On the Modular Theory of von Neumann Algebras

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Pure Mathematics

Waterloo, Ontario, Canada, 2010

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The purpose of this thesis is to provide an exposition of the *modular theory* of von Neumann algebras. The motivation of the theory is to classify and describe von Neumann algebras which do not admit a trace, and in particular, type III factors. We replace traces with weights, and for a von Neumann algebra $\mathcal{M}$ which admits a weight ϕ, we show the existence of an automorphic action $\sigma^{\phi} : \mathbb{R} \to \mathrm{Aut}(\mathcal{M})$. After showing the existence of these actions we can discuss the crossed product construction, which will then allow us to study the structure of the algebra.

Acknowledgements

I would like to thank my supervisor Laurent Marcoux for lending me his time and wisdom, and for his diligent reading and correcting of early drafts of this thesis. I would also like to thank my readers Brian Forrest and Nicolaas Spronk. Lastly, I would like to acknowledge Erin, who was a constant source of inspiration throughout, and without whose support, this would not have been possible.

Contents

1 Introduction

This thesis is an exposition of the *modular theory* of von Neumann algebras. The motivation of the theory is to classify and describe von Neumann algebras which do not admit a trace, and in particular, type III factors. We replace traces with weights, and for a von Neumann algebra $\mathcal{M}$ which admits a weight ϕ, we show the existence of an automorphic action $\sigma^\phi : \mathbb{R} \to \mathrm{Aut}(\mathcal{M})$. These automorphism groups will then allow us to study the structure of the algebra.

In Section 2 we provide the necessary background on unbounded operators. Unbounded operators are useful for their connection with one-parameter unitary groups given in Stone's Theorem. They allow us to define a broader functional calculus, providing a powerful tool in the study of abelian von Neumann algebras.

In Section 3 we study the representation theory of weights, generalizing the theory of traces. In particular, we are interested in the (generally unbounded) involution on the representation space obtained from the adjoint operation. Section 4 provides an abstract characterization of this representation space given by left Hilbert algebras. We prove Tomita's Theorem, which states that the involution yields one-parameter automorphism group on the von Neumann algebra. Then in Section 5 we study in depth the connection between weights and their associated automorphism groups. This is done by showing that a weight ϕ satisfies a trace-like condition, called the *modular condition*, with respect to the action σ^ϕ, the *modular automorphism group*. In turn, the modular condition completely determines the action. Section 6 gives the reverse construction of a weight from a left Hilbert algebra.

We begin the study of von Neumann algebra crossed products in Section 7. After providing some technical background on the construction, we show how to obtain a weight on the crossed product algebra coming from the original von Neumann algebra. In the case of the crossed product with the modular automorphism group, we obtain a weight which can be perturbed by a (generally unbounded) positive operator to obtain a trace on the crossed product. In Section 8 we shed more light on the structural implications of this trace. We do this by generalizing the Pontryagin duality of locally compact abelian groups to duality of crossed products by locally compact abelian groups, and construct an action of the dual group on the crossed product for which this trace satisfies a semi-invariance property. Finally in Section 9 we use this semi-invariance to give a structure theorem for type III von Neumann algebras.

All Hilbert spaces are complex, and unless otherwise stated, infinite dimensional. For a Hilbert space $\mathcal{H}$, $B(\mathcal{H})$ denotes the space of all bounded linear operators on $\mathcal{H}$. On a von Neumann algebra $\mathcal{M}$ we will use the abbreviations SOT, WOT, and σ-WOT for the strong-operator, weak-operator, and σ-weak operator topologies respectively. For a von Neumann algebra $\mathcal{M}$ we denote the center by $C_{\mathcal{M}}$. Unless otherwise indicated, all integration of Banach space valued functions are to be understood with respect to the definition of the Pettis integral.

2 Unbounded Operators

In this Section we summarize some of the basics of the theory of unbounded operators. In particular, we attempt to recover as much of the workable theory of bounded operators as possible, and we will extend the Spectral Theorem for normal operators to a nice class of unbounded operators, which are also called normal. In the interest of keeping this Section to within a reasonable size, most proofs will not be given. The material is taken from Chapter 10 of [2], except for the Polar Decomposition Theorem from page 401 of [7], and the Generalized Polar Decomposition Theorem and Lemma 2.30 which are respectively on pages 43 and 22 of [13].

Definition 2.1. *By an operator on a Hilbert space $\mathcal{H}$ we mean a linear function $A : \mathcal{K} \to \mathcal{H}$ where $\mathcal{K} \leq \mathcal{H}$ is a not-necessarily closed linear subspace. We write $\mathcal{D}(A)$ for the domain of A.*

We say that an operator A is **densely defined** if $\mathcal{D}(A)$ is dense in $\mathcal{H}$. We say that A is **closed** if its graph $\mathcal{G}(A)$ is closed in $\mathcal{H} \oplus \mathcal{H}$, and define $\mathscr{C}(\mathcal{H})$ to be the set of all closed densely defined operators on $\mathcal{H}$. We note that if A is closed and $\mathcal{D}(A) = \mathcal{H}$ then by the closed graph theorem, A will be bounded. Hence the content of this Section is the study of operators with proper domains. More generally we say that an operator is **closeable** if the closure $\overline{\mathcal{G}(A)}$ is the graph of an operator, and we call this operator the **closure** of A denoted by $\overline{A}$. Let A be a closed operator, and let $\mathcal{K} \leq \mathcal{D}(A)$ be a linear manifold. We say $\mathcal{K}$ is a **core** for A if $A = \overline{A|_{\mathcal{K}}}$. Lastly, if A is an operator, we define the **graph norm** of A to be the norm on $\mathcal{D}(A)$ given by the inclusion $\xi \in \mathcal{D}(A) \mapsto (\xi, A\xi) \in \mathcal{G}(A)$.

Definition 2.2. *Let A be a densely defined operator on $\mathcal{H}$. If $\xi \in \mathcal{H}$ is such that the function $\eta \mapsto < A\eta, \xi >$ is bounded on $\mathcal{D}(A)$, then there exists a unique element $A^*\xi \in \mathcal{H}$ such that for all $\eta \in \mathcal{D}(A)$ we have $< A\eta, \xi > = < \eta, A^*\xi >$. This defines the* ***adjoint*** *operator A^* on $\mathcal{H}$.*

Note that for the above definition it is important that A be densely defined since otherwise the adjoint would not be uniquely definable.

Lemma 2.3. *Let A be a densely defined operator. Then*

1) A^* is closed;
2) A^* is densely defined if and only if A is closable;
3) if A is closeable, then $\overline{A} = A^{**}$ and $A^{***} = A^*$;
4) $(ran\ A)^\perp = ker\ A^*$, and if A is closed then $(ran\ A^*)^\perp = ker\ A$.

It is therefore convenient for the development of the theory to assume that all operators are densely defined and closeable.

We now define operations. Let A, B be operators on $\mathcal{H}$. We define their sum by setting $\mathcal{D}(A+B) = \mathcal{D}(A) \cap \mathcal{D}(B)$ and define $(A+B)\xi = A\xi + B\xi$ for $\xi \in \mathcal{D}(A+B)$. We define the product AB by setting $\mathcal{D}(AB) = \{\xi \in \mathcal{D}(B) : B\xi \in \mathcal{D}(A)\}$, and define $AB\xi = A(B\xi)$ for $\xi \in \mathcal{D}(AB)$. Of course, neither $A+B$ nor AB need be densely defined or closable even if A, B are, so care will be taken accordingly. We write $A \subseteq B$ to mean $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ and $A\xi = B\xi$ for all $\xi \in \mathcal{D}(A)$. For $\lambda \in \mathbb{C}$ we write λ to denote the operator $\lambda \cdot I$, where I is the identity operator.

The usual notion of invertibility is too restrictive since we will not in general expect an inverse to be surjective. Instead we consider the following.

Definition 2.4.

1) *A closed operator A is* ***non-singular*** *if there exists a closed operator B such that $\mathcal{D}(B) = ran\ A, \mathcal{D}(A) = ran\ B$ and $AB \subseteq 1$ and $BA \subseteq 1$.*
2) *An operator A is* ***boundedly invertible*** *if there exists an operator $B \in B(\mathcal{H})$ such that $AB = 1$ and $BA \subseteq 1$.*
3) *We define the* ***spectrum*** *of an operator A, denoted $\sigma(A)$, to be the set of $\lambda \in \mathbb{C}$ such that $A - \lambda$ is not boundedly invertible.*

We note that the inverse B of the operator A for A non-singular or boundedly invertible, is unique and write $B = A^{-1}$. The requirement in 1) that A is closed was made partly for this reason.

Lemma 2.5.

1) *A closed operator A is non-singular if and only if $ker\ A = 0$ and $ran\ A$ is dense.*
2) *An operator A is boundedly invertible if and only if it is non-singular, and $ran\ A = \mathcal{H}$.*

Lemma 2.6. *The spectrum $\sigma(A)$ is a closed subset of $\mathbb{C}$.*

Note that if an operator A is closed if and only if $A - \lambda$ is closed for all $\lambda \in \mathbb{C}$ so that if A is not closed, $\sigma(A) = \mathbb{C}$.

We now define the natural generalizations of normal and self-adjoint operators.

Definition 2.7. *Let A be an operator on $\mathcal{H}$. Then,*

1) *A is called* ***normal*** *if A is closed and $A^*A = AA^*$.*
2) *A is* ***self-adjoint*** *if $A = A^*$.*
3) *A is* ***symmetric*** *if $A \subseteq A^*$*
4) *A is* ***positive****, and we write $A \geq 0$, if $< A\xi, \xi > \geq 0$ for all $\xi \in \mathcal{D}(A)$.*

Note that if A is symmetric then since $A \subseteq A^*$, it is closeable, and if A is self-adjoint, A is closed since A^* is. We also have the following.

Lemma 2.8. *An operator A is symmetric if and only if $< A\xi, \xi > \in \mathbb{R}$ for all $\xi \in \mathcal{D}(A)$*

Hence a positive operator is automatically symmetric. The following theorem summarizes the spectral properties of symmetric and self-adjoint operators.

Theorem 2.9. *Let A be a closed symmetric operator.*

1) The following are equivalent:

a) A is self-adjoint;
b) $\sigma(A) \subseteq \mathbb{R}$;
c) ker $(A^* - i) =$ ker $(A^* + i) = \{0\}$.

2) If $\sigma(A)$ does not contain $\mathbb{R}$, then A is self-adjoint.

3) If A is positive and self-adjoint, then $\sigma(A) \subseteq [0, \infty)$.

If $\mathcal{H}$ is a Hilbert space it is a well-known fact that the positive operators $T \in B(\mathcal{H})$ are those of the form $T = A^*A$ for some operator $A \in B(\mathcal{H})$. We have a partial analogue of this result for closed operators.

Theorem 2.10. *If A is closed then A^*A is positive self-adjoint, and $\mathcal{D}(A^*A)$ is a core for A.*

We note one last result about normal operators, which separates self-adjoint operators from general symmetric operators.

Lemma 2.11. *If N is normal then $\mathcal{D}(N) = \mathcal{D}(N^*)$ and $\|N\xi\| = \|N^*\xi\|$ for every $\xi \in \mathcal{D}(N)$. Therefore a closed symmetric operator A is normal if and only if A is self-adjoint.*

We now come to the spectral theory of unbounded operators.

Definition 2.12. *Let X be a set, Ω be a σ-algebra on X and let $\mathcal{H}$ be a Hilbert space. A projection valued function $E : \Omega \to B(\mathcal{H})$ is called a* ***spectral measure*** *for the triple $(X, \Omega, \mathcal{H})$ if it satisfies the following:*

1) $E(\varnothing) = 0$ and $E(X) = 1$;
2) for any $S_1, S_2 \in \Omega$ we have $E(S_1 \cap S_2) = E(S_1)E(S_2)$;
3) for any countable collection of disjoint sets $\{S_n\} \subseteq \Omega$ we have $E(\cup_n S_n) = \sum_n E(S_n)$, where this sum convergences in the SOT.

For $\xi, \eta \in \mathcal{H}$ and a spectral measure E, we define the measure $E_{\xi,\eta} \in M(X)$ by

$$E_{\xi,\eta}(S) =< E(S)\xi, \eta > .$$

For short we write E_ξ for $E_{\xi,\xi}$. If ϕ is a bounded Ω-measurable function we can define $\int \phi dE \in B(\mathcal{H})$ to be the unique operator satisfying

$$< \left(\int \phi dE \right) \xi, \eta >= \int \phi dE_{\xi,\eta}, \qquad \text{for } \xi, \eta \in \mathcal{H}.$$

The following result allows us to pass to the unbounded case.

Lemma 2.13. *Let $\{\mathcal{H}_n\}_{n=1}^\infty$ be Hilbert spaces and let $A_n \in B(\mathcal{H}_n)$. Define the operator $\oplus_{n=1}^\infty A_n$ with domain $\mathcal{D} = \{(\xi_n) \in \oplus_{n=1}^\infty \mathcal{H}_n : (A_n \xi_n) \in \oplus_{n=1}^\infty \mathcal{H}_n\}$ by $\oplus_{n=1}^\infty A_n(\xi_n) = (A_n \xi_n)$. Then $\oplus_{n=1}^\infty A_n$ is a densely defined closed operator. Moreover, $\oplus_{n=1}^\infty A_n$ is normal if and only if each A_n is normal.*

Now suppose we have a spectral measure E on the measurable space $(X, \Omega, \mathcal{H})$. If $\phi : X \to \mathbb{C}$ is an Ω-measurable function, set $X_n = \{x \in X : n - 1 \leq |\phi(x)| < n\}$ for $n \geq 1$ so that X is the disjoint, measurable union of the X_n. Define $\mathcal{H}_n = E(X_n)\mathcal{H}$ so that $\mathcal{H} = \oplus_{n=1}^\infty \mathcal{H}_n$, and set $\Omega_n = \{S \cap X_n : S \in \Omega\}$. We define a spectral measure E_n on the measurable space $(X_n, \Omega_n, \mathcal{H}_n)$ by restriction of E. We define the operator

$$\int_X \phi dE = \bigoplus_{n=1}^\infty \int_{X_n} \phi_n dE_n$$

where $\phi_n = \phi|_{X_n}$. Then $\int_X \phi dE$ is a normal operator and has domain

$\mathcal{D}_\phi = \{\xi \in \mathcal{H} : \sum_{n=1}^\infty \|(\int_X \phi_n dE_n) E(X_n)\xi\|^2 < \infty\}$.

Theorem 2.14. *Let E be a spectral measure on the measurable space $(X, \Omega, \mathcal{H})$ and let $\phi : X \to \mathbb{C}$ be an Ω-measurable function. Then,*

1) $\mathcal{D}_\phi = \{\xi \in \mathcal{H} : \int_X |\phi(x)|^2 dE_\xi < \infty\}$;

2) if $\xi \in \mathcal{D}_\phi, \eta \in \mathcal{H}$, then

a) $\displaystyle \int |\phi| d|E_{\xi,\eta}| \leq \|\eta\| \left(\int |\phi|^2 dE_\xi \right)^{1/2};$

b) $\displaystyle < \left(\int_X \phi dE \right) \xi, \eta >= \int_X \phi dE_{\xi,\eta}.$

From 1) we see that the domain $\mathcal{D}_\phi$ can be determined intrinsically in terms of the spectral measure E without requiring us to look at a specific decomposition of X. Moreover, from 2)a) we see that for $\xi \in \mathcal{D}_\phi, \eta \in \mathcal{H}$ we have the inequality

$$\left| \int \phi dE_{\xi,\eta} \right| \leq \|\eta\| \left(\int |\phi|^2 dE_\xi \right)^{1/2},$$

so that $\eta \mapsto \int \phi dE_{\xi,\eta}$ is a well-defined, bounded linear functional on $\mathcal{H}$. Moreover, by b), $(\int \phi dE)\xi$ is the unique element of $\mathcal{H}$ which satisfies

$$< \left(\int_X \phi dE \right) \xi, \eta >= \int_X \phi dE_{\xi,\eta}.$$

Hence we have the following more natural definition.

Definition 2.15. *Let $\phi : X \to \mathbb{C}$ be Ω-measurable, and E be a spectral measure on $(X, \Omega, \mathcal{H})$. Then for $\xi \in \mathcal{D}_\phi = \{\eta \in \mathcal{H} : \int |\phi|^2 dE_\eta < \infty\}$ we define $(\int \phi dE)\xi \in \mathcal{H}$ to be the unique vector satisfying*

$$< \left(\int \phi dE \right) \xi, \eta >= \int \phi dE_{\xi,\eta}$$

for all $\eta \in \mathcal{H}$. This defines a normal operator $\int \phi dE$ with domain $\mathcal{D}_\phi$.

Theorem 2.16 (The Spectral Theorem)**.** *Let N be a normal operator on a Hilbert space $\mathcal{H}$. Then there exists a unique spectral measure E on the Borel subsets of $\mathbb{C}$ supported on $\sigma(N)$ such that $N = \int z dE$ and such that if $G \neq \varnothing$ is open in $\sigma(N)$ then $E(G) \neq 0$. Moreover, we have the following properties.*

1) *For $A \in B(\mathcal{H})$, we have $AN \subseteq NA$ and $AN^* \subseteq N^*A$ if and only if $AE(S) = E(S)A$ for every Borel set S.*
2) *If $\phi : \mathbb{C} \to \mathbb{C}$ is Borel and if $\{X_n\}$ is an increasing sequence of Borel sets such that $\phi|_{X_n}$ is bounded for each $n \geq 1$, and if $E(X_n)$ converges to I in the SOT as $n \to \infty$, then $\mathfrak{M} = \bigcup_{n=1}^{\infty} E(X_n)\mathcal{H}$ is a core for $\int \phi dE$.*
3) *If $\{\phi_i\}$ is an increasing net of real-valued Borel functions such $\phi = \sup_i \phi_i$ is finite valued, then $(\int \phi dE)\xi = \lim_i (\int \phi_i dE)\xi$ for every $\xi \in \mathcal{D}_\phi$.*
4) *If ϕ is continuous, we have $\sigma(\int \phi dE) = \overline{\phi(\sigma(N))}$.*

The proof of the main statement of the Spectral Theorem can be found in [2]. We will just verify properties 1), 2), 3), and 4). First we consider what sort of functional calculus the spectral decomposition allows.

Theorem 2.17. *Let E be a spectral measure on the measurable space $(X, \Omega, \mathcal{H})$, and let $\Phi(X, \Omega)$ be the set of all Ω-measurable functions $\phi : X \to \mathbb{C}$. Define $\rho : \Phi(X, \Omega) \to \mathscr{C}(\mathcal{H})$ by $\rho(\phi) = \int_X \phi dE$. Then we have the following properties:*

1) $\rho(\overline{\phi}) = \rho(\phi)^*$;
2) $\rho(\phi\psi) \supseteq \rho(\phi)\rho(\psi)$ and $\mathcal{D}(\rho(\phi)\rho(\psi)) = \mathcal{D}_\psi \cap \mathcal{D}_{\phi\psi}$;
3) if ψ is bounded then $\rho(\phi)\rho(\psi) = \rho(\phi\psi)$;
4) $\rho(\phi)^*\rho(\phi) = \rho(|\phi|^2)$.

Proof.

Let $\phi, \psi \in \Phi(X, \Omega)$. First note that there exists a sequence of measurable sets $\{X_n\}$ such that $X = \cup_n X_n$, and the functions ϕ, ψ are bounded on each X_n. This follows after choosing partitions $\{Y_n\}, \{Z_n\}$ such that ϕ, ψ are respectively bounded, and then letting $\{X_n\}$ be the set of all non-empty intersections of the form $Y_n \cap Z_m$, which is clearly countable, partitions X, and for which both ϕ, ψ are bounded.

Let $\{\rho_n\}$ be the respective representations of $\Phi(X_n, \Omega_n)$ on $B(E(X_n)\mathcal{H})$, where Ω_n is the restriction of Ω to X_n, and where $B(E(X_n)\mathcal{H})$ is viewed as a subspace of $B(\mathcal{H})$ and ϕ_n, ψ_n the restrictions of ϕ, ψ to X_n. Let $\xi \in \mathcal{D}(\rho(\phi)\rho(\psi))$. Then

$$\begin{aligned}\infty &> \|\rho(\phi)\rho(\psi)\xi\|^2 \\ &= \sum_n \|\rho_n(\phi_n)[E(X_n)\rho(\psi)\xi]\|^2 \\ &= \sum_n \|\rho_n(\phi_n)[\rho_n(\psi_n)E(X_n)\xi]\|^2 \\ &= \sum_n \|\rho_n(\phi_n\psi_n)E(X_n)\xi]\|^2,\end{aligned}$$

which says that $\xi \in \mathcal{D}(\rho(\phi\psi))$ and $\rho(\phi\psi)\xi = \rho(\phi)\rho(\psi)\xi$. Hence $\rho(\phi\psi) \supseteq \rho(\phi)\rho(\psi)$, and $\mathcal{D}(\rho(\phi)\rho(\psi)) \subseteq \mathcal{D}(\rho(\psi)) \cap \mathcal{D}(\rho(\phi\psi))$. On the other hand, if $\xi \in \mathcal{D}(\rho(\psi)) \cap \mathcal{D}(\rho(\phi\psi))$ then the last two sums above are finite, and hence $\xi \in \mathcal{D}(\rho(\phi)\rho(\psi))$. Therefore 2) follows. Moreover, by the functional calculus for bounded operators we have $\rho_n(\phi)^* = \rho_n(\overline{\phi})$ so that

$$\rho(\phi)^* = \oplus_{n=1}^\infty \rho_n(\phi)^* = \oplus_{n=1}^\infty \rho_n(\overline{\phi}) = \rho(\overline{\phi}),$$

so we have 1). If ψ is bounded, then $\mathcal{D}_\psi = \mathcal{H}$ so that $\mathcal{D}(\rho(\phi)\rho(\psi)) = \mathcal{D}_{\phi\psi} = \mathcal{D}(\rho(\phi\psi))$ and so 3) follows. Lastly by 1) we have $\rho(\phi)^* = \rho(\overline{\phi})$ so that by 2), $\rho(\phi)^*\rho(\phi) = \rho(\overline{\phi})\rho(\phi) \subseteq \rho(|\phi|^2)$ and $\mathcal{D}(\rho(\phi)^*\rho(\phi)) = \mathcal{D}_\phi \cap \mathcal{D}_{|\phi|^2} = \mathcal{D}_{|\phi|^2}$, so the result follows.

□

By 2) we fail to have $\rho(\phi)\rho(\psi) = \rho(\phi\psi)$ precisely when the domain of $\rho(\psi)$ is not large enough. In other words, there has to be some vector $\xi \in \mathcal{H}$ such that $\int |\psi|^2 dE_\xi = \infty$

and $\int |\phi|^2|\psi|^2 dE_\xi < \infty$. For instance, this will happen when ϕ is a characteristic function of a bounded set and ψ is continuous and unbounded, so that ψ is unbounded, but $\phi\psi$ is bounded.

proof of the Spectral Theorem. As mentioned earlier, we will only verify properties 1), 2), 3), and 4) and assume the existence of a spectral decomposition $N = \int z dE$ of N.

1) Let $A \in B(\mathcal{H})$. Choosing an integer $n \geq 1$, let $X_n = \{\alpha \in \mathbb{C} : n-1 \leq |\alpha| < n\}$. Then $N|_{E(X_n)\mathcal{H}}$ is bounded, so that by the bounded version of the Spectral Theorem, we have

$$AN|_{E(X_n)\mathcal{H}} = N|_{E(X_n)\mathcal{H}}A,$$

and

$$A(N|_{E(X_n)\mathcal{H}})^* = (N|_{E(X_n)\mathcal{H}})^*A$$

if and only if

$$AE(S \cap X_n) = E(S \cap X_n)A$$

for every Borel set $S \subseteq \mathbb{C}$. Since $N = \oplus_{n=1}^{\infty} N|_{E(X_n)\mathcal{H}}$ and $E(S) = \oplus_{n=1}^{\infty} E(S \cap X_n)$, property 1) follows.

