# Self-Dual Graphs

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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# Abstract

The study of self-duality has attracted some attention over the past decade. A good deal of research in that time has been done on constructing and classifying all self-dual graphs and in particular polyhedra. We will give an overview of the recent research in the first two chapters. In the third chapter, we will show the necessary condition that a self-complementary self-dual graph have $n \equiv 0, 1 \pmod 8$ vertices and we will review White's infinite class (the Paley graphs, for which $n \equiv 1 \pmod 8$). Finally, we will construct a new infinite class of self-complementary self-dual graphs for which $n \equiv 0 \pmod 8$.

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# Introduction

The idea of duality in graph theory may be seen in objects as old as the Platonic solids, where we can see that the hexahedron (or cube) is dual to the octahedron, the dodecahedron is dual to the icosahedron, and the tetrahedron is dual to itself. However, until the past 30 years, not much research has been done in the area of self-dual graphs.

The purpose of this thesis is to consider the family of self-dual graphs. In order to properly study a family of graphs, we must first generate a large number of examples. As a result, most of the recent research has been on construction and classification techniques, both for general self-dual graphs, and specific subfamilies such as self-dual polyhedra. These techniques are summarized in Chapter 2, along with operations on maps which are closed with respect to self-duality.

Another important aspect of any family of graphs is its intersection with other related families. Towards that end, Chapter 3 is dedicated to studying the intersection of the families of self-dual and self-complementary graphs, by first reviewing the Paley graphs, and then creating a new infinite class of self-complementary self-dual graphs.

Finally, Chapter 4 discusses some open problems and conjectures. First, however, we introduce some preliminaries associated with graph theory in general, and duality in particular.

## 1.1 Graphs and Surfaces

A *graph* $G$ consists of a finite set $V$, whose elements are *vertices*, and a list $E$ of 1- and 2-subsets of $V$, whose elements are *edges*. *Parallel edges* are 2-subset edges which appear twice or more in the list of all edges. A 1-subset is a *loop*.

For this work, an *orientable surface* is a homeomorph of the sphere with $k$ handles added to it. Such a surface is denoted $S_k$. The *genus* of the surface $S_k$ is the number $k$. In this work, we are primarily concerned with orientable surfaces. Thus, unless otherwise specified, we shall refer to an orientable surface as simply a surface. A *homeomorphism* is a bijection $h : A \to B$ such that both the function and the inverse function are continuous. If such a function exists, then $A$ and $B$ are *homeomorphic*.

An *embedding* of a graph on a surface is a drawing of the graph on the surface so that no two edges cross. This divides the surface into *faces* (or *regions*). If all the faces are homeomorphic to an open disk, the embedding is a *2-cell embedding*. A graph $G = (V, E)$ together with an embedding is a *map*, denoted $M = (V, E, F)$ where $F$ is the set of faces. Unless otherwise specified, all embeddings will be assumed to be 2-cell embeddings.

The *genus* of a graph $G$ is the smallest number $k$ such that $G$ is embeddable on an orientable surface of genus $k$.

**Theorem 1.1 (Euler)** *Let $M = (V, E, F)$ be a 2-cell embedding of a graph in the orientable surface of genus $k$. Then*

$$|V| - |E| + |F| = 2 - 2k.$$

## 1.2 Duality

There are many forms of duality in graph theory. The purpose of this section is to introduce some of them and their associated self-dual examples.

Given a map $M = (V, E, F)$ (sometimes referred to as the *primal map*), a *geometric dual* is the map $M^* = (F^*, E^*, V^*)$ (referred to as the *dual map*) created by placing a new vertex $f^*$ inside each face $f \in F$. For each edge $e$ of $M$, there is an edge $e^*$ of $M^*$ joining the vertices of $M^*$ in the faces of $M$ separated by $e$. We draw $e^*$ to cross $M$ exactly once, that crossing being with $e$. Note that $f_1$ and $f_2$ need not be distinct faces, implying that $M^*$ may have loops, and that $f_1$ and $f_2$ may be adjacent along several edges, implying that $M^*$ may have parallel edges. As well, if the graph $G$ has distinguishable embeddings, then $G$ may have more than one dual graph $G^*$. See Figure 1.1 due to Servatius and Servatius [23]. We will use a $*$ to indicate the geometric dual operation. A *geometric duality* is a bijection $g : E(G) \rightarrow E(G^*)$ such that $e \in E$ is the edge dual to $g(e) \in E(G^*)$. If $M$ is 2-cell, then $M$ is connected; so if $M$ is a 2-cell embedding, then $(M^*)^* \cong M$.

An *algebraic duality* is a bijection $g : E(G) \to E(\hat{G})$ such that $p$ is a circuit of $G$ if and only if $g(p)$ is a minimal edge-cut of $\hat{G}$. Given a graph $G = (V, E)$, an *algebraic dual* of $G$ is a graph $\hat{G}$ for which there exists an algebraic duality $g : E(G) \to E(\hat{G})$.

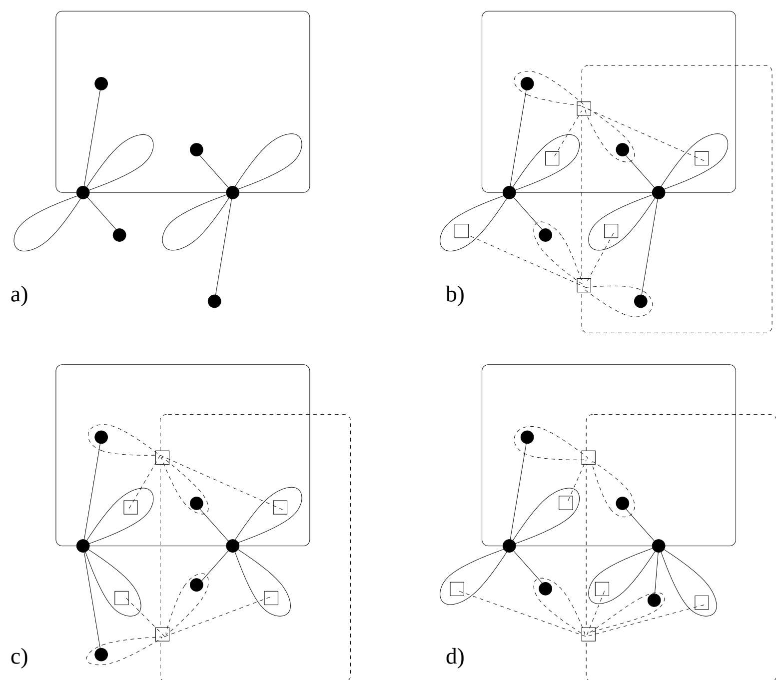


Figure 1.1: A graph and several of its embeddings. The geometric duals are shown in dotted lines. Embedding b) is map self-dual, c) is graphically self-dual and c) is algebraically self-dual.

We now define several forms of self-duality. Let $G = (V, E)$ be a graph and let $M = (V, E, F)$ be a fixed map of $G$, with geometric dual $M^* = (F^*, E^*, V^*)$.

**Definition 1.1** *1. $M$ is* map self-dual *if $M \cong M^*$.*

*2. $M$ is* graphically self-dual *if $(V, E) \cong (F^*, E^*)$.*

*3. $G$ is* algebraically self-dual *if $G \cong \hat{G}$, where $\hat{G}$ is some algebraic dual of $G$.*

**Remark.** In the literature, the term matroidal or abstract is sometimes used where we use algebraic.

Note that in Figure 1.1, embedding b) is map self-dual, embedding c) is graphically self-dual, and embedding d) is not self-dual even as a graph (it is however, algebraically self-dual).

For the purpose of this thesis, we will use the geometric duality operation and, unless specified, we will describe a graph as *self-dual* if it is graphically self-dual. Since the dual of a graph is always connected, we know that a self-dual graph is connected. Therefore, it suffices to consider connected graphs. The following are a few known results about self-dual graphs.

**Corollary 1.2** *Let $M = (V, E, F)$ be a 2-cell embedding on an orientable surface. If $M$ is self-dual, then $|E|$ is even.*

**Proof.** Since $M$ is self-dual, $|V| = |F|$. By Theorem 1.1, $|E| = 2-2k-|V|-|F| = 2(1-k-|V|)$. □

**Theorem 1.3** [27] *The complete graph $K_n$ has a self-dual embedding on an orientable surface if and only if $n \equiv 0$ or $1 \pmod 4$.*

**Theorem 1.4** [27] *For $w \geq 1$, there exists a self-dual embedding of some graph $G$ of order $n$ on $S_{n(w-1)+1}$ if and only if $n \geq 4w+1$.*

Note that a self-dual graph need not be self-dual on the surface of its genus. A single loop is planar; however it has a (non 2-cell) self-dual embedding on the torus.

Also note that there are infinitely many self-dual graphs. One such infinite family for the plane is the *wheels*. A wheel $W_n$ consists of a cycle of length $n$ and a single vertex adjacent to each vertex on the cycle by means of a single edge called a *spoke*. The complete graph on four vertices is also $W_3$. See Figure 1.2 for $W_6$.

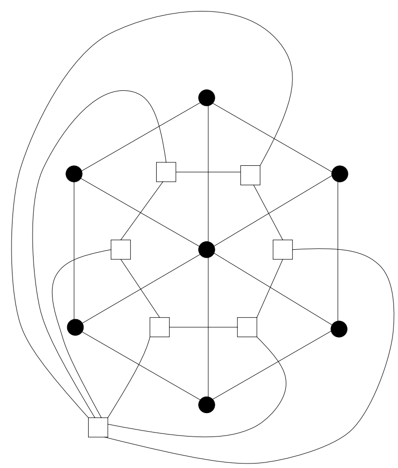

Figure 1.2: The 6-wheel and its dual.

## 1.3 Matroids

Matroids may be considered a natural generalization of graphs. Thus when discussing a family of graphs, we should also consider the matroidal implications.

**Definition 1.2** *Let $S$ be a finite set, the* ground set*, and let $\mathcal{I}$ be a set of subsets of $S$, the* independent sets*. Then $\mathcal{M} = (S, \mathcal{I})$ is a* matroid *if:*

1. *$\varnothing \in \mathcal{I}$;*

2. *if $J' \subseteq J \in \mathcal{I}$, then $J' \in \mathcal{I}$; and*

3. *for all $A \subseteq S$, all maximal independent subsets of $A$ have the same cardinality.*

A *isomorphism* between two matroids $\mathcal{M}_1 = (S_1, \mathcal{I}_1)$ and $\mathcal{M}_2 = (S_2, \mathcal{I}_2)$ is a bijection $\chi : S_1 \to S_2$ such that $I \in \mathcal{I}_1$ if and only if $\chi(I) \in \mathcal{I}_2$. If such a $\chi$ exists, then $\mathcal{M}_1$ and $\mathcal{M}_2$ are *isomorphic*, denoted $\mathcal{M}_1 \cong \mathcal{M}_2$.

Given a graph $G = (V, E)$, the *cycle matroid* $\mathcal{M}(G)$ of $G$ is the matroid with ground set $E$, and $F \subseteq E$ is independent if and only if $F$ is a forest (that is, $F$ contains no cycles). A matroid $\mathcal{M}$ is *graphic* if there exists a graph $G$ such that $\mathcal{M} \cong \mathcal{M}(G)$.

For a matroid $\mathcal{M} = (S, \mathcal{I})$, the *dual matroid* $\mathcal{M}^* = (S, \mathcal{I}^*)$ has ground set $S$ and $I \subseteq S$ is in $\mathcal{I}^*$ if there is a maximal independent set $B$ in $\mathcal{M}$ such that $I \subseteq S \setminus B$. A matroid $\mathcal{M}$ is *cographic* if $\mathcal{M}^*$ is graphic. It is easily shown that if $G$ is a connected planar graph, then $\mathcal{M}^*(G) = \mathcal{M}(G^*)$.

It is well known that $G$ is algebraically self-dual if and only if the cycle matroids of $G$ and $G^*$ are isomorphic (see [16] for more matroid properties).

## 1.4 Comparing Forms of Self-Duality

There are subtle differences between the forms of self-duality discussed in the previous section. We now wish to clarify these subtleties. The results of this section may be found in [23] and [10]. In both papers, the authors seek to classify and compare some of the forms through the connectivities of the graphs.

A theorem due to Whitney [30] says that if $G$ is 2-connected and $f$ is an algebraic duality, then $f$ is also a planar geometric duality. The distinction will be clarified further later on in this section.

Now we will proceed to compare map, graphically and algebraically self-duality. First we give a well known matroidal result.

**Theorem 1.5** [26] *Let $\mathcal{M}(G)$ be a graphic matroid. Then $\mathcal{M}(G)$ is cographic if and only if $G$ is planar.*

Thus, we limit our attention to planar graphs for the purposes of comparing the three self-dualities. Let $M = (V, E, F)$ be a planar map. It is clear for $M$ that

$$\text{map self-dual} \Rightarrow \text{graphic self-dual} \Rightarrow \text{algebraically self-dual.}$$

However, in general, these implications can not be reversed, as shown by Figure 1.1.

