \mathcal{G}_1^+, \succeq, e\rangle$ is said to have a *separable representation* if and only if there exist $U : \mathcal{C}^+ \to R_+^*$ and $W_E : \mathcal{E}_E \to [0, 1]$ such that the product UW_E is order preserving, that is,

$$(x, C; e, \bar{C}) \succeq (y, D; e, \bar{D}) \Longleftrightarrow U(x)W_E(C) \geq U(y)W_E(D) \tag{1.18}$$

and $W_E(\Omega_E) = 1$.

This property comes straightforward from the definition of the rank-dependent utility, in particular, from equation (1.18) with $h = e$. There is a lot of empirical data showing that this property of a representation commonly holds, for example see [8], pages 67-72.

2. Rank-dependent additivity of consequences.

A structure $\mathbf{G}_1^+$ is said to have a *rank-dependent additive representation* (RDA) if and only if for each $E \in \mathcal{E}$ there exists $U_{1,E}, U_{2,E} : \mathcal{C}^+ \times \mathcal{E}_E \to R_+^*$, such that the following additive order-preserving representation holds: For $x, x', y, y' \in \mathcal{C}^+$ with $x \succeq y$ and $x' \succeq y'$, and $C \in \mathcal{E}_E$,

$$(x, C; y) \succeq (x', C; y')$$

if and only if

$$U_{1,E}(x, C) + U_{2,E}(y, C) \geq U_{1,E}(x', C) + U_{2,E}(y', C),$$

and $U_{1,E}(e, C) = U_{2,E}(e, C) = 0$.

Further, we can define U over G_1^+ as

$$U(x, C; y) = U_{1,E}(x, C) + U_{2,E}(y, C), \qquad x \succeq y. \tag{1.19}$$

The representation for $x \prec y$ follows from the complementarity property

$$U(x, C; y) = U(y, \bar{C}; x) = U_{1,E}(x, \bar{C}) + U_{2,E}(y, \bar{C}), \quad x \prec y.$$

The definition extends to $\mathbf{G}_2^+$ in the obvious way.

If we compare the rank-dependent additivity of consequences and the gain-loss decomposition axiom we notice their similarity, indeed, in RDA representation the utility function is expressed as the sum of two terms. Psychologically, it means that people distinguish not only gains and losses but also less gains and larger gains and treat them independently. This property has also been tested and proved empirically in the works of few authors and some references can be found on the pages 71-72 of [8].

3. Event commutativity.

The third consequence of RDU does not involve trade-off, but rather compound games and shows a special kind of indifference.

A structure $\mathcal{G}_2^+$ satisfies *event commutativity* if for all $x, y \in \mathcal{C}^+$, $E, F \in \mathcal{E}, C \subset \Omega_E$ and $D \subset \Omega_F$,

$$((x, C; y, \Omega_E \backslash C), D; y, \Omega_F \backslash D) \sim ((x, D; y, \Omega_F \backslash D), C; y, \Omega_E \backslash C).$$

When this is true just for $y = e$, we say that *status-quo event commutativity* holds. This property seems to be quite rational: When two independent experiments E and F are run, the outcome x occurs when both C happens in Ω_E and D happens in

Ω_F, and the outcome y occurs, when either $\Omega_E \backslash C$ happens or $\Omega_F \backslash D$ happens. Thus, for the decision maker it is indifferent in which order these two experiments are run. The event commutativity follows from the rank-dependent utility representation if we suppose $x \succeq y$ and therefore $(x, C; y) \succeq y$.Thus,

$$U((x, C; y, E\backslash C), D; y, F\backslash D) = U(x, C; y, C\backslash E)W_F(D) + U(y)[1 - W_F(D)]$$

$$= U(x)W_E(C)W_F(D) + U(y)[1 - W_E(C)]W_F(D) + U(y)[1 - W_F(D)]$$

$$= U(x)W_E(C)W_F(D) + U(y)[1 - W_E(C)W_F(D)],$$

which is symmetrical in (C, E) and (D, F). Therefore the utility function for both gambles are equal, and thus two gambles are equivalent.

Even though this property seems to be quite rational, it is not always true according to some analysis (see [8], pages 72-74). However, in most cases it is true.

Chapter 2

First axiomatization of RDU

After building our mathematical model the goal for the rest of this thesis is to understand how the RDU representation onto an interval comes from behavioral properties that we can observe. We will investigate three approaches that involves examining some combinations of major features implied by the RDU representation, deriving functional equations and based on their solutions proving some major theorems for representations including RDU. In the first two approaches the separability, rank-dependent additivity and event commutativity assumptions together with some structural conditions will give rise to a new representation and RDU is a special case of it.

2.1 Introduction

In this section we assume that both additive and separable representations are satisfied and they both are onto intervals. We prove that the RDU representation exists. Returning to the definition of rank-dependent additive representation we can write it in the form

$$U(x, C; y) = U_{1,E}(x, C) + U_{2,E}(y, C) \qquad x \succeq y \succeq e, \tag{2.1}$$

where $U_{1,E}, U_{2,E} : \mathcal{C}^{+} \times \mathcal{E}_E \to [0, k[$ are surjective and $U_{1,E}(e, C) = U_{2,E}(e, C) = 0$. Letting $y = e$, we get

$$U(x, C; e) = U_{1,E}(x, C).$$

Because the utility function preserves the order of preference restricted to gambles of the form $(x, C; e)$, $U_{1,E}(x, C)$ also preserves the order of preference. Now we recall that the separable representation of RDU defined in chapter 1 preserves the order of preference, i.e.

$$(x, C; e, \bar{C}) \succeq (y, D; e, \bar{D}) \Longleftrightarrow U^*(x)W_E^*(C) \geq U^*(y)W_E^*(D)$$

and $W_E^*(\Omega_E) = 1$, where W_E^* and U^* play the role of W_E and U in the definition of the separable representation of RDU (equation (1.18)). The assumptions that this representation holds for $U_{1,E}(x, C)$ and that both additive and separable representation are onto an interval, give the conclusion that there exists a strictly increasing function Ψ with $\Psi(0) = 0$ such that $U_{1,E} = \Psi[U^*(x)W_E^*(C)]$. Thus, we justified the following expression for $U(x, C; y)$

$$U(x, C; y) = \Psi[U^*(x)W_E^*(C)] + U_{2,E}(y, C) \qquad x \succeq y \succeq e. \tag{2.2}$$

Further we let $C = \Omega_E$ and $y = e$. Because $W_E^*(\Omega_E) = 1$, $U_{2,E}(e, C) = 0$ and the certainty axiom is satisfied, we have

$$U(x) = \Psi[U^*(x)].$$

Thus, using certainty equivalence (1.5) and substituting this expression for $U(x)$ to the equation (2.2) we get

$$\Psi[U^*(x,C;y)] = \Psi[U^*(x)W_E^*(C)] + U_{2,E}(y,C) \qquad x \succeq y \succeq e, \tag{2.3}$$

where $\Psi(0) = 0, U_{2,E}(e,C) = 0$ and Ψ is strictly increasing.

Because of the rank dependence constraint $x \succeq y$, we cannot use the same argument to conclude that $U_{2,E}(y,C)$ has a separable representation. In [9], p.104 this mistake was made, however, already in [11], p.283, D. Luce and A. Marley discussed this case and made corrections by assuming, what they called the *double separable additive utility*, that there also exist a strictly increasing function Φ with $\Phi(0) = 0$, and functions U^{**}, W_E^{**} with $U^{**}(e) = 0, W_E^{**}(\emptyset) = 0$ and $W_E^{**}(\Omega_E) = 1$ such that the second term in (2.3) can be expressed in the form

$$U_{2,E}(y,C) = \Phi[U^{**}(y)W_E^{**}(\bar{C})]. \tag{2.4}$$

The authors could not justified this assumption in a fully satisfactory fashion, although they outlined two approaches to this justification and stated their drawbacks.

Collecting (2.3) and (2.4) together we get

$$\Psi[U^*(x,C;y)] = \Psi[U^*(x)W_E^*(C)] + \Phi[U^{**}(y)W_E^{**}(\bar{C})], \tag{2.5}$$

for all $x \succeq y \succeq e$. Setting $C = \emptyset$ in (2.5) proves that

$$\Psi[U^*(y)] = \Phi[U^{**}(y)] \qquad x \succeq y \succeq e. \tag{2.6}$$

Furthermore, letting $v = U^*(y), w = W_E^*(C), f = \Psi$, and defining the function g by $U^{**}(y) = g[U^*(y)] = g(v)$, and the function q by $q(w) = W_E^{**}(\bar{C})$, we can write the equation (2.6) in the form

$$f(v) = \Phi[g(v)] \quad \text{or} \quad \Phi = f \circ g^{-1}, \tag{2.7}$$

where we know that g^{-1} is defined because $g : [0, l[\to [0, l[$ is strictly monotonic and surjective.

Finally, letting $x = y$ in (2.5) and using (2.7) together with our new notations, we derive the functional equation

$$f(v) = f(vw) + f(g^{-1}[g(v)q(w)]) \quad (v \in [0, l[; w, q(w) \in [0, 1]). \tag{2.8}$$

Even though we know that f and q are strictly monotonic and continuous, in solving this equation we will only suppose that f is nonnegative on $[0, l[$.

2.2 A Functional Equation Arising from the First Axiomatization

In this section we will solve the equation, which arises from the additive and separable utility as it was shown above i.e., the equation

$$f(v) = f(vw) + f(g^{-1}(g(v)q(w))). \tag{2.9}$$

It has been solved by Aczél, Maksa, Ng and Páles (cf. [3]) under the assumptions that f : $[0,l[\to [0,+\infty[$, q : $[0,1] \to [0,1]$, and g^{-1} is the inverse of g: $[0,l[\to [0,l[$, strictly monotonic and surjective.

Let $k = \ln(l) \geq -\infty, I =]k, +\infty[, R_+ =]0, +\infty[$. Introducing new variables $t = -\ln(v)$ and $s = -\ln(w)$ we can write equation (2.9) on the interior in the form

$$f(e^{-t}) = f(e^{-(t+s)}) + f(g^{-1}(q(e^{-s})g(e^{-t}))).$$

Define

$$\begin{cases} F(t) = f(e^{-t}), \;\; G(t) = -ln\, g\,(e^{-t}) \;\; (t \in I) \\ Q(s) = -ln\,(e^{-s}), \;\; H(\psi) = f(g^{-1}(e^{-\psi}))(s \in R_+, \psi \in] - ln\, k, +\infty[) \end{cases} \tag{2.10}$$

so that we can write equation (2.9) on the subdomain, where $v \neq 0, w \neq 0$ in the additive form

$$F(t) - F(t+s) = H(G(t) + Q(s)) \;\; (t \in I, s \in R_+). \tag{2.11}$$

We are going to find the solutions of this equation under the following assumptions

$(a)\, F : I \to \mathbf{R}$,

$(b)\, G : I \to \mathbf{R}$ is strictly monotonic,

$(c)\, Q : \mathbf{R}_+ \to \mathbf{R}$,

$(d)\, H : G(I) + Q(R_+) \to \mathbf{R}_+$ is strictly monotonic.

We note that the strictly monotonicity of H can be assumed only if f is strictly monotone, that will follow from theorem 5. The condition (d) also implies that the function F is strictly decreasing, because the values of the function H and the argument s are both positive.