2) Let ϕ be a Borel function on $\mathbb{C}$, let $\{X_n\}$ be a sequence of Borel sets such that $\phi|_{X_n}$ is bounded and such that $E(X_n)$ converges to I in the SOT as $n \to \infty$. If $\xi \in \mathcal{D}_\phi$, then

$$\xi = \lim_{n\to\infty} E(X_n)\xi,$$

and by 2) and 3) of Theorem 2.17 we have

$$\begin{aligned}\left(\int \phi dE\right)\xi &= \lim_{n\to\infty} E(X_n)\left(\int \phi dE\right)\xi \\ &= \lim_{n\to\infty}\left(\int \phi dE\right)E(X_n)\xi,\end{aligned}$$

so that $\cup_{n=1}^{\infty} E(X_n)\mathcal{H}$ is a core for $\int \phi dE$.

3) Let $\{\phi_i\}$ be a bounded, increasing net of real-valued Borel functions such $\phi = \sup_i \phi_i$ is finite valued. By the Monotone Convergence Theorem, for $\xi \in \mathcal{H}$ we have

$$\begin{aligned}0 &= \lim_i \int (\phi - \phi_i)^2 dE_\xi \\ &= \lim_i < \left(\int (\phi - \phi_i) dE\right)^2 \xi, \xi > \\ &= \lim_i \| \left(\int \phi dE - \int \phi_i dE\right)\xi\|^2,\end{aligned}$$

and the result follows.

4) Let ϕ be a continuous function, and for $n \geq 1$ let $X_n = \{\alpha \in \mathbb{C} : n-1 \leq |\alpha| < n\}$. Then we have $N = \oplus_{n=1}^{\infty} N|_{E(X_n)\mathcal{H}}$. For each n, by the Spectral Mapping Theorem we have $\sigma(\phi(N|_{E(X_n)\mathcal{H}})) = \phi(X_n)$. Clearly $\sigma(\phi(N|_{E(X_n)\mathcal{H}})) \subseteq \sigma(\phi(N))$ so that

$$\sigma(\phi(N)) \supseteq [\cup_{n=1}^{\infty} \phi(X_n)]^- = [\phi(X)]^-.$$

On the other hand, suppose that for some $\lambda \in \mathbb{C}$, there exists $\delta > 0$ such that $|\lambda - \alpha| > \delta$ whenever $\alpha \in \phi(X)$. Then if $\xi \in \mathcal{D}_\phi$ we have

$$\begin{aligned} \left\| \left(\int \phi dE - \lambda \right) \xi \right\|^2 &= \int |\phi - \lambda|^2 dE_\xi \\ &\geq |\lambda|^2 \|\xi\|. \end{aligned}$$

Therefore, $\int \phi dE - \lambda$ is injective and has close range. Moreover, by part 1) of Lemma 2.17 and part 4) of Lemma 2.3 we have $[\text{ran } (\int \phi dE - \lambda)]^\perp = \ker (\int \phi dE - \lambda)^*$. But $\int \phi dE - \lambda$ is normal, so that by Lemma 2.11, we have $\ker (\int \phi dE - \lambda)^* = \ker (\int \phi dE - \lambda) = \{0\}$. By Lemma 2.5, $\int \phi dE - \lambda$ is boundedly invertible. Therefore $\lambda \notin \sigma(\int \phi dE)$, completing the proof.

□

We state the following unbounded version of the Fuglede-Putnam Theorem.

Theorem 2.18. *If A is a bounded operator, and N, M are normal operators such that $AN \subseteq MA$, then $AN^* \subseteq M^*A$.*

In particular, 1) of the Spectral Theorem can be replaced by the following:

1′) for $A \in B(\mathcal{H})$, we have $AN \subseteq NA$ if and only if $AE(S) = E(S)A$ for every Borel set S.

We now begin the study of the relation between unbounded operators and von Neumann algebras.

Definition 2.19. *We say a closed, densely defined operator A is* ***affiliated*** *with a von Neumann algebra $\mathcal{M}$ if for all $T \in \mathcal{M}'$ we have $AT = TA$.*

Of course if A is bounded this says precisely that $A \in \mathcal{M}$. The next lemma clarifies to what extent A belongs to $\mathcal{M}$ in the case where A is normal and unbounded.

Lemma 2.20. *Let N be a normal operator affiliated to a von Neumann algebra $\mathcal{M}$ with spectral decomposition $N = \int z dE$. If $\phi : \mathbb{C} \to \mathbb{C}$ is Borel we have that $\int \phi dE$ is affiliated with $\mathcal{M}$.*

Proof.

Claim: If A is closed, $\mathfrak{M}$ is a core for A, and $T \in B(\mathcal{H})$ is invertible such that $TA\xi = AT\xi$ and $T^{-1}A\xi = AT^{-1}\xi$ for every $\xi \in \mathfrak{M}$ then $TA = AT$ and $T^{-1}A = AT^{-1}$.

Let $\xi \in \mathcal{D}(A)$ and let $\{\xi_n\}$ be a sequence in $\mathfrak{M}$ converging to ξ in the graph norm of A. Then

$$\lim_n AT\xi_n = \lim_n TA\xi_n = TA\xi,$$

so that $T\xi \in \mathcal{D}(A)$ and $AT\xi = TA\xi$. Therefore $TA \subseteq AT$. On the other hand, if $\xi \in \mathcal{D}(AT) = T^{-1}\mathcal{D}(A)$ so that $\xi = T^{-1}\eta$ for some $\eta \in \mathcal{D}(A)$, then choosing a sequence $\{\eta_n\}$ in $\mathfrak{M}$ converging to η in the graph norm of A we have

$$\lim_n A(T^{-1}\eta_n) = \lim_n T^{-1}A\eta_n = T^{-1}A\eta = T^{-1}AT\xi.$$

Hence $\xi \in \mathcal{D}(A)$ and $A\xi = T^{-1}AT\xi$ so that $TA\xi = AT\xi$. Therefore, $TA = AT$ and by symmetry we also have $T^{-1}A = AT^{-1}$.

Returning to the normal operator N, we define for each $n \geq 1, X_n = \{z \in \mathbb{C} : |\phi(z)| \leq n\}$. Then by 2) of the Spectral Theorem $\mathfrak{M} = \cup_{n=1}^{\infty} E(X_n)\mathcal{H}$ is a core for $\int \phi dE$. Therefore, by the claim, to show that $\int \phi dE$ is affiliated with $\mathcal{M}$ it suffices to show that for any unitary $U \in \mathcal{M}'$ and for every $\xi \in \mathfrak{M}$ we have $U(\int \phi dE)\xi = (\int \phi dE)U\xi$. But $UN = NU$ so that by the unbounded Fuglede-Putnam Theorem $UN^* \subseteq N^*U$, and hence U commutes with the spectral projections of N. Then writing $\xi = E(X_n)\xi$, setting E_n to be the restricted spectral projection on X_n and $\phi_n = \phi|_{X_n}$ it follows from the bounded Spectral Theorem that $U(\int \phi_n dE_n) = (\int \phi_n dE_n)U$, proving the lemma.

□

As an application we now give the polar decomposition for unbounded, closed operators. We start with two lemmas, the first of which recovers some of the usual anticommutation of the adjoint operation, and the second gives the existence and uniqueness of n^{th} roots of positive operators.

Lemma 2.21. *Let A, C be closed operators on $\mathcal{H}$, and $B \in B(\mathcal{H})$. If $A = BC$ then $A^* = C^*B^*$.*

Proof. Let $\xi \in \mathcal{D}(A^*)$. Then for $\eta \in \mathcal{D}(A) = \mathcal{D}(C)$ we have

$$< A\eta, \xi >=< BC\eta, \xi >=< C\eta, B^*\xi >,$$

so that $B^*\xi \in \mathcal{D}(C^*)$ and $C^*(B^*\xi) = A^*\xi$. Hence $A^* \subseteq C^*B^*$. On the other hand, the exact same calculation says that if $\xi \in \mathcal{D}(C^*B^*)$ and $\eta \in \mathcal{D}(A)$ then $\xi \in \mathcal{D}(A^*)$ and $A^*\xi = C^*B^*\xi$, and we have the reverse inclusion.

□

Lemma 2.22. *If A is a positive self-adjoint operator then for each integer $n > 1$ there exists a unique positive self-adjoint operator B such that $A = B^n$.*

Proof. Let $A = \int z dE$ be the spectral decomposition of A. Since $\sigma(A) \subseteq [0, \infty)$, the existence follows by letting $B = \int z^{1/n} dE$. On the other hand, if C is another positive self-adjoint operator such that $A = C^n$, then since $AC = C^n C = CC^n = CA$, part 1) of the Spectral Theorem implies that C commutes with the spectral projections of A. In particular, for each $r > 0$ we have $CE([0, r]) = E([0, r])C$ and so

$$(C|_{E([0,r])\mathcal{H}})^n = (C^n)|_{E([0,r])\mathcal{H}} = A|_{E([0,r])\mathcal{H}}.$$

By the uniqueness of positive nth root for bounded operators we have $C|_{E([0,r])\mathcal{H}} = B|_{E([0,r])\mathcal{H}}$. By 3) of the Spectral Theorem, $\cup_{r=1}^{\infty} E([0, r])\mathcal{H}$ is a core for B so that $B \subseteq C$. But then $C = C^* \subseteq B^* = B$ and hence $B = C$.

□

Theorem 2.23 (The Polar Decomposition Theorem)**.** *Let A be a closed operator on $\mathcal{H}$ and write $|A| = (A^*A)^{1/2}$. Then there exists a unique partial isometry V such that $A = V|A| = |A^*|V$. If $A = UB$ where B is positive, self-adjoint and U is a partial isometry with initial space $\overline{\text{ran }(B)}$ then $U = V$ and $B = |A|$. Moreover, if A is affiliated with some von Neumann algebra $\mathcal{M}$, then $|A|$ is also affiliated with $\mathcal{M}$ and $V \in \mathcal{M}$.*

Proof. Define an operator $V_0 : |A|\mathcal{D}(A^*A) \to A\mathcal{D}(A^*A)$ by $V_0|A|\xi = A\xi$. Since

$$\||A|\xi\|^2 =< |A|\xi, |A|\xi >=< A^*A\xi, \xi >= \|A\xi\|^2,$$

it follows that V_0 is a well-defined isometry, and so extends to a partial isometry V with initial space $\overline{\text{ran }(|A|)}$ and final space $\overline{\text{ran }(A)}$. It has already been shown that $\mathcal{D}(A^*A)$ is a core for A, and by 2) of the Spectral Theorem it is easy to see that it is also a core for $|A|$. If $\xi \in \mathcal{D}(A)$ we can choose a sequence $\{\xi_n\}$ in $\mathcal{D}(A^*A)$ converging in the graph norm of A to ξ. Then

$$\lim_{n\to\infty} |A|\xi_n = \lim_{n\to\infty} V^*A\xi_n = V^*A\xi,$$

and since $|A|$ is closed, we have $\xi \in \mathcal{D}(A)$ and $|A|\xi = V^*A\xi$, so that $V|A|\xi = A\xi$. Therefore $A \subseteq V|A|$. On the other hand, if $\xi \in \mathcal{D}(|A|)$, and we choose a sequence $\{\xi_n\}$ converging to ξ in the graph norm of $|A|$ so that

$$\lim_{n\to\infty} A\xi_n = \lim_{n\to\infty} V|A|\xi_n = V|A|\xi.$$

Since A is closed we have $\xi \in \mathcal{D}(A)$ and $A\xi = V|A|\xi$ so that $A \supseteq V|A|$. Therefore $A = V|A|$ and by construction V is unique. By Lemma 2.21 we have $A^* = |A|V^*$ and so $AA^* = V(A^*A)V^*$. But since we also have $(V|A|V^*)^2 = V(A^*A)V^*$ the uniqueness of positive self-adjoint square roots we have $|A^*| = V|A|V^* = VA^*$ so that $A^* = V^*|A^*|$. Applying Lemma 2.21 again we have $A = |A^*|V$.

If we also have $A = UB$ where B is positive self-adjoint operator and U is a partial isometry with initial space $\overline{\text{ran } (B)}$, then by Lemma 2.21 we have $A^* = BU^*$ and by assumption on U we have $A^*A = BU^*UB = B^2$. By uniqueness of positive self-adjoint square roots we have $B = |A|$, and so we must also have $U = V$, giving uniqueness of the polar decomposition.

Now suppose that A is affilated with a von Neumann algebra $\mathcal{M}$, and let $U \in \mathcal{M}'$ be a unitary. Then $A = UAU^* = (UVU^*)(U|A|U^*)$. But $U|A|U^*$ is a positive self-adjoint operator and UVU^* is a partial isometry with initial projection $(UVU^*)^*(UVU^*) = UV^*VU^*$ which has range $U\overline{\text{ran } (|A|)}$, so coincides with the range projection of $U|A|U^*$. By uniqueness of the polar decomposition. we have $U|A|U^* = |A|$ and $UVU^* = V$, so that $|A|$ is affiliated with $\mathcal{M}$ and $V \in \mathcal{M}$.

□

We will mention briefly here the use of conjugate linear operators. If A is a conjugate linear operator, we can linearize it by considering it as an operator from $\mathcal{H}$ to the dual Hilbert space $\mathcal{H}^*$. To be more precise, we define $\mathcal{D}(A^*)$ to be the set of $\xi \in \mathcal{H}$ such that the conjugate-linear functional $\eta \mapsto < A\eta, \xi >$ is bounded on $\mathcal{D}(A)$, and define $A^*\xi \in \mathcal{H}$ to be the unique vector such that for all $\eta \in \mathcal{D}(A)$

$$< A\eta, \xi > = \overline{< \eta, A^*\xi >} = < A^*\xi, \eta > .$$

Then A^* is a closed conjugate-linear operator, and we note furthermore that A^*A is a self-adjoint linear operator on $\mathcal{H}$. We define a conjugate linear partial isometry to be a conjugate-linear operator V that linearizes to a partial isometry.

Theorem 2.24. *If A is a closed conjugate-linear operator on $\mathcal{H}$, then the conclusions of the Polar Decomposition Theorem hold with the linear partial isometry V replaced by a conjugate-linear partial isometry.*

We turn now to bounded operators to offer a more generalized polar decomposition for operators in a von Neumann algebra.

Theorem 2.25 (The Generalized Polar Decomposition Theorem)**.** *Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$.*

1) If $x, y \in \mathcal{M}$ such that $y^*y \leq x^*x$ then there exists a unique $s \in \mathcal{M}$ such that $y = sx$, and ker $s \supseteq$ (ran $x)^\perp$. Moreover, $\|s\| \leq 1$.

2) Let $\{x_i\}_{i \in I}$ be a family in $\mathcal{M}$ such that $\sum_i x_i^* x_i$ converges to an operator $a \in \mathcal{M}$ in the SOT. If we let $s_i \in \mathcal{M}$ be as in 1) such that $x_i = s_i a^{1/2}$, then $\sum_i s_i^* s_i$ converges in the SOT to the range projection p of a.

Proof.

1) Define a map $s_0 : \text{ran } x \to \text{ran } y$ by $s_0(x\xi) = y\xi$. The map is well-defined since if $x\xi = 0$, then $\|y\xi\|^2 =< y^*y\xi, \xi >\leq< x^*x\xi, \xi >= 0$. Moreover, this calculation implies that s_0 is bounded and that $\|s_0\| \leq 1$. We then extend this map by continuity onto the closure of ran x and finally extend it to a bounded operator on $\mathcal{H}$ by setting $s \equiv 0$ on $(\text{ran } x)^\perp$. The uniqueness is by construction, and it remains to prove that $s \in \mathcal{M}$. Now if $u \in \mathcal{M}'$ is unitary, then $y = uyu^* = usxu^* = (usu^*)x$. Since the range projection p of x lies in $\mathcal{M}$, the subspace $(1-p)\mathcal{H}$ is invariant under u^*, so $usu^*(1-p)\mathcal{H} = \{0\}$. By uniqueness we have $usu^* = s$, so that $s \in \mathcal{M}$.

2) For each finite subset $J \subseteq I$ set $p_J = \sum_{i\in J} s_i^* s_i$. Then $\{p_J\}$ is an increasing net in $\mathcal{M}^+$, and moreover, if $\xi \in \mathcal{H}, \eta = a^{1/2}\xi$ we have

$$\begin{aligned}
< p_J\eta, \eta > &=< a^{1/2} \sum_{i\in J} s_i^* s_i a^{1/2}\xi, \xi > \\
&=< \sum_{i\in J} x_i^* x_i \xi, \xi > \\
&\leq< a\xi, \xi > \\
&=< p\eta, \eta > .
\end{aligned}$$

Since $s_i(\text{ran } a^{1/2})^\perp = 0$, it follows that $p_J \leq p$. Hence $\{p_J\}$ converges to some $p_0 \in \mathcal{M}^+$ in the SOT such that $p_0 \leq p$. But then if we let ξ, η be as above, then

$$\begin{aligned}
< p_0\eta, \eta > &=< a^{1/2} \sum_{i\in I} s_i^* s_i a^{1/2}\xi, \xi > \\
&=< \sum_{i\in J} x_i^* x_i \xi, \xi > \\
&=< a\xi, \xi > \\
&=< p\eta, \eta > .
\end{aligned}$$

Therefore $p_0 = p$.

□

We end the exposition of unbounded operators by giving an application of unbounded operators to one-parameter unitary groups.

Theorem 2.26. *Let A be a self-adjoint operator, and set $U(t) = exp(itA)$ for $t \in \mathbb{R}$. Then we have the following:*

1) *the function $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ is a group homomorphism and is continuous in the SOT;*
2) *if $\xi \in \mathcal{D}(A)$ then $\lim_{t\to 0} \frac{1}{t}[U(t)\xi - \xi] = iA\xi$. Moreover, if $\xi \in \mathcal{H}$ and this limit exists, then $\xi \in \mathcal{D}(A)$. Consequently $\mathcal{D}(A)$ is invariant under each $U(t)$.*

We have the following relation to von Neumann algebras.

Corollary 2.27. *Let A be a self-adjoint operator and for each $t \in \mathbb{R}$ let $U(t) = exp(itA)$. Let $\mathcal{M}$ be the von Neumann algebra generated by $\{U(t)\}_{t\in\mathbb{R}}$. Then A is affiliated with $\mathcal{M}$.*

Proof. Let $\xi \in \mathcal{D}(A)$, and let $T \in \mathcal{M}'$. By 2) of Theorem 2.26 we have that $iA\xi = \lim_{t\to 0} \frac{1}{t}[U(t) - 1]\xi$ and since $U(t), T$ commute,

$$\lim_{t\to 0} \frac{1}{t}[U(t) - 1]T\xi = T \lim_{t\to 0} \frac{1}{t}[U(t) - 1]\xi.$$

Hence by 2) of Theorem 2.26 we have that $T\xi \in \mathcal{D}(A)$ and $AT\xi = TA\xi$. Hence $AT \subseteq TA$, and the reverse inclusion follows automatically since $\mathcal{D}(TA) = \mathcal{D}(A)$. Therefore A is affiliated with $\mathcal{M}$.

□

It is a remarkable fact is that the converse of Theorem 2.26 also holds.

Theorem 2.28 (Stone's Theorem)**.** *If $\{U(t)\}_{t\in\mathbb{R}}$ is a one-parameter group of unitaries which is continuous in the SOT, then there exists a self-adjoint operator A such that $U(t) = exp(itA)$. Moreover, if $\mathcal{M}$ is a von Neumann algebra acting on $\mathcal{H}$, then A is affiliated with $\mathcal{M}$ if and only it $U(t) \in \mathcal{M}$ for all t.*

The operator A given above is often called the *infinitessimal generator* of $\{U_t\}$ or just the *generator*. We note the following result relating the continuity of the unitary group, and the boundedness of the operator.

Theorem 2.29. *If A is a self-adjoint operator, then A is bounded if and only if the family of unitaries $\{exp(itA)\}_{t\in\mathbb{R}}$ is norm continuous.*

We will be applying these results in the following way. If A is a self-adjoint, positive, injective operator, then we can define the operator $\log(A)$, where log is the principal branch of the logarithm, and we define $\log(0) = 0$. This will yield a well-defined self-adjoint operator since $\ker(A) = \{0\}$. Then we have $A^{it} = exp(it\log(A))$ by the composition rule and the family $\{A^{it}\}$ is a one-parameter group of unitaries which is continuous in the SOT. Lastly, we give a result about the analyticity of such automorphism groups.

Lemma 2.30. *Let H be a positive, self-adjoint, injective operator on a Hilbert space $\mathcal{H}$. For a vector $\xi \in \mathcal{H}$ the following are equivalent:*

1) $\xi \in \mathcal{D}(H)$;
2) the function $t \mapsto H^{it}\xi$ can be extended from $\mathbb{R}$ to the closed strip $\overline{\mathbb{D}} \subseteq \mathbb{C}$ bounded by $-i$ and 0, yielding a bounded, continuous function which is holomorphic on the interior.

Proof.

1)$\Longrightarrow$2). The extension we are going to be considering is the function $\xi(\alpha) = H^{i\alpha}\xi$. If $H = \int \lambda dE(\lambda)$ is the spectral decomposition then

$$\int |\lambda^{i\alpha}|^2 dE_\xi \le \int |\lambda|^2 dE_\xi < \infty$$

which says that $\xi \in \mathcal{D}(H^{i\alpha})$ and $\|H^{i\alpha}\xi\| \le \|H\xi\|$ so that $\xi(\cdot)$ is well-defined and bounded. Moreover, if $\beta \to \beta_0$ then

$$\|H^{i\beta}\xi - H^{i\beta_0}\xi\|^2 = \int |\lambda^{i\beta} - \lambda^{i\beta_0}|^2 dE_\xi(\lambda).$$

This converges to 0 by the Lebesgue dominated convergence theorem with dominating function $\lambda \mapsto |\lambda|^2$.

Set $\mathfrak{M} = \bigcup_{n=1}^{\infty} E[1/n, n]\mathcal{H}$. Then $E[1/n, n] \to I$ SOT so $\mathfrak{M}$ is dense, and it is contained in the domain of $\phi(H)$ for every continuous function ϕ on $\mathbb{C}$. In particular, if $\eta \in E[1/n, n]\mathcal{H}, \zeta \in \mathcal{H}$ we have

$$< H^{i\beta}\eta, \zeta >= \int_{1/n}^{n} \lambda^{i\beta} dE_{\eta,\zeta}(\lambda),$$

which says that the function $\eta(\beta) = H^{i\beta}\eta$ is entire. Then if α belongs to the the interior of $\overline{\mathbb{D}}$ we have

$$\begin{aligned} < \xi(\alpha), \eta > &=< H^{i\beta}\xi, \eta > \\ &=< \xi, H^{-i\overline{\alpha}}\eta > \\ &=< \xi, \eta(-\overline{\alpha}) >, \end{aligned}$$

so $\xi(\cdot)$ is holomorphic on the interior.

2)$\Longrightarrow$1). Suppose that the function $t \in \mathbb{R} \mapsto H^{it}\xi$ has such an extension to a function F on $\overline{\mathbb{D}}$. By the implication 2)$\Longrightarrow$1) we have that for $\eta \in \mathcal{D}(H)$, the function $\eta(\alpha) = H^{i\alpha}\eta$ is defined with domain $\overline{D}(H)$, is bounded, continuous and holomorphic in the interior of its domain. Then the two function g_1, g_2 defined on $\overline{\mathbb{D}}$ by

$$g_1(\alpha) =< F(\alpha), \eta >, \qquad g_2(\alpha) =< \xi, \eta(-\overline{\alpha}) >,$$

are bounded, continuous, homorphic on the interior of their domain, and agree on the real line $\mathbb{R}$. By the Schwarz Reflection Principle we infer that they must agree everywhere. In particular we have

$$< F(i), \eta >=< \xi, H\eta > .$$

Therefore $\xi \in \mathcal{D}(H)$.

$\square$

3 Weights on a von Neumann Algebra

In this Section we consider a certain unbounded analogue of a positive linear functional on a von Neumann algebra, which is called a weight. We will first apply the GNS construction using weights. After mentioning some of the highlights of the theory of traces, we will study the topological properties of the adjoint operation on the GNS representation space. The material for this Section is from Chapter 7 of [13], except for the subsection on traces which is from Chapter 5 of [12] and Theorem 3.5 which is from Chapter 1 of [8].