By Steinitz's Theorem [25], a planar 3-connected simple graph has a unique embedding on the sphere. Thus, for a 3-connected graph

$$\text{map self-duality} \Leftarrow \text{graphic self-duality} \Leftarrow \text{algebraically self-duality.}$$

Thus, it is clear for planar graphs of high-connectivity (3 and above) that these self-dualities are identical. Figure 1.1 shows that in the connectivity 1 case, they may be all distinct. Thus we must still consider the 2-connected case, as well as further explore the 1-connected case.

**Theorem 1.6** [23] *There exists a 2-connected map $(V, E, F)$ which is graphically self-dual so that, if $M' = (V', E', F')$ is any map such that $\mathcal{M}(V, E) \cong \mathcal{M}(V', E')$, then $M'$ is not map self-dual.*

As the next theorem shows, it is not sufficient to have well connected pieces either.

**Theorem 1.7** [23] *There is a graphically self-dual map* $(V, E, F)$ *with* $(V, E)$ *1-connected and having only 3-connected blocks so that, if* $M' = (V', E', F')$ *is any map such that* $\mathcal{M}(V, E) \cong \mathcal{M}(V', E')$, *then* $M'$ *is not map self-dual.*

In the case of 2-connected graphs, the next theorem asserts that algebraic self-duality is sufficient up to embedding to imply graphic self-duality.

**Theorem 1.8** [23] *If* $G = (V, E)$ *is a planar 2-connected graph such that* $\mathcal{M}(G) \cong \mathcal{M}(G)^*$, *then* $G$ *has an embedding* $(V, E, F)$ *such that* $(V, E) \cong (F^*, E^*)$.

However, as shown in the next theorem, not all self-dual graphic matroids come from self-dual graphs.

**Theorem 1.9** [23] *There exists a self-dual graphic matroid* $\mathcal{M}$ *such that, for any graph* $G = (V, E)$ *with* $\mathcal{M}(G) = \mathcal{M}$, *and embedding* $(V, E, F)$ *of* $G$, $(V, E) \ncong (F^*, E^*)$.

For example, consider the 4-wheel. The 4-cycle bounds a face. In any self-duality, the centre vertex is mapped to the face bounded by the 4-cycle. Now consider the graph formed by identifying the centre vertices of two 4-wheels. There can be no graphic self-duality of this new graph as both 4-cycles bound faces, and the centre vertex cannot be mapped to two different faces. However, algebraic self-duality is not affected by vertex identification, and thus the matroid of the new graph is still algebraically self-dual.

More formally, let $\mathcal{M}_1$ and $\mathcal{M}_2$ be the cycle matroids of two distinct 3-connected self-dual maps $M_1$ and $M_2$. Suppose that $M_1$ and $M_2$ have vertices $v_1 \in V(M_1), v_2 \in V(M_2)$ and faces $f_1 \in F(M_1), f_2 \in F(M_2)$ such that every self-duality of $M_1$ maps $v_1$ to $f_1$ and every self-duality of $M_2$ maps $v_2$ to $f_2$. The matroid $\mathcal{M}_1 \oplus_{v_1,v_2} \mathcal{M}_2$ (that is, the cycle matroid of the graph created by identifying the vertices $v_1$ and $v_2$) is algebraically self-dual since vertex identification does not affect algebraic self-duality, but its only map realizations are as the 1-vertex union of $M_1$ and $M_2$, which cannot be self-dual since the cut vertex cannot simultaneously be sent to both faces $f_1$ and $f_2$.

The next two theorems generalize Whitney's theorem and formalize for all connectivities when an algebraic duality can be obtained as a geometric duality. It is divided into two cases, connected and disconnected graphs. Let $\delta(v)$ be the set of edges incident with $v$.

**Theorem 1.10** [10] *Let $G$ and $\hat{G}$ be connected graphs and let $g : E(G) \to E(\hat{G})$ be an algebraic duality. Then $g$ is attained as a geometric duality if and only if, for each cutpoint $v'$ of $\hat{G}$, the subgraph of $G$ induced by $g^{-1}(\delta(v'))$ is connected.*

**Theorem 1.11** [10] *Let $G$ have no isolated vertices and let $g : E(G) \to E(\hat{G})$ be an algebraic duality. For each component $K$ of $G$, let $\hat{K}$ denote the subgraph of $\hat{G}$ induced by $g(E(K))$. Then $g$ is a geometric duality if and only if:* (1) *$\hat{G}$ is connected and* (2) *for each component $K$ of $G$ and each cutpoint $v'$ of $\hat{K}$, $g^{-1}(\delta(v')) \cap E(K)$ induces a connected subgraph of $K$.*

## 1.5 Rank and the Self-Duality Permutation

We often desire a magnitude or measure of complexity within a given family of graphs. We define the rank of a self-dual map as in [12].

**Definition 1.3** *A* self-dual permutation *is a permutation of the vertices and faces of the map defined by composing an isomorphism onto the dual with the dual correspondence. The* rank *of a self-dual map is the smallest order of any of its self-dual permutations.*

The square of a self-dual permutation is an automorphism, and a mistake made has been to assume that the square is actually the identity automorphism. Grünbaum and Shephard [12] defined the rank as above and asked if the rank of a self-dual polyhedron (see Section 2.3) could be greater than 2. Jendroľ [14] answered this question in the affirmative. The example in Figure 1.3 was provided by Servatius and Servatius [21] where all self-dualities correspond to a rotation of $\pi/2$ or $3\pi/2$. Later, McCanna [15] showed that in fact, a polyhedron of rank $2^n$ existed for all $n$ and that these were the only possible ranks. A self-duality of order 2 is *involutory*. Self-dualities of order greater than 2 are *non-involutory*. A graph which admits an order 2 self-duality is *involutory*. Otherwise, the self-dual graph is *non-involutory*.

The collection of all self-duality permutations generates a group $Dual(M)$ in which the map automorphism group $Aut(M)$ of $(V, E, F)$ is contained as a subgroup of index 2 [23].

Normally, a self-dual permutation is only defined for maps. We can extend our definition to include graphs. Given an algebraically self-dual graph $G$, we define

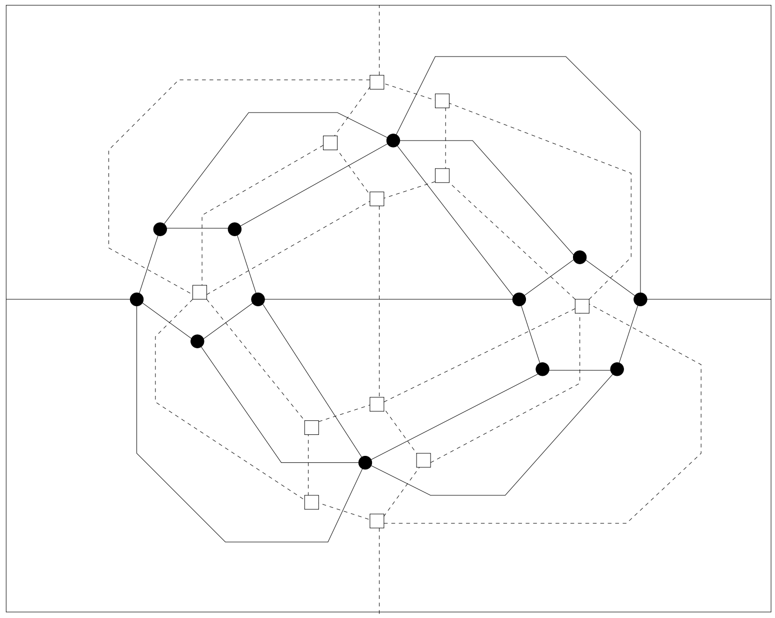

Figure 1.3: A polyhedron with all self-dualities of order 4.

a self-dual permutation of the edges of $G$ such that the permutation sends cycles to cocycles (where cocycles are a set of edges which form a cycle in the dual). More generally, given a graphic matroid $\mathcal{M}$, the set $Dual(\mathcal{M}(G))$ of all bijections $f : \mathcal{M} \to \mathcal{M}$ taking cycles to cocycles and vice-versa form a group, called the *self-duality group of* $\mathcal{M}$ which contains the automorphism group of the matroid as a subgroup of order 2.

In [23], Servatius and Servatius considered the normal subgroups $Dual(M) \rhd Aut(M)$ and $Dual(\mathcal{M}) \rhd Aut(\mathcal{M})$, enumerated all such pairings for graphically self-dual maps on the sphere and used them to classify all graphically self-dual maps on the sphere.

## 1.6 Cayley and Voltage Graphs

One of the main difficulties in studying graphic self-duality is describing in an economical fashion the map and dual map. One means of overcoming these difficulties lies within Cayley graphs and a generalization of Cayley graphs known as voltage graphs. Their algebraic nature not only allows for an easy description of embeddings, but also allows us to employ algebraic techniques.

**Definition 1.4** *Given a group $X$, called the* ground set*, and a subset $\Delta$ of $X$, the* Cayley graph $C(X, \Delta)$ *is the directed graph having $X$ as its vertex set, and two vertices $x, y$ are adjacent by a directed edge from $x$ to $y$ if and only if there exists an element $a \in \Delta$ such that $x \cdot a = y$. A* Cayley map *is a Cayley graph embedded on a surface.*

If a Cayley graph contains both the directed edges $(x, y)$ and $(y, x)$, then we replace the two directed edges with an undirected edge $\{x, y\}$. It is easy to show that the Cayley graph $C(X, \Delta)$ is connected if and only if $\Delta$ generates the group $X$, and that $C(X, \Delta)$ is undirected if and only if for all elements $a \in \Delta$, $a^{-1} \in \Delta$.

It has long been known that $K_{4n}$ and $K_{4n+1}$ admit Cayley maps that are map or graphically self-dual. In [4], Archdeacon showed that the $4n$-dimensional cube, $Q_{4n}$, is also self-dual. These are restated in [2], where it is also shown that $Q_{4n+2}$ has a Cayley map that is map self-dual.

In [24], Stahl showed the following.

**Theorem 1.12** [24] *A finitely generated abelian group is self-dual (for some Cayley graph of that group) on an orientable surface if and only if its order is neither 2* *nor 3. A finitely generated abelian group is self-dual on a nonorientable surface if and only if it has order at least six.*

Later, Archdeacon [4] proved the result in general for all finite groups.

**Definition 1.5** *Let $G = (V, E)$ be a multigraph for which every edge has been assigned a direction, let $X$ be a group and let $h : E \to X$. Then $(G, X, h)$ is a* voltage graph*. The value $h$ assigns to an edge is its* voltage assignment*. The graph $G$ is the* base graph*. The* (right) lift (or (right) derived graph) *of a voltage graph $(G, X, h)$ is the graph with vertex set $V \times X$, where the vertex $(u, x)$ is adjacent to the vertex $(v, y)$ by a directed edge in the lift if and only if $u \sim v$ by a directed edge $e$ from $u$ to $v$ in $G$ and $x \cdot h(e) = y$. The* left lift *is defined identically except that $h(e) \cdot x = y$.*

Note $u$ and $v$ need not be distinct, $x$ and $y$ need not be distinct (if $h(u, v) = id$, where $id$ is the group identity), and that, if $X$ is commutative, the left and right lifts are identical.

A *voltage map* is a voltage graph embedded on a surface. A voltage map may induce an embedding of the lift. Let $v \in V(G)$ be on a face with boundary face walk of length $n$ with edge sequence $(e_1, e_2, \ldots, e_n)$. Suppose that $h(e_1)h(e_2)\cdots h(e_n) = a$ has order $k$. Then all vertices of the form $(v, x)$ from the lift are on a face of length $kn$, with edge sequence $((e_1, x), (e_2, xh(e_1)), \ldots, (e_n, xa^{k-1}h(e_1)\cdots h(e_{n-1})))$.

Voltage graphs may be thought of as a generalization of Cayley graphs in the sense that all Cayley graphs may be represented as a voltage graph whose base graph is a single vertex with loops.

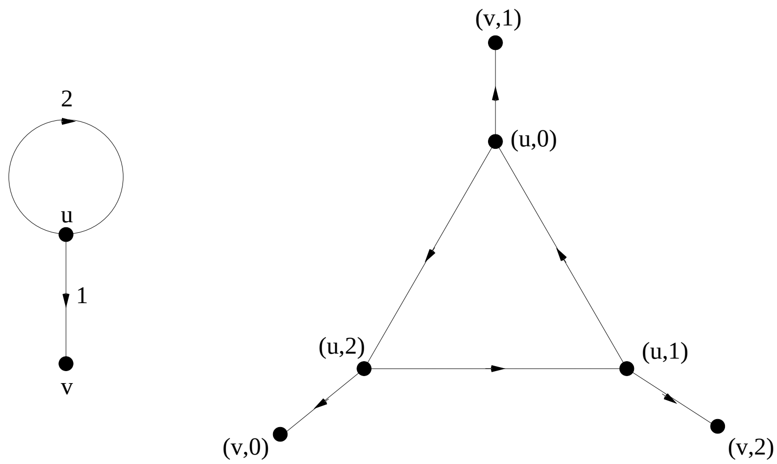


Figure 1.4: An example of a voltage graph and its lift. The group is $\mathbb{Z}_3$.