Theorem 2 *Suppose the equation* (2.11) *holds for all* $t \in I$, $s \in R_+$, *where the functions* F, G, Q, *and* H *satisfy the assumptions (a) - (d). Then*

(i) F has right and left derivative everywhere on I , Q is differentiable everywhere on R_+ and G'_+ exists everywhere on I,

(ii) Q', F'_+ and G'_+ satisfy the differential-functional equation

$$Q'(s)[F'_+(t+s) - F'_+(t)] = G'_+(t)F'_+(t+s)(t \in I, s \in R_+), \tag{2.12}$$

(iii) F'_+ *,* G'_+*, and* Q *preserve signs on their domains.*

Proof. (i) For all fixed $s \in R_+$ (2.7), (b), and (d) imply that the function $t \mapsto F(t) - F(t+s)\,(t \in I)$ is strictly monotonic. Therefore in the strictly decreasing case,

$$F(t) - F(t+s) > F(t+s) - F((t+s)+s),$$

or

$$2F(t+s) < F(t) + F(t+2s) \quad (t \in I, s \in R_+)$$

and thus F is strictly Jensen convex. Since F is locally bounded , F is strictly convex (or strictly concave if we had chosen that $F(t) - F(t+s)$ is strictly increasing), therefore, F is differentiable except for at most countably many points and $F'_-(t)$ and $F'_+(t)$ exist and are strictly monotone (cf. [7], page 156). According to (2.7) , $F(t) - F(t+s)$ is in the codomain of H for all $t \in I, s \in R_+$ and since H is strictly monotonic , we can write (2.7) in the form

$$H^{-1}(F(t) - F(t+s)) = G(t) + Q(s) \quad (t \in I, s \in R_+). \tag{2.13}$$

The set $J = \{F(t) - F(t+s) : t \in I, s \in R_+\}$ is an open interval of positive length in R_+, due to strict monotonicity and convexity of F . Also H^{-1} is strictly monotonic and by Lebesgue's theorem (cf. [5], page 264), H^{-1} is differentiable almost everywhere on J. Having the differentiability of F almost everywhere, for each $s \in R_+$ we can choose such $t_0 \in I$ that F is differentiable at $t_0 + s$, and H^{-1} is differentiable at $F(t_0) - F(t_0 + s)$. Thus the left-hand side of (2.13) is differentiable with respect to s, and therefore Q is differentiable everywhere, since s has been taken arbitrarily. Choose $z_0 = F(t_0) - F(t_0 + s_0) \in J$ for arbitrary $t_0 \in I,$ and $s_0 \in R_+$. We note that $s_0 = F^{-1}[F(t_0) - z_0] - t_0$. Since the map $(t, z) \to F^{-1}[F(t) - z] - t$ is well defined and jointly continuous on a neighbourhood of (t_0, z_0) and $s_0 \in R_+$, there exists a neighbourhood $T_0 \times Z_0$ of (t_0, z_0) such that $F^{-1}[F(t) - z] - t \in R_+$ for all $(t, z) \in T_0 \times Z_0$. We can take $t_1 \in T_0$ such that the strictly decreasing F^{-1} is differentiable at $F(t_1) - z$, and $s = F^{-1}[F(t_1) - z] - t_1$ in (2.13). We get

$$H^{-1}(z) = G(t_1) + Q(F^{-1}[F(t_1) - z] - t_1) \quad (z \in Z_0). \tag{2.14}$$

By the differentiability of Q and by the choice of t_1 , the right side of (2.14) is differentiable with respect to z at z_0, therefore H^{-1} is differentiable at z_0, and therefore everywhere on J. The existence of G'_+ everywhere on I follows from existence of F'_+ everywhere and H^{-1} on J, and the chain rule.

We prove (ii) by differentiating (2.13) with respect to s and t , getting

$$-(H^{-1})'\Big(F(t) - F(t+s)\Big)F'_+(t+s) = Q'(s),$$

$$(H^{-1})'\Big(F(t) - F(t+s)\Big)(F'_+(t) - F'_+(t+s)) = G'_+(t),$$

and eliminating $(H^{-1})'\Big(F(t) - F(t+s)\Big)$ from these two equations.

(iii) Because F is strictly decreasing and strictly convex or concave , we have (see [14], page 5)

$$F'_+(t) < 0,$$

and

$$(F'_+(t) - F'_+(t+s)) \text{ preserves its sign for all } t \in I, s \in R_+.$$

Since G is strictly monotone, there exists such t_1 that $G'_+(t_1) \neq 0$, therefore $Q'(s) \neq 0$ for some s. Hence, fixing this s, we conclude that G'_+ preserves sign. Now letting s vary again, Q' also preserves sign.

□

Introducing new notations

$$\gamma(s) = Q'(s) \quad (s \in R_+), \quad \varphi(t) = G'_+(t) \text{ and } \psi(t) = F'_+(t) \quad (t \in I) \tag{2.15}$$

we are able to rewrite equation (2.8) in the form

$$\gamma(s)(\psi(t+s) - \psi(t)) = \varphi(t)\psi(t+s)(t \in I, s \in R_+), \tag{2.16}$$

where the functions γ, ψ, and φ do not change signs on their domain.

We introduce the set $P(I)$ of all pairs (c, μ) $(c \neq 0)$ for which the function

$$t \mapsto \mu + e^{ct}$$

does not equal to zero for all $t \in I$.

Theorem 3 *The sign preserving functions* $\gamma : R_+ \to R, \psi, \varphi : I \to R$ *satisfy* (2.16) *if , and only if, either*

$$\varphi(t) = \frac{p}{t+r}, \ \psi(t) = \frac{q}{t+r}, \gamma(s) = -\frac{p}{s} \quad (t \in I \neq R, s \in R_+), \tag{2.17}$$

where $p, q,$ *and* r *are real constants,* $pq \neq 0, -r \notin I$ *; or*

$$\varphi(t) = \frac{ae^{ct}}{\mu + e^{ct}}, \psi(t) = \frac{b}{\mu + e^{ct}}, \gamma(s) = \frac{a}{1 - e^{ct}} \quad (t \in I, s \in R_+), \tag{2.18}$$

where $a, b, c,$ *and* μ *are constants,* $abc \neq 0, (c, \mu) \in P(I)$.

Proof. Equation (2.16) can be in the form

$$l(t+s) - l(t) = m(t)n(s) \tag{2.19}$$

if we denote

$$l := 1/\psi, m := \varphi/\psi \text{ and } n := -1/\gamma, \tag{2.20}$$

that can be done since φ, ψ, and γ are sign preserving, and therefore l, n, and m also preserve signs. Since $n(s)m(t)$ is either positive or negative for all s and t, l is strictly monotonic, and thus n is strictly monotone as well. From the proof of theorem 2 it is obvious that l has right and left derivative everywhere, and it implies that n also has right and left derivative. Now we differentiate (2.19) with respect to s from the right to get the Pexider equation ([1])

$$l'_+(t+s) = m(t)n'_+(s) \quad (t \in I, s \in R_+) \tag{2.21}$$

with solution

$$l'_+(t) = a_1e^{ct}, m(t) = a_2e^{ct}, \quad and \quad n'_+(t) = a_3e^{ct},$$

where $a_1a_2a_3 \neq 0$ and $a_1 = a_2a_3$. Since l and n have continuous right derivatives, by [7], page 156, they are differentiable everywhere. Therefore if $c \neq 0$ by integration we get

$$l(t) = \frac{a_1}{c}e^{ct} + b_1, \quad n(s) = \frac{a_3}{c}e^{ct} + b_3; \tag{2.22}$$

or

$$l(t) = a_1t + b_1, \quad n(s) = a_3t + b_3 \tag{2.23}$$

when $c = 0$. Now substitute these solutions to (2.19) to get $b_2 = 0$. Furthermore, if $c = 0$, then $b_3 = 0$, while if $c \neq 0$, then $b_3 = a_1/a_2$. It is easy to check that we get (2.17) and (2.18) using (2.20) and relabelling the constants.

Theorem 4 *The functions $Q, F, G,$ and H with the properties (a) -(d) are the solutions of (2.11) for all $t \in I$ and $s \in R_+$ if and only if either*

$$\begin{cases} Q(s) = -p\ \ln(s) + C_1, \quad F(t) = q\ \ln(t+r) + B_1, \\ G(t) = p\ \ln(t+r) + A_1, \quad H(\xi) = -q\ \ln(1 + e^{-\frac{1}{p}(\xi - A_1 - C_1)}), \end{cases} \tag{2.24}$$

where A_1, B_1, C_1 and $p, q, r,$ are constants with $p \neq 0, q < 0, -r \notin I,$ or

$$Q(s) = -d \ln \frac{1 - e^{-cs}}{\beta} \quad (s \in R_+), \tag{2.25}$$

$$G(t) = d \ln |\mu + e^{ct}| + A_2 \quad (t \in I), \tag{2.26}$$

$$F(t) = \begin{cases} \frac{\alpha}{\mu} \ln |\mu e^{-ct} + 1| + B_2 & \text{if } \mu \neq 0 \\ \alpha e^{-ct} + B_2 & \text{if } \mu = 0 \end{cases} \qquad (t \in 0), \tag{2.27}$$

$$H(\xi) = \begin{cases} -\frac{\alpha}{\mu} \ln |1 - \varepsilon(c, \mu)\beta\mu e^{-\frac{1}{d}(\xi - A_2}| & \text{if } \mu \neq 0 \\ \alpha\beta e^{-\frac{1}{d}(\xi - A_2} & \text{if } \mu = 0 \end{cases} \quad (\xi \in G(I) + Q(R_+)), \tag{2.28}$$

where

$$\varepsilon(c, \mu) = \begin{cases} +1 \quad if \quad (c, \mu) \in P_+(I) \\ -1 \quad if \quad (c, \mu) \in P_-(I), \end{cases} \tag{2.29}$$

where $P_+(I)$ and $P_-(I)$ are the sets of all pairs (c, μ) for which the function $t \mapsto \mu + e^{ct}$ is everywhere positive or everywhere negative, respectively. Here $d, \alpha, c, \beta, \mu, A_2, B_2$ are constants constrained by
$d \neq 0, \beta c < 0, (c, \mu) \in P(I), \varepsilon(c, \mu)\alpha\beta > 0$ (We have $\varepsilon(c, 0) > 0$ and so $\alpha\beta > 0 \quad if \quad \mu = 0$).

Proof. The result follows straight from theorem 2 and (2.15). Indeed, we have either

$$G'_+(t) = \frac{p}{t+r}, \ F'_+(t) = \frac{q}{t+r}, Q'(s) = -\frac{p}{s} \quad (t \in I \neq R, s \in R_+) \tag{2.30}$$

where p, q, and r are real constants, $pq \neq 0, -r \notin I$; or

$$G'_+(t) = \frac{ae^{ct}}{\mu + e^{ct}}\ , F'_+(t) = \frac{b}{\mu + e^{ct}}\ , Q'(s) = \frac{a}{1 - e^{ct}} \quad (t \in I, s \in R_+) \qquad (2.31)$$

where a, b, c, and μ are constants, $abc \neq 0, (c, \mu) \in P(I)$. Since $F'_+(t)$ and $G'_+(t)$ are continuous, F and G are differentiable everywhere (see [7], page 157). After integrating (2.30) and substituting into the equation (2.11) we get (2.24). Since $F'_+(t) < 0$, we have additional restriction $q < 0$. Next, by integrating (2.31) and putting the resulting forms of Q, F, G into (2.11), we get (2.25)-(2.29) with $d = a/c$ and $\alpha = -b/c$. Since H is positive, $\varepsilon(c, \mu)\alpha\beta > 0$. Direct computation shows that the functions (2.25)-(2.29) indeed satisfy equation (2.11) under the given restrictions on constants.

□

The solution of the original equation

Let $0 < k \leq +\infty$ be fixed. We consider our original equation

$$f(v) = f(vw) + f(g^{-1}(g(v)q(w))) \qquad (v \in [0, k[, w \in [0, 1]) \qquad (2.9)$$

under the assumptions

(A) $f : [0, k[\to [0, +\infty]$,

(B) $q : [0, 1] \to [0, 1]$,

(C) $g : [0, k[\to [0, k'[$ is strictly monotonic and surjective.