Definition 3.1. *A* ***weight*** *on a von Neumann algebra $\mathcal{M}$ is a function $\phi : \mathcal{M}^+ \to [0, \infty]$ satisfying, for $x, y \in \mathcal{M}^+, \lambda > 0$,*

$$\phi(x + y) = \phi(x) + \phi(y); \qquad \phi(\lambda x) = \lambda\phi(x).$$

If we have that for any $x \in \mathcal{M}$,

$$\phi(x^* x) = \phi(xx^*),$$

then we say that ϕ is a ***trace****. In addition we say that:*

a) ϕ is ***faithful*** *if $\phi(x) = 0$ only if $x = 0$;*

b) ϕ is ***normal*** *if $\phi(\sup_i x_i) = \sup_i \phi(x_i)$ whenever $\{x_i\}$ is an increasing bounded net;*

c) ϕ is ***finite*** *if $\phi(1) < \infty$.*

d) ϕ is ***semifinite*** *if the *-algebra generated by the set $\{x \in \mathcal{M}^+ : \phi(x) < \infty\}$ generates $\mathcal{M}$.*

Let ϕ be finite. Since for $x \in \mathcal{M}^+$ we have $x \leq \|x\|1$, it follows from additivity that $\phi(x) < \infty$. In this case, the normality condition is equivalent to saying that ϕ can be uniquely extended to an element of the predual $\mathcal{M}_*$. If ϕ, ω are weights on $\mathcal{M}$ then we write $\omega \leq \phi$ if for all $x \in \mathcal{M}^+$ we have $\omega(x) \leq \phi(x)$. The next Theorem says that in general, a normal weight is just the pointwise limit of an increasing net of positive elements from $\mathcal{M}_*^+$.

Theorem 3.2. *For a weight ϕ on a von Neumann algebra $\mathcal{M}$ the following are equivalent:*

1) ϕ is normal;
2) Setting $\Phi = \{\omega \in \mathcal{M}_*^+ : \omega \leq \phi\}$ we have for $x \in \mathcal{M}^+$

$$\phi(x) = \sup_{\omega \in \Phi} \omega(x).$$

We now begin to study the GNS representation of $\mathcal{M}$ with respect to a weight ϕ. For this purpose we must first extend the weight ϕ to a positive linear functional, also denoted by ϕ, acting on a *-subalgebra of $\mathcal{M}$. We consider the following spaces:

$$n_\phi = \{x \in \mathcal{M} : \phi(x^*x) < \infty\};$$
$$m_\phi = n_\phi^* n_\phi = \left\{ \sum_i x_i^* y_i : x_i, y_i \in n_\phi \right\}.$$

Lemma 3.3. *The set n_ϕ is a left ideal, and m_ϕ is hereditary *-subalgebra linearly spanned by $m_\phi^+ = \{x \in \mathcal{M}^* : \phi(x) < \infty\}$. Hence ϕ extends uniquely to a positive linear functional on m_ϕ.*

Proof. If $a \in \mathcal{M}, x \in n_\phi$ then the inequality $(ax)^*ax \le \|a\|^2 x^*x$ shows that n_ϕ is a left ideal. Suppose $z = \sum_{j=1}^n x_j^* y_j$ with $x_j, y_j \in [n_\phi \bigcap \mathcal{M}^+]$. Then by polarizing,

$$\begin{aligned} z &= \frac{1}{2}(z + z^*) \\ &= \frac{1}{8} \sum_{j=1}^n \sum_{k=0}^3 [i^k (x_j + i^k y_j)^* (x_j + i^k y_j) + (-i)^k (x_j + i^k y_j)^* (x_j + i^k y_j)] \\ &= \frac{1}{4} \sum_{j=1}^n [(x_j + y_j)^*(x_j + y_j) - (x_j - y_j)^*(x_j - y_j)] \\ &\le \frac{1}{4} \sum_{j=1}^n (x_j + y_j)^*(x_j + y_j), \end{aligned}$$

so $\phi(z) < \infty$ and it follows that $m_\phi^+ \subseteq \{x \in \mathcal{M}^+ : \phi(x^*x) < \infty\}$. The reverse inclusion is clear, so we have equality. For $z \in m_\phi$, the equality

$$z = \frac{1}{4} \sum_{j=1}^n \sum_{k=0}^3 i^k (x_j + i^k y_j)^* (x_j + i^k y_j)$$

implies that m_ϕ is spanned by its positive elements and the lemma follows.

□

Now set $N_\phi = \{x \in \mathcal{M} : \phi(x^*x) = 0\}$, and let $q_\phi : n_\phi \to n_\phi / N_\phi$ be the canonical quotient map. Define an inner product on $q_\phi(n_\phi)$ by $< q_\phi(x), q_\phi(y) >= \phi(y^*x)$. The representation space $\mathcal{H}_\phi$ is the completion of $q_\phi(n_\phi)$ and we obtain a representation $\pi_\phi : \mathcal{M} \to B(\mathcal{H}_\phi)$ by $\pi_\phi(a) q_\phi(x) = q_\phi(ax)$.

Definition 3.4. *Let π be a representation of a von Neumann algebra $\mathcal{M}$ on a Hilbert space $\mathcal{H}$. Then we say that π is* ***normal*** *if it is σ-weakly continuous. Equivalently, whenever $\{x_i\}$ is a bounded increasing net of self-adjoint elements in $\mathcal{M}$ we have*

$$\pi(\sup_i x_i) = \sup_i \pi(x_i).$$

Theorem 3.5. *If π is a non-degenerate normal representation of a von Neumann algebra $\mathcal{M}$, then $\pi(\mathcal{M})$ is a von Neumann algebra.*

We now have the following result on the representation π_ϕ.

Proposition 3.6. *If ϕ is a semifinite normal weight, then $(\pi_\phi, \mathcal{H}_\phi)$ is a non-degenerate, normal representation. In particular, $\pi_\phi(\mathcal{M})$ is a von Neumann algebra. If ϕ is faithful, then so is π_ϕ.*

Proof. Since we have $\pi_\phi(1) = 1_{\mathcal{H}_\phi}, \pi_\phi$ is non-degenerate. Let $\{x_i\}$ be an increasing net in $\mathcal{M}^+$ with $x = \sup_i x_i$, let $y \in n_\phi$ and consider the functional $\omega_y(T) =< Tq_\phi(y), q_\phi(y) >$ which belongs to $\pi_\phi(\mathcal{M})_*^+$. Then

$$\begin{aligned}
\lim_i \omega_{q_\phi(y)} \circ \pi_\phi(x - x_i) &= \lim_i < \pi_\phi(x - x_i) q_\phi(y), q_\phi(y) > \\
&= \lim_i < q_\phi((x - x_i)^{1/2} y), q_\phi((x - x_i)^{1/2} y) > \\
&= \lim_i \phi(y^*(x - x_i)y) \\
&= 0.
\end{aligned}$$

Hence $\omega_{q_\phi(y)} \circ \pi_\phi$ is normal, and since the set $\{\omega_y : y \in n_\phi\}$ is total in $\pi_\phi(\mathcal{M})_*^+$ it follows that π_ϕ is normal.

If ϕ is faithful and $0 \neq x \in \mathcal{M}$, and if we choose a net $\{y_i\}$ in n_ϕ which converges to 1 in the SOT, then $(xy_i)^*(xy_i)$ converges weakly to x^*x. In particular, there exists some i such that $y_i^* x^* x y_i \neq 0$. Then

$$\|\pi_\phi(x) q_\phi(y_i)\|^2 = \phi(y_i^* x^* x y_i) > 0,$$

so π_ϕ is faithful.

□

Before continuing the general discussion we now summarize some of the key features of the theory of traces. Since the purpose is mainly to motivate the development of the theory of weights, full proofs will not be given.

Let τ be an faithful, normal, semifinite (fns) trace on $\mathcal{M}$ and let n_τ be as before. If $x \in n_\tau$, we have

$$\tau(xx^*) = \tau(x^*x) < \infty,$$

so that $x^* \in n_\tau$. Therefore n_τ is a self-adjoint left ideal, and consequently, also a right ideal. It follows that we can consider an (anti-homomorphic) representation π'_τ of $\mathcal{M}$ on $\mathcal{H}_\tau$ given by

$$\pi'_\tau(a)q_\tau(x) = q_\tau(xa).$$

As before, this defines a bounded operator since

$$\|q_\tau(xa)\|^2 = \tau((xa)^*(xa)) = \tau((xa)(xa)^*) \leq \|a\|^2\tau(xx^*) = \|a\|^2\|q_\tau(x)\|^2.$$

Moreover, since

$$\|q_\tau(x^*)\|^2 = \tau(xx^*) = \tau(x^*x) = \|q_\tau(x)\|^2,$$

the involution on n_τ induced by the adjoint extends to a conjugate linear unitary on $\mathcal{H}_\tau$, denoted by J. For $a \in \mathcal{M}$, we have

$$(J\pi_\tau(a)J)q_\phi(x) = q_\tau((ax^*)^*) = q_\tau(xa^*) = \pi'_\tau(a^*)q_\phi(x)$$

so that $J\pi_\tau(a)J = \pi'_\tau(a^*)$. Hence J induces an anti-isomorphism between $\pi_\tau(\mathcal{M})$ and $\pi'_\tau(\mathcal{M})$, and in particular, $\pi'_\tau(\mathcal{M})$ is a von Neumann algebra. Moreover, we note that if $a, b \in \mathcal{M}$, $x \in n_\tau$, we have

$$\pi_\tau(a)\pi'_\tau(b)q_\tau(x) = q_\tau(axb) = \pi'_\tau(b)\pi_\tau(a)q_\phi(x),$$

so that the representations π_τ, π'_τ commute. Therefore $\pi'_\tau(\mathcal{M}) \subseteq \pi_\tau(\mathcal{M})'$. In fact, we also have the reverse inclusion. The above is summarized in the following theorem.

Theorem 3.7. *The representations π_τ, π'_τ of $\mathcal{M}$ obtained by left and right multiplication operators on the Hilbert space completion $\mathcal{H}_\tau$ of n_τ are faithful, normal and satisfy the following:*

$$J\pi_\tau(\mathcal{M})J = \pi_\tau(\mathcal{M})' = \pi'_\tau(\mathcal{M}).$$

The condition that a weight be a trace is quite restrictive, and the existence of a trace turns out to be dependent on the type of the algebra.

Theorem 3.8. *Let $\mathcal{M}$ be a von Neumann algebra. Then there exists a unique decomposition of $\mathcal{M}$ into a direct sum*

$$\mathcal{M} = \mathcal{M}_I \oplus \mathcal{M}_{II_1} \oplus \mathcal{M}_{II_\infty} \oplus \mathcal{M}_{III},$$

where each $\mathcal{M}_j$ is of type j.

We say that a von Neumann algebra $\mathcal{M}$ is **semifinite** if the type III summand is trivial. Suppose that $\mathcal{M}$ admits an fns trace τ. If we choose any projection $e \in \mathcal{M}$ such that $\tau(e) < \infty$, it follows that τ restricts to a faithful finite trace on the corner algebra $e\mathcal{M}e$. Then if we choose any isometry $u \in e\mathcal{M}e$, we have $0 = \tau(e - u^*u) = \tau(e - uu^*)$ so that $e = uu^*$. Therefore the algebra $e\mathcal{M}e$ is finite, so that e is a finite projection. On the other hand, if $e \in \mathcal{M}$ is an arbitrary projection, then the semifiniteness of τ ensures that there exists a non-zero element $x \in (e\mathcal{M}e)^+$ such that $\tau(x) < \infty$. If we let $x = \int \lambda dE(\lambda)$ be the spectral decomposition of x, and if we choose a number ϵ satisfying $0 < \epsilon < \|x\|$ we have that $\epsilon \cdot E[\epsilon, \|x\|] \leq x$, so that $\tau(E[\epsilon, \|x\|]) \leq \epsilon^{-1}\tau(x) < \infty$. Therefore $E[\epsilon, \|x\|] \leq e$ and $E[\epsilon, \|x\|]$ is a non-zero finite projection. Since e was arbitary, $\mathcal{M}$ is semifinite. In fact, the converse is also true.

Theorem 3.9. *A von Neumann algebra is semifinite if and only if it admits an fns trace.*

For a von Neumann algebra that does not admit a trace we can still obtain a similar picture to the one above, and which is in fact more useful. To begin, we state the following result from [13].

Theorem 3.10. *Every von Neumann algebra admits an fns weight.*

Now let $\mathcal{M}$ be a von Neumann algebra with an fns weight ϕ. We note that if ϕ is not a trace, the difference between the GNS representation $(\pi_\phi, \mathcal{H}_\phi, q_\phi)$ and that of a trace is that we no longer have a conjugate unitary J on $\mathcal{H}_\phi$ coming from the adjoint operation. However, we define $\mathcal{U}_\phi = q_\phi(n_\phi \cap n_\phi^*)$, and give it an involution $^\sharp$ by

$$q_\phi(x)^\sharp = q_\phi(x^*).$$

In the sequel we begin a more in depth study of this involution by showing that it has a closed extension S. This allows us to take the polar decomposition $S = J\Delta^{1/2}$, yielding as before a conjugation J which relates the represented algebra to its commutant, and more importantly a positive self-adjoint non-singular operator Δ, which will be used to define a one-parameter automorphism group on the von Neumann algebra.

We can now state the main theorem of this Section.

Theorem 3.11. *The pre-Hilbert space $\mathcal{U}_\phi = q_\phi(n_\phi \cap n_\phi^*)$ is dense in $\mathcal{H}_\phi$. We define an involution and product by*

$$q_\phi(x)^\sharp = q_\phi(x^*), \qquad\qquad q_\phi(x)q_\phi(y) = q_\phi(xy).$$

For $\xi \in \mathcal{U}_\phi$, denote by $\pi_\ell(\xi)$ the operator on the closure $\mathcal{H}_\phi$ extending left multiplication by ξ. Then the set of left multiplication operators generate $\pi_\phi(\mathcal{M})$. Moreover, the involution $^\sharp$ is a densely defined, closeable, conjugate linear operator on $\mathcal{H}_\phi$.

First we want to show that $\mathcal{U}_\phi$ is dense in $\mathcal{H}_\phi$. Since ϕ is semifinite the subalgebra $\pi_\phi(n_\phi \cap n_\phi^*)$ generates $\pi_\phi(\mathcal{M})$ so that in particular it acts non-degenerately on $\mathcal{H}_\phi$. But if $q_\phi(x) \in \mathcal{U}_\phi^\perp \cap q_\phi(n_\phi)$, then for any $y, z \in n_\phi$ we have

$$< \pi_\phi(y) q_\phi(x), q_\phi(z) >=< q_\phi(x), \pi_\phi(y)^* q_\phi(z) >=< q_\phi(x), q_\phi(y^* z) >= 0,$$

and so $q_\phi(x) = 0$. Therefore $\mathcal{U}_\phi^\perp = \{0\}$, and so the left multiplication operators $\pi_\ell(q_\phi(x))$ coincide with the operators $\pi_\phi(x)$. It now suffices to show that the involution is closeable. Recall that an operator is closeable if and only if it has a densely defined adjoint. We will use the characterization of normality given in Theorem 3.2 to show that the domain of the adjoint is large enough.

Theorem 3.12. *Set $E_\phi = \{\omega \in \mathcal{M}_*^+ : \omega \leq \lambda\phi$ for some $\lambda > 0\}$. Then for $\omega \in E_\phi$ there exists a vector $\eta_\omega \in \mathcal{H}_\phi$ and a positive operator $h_\omega \in \pi_\phi(\mathcal{M})'$ such that $h_\omega^{1/2} q_\phi(x) = \pi_\phi(x)\eta_\omega$ and $\omega(x) =< \pi_\phi(x)\eta_\omega, \eta_\omega >$.*

Proof. Let $\omega \in E_\phi$. Since $\omega \leq \lambda\phi$ for some $\lambda > 0$, this gives that for $x, y \in n_\phi$,

$$\begin{aligned} |\omega(y^* x)| &\leq \omega(x^* x)^{1/2} \omega(y^* y)^{1/2} \\ &\leq \lambda\phi(x^* x)^{1/2} \phi(y^* y)^{1/2}, \end{aligned}$$

so that there exists an operator $h_\omega \in B(\mathcal{H}_\phi)$ such that $< h_\omega q_\phi(x), q_\phi(y) >= \omega(y^* x)$. Moreover, for any $a \in \mathcal{M}$ we have

$$\begin{aligned} < h_\omega \pi_\phi(a) q_\phi(x), q_\phi(y) &= \omega(y^* a x) > \\ &= \omega((a^* y)^* x) \\ &=< h_\omega q_\phi(x), q_\phi(a^* y) > \\ &=< \pi_\phi(a) h_\omega q_\phi(x), q_\phi(y) >, \end{aligned}$$

so that $h_\omega \in \pi_\phi(\mathcal{M})'$. Now let $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$ denote the cyclic representation associated to ω. Then the same inequality $\omega \leq \lambda\phi$ implies that the function $q_\phi(x) \mapsto \pi_\omega(x)\xi_\omega$ is well-defined and extends to a bounded operator $t_\omega : \mathcal{H}_\phi \to \mathcal{H}_\omega$. That is, we have

$$\|\pi_\omega(x)\xi_\omega\|^2 = \omega(x^* x) \leq \lambda\phi(x^* x) = \lambda \|q_\phi(x)\|^2.$$

Note that for any $a \in \mathcal{M}$ we have

$$\begin{aligned} t_\omega \pi_\phi(a) q_\phi(x) &= t_\omega q_\phi(ax) \\ &= \pi_\omega(ax)\xi_\omega \\ &= \pi_\omega(a) t_\omega q_\phi(x), \end{aligned}$$

so that $t_\omega \pi_\phi(a) = \pi_\omega(a) t_\omega$. Also note that for $x \in n_\phi$

$$\begin{aligned} < t_\omega q_\phi(x), \pi_\omega(y)\xi_\omega > &=< \pi_\omega(x)\xi_\omega, \pi_\omega(y)\xi_\omega > \\ &= \omega(y^* x) \\ &=< h_\omega q_\phi(x), q_\phi(y) > \\ &=< q_\phi(x), h_\omega q_\phi(y) >, \end{aligned}$$

so $t_\omega^*(\pi_\omega(y)\xi_\omega) = h_\omega q_\phi(y)$ and $t_\omega^* t_\omega = h_\omega$. Now let $t_\omega = u_\omega h_\omega^{1/2}$ be the polar decomposition of t_ω. For $a \in \mathcal{M}$, the above commutation relation for t_ω gives

$$\begin{aligned} \pi_\omega(a) u_\omega h_\omega^{1/2} &= u_\omega h_\omega^{1/2} \pi_\phi(a) \\ &= u_\omega \pi_\phi(a) h_\omega^{1/2}, \end{aligned}$$

so that $\pi_\omega(a)u_\omega, u_\omega\pi_\phi(a)$ agree on $[h_\omega^{1/2}\mathcal{H}]$. But u_ω vanishes on $[h_\omega^{1/2}\mathcal{H}]^\perp$ and this space is invariant under $\pi_\phi(a)$ so that $u_\omega\pi_\phi(a)$ also vanishes on $[h_\omega^{1/2}\mathcal{H}]^\perp$. Hence, $\pi_\omega(a)u_\omega = u_\omega\pi_\phi(a)$.

Set $\eta_\omega = u_\omega^* \xi_\omega \in \mathcal{H}_\phi$. Then for $x \in n_\phi$, we have

$$\begin{aligned} \pi_\phi(x)\eta_\omega &= \pi_\phi(x) u_\omega^* \xi_\omega \\ &= (u_\omega \pi_\phi(x^*))^* \xi_\omega \\ &= (\pi_\omega(x^*) u_\omega)^* \xi_\omega \\ &= u_\omega^* \pi_\omega(x) \xi_\omega \\ &= u_\omega^* t_\omega q_\phi(x) \\ &= h_\omega^{1/2} q_\phi(x). \end{aligned}$$

Moreover, if $x = z^* y$ for $y, z \in n_\phi$, we have

$$\begin{aligned} < \pi_\phi(x)\eta_\omega, \eta_\omega > &=< \pi_\phi(y)\eta_\omega, \pi_\phi(z)\eta_\omega > \\ &=< h_\omega^{1/2} q_\phi(y), h_\omega^{1/2} q_\phi(z) > \\ &= \omega(x), \end{aligned}$$

which completes the proof.

□

proof of Theorem 3.11. Retaining the notation from Lemma 3.12 we have that for $x \in n_\phi \cap n_\phi^*, b \in \pi_\phi(\mathcal{M})', \omega_1, \omega_2 \in E_\phi$,

$$\begin{aligned} < q_\phi(x)^\sharp, h_{\omega_1}^{1/2} b \eta_{\omega_2} > &=< b^* h_{\omega_1}^{1/2} q_\phi(x^*), \eta_{\omega_2} > \\ &=< b^* \pi_\phi(x)^* \eta_{\omega_1}, \eta_{\omega_2} > \\ &=< b^* \eta_{\omega_1}, \pi_\phi(x) \eta_{\omega_2} > \\ &=< b^* \eta_{\omega_1}, h_{\omega_2}^{1/2} q_\phi(x) > \\ &=< h_{\omega_2}^{1/2} b^* \eta_{\omega_1}, q_\phi(x) > . \end{aligned}$$

Therefore letting F denote the adjoint of $\sharp$ we have that $h_{\omega_1}^{1/2} b\eta_{\omega_2} \in \mathcal{D}(F)$ and $F(h_{\omega_1}^{1/2} b\eta_{\omega_2}) = h_{\omega_2}^{1/2} b^*\eta_{\omega_1}$. Hence to show that $\sharp$ is closeable it suffices to show that the set

$$\{h_{\omega_1}^{1/2}\pi_\phi(\mathcal{M})'\eta_{\omega_2} : \omega_1, \omega_2 \in \Phi\}$$

is total in $\mathcal{H}_\phi$.

Claim: The net $\{h_\omega : \omega \in \Phi\}$ converges to 1 in the SOT. Hence for fixed $\omega_2 \in \Phi$, the set $\{h_{\omega_1}^{1/2}\pi_\phi(\mathcal{M})'\eta_{\omega_2} : \omega_1 \in \Phi\}$ is total in $\pi_\phi(\mathcal{M})'\eta_2$.

Since the map $\omega \mapsto h_\omega$ is additive, the net $\{h_\omega : \omega \in \Phi\}$ is increasing. Then by Theorem 3.2,

$$\begin{aligned}
\|q_\phi(x)\|^2 &= \phi(x^*x) \\
&= \sup_{\omega \in \Phi} \omega(x^*x) \\
&= \sup_{\omega \in \Phi} < h_\omega q_\phi(x), q_\phi(x) >,
\end{aligned}$$

proving the claim.

Claim: The set $\{\pi_\phi(\mathcal{M})\eta_\omega : \omega \in \Phi\}$ is total in $\mathcal{H}_\phi$.

Let e be the projection onto the closed span of $\{\pi_\phi(\mathcal{M})\eta_\omega : \omega \in \Phi\}$. Then $e \in \pi_\phi(\mathcal{M})$ and $(1-e)\eta_\omega = 0$ for every $\omega \in \Phi$. Now let $f \in \mathcal{M}$ be the projection such that $\pi_\phi(f) = 1 - e$. Then

$$\begin{aligned}
\phi(f) &= \sup_{\omega \in \Phi} \omega(f) \\
&= \sup_{\omega \in \Phi} < (1-e)\eta_\omega, \eta_\omega > \\
&= 0.
\end{aligned}$$

Since ϕ is faithful, $f = 0$ so $1 - e = 0$, which completes the proof of the theorem.

$\square$

4 Left Hilbert Algebras and Tomita's Theorem

In Section 3 we studied the GNS construction corresponding to an fns weight, and obtained the important result that the adjoint acting on the representation space is closeable. In this Section we continue the study of this representation space by considering the following abstract characterization. The material for this Section is from Chapter 6 of [13].

Definition 4.1. *Let $\mathcal{U}$ be a an associative algebra with a scalar-valued inner product $< \cdot, \cdot >$ and an involution denoted $^\sharp$. Let $\mathcal{H}$ be its Hilbert space completion. We say that $\mathcal{U}$ is a **left (right) Hilbert algebra** if it satisfies the following.*

1) For each $\xi \in \mathcal{U}$ the map $\eta \in \mathcal{U} \mapsto \xi\eta \in \mathcal{U}$ (respectively, $\eta \in \mathcal{U} \mapsto \eta\xi \in \mathcal{U}$) is continuous.
2) For $\xi, \eta, \zeta \in \mathcal{U}, < \xi\eta, \zeta >=< \eta, \xi^\sharp\zeta >$ (respectively, $< \eta\xi, \zeta >=< \eta, \zeta\xi^\sharp >$).
3) The involution $^\sharp$ is closable.
4) The set $\{\xi\eta : \xi, \eta \in \mathcal{U}\}$ is total in $\mathcal{H}$.