For more background on Cayley graphs and voltage graphs, see [9] and [11] respectively.

# Chapter 2

# Constructions

A central problem in studying self-dual graphs (or any family of graphs) is finding means of constructing a large number examples and creating subfamilies. In this chapter we describe some constructions of self-dual maps and graphs.

## 2.1 Adhesion and Explosion

The notions of adhesion and explosion were first given in [20] and were only defined for the plane. Thus for this section, when we say a map we mean a graph embedded on the plane. Adhesion is the most straightforward method of creating a self-dual graph, consisting of gluing together a graph and its dual.

**Definition 2.1** *Let $M = (V, E, F)$ be a map and let $v \in V$ be a vertex incident with the face $f \in F$. Let $v^*$ and $f^*$ be the face and vertex of $M^*$ corresponding to $v$ and $f$ respectively. The* adhesion $M \overset{v,f}{\circ} M^*$ *of $M$ and $M^*$ is the planar graph* *created by embedding both $M$ and $M^*$ on the plane such that $v$ and $f^*$ are in the same face and then identifying the vertices $v$ and $f^*$.*

**Theorem 2.1** [20] *The graph created from the adhesion of a planar map $M$ and its dual is a self-dual graph.*

**Proof.** Let $M$ be a map with vertex $v$ and face $f$. Let $v^*$ and $f^*$ be the corresponding face and vertex in $M^*$. Embed the map $M^*$ such that $v^*$ is the outer face inside the face $f$ of the map $M$. Identify the vertices $v$ and $f^*$. Label the vertices and faces from $M \overset{v,f}{\circ} M^*$ with their inherited labels from $M$ and $M^*$ respectively and label the new face $(v^*, f)$ and the new vertex $(v^*, f)^*$. Given a vertex of $(M \overset{v,f}{\circ} M^*)^*$ corresponding to a face $r$ of $M$, label the vertex of $(M \overset{v,f}{\circ} M^*)^*$ with $(r^*)'$. Similarly, given a vertex of $(M \overset{v,f}{\circ} M^*)^*$ corresponding to the face $s^*$ of $M^*$, label the vertex with $s'$. The desired isomorphism between $(M \overset{v,f}{\circ} M^*)^*$ and $M \overset{v,f}{\circ} M^*$ is obtained by mapping vertices $\alpha' \in (M \overset{v,f}{\circ} M^*)^*$ to vertices $\alpha \in M \overset{v,f}{\circ} M^*$. $\square$

Let $M$ be a planar map with a face $f$ bounded by a cycle. Let $f^*$ be the vertex of $M^*$ inside $f$. Label the vertices of $f$ consecutively as $1, 2, \ldots, n$ and let $e_i$ be the edge joining $i$ and $i+1 \pmod n$. Label the edge of $M^*$ dual to $e_i$ as $e_i^*$. Observe that $e_i^*$ is incident in $M^*$ with $f^*$. Let $t$ be a positive integer. The map $M \overset{f}{\square} M^*$ is obtained from $M$ by placing a copy of $M^*$ in $f$, with $f^*$ on the boundary of the face of $M^*$ containing $M$, then erasing the single point $f^*$ and joining, for $i = 1, 2, \ldots, n$, the edge $e_i^*$ of $M^*$ to the vertex $t - i \pmod n$ of $M$."

**Definition 2.2** *The map $M \overset{f}{\square} M^*$ is the* explosion *of $M$ and $M^*$.*

**Theorem 2.2** [20] *If it is defined, the graph created from the explosion of a map $M$ in its dual is a self-dual graph.*

**Proof.** Choose a map $M$ with a face $f$ whose boundary is a cycle of length $n$. Embed the dual map $M^*$ so that $f^*$ is on the outside face. Then embed $M$ so that $M^*$ is contained within the face $f$ and perform explosion as defined above so that $i + j \equiv t \pmod{n}$. This divides the face $f$ into $n$ regions. These faces inherit their labels from $M^*$. All other faces remain unchanged and inherit their labels from $M$ or $M^*$ respectively. Given a vertex from $(M \overset{f}{\square} M^*)^*$, if it is in a face labeled $r$, label the vertex $(r^*)'$. Otherwise, it is in a face labeled $s^*$, in which case we label the vertex $s'$.

We claim mapping vertices $\beta' \in V((M \overset{f}{\square} M^*)^*)$ to $\beta \in V(M \overset{f}{\square} M^*)$ is an isomorphism between the explosion and its dual. To verify this, we must check that adjacency is preserved. Obviously, edges which didn't have both endpoints involved in the explosion are preserved by our labeling. Thus, it remains to be shown that the adjacencies performed by the edges $e_i$ and $e_i^*$ are preserved. Consider the face of $M \overset{f}{\square} M^*$ containing the edges $e_{i-1}^*$ and $e_i^*$ of $M^*$ and the edge $e_j$ of $M$. This face corresponds to vertex $i$ of $M$ after the dual operation, and an edge $e_j^*$ which crosses the edge $e_j$. Therefore, our rule $i + j \equiv t \pmod{n}$ is satisfied and the mapping is indeed an isomorphism. $\square$

Note that if the boundary face is not a cycle, the explosion of a graph and its dual is not well defined. The wheels may be derived from explosion by letting $M$ be an $n$-cycle and choosing $f$ to be the outside face.

## 2.2 Tilings

One subfamily of self-dual graphs that has been known and studied for some time consists of the self-dual tilings. Here we give a brief review of the usual plane tilings and then consider tilings of the sphere and other surfaces.

A *closed topological disk* is any plane set which is the image of a closed circular disk under a homeomorphism. An *open topological disk* is defined analogously.

As in [13], a *plane tiling* $\mathcal{T}$ is a countable family of closed sets $\mathcal{T} = \{T_1, T_2, \ldots\}$, known as *tiles*, whose union is the whole plane, and the interiors of the tiles are pairwise disjoint. Tiles intersect at points or arcs. The points will be called *vertices* and the arcs will be called *edges*, coinciding with the usual sense of vertices and edges. If three or more tiles all mutually intersect at a common point, that point shall be a vertex.

A *sphere tiling* is a family $\mathcal{T}^f = \{T_1, T_2, \ldots T_n\}$ of finitely many closed sets, known as *tiles*, whose union is the whole sphere, and the interiors of the tiles are pairwise disjoint. A *plane tiling with m-gons* is a plane tiling $\mathcal{T}_m$ such that $\mathcal{T}_m$ is a set of polygons $T$ with $m$ sides. A *sphere tiling with m-gons* $\mathcal{T}_m^f$ is defined in the obvious way. A plane tiling is *normal* if each tile is a closed topological disk, the intersection of any two tiles is either empty, a single point or a single arc, and if the diameters of the tiles are uniformly bounded (that is, there exist positive numbers $u$ and $U$ such that every tile contains a circular disk of radius $u$ and is contained in a circular disk of radius $U$). A sphere tiling is *normal* if each tile is a closed topological disk, and the intersection of any two tiles is either empty, a single point or a single arc. The *underlying map* $M = (V, E, F)$ of a tiling is the map, if it exists, whose faces are the interiors of the tiles, and whose vertices and edges correspond to tiles intersecting at points and lines respectively. Note the underlying map may not be well defined if, for example, two tiles intersect at a closed curve. An $S_k$*-tiling* is defined analogously for the surface $S_k$.

Infinite self-dual tilings are explored in [7]. Of note, Ashley et al [7] show that the rank of any self-dual normal plane tiling is either 2, 4 or $\infty$. An example of a self-dual plane tiling is the infinite grid.

Note that the embedding of the underlying graph of a normal tiling must be a 2-cell embedding since all the tiles are closed topological disks, which implies that all the faces are open topological disks.

The following research was motivated by the observation that the tetrahedron is a triangulation of the plane. That is, $K_4$ is a sphere tiling with 3-gons. One might wonder if other values of $m$ might be feasible for the sphere. We show there are none. First we require a lemma.

**Lemma 2.3** *Let $\mathcal{T}_m^f$ be a self-dual normal $S_k$-tiling with $m$-gons. Let $M = (V, E, F)$ be the underlying map of the tiling, with $n = |V|$. Then*

$$(4 - m)\, n = 4 - 4k.$$

**Proof.** Since $M$ is self-dual and all the faces have degree $m$, we know that every vertex has degree $m$. For any graph, we know that

$$\sum_{v \in V} deg(v) \;=\; 2|E|.$$

Since every vertex has degree $m$, $mn = 2|E|$, or $(m/2)n = |E|$. Since $M$ is self-dual, we know that $|V| = |F|$. Therefore, by Euler's Formula Theorem 1.1 and the fact that $n = |V|$, we have $2 - 2k = |V| - |E| + |F| = 2n - (m/2)n$, or $(4 - m)n = 4 - 4k$. $\square$

**Theorem 2.4** *If $\mathcal{T}_m^f$ is a self-dual normal sphere tiling with $m$-gons, then the underlying map of $\mathcal{T}_m^f$ is homeomorphic to $K_4$.*

**Proof.** Let $\mathcal{T}_m^f$ be a self-dual normal sphere tiling with $m$-gons. Let $M = (V, E, F)$ be the underlying map. Let $n = |V|$. By Lemma 2.3, $4 - m = 2/n > 0$, so $m < 4$.

Since there are no polygons with 0 or 1 sides, the only possibilities for $m$ are 2 and 3. If $m = 3$, Lemma 2.3 tells us that $n = 4$. Since vertices of degree 1 and 2 are not obtainable through normal tilings, dually this means that any underlying graph must have no loops or parallel edges. Since every vertex has degree 3 and the graph must be simple, the graph must be $K_4$. If $m = 2$, then $n = 2$. The only two self-dual graphs on the plane with two vertices are a loop and a single edge, and 2 parallel lines. The first does not have 2-gons as faces, and the second cannot be obtained as a tiling, since the two closed sets would have to be homeomorphic to two closed disks, whose intersection is just one closed curve, with no vertices. $\square$

**Theorem 2.5** *Let $\mathcal{T}_m^f$ be a self-dual normal $S_k$-tiling with $m$-gons with underlying map $M = (V, E, F)$. If $k \neq 1$, then $n = |V|$ is fixed and depends only on $m$ and $k$.*

*If $k = 1$, then $m = 4$ and $|V|$ can be arbitrarily large.*

**Proof.** From Lemma 2.3, we know that $(4 - m)\,n = 4 - 4k$. If $4 - m \neq 0$, then $n = \frac{4-4k}{4-m}$. Otherwise, $m = 4$, so $4 - 4k = 0$, and therefore $k = 1$. If $k = 1$, then $m = 4$. The graph can be arbitrarily large since the toroidal $i \times j$ grids are self-dual quadrangulations of the torus (self-dual normal $S_1$-tiling with 4-gons). The toroidal $3 \times 3$ grid is shown with a self-dual embedding on the torus in Figure 2.1. □

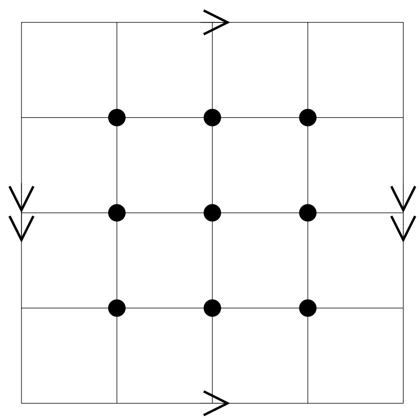

Figure 2.1: The toroidal $3 \times 3$ grid.

## 2.3 Polyhedra

One of the earliest concepts associated with polyhedra is that of duality. As mentioned earlier, the Platonic solids are closed under duality, with the tetrahedron being self-dual. As a result of this long association and their general importance in combinatorics, much of the work in recent years has focused on self-dual polyhedra. This section will summarize some of the recent work, which includes complete characterizations of self-dual spherical and projective polyhedra.

**Definition 2.3** *A* polyhedron *is a map in which every vertex has degree at least* 3*, every face is homeomorphic to a closed disk in the surface, and any two faces intersect either not at all, in a vertex, or in an edge.*

Note that the graph of any polyhedron is 3-connected (c.f., Barnette [8]).

The central problem is to construct and classify all self-dual polyhedra. The recent work has used covering spaces (see [11]), for which we require some background.

**Definition 2.4** *Let $M$ be a map. The* radial graph $R(M)$ *of $M$ has as vertices the vertices and faces of $M$ and every vertex-face incidence of $M$ gives an edge of $R(M)$ joining the incident elements of $M$.*

Any radial graph's vertices may be divided into vertices and faces of the original graph, giving an obvious bipartition. The graph $R(M)$ may also be embedded in the same surface as $M$. Also, for any map $M$, $R(M) \cong R(M^*)$ as vertices in the primal become faces in the dual and vice-versa. The faces of any radial graph are all quadrilaterals which correspond to edges in the original graph, as every edge of $M$ is incident with two vertices and two faces. Conversely, it can be shown that any bipartite quadrangulation is a radial graph.