Theorem 5 *The functions f, g , and q with the properties (A), (B) , and (C) satisfy (2.9) if and only if*

- *either* $f \equiv 0$ *on* $[0, k[$ *, and* g*,* q *are arbitrary,*

- *or* g *is arbitrary and there exists a constant* $c > 0$ *such that*

$$f(0) = 0, \qquad \textit{and} \quad f(v) = c \quad (v \in]0, k[),$$

$$0 < q(0) \leq 1 \quad q(w) = 0 \quad (w \in]0, 1]),$$

- *or there exist constants* $\alpha > 0,\ c > 0,\ d > 0$ *and* $\mu \geq -k^{-c}$ *such that*

$$q(w) = (1 - w^c)^d \quad (w \in [0, 1]),$$

$$g(0) = f(0) = 0, \quad g(v) = \delta(\mu + v^{-c})^{-d}, \quad \textit{and}$$

$$f(v) = \begin{cases} \frac{\alpha}{\mu} \ln(1 + \mu v^c) & \textit{if } \mu \neq 0 \\ \\ \alpha v^c & \textit{if } \mu = 0 \end{cases} \qquad (v \in]0, k[),$$

where the convention $k^{-c} = 0$ *if* $k = +\infty$ *is assumed and*

$$\textit{if } k' = +\infty, \textit{then } \mu = -k^{-c} \textit{and } \delta > 0 \textit{ is arbitrary };$$

$$\textit{if } k' < +\infty, \textit{then } \mu > -k^{-c} \textit{and } \delta = k'(\mu + k^{-c})^d.$$

Proof. **1**. It is obvious that if $f \equiv 0$ on $[0, k[$, and g, q are arbitrary, then such functions are the first family of solutions. Also we note here that even though g is strictly monotonic in this theorem, in fact because of condition (C) we have only the strictly increasing case as a possibility.

2. Suppose that f is not identically zero on [0,k[. Since g is strictly increasing and surjective , it and its inverse equal to zero only if $v = 0$. Let $v = 0$, then (2.9) immediately implies that $f(0) = 2f(0) = 0$. Now let $w = 0$. If $q(0)$ were equal to zero, we would get zero solution $f(v) = 0$ for all $v \in I$. Therefore we would be in the first case. Thus, we have $q(0) > 0$. Now we suppose that $q(w) = 0$ for all w on $]0, 1]$, so we have $f(v) = f(vw)$ for all $v \in I$ and $w \in]0, 1]$, and therefore $f(v)$ is constant on $]0, k[$. This yields the second family of solutions.

3. So far we have that $g(0) = g^{-1}(0) = f(0) = 0 \quad q(0) > 0$ and $f \not\equiv 0$ on $[0, k[$ and $q \not\equiv 0$ on $[0, 1]$. Since f is non-negative function and $w \in [0, 1]$, the equation (2.9) implies that f is increasing, but not necessarily strictly increasing.

Fact 1. There exists a $w_1 \in]0, 1[$ such that $q(w_1) < 1$.

Suppose that $q(w) = 1$ for all w $\in]0, 1[$. Then equation (2.9) gives that $f(vw) = 0$ for all $v \in [0, k[$ and $w \in]0, 1[$. Therefore f is identically zero on $[0, k[$.

Fact 2. $f(v) > 0$ for all $k > v > 0$.

Suppose that $f(v_0) = 0$ for some $v_0 \neq 0$ and $f(v) > 0$ for all $v > v_0$. Since f is increasing function, we have that $f(v) = 0$ for all $0 \leq v \leq v_0$. By Fact 1 there exists w_1 such that $q(w_1) < 1$. Choose $v_1 > v_0$ such that $v_1 w_1 < v_0$, thus $f(v_1 w_1) = 0$. Since g is strictly increasing and $q(w_1) < 1$, we can take $v_2 > v_0$ such that $g^{-1}(g(v_2)q(w_1)) < v_0$. Thus $f(g^{-1}(g(v_2)q(w_1))) = 0$. By the choice of v_1 and v_2 we have that

$$f(v) = f(vw_1) + f(g^{-1}(g(v)q(w_1))) = 0$$

for all $v \in [0, \min(v_1, v_2)]$. This contradicts the definition of v_0, since both v_1 and v_2 are greater than v_0.

Fact 3. If $q(w) = 0$ for some $w \in]0,1[$, then $q(w) = 0$ for all $w \in]0,1[$.

Suppose that $q(w_0) = 0$ for some $w_0 \in]0,1[$. Then from equation (2.9) we see that $f(v) = f(vw_0)$ for all $v \in]0,k[$ and $0 < w_0 < 1$. This proves the constancy of f on]0,k[, thus, together with (2.9), it follows that $q(w) = 0$ for all $w \in]0,1[$. We note here that if $q(w) = 0$ for some $w \in]0,1[$, then we get the second family of solutions. Therefore we can disregard this case.
Now we suppose that q is nowhere zero on $]0,1[$. Thus we have that $f(v) \neq f(vw)$ for all $w \in]0,1[$ and $v \in]0,k[$ and therefore f is injective, and thus, strictly increasing on $[0,k[$. This in turn implies that we can use theorem 3 to find a third solution of equation (2.9). Also, we note that $q(0) = 1$ because of the strict monotonicity of f. Substituting in (2.10), we conclude that all of the conditions of theorem 3 are satisfied and hence F, G, Q, and H must be of the forms (2.25)-(2.29). However, function F in (2.25) cannot be the solution, because this would give us that

$$\lim_{t \to 0} q \ \ln(-\ln\ t + r) + B_1 = -\infty$$

with constants r, B_1, and $q < 0$, but not 0.
Thus we are only left with (2.26) - (2.29). Using (2.10) we get

$$g(v) = e^{A_2} |\mu + v^{-c}|^{-d}, \tag{2.32}$$

$$f(v) = \begin{cases} \frac{\alpha}{\mu} \ln |1 + \mu v^c| + B_2 & if \quad \mu \neq 0 \\ \alpha v^c + B_2 & if \quad \mu = 0 \end{cases} \qquad (v \in I), \tag{2.33}$$

and

$$q(w) = (\frac{1 - w^c}{\beta})^d \qquad (w \in]0, 1[), \tag{2.34}$$

where $d, \alpha, c, \beta, \mu, A_2, B_2$, are constants with

$$d \neq 0, \quad \varepsilon(c, \mu)\alpha\beta > 0, \quad \beta c > 0, \text{ and } \quad (c, \mu) \in P(I).$$

By comparing the functions H in (2.28) to that obtained from (2.33) and (2.34) we get

$$B_2 = 0, \quad \beta = 1,$$

and therefore $c > 0$. Hence $\mu + e^{ct} > 0$ for sufficiently large t. This implies $\varepsilon(c, \mu) = 1$, i.e. $\mu + e^{ct} = \mu + v^{-c} > 0$ for all $v \in]0, k[$. Further, $\alpha > 0$ and $d > 0$ since $q : [0, 1] \to [0, 1]$. Also taking into consideration the boundary conditions $f(0) = q(1) = 0$ and $q(0) = 1$ and that g is surjective, we obtain the third family of solutions.

□

Chapter 3

Second Axiomatization of RDU

3.1 Introduction

We begin as in the introductory part of chapter 2 by assuming that the additive and separable representations holds for $U_{1,E}(x, C)$, that is,

$$\Psi[U^*(x, C; y)] = \Psi[U^*(x)W_E^*(C)] + U_{2,E}(y, C), \tag{3.1}$$

where $\Psi(0) = 0, U_{1,E}(e, C) = 0, U_{2,E}(y, \Omega_E) = 0$, and Ψ is strictly increasing. Setting $x = y$ in the equation (3.1) and using the idempotence axiom we get

$$\Psi[U^*(y)] - \Psi[U^*(y)W_E^*(C)] = U_{2,E}(y, C).$$

Substituting it back to (3.1) gives us for $x \succeq y \succ e$

$$\Psi[U^*(x, C; y)] = \Psi[U^*(x)W_E^*(C)] + \Psi[U^*(y)] - \Psi[U^*(y)W_E^*(C)], \tag{3.2}$$

or, since Ψ is strictly increasing

$$U^*(x, C; y) = \Psi^{-1}(\Psi[U^*(x)W_E^*(C)] + \Psi[U^*(y)] - \Psi[U^*(y)W_E^*(C)]). \tag{3.3}$$

Assuming event commutativity that is for each $x \succeq y \succeq e$, and events C, D,

$$((x, C; y), D; y) \sim ((x, D; y), C; y)$$

and taking $x' = U^*(x), w' = W_E^*(C), y' = U^*(y)$, and $z = W_E^*(D)$ and changing the function Ψ to φ, we get the functional equation

$$\varphi(\varphi^{-1}[\varphi(x'w') + \varphi(y') - \varphi(y'w')]z') - \varphi(y'z')$$

$$= \varphi(\varphi^{-1}[\varphi(x'z') + \varphi(y') - \varphi(y'z')]w') - \varphi(y'w')$$

$$(0 \leq y' \leq x' < K; z', w' \in [0, 1]). \tag{3.4}$$

Here $y' \leq x'$ since this equation was derived from $y \preceq x$.

3.2 A Functional Equation Arising from the Second Axiomatization

By analogy with the second chapter, we first find the solutions of the functional equation, that was originally solved by J.Aczél and G.Maksa (see [2]). Consider equation (3.4), where for simplicity we omit the primes but bear in mind that these are real variables and not the gambles and events,

$$\varphi(\varphi^{-1}[\varphi(xw) + \varphi(y) - \varphi(yw)]z) - \varphi(yz) =$$

$$\varphi(\varphi^{-1}[\varphi(xz) + \varphi(y) - \varphi(yz)]w) - \varphi(yw), \tag{3.5}$$

where $0 \leq y \leq x < K; z, w \in [0, 1[$.

Function $\varphi : [0, K[\rightarrow [0, +\infty[$ has the following properties:

(i) φ is mapping the domain $[0, K[$ onto an interval $[0, K^*[,\ K^* > 0$,

(ii) φ is twice differentiable on $]0, K[$ and $\varphi'(y) \neq 0$ for all $y \in]0, K[$ and

(iii) $\varphi(xw) + \varphi(y) - \varphi(yw)$ belongs to the range of φ for all

$y \in [0, K[, x \in [y, K[, w \in [0, 1[$.

It follows from (i) and (ii) that

(iv) φ is continuous, strictly decreasing on $[0, K[$ and $\varphi(0) = 0$.

The following theorem gives the solutions of the equation 3.5 under the described above assumptions for function φ.

Theorem 6 *Assuming properties (i) - (iii) the general solutions of equation (3.5) are given by :*

$$\varphi(x) = \alpha x^q \quad (q > 0, \alpha > 0)$$

and

$$\varphi(x)) = \gamma\ \ln(\mu x^q + 1) \quad (q > 0, \mu\gamma > 0, \mu > -K^{-q}).$$

(In the subcase $\mu < 0, \gamma < 0$ also $K < \infty$.)

STEP 1. The First Step of the proof.

(a) We can always assume that $K > 1$ in equation (3.5). Indeed, if we introduce new function $\tilde{\varphi}(t) = \varphi(\frac{t}{L})$ with $(0 < L < \infty)$, then it satisfies equation (3.5) and conditions (i) - (iv) for $x, y \in\ [0, K/L[, x \geq y;\ z, w \in [0, 1[$. Also the restrictions on constants remain the same.