For each $\xi \in \mathcal{U}$ denote by $\pi_\ell(\xi)$ the extension of the left multiplication operator to $\mathcal{H}$. By property 2) we have $< \pi_\ell(\xi)\eta, \zeta >=< \eta, \pi_\ell(\xi^\sharp)\zeta >$ so that $\pi_\ell(\xi)^* = \pi_\ell(\xi^\sharp)$. Consequently $\pi_\ell(\mathcal{U}) = \{\pi_\ell(\xi) : \xi \in \mathcal{U}\}$ is a *-subalgebra of $B(\mathcal{H})$. Moreover, since the set $\{\xi\eta : \xi, \eta \in \mathcal{U}\}$ is total in $\mathcal{H}$, the left-multiplication operators act non-degenerately on $\mathcal{H}$, and so $\pi_\ell(\mathcal{U})'' = \pi_\ell(\mathcal{U})^{-SOT}$. We call $\pi_\ell(\mathcal{U})''$ the **left von Neumann algebra** associated to the left Hilbert algebra $\mathcal{U}$, and denote it by $\mathcal{R}_\ell(\mathcal{U})$. Analogously, if $\mathcal{U}$ is a right Hilbert algebra, we write $\pi_r(\xi)$ for right multiplication operator, $\pi_r(\mathcal{U})$ for the set of all right multiplication operators, and write $\mathcal{R}_r(\mathcal{U}) = \pi_r(\mathcal{U})''$ the right von Neumann algebra associated to $\mathcal{U}$.

It was shown in Theorem 3.11 that if ϕ is an fns weight on a von Neumann algebra $\mathcal{M}$, then the the space $\mathcal{U}_\phi = q_\phi(n_\phi \cap n_\phi^*)$ is dense in the representation space $\mathcal{H}_\phi$, the set of left multiplication operators generate $\pi_\phi(\mathcal{M})$, and the adjoint is closeable. Therefore, $\mathcal{U}_\phi$ is a left Hilbert algebra.

Let $\mathcal{U}$ be a left Hilbert algebra, and let S be the closure of the involution. In the following lemma we introduce two important operators arising from S. First we make a remark on the use of the word involution. If a conjugate linear closed operator T satisfies $T^2 = 1|_{\mathcal{D}(T)}$, and $\mathfrak{M} \leq \mathcal{D}(T)$ is a core for T such that $\mathfrak{M}$ admits an algebra structure for which $T|_{\mathfrak{M}}$ is an involution on $\mathfrak{M}$ in the usual sense, then we will also say that T is an involution. The context will make it clear what definition we are using.

Lemma 4.2. *The operator S is an involution and admits polar decomposition $S = J\Delta^{1/2} = \Delta^{-1/2}J$ where J is an isometric involution and Δ is non-singular.*

Proof. Let $\mathcal{D}^\sharp$ be the domain of S. If $\xi \in \mathcal{D}^\sharp$, then by definition there exists a sequence $\{\xi_n\} \subseteq \mathcal{U}$ such that $\|\xi - \xi_n\| \to 0$ and $\|S\xi - S\xi_n\| \to 0$. But this immediately implies that the sequence $\{(S\xi_n, \xi_n) = (S\xi_n, S^2\xi_n)\}$ is Cauchy in the graph norm of S, so $S\xi \in \mathcal{D}^\sharp$ and $S^2\xi = \xi$.

Since S is non-singular, it follows immediately from the polar decomposition $S = J\Delta^{1/2}$ that Δ is non-singular and J is unitary. Moreover, since

$$\begin{aligned} J\Delta^{1/2} &= S \\ &= S^{-1} \\ &= (\Delta^{-1/2}J)^{-1} \\ &= J^{-1}\Delta^{1/2}, \end{aligned}$$

by the uniqueness of the polar decomposition we infer that $J = J^{-1}$.

□

We call J, Δ respectively the **modular conjugation** and **modular operator** associated with $\mathcal{U}$. As noted at the end of Section 2, the fact that Δ is injective means that we can unambiguously define the operator $\log(\Delta)$, and so obtain a one-parameter unitary group $\{\Delta^{it}\}_{t\in\mathbb{R}}$ in $\mathcal{H}$. This implements an automorphism group $\{\sigma_t\}_{t\in\mathbb{R}}$, on $B(\mathcal{H})$ by $\sigma_t(x) = \Delta^{it}x\Delta^{-it}$, and we will show that this restricts to an action on $\mathcal{R}_\ell(\mathcal{U})$. Moreover, the conjugation J implements a *-anti-ismorphism between $\mathcal{R}_\ell(\mathcal{U})$ and its commutant by $a \mapsto Ja^*J$. As a first step towards proving this we will come up with a description of the commutant of $\mathcal{R}_\ell(\mathcal{U})$.

Definition 4.3. *A vector $\eta \in \mathcal{H}$ is called **right bounded** if there exists $x \in B(\mathcal{H})$ such that $x\xi = \pi_\ell(\xi)\eta$ for all $\xi \in \mathcal{U}$. We write $x = \pi_r(\eta)$ and denote the subspace of all right bounded vectors by $\mathcal{B}'$.*

The use the prime symbol in $\mathcal{B}'$ is made in direct analogy to the commutant of a von Neumann algebra. As to be expected, we will at some point define a set $\mathcal{B}$ of left bounded vectors. Note that the operators $\pi_\ell(\xi), \pi_r(\eta)$ where $\xi \in \mathcal{U}, \eta \in \mathcal{B}'$ commute, and since $\pi_\ell(\mathcal{U})$ acts non-degenerately we conclude that $\pi_r(\mathcal{B}') \subseteq \mathcal{R}_\ell(\mathcal{U})'$. For convenience, we extend multiplication by defining

$$\begin{aligned} \xi\eta &= \pi_\ell(\xi)\eta, &\qquad \text{for } \xi \in \mathcal{U}, \eta \in \mathcal{H}; \\ \xi\eta &= \pi_r(\eta)\xi, &\qquad \text{for } \xi \in \mathcal{H}, \eta \in \mathcal{B}'. \end{aligned}$$

While the set $\pi_r(\mathcal{B}')$ might not be self-adjoint we can fix this dificiency if we consider the operator $F = S^*$. Note that if $\eta \in \mathcal{B}' \cap \mathcal{D}(F), \xi, \zeta \in \mathcal{U}$ then

$$\begin{aligned}
< \pi_r(\eta)\xi, \zeta > &=< \pi_\ell(\xi)\eta, \zeta > \\
&=< \eta, \xi^\sharp \zeta > \\
&=< \eta, S(\zeta^\sharp \xi) > \\
&=< \zeta^\sharp \xi, F\eta > \\
&=< \xi, \pi_\ell(\zeta) F\eta >,
\end{aligned}$$

so that $\pi_r(\eta)^*\zeta = \pi_\ell(\zeta)F\eta$. Hence $F\eta$ is right bounded and $\pi_r(F\eta) = \pi_r(\eta)^*$. By analogy with the involution S we write $\mathcal{D}(F) = \mathcal{D}^\flat$, and set $\mathcal{U}' = \mathcal{B}' \cap \mathcal{D}^\flat$.

Lemma 4.4. *The operator $F = S^*$ satisfies $F^2 = 1|_{\mathcal{D}^\flat}$.*

Proof. If $\eta \in \mathcal{D}^\flat, \xi \in \mathcal{D}^\sharp$ then

$$< S\xi, F\eta > =< \eta, S^2\xi > =< \eta, \xi >,$$

so that $F\eta \in \mathcal{D}^\flat$ and $F^2\eta = \eta$.

□

Theorem 4.5. *The set $\mathcal{U}'$ is a right Hilbert algebra with involution given by $F|_{\mathcal{U}'}$. Moreover, $\mathcal{U}'$ is dense in $\mathcal{H}$ and $\mathcal{R}_r(\mathcal{U}') = \mathcal{R}_\ell(\mathcal{U})'$.*

This theorem will follow from a series of lemmas. To simplify notation we will often write $\xi^\sharp, \eta^\flat$ in place of $S\xi, F\eta$ for $\xi \in \mathcal{D}^\sharp, \eta \in \mathcal{D}^\flat$.

Lemma 4.6.

1) *For $a \in \mathcal{R}_\ell(\mathcal{U})', \eta \in \mathcal{B}'$, we have that $a\eta \in \mathcal{B}'$ and $\pi_r(a\eta) = a\pi_r(\eta)$. Hence $\mathcal{B}'$ is invariant under $\mathcal{R}_\ell(\mathcal{U})'$ and $\pi_r(\mathcal{B}')$ is a left ideal in $\mathcal{R}_\ell(\mathcal{U})'$.*

2) *For $\eta_1, \eta_2 \in \mathcal{B}'$ we have $\pi_r(\eta_1)^*\eta_2 \in \mathcal{D}^\flat$ and $(\pi_r(\eta_1)^*\eta_2)^\flat = \pi_r(\eta_2)^*\eta_1$. Hence, $\pi_r(\mathcal{B}')^*\mathcal{B}' \subseteq \mathcal{U}'$.*

Proof.

1) Let $a \in \mathcal{R}_\ell(\mathcal{U})', \eta \in \mathcal{B}'$. Then for any $\xi \in \mathcal{U}$ we have

$$\pi_\ell(\xi)a\eta = a\pi_\ell(\xi)\eta = a\pi_r(\eta)\xi.$$

It follows that $a\eta$ is right bounded and $\pi_r(a\eta) = a\pi_r(\eta)$.

2) If $\eta_1, \eta_2 \in \mathcal{B}'$ then for $\xi \in \mathcal{U}$ we have

$$\begin{aligned} < \pi_r(\eta_1)^*\eta_2, \xi^\sharp > &=< \eta_2, \pi_r(\eta_1)\xi^\sharp > \\ &=< \eta_2, \pi_\ell(\xi^\sharp)\eta_1 > \\ &=< \pi_\ell(\xi)\eta_2, \eta_1 > \\ &=< \pi_r(\eta_2)\xi, \eta_1 > \\ &=< \xi, \pi_r(\eta_2)^*\eta_1 > . \end{aligned}$$

Since $\mathcal{U}$ is a core for S the result follows.

□

Since $\pi_r(\mathcal{U}')$ is self-adjoint, part2) from Lemma 4.6 implies that

$$\pi_r(\mathcal{U}')\mathcal{U}' \subseteq \pi_r(\mathcal{B}')^*\mathcal{B}' \subseteq \mathcal{U}',$$

and that for $\eta, \zeta \in \mathcal{U}'$ we have $(\eta\zeta)^\flat = \zeta^\flat\eta^\flat$ so that $\mathcal{U}'$ is an involutive algebra with involution given by $^\flat$. Property 2) of Definition 4.1 also follows easily. We now want to show that $\mathcal{U}'^2$ is dense in $\mathcal{H}$. The strategy is to obtain a larger class of not necessarily bounded operators affiliated to $\mathcal{R}_\ell(\mathcal{U})'$ by extending the definition of $\pi_r(\eta)$ to vectors $\eta \in \mathcal{D}^\flat$. We then apply the functional calculus developed in Section 2.

Lemma 4.7. *Let $\eta \in \mathcal{D}^\flat$. Define the operators a_0, b_0 on $\mathcal{U}$ by*

$$a_0\xi = \pi_\ell(\xi)\eta, \qquad b_0\xi = \pi_\ell(\xi)\eta^\flat.$$

Then a_0, b_0 are preclosed and $a_0 \subseteq b_0^*, b_0 \subseteq a_0^*$. We denote their closures respectively by $\pi_r(\eta), \pi_r(\eta^\flat)$ and note that they are affiliated with $\mathcal{R}_\ell(\mathcal{U})'$. Moreover, $\eta \in \overline{\pi_r(\eta)\mathcal{H}}$.

Proof. If $\xi, \zeta \in \mathcal{U}$, then we have

$$\begin{aligned} < a_0\zeta, \xi > &=< \pi_\ell(\zeta)\eta, \xi > \\ &=< \eta, \pi_\ell(\zeta)^*\xi > \\ &=< \eta, (\pi_\ell(\xi)^*\zeta)^\sharp > \\ &=< \pi_\ell(\xi)^*\zeta, \eta^\flat > \\ &=< \zeta, \pi_\ell(\xi)\eta^\flat >, \end{aligned}$$

which is clearly bounded on $\mathcal{U}$ as a function in ζ. Hence $\xi \in \mathcal{D}(a_0^*)$ and $a_0^*\xi = \pi_\ell(\xi)\eta^\flat = b_0\xi$. Hence, $b_0 \subseteq a_0^*$ so that a_0 is closable. Similarly, we have that b_0 is closeable.

Note that if a_0^* is affiliated with $\mathcal{R}_\ell(\mathcal{U})'$, then since $\pi_r(\eta) = (a_0^*)^*$ and $\pi_r(\eta^\flat) = (b_0^*)^*$, for any self-adjoint $x \in \mathcal{R}_\ell(\mathcal{U})$, we will have

$$x\pi_r(\eta) = x(a_0^*)^* = (a_0^*x)^* = (xa_0^*)^* = (a_0^*)^*x = \pi_r(\eta)x,$$

and so $\pi_r(\eta)$ is also affiliated with $\mathcal{R}_\ell(\mathcal{U})'$. Then if $\zeta \in \mathcal{D}(a_0^*)$, $x \in \mathcal{R}_\ell(\mathcal{U}), \xi_1, \xi_2 \in \mathcal{U}$ we have

$$\begin{aligned} < a_0\xi_1, \pi_\ell(\xi_2)\zeta > &=< \pi_\ell(\xi_1)\eta, \pi_\ell(\xi_2)\zeta > \\ &=< \pi_\ell(\xi_2)^*\pi_\ell(\xi_1)\eta, \zeta > \\ &=< a_0\xi_2^\sharp\xi_1, \zeta > \\ &=< \xi_1, \pi_\ell(\xi_2)a_0^*\zeta > . \end{aligned}$$

Hence, $\pi_\ell(\xi_2)\zeta \in \mathcal{D}(a_0^*)$ and $a_0^*\pi_\ell(\xi_2)\zeta = \pi_\ell(\xi_2)a_0^*\zeta$. Now let $x \in \mathcal{R}_\ell(\mathcal{U})$ and let $\{\xi_i\} \subseteq \mathcal{U}$ be chosen such that $\pi_\ell(\xi_i) \to x$ in the SOT. Then for $\zeta \in \mathcal{D}(a_0^*)$ we have

$$\lim_i a_0^*\pi_\ell(\xi_i)\zeta = \lim_i \pi_\ell(\xi_i)a_0^*\zeta = xa_0^*\zeta$$

Since a_0^* is closed, $x\zeta \in \mathcal{D}(a_0^*)$ and $a_0^*x\zeta = xa_0^*\zeta$. On the other hand, if $\zeta \in \mathcal{H}$ is chosen such that $x\zeta \in \mathcal{D}(a_0^*)$ and x is unitary, then $\xi \in x^*\mathcal{D}(a_0^*) \subseteq \mathcal{D}(a_0^*)$, so that the above calculation yields $xa_0^*\xi = a_0^*x\xi$.

The last claim follows by choosing a net $\{\pi_\ell(\xi_i)\}$ in $\pi_\ell(\mathcal{U})$ converging to 1 in the SOT, so that $\eta = \lim_i \pi_\ell(\xi_i)\eta = \lim_i \pi_r(\eta)\xi_i$.

□

Lemma 4.8. *Let $\eta \in \mathcal{D}^\flat$, and let $\pi_r(\eta) = uh = ku$ be the left and right polar decompositions of $\pi_r(\eta)$. If $f \in C_c(0,\infty)$ then $f(h)\eta^\flat, f(k)\eta$ are right-bounded and*

$$\begin{aligned} \pi_r(f(h)\eta^\flat) &= hf(h)u^*, \\ \pi_r(f(k)\eta) &= kf(k)u. \end{aligned}$$

Proof. Let $\xi \in \mathcal{U}$. Since $\pi_r(\eta)$ is affiliated with $\mathcal{R}_\ell(\mathcal{U})'$, by the Polar Decomposition Theorem so are h, k and $u \in \mathcal{R}_\ell(\mathcal{U}')$. By Lemma 2.20, $f(h), f(k)$ belong to $\mathcal{R}_\ell(\mathcal{U})'$. Then

$$\begin{aligned} \pi_\ell(\xi)f(h)\eta^\flat &= f(h)\pi_\ell(\xi)\eta^\flat \\ &= f(h)\pi_r(\eta^\flat)\xi \\ &= f(h)\pi_r(\eta)^*\xi \\ &= f(h)hu^*\xi \\ &= hf(h)u^*\xi. \end{aligned}$$

But $hf(h)$ is bounded, and it follows that $f(h)\eta^\flat$ is right-bounded and $\pi_r(f(h)\eta^\flat) = hf(h)u^*$. The other part follows similarly.

□

Lemma 4.9. *Both $\mathcal{U}'$,$\mathcal{U}'^2$ are cores for F.*

Proof.

Claim: Let $f \in C_c(0,\infty)$. With η, h, k be as before, we have $f(h)\eta^\flat, f(k)\eta \in \mathcal{U}'^2$.

If p is any polynomial, it is clear that $hp(h) = u^*kp(k)u$. Then if $K = \text{supp } f$, we can choose a sequence of polynomials p_n such that $f = \lim \chi_K p_n$. By the functional calculus it follows that $hf(h) = u^*kf(k)u$. Then

$$\begin{aligned} hf(h) &= u^*kf(k)u \\ &= u^*\pi_r(f(k)\eta) \\ &= \pi_r(u^*f(k)\eta), \end{aligned}$$

and similarly, $kf(k) = \pi_r(uf(h)\eta^\flat)$ so that $hf(h), kf(k) \in \pi_r(\mathcal{B}')$.

Choose a function $g \in C_c(0, \infty)$ such that for any λ in the support of f we have $f(\lambda) = \lambda g(\lambda) f(\lambda)$. Then $f(h) = g(h)hf(h)$ and $f(k) = g(k)kf(k)$, both of which lie in $\pi_r(\mathcal{B}')$ since it is a left ideal in $\mathcal{R}_\ell(\mathcal{U})'$.

Now choose $f_1, f_2 \in C_c(0, \infty)$ such that $f = \overline{f_1} f_2$. Then

$$f(h) = f_1(h)^* f_2(h) \in \pi_r(\mathcal{B}')^* \pi_r(\mathcal{B}') \subseteq \pi_r(\mathcal{U}').$$

Hence, we also have that $\overline{f_1}(h), f_2(h) \in \pi_r(\mathcal{U}')$ so that $f(h) \in \pi_r(\mathcal{U}'^2)$. Moreover,

$$\begin{aligned} \pi_r(f(h)\eta^\flat) &= \pi_r(\overline{f}_1(h) f_2(h)\eta^\flat) \\ &= \overline{f}_1(h)\pi_r(f_2(h)\eta^\flat) \\ &\in \pi_r(\mathcal{B}')^*\pi_r(\mathcal{B}') \\ &\subseteq \pi_r(\mathcal{U}'), \end{aligned}$$

so $f(h)\eta^\flat \in \mathcal{U}'$. Repeating the last argument (noting that we now have that $\overline{f}_1(h), \pi_r(f_2(h)\eta^\flat) \in \pi_r(\mathcal{U}')$) we see that $f(h)\eta^\flat \in \mathcal{U}'^2$, and similarly $f(k)\eta \in \mathcal{U}'^2$.

Let $\{f_n\}$ be a sequence of non-negative functions in $C_c(0, \infty)$ which increases pointwise to $\chi_{(0,\infty)}$. By the Spectral Theorem, $f_n(h), f_n(k)$ converge in the SOT to the range projections p, q of h, k respectively. Then p and q are respectively the range projections for $\pi_r(\eta)^*$ and $\pi_r(\eta)$, and as shown in Lemma 4.7 we have $\eta^\flat \in \overline{\pi_r(\eta)^*\mathcal{H}}$ and $\eta \in \overline{\pi_r(\eta)\mathcal{H}}$ so that $q\eta = \eta$ and $p\eta^\flat = \eta^\flat$. Hence $\eta = q\eta = \lim_n f_n(k)\eta$ and $\eta^\flat = p\eta^\flat = \lim_n f_n(h)\eta^\flat$, and by the claim we have $f_n(k)\eta \in \mathcal{U}'^2$, $f_n(h)\eta^\flat = (f_n(k)\eta)^\flat$, which completes the proof.

□

proof of Theorem 4.5. First note that since $\mathcal{U}'^2$ is dense in $\mathcal{H}$ the operators $\pi_r(\eta)$ for $\eta \in \mathcal{U}'$ are precisely the operators obtained by right multiplication on $\mathcal{U}'$. We have $\pi_r(\mathcal{U}') \subseteq \mathcal{R}_\ell(\mathcal{U})'$, so that $\mathcal{R}_r(\mathcal{U}') = \pi_r(\mathcal{U}')'' \subseteq \mathcal{R}_\ell(\mathcal{U})'$. But by Lemma 4.9, $\pi_r(\mathcal{U}')$ acts non-degenerately on $\mathcal{H}$, so it contains a bounded net $\{a_i\}$ which converges to 1 in the SOT. Then if $x \in \mathcal{R}_\ell(\mathcal{U})'^+$, we have that $x^{1/2}a_i$ converges to $x^{1/2}$ in the SOT, so that $(x^{1/2}a_i)^*(x^{1/2}a_i)$ converges to x in the WOT. But $(x^{1/2}a_i)^*(x^{1/2}a_i) \in \pi_r(\mathcal{B}')^*\pi_r(\mathcal{B}') \subseteq \pi_r(\mathcal{U}')$, so that $x \in \mathcal{R}_r(\mathcal{U}')$. Hence $\mathcal{R}_\ell(\mathcal{U})' \subseteq \mathcal{R}_r(\mathcal{U}')$, completing the proof.

□

The following lemma shows that our choice of $\mathcal{U}'$ is in fact as large as possible.

Lemma 4.10. *We have that $\pi_r(\mathcal{U}') = \pi_r(\mathcal{B}') \cap \pi_r(\mathcal{B}')^*$. That is, $\pi_r(\mathcal{U}')$ is the set of right-multiplication operators whose adjoint is also a right multiplication operator.*

Proof.

Claim: The subspace $\mathcal{U}^2$ is a core for S.

Since $\mathcal{U}$ is a core for S it suffices to approximate $\xi \in \mathcal{U}$ in the graph norm of S. Assume WLOG that $\|\pi_\ell(\xi)\| \leq 1$. Then setting $p_n(t) = 1 - (1-t)^n$ we have that

$$p_n(\pi_\ell(\xi)\pi_\ell(\xi)^\sharp) = p_n(\xi\xi^\sharp)$$

converges to the range projection of $\pi_\ell(\xi)$ in the SOT. Then

$$\xi = \lim_{n\to\infty} p_n(\xi\xi^\sharp)\xi,$$

and

$$\xi^\sharp = \lim_{n\to\infty} p_n(\xi^\sharp\xi)\xi^\sharp = \lim_{n\to\infty} (p_n(\xi\xi^\sharp)\xi)^\sharp,$$

which proves the claim.

Now suppose $\eta_1, \eta_2 \in \mathcal{B}'$ such that $\pi_r(\eta_1)^* = \pi_r(\eta_2)$. Then for $\xi_1, \xi_2 \in \mathcal{U}$, we have

$$\begin{aligned} < \eta_1, \xi_1^\sharp \xi_2 > &=< \pi_\ell(\xi_1)\eta_1, \xi_2 > \\ &=< \pi_r(\eta_1)\xi_1, \xi_2 > \\ &=< \xi_1, \pi_r(\eta_2)\xi_2 > \\ &=< \xi_2^\sharp \xi_1, \eta_2 > \\ &=< (\xi_1^\sharp \xi_2)^\sharp, \eta_2 > . \end{aligned}$$

But since $\mathcal{U}^2$ is a core for S, it follows that $\eta_1 \in \mathcal{D}^\flat$ and $\eta_1^\flat = \eta_2$. Hence,

$$\pi_r(\mathcal{B}') \cap \pi_r(\mathcal{B}')^* \subseteq \pi_r(\mathcal{U}').$$

But by part 2) of Lemma 4.6, the reverse inclusion holds, so that $\pi_r(\mathcal{U}') = \pi_r(\mathcal{B}') \cap \pi_r(\mathcal{B}')^*$. □

We now have the dual version of Definition 4.3.