Given any map isomorphism from $M$ to $M^*$, we may induce a part-reversing (that is, vertices of $M$ are sent to faces of $M$ and vice-versa) automorphism of the radial graph $R(M)$. Similarly, given any part-reversing automorphism of $R(M)$, we can induce a self-duality of $M$. Therefore, we may restate the original problem as:

*Construct and classify all bipartite quadrangulations with a part-reversing map automorphism.*

**Spherical Polyhedra**

For the sphere, the problem was solved by Archdeacon and Richter [6], who gave six constructions, three for involutory polyhedra and three for non-involutory polyhedra, and showed that any self-dual spherical polyhedron comes from one of their six constructions. The authors also have examples for each construction yielding polyhedra that cannot arise from any of the other constructions. As shown by their constructions and independently by McCanna [15], there exists a self-dual spherical polyhedron of rank $2^n$ for every $n$ and these are the only possible ranks.

Servatius and Servatius [21] used a variation on the radial graph as follows to derive Theorem 2.6 below. Let $M$ be a self-dual map. Create a new map $M_2$ by superimposing the dual map with the original map as prescribed by the geometric dual operation. The vertices of $M_2$ are the vertices of $M$, the vertices of $M^*$ and the intersection points of an edge $e$ and its dual edge $e^*$, the set of which is denoted $E \cap E^*$. The edges consist of the subdivided edges of $M$ and $M^*$.

Again, the faces of $M_2$ are quadrilaterals. We colour the edges of $M_2$ by two colours, say red and blue, colouring all edges that are incident with a vertex from $M$ red and all edges that are incident with a vertex of $M^*$ blue. As with the radial graph, every map isomorphism $\varphi$ from $M$ to $M^*$ induces a unique colour-reversing map automorphism $\varphi_2$ of $M_2$ and conversely.

**Theorem 2.6** [21] *Let $M$ be a self-dual map on the sphere with self-duality $\varphi$. Let $\varphi_2 : M_2 \to M_2$ be the corresponding colour-reversing automorphism of $M_2$. Then*

*$\varphi_2$ is realized by one of the following:*

1. *a rotation of order 4, the poles being two elements in $E \cap E^*$;*
2. *a rotation of order 2, the poles lying in the interiors of two quadrilaterals;*
3. *the antipodal map;*
4. *a simple reflection with equator intersecting the graph of $M_2$ only at vertices in $E \cap E^*$;*
5. *a rotatory reflection of order 4 with poles at two vertices in $E \cap E^*$;*
6. *a rotatory reflection of order $2k > 2$ which has one pole in $V$ and one pole in $F^*$ and for which $\varphi_2^k$ is the antipodal map, a rotation, or a reflection.*

Servatius and Servatius then noted that these six transformations correspond exactly to the six constructions of Archdeacon and Richter, and proceeded to recursively construct all self-dual maps on the sphere together with their self-dualities.

**Projective Polyhedra**

Another natural surface to consider is the projective plane. In [5], Archdeacon and Negami found two constructions, and showed that all possible projective polyhedra arises from one of these two construction. Due to the two constructions, they proved that the only possible ranks for self-dual projective polyhedra are 2 and 4.

**General Polyhedra**

The natural question is, can we generalize these results for all surfaces? In his survey paper on self-dual polyhedra [3], Archdeacon does give one general construction for an arbitrary surface, along with various other general construction techniques, some of which will be discussed in the next section. The problem in general is still open.

## 2.4 Operations on self-dual maps

An obvious question for a given family of graphs is, what operations are closed within the family. Thus, given a self-dual graph, what operations may we perform on that graph and still obtain a self-dual graph. In this section we discuss several possible operations we can perform on a given self-dual graph to obtain larger or smaller self-dual graphs.

Let $G$ be an involutory self-dual graph with self-dual map $M$ and with map isomorphism $\varphi : M \to M^*$ of order 2.

**Definition 2.5** *Let $e$ be an edge of $M$. The* mate *of $e$, denoted $\overline{e}$, is the edge of $M$ dual to $\varphi(e)$.*

Thus, if $e$ is incident with vertices $u$ and $v$, then $\overline{e}$ is incident with faces $\varphi(u)$ and $\varphi(v)$. By our assumption, $\varphi$ has order 2. Thus, $\varphi(\overline{e}) = e$ and if $\overline{e}$ has ends $a$ and $b$, then $\{\varphi(a), \varphi(b)\} = \{u, v\}$. If an edge is mated with itself, the edge is called *self-mated.*

We use the notation $M/e$ and $M\backslash e$ to denote the maps obtained by contracting the edge $e$ and deleting the edge $e$ respectively.

**Definition 2.6** *Let $M$ be an involutory self-dual map with an edge $e$ such that $e \neq \overline{e}$. The* squash *of $M$ is the map $M' = M \setminus \overline{e}/e$. If $M'$ is the squash of $M$, then $M$ is obtained from $M'$ by* cleaving*.*

That is, cleaving $M$ from $M'$ is accomplished by adding an edge across a face (the inverse of deletion) and we split the corresponding mated vertex (the inverse of contraction).

Suppose $v$ is a degree 2 vertex in a self-dual map $M$ with self-duality $\phi : M \to M^*$. Suppose further that $v$ is incident with $\phi(v)$. Then $\phi(v)$ is a digon; let $u$ be the other vertex incident with $\phi(v)$. Then $\phi(u)$ is the other face incident with $v$.

**Definition 2.7** *The* trim *$M'$ is created by deleting $v$, which simultaneously deletes the face $\varphi(v)$. If a map $M'$ is the trim of the map $M$, then $M$ is obtained from $M'$ by* dangling*.*

B. Servatius first defined cleaving and dangling in [19]. In [3], Archdeacon states that the set of involutory self-duality maps are exactly those formed from a map with one vertex by some sequence of cleavings and/or danglings.

**Duality Reduction**

Servatius and Servatius generalized squashing in [21].

**Lemma 2.7** [21] *Let $G$ be a self-dual graph with self-dual map $M$. Suppose the self-duality $\delta$ of $M$ has order $2k$. Define the edge permutation $\Delta$ by $\Delta(e) = \delta(e)^*$. Chose an edge $e$ such that $e$ has orbit size $2k$ in the permutation. Then the sets $\{\Delta^{2i}(e) : i = 1, 2, \ldots, k\}$ and $\{\Delta^{2i-1}(e) : i = 1, 2, \ldots, k\}$ do not both separate $G$.*

Let $G$ be a self-dual graph with self-duality $\delta$ of order $2k$. Suppose the edge $e$ has orbit size $2k$.

**Definition 2.8** *Let $\nabla$ be one of $\{\Delta^{2i}(e) : i = 1, 2, \ldots, k\}$ and $\{\Delta^{2i-1}(e) : i = 1, 2, \ldots, k\}$ such that $G \backslash \nabla$ is connected and let $\nabla'$ be the other. The* duality reduction of $G$ along $\nabla$ *is $G \backslash \nabla / \nabla'$.*

Servatius and Servatius proved in [21] that the map isomorphism $\delta$ induces a map isomorphism for the new graph and its dual. They also proved the following categorization theorem.

**Theorem 2.8** [21] *Any self-dual planar map $M$ with graphic self-duality of order $2k$ can be reduced through a series of duality reductions to a thorned rose, a wheel, or a dipole tree. (See Figure 2.2).*

### The Addition Construction

This construction is due to Archdeacon [3].

**Definition 2.9** *Given two graphs $G_1$ and $G_2$ with vertices $v_1, \ldots, v_k$ and $u_1, \ldots, u_k$ respectively, the* amalgamation $G = G_1 \cup_k G_2$ *along these vertices is the graph formed by identifying the vertices $v_i$ and $u_i$ for all $i$.*

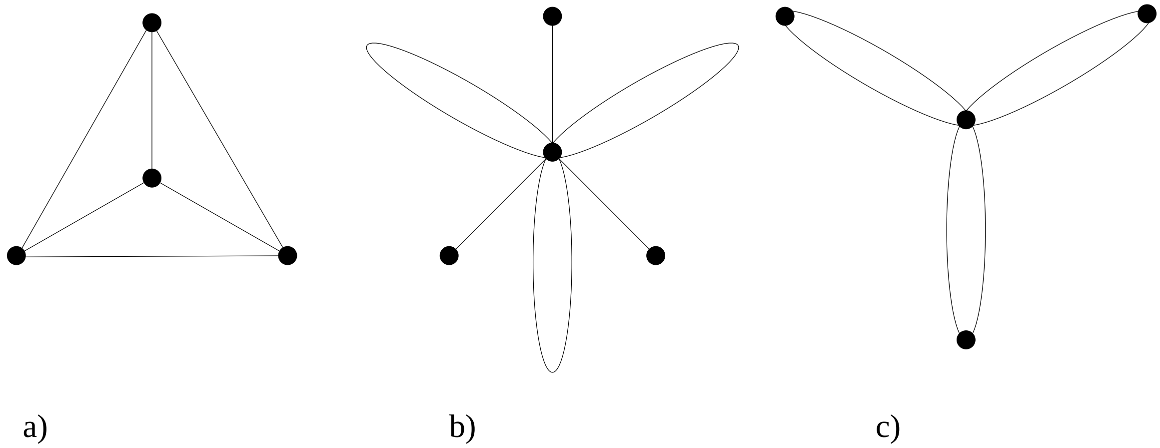


Figure 2.2: (a) The 3-sided wheel, (b) the 3-thorned rose and (c) the 3-leaved dipole tree.

**Definition 2.10** *Given a self-dual map $M$ with map isomorphism $\varphi$, a vertex is* reflexive *if it is incident with its dual face.*

**Theorem 2.9** [3] *Let $G_1$ be the graph of a graphically self-dual map which has reflexive vertices $v_1, \ldots, v_k$. Similarly, let $G_2$ be the graph of a graphically self-dual map which has reflexive vertices $u_1, \ldots, u_k$. Then the amalgamation $G$ along these vertices has a graphically self-dual map. Furthermore, in the self-dual map of $G$, the vertices of identification are all reflexive.*

Archdeacon then uses this construction to give self-dual embeddings of $K_{4,4n}$. Later, Archdeacon used a similar construction to arrive at the following result, where the notation $K_{m(n)}$ is the complete multipartite graph with $m$ sets of $n$ independent vertices.

**Theorem 2.10** [4] *The complete multipartite graph $K_{m(n)}$, $m \geq 2$ has both orientable and nonorientable self-dual embeddings, except:*

1. *in the orientable case, when $n$ is odd and $m$ is $2$ or $3$ modulo $4$, or when the graph is $K_{6,6}$, or possibly when the graph is $K_{4(2)}$ or $K_{6(2)}$;*

2. *in the nonorientable case, when the graph is $K_n$, $n \leq 5$, or possibly when the graph is $K_{6(3)}$.*

Thus, there remain only three open cases for the complete multipartite graphs.

# Chapter 3

# Self-Complementary Self-Dual Graphs

A natural question that arises when considering classes of graphs is the intersection of similar families. To this end, we shall consider when graphs can be both self-complementary and self-dual. White ([28] or [29]) first considers this problem within the context of strongly symmetric maps and gives an infinite class of graphs where $|V(G)| \equiv 1 \pmod 8$. We will then construct a new infinite class where $|V(G)| \equiv 0 \pmod 8$.

## 3.1 Self-Complementary Graphs

**Definition 3.1** *Let $G = (V, E)$ be a simple graph. The* complement $\overline{G}$ *of $G$ is the graph with vertex set $V$, two vertices being adjacent if and only if they are not adjacent in $G$. A graph $G$ is* self-complementary *if $G \cong \overline{G}$.*

Note the complement of a graph is only defined for graphs with no loops or parallel edges. The following follows directly from the definition of the complement of $G$.

**Theorem 3.1** *Let $G = (V, E)$ be a simple graph. Then $|E(G)| + |E(\overline{G})| = \binom{n}{2}$.*

Thus, for a self-complementary graph with $n$ vertices and $|E|$ edges, $|E| = n(n-1)/4$.

**Theorem 3.2** [18] *There exists a self-complementary graph of order $n$ if and only if $n \equiv 0, 1 \pmod 4$.*

**Proof.** If $G$ is a self-complementary graph of order $n$, then $|E| = n(n-1)/4$. Since $|E|$ is an integer, $n \equiv 0, 1 \pmod 4$.

Conversely, let $n \equiv 0, 1 \pmod 4$. If $n \equiv 0 \pmod 4$, we may construct a self-complementary graph on $n$ vertices by taking two copies of any graph on $n/4$ vertices $H$, labelled $H_1$ and $H_2$, and two copies of its complement $\overline{H}$, labelled $\overline{H_1}$ and $\overline{H_2}$. The desired graph is obtained by joining every vertex in $\overline{H_1}$ to every vertex in $H_1$, joining every vertex in $\overline{H_1}$ to every vertex in $\overline{H_2}$, and also joining every vertex in $\overline{H_2}$ to every vertex in $H_2$. When $n = 4$, this graph is the path on 4 vertices. For $n \equiv 1 \pmod 4$, create a self-complementary graph on $n - 1$ vertices as above and add a new vertex adjacent to all the vertices of the two copies of $\overline{H}$. □

## 3.2 Paley Graphs

In [28], White first considered the problem of when a graph could be both self-complementary and self-dual. He considers this question in the context of graphs he labels ‘strongly symmetric’. We first need to define a symmetric graph. We proceed as in [28].