(b) We introduce new variables and functions:

$$f(u) = \varphi(e^{-u}), \quad g(u, z) = -\ln \varphi^{-1}[\varphi(e^{-u}z) - \varphi(z) + \varphi(1)]$$

$$(z \in [0, 1[, u \in I), \tag{3.6}$$

where $I :=] - \ln K, \infty[$,

$$h(t, z) = \varphi\Big(\varphi^{-1}(t + \varphi(1))z\Big) - \varphi(z) \qquad (z \in [0, 1[, t + \varphi(1) \in [0, K^*[. \tag{3.7}$$

Now we choose $y = 1, x = e^{-u} \in [1, K[$ (excluding values of y such that $y = x$ since equation (3.5) is identically satisfied for such values) so $x > y = 1$, that is, $u \in I_0$ where $I_0 :=] - \ln K, 0[$, and write $w = e^{-v}$ $(v \in R_+ = \{\alpha|\alpha > 0\})$. Then we can rewrite equation (3.5) in the form of

$$h(f(u + v) - f(v), z) = f(g(u, z) + v) - f(v) \quad (u \in I_0, v \in R_+, z \in [0, 1[). \tag{3.8}$$

STEP 2. Functional differential equation and its solution.

We can differentiate (3.8) with respect to u and v since φ is twice differentiable and therefore f, g, h are twice differentiable too. Respectively we get for all $u \in I_0, v \in R_+$, and $z \in]0, 1[$

$$\partial_1 h(f(u + v) - f(v), z)f'(u + v) = f'(g(u, z) + v)\partial_1 g(u, z)$$

and

$$\partial_1 h(f(u + v) - f(v), z)(f'(u + v) - f'(v)) = f'(g(u, z) + v) - f'(v).$$

Elimination of $\partial_1 h(f(u + v) - f(v), z)$ gives

$$f'(g(u, z) + v)[f'(u + v) + \partial_1 g(u, z)(f'(v) - f'(u + v))] = f'(v)f'(u + v)$$

$$(u \in I_0, v \in R_+, z \in]0, 1[). \tag{3.9}$$

Again differentiating with respect to u and to v and eliminating $f''(g(u, z) + v)$ from received two equalities we get the functional differential equation

$$\partial_1 g(u, z)[\partial_1 g(u, z) - 1]\Big[f''(v)f'(u + v)^2 - f'(v)^2 f''(u + v)\Big] =$$

$$\partial_1^2 g(u,z) f'(v) f'(u+v)[f'(v) - f'(u+v)]. \tag{3.10}$$

Introducing new functions F and G (we are allowed to do that since $\varphi'(y) \neq 0$ for all $y \in]0, K[$ and therefore by (3.6) $f'(u) \neq 0$ and $\partial_1 g(u,z) \neq 0$ for all $u \in I, z \in]0,1[$) by

$$F(u) = \frac{1}{f'(u)}, \quad G(u,z) = \frac{1}{\partial_1 g(u,z)} \quad (u \in I =]-\ln K, \infty[, z \in]0,1[) \tag{3.11}$$

we rewrite equation (3.10) in a different form

$$[G(u,z) - 1][F'(u+v) - F'(v)] = \partial_1 G(u,z)[F(u+v) - F(v)]$$

$$(u \in I_0, v \in R_+, z \in]0,1[). \tag{3.12}$$

Suppose that $G(u,z) = 1$ for all $u \in I_0, z \in]0,1[$. Then, by (3.11) it follows that $\partial_1 g(u,z) = 1$ for $u \in I_0$ and $z \in]0,1[$, and moreover by (3.6),

$$g(u,z) = \begin{cases} u + d(z) & \text{for } u \in I_0, z \in]0,1[\\ 0 & \text{for } u \in I_0, z = 0. \end{cases} \tag{3.13}$$

with some twice differentiable function $d :]0,1[\to R$. Also by (3.6) it follows that $g(u,z)$ is continuous for all $z \in [0,1[$ with the arbitrary fixed $u \in I_0$. Thus, by (3.13)

$$\lim_{z \to 0} g(u,z) = g(u,0) = 0,$$

or,

$$\lim_{z \to 0} d(z) = -u,$$

which is impossible because u can be any value in I_0. This shows that there exist $z_0 \in]0,1[$ and $u_0 \in I_0$ such that $G(u_0, z_0) \neq 1$. Further, since G is continuous, there also exists a neighbourhood U_0 of point u_0 such that $G(u, z_0) \neq 1$ for $u \in U_0$.

Therefore for all $u \in U_0, v \in R_+$ and for chosen z_0 we get from (3.12)

$$F'(v) - F'(u+v) = \frac{\partial_1 G(u, z_0)}{G(u, z_0) - 1}[F(v) - F(u+v)]. \tag{3.14}$$

This is a first order homogeneous differential equation for the function $v \mapsto F(v) - F(u+v)$ whose general solution is

$$F(v) - F(u+v) = k(u)e^{p(u)v} \quad (u \in U_0, v \in R_+), \tag{3.15}$$

where

$$p(u) = \frac{\partial_1 G(u, z_0)}{G(u, z_0) - 1}.$$

Since $v = 0$ is in the domain of F and F is continuous function, we can take a limit of (3.15) when $v \to 0$ to get

$$k(u) = F(0) - F(u) \qquad (u \in U_0). \tag{3.16}$$

Since F is differentiable, we have that $k(u)$ is differentiable, and therefore by (3.15) so is $p(u)$.

If we had $k(u) = 0$ for all u in some interval $\tilde{U}_0$ of positive length of U_0, then by (3.15), we would have $F(v) - F(u+v) = 0$ for all $u \in \tilde{U}_0$ and $v \in R_+$. Therefore, $F(v) = D$ for some nonzero constant D and all $v \in R_+$. By (3.11) this would give us $f'(v) = 1/D$ for all v in R_+, thus $f(v) = (1/D)v + E$ for all $v \in R_+$ and some constant E. Finally, by (3.6) it follows that $\varphi(t) = -(1/D)\ln t + E$ for all $t \in]0, 1[$, which contradicts the fact that φ is continuous on $[0, K[$, therefore on $[0, 1[$ and equal to zero in point $u = 0$. Therefore there exists a point $u_1 \in U_0$, and by continuity a nonempty neighborhood $U_1 \in U_0$ such that $k(u) \neq 0$ for all $u \in U_1$.

Substituting (3.16) into (3.15) and differentiating the result with respect to u we get

$$-F'(u+v) = -F'(u)\, e^{p(u)v} + (F(0) - F(u))p'(u)v\, e^{p(u)v} =$$

$$e^{p(u)v}\Big(-F'(u) + (F(0) - F(u))p'(u)v\Big)$$

$$u \in U_1, v \in R_+).$$

Fixing $u = u_1 \in U_1$ and writing u in place of $u_1 + v$ we get

$$-F'(u) = e^{p(u_1)u} e^{-p(u_1)u_1}\Big(-F'(u_1) + (F(0) - F(u_1))p'(u_1)(u - u_1)\Big)$$

$$(u_1 < u \in U_1, v \in R_+)$$

or with new notations

$$F'(u) = Q^u(Au + B) \quad (u_1 < u \in U_1)$$

(A, B, Q, u_1 are constants and $Q > 0$). Integrating we get

$$F(u) = Q^u(Au + B) + C \quad (u_1 < u \in U_1).$$

After substituting it into (3.15) and using (3.16) we have

$$Q^v[A(1 - Q^u)v - (Au + B)Q^u + B] = k(u)e^{p(u)v} =$$

$$[(B + C) - (Q^u(Au + B) + C)]e^{p(u)v} \quad (u_1 < u \in U_1).$$

Thus $e^{p(u)} = Q$ and $A(1 - Q^u)v = 0$, that is , either $A = 0$ or $Q^u = 1$, i.e. $Q = 1$ since $Q > 0$. So we have either

$$F(u) = Au + B \quad (A \neq 0, \text{ and } u_1 < u \in U_1) \tag{3.17}$$

or

$$F(u) = BQ^u + C \quad (B \neq 0, Q > 0, Q \neq 1, \text{ and } u_1 < u \in U_1). \tag{3.18}$$

STEP 3. Solutions of the original equation.

Suppose $]-l, -m\,[:= \{u \in U_1 : u > u_1\}$ for convenience. Now substituting (3.17) and (3.18) back to (3.16) gives $k(u) = -Au,\ p(u) = 1$ or

$k(u) = B(1 - Q^u),\ p(u) = Q$ on $]-l, -m[$, respectively. Thus (3.15) implies

$$F(v) - Av = F(u+v) - A(u+v) \quad (u \in]-l, -m[, v \in R_+)$$

or

$$F(v) - BQ^v = F(u+v) - BQ^{u+v} \quad (u \in]-l, -m[, v \in R_+),$$

respectively. Introducing $\Phi(t) = F(t) - At$, and respectively $\Phi(t) = F(t) - BQ^t$ gives us that $\Phi(u+v) = \Phi(v)$ for all $v \in R_+$ and for all $u \in]-l, -m[$. Therefore by fixing an arbitrary $v = v_0 \in R_+$ and letting $\Phi(v_0) = \Gamma$, we have

$$\Phi(t) = \Gamma \quad \text{for } t \in]v_0 - l, v_0 - m[.$$

Now choose $t = v_1 = v_0 - 1/2l$, which is still in the domain of Φ. Then $\Phi(v_1) = \Gamma$ and

$$\Phi(t) = \Phi(v_1) = \Gamma \quad \text{for } t \in]v_1 - l, v_1 - m[=]v_0 - 3/2l, v_0 - m - 1/2l[.$$

Continuing we get

$$\Phi(t) = \Gamma \quad \text{for all } t \in]-\ln K, v_0 - m[.$$

Since we can take v_0 as large as we want to, we have shown that

$$\Phi(t) = \text{ constant for all } t \in I =]-\ln K, \infty[.$$

Therefore (3.17) and (3.18) hold for all $u \in I$. More precisely

$$F(u) = Au + B \quad (A \neq 0, u \in I) \tag{3.19}$$

or

$$F(u) = BQ^u + C \quad (B \neq 0, Q > 0, Q \neq 1, u \in I). \tag{3.20}$$

Finally, we find all necessary forms of solutions of (3.5) on $[0, K[$ for which the conditions (i),(ii), and (iii) hold.

By (3.19) and by (3.11) , $f'(u) = \frac{1}{Au+B}$, therefore integrating with the constant of integrating $\ln |R|$ we get $f(u) = \frac{1}{A} \ln |RAu + RB|$ on I . Since φ and also f are strictly monotonic, we have either $RAu + RB > 0 \quad (u \in I)$ or $-RAu - RB > 0 \quad (u \in I)$. From (3.6), with $\alpha = \pm RA, \beta = \pm RB, \gamma = 1/A$, we get

$$\varphi(x) = \gamma \ln (\alpha \ln x + \beta) \tag{3.21}$$

and since $\varphi(0)$ is not defined, this is not one of the possible forms of solutions. For (3.20) we consider two cases $C = 0$ and $C \neq 0$. If $C = 0$ then, using (3.11),

$$f'(u) = \frac{1}{B} Q^{-u}, f(u) = -\frac{1}{B \ln Q} Q^{-u} + \beta$$

and using (3.6), with $q = \ln Q, \alpha = -1/(B \ln Q)$, we get

$$\varphi(x) = \alpha x^q + \beta \quad (x \in [0, K[, \alpha \neq 0) \tag{3.22}$$

and since $\varphi(0) = 0$, $\beta = 0$. Thus the first group of possible solutions is

$$\varphi(x) = \alpha x^q \qquad (x \in [0, K[, \alpha \neq 0. \tag{3.23}$$

If the second case takes place when $C \neq 0$ in (3.20) then, using (3.11), we get

$$f'(u) = \frac{1}{BQ^u + C} = \frac{Q^{-u}}{B + CQ^{-u}}, \quad f(u) = -\frac{1}{B \ln Q} \ln |R(CQ^{-u} + B|.$$

Since f is strictly monotone, we have that either

$$RCQ^{-u} + RB > 0 \quad (u \in I) \text{ or } RCQ^{-u} + RB < 0 \quad (u \in I).$$

At last using (3.6) and taking $q = \ln Q$, $\gamma = -\frac{1}{B \ln Q}$, $\mu = \pm CR$, and $\beta = \pm BR$, we have

$$\varphi(x) = \gamma \ln(\mu x^q + \beta) \quad (x \in]0, K[; \mu\gamma \neq 0) \tag{3.24}$$

and with the fact that $\varphi(0) = 0$ we get

$$\varphi(x) = \gamma \ln(\mu x^q + 1) \quad (x \in [0, K[; \mu\gamma \neq 0). \tag{3.25}$$

The final step of the proof is to find all restrictions on the constants which have been found.

a) Since φ is strictly increasing and maps into R_+, for (3.23) we must have $q > 0$ and $\alpha > 0$. The only condition we have to check is (iii). Indeed, the range of φ is $[0, \alpha K^q[$ and $\alpha(x^q w^q + y^q - y^q w^q)$ obviously belongs to this range for all $y \in [0, K[, x \in [y, K[, w \in [0, 1[$.

b) For (3.25) again since φ is strictly increasing and maps into R_+, we must have $\mu\gamma > 0$, $q > 0$ and $\mu > -K^{-q}$. In the subcase $\mu < 0, \gamma < 0$ also $K < \infty$.