Definition 4.11. *A vector $\xi \in \mathcal{H}$ is called **left bounded** if there exists $x \in B(\mathcal{H})$ such that $x\eta = \pi_r(\eta)\xi$ for all $\eta \in \mathcal{U}'$. We write $x = \pi_\ell(\xi)$ and denote the space of all left bounded vectors by $\mathcal{B}$.*

Setting $\mathcal{U}'' = \mathcal{B} \cap \mathcal{D}^\sharp$ the above arguments can easily be adapted to show that $\mathcal{U}''$ is a left Hilbert algebra containing $\mathcal{U}$, and that

$$\mathcal{R}_\ell(\mathcal{U}'') = \mathcal{R}_r(\mathcal{U}')' = \mathcal{R}_\ell(\mathcal{U}).$$

Moreover, $\mathcal{U}'' = \mathcal{U}''''$. We say that a left Hilbert algebra is **full** if $\mathcal{U} = \mathcal{U}''$.

Let ϕ be an fns weight on a von Neumann algebra $\mathcal{M}$. We have already seen that $\mathcal{U}_\phi = q_\phi(n_\phi \cap n_\phi^*)$ is a left Hilbert algebra such that $\mathcal{M} \cong \mathcal{R}_\ell(\mathcal{U}_\phi)$. Retaining the notation of Lemma 3.12 we note that each vector $\eta_\omega, \omega \in \Phi$ is right-bounded with $\pi_r(\eta_\omega) = h_\omega^{1/2}$. Then if $\xi \in \mathcal{H}_\phi$ is left-bounded, $x = \pi_\ell(\xi)$ we have

$$\begin{aligned}
\phi(x^*x) &= \sup_{\omega \in \Phi} \omega(x^*x) \\
&= \sup_{\omega \in \Phi} \|\pi_\phi(x)\eta_\omega\|^2 \\
&= \sup_{\omega \in \Phi} \|\pi_r(\eta_\omega)\xi\|^2 \\
&= \sup_{\omega \in \Phi} \|h_\omega^{1/2}\xi\|^2 \\
&= \|\xi\|^2 \\
&< \infty,
\end{aligned}$$

so $x \in n_\phi$. It follows that $\mathcal{U}_\phi'' \subseteq n_\phi \cap n_\phi^* = \mathcal{U}_\phi$, so $\mathcal{U}_\phi$ is full.

It will be convenient for the rest of this Section to assume all left Hilbert algebras are full. We now come to the main theorem of this Section.

Theorem 4.12 (Tomita's Theorem)**.** *Let $\mathcal{U}$ be a full left Hilbert algebra with associated left von Neumann algebra $\mathcal{R}_\ell(\mathcal{U})$, modular operator Δ, and modular conjugation J. Then we have the following:*

1) for all $t \in \mathbb{R}$, $\Delta^{it}\mathcal{R}_\ell(\mathcal{U})\Delta^{-it} = \mathcal{R}_\ell(\mathcal{U})$, and $\Delta^{it}\mathcal{U} = \mathcal{U}, \Delta^{it}\mathcal{U}' = \mathcal{U}'$;
2) $J\mathcal{R}_\ell(\mathcal{U})J = \mathcal{R}_\ell(\mathcal{U})'$ and $J\mathcal{U} = \mathcal{U}'$.

The proof will follow from a series of Lemmas. We begin by showing how the operator Δ acts as a map from $\mathcal{U}'$ into $\mathcal{U}$.

Lemma 4.13. *Let $\omega \in \mathbb{C} - [\mathbb{R}^+ \cup \{0\}]$ and set $\gamma(\omega) = \frac{1}{\sqrt{2(|\omega| - Re\ \omega)}}$. Then $(\Delta - \omega)^{-1}\mathcal{U}' \subseteq \mathcal{U}$ and for $\eta \in \mathcal{U}'$ we have $\|\pi_\ell((\Delta - \omega)^{-1}\eta)\| \le \gamma(\omega)\|\pi_r(\eta)\|$.*

Proof. First note that $\xi = (\Delta - \omega)^{-1}\eta \in \mathcal{D}(\Delta - \omega) \subseteq \mathcal{D}^\sharp$. Now let $\pi_\ell(\xi) = uh = ku$ be the left, right polar decompositions of $\pi_\ell(\xi)$ respectively, and let $h = \int \lambda dE(\lambda)$ be the spectral decomposition of h. As shown in the proof of Lemma 4.9, for any $f \in C_c(0, \infty)$ we have $f(k)\xi \in \mathcal{U}$ and $(f(k)\xi)^\sharp = \overline{f}(h)\xi^\sharp$.

Claim: We have the estimate $\|hf(h)\xi^\sharp\|^2 \leq \gamma(\omega)^2\|\pi_r(\eta)\|^2\|\|f(h)\xi^\sharp\|^2$, and therefore the measure $E_{\xi^\sharp}$ is supported on the interval $[0,c]$, where $c = \gamma(\omega)\|\pi_r(\eta)\|$. In particular, $\xi^\sharp \in E([0,c])\mathcal{H}$.

To see this we note that

$$\begin{aligned}\|hf(h)\xi^\sharp\|^2 &=< hf(h)\xi^\sharp, hf(h)\xi^\sharp > \\ &=< \xi^\sharp, h\overline{f}(h)hf(h)\xi^\sharp > \\ &=< \xi^\sharp, (k\overline{f}(k)kf(k)\xi)^\sharp > \\ &=< k\overline{f}(k)kf(k)\xi, \Delta\xi > \\ &=< kf(k)\xi, kf(k)\Delta\xi > .\end{aligned}$$

Since the last term must be a real number, we have

$$\begin{aligned}&2(|\omega| - \mathrm{Re}(\omega))\|hf(h)\xi^\sharp\|^2 \\ &\quad\leq 2\|kf(k)\omega\xi\|\|kf(k)\Delta\xi\| - 2\mathrm{Re}(< kf(k)\omega\xi, kf(k)\Delta\xi >) \\ &\quad= \|kf(k)(\omega - \Delta)\xi\|^2 - (\|kf(k)\omega\xi\| + \|kf(k)\Delta\xi\|)^2 \\ &\quad\leq \|kf(k)(\Delta - \omega)\xi\|^2 \\ &\quad= \|kf(k)\eta\|^2 \\ &\quad= \|f(k)k\eta\|^2 \\ &\quad= \|f(k)u\pi_\ell(\xi^\sharp)\eta\|^2 \\ &\quad= \|f(k)u\pi_r(\eta)\xi^\sharp\|^2 \\ &\quad= \|\pi_r(\eta)u^*f(h)\xi^\sharp\|^2 \\ &\quad\leq \|\pi_r(\eta)\|^2\|f(h)\xi^\sharp\|^2,\end{aligned}$$

and so we have the inequality. But this means precisely that for any $f \in C_c(0,\infty)$,

$$\int_0^\infty \lambda^2 f(\lambda)^2 dE_{\xi^\sharp}(\lambda) \leq c^2 \int_0^\infty f(\lambda)^2 dE_{\xi^\sharp}(\lambda),$$

and hence, the measure $E_{\xi^\sharp}$ must be supported on the interval $[0,c]$. In particular, we have

$$\|\xi^\sharp\|^2 =< E[0,\infty]\xi^\sharp, \xi^\sharp >= \int_0^\infty 1 dE_{\xi^\sharp} = \int_0^c 1 dE_{\xi^\sharp} =< E[0,c]\xi^\sharp, \xi^\sharp >= \|E[0,c]\xi^\sharp\|^2$$

so that $E[0,c]\xi^\sharp = \xi^\sharp$.

But $E([0,c]) \in \mathcal{R}_\ell(\mathcal{U})$ and since $\|h|_{E([0,c])\mathcal{H}}\| \leq c$, for $\zeta \in \mathcal{U}'$ we have

$$\begin{aligned}\|\pi_r(\zeta)\xi^\sharp\| &= \|\pi_r(\zeta)E([0,c])\xi^\sharp\| \\ &= \|E([0,c])\pi_r(\zeta)\xi^\sharp\| \\ &= \|E([0,c])\pi_\ell(\xi^\sharp)\zeta\| \\ &= \|E([0,c])hu^*\zeta\| \\ &= \|hE([0,c])u^*\zeta\| \\ &\leq c\|\zeta\|.\end{aligned}$$

Thus $\xi^\sharp$ is left-bounded. Hence $\xi \in \mathcal{U}$.

□

The above lemma shows that for fixed $\eta \in \mathcal{U}'$ we have a function $s \in \mathbb{R} \mapsto (\Delta + e^s)\eta \in \mathcal{U}$. We now show how the operators $\pi_r(\eta) \in \mathcal{R}_\ell(\mathcal{U})'$ and $\pi_\ell((\Delta + e^s)\eta) \in \mathcal{R}_\ell(\mathcal{U})$ are related.

Lemma 4.14. *Let $\eta \in \mathcal{U}', s \in \mathbb{R}$. Setting $\xi = (\Delta + e^s)^{-1}\eta \in \mathcal{U}$, we have that for any $\zeta_1, \zeta_2 \in \mathcal{D}^\sharp \cap \mathcal{D}^\flat$,*

$$< \pi_r(\eta)\zeta_1, \zeta_2 >=< (J\pi_\ell(\xi^\sharp)J)\Delta^{-1/2}\zeta_1, \Delta^{1/2}\zeta_2 > +e^s < (J\pi_\ell(\xi^\sharp)J)\Delta^{1/2}\zeta_1, \Delta^{-1/2}\zeta_2 > .$$

Proof. First suppose $\zeta_1, \zeta_2 \in \mathcal{U} \cap \mathcal{D}^\flat$. Then

$$\begin{aligned}< \pi_r(\eta)\zeta_1, \zeta_2 > &=< \pi_r((\Delta + e^s)\xi)\zeta_1, \zeta_2 > \\ &=< \pi_\ell(\zeta_1)\Delta\xi, \zeta_2 > +e^s < \pi_\ell(\zeta_1)\xi, \zeta_2 > .\end{aligned}$$

Working with the first term, we have

$$\begin{aligned}< \pi_\ell(\zeta_1)\Delta\xi, \zeta_2 > &=< \xi^{\sharp\flat}, \zeta_1^\sharp\zeta_2 > \\ &=< \zeta_2^\sharp\zeta_1, \xi^\sharp > \\ &=< \zeta_1, \zeta_2\xi^\sharp > \\ &=< \zeta_1, (\xi\zeta_2^\sharp)^\sharp > \\ &=< \zeta_1, \Delta^{-1/2}J\pi_\ell(\xi)\Delta^{1/2}\zeta_2 > \\ &=< J\pi_\ell(\xi^\sharp)J\Delta^{-1/2}\zeta_1, \Delta^{1/2}\zeta_2 >,\end{aligned}$$

and similarly, we have

$$< \pi_\ell(\zeta_1)\xi, \zeta_2 >=< (J\pi_\ell(\xi^\sharp)J)\Delta^{1/2}\zeta_1, \Delta^{-1/2}\zeta_2 > .$$

Therefore, in order to complete the proof we need to be able to approximate an arbitrary vector $\zeta \in \mathcal{D}^\sharp \cap \mathcal{D}^\flat$ with a sequence $\{\zeta_n\}$ in $\mathcal{U} \cap \mathcal{D}^\flat$ converging to ζ simultaneously in the graph norms of $\Delta^{1/2}$ and $\Delta^{-1/2}$. First, note that

$$J\mathcal{U}' = JF\mathcal{U}' = J(J\Delta^{-1/2})\mathcal{U}' = \Delta^{-1/2}\mathcal{U}',$$

so that $\Delta^{-1/2}\mathcal{U}'$ is dense in $\mathcal{H}$. Hence, we can choose a sequence $\{\eta_n\}$ in $\mathcal{U}'$ such that

$$(\Delta^{1/2} + \Delta^{-1/2})\zeta = \lim_{n\to\infty} \Delta^{-1/2}\eta_n.$$

Then $\zeta_n = (1+\Delta)^{-1}\eta_n \in \mathcal{U} \cap \mathcal{D}^\flat$, and simple calculations show that we have

$$\zeta = \lim_{n\to} \zeta_n, \qquad \Delta^{1/2}\zeta = \lim_{n\to} \Delta^{1/2}\zeta_n, \qquad \Delta^{-1/2}\zeta = \lim_{n\to} \Delta^{-1/2}\zeta_n.$$

□

To avoid issues of domain, for the moment we will suppose that Δ is bounded and invertible. Then the above relation implies

$$\pi_r(\eta) = \Delta^{1/2}(J\pi_\ell(\xi)^*J)\Delta^{-1/2} + e^s\Delta^{-1/2}(J\pi_\ell(\xi)^*J)\Delta^{1/2}.$$

For $\alpha \in \mathbb{C}$ consider the operator $\sigma_\alpha \in B(B(\mathcal{H}))$ given by

$$\sigma_\alpha(x) = \Delta^{i\alpha}x\Delta^{-i\alpha}.$$

The above calculation says that

$$\pi_r(\eta) = [\sigma_{-i/2} + e^s\sigma_{i/2}](J\pi_\ell(\xi)^*J.)$$

In other words, we have the equation

$$\pi_\ell(S(e^s+\Delta)^{-1}\eta) = J[(\sigma_{-i/2} + e^s\sigma_{i/2})^{-1}(\pi_r(\eta))]J.$$

The following proposition sheds some light on the nature of this equation.

Proposition 4.15. *Let $\mathcal{A}$ be a unital Banach algebra, and let $u : \mathbb{C} \to GL(\mathcal{A})$ be a holomorphic group homomorphism such that $\sup_{t\in\mathbb{R}} \|u(t)\| = M < \infty$. Then for any $s \in \mathbb{R}$, the element $e^{-s/2}u(-i/2) + e^{s/2}u(i/2)$ is invertible and*

$$[e^{-s/2}u(-i/2) + e^{s/2}u(i/2)]^{-1} = \int_{-\infty}^{\infty} \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}} u(t)dt.$$

Proof. Set $f(\alpha) = \frac{e^{is\alpha}}{e^{\pi\alpha} - e^{-\pi\alpha}} u(\alpha)$. Then f is holomorphic on its domain $D = \mathbb{C} - i\mathbb{Z}$ and if $\alpha = t + ir$ we have the following estimate:

$$\begin{aligned}\|f(\alpha)\| &= \|\frac{e^{ist}e^{-sr}}{e^{\pi t}e^{i\pi r} - e^{-\pi t}e^{-i\pi r}} u(t)u(ir)\| \\ &\leq Me^{-sr} \frac{1}{|e^{\pi t}e^{i\pi r} - e^{-\pi t}e^{-i\pi r}|} \|u(ir)\|.\end{aligned}$$

For $t \neq 0$, this gives

$$\|f(\alpha)\| \leq Me^{-sr}\frac{1}{||e^{\pi t}| - |e^{-\pi t}||}\|u(ir)\|.$$

Now let $R > 0$ and consider the boundary of the rectangular region

$$c_R = \{\alpha \in \mathbb{C} : |\mathrm{Re}(\alpha)| \leq R, |\mathrm{Im}(\alpha)| \leq 1/2\}$$

given a counterclockwise orientation. From the above estimate, we have

$$\| \int_{-1/2}^{1/2} f(R + ir)dr\| \leq MKe^{-s}\frac{1}{||e^{\pi R}| - 1||},$$

and

$$\| \int_{1/2}^{-1/2} f(-R + ir)dr\| \leq MKe^{-s}\frac{1}{||e^{\pi R}| - 1||},$$

where $K = \sup_{-1/2 \leq r \leq 1/2} \|u(ir)\|$. Letting $R \to \infty$, the integral of f on the left/right sides of the rectangle converges to 0. Moreover, from the above estimate it is easy to see that the integrals $\int_{-\infty}^{\infty} f(t + i/2)dt, \int_{-\infty}^{\infty} f(t - i/2)dt$ exist and are finite, and that

$$\int_{-\infty}^{\infty} f(t - i/2)dt - \int_{-\infty}^{\infty} f(t + i/2)dt = \lim_{R \to \infty} \int_{c_R} f(\alpha).$$

Now, since

$$\lim_{\alpha \to 0} \frac{e^{\pi\alpha} - e^{-\pi\alpha}}{\alpha} = \pi e^{\pi 0} - (-\pi e^{-\pi 0}) = 2\pi,$$

we have

$$\lim_{\alpha \to 0} \alpha f(\alpha) = \lim_{\alpha \to 0} \alpha \frac{e^{-is\alpha}}{e^{\pi\alpha} - e^{-\pi\alpha}} u(\alpha) = 1/2\pi.$$

Hence f has a simple pole at 0, with residue $1/2\pi$. By the residue theorem, for any $R > 0$, we have $\int_{c_R} f(\alpha) = i$, and so

$$\int_{-\infty}^{\infty} f(t - i/2)dt - \int_{-\infty}^{\infty} f(t + i/2)dt = i.$$

But writing this out, we have

$$\begin{aligned}
i &= \int_{-\infty}^{\infty} \frac{e^{is(t-i/2)}}{e^{\pi(t-i/2)} - e^{-\pi(t-i/2)}} u(t - i/2)dt - \int_{-\infty}^{\infty} \frac{e^{is(t+i/2)}}{e^{\pi(t+i/2)} - e^{-\pi(t+i/2)}} u(t + i/2)dt \\
&= \int_{-\infty}^{\infty} [\frac{e^{ist}e^{s/2}}{(-i)(e^{\pi t} + e^{-\pi t})} u(t)u(-i/2)) - \frac{e^{ist}e^{-s/2}}{i(e^{\pi t} + e^{-\pi t})} u(t)u(i/2)]dt \\
&= i[e^{s/2}u(-i/2) + e^{-s/2}u(i/2)] \int_{-\infty}^{\infty} \frac{e^{ist}u(t)}{e^{\pi t} + e^{-\pi t}} dt.
\end{aligned}$$

Replacing s with $-s$ gives the desired result.

□

Corollary 4.16. *If $x, y \in B(\mathcal{H}), s \in \mathbb{R}$ are chosen such that for all $\zeta_1, \zeta_2 \in \mathcal{D}^\sharp \cap \mathcal{D}^\flat$ we have*

$$< x\zeta_1, \zeta_2 >=< y\Delta^{-1/2}\zeta_1, \Delta^{1/2}\zeta_2 > +e^s < y\Delta^{1/2}\zeta_1, \Delta^{-1/2}\zeta_2 >,$$

then

$$y = e^{-s/2}\int_{-\infty}^{\infty} \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}} \Delta^{it} x \Delta^{-it} dt.$$

In particular, if $\eta \in \mathcal{U}'$, we have

$$\pi_\ell((\Delta + e^s)^{-1}\eta)^* = e^{-s/2}\int_{-\infty}^{\infty} \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}} J\Delta^{it}\pi_r(\eta)\Delta^{-it}J dt.$$

Proof. Let $\Delta = \int \lambda dE(\lambda)$ be the spectral resolution of Δ, and for each $r > 0$ set $E(r) = E[1/r, r]$. Consider the one-parameter holomorphic subgroup $\{\sigma_\alpha : \alpha \in \mathbb{C}\}$ of $B(B(E(r)\mathcal{H}))$ given by $\sigma_\alpha(a) = \Delta^{i\alpha} a \Delta^{-i\alpha}$. Then since $E(r)\zeta_1, E(r)\zeta_2 \in \mathcal{D}^\sharp \cap \mathcal{D}^\flat$, we have

$$\begin{aligned}
< E(r)xE(r)\zeta_1, \zeta_2 > &=< xE(r)\zeta_1, E(r)\zeta_2 > \\
&=< y\Delta^{-1/2}E(R)\zeta_1, \Delta^{1/2}E(r)\zeta_2 > +e^s < y\Delta^{1/2}E(r)\zeta_1, \Delta^{-1/2}E(r)\zeta_2 > \\
&=< \Delta^{1/2}E(r)yE(R)\Delta^{-1/2}\zeta_1, \zeta_2 > +e^s < \Delta^{-1/2}E(r)yE(r)\Delta^{1/2}\zeta_1, \zeta_2 > \\
&=< [\sigma_{1/2}(E(r)yE(r)) + e^s\sigma_{-1/2}(E(r)yE(r))]\zeta_1, \zeta_2 > .
\end{aligned}$$

Hence,

$$\begin{aligned}
E(r)xE(r) &= \sigma_{1/2}(E(r)yE(r)) + e^s\sigma_{-1/2}(E(r)yE(r)) \\
&= e^{s/2}(e^{-s/2}\sigma_{1/2} + e^{s/2}\sigma_{-1/2})(E(r)yE(r)),
\end{aligned}$$

or equivalently,

$$e^{s/2}E(r)yE(r) = (e^{-s/2}\sigma_{1/2} + e^{s/2}\sigma_{-1/2})^{-1}(E(r)xE(r)).$$

By Proposition 4.15,

$$(e^{-s/2}\sigma_{1/2} + e^{s/2}\sigma_{-1/2})^{-1} = \int_{-\infty}^{\infty} \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}} \sigma_t dt,$$

so that

$$e^{s/2}E(r)yE(r) = \int_{-\infty}^{\infty} \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}} \Delta^{it}E(r)xE(r)\Delta^{-it} dt.$$

Taking the limit in the SOT as $r \to \infty$ gives the desired result.

$\square$

We can now prove part of Tomita's Theorem. Corollary 4.16 says that for any $\eta \in \mathcal{U}'$, $\zeta_1, \zeta_2 \in \mathcal{D}^\sharp \cap \mathcal{D}^\flat$, we have

$$< J\pi_\ell((\Delta + e^s)^{-1}\eta)^* J\zeta_1, \zeta_2 >= e^{-s/2} \int_{-\infty}^{\infty} \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}} < \Delta^{it}\pi_r(\eta)\Delta^{-it}\zeta_1, \zeta_2 > dt.$$

Since $\mathcal{D}^\sharp \cap \mathcal{D}^\flat$ is dense in $\mathcal{H}$, the linear functionals of the form ω_{ζ_1,ζ_2} for $\zeta_1, \zeta_2 \in \mathcal{D}^\sharp \cap \mathcal{D}^\flat$ are total in $B(\mathcal{H})_*$. Hence, if $\omega \in B(\mathcal{H})_*$, the above calculation implies that

$$\omega(J\pi_\ell((\Delta + e^s)^{-1}\eta)^* J) = e^{-s/2} \int_{-\infty}^{\infty} \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}} \omega(\Delta^{it}\pi_r(\eta)\Delta^{-it}) dt.$$

In particular, if $\omega \in B(\mathcal{H})_*$ vanishes on $J\mathcal{R}_\ell(\mathcal{U})'J$, then

$$\int_{-\infty}^{\infty} \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}} \omega(\Delta^{it}\pi_r(\eta)\Delta^{-it}) dt = 0.$$

By the uniqueness of the Fourier transform, and since the function $t \mapsto \frac{1}{e^{\pi t} + e^{-\pi t}}$ is non-vanishing, it follows that $\omega(\Delta^{it}\pi_r(\eta)\Delta^{-it}) = 0$. Since $\pi_r(\mathcal{U}')$ is σ-weakly dense in $\mathcal{R}_\ell(\mathcal{U})'$ we also have $\omega(\Delta^{it}x\Delta^{-it}) = 0$ for all $x \in \mathcal{R}_\ell(\mathcal{U})'$. Therefore,

$$\Delta^{it}\mathcal{R}_\ell(\mathcal{U})'\Delta^{-it} \subseteq J\mathcal{R}_\ell(\mathcal{U})J.$$

For $t = 0$, this gives $\mathcal{R}_\ell(\mathcal{U})' \subseteq J\mathcal{R}_\ell(\mathcal{U})J$. By symmetry, we also have that

$$\Delta^{it}\mathcal{R}_\ell(\mathcal{U})\Delta^{-it} \subseteq J\mathcal{R}_\ell(\mathcal{U})'J.$$

Again, using $t = 0$, this gives that $\mathcal{R}_\ell(\mathcal{U}) \subseteq J\mathcal{R}_\ell(\mathcal{U})'J$, and therefore $\mathcal{R}_\ell(\mathcal{U}) = J\mathcal{R}_\ell(\mathcal{U})'J$, and $\Delta^{it}\mathcal{R}_\ell(\mathcal{U})\Delta^{-it} \subseteq \mathcal{R}_\ell(\mathcal{U})$. Since this holds for all t, we have in fact have

$$\Delta^{it}\mathcal{R}_\ell(\mathcal{U})\Delta^{-it} = \mathcal{R}_\ell(\mathcal{U}).$$

To finish the proof, we need to look at what is happening at the level of the Hilbert space, and for that we consider the following result.