Let $\mathcal{G}$ be a group and let $X$ be a set. An *action of* $\mathcal{G}$ *on* $X$ is a homomorphism $\phi : \mathcal{G} \to Sym(X)$, where $Sym(X)$ is the set of all permutations of $X$. For each $g \in \mathcal{G}$ and $x \in X$, define $g(x) = (\phi(g))(x)$. Then:

1. for all $x \in X$, $id(x) = x$; and

2. for all $g, h \in \mathcal{G}, x \in X, g(h(x)) = (gh)(x)$.

The action is *transitive* if, for every $x, y \in X$, there is a $g \in \mathcal{G}$ such that $g(x) = y$. The action is *regular* if, for every $x, y \in X$, there is a unique $g \in \mathcal{G}$ such that $g(x) = y$.

If $\mathcal{G}$ acts on $X$, then there is an induced action of $\mathcal{G}$ on the subsets of $X$, defined by $g(Y) = \{g(y) : y \in Y\}$. Similarly, $\mathcal{G}$ acts on ordered pairs of elements of $X$, and so on.

Given a graph $G = (V, E)$ with map $M$, let $S = \{(u, v) : \{u, v\} \in E(G)\}$ denote the set of arcs of $G$ (that is, replace each edge of $G$ by two directed edges). Let $Aut(M)$ denote the set of automorphisms of the map $M$. Then the action of $Aut(M)$ on $V(G)$ induces an action on $S$, by $\alpha(u, v) = (\alpha(u), \alpha(v))$, where $\alpha \in Aut(M)$. If $Aut(M)$ acts transitively on $S$ and $|Aut(M)| = |S|$, then $(Aut(M), S)$ is a regular action on $S$. In this case, $M$ is *regular*. If $M$ is regular, then $Aut(M)$ is transitive on the vertices and edges of $G$ and on the regions of $M$. Regular maps are also called *symmetrical maps*.

Two permutation groups $(\mathcal{G}, \mathcal{X})$ and $(\mathcal{G}', \mathcal{X}')$ are said to be *equivalent* if there exists a bijection $\beta : X \to X'$ and a group isomorphism $\phi : \mathcal{G} \to \mathcal{G}'$ satisfying, for each $x \in X$ and $g \in \mathcal{G}$, $\phi(g)(\beta(x)) = \beta(g(x))$. For example, if $M$ is any map and $M^*$ is its dual, then the actions of $Aut(M)$ on $S$ and $Aut(M^*)$ on $S^*$ are equivalent under $\beta : S \to S^*$, where $\beta$ assigns to each arc $s \in S$ the arc $s^* \in S^*$ which crosses $s$ and has the same direction after a 90 degree rotation clockwise.

**Definition 3.2** *Let $\beta : S \to S^*$ be as described above. The map $M(G)$ is* strongly symmetric *if*

1. *$G \cong \overline{G}$;*

2. *$G \cong G^*$;*

3. *$M$ is symmetrical; and*

4. *$Aut(M)$ and $Aut(M^*)$ are equivalent under $\beta$.*

White proves the following theorem.

**Theorem 3.3** [28] *There exists a strongly symmetric map of order $n$ if and only if $n$ is a prime power congruent to* $1 \pmod 8$.

For existence, White uses Paley graphs, defined as follows. (See also [17] for the original paper by Paley.)

**Definition 3.3** *Let $n = p^r \equiv 1 \pmod 8$, $p$ a prime. A* Paley graph *is a Cayley graph $P_n = C(X_n, \Delta_n)$, where $X_n = (\mathbb{Z}_p)^r$ is the additive group of the Galois field $GF(p^r)$ and $\Delta_n = \{1, x^2, x^4, \ldots, x^{n-3}\}$ for a primitive element $x$ of $GF(p^r)$.*

Note that $\Delta_n$ is the set of all squares in $GF(p^r)$. (Equivalently, $\{u, v\} \in E$ if and only if $v - u$ is a square in $GF(p^r)$.)

**Definition 3.4** *Given a Paley graph $P_n$, the* Paley map *is the embedding of $P_n$ with the clockwise vertex rotation at $v \in X_n$ being* $(\{v, v \oplus 1\}, \{v, v \oplus x^2\}, \{v, v \oplus x^4\}, \ldots, \{v, v \oplus x^{n-3}\})$.

The dual turns out to be a Cayley map with generators either the squares or nonsquares, in which case it is isomorphic to either the Paley graph or its complement, respectively. White goes on to prove that the Paley maps are an infinite class of strongly symmetric graphs. The Paley graph on 9 vertices is shown in Figure 2.1.

## 3.3 A New Infinite Class of Self-Complementary Self-Dual Graphs

In Section 3.2, we saw that a strongly symmetric graph of order $n$ exists if and only if $n \equiv 1 \pmod 8$ and $n$ is a prime power. A question that naturally arises is, if we require only that a graph be self-complementary and self-dual, do we arrive at a larger class of graphs? As this section shows, we do indeed, and we constructively prove this by giving a new infinite class of self-complementary self-dual non-symmetric graphs. We first characterize self-complementary self-dual planar graphs.

**Theorem 3.4** *Let $G = (V, E)$ be a planar self-complementary self-dual graph. Then $G$ is either $K_1$ or one of the three graphs in Figure 3.1.*

**Proof.** It can be verified that graphs A, B and C from Figure 3.1 are self-complementary and self-dual. $K_1$ is trivially self-complementary and self-dual.

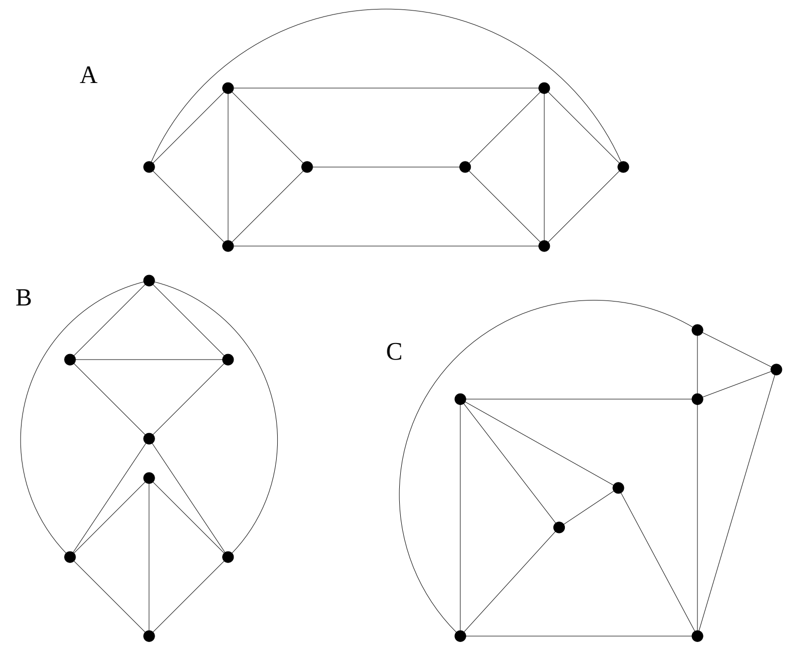


Figure 3.1: The 3 self-complementary self-dual graphs on 8 vertices

Thus it remains to be shown that this is the complete set of graphs.

Let $G = (V, E)$ be a planar self-complementary self-dual graph. Let $n = |V|$. By Euler's Formula (Theorem 1.1), $|V| - |E| + |F| = 2$. Since $G$ is self-dual, we know that $|V| = |F|$, and since $G$ is self-complementary, we know that $|E| = \frac{n(n-1)}{4}$. Thus, $\frac{n(n-1)}{4} = 2n - 2$, so $n$ is either 1 or 8. If $|V| = 1$, then $|E| = 0$, which implies that $G = K_1$.

Therefore, suppose $G$ has 8 vertices. We may immediately discard any graphs that have a vertex of degree 1 or 2, since dually these yield loops and parallel edges, and self-complementary graphs are simple. From [1], we know there are only 10 self-complementary graphs on 8 vertices, and only 4 of these have no degree 1 or 2 vertices. They consist of graphs A, B and C from Figure 3.1 and graph D from

Figure 3.2, which is 3-connected (thus is uniquely embeddable on the plane up to homeomorphism) and is not self-dual, as in the dual no degree 3 vertices are adjacent but in the primal there exist adjacent degree 3 vertices. □

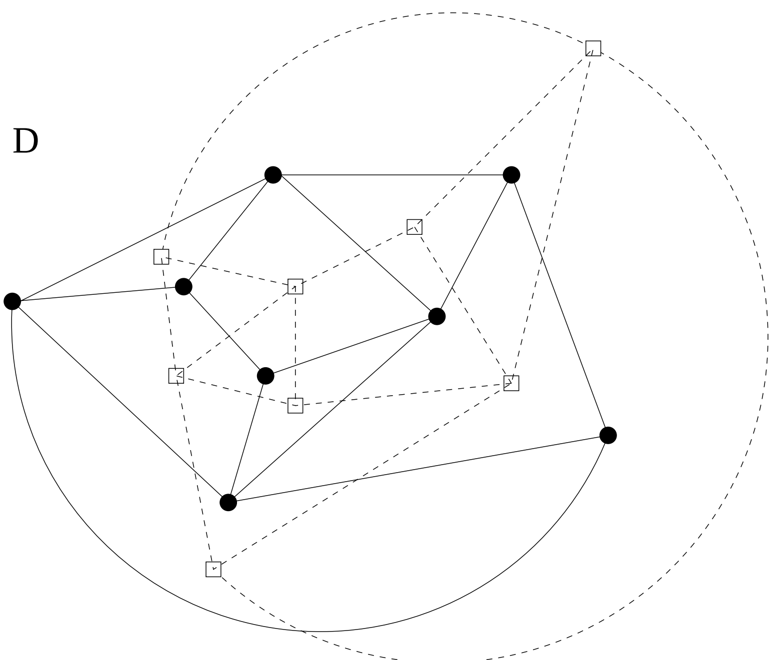


Figure 3.2: A self-complementary non-self-dual graph on 8 vertices. The dual is illustrated with dotted lines.

We now complete the same calculation from the beginning of the last proof for a general surface $S_k$.

**Theorem 3.5** *Let $G = (V, E)$ be a self-complementary self-dual graph on $n$ vertices with a self-dual embedding on an orientable surface of genus $k$. Then $n \equiv 0, 1 \pmod 8$. In particular, if $n = 8 + 8k'$, then $k = 8(k')^2 + 7k'$, and if $n = 9 + 8k'$, then $k = 8(k')^2 + 9k' + 1$.*

**Proof.** Let $G = (V, E)$ be a self-complementary self-dual graph on $n$ vertices. Since $G$ is self-complementary, we know that $|E| = \frac{n(n-1)}{4}$ edges. Also, from Theorem 1.1, we know that $n - |E| + |F| = 2 - 2k$. In a self-dual embedding, we know that $n = |F|$. Therefore, $|E| = 2n - 2 + 2k$. Thus, $n(n-1)/4 = 2n - 2 + 2k$, so $n = (9 \pm \sqrt{49 + 32k})/2$. Since $n$ must be an integer, $49 + 32k$ must be a perfect square. Thus, $x^2 - 49 = 32k$, or $(x + 7)(x - 7) = 32k$.

For integers $a, b$, the notation $a \mid b$ means $a$ divides $b$ and $2 \parallel a$ means $2 \mid a$ but $4 \nmid a$. Therefore, for such an $x$,

$$\begin{aligned} 2^4 \mid x - 7 &\Leftrightarrow 2 \parallel x + 7 \\ 2^4 \mid x + 7 &\Leftrightarrow 2 \parallel x - 7. \end{aligned}$$

Therefore, there is a non-negative integer $k'$ such that either $x = 7 + 2^4 k'$ or $x = 9 + 2^4 k'$.

For $k \geq 1$, $\frac{9 - \sqrt{49+32k}}{2} \leq 0$. If $k = 0$, $n = \frac{9-7}{2} = 1$ yielding the graph $K_1$. Thus, we may assume $n = (9 + \sqrt{49 + 32k})/2$ and either $x = 7 + 2^4 k'$, in which case $n = (9 + x)/2 = 8 + 8k'$ and $k = (x^2 - 49)/32 = 8(k')^2 + 7k'$, or $x = 9 + 2^4 k'$, in which case $n = 9 + 8k'$ and $k = 8(k')^2 + 9k' + 1$. $\square$

Theorem 3.5 gives us the following easy corollary.

**Corollary 3.6** *For all $k$, there exist at most two values of $n$ for which there exists a self-dual self-complementary graph $G$ in $S_k$ on $n$ vertices.*

We now proceed with the main result of this section, constructing an infinite class of self-complementary self-dual graphs.