□

3.3 RDU Representation in the First Two Axiomatizations

The study of the RDU representation is completed in the following theorem, where the functions Ψ, U^* and W_E^* are from the introductory part of this chapter:

Theorem 7 *Suppose that separable and additive representations and event commutativity hold, and the function Ψ is strictly increasing and twice differentiable. Also assume that the image of U^* is the real interval $[0, k[$, and W_E^* is onto $[0, 1]$. Then letting $U = (U^*)^q$ and $W = (W^*)^q$ gives for some real constant $\mu > -\frac{1}{k^q}$*

$$U(x, C; y) = \begin{cases} \frac{U(x)W(C)+U(y)[1-W(C)]+\mu U(x)U(y)W(C)}{1+\mu U(y)W(C)} & \textit{it } x \succ y \succeq e \\ U(X) & \textit{if } x \sim y \succeq e \\ \frac{U(x)[1-W(\bar{C})]+U(y)W(\bar{C})+\mu U(x)U(y)W(\bar{C})}{1+\mu U(x)W(\bar{C})} & \textit{if } y \succ x \succeq e. \end{cases} \tag{3.26}$$

This equation is called ratio rank-dependent utility. The case when $\mu = 0$ is the standard rank-dependent utility model.

Proof. According to theorem 6 the solutions of equation (3.5) are :

$$\varphi(x) = \alpha x^q \quad (q > 0, \alpha > 0) \tag{3.27}$$

and

$$\varphi(x) = \gamma \ln(\mu x^q + 1) \quad (q > 0, \mu\gamma > 0, \mu > -K^{-q}). \tag{3.28}$$

Substituting the first family of solutions (3.27) into equation (3.2) gives

$$\alpha(U^*(x, C; y))^q = \alpha(U^*(x)W_E^*(C))^q + \alpha(U^*(y))^q - \alpha(U^*(y)W_E^*(C))^q.$$

Therefore, introducing

$$U = (U^*(x))^q \text{ and } W = (W_E^*(C))^q \tag{3.29}$$

we get the standard RDU model

$$U(x, C; y) = U(x)W(C) + U(u)[1 - W(C)].$$

If we substitute the second family (3.28) into (3.2) we get, using (3.29)

$$\gamma \ln(\mu U(x, C; y) + 1) =$$

$$\gamma \ln(\mu U(x)W(C) + 1) + \gamma \ln(\mu U(y) + 1) - \gamma \ln(\mu U(y)W(C) + 1) =$$
$$\gamma \ln\Big(\frac{(\mu U(x)W(C) + 1)(\mu U(y) + 1)}{\mu U(y)W(C) + 1}\Big)$$

and thus,

$$U(x, C; y) = \frac{U(x)W(C) + U(y)[1 - W(C)] + \mu U(x)U(y)W(C)}{1 + \mu U(y)W(C)}.$$

As we said repeatedly before, the case, when $y \succeq x \succeq e$, is obtained from the complementarity axiom and the case when $y \sim x$ from the idempotence axiom. The standard RDU representation is received when $\mu = 0$. The condition $1 + \mu v^q$ implies that for $\mu < 0$, $U(x) < \frac{1}{|\mu|}$.

$\square$

Here we also notice that the same theorem is true for the case of the additive and separable representations, which were considered in chapter 2 and gave rise to different functional solution, but with nontrivial solutions in the forms (3.27) and (3.28).

Chapter 4

Third Axiomatization of RDU

In our third approach we will use separability and gain partition properties together with some structural conditions, which we will derive in this chapter, to get an RDU representation. This may be the most efficient approach. However, in this axiomatization, the consequences and the events have to be dense, as is shown in [12] .

4.1 Additional Necessary Conditions of RDU representation

There are a few more RDU representation properties that are required for the third axiomatization. The main property is the Thomsen condition for binary gambles.

a) The Thomsen condition:

A structure $\mathbf{G}_1^+$ is said to satisfy the Thomsen condition if for all $x, y, z \in \mathcal{C}^+$ and for all $E \in \mathcal{E}$ and $B, C, D \subseteq \Omega_E$,

$$\text{if } (x, B; e) \sim (z, D; e) \text{ and } (z, C; e) \sim (y, B; e), \text{ then } (x, C; e) \sim (y, D; e).$$

Though it does not seem to be very natural, the following proposition gives us a proof that the Thomsen condition is a property of RDU representation.

Proposition 2 *Suppose the binary gambles from $\mathcal{G}_2^+$ satisfy transitivity, consequence monotonicity, and status-quo event commutativity, then Thomsen condition holds.*

Proof. Suppose $B, C, D \subseteq E, (x, B; e) \sim (z, D; e)$ and $(z, C; e) \sim (y, B; e)$. By the assumptions of status-quo event commutativity and consequence monotonicity, we get:

$$((x, C; e), B; e) \sim ((x, B; e), C; e) \sim ((z, D; e), C; e)$$

$$\sim ((z, C; e), D; e) \sim ((y, B; e), D; e) \sim ((y, D; e), B; e)$$

whence by transitivity and consequence monotonicity, $(x, C; e) \sim (y, D; e)$. This proves that the Thomsen condition holds.

□

b) Restricted solvability:

This property is said to hold if and only if, for all $x^*, x, x_*, y, z \in \mathcal{G}_1^+$ and $D, C^*, C_* \in \mathcal{E}_E$,

$$\text{if } (x^*, C; y) \succeq (z, D; y) \succeq (x_*, C; y),$$

$$\text{then there exists } v \in \mathcal{C}^+ \text{ such that } (v, C; y) \sim (z, D; y),$$

and

$$\text{if } (x, C^*; y) \succeq (z, D; y) \succeq (x, C_*; y),$$

$$\text{then there exists } A \in \mathcal{E}_E \text{ such that } (x, A; y) \sim (z, D; y).$$

Proposition 3 *Suppose $\boldsymbol{G}_2^+$ is an elementary rational structure that satisfies restricted solvability. Then for any gamble $(x, C; y) \in \mathcal{G}_2^+, x \succeq y$, there exists $CE(x, C; y) \in \mathcal{C}$ such that*

$$CE(x, C; y) \sim (x, C; y).$$

Proof. Suppose an elementary rational structure satisfies restricted solvability. Then for all $x, y \in \mathcal{G}_1^+$ with $x \succeq y$ and $C \in \mathcal{E}_E$ by complementarity, certainty, idempotence and consequence monotonicity

$$(x, \Omega_E; y) \sim (y, \emptyset; x) \sim x \sim (x, C; x) \succeq (x, C; y)$$

$$\succeq (y, C; y) \sim y \sim (y, \emptyset; y) \sim (y, \Omega_E; y).$$

Further, by restricted solvability there exists $z \in \mathcal{C}^+$ such that $(z, \Omega_E; y) \sim (x, C; y)$. We use certainty to get

$$z \sim (z, \Omega_E; y) \sim (x, C; y)$$

and taking $CE(x, C; y) = z$ proves the proposition.

□

This simple proposition means that any second-order gamble is indifferent to a pure consequence.

Before introducing the next property we need a few definitions:

Definition. Sequences $\{x_i\}$ from $\mathcal{G}_1^+$ and $\{C_j\}$ from $\mathcal{E}_E$ are said to form *standard sequences* if and only if respectively

$$(x_i, C; z) \sim (x_{i+1}, D; z),$$

for some $z \in \mathcal{G}_1^+$ and some $C, D \in \mathcal{E}_E$ with $C \succ_{\mathcal{E}} D$, where $\succ_{\mathcal{E}_E}$ denotes the ordering induced by the assumption of order independence of events, and

$$(x, C_i; z) \sim (y, C_{i+1}; z),$$

for some $x, y, z \in \mathcal{G}_1^+$ with $x \succ y$.

If for some $y, x \in \mathcal{G}_1^+$ and all x_i in a standard sequence $y \succeq x_i \succeq z$ holds, then the sequence is said to be *bounded*.

c) Archimedeanness :

An elementary rational structure is said to be *archimedean* if every bounded standard sequence is finite.

Let $\mathcal{G}_1^{+*} = \{g : g = (x, C; e)$ where $x \in \mathcal{G}_0^+\}$.

Definition. An ordering $\succeq$ is *nontrivial* if and only if there exist elements x, y in the domain such such that $x \succ y$, and it is *dense* if and only if whenever $x \succ y$, there exists z in the domain such that $x \succ z \succ y$.

Theorem 8 *Suppose an elementary rational structure of binary gambles satisfies:*

- *The ordering over $\mathcal{G}_2^+$ and $\mathcal{E}_E$ are both dense and nontrivial.*
- *Restricted solvability holds.*
- *Structure is Archimedean.*

- *Status-quo event commutativity holds.*

Then there exists a set I that is dense in an interval $[0, a[, a > 0$ such that $U : \mathcal{G}_1^{+*} \to I$ is onto, and a set J that is dense in $[0, 1]$ such that $W : \mathcal{E}_E \to J$ is onto and UW is a separable order preserving representation of $\langle \mathcal{G}_1^{+*}, \succeq \rangle$.

Proof. By proposition 2 the Thomsen condition holds for the given elementary rational structure. Therefore by the theorem of additive conjoint representation, that can be found in the book [6], chapter 6, it follows that an order preserving multiplicative representation exists, but is not defined on elements $x \sim e \in \mathcal{G}_1$ and $C \sim_E \emptyset \in \mathcal{E}_E$. By taking $U(e) = W(\emptyset) = 0$ we include these points while preserving multiplicative representation. Indeed, by using certainty and idempotence, we get the required multiplicative forms

$$U(x, \emptyset; e) = U(e) = 0 = U(x)W(\emptyset)$$

and

$$U(e, C; e) = U(e) = 0 = U(e)W(C).$$

The density of the order guarantees that the image of U is dense in an interval $[0, l[$, where $l = \sup\{ U(g) : g \in \mathcal{G}_1^* \}$, and the image of W is dense in [0,1[.

□

We need another necessary property to get an RDU representation.

d) Gain Partition:

Gain partition is said to hold if there exists a bijection $M : \mathcal{E}_E \to \mathcal{E}_E$ that inverts the order $\succeq_{\mathcal{E}_E}$ and for $x, x', y, y' \in \mathcal{G}_0^+$, with $x \succeq y, x' \succeq y'$, and $C, C' \in \mathcal{E}_E$,

$$(x, C; e) \sim (x', C'; e) \text{ and } (y, M(C); e) \sim (y', M(C'); e)$$

imply

$$(x, C; y) \sim (x', C'; y').$$

4.2 Third Functional Equation

At first we solve a functional equation which is used in the proof of the main result in this section.