Corollary 4.17. *Let $s \in \mathbb{R}$. Then*

$$e^{s/2}\Delta^{1/2}(\Delta + e^s)^{-1} = \int_{-\infty}^{\infty} \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}} \Delta^{it} dt.$$

Proof. Letting u be as in Proposition 4.15, we first note that

$$e^{-s/2}u(-i/2)+e^{s/2}u(i/2)=e^{s/2}u(-i/2)(u(-i)+e^s).$$

Let $E(r)$ be as in the proof of Corollary 4.16. Then $\Delta|_{E(r)}$ is a positive, bounded, injective operator on $B(E(r)\mathcal{H})$, so that $\{(\Delta E(r))^{i\alpha} : \alpha \in \mathbb{C}\}$ is a one-parameter, holomorphic subgroup of $GL(B(E(r)\mathcal{H}))$ satisfying the conditions of Proposition 4.15. Hence,

$$e^{s/2}(\Delta|_{E(r)})^{1/2}(\Delta|_{E(r)}+e^s)^{-1}=\int_{-\infty}^{\infty}\frac{e^{-ist}}{e^{\pi t}+e^{-\pi t}}(\Delta|_{E(r)})^{it}dt.$$

Since $E(r)\mathcal{H}$ is reducing for $\Delta^{i\alpha}$, we in fact have

$$E(r)e^{s/2}\Delta^{1/2}(\Delta+e^s)^{-1}=E(r)\int_{-\infty}^{\infty}\frac{e^{-ist}}{e^{\pi t}+e^{-\pi t}}\Delta^{it}dt.$$

Since $E(r)$ converges to 1 in the SOT as $r\to\infty$, the result follows.

□

We now finish the proof of Tomita's Theorem. Recall that for fixed $\eta\in\mathcal{U}'$, we have a function $s\in\mathbb{R}\mapsto S(\Delta+e^s)^{-1}\eta\in\mathcal{U}$. We can rewrite this as $S(\Delta+e^s)^{-1}\eta=J\Delta^{1/2}(\Delta+e^s)^{-1}\eta$, so that by Corollary 4.17, for $\zeta\in\mathcal{U}'$ we have

$$\begin{aligned}\pi_\ell(J\Delta^{1/2}(\Delta+e^s)^{-1}\eta)\zeta &= \pi_r(\zeta)J\Delta^{1/2}(\Delta+e^s)^{-1}\eta\\ &= e^{-s/2}\int\frac{e^{-ist}}{e^{\pi t}+e^{-\pi t}}\pi_r(\zeta)J\Delta^{it}\eta.\end{aligned}$$

On the other hand, from before we had

$$\pi_\ell(J\Delta^{1/2}(\Delta+e^s)^{-1}\eta)\zeta=e^{-s/2}\int_{-\infty}^{\infty}\frac{e^{-ist}}{e^{\pi t}+e^{-\pi t}}J\Delta^{it}\pi_r(\eta)\Delta^{-it}J\zeta dt.$$

We conclude by the uniqueness of the Fourier transform, and from the fact that $t\mapsto\frac{1}{e^{\pi t}+e^{-\pi t}}$ is non-vanishing, that

$$\pi_r(\zeta)J\Delta^{it}\eta=J\Delta^{it}\pi_r(\eta)\Delta^{-it}J\zeta.$$

Therefore, $J\Delta^{it}\eta$ is left-bounded and $\pi_\ell(J\Delta^{it}\eta)=J\Delta^{it}\pi_r(\eta)\Delta^{-it}J$. Moreover, since $\pi_\ell(J\Delta^{it}\eta)^*=J\Delta^{it}\pi_r(\eta^\flat)\Delta^{-it}J=\pi_\ell(J\Delta^{it}\eta^\flat)$, it follows that

$$\pi_\ell(J\Delta^{it}\eta)\in\pi_r(\mathcal{B})\cap\pi_r(\mathcal{B})^*=\pi_r(\mathcal{U}),$$

so that $J\Delta^{it}\mathcal{U}'\subseteq\mathcal{U}$. By replacing the roles of $\mathcal{U}$ and $\mathcal{U}'$, we have $J\Delta^{it}\mathcal{U}\subseteq\mathcal{U}'$. Setting $t=0$, and combining these containments, gives $J\mathcal{U}'=\mathcal{U}$ which in turn gives $\Delta^{it}\mathcal{U}=\mathcal{U}$ and $\Delta^{it}\mathcal{U}'=\mathcal{U}'$.

We summarize below some useful computational formulas that arose in the proof.

Corollary 4.18. *For $\xi \in \mathcal{U}, \eta \in \mathcal{U}', t \in \mathbb{R}$ we have:*

1) $\pi_\ell(\Delta^{it}\xi) = \Delta^{it}\pi_\ell(\xi)\Delta^{-it}$*;*
2) $\pi_\ell(\Delta^{it}\eta) = \Delta^{it}\pi_\ell(\eta)\Delta^{-it}$*;*
3) $\pi_r(J\xi) = J\pi_\ell(\xi)J$*;*
4) $\pi_\ell(J\eta) = J\pi_r(\eta)J$*.*

These equations will be useful in Section 5 when we look at analytic functions coming from this one-parameter automorphism group. More specifically, we can derive a relation between holomorphic functions of the form $\alpha \mapsto \Delta^{i\alpha}\xi$ for $\xi \in \mathcal{U}$, and those of the form $\alpha \mapsto \sigma_\alpha(x)$ for $x \in \mathcal{R}_\ell(\mathcal{U})$.

5 Modular Condition of a Weight

Let $\mathcal{M}$ be a von Neumann algebra with an fns weight ϕ. Then we can identify $\mathcal{M}$ isomorphically with the left von Neumann algebra $\mathcal{R}_\ell(\mathcal{U}_\phi)$ of the full left Hilbert algebra $\mathcal{U}_\phi = q_\phi(n_\phi \cap n_\phi^*)$. If we let Δ be the modular operator associated with $\mathcal{U}_\phi$, then by Tomita's Theorem, for all $t \in \mathbb{R}, x \in \mathcal{M}$ we have $\Delta^{it} x \Delta^{-it} \in \mathcal{M}$ so that we can define an action on $\mathcal{M}$ by $t \in \mathbb{R} \mapsto \sigma_t^\phi = \mathrm{Ad}(\Delta^{it})$. We call σ^ϕ the **modular automorphism group** associated to the weight ϕ. In this Section we explore the relationship between ϕ and σ^ϕ. The material follows Chapter 8 of [13]. Lemma 5.20 follows the proof given in Section 3.6 of [9].

Let $\mathbb{D} = \{\alpha \in \mathbb{C} : 0 \leq \mathrm{Im}(\alpha) \leq 1\}$, and define $\mathcal{A}(\mathbb{D})$ to be the set of functions which are bounded, continuous on $\mathbb{D}$ and holomorphic on the interior of $\mathbb{D}$.

Definition 5.1. *Let ϕ be an fns weight on $\mathcal{M}$, and let $\{\alpha_t : t \in \mathbb{R}\}$ be a one-parameter automorphism group of $\mathcal{M}$. Then ϕ satisfies the **modular condition** with respect to α if the following conditions hold.*

1) The weight ϕ is invariant under α. That is, $\phi = \phi \circ \alpha_t$ for every $t \in \mathbb{R}$.
2) For each $x, y \in n_\phi \cap n_\phi^*$ there exists $F_{x,y} \in \mathcal{A}(\mathbb{D})$ satisfying the boundary condition for $t \in \mathbb{R}$,

$$F_{x,y}(t) = \phi(\alpha_t(x)y), \qquad F_{x,y}(t+i) = \phi(y\alpha_t(x))$$

Condition 2) says that ϕ satisfies a trace-like condition with respect to the action in the sense that the function $F_{x,y}$ relates $\phi(\alpha_t(x)y)$ with $\phi(y\alpha_t(x))$.

The importance of the modular condition in applications lies in the following theorem.

Theorem 5.2. *Let $\mathcal{M}$ be a von Neumann algebra and let ϕ be an fns weight on $\mathcal{M}$. Then the modular automorphism group $\{\sigma_t^\phi\}$ is the unique one-parameter automorphism group on $\mathcal{M}$ satisfying the modular condition with respect to ϕ.*

Proof. By Tomita's theorem we know that $\sigma_t^\phi(n_\phi \cap n_\phi^*) = n_\phi \cap n_\phi^*$, and since

$$m_\phi = n_\phi^* n_\phi = (n_\phi \cap n_\phi^*)^2,$$

we also have that $\sigma_t^\phi(m_\phi) = m_\phi$. If $x, y \in n_\phi \cap n_\phi^*$, we have

$$\begin{aligned} \phi(\sigma_t^\phi(y^*x)) &= < \Delta^{it} q_\phi(x), \Delta^{it} q_\phi(y) > \\ &= < q_\phi(x), q_\phi(y) > \\ &= \phi(y^*x), \end{aligned}$$

and so ϕ is σ^{ϕ}-invariant.

Let $\xi, \eta \in \mathcal{U}_\phi$ and set $x = \pi_\ell(\xi), y = \pi_\ell(\eta)$. Then by Corollary 4.18,

$$\begin{aligned}
\phi(\sigma_t^{\phi}(x)y) &= \phi(\pi_\ell(\Delta^{it}\xi)\pi_\ell(\eta)) \\
&=< \eta, \Delta^{it}\xi^{\sharp} > \\
&=< \Delta^{it+1}\xi, \eta^{\sharp} > \\
&=< \Delta^{\frac{-i}{2}(-t+i)}\xi, \Delta^{\frac{i}{2}(-t-i)}\eta^{\sharp} >, \\
\phi(y\sigma_t(x)) &= \phi(\pi_\ell(\eta)\pi_\ell(\Delta^{it}\xi)) \\
&=< \Delta^{it}\xi, \eta^{\sharp} > \\
&=< \Delta^{\frac{-i}{2}(-t)}\xi, \Delta^{\frac{i}{2}(-t)}\eta^{\sharp} > .
\end{aligned}$$

Therefore the function $F(\alpha) =< \Delta^{\frac{-i}{2}(-\alpha+i)}\xi, \Delta^{\frac{i}{2}(-\overline{\alpha}-i)}\eta^{\sharp} >$ satisfies the necessary boundary conditions. The fact that $F \in \mathcal{A}(\mathbb{D})$ follows from Lemma 2.30 and the fact that $\xi, \eta \in \mathcal{D}(\Delta^{1/2})$.

Now suppose that $\{\alpha_t\}_{t\in\mathbb{R}}$ is another one-parameter automorphism group satisfying the modular condition with respect to ϕ. For $t \in \mathbb{R}, x \in n_\phi$, define $u_t q_\phi(x) = q_\phi(\alpha_t(x))$. Then since ϕ is α_t-invariant it follows that $\alpha_t(n_\phi) = n_\phi$, and

$$\|q_\phi(\alpha_t(x))\|^2 = \phi(\alpha_t(x^*x)) = \phi(x^*x) = \|q_\phi(x)\|^2,$$

so u_t is well-defined and extends to a unitary on $\mathcal{H}_\phi$, also denoted by u_t. Let $x \in n_\phi \cap n_\phi^*$, and let $F_{x^*,x} \in \mathcal{A}(\mathbb{D})$ be the function satisfying the modular condition for x^* and x. Then

$$\lim_{t\to 0} \|u_t q_\phi(x) - q_\phi(x)\|^2 = \lim_{t\to 0}[F_{x^*,x}(0) - F_{x^*,x}(t) - F_{x^*,x}(t+i) + F_{x^*,x}(i)] = 0.$$

Hence $u_t q_\phi(x) \to q_\phi(x)$ for any $x \in n_\phi \cap n_\phi^*$, and since the $\{u_t\}$ are uniformly bounded and $q_\phi(n_\phi \cap n_\phi^*)$ is dense in $\mathcal{H}_\phi$, we conclude that the family $\{u_t\}_{t\in\mathbb{R}}$ is SOT continuous. By Stone's Theorem, there exists a self-adjoint operator K on $\mathcal{H}_\phi$ such that $u_t = e^{itK}$. To finish the proof, we need to show that $\Delta = e^K$, for then we will have that for $x \in n_\phi \cap n_\phi^*$,

$$\alpha_t(x) = \pi_\ell(q_\phi(\alpha_t(x))) = \pi_\ell(u_t q_\phi(x)) = \sigma_t^{\phi}(x).$$

The result then follows since σ_t^{ϕ} and α_t are continuous in the σ-WOT.

Claim: Each u_t commutes with S, J and Δ. Therefore Δ commutes with the spectral projections of K.

For $x \in n_\phi \cap n_\phi^*$, we have

$$\begin{aligned}
Su_t q_\phi((x)) &= Sq_\phi(\alpha_t(x)) \\
&= q_\phi(\alpha_t(x^*)) \\
&= u_t S q_\phi(x).
\end{aligned}$$

Since $\mathcal{U}_\phi$ is a core for S, $Su_t = u_tS$ as shown in the proof of Lemma 2.20. By polarizing we have

$$\begin{aligned} <\Delta^{1/2}u_tq_\phi(x), \Delta^{1/2}u_tq_\phi(y)> &=< Su_tq_\phi(y), Su_tq_\phi(x)> \\ &=< u_tSq_\phi(y), u_tSq_\phi(x)> \\ &=< Sq_\phi(y), Sq_\phi(x)> \\ &=< \Delta^{1/2}q_\phi(x), \Delta^{1/2}q_\phi(y)>, \end{aligned}$$

so again we have $\Delta^{1/2}u_t = u_t\Delta^{1/2}$. By Corollary 2.27, $\Delta^{1/2}$ is affiliated with the von Neumann algebra generated by the spectral projections of K, so that by Lemma 2.20, Δ is as well. Lastly,

$$Ju_tq_\phi(x) = JS^2u_tq_\phi(x) = \Delta^{1/2}u_tSq_\phi = u_t\Delta^{1/2}Sq_\phi = u_tJS^2q_\phi(x) = u_tJq_\phi(x),$$

completing the proof of the claim.

For $x, y \in n_\phi \cap n_\phi^*$ let $F_{x,y} \in \mathcal{A}(\mathbb{D})$ be the function satisfying the modular condition for α. By definition we have

$$\begin{aligned} F_{x,y}(t) &= \phi(\alpha_t(x)y) \\ &=< q_\phi(y), Su_tq_\phi(x)> \\ &=< q_\phi(y), u_tSq_\phi(x)> \\ &=< u_t^*q_\phi(y), Sq_\phi(x)>, \\ F_{x,y}(t+i) &= \phi(y\alpha_t(x)) \\ &=< u_tq_\phi(x), Sq_\phi(y)> \\ &=< JSq_\phi(y), Ju_tq_\phi(x)> \\ &=< \Delta^{1/2}q_\phi(y), u_tJq_\phi(x)> \\ &=< u_t^*\Delta^{1/2}q_\phi(y), \Delta^{1/2}Sq_\phi(x)> \\ &=< \Delta^{1/2}u_t^*q_\phi(y), \Delta^{1/2}Sq_\phi(x)> . \end{aligned}$$

Then for elements $\xi, \eta \in \mathcal{D}^\sharp$, we choose sequences $\{\xi_n\}, \{\eta_n\} \subseteq \mathcal{U}_\phi$ which converge to ξ, η in the graph norm of S and functions $F_n \in \mathcal{A}(\mathbb{D})$ satisfying the boundary conditions

$$F_n(t) =< u_t^*\eta_n, \xi_n^\sharp>, \qquad F_n(t+i) =< \Delta^{1/2}u_t^*\eta_n, \Delta^{1/2}\xi_n^\sharp> .$$

The sequence $\{F_n\}$ is uniformly Cauchy on the boundary and these functions are bounded on the strip, so by the Phragmen-Lindelof Theorem they converge uniformly everywhere to some function $F_{\xi,\eta} \in \mathcal{A}(\mathbb{D})$ satisfying the boundary conditions

$$F_{\xi,\eta}(t) =< u_t^*\eta, \xi^\sharp>, \qquad F_{\xi,\eta}(t+i) =< \Delta^{1/2}u_t^*\eta, \Delta^{1/2}\xi^\sharp> .$$

Now let $K = \int \lambda dE(\lambda)$ be the spectral decomposition of K, and set $E_n = E[-n, n]$. Let $\mathcal{D}_0 = \bigcup_{n=1}^{\infty} E_n \mathcal{D}^\sharp$. Then for $\xi \in \mathcal{D}^\sharp, \eta \in E_n \mathcal{D}^\sharp$, we have

$$F_{\xi,\eta}(t) = \int_{-n}^{n} e^{-it\lambda} dE_{\eta,\xi^\sharp}.$$

Hence $F_{\xi,\eta}$ has a unique entire extension satisfying

$$\begin{aligned} F_{\xi,\eta}(t+i) &= \int e^{-i(t+i)\lambda} dE_{\eta,\xi^\sharp} \\ &= < e^K u_t^* \eta, \xi^\sharp > . \end{aligned}$$

With $t = 0$, this gives $< \Delta^{1/2}\eta, \Delta^{1/2}\xi^\sharp > = < e^K \eta, \xi^\sharp >$, so that $\mathcal{D}_0 \subseteq \mathcal{D}(\Delta)$ and for any $\eta \in \mathcal{D}_0$, we have $\Delta\eta = e^K \eta$. If we can show that $\mathcal{D}_0$ is a common core for both operators, it will follow that $\Delta = e^K$. From the claim we have that Δ commutes with each E_n. Then since $E_n \to 1$ in the SOT, for any $\xi \in \mathcal{D}(\Delta)$, we have

$$\lim_{n\to\infty} E_n \xi = \xi, \qquad \lim_{n\to\infty} \Delta E_n \xi = \lim_{n\to\infty} E_n \Delta \xi = \Delta \xi,$$

so $\xi_n \to \xi$ in the graph norm of Δ. Therefore $\mathcal{D}_0$ is a core for Δ. From the Spectral Theorem we know that $\cup_{n=1}^{\infty} E_n \mathcal{H}$ is a core for e^K. Since $1 + e^K$ is self-adjoint and injective, it follows that $(1 + e^K) E_n \mathcal{D}^\sharp$ is dense in $E_n \mathcal{H}$. Then given $\xi \in E_n \mathcal{H}$ we can choose a sequence $\{\xi_m\}$ in $E_n \mathcal{D}^\sharp$ such that $(1 + e^K) E_n \xi_m \to \xi$. Moreover, $(1 + e^K)|_{E_n \mathcal{H}}$ is bounded, invertible, so that $\xi_m \to \xi$, and consequently we also have $e^K \xi_m \to e^K \xi$. Hence $\mathcal{D}_0$ is also a core for e^K.

□

The uniqueness coming from the modular condition immediately gives the following corollary.

Corollary 5.3. *Let ϕ be an fns weight on $\mathcal{M}$, $\theta \in Aut(\mathcal{M})$. Then $\sigma_t^{\phi\circ\theta} = \theta^{-1} \circ \sigma_t^\phi \circ \theta$.*

Proof. By the σ^ϕ-invariance of ϕ, for $x \in \mathcal{M}^+$, we have

$$\begin{aligned} \phi \circ \theta(\theta^{-1} \circ \sigma_t^\phi \circ \theta(x)) &= \phi \circ (\sigma_t^\phi \circ \theta(x)) \\ &= \phi \circ \theta(x), \end{aligned}$$

so $\phi \circ \theta$ is $\theta^{-1} \circ \sigma_t^\phi \circ \theta$-invariant. If $x, y \in n_{\phi\circ\theta} = \theta^{-1} n_\phi$, and if $F_{\theta(x),\theta(y)} \in \mathcal{A}(\mathbb{D})$ satisfies the boundary conditions for $\theta(x), \theta(y)$ with respect to σ^ϕ, then for $t \in \mathbb{R}$,

$$\begin{aligned} F_{\theta(x),\theta(y)}(t) &= \phi(\sigma_t^\phi(\theta(x))\theta(y)) = \phi \circ \theta(\theta^{-1}\sigma_t^\phi \theta(x) y), \\ F_{\theta(x),\theta(y)}(t+i) &= \phi(\theta(y)\sigma_t^\phi(\theta(x))) = \phi \circ \theta(y \theta^{-1} \sigma_t^\phi \theta(x)), \end{aligned}$$

so $F_{\theta(x),\theta(y)}$ satisfies the boundary conditions for x, y with respect to $\theta^{-1}\sigma^\phi\theta$. By Theorem 5.2, $\sigma_t^{\phi\circ\theta} = \theta^{-1} \circ \sigma_t^\phi \circ \theta$.

□

We will now explore various applications of the uniqueness of the modular condition. We continue by considering the problem of relating the modular automorphism groups of two fns weight ϕ, ψ on $\mathcal{M}$, which results in the Connes Cocycle Derivative Theorem. Roughly speaking, it is a uniqueness result for modular automorphism groups which will become crucial when we study von Neumann algebra crossed products in Section 7.

Consider the von Neumann algebra $\mathcal{N} = \mathcal{M} \otimes M_2(\mathbb{C})$ and define the function ρ on $\mathcal{N}^+$ by

$$\rho \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} = \phi(x_{1,1}) + \psi(x_{2,2}).$$

Lemma 5.4. *The function ρ defines an fns weight on $\mathcal{N}$ such that*

$$n_\rho = \{ \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \in \mathcal{N} : x_{1,1}, x_{2,1} \in n_\phi, x_{1,2}, x_{2,2} \in n_\psi \}.$$

Proof. Let $x \in \mathcal{N}$ be as above and note that the $(1,1)$-entry of x^*x is $x_{1,1}^* x_{1,1} + x_{2,1}^* x_{2,1}$ and the $(2,2)$-entry is $x_{1,2}^* x_{1,2} + x_{2,2}^* x_{2,2}$, which immediately give the identification of n_ρ as above. Moreover we infer that $\rho(x^*x) = 0$ if and only if $x^*x = 0$ by the faithfulness of ϕ, ψ, so that ρ is faithful.

To see that ρ is normal, note that by Theorem 3.2,

$$\begin{aligned} \rho(x) &= \phi(x_{1,1}) + \psi(x_{2,2}) \\ &= \sup_{\omega \le \phi} \omega(x_{1,1}) + \sup_{\omega' \le \psi} \omega'(x_{2,2}) \\ &= \sup_{\omega \le \phi, \omega' \le \psi} \omega(x_{1,1}) + \omega'(x_{2,2}) \\ &\le \sup_{\omega'' \le \rho} \omega''(x). \end{aligned}$$

Since the reverse inequality is clear, we have that ρ is normal.

To see that ρ is semifinite, let $\{x_i\} \subseteq m_\phi, \{y_j\} \subseteq m_\psi$ be monotone increasing nets which converge to $1_\mathcal{M}$ in the SOT. Then for each i and j, we have $x_i \otimes e_{1,1} + y_j \otimes e_{2,2} \in m_\rho$ and the net $\{x_i \otimes e_{1,1} + y_j \otimes e_{2,2}\}$ converges to $1_\mathcal{N}$ in the SOT.