The construction will be based on voltage graphs. See Section 1.6 or [11]. For the duration of this construction, all counting or enumeration begins at 1.

**Notation 1** *Let* $\mathbb{Z}_2^t$ *denote* $\underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{t}$.

**Remark.** On the group $(\mathbb{Z}_2^t, \oplus)$, when discussing order, we will impose binary ordering (i.e. treating the vectors as the binary representations of the numbers $0 \dots 2^t - 1$ with possibly leading zeros; for vectors $a, b \in \mathbb{Z}_2^t$, $a \leq b$ if and only if the integer represented by $a$ is at most the integer represented by $b$).

**Definition 3.5** *A binary vector has* even weight *if it has an even number of 1's and has* odd weight *otherwise. An* $\varepsilon$-vector *is a vector in* $\mathbb{Z}_2^{t-1}$ *with even weight. A* $\sigma$-vector *is a vector in* $\mathbb{Z}_2^{t-1}$ *with odd weight. We label the* $\varepsilon$*-vectors so that* $\varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_{2^{t-2}}$*. Similarly, label the* $\sigma$*-vectors so that* $\sigma_1 < \sigma_2 < \cdots < \sigma_{2^{t-2}}$*.*

**Definition 3.6** *A* $1\varepsilon$-vector *is a vector in* $\mathbb{Z}^t$ *where the first entry is a one and the remainder of the vector is an* $\varepsilon$*-vector. We define* $1\sigma$*-,* $0\varepsilon$*- and* $0\sigma$-vectors *similarly. A* $1\varepsilon$-edge *is an edge with a* $1\varepsilon$*-vector as a voltage assignment. We define* $1\sigma$*-,* $0\varepsilon$*- and* $0\sigma$-edges *similarly.*

For example, when $t = 4$,

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $0\varepsilon_1$ | = | 0000 | = | 0 | $0\sigma_1$ | = | 0001 | = | 1 |
| $0\varepsilon_2$ | = | 0011 | = | 3 | $0\sigma_2$ | = | 0010 | = | 2 |
| $0\varepsilon_3$ | = | 0101 | = | 5 | $0\sigma_3$ | = | 0100 | = | 4 |
| $0\varepsilon_4$ | = | 0110 | = | 6 | $0\sigma_4$ | = | 0111 | = | 7 |
| $1\varepsilon_1$ | = | 1000 | = | 8 | $1\sigma_1$ | = | 1001 | = | 9 |
| $1\varepsilon_2$ | = | 1011 | = | 11 | $1\sigma_2$ | = | 1010 | = | 10 |
| $1\varepsilon_3$ | = | 1101 | = | 13 | $1\sigma_3$ | = | 1100 | = | 12 |
| $1\varepsilon_4$ | = | 1110 | = | 14 | $1\sigma_4$ | = | 1111 | = | 15 |

**Definition 3.7** *A* link *is an edge which is incident with* 2 *different vertices. A* loop *is an edge which has two incidences with the same vertex. A* half edge *is an edge with a single incidence with a vertex.*

Loops and half edges and their lifts will be denoted with a '+'.

The construction is based on a generalization of graph $A$ from Figure 3.1. The construction actually generalizes the voltage graph $(A', \mathbb{Z}_2 \times \mathbb{Z}_2)$ associated with graph $A$, illustrated in Figure 3.3.

**Definition 3.8** *Let $t \geq 3$. Let $H_t$ be the voltage graph defined as follows over the group $(\mathbb{Z}_2^t, \oplus)$: $H_t$ has two vertices, $u$ and $v$. There are $2^{t-1}$ links between $u$ and $v$, with voltage assignments $0\sigma_1, \ldots, 0\sigma_{2^{t-2}}$ and $1\varepsilon_1, \ldots, 1\varepsilon_{2^{t-2}}$ (equivalently, all possible vectors in $\mathbb{Z}_2^t$ with odd weight). There are $2^{t-1}$ half edges about $v$ with voltage assignments $0\sigma_1, \ldots, 0\sigma_{2^{t-2}}$ and $1\sigma_1, \ldots, 1\sigma_{2^{t-2}}$. Similarly, there are $2^{t-1}-1$ half edges about $u$ with voltage assignments $0\varepsilon_2, 0\varepsilon_3, \ldots, 0\varepsilon_{2^{t-2}}$ and $1\varepsilon_1, \ldots, 1\varepsilon_{2^{t-2}}$.*

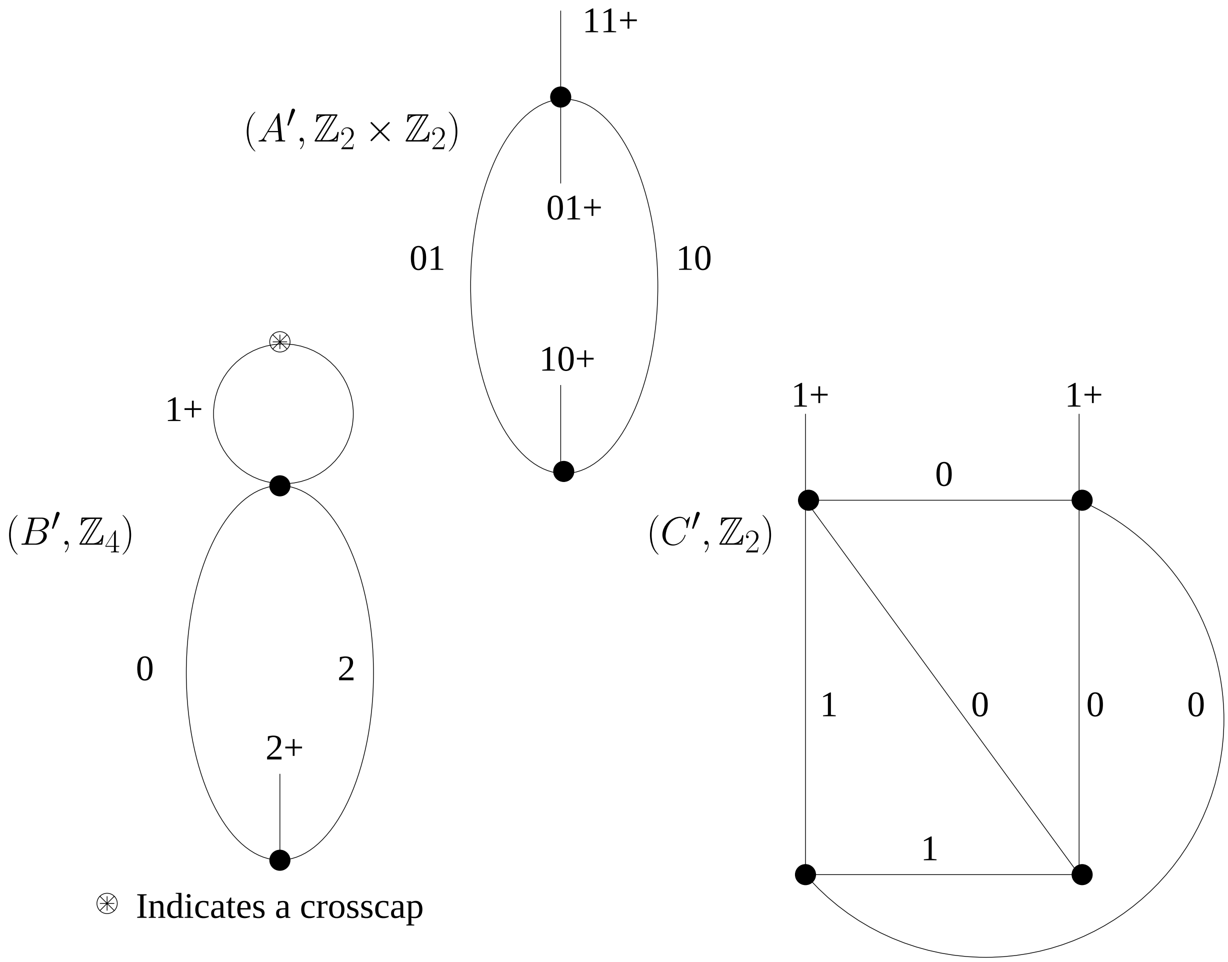


Figure 3.3: Associated voltage graphs for Figure 3.1

Let $G_t$ be the lift of $H_t$. Note that $|V(G_t)| = |V(H_t)| \times |\mathbb{Z}_2^t| = 2 \times 2^t$. Since $t \geq 3$, $2^{t+1} \equiv 0 \pmod 8$.

Let $\theta : V(G_t) \to V(G_t)$ be defined by $\theta(u,x) = (v,x)$ and $\theta(v,y) = (u, y + 10\ldots0)$, where $x, y$ and $10\ldots0 \in \mathbb{Z}_2^t$.

**Theorem 3.7** *$G_t$ is self-complementary, with self-complementary isomorphism $\theta$.*

**Proof.** Obviously, $\theta$ is 1-1 and onto, and thus a bijection. Therefore, we need only prove adjacency and non-adjacency are preserved. By our construction, our graph has $|E| = \left(\frac{n(n-1)}{4}\right)$ edges, thus we need only check that adjacency is preserved. There are three types of edges which require consideration. Lifts of links, half edges about $u$, and half edges about $v$.

Let $x_1, x_2, y_1, y_2 \in \mathbb{Z}_2^t$. Suppose $(u, x_1) \sim (v, y_1)$ in $G_t$. Then $x_1 \oplus y_1$ has odd weight. Thus $x_1 \oplus y_1 \oplus 10\ldots0$ has even weight, so $(v, x_1) = \theta(u, x_1)$ and $(u, (y_1 \oplus 10\ldots0)) = \theta(v, y_1)$ are non-adjacent in $G_t$ and therefore are adjacent in $\overline{G_t}$.

Suppose $(u, x_1) \sim (u, x_2)$ in $G_t$. Therefore $x_1$ and $x_2$ differ by either a $1\varepsilon$- or a $0\varepsilon$-vector (that is, $x_1 \oplus x_2 = 1\varepsilon$ or $0\varepsilon$). Now, $\theta(u, x_1) = (v, x_1)$ and $\theta(u, x_2) = (v, x_2)$. Since $x_1$ and $x_2$ differ by either a $1\varepsilon$- or $0\varepsilon$-vector, $(v, x_1)$ and $(v, x_2)$ are non-adjacent in $G_t$ and therefore adjacent in $\overline{G_t}$ as required.

Suppose $(v, y_1) \sim (v, y_2)$ in $G_t$. Therefore $y_1$ and $y_2$ differ by either a $0\sigma$- or a $1\sigma$-vector (that is, $y_1 \oplus y_2 = 0\sigma$ or $1\sigma$). Now, $\theta(v, y_1) = (u, (y_1 \oplus 10\ldots0))$ and $\theta(v, y_2) = (u, (y_2 \oplus 10\ldots0))$. Since $y_1$ and $y_2$ differ by either a $0\sigma$- or $1\sigma$-vector, so do $(y_1 \oplus 10\ldots0)$ and $(y_2 \oplus 10\ldots0)$. Thus $(u, (y_1 \oplus 10\ldots0))$ and $(u, (y_2 \oplus 10\ldots0))$ are non-adjacent in $G_t$ and therefore adjacent in $\overline{G_t}$ as required.

Therefore, $G_t$ is self-complementary. □

For self-duality, embed $H_t$ in the plane such that, about $v$, we alternate links and half edges, and about $u$ we again alternate links and half edges except that there is no half edge on the outside face.

We now assign voltage assignments as follows: label the links with all possible $0\sigma$- and $1\varepsilon$-vectors, clockwise around $u$, beginning from the outside face, smallest to largest in the binary ordering. About $u$, proceeding counterclockwise starting from the outside face, we label the half edges with all possible $0\varepsilon$- and $1\varepsilon$-vectors, excluding the all zeros vector (as that would result in a loop in the lift), from smallest to largest. About $v$, proceeding clockwise starting from the outside face, we label the half edges with all possible $1\sigma$-vectors in increasing order, followed by the $0\sigma$-vectors in increasing order.

This embedding is on the plane. However, in reality we wish to embed $H_t$ so that we obtain only 2 faces, a face of degree $2^t$ and a face of degree $2^t - 1$, including half edges. This is the embedding obtained by keeping the cyclic rotation at $u$ and inverting the cyclic rotation at $v$. The face of degree $2^t$ will begin at vertex $v$, traverse the $0\sigma_1$-link, traverse the $1\varepsilon^+_{2^{t-2}}$-edge at $u$, traverse the $0\sigma_2$-link, traverse the $0\sigma^+_{2^{t-1}-1}$-edge at $v$ and so forth. To summarize, edge $i$ in the boundary face walk of the face of degree $2^t$, starting at vertex $v$, $i = 1$, is:

if $i \leq 2^{t-1}$, then

$$i \equiv 1 \pmod 4 \Leftrightarrow i \text{ is the } 0\sigma_{(i+1)/2}\text{-link}$$

$$i \equiv 2 \pmod 4 \Leftrightarrow i \text{ is the } 1\varepsilon^+_{2^{t-2}-(i-2)/2}\text{-edge}$$

$i \equiv 3 \pmod 4 \Leftrightarrow i$ is the $0\sigma_{(i+1)/2}$-link

$i \equiv 0 \pmod 4 \Leftrightarrow i$ is the $0\sigma^{+}_{2^{t-2}-(i-2)/2}$-edge

if $i > 2^{t-1}$, then let $j = i - 2^{t-1}$

$i \equiv 1 \pmod 4 \Leftrightarrow i$ is the $1\varepsilon_{(j+1)/2}$-link

$i \equiv 2 \pmod 4 \Leftrightarrow i$ is the $0\varepsilon^{+}_{2^{t-2}-(j-2)/2}$-edge

$i \equiv 3 \pmod 4 \Leftrightarrow i$ is the $1\varepsilon_{(j+1)/2}$-link

$i \equiv 0 \pmod 4 \Leftrightarrow i$ is the $1\sigma^{+}_{2^{t-2}-(j-2)/2}$-edge.