We consider the functional equation

$$\frac{z}{p}\gamma^{-1}[z\gamma(p)] = \varphi^{-1}[\varphi(z)\psi(p)] \quad (z, p \in]0, 1[) \tag{4.1}$$

with the following assumptions

(i) $\varphi :]0, 1[\to R_+$ is onto and strictly increasing,

(ii) $\psi :]0, 1[\to]1, \infty[$ is strictly decreasing,

(iii) $\gamma :]0, 1[\to R_+$ is onto, strictly decreasing and that

$$\lim_{p \to 0} p\gamma(p) = 1 \text{ and } \lim_{z \to 1} \varphi(z) = \infty.$$

Since φ is onto and strictly monotonic we can write equation (4.1) in the form

$$\varphi\Big((z/p)\gamma^{-1}[z\gamma(p)]\Big) = \varphi(z)\psi(p) \quad (z, p \in]0, 1[). \tag{4.2}$$

We will try to find solutions of (4.2) with more general assumptions that φ and ψ are only strictly monotonic and map $]0, 1[$ into R_+, thus the solutions of original problem will follow from it.

Introducing new variables u and v with $z = e^{-u}$, $p = \gamma^{-1}(e^{-v})$ and taking logarithms of both sides in (4.2), we get

$$\ln\Big(\varphi(e^{-u})\psi(\gamma^{-1}(e^{-v}))\Big) = \ln\varphi\Big(e^{-u+\ln\gamma^{-1}(e^{-u-v})+\ln\gamma^{-1}(e^{-v})}\Big).$$

Again with new notations of functions $F, G : R \to R$ and $\Phi : R_+ \to R$ defined by

$$\Phi(u) = \ln \varphi(e^{-u}) \quad (u \in R_+), \tag{4.3}$$

$$F(v) = \ln \psi(\gamma^{-1}(e^{-v})) \quad (v \in R) \tag{4.4}$$

and

$$G(v) = v - \ln \gamma^{-1}(e^{-v}) \quad (v \in R), \tag{4.5}$$

the last equation can be written in the form

$$\Phi(u) + F(v) = \Phi\Big(G(u+v) - G(v)\Big) \quad (u \in R_+, v \in R). \tag{4.6}$$

We notice that since the logarithmic and exponential functions preserve monotonicity , the new functions Φ and F are strictly monotonic. Furthermore, since $\gamma^{-1}(e^{-v})$ is strictly decreasing by (iii), G is strictly increasing.
The solutions of the equation (4.6) are given by the following theorem.

Theorem 9 *Let $F : R \to R, \Phi : R_+ \to R$ be strictly monotonic, $G : R \to R$ strictly increasing. They satisfy (4.6), if and only if, there exist nonzero constants a', λ' and positive constants α', β', μ' such that*

$$F(v) = \frac{a'}{\lambda'} \ln\Big(1 + \frac{1}{\mu'} e^{\lambda' v}\Big) \quad (v \in R), \tag{4.7}$$

$$G(v) = -\frac{1}{\lambda'} \ln(\beta'(1 + \mu' e^{-\lambda' v})) \quad (v \in R), \tag{4.8}$$

and

$$\Phi(u) = -\frac{a'}{\lambda'} \ln(\alpha' |e^{-\lambda' u} - 1|) \quad (u \in R_+). \tag{4.9}$$

Proof. Considering equation (2.11) in chapter 2

$$H^{-1}(F(t) - F(t+s)) = G(t) + Q(s) \quad (t \in I =]k, +\infty[, s \in R_+, k \geq -\infty). \tag{4.10}$$

with G and H strictly monotonic and comparing it with equation (4.6) in this chapter

$$\Phi(u) + F(v) = \Phi\Big(G(u+v) - G(v)\Big) \quad (u \in R_+, v \in R),$$

we can notice that (4.6) is the special case of equation (4.10). Indeed, the variables t, s in equation (4.10) correspond to the variables v, u respectively, and the functions H^{-1}, F, Q and G in equation (4.10) correspond to the functions $\Phi, -G, \Phi$, and F respectively. Thus, if we assume that

$$H^{-1} = Q \qquad \text{and } k = -\infty \tag{4.11}$$

in equation (4.10), then we have equation (4.6) with all assumptions on functions (theorem 2) and variables matching.

Theorem 4 gives all solutions of equation (4.10), and therefore it remains to check if there are solutions of equations (4.6) in the form of solutions of equation (4.10) with restriction (4.11). At first we check the first family of solutions

$$\begin{cases} Q(s) = -p\,\ln(s) + C_1, \quad F(t) = q\,\ln(t+r) + B_1, \\ G(t) = p\,\ln(t+r) + A_1, \quad H(\xi) = -q\,\ln(1 + e^{-\frac{1}{p}(\xi - A_1 - C_1)}), \end{cases} \tag{4.12}$$

where A_1, B_1, C_1 and p, q, r, are constants with $p \neq 0, q < 0, -r \notin I$.

The restriction (4.11) gives that

$$-r \notin]-\infty, \infty[,$$

which is impossible, therefore there are no solutions of equation (4.6) in the form (4.12).

We now turn to check the second family of solutions of equations (4.10) given by theorem 4

$$Q(s) = -d \ln \frac{1 - e^{-cs}}{\beta} \quad (s \in R_+), \tag{4.13}$$

$$G(t) = d \ln |\mu + e^{ct}| + A_2 \quad (t \in I), \tag{4.14}$$

$$F(t) = \begin{cases} \frac{\alpha}{\mu} \ln |\mu e^{-ct} + 1| + B_2 & \text{if } \mu \neq 0 \\ \alpha e^{-ct} + B_2 & \text{if } \mu = 0 \end{cases} \quad (t \in 0), \tag{4.15}$$

$$H(\xi) = \begin{cases} -\frac{\alpha}{\mu} \ln |1 - \varepsilon(c, \mu)\beta\mu e^{-\frac{1}{d}(\xi - A_2)}| & \text{if } \mu \neq 0 \\ \alpha\beta e^{-\frac{1}{d}(\xi - A_2)} & \text{if } \mu = 0 \end{cases} \quad (\xi \in G(I) + Q(R_+)), \tag{4.16}$$

where

$$\varepsilon(c, \mu) = \begin{cases} +1 & if \quad (c, \mu) \in P_+(I) \\ -1 & if \quad (c, \mu) \in P_-(I), \end{cases} \tag{4.17}$$

where $P_+(I)$ and $P_-(I)$ are the sets of all pairs (c, μ) for which the function $t \mapsto \mu + e^{ct}$ is everywhere positive or everywhere negative, respectively. Also here $d, \alpha, c, \beta, \mu, A_2, B_2$ are constants constrained by $d \neq 0, \beta c < 0$, $(c, \mu) \in P(I), \varepsilon(c, \mu)\alpha\beta > 0$ (We have $\varepsilon(c, 0) > 0$ and so $\alpha\beta > 0 \quad if \quad \mu = 0$).

The restriction (4.11) gives $H^{-1} = Q$ or equivalently $H = Q^{-1}$, thus considering the first limb of (4.16) gives

$$-\frac{\alpha}{\mu} \ln |1 - \varepsilon(c, \mu)\beta\mu e^{-\frac{1}{d}(\xi - A_2)}| = -\frac{1}{c} \ln(1 - \beta e^{-\frac{1}{d}\xi}),$$

which gives two conditions on the constants

$$c = \frac{\alpha}{\mu} \text{ and } \varepsilon(c, \mu)\mu e^{A_2/d} = 1. \tag{4.18}$$

We will get solution (4.7) from (4.14) with

$$d = \frac{a'}{\lambda'}, c = \lambda', \mu = \mu', A_2 = \frac{a'}{\lambda'} \ln(1/\mu').$$

Also the second condition of (4.11) gives that $|\mu + e^{ct}| = \mu + e^{ct}$ with $\mu > 0$.

Solution (4.8) is received from (4.13) if $\frac{\mu}{\alpha} = \lambda', \mu = \mu', c = \lambda'$, and $B_2 = \frac{1}{\lambda'} \ln \beta'$.

Again since $t \in R$, μ is positive.

And finally, if $d = \frac{a'}{\lambda'}, c = \lambda'$, and since $\beta c > 0$ guarantees that $\frac{1-e^{-ct}}{\beta} > 0$, $|\beta| = \frac{1}{\alpha'}$, then (4.15) gives us solution (4.9).

To finish the proof of this theorem we notice that the second family of solutions of equation (4.10) when $\mu = 0$ will not give any solutions for equation (4.6) since the condition $H^{-1} = Q$ will not be satisfied.

□

Returning to our old notations in equation (4.6) with

$$f(v) = F'(v), \quad g(v) = G'(v) \quad (v \in R), \quad h(u) = \Phi'(u) \quad (u \in R_+)$$

and using (4.3), (4.4) and (4.5) we get

$$F'(v) = \frac{ae^{\lambda v}}{\mu + e^{\lambda v}}, \quad G'(v) = \frac{b}{\mu + e^{\lambda v}}, \quad (v \in R) \quad \text{and } \Phi'(u) = \frac{a}{1 - e^{\lambda u}} \quad (u \in R_+), \tag{4.19}$$

where $\mu \geq 0, \lambda \neq 0, a \neq 0, b \neq 0$ are constants. Moreover, since $G'(v) > 0$, we have stronger restriction

$$b > 0. \tag{4.20}$$

We integrate (4.19) to get

$$\Phi(u) = -\frac{a}{\lambda} \ln(\alpha|e^{-\lambda u} - 1|) \quad (u \in R_+), \tag{4.21}$$

$$F(v) = \frac{a}{\lambda} ln(\delta(\mu + e^{\lambda v})) \quad (v \in R) \tag{4.22}$$

with the constants of integration $-\frac{a}{\lambda} \ln \alpha$ and $\frac{a}{\lambda} \ln \delta$.

We now prove that $\mu \neq 0$. Indeed, if we had $\mu = 0$, then (4.19) would give

$$F(v) = av + C_1, \quad G'(v) = be^{-\lambda v}, \quad \text{or} \quad G(v) = -\frac{b}{\lambda} e^{-\lambda v} + C_2.$$

Substitution into (4.6) with $v = 0$ would yield

$$\ln(\alpha |e^{-\lambda u} - 1|) - \ln \delta = \ln(\alpha |e^{b(e^{-\lambda u} - 1)} - 1|),$$

where $\delta = e^{C_2}$ and with $x = e^{-\lambda u} - 1$ we would get

$$|x| = \delta |e^{bx} - 1|$$

which is not true for all $x > 0$ (if $\lambda < 0$) or for all $x < 0$ (if $\lambda > 0$). Thus we have $\mu > 0$ and , integrating (4.19), we obtain

$$G(v) = -\frac{b}{\lambda \mu} \ln(\beta(1 + \mu e^{-\lambda v})) \quad (v \in R) \tag{4.23}$$

with $-\frac{b}{\lambda} \ln \beta$ as a integration constant.

To find necessary restriction on the constants of integrating we substitute (4.21), (4.22) and (4.23) into (4.6) to get after obvious simplifications

$$\frac{\alpha |e^{-\lambda u} - 1|}{\delta(\mu + e^{\lambda v})} = \alpha \Big| \Big(\frac{e^{\lambda v} + \mu e^{-\lambda u}}{e^{\lambda v} + \mu} \Big)^{b/\mu} - 1 \Big|. \tag{4.24}$$

Letting $e^{\lambda v} \to 0$ (i.e. $v \to +\infty$ if $\lambda < 0$, while $v \to -\infty$ if $\lambda < 0$), we have:

$$\frac{|e^{-\lambda u} - 1|}{\delta \mu} = \Big| (e^{-\lambda u})^{b/\mu} - 1 \Big|.$$

From here it follows that $\delta = \frac{1}{\mu}$ and $b = \mu$, and thus the proof is complete.