□

We will now identify the operators S, F, Δ, J associated to the left Hilbert algebra $\mathcal{U}_\rho = q_\rho(n_\rho \cap n_\rho^*)$, with the goal of finding an intertwining operator for the representations π_ϕ, π_ψ. Using the previous lemma,

$$n_\rho \cap n_\rho^* = \{ \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \in \mathcal{N} : x_{1,1} \in n_\phi \cap n_\phi^*, x_{2,1} \in n_\phi \cap n_\psi^*, x_{1,2} \in n_\phi^* \cap n_\psi, x_{2,2} \in n_\psi \cap n_\psi^* \}.$$

Note that if $x, y \in n_\rho$, then

$$\begin{aligned} < x, y > &= \rho(y^* x) \\ &= \phi(y_{1,1}^* x_{1,1}) + \phi(y_{2,1}^* x_{2,1}) + \psi(y_{1,2}^* x_{1,2}) + \psi(y_{2,2}^* x_{2,2}), \end{aligned}$$

so that we have an orthogonal direct sum

$$\begin{aligned} \mathcal{H}_\rho &= [n_\phi \cap n_\phi^* \otimes e_{1,1}]^- \oplus [n_\phi \cap n_\psi^* \otimes e_{2,1}]^- \oplus [n_\psi \cap n_\phi^* \otimes e_{1,2}]^- \oplus [n_\psi \cap n_\psi^* \otimes e_{2,2}]^- \\ &= \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4. \end{aligned}$$

Hence we can represent operators on $\mathcal{H}_\rho$ using 4×4 matrices preserving the above decomposition. Now the involution S is the closure of the operator on $n_\rho \cap n_\rho^*$ given by

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \mapsto \begin{pmatrix} x_{1,1}^* & x_{2,1}^* \\ x_{1,2}^* & x_{2,2}^* \end{pmatrix},$$

so that the involutions have matrices of the form

$$S = \begin{pmatrix} S_\phi & 0 & 0 & 0 \\ 0 & 0 & S_{\phi,\psi} & 0 \\ 0 & S_{\psi,\phi} & 0 & 0 \\ 0 & 0 & 0 & S_\psi \end{pmatrix}, \qquad F = \begin{pmatrix} S_\phi^* & 0 & 0 & 0 \\ 0 & 0 & S_{\psi,\phi}^* & 0 \\ 0 & S_{\phi,\psi}^* & 0 & 0 \\ 0 & 0 & 0 & S_\psi^* \end{pmatrix}.$$

The modular operator and modular conjugation associated to ρ are then given by

$$\begin{aligned} \Delta &= \begin{pmatrix} S_\phi^* S_\phi & 0 & 0 & 0 \\ 0 & S_{\psi,\phi}^* S_{\psi,\phi} & 0 & 0 \\ 0 & 0 & S_{\phi,\psi}^* S_{\phi,\psi} & 0 \\ 0 & 0 & 0 & S_\psi^* S_\psi \end{pmatrix} = \begin{pmatrix} \Delta_\phi & 0 & 0 & 0 \\ 0 & \Delta_{\psi,\phi} & 0 & 0 \\ 0 & 0 & \Delta_{\phi,\psi} & 0 \\ 0 & 0 & 0 & \Delta_\psi \end{pmatrix}, \\ J &= \begin{pmatrix} J_\phi & 0 & 0 & 0 \\ 0 & 0 & J_{\phi,\psi} & 0 \\ 0 & J_{\psi,\phi} & 0 & 0 \\ 0 & 0 & 0 & J_\psi \end{pmatrix}. \end{aligned}$$

For $x \in \mathcal{N}$, we can write

$$\pi_\rho(x) = \begin{pmatrix} \pi_\phi(x_{1,1}) & \pi_\phi(x_{1,2}) & 0 & 0 \\ \pi_\phi(x_{2,1}) & \pi_\phi(x_{2,2}) & 0 & 0 \\ 0 & 0 & \pi_\psi(x_{1,1}) & \pi_\psi(x_{1,2}) \\ 0 & 0 & \pi_\psi(x_{2,1}) & \pi_\psi(x_{2,2}) \end{pmatrix},$$

so that in particular,

$$
\begin{aligned}
J\pi_\rho\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}J &= \begin{pmatrix}J_\phi & 0 & 0 & 0\\ 0 & 0 & J_{\phi,\psi} & 0\\ 0 & J_{\psi,\phi} & 0 & 0\\ 0 & 0 & 0 & J_\psi\end{pmatrix}\begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}J_\phi & 0 & 0 & 0\\ 0 & 0 & J_{\phi,\psi} & 0\\ 0 & J_{\psi,\phi} & 0 & 0\\ 0 & 0 & 0 & J_\psi\end{pmatrix}\\
&= \begin{pmatrix}J_\phi & 0 & 0 & 0\\ 0 & 0 & J_{\phi,\psi} & 0\\ 0 & J_{\psi,\phi} & 0 & 0\\ 0 & 0 & 0 & J_\psi\end{pmatrix}\begin{pmatrix}0 & 0 & J_{\phi,\psi} & 0\\ J_\phi & 0 & 0 & 0\\ 0 & 0 & 0 & J_\psi\\ 0 & J_{\psi,\phi} & 0 & 0\end{pmatrix}\\
&= \begin{pmatrix}0 & 0 & J_\phi J_{\phi,\psi} & 0\\ 0 & 0 & 0 & J_{\phi,\psi}J_\psi\\ J_{\psi,\phi}J_\phi & 0 & 0 & \\ 0 & J_\psi J_{\psi,\phi} & 0 & 0\end{pmatrix}.
\end{aligned}
$$

But this operator belongs to $\pi_\rho(\mathcal{N})'$, so that for $x \in \mathcal{N}$ we have

$$
J_\phi J_{\phi,\psi}\pi_\psi(x_{1,1}) = [(J\pi_\rho\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}J)\pi_\rho(x)]_{1,3} = [\pi_\rho(x)(J\pi_\rho\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}J)]_{1,3} = \pi_\phi(x_{1,1})J_\phi J_{\phi,\psi}.
$$

Therefore $U_{\phi,\psi} = J_\phi J_{\phi,\psi}$ is an isomorphism of $\mathcal{H}_\psi$ onto $\mathcal{H}_\phi$ such that for $x \in \mathcal{M}$ we have

$$
\pi_\phi(x) = U_{\phi,\psi}\pi_\psi(x)U^*_{\phi,\psi}.
$$

Identifying these representations, we write

$$
\pi_\rho(x) = \begin{pmatrix}x_{1,1} & x_{1,2} & 0 & 0\\ x_{2,1} & x_{2,2} & 0 & 0\\ 0 & 0 & x_{1,1} & x_{1,2}\\ 0 & 0 & x_{2,1} & x_{2,2}\end{pmatrix}.
$$

In particular we have a diagonal representation of $\mathcal{N}$ of multiplicity 2, and since the operator Δ is diagonal, we only need to consider the first two diagonal entries to define the automorphisms σ_t^ρ. That is, if we write $\sigma_t^{\phi,\psi}(x) = \Delta_\phi^{it}x\Delta_{\psi,\phi}^{-it}$ and $\sigma_t^{\psi,\phi}(x) = \Delta_{\psi,\phi}^{it}x\Delta_\phi^{-it}$, we have

$$
\sigma_t^\rho\left[\begin{pmatrix}x_{1,1} & x_{1,2}\\ x_{2,1} & x_{2,2}\end{pmatrix}\right] = \begin{pmatrix}\sigma_t^\phi(x_{1,1}) & \sigma_t^{\phi,\psi}(x_{1,2})\\ \sigma_t^{\psi,\phi}(x_{2,1}) & \sigma_t^\psi(x_{2,2})\end{pmatrix}.
$$

Set $u_t = \sigma_t^{\phi,\psi}(1)$. Since

$$
\begin{pmatrix}x & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & x\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},
$$

it follows that $\sigma_t^\phi(x) = u_t\sigma_t^\psi(x)u_t^*$. Since

$$
\begin{pmatrix}0 & xy\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0 & x\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & y\end{pmatrix},
$$

we have that

$$\sigma_t^{\phi,\psi}(xy) = \sigma_t^{\phi,\psi}(x)\sigma_t^{\psi}(y),$$

and hence,

$$u_{s+t} = \sigma_s^{\phi,\psi}(1\sigma_t^{\phi,\psi}(1)) = \sigma_s^{\phi,\psi}(1)\sigma_s^{\psi}(\sigma_t^{\phi,\psi}(1)) = u_s\sigma_s^{\psi}(u_t).$$

We now have 1) and 2) of the following Theorem.

Theorem 5.5 (Connes Cocycle Derivative Theorem)**.** *If ϕ, ψ are weights on $\mathcal{M}$ then there exists a unique family of unitaries $\{u_t\}_{t\in\mathbb{R}}$ in $\mathcal{M}$ which is continuous in the SOT and satisfies the following:*

1) for each $t \in \mathbb{R}$ we have $\sigma_t^{\phi}(x) = u_t\sigma_t^{\psi}(x)u_t^*$;
2) for each $s, t \in \mathbb{R}$ we have $u_{s+t} = u_s\sigma_t^{\psi}(u_t)$;
3) for each $x \in n_\phi \cap n_\psi^*$ and $y \in n_\phi^* \cap n_\psi$ there exists an $F \in \mathcal{A}(\mathbb{D})$ satisfying the boundary conditions

$$F(t) = \phi(u_t\sigma_t^{\psi}(y)x), \qquad F(t+i) = \psi(xu_t\sigma_t^{\psi}(y)).$$

Proof.

3) Let $x \in n_\phi \cap n_\psi^*, y \in n_\phi^* \cap n_\psi$. Then by the modular condition for σ_t^{ρ}, there exists an $F \in \mathcal{A}(\mathbb{D})$ such that

$$F(t) = \rho(\sigma_t^{\rho}(y)x), \qquad F(t+i) = \rho(x\sigma_t^{\rho}(y)).$$

Then $\sigma_t^{\rho}(y) \in n_\phi^* \cap n_\psi$, so $\sigma_t^{\rho}(y) = \sigma_t^{\phi,\psi}(y) = u_t\sigma_t^{\psi}(y)$. Therefore, $\rho(\sigma_t^{\rho}(x)y) = \psi(u_t\sigma_t^{\psi}(y)x)$ and $\rho(x\sigma_t^{\rho}(y)) = \phi(xu_t\sigma_t^{\psi}(y))$. The proof of uniqueness can be found in [13].

□

Definition 5.6. *If ϕ, ψ are fns weights and $\{u_t\}$ a family of unitaries satisfying 1), 2), 3) of the theorem, we call $\{u_t\}$ the **Connes cocycle derivative** of σ_t^{ϕ} with respect to σ_t^{ψ} and write $u_t = (D_\phi : D_\psi)_t$. Condition 3) in the theorem is called the **relative modular condition**.*

The Connes cocycle derivative satisfies the following chain-rule property.

Lemma 5.7. *Let ϕ, ψ, χ be fns weights on a von Neumann algebra $\mathcal{M}$. Then for $t \in \mathbb{R}$ we have*

$$(D_\phi : D_\psi)_t = (D_\phi : D_\chi)_t(D_\chi : D_\psi)_t.$$

The following elaborates on Corollary 5.3.

Corollary 5.8. *For any automorphism $\theta \in Aut(\mathcal{M})$ we have*

$$(D_{\psi\circ\theta} : D_{\phi\circ\theta}) = \theta^{-1}[(D_\psi : D_\phi)].$$

Proof. Let $(D_\psi : D_\phi) = u_t$. By Corollary 5.3, for $x \in \mathcal{M}$, we have

$$\begin{aligned}
\sigma_t^{\psi\circ\theta}(x) &= \theta^{-1} \circ \sigma_t^{\psi} \circ \theta(x) \\
&= \theta^{-1}(u_t \sigma_t^{\phi} \circ \theta(x) u_t^*) \\
&= \theta^{-1}(u_t)\theta^{-1} \circ \sigma_t^{\phi} \circ \theta(x)\theta^{-1}(u_t^*) \\
&= \theta^{-1}(u_t)\sigma_t^{\phi\circ\theta}(x)\theta^{-1}(u_t)^*.
\end{aligned}$$

Hence, to prove the claim it suffices to check the relative modular condition. Now if we let $x \in n_{\psi\circ\theta} \cap n^*_{\phi\circ\theta}, y \in n_{\phi\circ\theta} \cap n^*_{\psi\circ\theta}$ and let $F_{\theta(x),\theta(y)} \in \mathcal{A}(\mathbb{D})$ be the function satisfying the relative modular condition for $\theta(x), \theta(y)$ with respect to u_t, we have

$$\begin{aligned}
F_{\theta(x),\theta(y)}(t) &= \psi(u_t \sigma^{\phi}(\theta(y))\theta(x)) \\
&= \psi \circ \theta \circ \theta^{-1}(u_t \sigma^{\phi}\theta(y)\theta(x)) \\
&= \psi \circ \theta(\theta^{-1}(u_t)\theta^{-1} \circ \sigma^{\phi} \circ \theta(y)x) \\
&= \psi \circ \theta(\theta^{-1}(u_t)\sigma_t^{\phi\circ\theta}(y)x), \\
F_{\theta(x),\theta(y)}(t+i) &= \phi(\theta(x)u_t \sigma^{\psi}(\theta(y))) \\
&= \phi \circ \theta \circ \theta^{-1}(\theta(x)u_t \sigma^{\psi}\theta(y)) \\
&= \phi \circ \theta(x\theta^{-1}(u_t)\theta^{-1} \circ \sigma^{\psi} \circ \theta(y)) \\
&= \phi \circ \theta(x\theta^{-1}(u_t)\sigma^{\psi\circ\theta}(y)),
\end{aligned}$$

proving the claim.

□

We state here a converse of the Connes Cocycle Derivative Theorem.

Theorem 5.9. *If ϕ is an fns weight on $\mathcal{M}$, and if $\{u_t\}_{t\in\mathbb{R}}$ is a family of unitaries in $\mathcal{M}$ which is continuous in the SOT and satisfies $u_{s+t} = u_s\sigma_s^{\phi}(u_t)$ for all s and t, then there exists an fns weight ψ on $\mathcal{M}$ such that $(D_\psi : D_\phi) = u_t$ for all $t \in \mathbb{R}$.*

For the remainder of the Section we will work towards a characterization of semifinite von Neumann algebras in terms of modular automorphism groups. Note that if τ is a trace on a von Neumann algebra $\mathcal{M}$, then the modular operator Δ_τ is just the identity, so that modular automorphism group σ^τ is trivial. In general, we will see that the modular automorphism groups of semifinite algebras are inner, with implementation given by the Connes cocycle derivative with respect to the trivial action.

We continue the study of weights by fixing an fns weight ϕ and characterizing those weights ψ which are invariant under the action σ^ϕ. Let $h \in \mathcal{M}^+$, and consider a new weight $\phi_h : \mathcal{M}^+ \to [0, \infty]$ given by $\phi_h(x) = \phi(h^{1/2}xh^{1/2})$. This weight is clearly normal because ϕ is, and if we require furthermore that h be invertible, then ϕ_h will be faithful and semifinite.

We may initially be tempted to consider the automorphism $x \in \mathcal{M} \mapsto h^{it}\sigma_t^\phi(x)h^{-it}$, however, there is no guarantee that ϕ_h is invariant under such an automorphism. We define

$$\mathcal{M}_\phi = \{x \in \mathcal{M} : \sigma_t^\phi(x) = x \text{ for all } t \in \mathbb{R}\}.$$

Note that the weights of the form ϕ_h for $h \in \mathcal{M}_\phi$ are automatically invariant under the automorphisms $x \mapsto h^{it}\sigma_t^\phi(x)h^{-it}$, and moreover, are σ_t^ϕ-invariant. In order to make sure we consider a large enough class of weights, we also want to define weights of the form ϕ_h where h is positive self-adjoint injective and affiliated with $\mathcal{M}_\phi$. Ideally, these weights will be of the form

$$\phi_h = \sup_{h_i} \phi_{h_i},$$

where $\{h_i\} \subseteq \mathcal{M}_\phi^+$ is an increasing net converging pointwise to h on its domain. It is not yet clear that this makes sense, so to this end we will take a diversion into analytic subalgebras, which leads to the following characterization of elements of $\mathcal{M}_\phi$ in terms of the weight ϕ.

Theorem 5.10. *An element $a \in \mathcal{M}$ belongs to $\mathcal{M}_\phi$ if and only if the following conditions hold:*

1) $am_\phi \subseteq m_\phi, m_\phi a \subseteq m_\phi$;
2) $\phi(az) = \phi(za)$ for all $z \in m_\phi$.

The proof of the Theorem will follow after Lemma 5.16 below.

Definition 5.11. *An element $x \in \mathcal{M}$ is said to be **analytic** if the funtion $t \mapsto \sigma_t^\phi(x)$ extends to an entire function $\alpha \in \mathbb{C} \mapsto \sigma_\alpha(x)$. Equivalently, for all $\omega \in \mathcal{M}_\phi$ the function $t \mapsto \omega(\sigma_t^\phi(x))$ has an entire extension. We denote by $\mathcal{M}_a^\phi$ the set of all analytic elements.*

It is immediate that $\mathcal{M}_\phi \subseteq \mathcal{M}_a^{\sigma^\phi}$, with the obvious extension $\sigma_\alpha(x) = x$. Now let $\mathcal{U}$ be the full left Hilbert algebra associated with ϕ, and define

$$\mathcal{U}_0 = \{\xi \in \cap_{\alpha \in \mathbb{C}} \mathcal{D}(\Delta^\alpha) : \Delta^\alpha \xi \in \mathcal{U} \text{ for all } \alpha\}.$$

We have the following result on analyticity of left multiplication operators.

Lemma 5.12. *For a full left Hilbert algebra $\mathcal{U}$, the subspace $\mathcal{M}_a^\phi \cap \pi_\ell(\mathcal{U})$ coincides with $\pi_\ell(\mathcal{U}_0)$.*

Proof. The claim is that for $\xi \in \mathcal{U}$, the function $t \in \mathbb{R} \mapsto \pi_\ell(\Delta^{it}\xi)$ has an entire extension if and only if the function $t \in \mathbb{R} \mapsto \Delta^{it}\xi$ has an entire extension. For $\xi, \eta \in \mathcal{H}$ we have

$$\omega_{\eta,\zeta}(\pi_\ell(\Delta^{it}\xi)) =< \pi_\ell(\Delta^{it}\xi)\eta, \zeta >=< \pi_r(\eta)\Delta^{it}\xi, \zeta >=< \Delta^{it}\xi, \zeta\eta^\flat >,$$

so the function $t \mapsto \omega_{\eta,\zeta}(\pi_\ell(\Delta^{it}\xi))$ has an entire extension if and only if the function $t \mapsto< \Delta^{it}\xi, \zeta\eta^\flat >$ has an entire extension. Since the sets $\{\omega_{\eta,\zeta} : \eta, \zeta \in \mathcal{U}'\}, \mathcal{U}'^2$ are respectively total in $\mathcal{M}_*, \mathcal{H}^*$, the claim follows. By Lemma 2.30, $t \mapsto \Delta^{it}\xi$ has an entire extension if and only if $\xi \in \cap_{\alpha\in\mathbb{C}}\mathcal{D}(\Delta^{\alpha})$, in which case the extension is $\alpha \in \mathbb{C} \mapsto \Delta^{i\alpha}\xi$. Consequently, if $\pi_\ell(\xi) \in \mathcal{M}_a^{\phi} \cap \pi_\ell(\mathcal{U})$, for $\eta, \zeta \in \mathcal{U}'$,

$$< \sigma_\alpha^{\phi}(\pi_\ell(\xi))\eta, \zeta >=< \Delta^{i\alpha}\xi, \zeta\eta^{\flat} >=< \pi_r(\eta)\Delta^{i\alpha}\xi, \zeta >,$$

so that $\Delta^{i\alpha}\xi$ is left-bounded and $\pi_\ell(\Delta^{i\alpha}\xi) = \sigma_\alpha^{\phi}(\pi_\ell(\xi))$. Since $\Delta^{i\alpha}\xi \in \mathcal{D}(\Delta^{1/2}) = \mathcal{D}^{\sharp}$ it follows that $\xi \in \mathcal{U}_0$.

□

Theorem 5.13. *Let $\mathcal{U}$ be a full left Hilbert algebra and let $\mathcal{U}_0$ be as defined above. Then $\mathcal{U}_0$ is a left Hilbert algebra satisfying $\mathcal{U}_0' = \mathcal{U}'$ and $J\mathcal{U}_0 = \mathcal{U}_0$. Moreover, $\{\Delta^{i\alpha}\}_{\alpha\in\mathbb{C}}$ acts on $\mathcal{U}_0$ as a one-parameter group of automorphisms.*

Combining Lemma 5.12 and Theorem 5.13 gives the following corollary.

Corollary 5.14. *The analytic subspace $\mathcal{M}_a^{\phi} \cap \pi_\ell(\mathcal{U})$ is SOT-dense in $\mathcal{M}$.*

proof of Theorem 5.13.

Claim: We have $\mathcal{U}_0 \subseteq \mathcal{U} \cap \mathcal{U}'$ and $J\mathcal{U}_0 = \mathcal{U}_0$.

By definition, we have that $\mathcal{U}_0 \subseteq \mathcal{D}(\Delta^{-1/2})$ and $\Delta^{-1/2}\mathcal{U}_0 = \mathcal{U}_0$, so that

$$J\mathcal{U}_0 = S\Delta^{-1/2}\mathcal{U}_0 = S\mathcal{U}_0 = \mathcal{U}_0.$$

Moreover, if $\xi \in \mathcal{U}_0$, then $(1 + \Delta^{-1})\xi \in \mathcal{U}$, so that by the dual form of Lemma 4.13, we have $\xi = (1 + \Delta^{-1})^{-1}(1 + \Delta^{-1})\xi \in \mathcal{U}'$, proving the claim.

By definition, each $\Delta^{i\alpha}$ gives an automorphism on $\mathcal{U}_0$. Moreover, if $\xi, \eta \in \mathcal{U}_0$, then the function $\alpha \mapsto (\Delta^{i\alpha}\xi)(\Delta^{i\alpha}\eta)$ is entire and extends the function $t \in \mathbb{R} \mapsto \Delta^{it}\xi\eta$, so that $\xi\eta \in \mathcal{U}_0$ and $\Delta^{i\alpha}$ is multiplicative.

Claim: The subalgebra $\mathcal{U}_0^2$ is a core for S.

Let $\xi \in \mathcal{U}$. Then for $r > 0$ set

$$\xi_r = \sqrt{\frac{r}{\pi}} \int_{\mathbb{R}} e^{-rt^2}\Delta^{it}\xi dt.$$

We note that $\xi_r \in \mathcal{D}(\Delta^{i\alpha})$ for all $\alpha \in \mathbb{C}$ by considering the entire function

$$\xi_r(\alpha) = \sqrt{\frac{r}{\pi}} \int_{\mathbb{R}} e^{-r(t-\alpha)^2}\Delta^{it}\xi dt,$$

which, by uniqueness of the extension, defines $\Delta^{i\alpha}\xi_r$. If $\eta \in \mathcal{U}'$, then

$$\begin{aligned} \pi_r(\eta)\Delta^{i\alpha}\xi_r &= \sqrt{\frac{r}{\pi}} \int_{\mathbb{R}} e^{-r(t-\alpha)^2}\pi_r(\eta)\Delta^{it}\xi dt \\ &= \sqrt{\frac{r}{\pi}} \left[\int_{\mathbb{R}} e^{-r(t-\alpha)^2}\sigma_t^{\phi}(\pi_\ell(\xi))dt\right]\eta, \end{aligned}$$

so that $\Delta^{it}\xi_r$ is left-bounded, and hence, belongs to $\mathcal{U}$. Therefore $\xi_r \in \mathcal{U}_0$. Lastly, note that as $r \to 0, \xi_r \to \xi$ and $\Delta^{1/2}\xi_r = (\Delta^{1/2}\xi)_r \to \Delta^{1/2}\xi$, so $\xi_r \to \xi$ in the graph norm of S. Since $\mathcal{U}$ is a core for S, it follows that $\mathcal{U}_0$ is also a core for S. Lastly, note that if $\xi, \zeta \in \mathcal{U}$ then $\mathcal{U}_0 \subseteq \mathcal{U} \cap \mathcal{U}'$ implies that we have that $\xi_r\zeta_r \to \xi\zeta$ in the graph norm of S, so that $\mathcal{U}_0$ is also a core for S. Therefore $\mathcal{U}_0$ is a left Hilbert algebra.