Similarly, for the $2^t - 1$ face, edge $i$ in the boundary face walk starting at vertex $u$, $i = 1$, is:

if $i \leq 2^{t-1}$, then

$i \equiv 1 \pmod 4 \Leftrightarrow i$ is the $0\sigma_{(i+1)/2}$-link

$i \equiv 2 \pmod 4 \Leftrightarrow i$ is the $0\sigma^{+}_{2^{t-2}-(i-2)/2}$-edge

$i \equiv 3 \pmod 4 \Leftrightarrow i$ is the $0\sigma_{(i+1)/2}$-link

$i \equiv 0 \pmod 4 \Leftrightarrow i$ is the $1\varepsilon^{+}_{2^{t-2}-(i-2)/2}$-edge

if $i > 2^{t-1}$, then let $j = i - 2^{t-1}$

$i \equiv 1 \pmod 4 \Leftrightarrow i$ is the $1\varepsilon_{(j+1)/2}$-link

$i \equiv 2 \pmod 4 \Leftrightarrow i$ is the $1\sigma^{+}_{2^{t-2}-(j-2)/2}$-edge

$i \equiv 3 \pmod 4 \Leftrightarrow i$ is the $1\varepsilon_{(j+1)/2}$-link

$i \equiv 0 \pmod 4 \Leftrightarrow i$ is the $0\varepsilon^{+}_{2^{t-2}-(j-2)/2}$-edge.

The situation is illustrated in Figure 3.4.

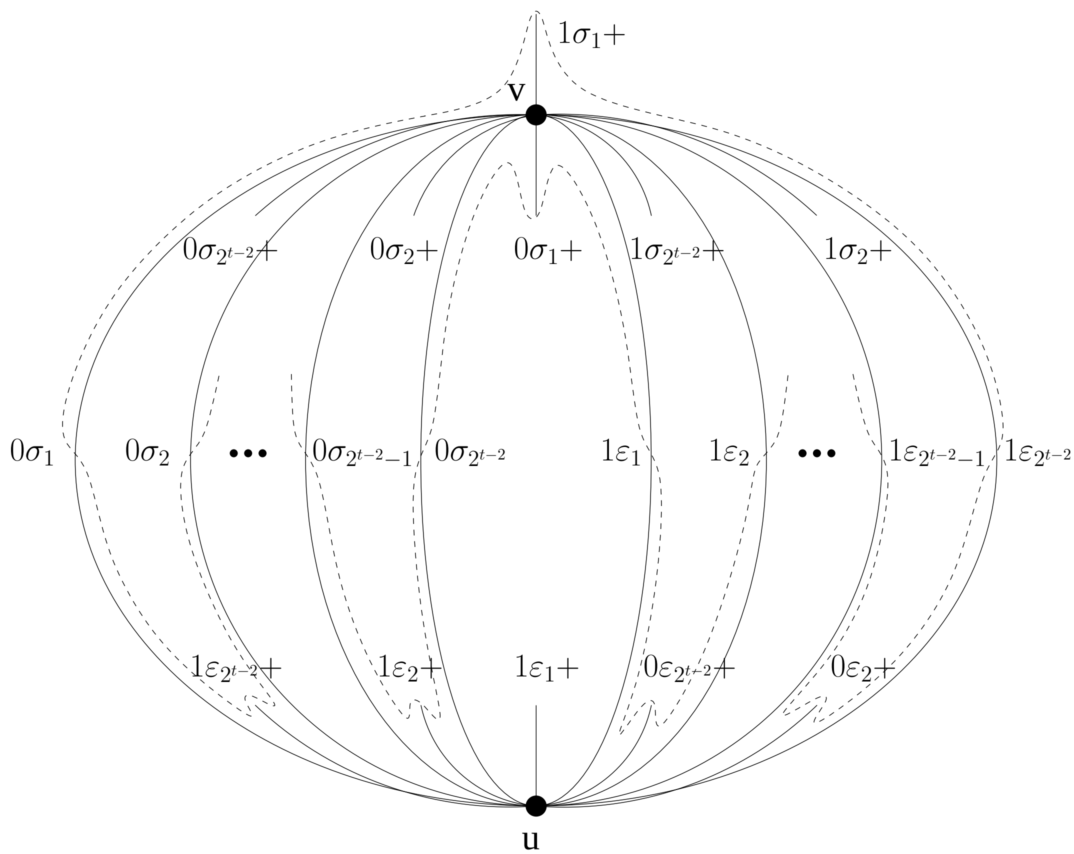


Figure 3.4: Generalized voltage graph $H_t$ with embedding. Boundary face walk of degree $2^t$ illustrated with dotted line

**Theorem 3.8** *$G_t$ with the lift of this embedding of $H_t$ is self-dual.*

**Proof.** Since the $v$-vertices of $G_t$ have degree $2^t$ and the $u$-vertices have degree $2^t-1$, a necessary condition is that half of the faces have degree $2^t$ and the other half have $2^t-1$. In the embedding of the voltage graph, we had faces of degree $2^t$ and $2^t-1$. Therefore in the lift, the faces have degree $\geq 2^t$ and $2^t-1$ respectively. There will be equality if and only if the net voltage in going around a face of the voltage graph is $00\ldots0$.

Consider a face in $G_t$ corresponding to a face of degree $2^t$ in $H_t$. Consider a walk around the face in $G_t$ corresponding to exactly one walk around the corresponding face in $H_t$. Consider all lifts of half edges along such a walk. Such a walk has all possible $1\sigma^+_{2i+1}$-, $0\sigma^+_{2i+1}$-,$1\varepsilon^+_{2i}$- and $0\varepsilon^+_{2i}$-edges.. For a particular $i$, the $1\sigma_{2i+1}$-vector and the $0\sigma_{2i+1}$-vector will be identical except for the first entry. Since $1\sigma_{2i+1}\oplus 0\sigma_{2i+1}=10\ldots0$ and there are $2^{t-3}$ such pairs with $t\geq 3$, cumulatively all these pairs contribute $10\ldots0$ if $t=3$, or $00\ldots0$ otherwise. Similarly, all such pairs of $1\varepsilon^+_{2i}$- and $0\varepsilon^+_{2i}$-edges around the face add a total of $10\ldots0$ if $t=3$ or $00\ldots0$ otherwise to the boundary face walk total, for a total of 0 from the half edges in either case.

Now consider all lifts of links of $H_t$ along this face. We have all $0\sigma_i$- and $1\varepsilon_i$-links. So, ignoring the first entry, we have all possible binary strings of length $2^{t-1}$. Consider position $j$ of all $2^{t-1}$ binary strings. A 1 appears in position $j$ exactly half of the time. Since $t\geq 3$, $2^{t-2}$ is even, so adding all possible binary strings of length $2^{t-1}$ yields a total of 0 in position $j$ for all $j$. Returning to the first entry, since as noted before there is an even number of $1\varepsilon_i$-links, the total of the first entries is also 0. Thus the total of all links is 0. Therefore, after traversing once around the face in the voltage graph, we have a net voltage of 0, so we have returned to the starting vertex, so indeed in $G_t$ we have a face of degree exactly $2^t$. A similar discussion yields equality for the face of degree $2^t - 1$. Thus the embedding has faces of the proper degrees required for self-duality.

We now wish to describe $G_t^*$ as the lift of a voltage graph which we will denote as $H_t^*$. We then intend to show $G_t \cong G_t^*$. Note that the lift of a half-edge is always incident with two faces of the same length and that lifts of links are always incident with faces of different lengths.

We label each face $F$ of $G_t$ as follows: find the $0\ldots01$-link incident with $F$. If $F$ has degree $2^t$, give $F$ the same label as the $v$-vertex incident with $0\ldots01$-link. If $F$ has degree $2^t-1$, give $F$ the same label as the $u$-vertex incident with the $0\ldots01$-link. We denote faces using a $*$. Thus, if $F$ is incident with the link joining $(u, x_1)$ to $(v, x_1 \oplus 0\ldots01)$, and $F$ has length $2^t$, then $F$ gets the label $(v, x_1 \oplus 0\ldots01)^*$, while if $F$ has length $2^t - 1$, then $F$ gets the label $(u, x_1)^*$.

We now describe each edge in the dual in terms of the edge it crosses in the primal. We do this so as to describe the dual as the lift of a voltage graph. For the sake of this discussion, we will always start our boundary face walk at the vertex used to label the face, and travel along the $0\ldots01$-link. Consider an edge incident with the face $(u, x_1)^*$ joining $(u, a)$ and $(u, a \oplus z)$. This is a lift of a half edge incident with $u$ having voltage $z$. Say $(u, x_1)^*$ is adjacent to the face $(u, x_2)^*$ along the $z$-edge. Suppose the $z$-edge is the $(i+1)^{st}$ edge in the face walk. Therefore, after traversing the first $i$ edges along the $(u, x_1)^*$ face, we arrive at the vertex

$(u, a)$. Similarly, after traversing the first $i$ edges along the $(u, x_2)^*$ face, we arrive at the vertex $(u, a \oplus z)$. But we traversed edges with the same voltage assignment on both walks, so the resulting vertex voltage assignments differ by $z$ if and only if $x_1$ and $x_2$ differ by exactly $z$. Therefore, if $(u, x_1)^* \sim (u, x_2)^*$ along the $z$-edge, then $x_1 = x_2 \oplus z$. That is, edges of $G_t^*$ corresponding to half edges of $H_t^*$ can be properly labeled with the identical label for the edge they cross. The same result holds for the lifts of half edges of $H_t^*$ in $(v, y)^*$ faces.

Now we must consider edges crossing lifts of links. Suppose $(u, x_1)^* \sim (v, y_1)^*$ by an edge crossing a lift of a link whose voltage assignment is $z'$. Now, starting at the $0 \ldots 01$-link, the walk along the boundary of the $(u, x_1)^*$ face must traverse the same number of edges as the walk along the boundary of the $(v, y_1)^*$ face to get to edge $z'$. In particular, the boundary face walks traverse edges which were lifts of the same links, and when the walks are traversing a $0\sigma_i^+$-edge on one face, they are traversing the $1\varepsilon_i^+$-edge on the other face, and, similarly, when one walk traverses a $1\sigma_i^+$-edge the other traverses the $0\varepsilon_i^+$-edge. (See Figure 3.4.) Note that $\sigma_i$ and $\varepsilon_i$ differ in only the last position. Thus, $\sigma_i \oplus \varepsilon_i = 0 \ldots 01$ for all $i$. Therefore, $0\sigma_i \oplus 1\varepsilon_i = 10 \ldots 01$. Similarly with the $1\sigma_i^+$- and $0\varepsilon_i^+$-edges. Thus, if during the boundary face walks up to $z'$, we have traversed an even number of lifts of half-edges (and thus an even number of lifts of links (which do not affect the partial sums of the boundary face walks)), then $x_1 \oplus y_1 = z'$. If we have traversed an odd number of lifts of half-edges (and thus an odd number of lifts of links), then $x_1 \oplus y_1 = z' \oplus 10 \ldots 01$. Therefore, if $z' = 0\sigma_{2i}$ or $1\varepsilon_{2i}$, then we have traversed an odd number of lifts of half-edges and the edge crossing $z'$ is labeled $1\varepsilon_{2i}$ or $0\sigma_{2i}$

respectively. Otherwise, we have traversed an even number of lifts of half-edges and the edge crossing $z'$ is labeled $z'$.

Suppose $j \in \{0,1\}$ and $\tau \in \{\varepsilon_i, \sigma_i\}$. Let $\phi : \mathbb{Z}_2^t \to \mathbb{Z}_2^t$ be defined by $\phi(j\tau) = 10\ldots01 \oplus j\tau$ if and only if $i \equiv 0 \pmod 2$; otherwise, $\phi(j\tau) = j\tau$. Furthermore, let $\psi : V(G_t^*) \to V(G_t)$ be defined by $\psi((s, j\tau)^*) = (s, \phi(j\tau))$, where $s \in \{u, v\}$.