Theorem 10 *The functions $\varphi, \psi :]0,1[\to R_+$ are strictly monotonic and $\gamma :]0,1[\to R_+$ strictly decreasing, surjective and they satisfy (4.2) if, and only if, there exist constants $A > 0, B > 0, K > 0$ and $c \neq 0$ such that for all $z, p \in]0,1[$*

$$\varphi(z) = A(z^{-k} - 1)^c,$$

$$\gamma(p) = B(p^{-k} - 1)^{1/k},$$

and

$$\psi(p) = p^{kc}.$$

Proof. From theorems 9 and 10 we have the following results:

$$\varphi(z) = e^{\Phi(-lnz)} = (\alpha|z^{\lambda} - 1|)^{-a/\lambda} = A|z^{-k} - 1|^c \quad (z \in]0,1[), \tag{4.25}$$

$$\begin{aligned} \gamma^{-1}(y) &= \frac{1}{y} e^{-G(-\ln y)} = \Big(\frac{\beta + \beta\mu y^{\lambda}}{y^{\lambda}}\Big)^{1/\lambda} \\ &= (\beta y^k + \beta\mu)^{-1/k} \quad (y \in R_+), \end{aligned} \tag{4.26}$$

$$\psi(\gamma^{-1}(y)) = e^{F(-lny)} = \Big(1 + \frac{1}{\mu} y^{-\lambda}\Big)^{a/\lambda} = \Big(1 + \frac{1}{\mu} y^k\Big)^{-c} \quad (y \in R_+) \tag{4.27}$$

(where $k = -\lambda \neq 0, c = -\frac{a}{\lambda} \neq 0, A = \alpha^c > 0$). Since γ is supposed to be decreasing and surjective, therefore $\lim_{y \to +\infty} \gamma^{-1}(y) = 0$ and $\lim_{y \to 0} \gamma^{-1}(y) = 1$. Thus, in (4.26), $k > 0$ and $\beta = \frac{1}{\mu}$. Hence, from (4.25),(4.26), and (4.27) with $B = \mu^{1/k} > 0$, we get

$$\varphi(z) = A(z^{-k} - 1)^c \quad (z \in]0,1[), \tag{4.28}$$

$$\begin{aligned} \gamma(p) &= (\mu p^{-k} - \mu)^{1/k} = \mu^{1/k}(p^{-k} - 1)^{1/k} \\ &= B(p^{-k} - 1)^{1/k} \quad (p \in]0,1[), \end{aligned} \tag{4.29}$$

$$\phi(p) = (1 + \frac{1}{\mu}\gamma(p)^k)^{-c} = (1 + \frac{1}{\mu}B^k(p^{-k} - 1)^{-c}$$

$$= p^{kc} \quad (z \in]0,1[). \tag{4.30}$$

Conversely , (4.2) is satisfied by (4.28),(4.29) and (4.30) with the constants $c \neq 0, k > 0, A > 0, B > 0$. Also,$\varphi, \psi :]0,1[\to R_+$ are strictly monotonic and $\gamma :]0,1[\to R$ is strictly decreasing and surjective.

□

It is easy to see that the next corollary holds.

Corollary 1 *1.If* $\lim_{p\to 0}[p\gamma(p)] = 1$*, then* $B = 1$*.*

2.If $\lim_{z\to 1}\varphi(z) = +\infty$ *is supposed, then* $c < 0$*. Thus,* φ *is strictly increasing and* ψ *is strictly decreasing. Furthermore the codomains of* φ *and* ψ *are* $]0, +\infty[$ *and* $]1, +\infty[$*, respectively.*

4.3 Main Theorem

One of the main results is due to Marley and Luce [12] and contained in the following theorem.

Theorem 11 *The following statements hold for the structure* $\langle \mathcal{G}_2^+, \succeq \rangle$*:*

(i) If $\langle \mathcal{G}_2^+, \succeq \rangle$ *has an RDU representation, then it is an Archimedean elementary rational structure that satisfies event commutativity and gains partition with* $W[M(C)] = 1 - W(C)$*.*

(ii) If in addition the representation is onto sets that are dense in intervals, then the density property holds.

(iii) If an Archimedean elementary rational structure $\langle \mathcal{G}_2^+, \succeq \rangle$ satisfies event commutativity and gains partition, as well as dense and restricted solvability properties, then it has a RDU representation with $W[M(C)] = 1 - W(C)$.

Proof. We start with a few propositions.

We define the set $A = \{(X, Y) : \exists\, x, y \in \mathcal{G}_0, C \in \mathcal{E}_E$ such that $x \succeq y, X = U(x)W(C), Y = U(y)W[M(C)]\}$.

Proposition 4 *Under the same assumptions as in theorem 8 together with the gain partition property and a separable representation UW, there exists a function $R : A \to I$ (we recall that the set I has been defined in theorem 8) that is surjective, continuous and strictly increasing in each argument and such that for all $x, y \in \mathcal{G}_0$ with $x \succeq y$,*

$$U(x, C; y) = R(U(x)W(C), U(y)W[M(C)]), \tag{4.31}$$

and

$$R(X, 0) = X, \quad R(0, Y) = Y. \tag{4.32}$$

Proof. We first extend U to $\mathcal{G}_1$ since by theorem 8, function U is defined only on $\mathcal{G}_1^{+*}$. To do that we use the property of the existence of certainty equivalents and set $U(x, C; y) = U[CE(x, C; y)]$. Then for all $(X, Y) \in A$, define $R(X, Y) = Z, Z \in I$ for $x, y \in \mathcal{G}_0, x \succeq y$ and $C \in \mathcal{E}_E$ with $X = U(x)W(C), Y = U(y)W[M(C)]$ and $Z = U(x, C; y)$. R is well defined because of gains partition and thus, (4.31) holds. By proving one of the equalities in (4.32) we will show that R is onto I. Let $X = U(x) = U(x)W(\Omega_E)$ and $Y = U(y)W(\emptyset) = U(y)W[M(\Omega_E)] = 0$ (here $M(\Omega_E) = \emptyset$ and also $M(\emptyset) = \Omega_E$ by theorem 8). Then by complementarity and certainty axioms and by proposition 3

$$R(X,0) = R(U(x)W(\Omega_E), U(y)W[M(\Omega_E)]) = U[CE(x,\Omega_E;y)]$$

$$= U[CE(y,\emptyset;x)] = U(y,\emptyset;x) = U(x) = U(x)W(\Omega_E) = X.$$

By choosing respectively $X = 0$ and $Y = U(y)$ and using the same method we will get

$$R(0,Y) = Y.$$

Therefore R is surjective. By consequence monotonicity, R is strictly increasing in each argument and because it is onto an interval, which is dense in $[0,a[$ for some $a > 0$, it is also continuous in each argument. The result follows.

$\square$

For each $P \in [0,1]$ such that there exists $C \in \mathcal{E}_E$ with $P = W(C)$ we introduce a function

$$\pi(P) = W[M(C)].$$

It is not difficult to show that $\pi : [0,1]^* \to [0,1]$ is a well defined strictly decreasing function (here we should mention that $[0,1]^* \subseteq [0,1]$ is the domain of the function π. This domain is not necessarily the whole interval $[0,1]$, since there may not exist $C \in \mathcal{E}_E$ for all $P \in [0,1]$, but later we will work with the extension of π, which has $[0,1]$ as the domain). Indeed, taking C and C' such that $P = W(C) = W(C')$ gives $C \sim_{\mathcal{E}_E} C'$ and $W[M(C)] = W[M(C')]$, because both M and W preserve equivalence with respect to the order $\succeq_{\mathcal{E}_E}$ (relation $\succeq_{\mathcal{E}_E}$ was introduced in the axiom 8). Since M inverts the order $\succeq_{\mathcal{E}_E}$, π is strictly decreasing.

We will use the following notations:

$$P = W(C) \quad P \in [0,1],$$

$$Z = \frac{U(y)}{U(x)} \quad Z \in [0,1] \text{ if } x \succeq y \succ e,$$

$$\gamma(P) = \frac{\pi(P)}{P} \quad \gamma :]0,1[\to]0,\infty[.$$

We note here that π can be extended to a continuous, strictly decreasing function on]0,1[because the range of W is dense in $[0,1]$, therefore γ is continuous on]0,1[, and R can be extended continuously to intervals for each of its two variables. For the remainder of the proof of the main theorem we will work with these extension.

Proposition 5 *Under the same assumption as in theorem 8 we have that for all* $x, y \in \mathcal{G}_0^+$ *with* $x \succeq y$

$$U(x,C;y) = \begin{cases} \frac{U(x)P}{\gamma^{-1}[Z\gamma(P)]}, & \textit{if} \quad U(x)P > 0 \\ U(y)\pi(P), & \textit{if} \quad U(x)P = 0, \end{cases} \tag{4.33}$$

where $P = W(C)$.

Proof. For the case $U(x)P = 0$,

$$U(x,C;y) = R[U(x)P, U(y)\pi(P)] = R[0, U(y)\pi(P)] = U(y)\pi(P).$$

Thus, we assume $U(x)P > 0$. Letting $v = CE(x,C;y)$ and $Q = \frac{U(x)}{U(v)}$, where $Q \in [0,1]$ by consequence monotonicity, gives

$$U(v) = U(x,C;y) \geq U(x,C;e) = R(U(x)W(C), U(e)W[M(C)])$$

$$= R(U(x)W(C), 0) = U(x)W(C) = U(x)P.$$

Let the image of W be J. Since J is dense in the unit interval, we may choose a descending sequence $Q_i \in J$ converging to Q. Let $D_i \in \mathcal{E}_E$ be such that $W(D_i) = Q_i$ and $W[M(D_i)] = \pi(Q_i)$. Therefore by the previous proposition

$$R[U(x)P, U(y)\pi(P)] = U(x, C; y) = U(v) = U(v, D_i; v)$$

$$= R(U(v)W(D_i), U(v)W[M(D_i)]) = R[U(v)Q_i, U(v)\pi(Q_i)].$$

By continuity of R we can take limits to get

$$R[U(x)P, U(y)\pi(P)] = \lim_{i\to\infty} R[U(v)Q_i, U(v)\pi(Q_i)]$$

$$= R[U(v)Q, U(v)\pi(Q)] = R[U(x)P, U(v)\pi(Q)].$$

Using the monotonicity of R, we see $U(y)\pi(P) = U(v)\pi(Q)$. Thus,

$$\gamma(Q) = \frac{\pi(Q)}{Q} = \frac{U(y)\pi(P)}{U(x)P} = Z\gamma(P).$$

Finally,

$$U(x, C; y) = U(v) = \frac{U(x)P}{Q} = \frac{U(x)P}{\gamma^{-1}[Z\gamma(P)]},$$

which proves this proposition.

Further, by monotonicity of consequences and idempotence it follows that for $x \succeq y$ and $C \in \mathcal{E}_E$,

$$x \succeq (x, C; y) \succeq y. \tag{4.34}$$

Therefore from (4.33) and (4.34) we have for $Z, P \in]0, 1[$

$$P \leq \gamma^{-1}[Z\gamma(P)] \leq \frac{P}{Z}. \tag{4.35}$$

We will introduce new notations,

$$F(Z, P) = \gamma^{-1}[Z\gamma(P)], \text{ where } \quad F :]0, 1[\times]0, 1[\to]0, \infty[, \tag{4.36}$$

$$G(Z,P)=\frac{Z}{P}F(Z,P), \text{ where } \quad Z,P\in]0,1[, \tag{4.37}$$

where we use the continuous extension of γ. Both F and G are continuous in both variables. From equation (4.35) we have two bound for each F and G:

$$P\leq F(Z,P)\leq\frac{P}{Z},\quad Z\leq G(Z,P)\leq 1. \tag{4.38}$$

Finally, introducing iteration of G by

$$G^1(Z,P)=G(Z,P),$$

$$G^i(Z,P)=G[G^{i-1}(Z,P),P]\quad i>1,$$

we can state the last proposition.