**Theorem 3.9** *The function $\psi$ is an isomorphism between $G_t$ and $G_t^*$.*

**Proof.** Note that $\phi(x)$ and $x$ have weights of the same parity. Thus, $(u,x)^* \sim (v,y)^*$ if and only if $\psi((u,x)^*)) \sim \psi((v,y)^*$. Thus, adjacency by a lift of a link is preserved. For preservation of adjacency along lifts of half-edges, we require a lemma.

**Definition 3.9** *A* flip vector *is a vector $x$ such that $\phi(x) \neq x$. A* flip edge *is an edge of $G_t^*$ with a flip vector assignment.*

**Lemma 3.10** *$(u,x_1)^* \sim (u,x_2)^*$ by a flip edge if and only if $|\{x_1,x_2\} \cap \{\phi(x_1),\phi(x_2)\}|$ is odd. Equivalently, $(u,x_1)^* \sim (u,x_2)^*$ by a non-flip edge if and only if $|\{x_1,x_2\} \cap \{\phi(x_1),\phi(x_2)\}|$ is even.*

**Proof.** Consider the elements of $\mathbb{Z}_2^t$ and their binary ordering. In that ordering, when $f_i$ is a vector which is the binary representation of an even integer, vectors $f_i$ and $f_{i+1}$ are identical except for the last digit. For vector $f_i$ the last digit is 0 and for $f_{i+1}$ it is 1. Similarly, when the integer $f_i$ represents is divisible by four, the vectors $f_i, f_{i+1}, f_{i+2}$ and $f_{i+3}$ are identical except in their last two positions, which are, respectively, $00, 01, 10$ and $11$. We divide the elements of $\mathbb{Z}_2^t$ into *duets* of two, taking consecutive elements (in the binary ordering) and *quartets* of four, taking two consecutive duets, starting with the all zeros vector.

Note that a duet has exactly one $0\varepsilon$-vector if and only if it has one $0\sigma$-vector. Similarly for $1\varepsilon$- and $1\sigma$-vectors. Also note that every second $0\varepsilon$-/$0\sigma$-/$1\varepsilon$-/$1\sigma$-vector is flipped. In terms of duets, this means that there is one element in the duet that was not flipped if and only if both elements in the duet were not flipped. Since $t \geq 3$, the leading entries in the vectors of a quartet are the same. Suppose for the sake of definiteness that the quartet in question has a 0 as the first entry. Within a given quartet, the first duet will contain the $i^{th}$ largest $0\varepsilon$- and $0\sigma$-vectors, where $i$ is odd. Thus, an edge is flipped by $\psi$ if and only if the second last entry is a 1, or equivalently, the vector ends in 10 or 11.

Now Lemma 3.10 may be restated as:

$x_1 \oplus x_2 = b10$ or $b11$ (for some value of $b \in \mathbb{Z}_2^{(t-2)}$) if and only if either ($x_1$ ends in 10 or 11) or ($x_2$ ends in 10 or 11), but not both.

$x_1 \oplus x_2 = b10$ or $b11 \Leftrightarrow x_1 \oplus (b10$ or $b11) = x_2$. Since we are doing group addition, this is true if and only if the second last entry of $x_1$ and $x_2$ are different. Therefore, exactly one of them has a 1 as the second last entry, and the lemma is proven. $\square$

Returning to the proof of Theorem 3.9, suppose $(u, x_1)^* \sim (u, x_2)^*$. Assume for sake of definiteness that $x_1$ is $0\varepsilon$. Now consider $\psi((u, x_1)^*)$ and $\psi((u, x_2)^*)$.

**CASE 1** $\psi((u, x_1)^*) = (u, x_1)$.

If $x_2$ is $1\sigma$ or $0\sigma$, then $(u, x_1)^* \sim (u, x_2)^*$ with $x_1$ being $0\varepsilon$ implies the adjacency is by a $1\sigma_i^+$- or $0\sigma_i^+$-edge (i.e., the lift of a half edge with voltage assignment $1\sigma_i$ or $0\sigma_i$). Since we are on a $(u,x)^*$ face, they must be of the form $1\sigma_{2i}^+$ or $0\sigma_{2i}^+$, which implies they are flip edges. Since $\psi((u,x_1)^*) = (u,x_1)$, Lemma 3.10 implies that $\psi((u,x_2)^*) \neq (u,x_2)$. Since $x_2$ is $1\sigma$ or $0\sigma$, $\phi(x_2)$ is $0\varepsilon$ or $1\varepsilon$ respectively. Thus $\psi((u,x_1)^*) \sim \psi((u,x_2)^*)$, because in $G_t$ all $(u,0\varepsilon)$ vertices are adjacent to all other possible $(u,0\varepsilon)$ and $(u,1\varepsilon)$ vertices.

Otherwise, $x_2$ is $1\varepsilon$ or $0\varepsilon$, which implies $(u,x_1)^* \sim (u,x_2)^*$ by a $0\varepsilon_i^+$- or a $1\varepsilon_i^+$-edge. Since we are on a $(u,x)^*$ face, they must be of the form $1\varepsilon_{2i+1}^+$ or $0\varepsilon_{2i+1}^+$, which implies they are not flip edges. Thus, by Lemma 3.10, $\psi((u,x_1)^*) = (u,x_1)$ implies that $\psi((u,x_2)^*) = (u,x_2)$. Again, $\psi((u,x_1)^*) \sim \psi((u,x_2)^*)$, because in $G_t$ all $(u,0\varepsilon)$ are adjacent to all possible $(u,0\varepsilon)$ and $(u,1\varepsilon)$ vertices.

**CASE 2** $\psi((u,x_1)^*) \neq (u,x_1)$. (This implies $\phi(x_1)$ is a $1\sigma$ vector.)
If $x_2$ is $1\sigma$ or $0\sigma$, then $(u,x_1)^* \sim (u,x_2)^*$, with $x_1$ being $0\varepsilon$, implies the adjacency is by a $1\sigma_i^+$- or $0\sigma_i^+$-edge. Since we are on a $(u,x)^*$ face, they must be of the form $1\sigma_{2i}^+$ or $0\sigma_{2i}^+$, which implies they are flip edges. Since $\psi((u,x_1)^*) \neq (u,x_1)$, Lemma 3.10 implies that $\psi((u,x_2)^*) = (u,x_2)$. Since $x_2$ was $1\sigma$ or $0\sigma$, $\psi((u,x_1)^*) \sim \psi((u,x_2)^*)$, because in $G_t$ all $(u,1\sigma)$ vertices are adjacent to all other possible $(u,0\sigma)$ and $(u,1\sigma)$ vertices.

Otherwise, $x_2$ is $1\varepsilon$ or $0\varepsilon$, which implies $(u,x_1)^* \sim (u,x_2)^*$ by a $0\varepsilon_i^+$- or a $1\varepsilon_i^+$-edge. Since we are on a $(u,x)^*$ face, they must be of the form $1\varepsilon_{2i+1}^+$ or $0\varepsilon_{2i+1}^+$, which implies they are not flip edges. Thus, by Lemma 3.10, $\psi((u,x_1)^*) \neq (u,x_1)$ implies that $\psi((u,x_2)^*) \neq (u,x_2)$. Therefore, $\phi(x_1)$ is a $1\sigma$-vector, and $\phi(x_2)$ is either a $0\sigma$- or a $1\sigma$-vector. Again, $\psi((u,x_1)^*) \sim \psi((u,x_2)^*)$, because in $G_t$ all $(u,1\sigma)$ are adjacent to all possible $(u, 0\sigma)$ and $(u, 1\sigma)$ vertices.

Similar arguments show $(v, y_1)^* \sim (v, y_2)^*$ implies that $\psi((v, y_1)^*) \sim \psi((v, y_2)^*)$. Since both $G_t$ and $G_t^*$ have the same number of edges, and $\psi$ is a bijection, $\psi$ is an isomorphism and $G_t$ and $G_t^*$ are isomorphic, as required. □

Therefore this embedding of $G_t$ is self-dual. □

Our main theorem now follows directly from Theorems 3.7 and 3.8.

**Theorem 3.11** $\{G_t : t \geq 3\}$ *is an infinite class of self-complementary self-dual graphs with* $|V(G_t)| \equiv 0 \pmod 8$.

Note that these graphs cannot possibly be strongly symmetric, as strongly symmetric graphs are vertex-transitive, and a self-complementary graph on an even number of vertices must have vertices of different degrees.

Figure 3.5 shows the overall structure of $G_t$, not including the embedding. A vertex is either a complete graph or an empty graph on $n = |V_t|/8$ vertices as indicated, and an edge indicates the join operation, namely, an edge $A \sim B$ implies that $a \sim b$ $\forall a \in A$ and $b \in B$.

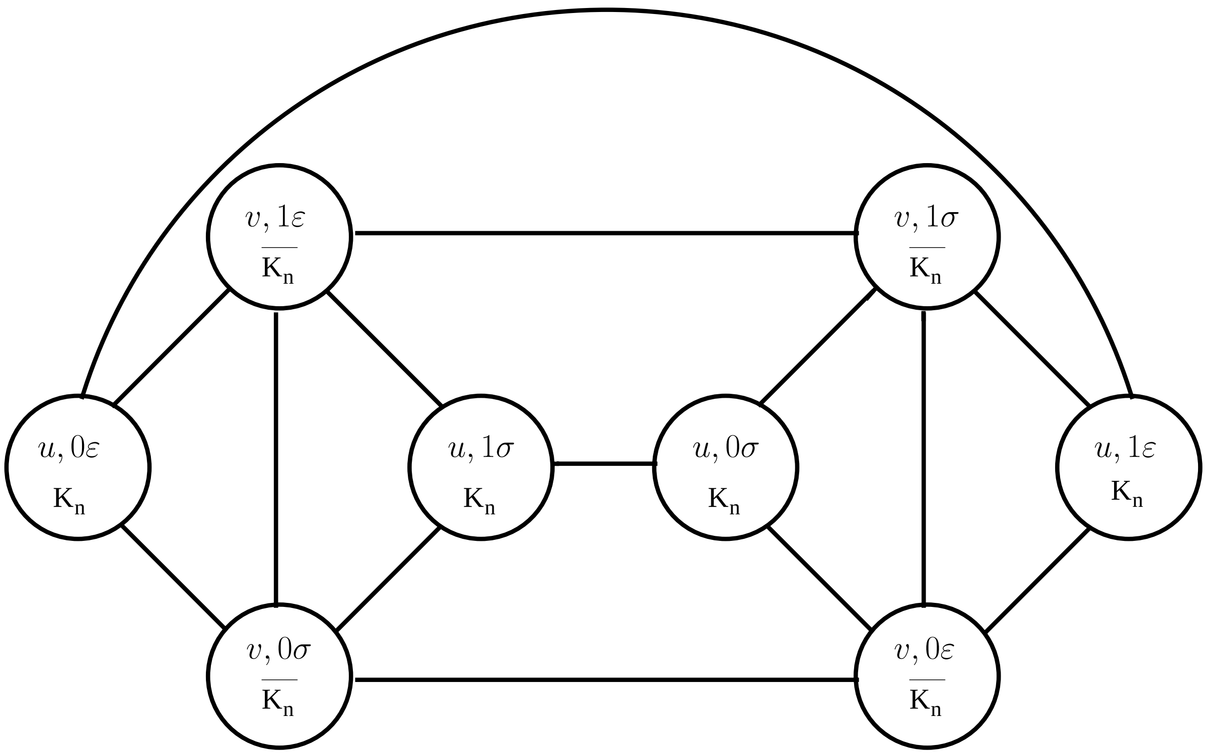


Figure 3.5: General structure of $G_t$

# Chapter 4

# Conclusion and Open Problems

There is still much work to be done in the context of self-dual graphs. One basic question seems to be, can we enumerate all self-dual graphs? That is, given an $n$, how many self-dual graphs on $n$ vertices are there? Is there an efficient way of proving a graph is not self-dual? If we know that a graph is self-dual, is there an efficient way of finding a self-dual embedding?

Other problems of a more specific nature exist. For example, I conjecture that all 3-connected self-dual graphs are Hamiltonian. One possible approach to that conjecture might be trying to use the radial graphs of Section 2.3. For self-dual planar maps, there is a chance of proving/disproving this using the fact that we know how to construct all such graphs. Another problem is the general problem of constructing and classifying self-dual polyhedra for a given surface.

My own research showed that if a self-complementary self-dual graph exists, then $|V(G)| \equiv 0, 1 \pmod 8$. Is this sufficient as well as necessary? The first open case for $|V(G)| \equiv 0 \pmod 8$ is 24 and for $|V(G)| \equiv 1 \pmod 8$ is 33. That is, does there exist a self-complementary self-dual graph on 24 vertices? Another question would be, is there a self-complementary self-dual graph on 9 vertices which is not vertex transitive? As well, what other symmetries may be exhibited by a self-dual self-complementary graph on $8k$ vertices?

This thesis has provided an overview of the family of self-dual graphs. It has presented multiple constructions for creating self-dual graphs from arbitrary graphs, constructions for particular subfamilies of self-dual graphs, and some possible minor operations. In addition, it has reviewed the previously known infinite class of self-complementary self-dual graphs and created a new infinite class using voltage graphs. As stated above, there is still much work to be done in the area.

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