Proposition 6 *Under the same assumption as in theorem 8 together with gains partition, G has the following properties:*

(i) *G is strictly increasing in the first variable and strictly decreasing in the second.*

(ii) *For all $Z,P,Q\in]0,1[$,*

$$G[G(Z,P),Q]=G[G(Z,Q),P]. \tag{4.39}$$

(iii) *G is Archimedean in the sense that for all $Y,Z,P\in]0,1[$, there exists a positive integer m such that $G^m(Z,P)\geq Y$.*

(iv) *G is solvable in the sense that for all $Y,Z,P\in]0,1[$ if there exists a nonnegative integer n such that*

$$G^{n+1}(Z,P)\geq Y>G^n(Z,P); \tag{4.40}$$

then there exists $Q\in]0,1[$ with $Y=G[G^n(Z,P),Q]$.

Proof. Since we have $Z, P \in]0,1[$ and $x \succ y \succ e$, we can assume that $U(x)P > 0$ and therefore we will consider the first limb of equation (4.33).

(i) Let $\tilde{I} = \{r | \exists x, y \in \mathcal{G}_0 \text{ with } r = \frac{U(y)}{U(x)}\}$. For $s, x, y \in \mathcal{G}_0$ with $s \succ x \succ y \succ e$ consider $Y, Z \in \tilde{I}$ such that $Y = \frac{U(y)}{U(s)}, Z = \frac{U(y)}{U(x)}$ and $Y < Z$. Fix P such that there exists $C \in \Omega_E$ with $W(C) = P$, then by the first limb of equation (4.33) we have

$$\begin{aligned} Y < Z \quad &\Leftrightarrow \quad U(s) > U(x) \quad \Leftrightarrow \quad U(s, C; y) > U(x, C; y) \\ &\Leftrightarrow \quad \frac{U(y)}{U(s, C; y)} < \frac{U(y)}{U(x, C; y)} \quad \Leftrightarrow \\ &\Leftrightarrow \quad \frac{Y}{P}F(Y, P) < \frac{Z}{P}F(Z, P) \quad \Leftrightarrow \quad G(Y, P) < G(Z, P). \end{aligned} \tag{4.41}$$

Now we consider arbitrary $Y, Z, P \in]0,1[$ with $Y < Z$, then by density and continuity of G there exist $Y'', Z'' \in \tilde{I}$ such that $Y < Y" < Z" < Z$ and $\max\{|G(Y'', P) - G(Y, P)|, |G(Z'', P) - G(Z, P)|\} < R/4$, where $R = G(Z", P) - G(Y", P)$. By (4.41) it follows that $G(Y, P) < G(Z, P)$ and by continuity of G it extends to any $P \in]0,1[$.

The fact that G is strictly decreasing in the second variable can be proved by analogy with fixing $Z = \frac{U(y)}{U(x)}$ and considering $C, D \in \Omega_E$ with $P = W(C), Q = W(D)$ and $P > Q$.

(ii) Since $P, Z \in]0,1[$, we are in the case where $x \succ y \succ e$ and $U(x)P > 0$. Therefore by equation (4.33)

$$U[((x, C; y), D; y)] = U[(CE(x, C; y), D; y)] = \frac{U[CE(x, C; y)]W(D)}{\gamma^{-1}(\frac{U(y)}{U[CE(x,C;y)]}\gamma[W(D)])}$$

$$\frac{U(x, C; y)W(D)}{\gamma^{-1}(\frac{U(y)}{U(x,C;y)}\gamma[W(D)])} = \frac{U(x)PQ}{F(Z, P)F[\frac{Z}{P}F(Z, P), Q]}.$$

By the event commutativity axiom and by the continuity of F in each variable we have

$$F(Z,P)F[\frac{Z}{P}F(Z,P,Q)] = F(Z,Q)F[\frac{Z}{Q}F(Z,Q),P]$$

holds for all $P,Q,Z \in]0,1[$. Multiplying this equality by $\frac{Z}{PQ}$ and writing it in terms of G gives us the desired result.

(iii) Because G is strictly increasing in the first variable, the Archimedean property is satisfied in all cases, except one when G^m approaches a limit $K < 1$. But in this case by definition of G

$$K = G(K,P) = \frac{K}{P}\gamma^{-1}[K\gamma(P)],$$

thus, $\gamma(P) = K\gamma(P)$ and finally, $K = 1$ because $\gamma(P) > 0$ for all $P \in]0,1[$.

(iv) Since $G(Z,P)$ is strictly decreasing in P, the equation (4.40) together with the fact that $G(Z,1) = 1$ gives

$$G[G^n(Z,P),P] = G^{n+1}(Z,P) \geq Y > G^n(Z,P) = G[G^n(Z,P),1].$$

By continuity of G in the second variable there exists Q such that
$Y = G[G^n(Z,P),Q]$. That concludes the proof of the proposition.

Now we are returning to the proof of the main theorem. We consider equation (4.39). According to Marley (see [13]), if G is commutative, strictly monotonic in Z, Archimedean and it is solvable, as we have in Proposition 6, there exist functions $\varphi :]0,1[\to]0,\infty[$ and $\varphi :]0,1[\to]1,\infty[$ such that

$$G(Z,P) = \varphi^{-1}[\varphi(Z)\psi(P)]. \tag{4.42}$$

Since G is strictly increasing in Z , φ is strictly increasing, and because G is strictly decreasing in P, ψ is strictly decreasing. From equation (4.38) we have

$$1 \leq \psi(P) \leq \frac{\varphi(1)}{\varphi(Z)}.$$

Putting together (4.36), (4.37), and (4.42) gives us

$$\frac{Z}{P}\gamma^{-1}[Z\gamma(P)] = \frac{Z}{P}F(X,Z)$$

$$= G(Z,P) = \varphi^{-1}[\varphi(Z)\psi(P)].$$

We have obtained the functional equation that has been solved in the previous subsection and based on those solutions we have for some constants

$A > 0, k > 0, c > 0$

$$\gamma(P) = \frac{(1-P^k)^{1/k}}{P},$$

$$\varphi(Z) = A(\frac{1-Z^k}{Z^k})^c,$$

and

$$\psi(P) = P^{kc}.$$

Thus,

$$\pi(P) = P\gamma(P) = (1-P^k)^{1/k},$$

and

$$\varphi(Z)\psi(P) = \varphi\Big(\frac{Z}{[(1-P^k)Z^k + P^k]^{1/k}}\Big). \tag{4.43}$$

We now consider expressions for $U(x, C; y)$, $x \succeq y$. Let $P = W(C)$ and consider three possibilities:

(i) $x \succ y \succ e$. Then $U(x) > U(y) > 0$, and so $Z = \frac{U(x)}{U(y)}$ is in $]0, 1[$. We again split this case into three cases for P.

For $P \in]0,1[$, by proposition (5) and the previous solutions, we have

$$U(x,C;y) = \frac{U(x)W(C)}{F\Big(\frac{U(y)}{U(x)}, \gamma[W(C)]\Big)} = \frac{U(x)P}{\frac{P}{Z}G(Z,P)}$$

$$= \frac{U(y)}{G(Z,P)} = \frac{U(y)}{\varphi^{-1}[\varphi(Z)\psi(P)} = \frac{U(y)}{\varphi^{-1}[\varphi(\frac{Z}{[(1-P^k)Z^k+P^k]^{1/k}})]}$$

$$= \frac{U(y)[(1-P^k)Z^k+P^k]^{1/k}}{Z}.$$

Thus, setting $U^* = U^k$ and $W^* = W^k$, we have

$$U^*(z,C;y) = U^*(x)W^*(C) + U^*(y)[1 - W^*(C)],$$

which is the binary rank-dependent form for the case $x \succeq y \succ e$.

For $P = 0$, i.e. $C \sim_{\mathcal{E}_E} \emptyset$, proposition 4 gives

$$U(x,C;y) = R(U(x)W(\emptyset), U(y)W[M(\emptyset)]) = R(0, U(y)W[M(\emptyset)])$$

$$U(y)W([M(\emptyset)] = U(y)W(\Omega) = U(y).$$

Taking k th power gives $U^*(x,C;y) = U^*(y)$ which is the special case of the binary RDU form.

For $P = 1$ i.e. $C \sim_{\mathcal{E}_E} \bar{\emptyset}$ complementarity gives

$$(x,C;y) = (x,\emptyset;y) \sim (y,\emptyset;x),$$

and the binary RDU form follows by applying the above argument for the case $P = 0$ to the term $(y,\emptyset;x)$.

(ii) $x \sim y \succ e$, i.e. $U(x) = U(y)$. Using idempotence, we have $U(x,C;y) = U(x) = U(y)$, and with $U^* = U^k$, we have

$$U^*(x,C;y) = U^*(x)W^*(C) + U^*(y)[1 - W^*(C)].$$

(iii) $y \sim e$, i.e. $U(y) = 0$. Using proposition 4

$$U(x, C; y) = R(U(x)W(C), U(y)W[M(C)]) = R[U(x)W(C), 0] = U(x)W(C).$$

Taking the k-th power gives us the binary RDU form.

Complementarity proves the case when $x \preceq y$.

Bibliography

[1] J.Aczél. *Lectures on Functional Equations and Their Approaches.* Academic Press, New York/London,1966.

[2] J.Aczél and G.Maksa. A functional equation generated by event commutativity in separable and additive utility theory. Aequationes Math. 62 (2001) 160-174.

[3] J.Aczél, G.Maksa, C.T.Ng and Z.Páles. A functional equation arising from ranked additive and separable utility. Proc.Amer. Math. Soc.,Vol.129, no. 4 (2000), 989-998.

[4] J.Aczél, G.Maksa, and Z.Páles. Solution of a functional equation arising in an axiomatization of the utility of binary gambles. Proc. Amer. Math. Soc. Vol. 129, no. 2 (2000), 483-493.

[5] E.Hewitt and K.Stromberg. *Real and Abstract Analysis.* Springer, New York, 1975.

[6] D.H.Krantz, R.D.Luce, P.Suppes and A.Tversky. *Foundations of Measurements* (Vol I) Academic Press, New York, 1971.

[7] M.Kuczma. *An Inroduction to teh Theory of Functional Equations and Inequalities* Pan. Wydaw. Nauk., Warszawa, 1985.

[8] R.D. Luce. *Utility of Gains and Losses: Measurement-Theoretical and Experimental Approaches.* LEA Publishers, 2000.

[9] R.D.Luce. Coalescing, event commutativity, and theories of utility. J. Risk and Uncert., 16 (1998),87-114.

[10] R.D.Luce. When four distinct ways to measure utility are the same. J. Math. Psych. 40 (1996), 297-317.

[11] R.D.Luce and A.A.J.Marley. Separable and additive representations of binary gambles of gains. Math. Social Sci. 40 (2000),no. 3, 227-295.

[12] A.A.J.Marley and R.D.Luce. A simple axiomatization of binary rank-dependent utility of gains (losses). J. Math. Psych. 46 (2002), 40-55.

[13] A.A.J.Marley and R.D.Luce. Ranked-weighted utilities and qualitative convolution. J. Risk and Uncert., 23:2,(2001), 135-163.

[14] A.W.Roberts and D.E.Varberg. *Convex Functions.* Academic Press, New York, 1973.