Since $J\mathcal{U}_0 = \mathcal{U}_0$, we also have that $\mathcal{U}_0^2$ is a core for F. Moreover, note that for all $r, \|\pi_\ell(\xi_r)\| \leq \|\pi_\ell(\xi)\|$. Hence $\pi_\ell(\xi_r) \to \pi_\ell(\xi)$ in the SOT. Let $\eta \in \mathcal{H}$ be right-bounded with respect to $\mathcal{U}_0$, so that for some $c > 0$, $\zeta \in \mathcal{U}_0$, implies that $\|\pi_\ell(\zeta)\eta\| \leq c\|\zeta\|$. Then for $\xi \in \mathcal{U}$,

$$\begin{aligned}\|\pi_\ell(\xi)\eta\| &= \lim_{r\to 0} \|\pi_\ell(\xi_r)\eta\| \\ &\leq \lim_{r\to 0} c\|\xi_r\| \\ &= c\|\xi\|,\end{aligned}$$

so that η is right-bounded with respect to $\mathcal{U}$. Since the closure of the involution $^\flat$ coincides for $\mathcal{U}'$ and $\mathcal{U}_0$, it follows that $\mathcal{U}_0' = \mathcal{U}'$.

□

We now turn to prove the characterization of the subalgebra $\mathcal{M}_\phi$ given in Theorem 5.10. The strategy is to use analytic elements to define functions which satisfy a modular conditon and apply this to the special case that $a \in \mathcal{M}_\phi$.

Lemma 5.15. *The analytic subset $\mathcal{M}_a^\phi$ is a *-subalgebra of $\mathcal{M}$ and for $x, y \in \mathcal{M}_a^\phi, \alpha, \beta \in \mathbb{C}$ we have the following:*

$$\begin{aligned}\sigma_\alpha^\phi(xy) &= \sigma_\alpha^\phi(x)\sigma_\alpha^\phi(y); \\ \sigma_{\alpha+\beta}^\phi(x) &= \sigma_\alpha^\phi(\sigma_\beta^\phi(x)); \\ \sigma_\alpha^\phi(x^*) &= \sigma_{\overline{\alpha}}^\phi(x)^*.\end{aligned}$$

Moreover, the subalegbras $\pi_\ell(\mathcal{U}_\phi), m_\phi$ are $\mathcal{M}_a^\phi$-bimodules, and $\pi_\ell(\mathcal{U}_0)$ is two-sided ideal in $\mathcal{M}_a^\phi$.

Proof. Let $x, y \in \mathcal{M}_a^\phi$. Note that by the usual argument for the product of analytic functions, the function $\alpha \mapsto \sigma_\alpha^\phi(x)\sigma_\alpha^\phi(y)$ is entire. Since this extends the function $t \mapsto \sigma_t^\phi(xy)$ on $\mathbb{R}$, this implies that $xy \in \mathcal{M}_a^\phi$, and that $\sigma_\alpha^\phi(xy) = \sigma_\alpha^\phi(x)\sigma_\alpha^\phi(y)$ by the uniqueness of an entire extension from $\mathbb{R}$. To see that $\mathcal{M}_a^\phi$ is self-adjoint, we note that the function $\alpha \mapsto \sigma_{\overline{\alpha}}^\phi(x)^*$ is entire by considering the series representation of the function $\alpha \mapsto \sigma_\alpha^\phi(x)$. Since this is an entire extension of the function $t \mapsto \sigma_t^\phi(x^*)$, it follows that $x^* \in \mathcal{M}_a^\phi$ and $\sigma_\alpha^\phi(x^*) = \sigma_{\overline{\alpha}}^\phi(x)^*$.

Claim: If $a \in \mathcal{M}_a^\phi, s \in \mathbb{R}$, and $\xi \in \mathcal{B} \cap \mathcal{D}(\Delta^s)$, then we also have $a\xi \in \mathcal{B} \cap \mathcal{D}(\Delta^s)$.

To see this consider the function $t \in \mathbb{R} \mapsto \xi(t) = \Delta^{it} a\xi$. Then $\pi_\ell(\xi(t)) = \sigma_t^\phi(a)\pi_\ell(\Delta^{it}\xi)$ so that $\xi(t) = \sigma_t^\phi(a)\Delta^{it}\xi$. By Lemma 2.30, this extends to the function $\alpha \mapsto \sigma_\alpha^\phi(a)\Delta^{i\alpha}\xi$, with domain $\{\alpha : -s \le \mathrm{Im}(\alpha) \le 0\}$, and which is bounded, continuous on its domain, and analytic on the interior. Therefore $a\xi \in \mathcal{B} \cap \mathcal{D}(\Delta^s)$ and $\Delta^{i\alpha} a\xi = \sigma_\alpha^\phi(a)\Delta^{i\alpha}\xi$.

From the claim we easily derive the bimodule properties. Moreover, we note that for $x \in \mathcal{M}_a^\phi, \alpha \in \mathbb{C}$ we have $\pi_\phi(\sigma_\alpha^\phi(x))|_{\mathcal{D}(\Delta^{-i\alpha})} = \Delta^{i\alpha}\pi_\phi(x)\Delta^{-i\alpha}$. The composition rule easily follows.

□

Lemma 5.16.

1) If $a \in \mathcal{M}$ such that $am_\phi, m_\phi a \subseteq m_\phi$, then for any $x, y \in \pi_\ell(\mathcal{U}_0^2)$ there exists an entire function $F \in \mathcal{A}(\mathbb{D})$ such that

$$F(t) = \phi(\sigma_t^\phi(a)xy^*), \qquad F(t+i) = \phi(xy^*\sigma_t^\phi(a)).$$

2) If $a \in \mathcal{M}_a^\phi$ and $z \in m_\phi$, then the function $F_z(\alpha) = \phi(\sigma_\alpha^\phi(a)z)$ is entire and satisfies $F(t+i) = \phi(z\sigma_t^\phi(a))$.

Proof.

1) Consider the function $F(\alpha) =< a\Delta^{-i\alpha}q_\phi(x), \Delta^{-i\overline{\alpha}+1}q_\phi(y) >$. By choice of x and y, it is entire and belongs to $\mathcal{A}(\mathbb{D})$, and by the assumption on a, we have

$$\begin{aligned} F(t) &=< a\Delta^{-it}q_\phi(x), \Delta^{-it+1}q_\phi(y) > \\ &=< \sigma_t^\phi(a)q_\phi(x), \Delta q_\phi(y) > \\ &=< \Delta^{1/2}\sigma_t^\phi(a)q_\phi(x), \Delta^{1/2}q_\phi(y) > \\ &=< Sq_\phi(y), S\sigma_t^\phi(a)q_\phi(x) > \\ &= \phi(\sigma_t^\phi(a)xy^*), \\ F(t+i) &=< a\Delta^{-it+1}q_\phi(x), \Delta^{-it}q_\phi(y) > \\ &=< \Delta^{1/2}q_\phi(x), \Delta^{1/2}\sigma_t^\phi(a^*)q_\phi(y) > \\ &=< S\sigma_t^\phi(a^*)q_\phi(y), Sq_\phi(x) > \\ &= \phi(xy^*\sigma_t^\phi(a)). \end{aligned}$$

2) Assume $z = xy^*$ where $x = \pi_\ell(\xi), y = \pi_\ell(\eta) \in \mathcal{U}$. Then from the proof of Lemma 5.15,

$$\begin{aligned} F_z(\alpha) &= \phi(\sigma_\alpha^\phi(a)\pi_\ell(\xi)\pi_\ell(\eta^\sharp)) \\ &=< S\eta, S\sigma_\alpha(a)\xi > \\ &=< \Delta^{1/2}\sigma_\alpha^\phi(a)\xi, \Delta^{1/2}\eta > \\ &=< \sigma_{-i/2}^\phi(\sigma_\alpha^\phi(a))\Delta^{1/2}\xi, \Delta^{1/2}\eta > \\ &=< \sigma_{\alpha-i/2}^\phi(a)\Delta^{1/2}\xi, \Delta^{1/2}\eta > . \end{aligned}$$

Hence,

$$\begin{aligned}
F_z(t+i) &=< \sigma^{\phi}_{t+i/2}(a)\Delta^{1/2}\xi, \Delta^{1/2}\eta > \\
&=< \Delta^{1/2}\xi, \sigma^{\phi}_{t-i/2}(a^*)\Delta^{1/2}\eta > \\
&=< \Delta^{1/2}\xi, \Delta^{1/2}\sigma^{\phi}_{t}(a^*)\eta > \\
&=< S\sigma_t(a^*)\eta, S\xi > \\
&= \phi(\pi_\ell(\xi)\pi_\ell(\eta^\sharp)\sigma^{\phi}_t(a)) \\
&= \phi(xy^*\sigma^{\phi}_t(a)).
\end{aligned}$$

□

proof of Theorem 5.10. Suppose $a \in \mathcal{M}_\phi$. Then a is analytic so part 1) of Theorem 5.10 holds by Lemma 5.15. If $z \in m_\phi$ and we let F_z be the function as in part 2) of Lemma 5.16, then F_z is constant and satisfies $F_z(0) = \phi(az)$ and $F_z(i) = \phi(za)$. Therefore, part 2) of Theorem 5.10 follows.

Now suppose the conditions hold. Let $\xi, \eta \in \mathcal{U}_0^2, x = \pi_\ell(\xi), y = \pi_\ell(\eta)$ and let F be as in 2) of Lemma 5.16. Since $xy^* \in m_\phi$ and m_ϕ is σ^{ϕ}_t-invariant, by condition 2) we have

$$\begin{aligned}
F(t) &= \phi(\sigma^{\phi}_t(a)xy^*) \\
&= \phi(a\sigma^{\phi}_{-t}(xy^*)) \\
&= \phi(\sigma^{\phi}_{-t}(xy^*)a) \\
&= \phi(xy^*\sigma^{\phi}_t(a)) \\
&= F(t+i).
\end{aligned}$$

Since F is entire and the holomorphic extension is unique it follows that F satisfies $F(\alpha) = F(\alpha + i)$ for $\alpha \in \mathbb{C}$. Since $F \in \mathcal{A}(\mathbb{D})$, t$F$ must be bounded, and so it is constant by Liouville's Theorem. Therefore for all t, $\phi(\sigma^{\phi}_t(a)xy^*) = \phi(axy^*)$. Equivalently

$$\begin{aligned}
0 &=< a\Delta^{-it}\xi, \Delta^{-it+1}\eta > - < a\xi, \Delta\eta > \\
&=< (\sigma^{\phi}_t(a) - a)\xi, \Delta\eta > .
\end{aligned}$$

The result will then follow if we can show that $\Delta\mathcal{U}_0^2$ is dense in $\mathcal{H}$. We already know that $\Delta\mathcal{U}_0 = \mathcal{U}_0 = \Delta^{-1}\mathcal{U}_0$. Moreover, since Δ^α is multiplicative for every α it follows that $\Delta^\alpha\mathcal{U}_0^2 = \mathcal{U}_0^2$ and the claim follows since $\mathcal{U}_0^2$ is dense in $\mathcal{H}$.

□

We can now begin to characterize the weights invariant under a fixed modular automorphism group action σ^ϕ. Let h be a positive, self-adjoint operator affiliated with $\mathcal{M}_\phi$. For

$\epsilon > 0$ the operator $h_\epsilon = h(1+\epsilon h)^{-1}$ belongs to $\mathcal{M}_\phi^+$ and moreover if $\delta < \epsilon$ we have $h_\delta \geq h_\epsilon$. The following lemma says that the set of normal weights $\{\phi_{h_\epsilon}\}_{\epsilon>0}$ with the reverse ordering on $\mathbb{R}_+$ is increasing, so that we can define ϕ_h as the pointwise limit as $\epsilon \to 0$. If h is bounded then $h_\epsilon \to h$ in the SOT so that by normality of each ϕ_{h_ϵ}, these definitions will be consistent.

Lemma 5.17. *The map $h \in \mathcal{M}_\phi^+ \mapsto \phi_h$ is additive, and its range consists of normal semifinite weights.*

Proof. If $x \in m_\phi^+$ then $h^{1/2}xh^{1/2}$ is also in m_ϕ^+ since m_ϕ is a $\mathcal{M}_\phi$-bimodule, so ϕ_h is semifinite. The normality of ϕ_h follows from the normality of ϕ.

Let $h, k \in M_\phi^+$. Since $h, k \leq h+k$, by the Generalized Polar Decomposition Theorem there exists $u, v \in \mathcal{M}_\phi$ such that $h^{1/2} = u(h+k)^{1/2}, k^{1/2} = v(h+k)^{1/2}$ and $u^*u + v^*v$ is the range projection for $h+k$. If $x \in m_{\phi_{h+k}}$, then by Lemma 5.15, the elements

$$u(h+k)^{1/2}x(h+k)^{1/2}, v(h+k)^{1/2}x(h+k)^{1/2}$$

belong to m_ϕ, and by Theorem 5.10,

$$\begin{aligned}
\phi_h(x) + \phi_k(x) &= \phi(h^{1/2}xh^{1/2}) + \phi(k^{1/2}xk^{1/2}) \\
&= \phi(u(h+k)^{1/2}x(h+k)^{1/2}u^*) + \phi(v(h+k)^{1/2}x(h+k)^{1/2}v^*) \\
&= \phi((u^*u + v^*v)(h+k)^{1/2}x(h+k)^{1/2}) \\
&= \phi((h+k)^{1/2}x(h+k)^{1/2}) \\
&= \phi_{(h+k)}(x).
\end{aligned}$$

Now suppose $\phi_h(x), \phi_k(x) < \infty$. Then

$$\begin{aligned}
&(h+k)^{1/2}x(h+k)^{1/2} \\
&= \lim_{\epsilon\to 0}(h+k+\epsilon)^{-1/2}(h+k)x(h+k)(h+k+\epsilon)^{-1/2} \\
&\leq 2(h+k+\epsilon)^{-1/2}(hxh + kxk)(h+k+\epsilon)^{-1/2} \\
&= 2(h+k+\epsilon)^{-1/2}(h+k)^{1/2}(u^*h^{1/2}xh^{1/2}u + v^*k^{1/2}xk^{1/2}v)(h+k)^{1/2}(h+k+\epsilon)^{-1/2} \\
&= u^*h^{1/2}xh^{1/2}u + v^*k^{1/2}xk^{1/2}v.
\end{aligned}$$

Since the element in the last line belongs to m_ϕ, it follows that $\phi_{h+k}(x) < \infty$. By the previous argument, $\phi_{h+k}(x) = \phi_h(x) + \phi_k(x)$. Therefore, $\phi_h + \phi_k = \phi_{h+k}$.

∎

Lemma 5.18. *Let h be a positive self-adjoint operator affiliated with $\mathcal{M}$. If for $x \in \mathcal{M}^+$ we define*

$$\phi_h(x) = \lim_{\epsilon\to 0} \phi_{h_\epsilon}(x),$$

then the function ϕ_h is a semifinite normal weight. It is faithful if and only if h is injective.

Proof. If $h = \int \lambda dE(\lambda)$ is the spectral decomposition of h, note that since h is affiliated to $\mathcal{M}_\phi$, the spectral projections $E_n = E[0,n]$ for $n > 0$ belong to $\mathcal{M}_\phi$. Then $\bigcup_n (E_n m_\phi E_n) \subseteq m_\phi$ is dense in $\mathcal{M}$ in the SOT, and for any $x \in m_\phi^+$, $E_n h$ is bounded, so $\phi_h(E_n x E_n) = \phi_{E_n h}(x) < \infty$. Hence, ϕ_h is semifinite.

The operator h is injective if and only if its range projection p is equal to 1. Since p is also the range projection for $h_\epsilon^{1/2}$ we have $\phi_{h_\epsilon}(1-p) = 0$ so that $\phi_h(1-p) = 0$. Thus ϕ_h is faithful only if h is injective. On the other hand, for $x \in \mathcal{M}^+$ we have $\phi_h(x) = 0$ if and only if $\phi(h_\epsilon^{1/2} x h_\epsilon^{1/2}) = 0$ for all $\epsilon > 0$. Suppose that h is injective, so that each h_ϵ is also injective with dense range. Then for any ϵ, $h_\epsilon^{1/2} x h_\epsilon^{1/2} = 0$ only if $x = 0$, so that ϕ_h is faithful.

□

By Lemmas 5.17 and 5.18, whenever we choose a positive self-adjoint injective operator affiliated with $\mathcal{M}_\phi$, we obtain an fns weight ϕ_h. We can now identify its modular automorphism group.

Theorem 5.19. *If h is a positive self-adjoint injective operator affiliated with $\mathcal{M}_\phi$, then*

$$\sigma_t^{\phi_h}(x) = h^{it}\sigma_t^\phi(x)h^{-it}.$$

Moreover, we have $(D_{\phi_h} : D_\phi)_t = h^{it}$.

Proof. First we assume that h is bounded and invertible. It suffices to check the modular condition for the automorphism group $\{\gamma_t = Ad(h^{it})\sigma_t^\phi\}_{t\in\mathbb{R}}$. First note that by the invertibility of $h^{1/2}$, we have $m_{\phi_h} = m_\phi$. The invariance of ϕ_h under γ follows easily from the fact that $h \in \mathcal{M}_\phi$. Now suppose that $\xi \in \mathcal{U}_0, \eta \in \mathcal{U}$. Using Lemma 4.6, and the fact that Δ^{it}, h commute we have

$$\begin{aligned}\phi_h(h^{it}\sigma_t^\phi(\pi_\ell(\xi))h^{-it}\pi_\ell(\eta)) &= \phi(\pi_\ell(\Delta^{it}h^{it+1}\xi))\pi_\ell(h^{-it}\eta)) \\ &=< h^{-it}\eta, S\Delta^{it}h^{it+1}\xi > .\end{aligned}$$

But from the proof of Lemma 5.15, we have that $h^{it+1}\xi \in \mathcal{U}_0$. In particular, $\Delta^{it}h^{it+1}\xi \in \mathcal{D}(\Delta)$. Hence,

$$\phi_h(h^{it}\sigma_t^\phi(\pi_\ell(\xi))h^{-it}\pi_\ell(\eta)) =< \Delta^{it+1}h^{it+1}\xi, Sh^{-it}\eta > .$$

By Theorem 5.10, we have

$$\begin{aligned}\phi_h(\pi_\ell(\eta)h^{it}\sigma_t^\phi(\pi_\ell(\xi))h^{-it}) &= \phi(\pi_\ell(h^{-it+1}\eta)\pi_\ell(\Delta^{it}h^{it}\xi)) \\ &=< \Delta^{it}h^{it}\xi, Sh^{-it+1}\eta > .\end{aligned}$$

Setting $G(\alpha) =< h^{i\alpha+1}\Delta^{i\alpha+1}\xi, Sh^{-i\alpha}\eta >$, the above calculations give

$$G(t) = \phi_h(\gamma_t(\pi_\ell(\xi))\pi_\ell(\eta)), \qquad G(t+i) = \phi_h(\pi_\ell(\eta)\gamma_t(\pi_\ell(\xi))).$$

Moreover, by choice of ξ and η, G is entire and bounded on $\mathbb{D}$. Then applying a similar argument as in the proof of Theorem 5.2, we can remove the condition that $\xi \in \mathcal{U}_0$. Hence, the theorem holds in this special case.

Now let h be an injective, positive, self-adjoint operator affiliated with $\mathcal{M}_*$. Let $h = \int \lambda dE(\lambda)$ be the spectral decomposition of h, and let $e_n = E[1/n, n]$. Then $\phi_n = \phi|_{\mathcal{M}_{e_n}}$ is an fns weight and by checking the modular condition, we see that the modular automorphism group for ϕ_n is just $\sigma^{\phi}|_{\mathcal{M}_{e_n}}$. Since $h|_{E_n\mathcal{H}}$ is bounded, invertible, by the special case above we have for $x \in \mathcal{M}_{e_n}$,

$$\sigma_t^{\phi\circ h}(x) = (he_n)^{it}x(he_n)^{-it} = h^{it}xh^{-it}.$$

The result follows since $\sigma_t^{\phi\circ h}, \sigma_t^{\phi}$ are continuous the σ-WOT. The last assertion can be easily verified using the construction of the Connes cocycle derivative.

□

Now that we have identified a large class of weights which come from perturbing a fixed weight ϕ, we use the Connes Cocycle Derivative Theorem to characterize the σ^{ϕ}-invariant weights. We begin with the following uniqueness result for cocycle derivatives.

Lemma 5.20. *If ϕ, ψ, χ are fns weights on $\mathcal{M}$, and if for all $t \in \mathbb{R}$ we have*

$$(D_\psi : D_\phi)_t = (D_\chi : D_\phi)_t,$$

then $\psi = \chi$.

Proof. First note that by Lemma 5.7, and the fact that $(D_\psi : D_\phi)_t^{-1} = (D_\phi : D_\psi)_t$, it suffices to prove that if for all $t \in \mathbb{R}$ we have $(D_\psi : D_\phi)_t = 1$, then $\phi = \psi$. But if we let ρ be the weight on $\mathcal{N} = \mathcal{M} \otimes M_2(\mathbb{C})$ as before, this condition implies that the element $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ belongs to the fixed point algebra $\mathcal{N}_\rho$. By Theorem 5.10, we have for $x \in \mathcal{M}^+$,

$$\phi(x) = \rho\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \psi(x),$$

completing the proof.

□

Theorem 5.21. *For two fns weights ϕ, ψ on a von Neumann algebra $\mathcal{M}$, the following are equivalent:*

1) ψ is σ^{ϕ}-invariant;
2) $\psi = \phi_h$ for some non-singular, positive, self-adjoint operator affiliated with $\mathcal{M}_\phi$.

Proof.

1)$\Longrightarrow$2): Combining Corollary 5.8 with the invariance assumption on ψ, we have

$$\begin{aligned}(D_\psi : D_\phi)_{-s}(D_\psi : D_\phi)_{s+t} &= \sigma_s^\phi[(D_\psi : D_\phi)_t] \\ &= (D_{\psi\circ\sigma_{-s}^\phi} : D_{\phi\circ\sigma_{-s}^\phi})_t \\ &= (D_\psi : D_\phi)_t.\end{aligned}$$

Therefore, $\{(D_\psi : D_\phi)_t\}$ is a one-parameter group of unitaries belonging to $\mathcal{M}_\phi$ which is continuous in the SOT. By Stone's Theorem there exists a positive, injective, self-adjoint operator h affiliated with $\mathcal{M}_\phi$ such that for all $t \in \mathbb{R}$, $(D_\psi : D_\phi)_t = h^{it}$. By Lemma 5.20 and Theorem 5.19 we have that $\psi = \phi_h$.

2)$\Longrightarrow$1): By Theorem 5.10, we have for $x \in m_{\phi_h}$,

$$\begin{aligned}\psi(x) &= \psi \circ h^{it}\sigma_t^\phi(x)h^{-it} \\ &= \psi \circ \sigma_t^\phi(x).\end{aligned}$$

□

Finally, we can apply the above results to semifinite von Neumann algebras.

Theorem 5.22. *For a von Neumann algebra $\mathcal{M}$ the following are equivalent:*

1) the algebra $\mathcal{M}$ is semifinite;
2) every modular automorphism group $\{\sigma_t^\phi\}_{t\in\mathbb{R}}$ is inner;
3) there exists a weight ϕ such that $\{\sigma_t^\phi\}_{t\in\mathbb{R}}$ is inner.

Proof.
1)$\Longrightarrow$2): Let τ be an fns trace on $\mathcal{M}$, ϕ an fns weight. Then the automorphism group σ^τ is trivial so that ϕ is σ^τ-invariant. By Theorem 5.21, it is the form τ_h for some positive self-adjoint operator h affiliated with $\mathcal{M}_\tau = \mathcal{M}$. Hence $h^{it} \in \mathcal{M}$ and $\sigma_t^\phi = \mathrm{Ad}(h^{it}) \circ \sigma_t^\tau = \mathrm{Ad}(h^{it})$.

2)$\Longrightarrow$3) is trivial.

3)$\Longrightarrow$1): Let ϕ be an fns weight such that $\sigma_t^\phi = Ad(u_t)$, where $\{u_t\}$ is a unitary group in $\mathcal{M}$ continuous in the SOT. Since $\{u_t\} \subseteq \mathcal{M}_\phi$, by Stone's theorem there exists a positive injective self-adjoint operator h affiliated with $\mathcal{M}_\phi$ such that $u_t = h^{it}$. Then

$$\sigma_t^{\phi_{h^{-1}}} = Ad(h^{-it})Ad(h^{it}) = id,$$

so that $\phi_{h^{-1}}$ is a trace.

□

To give an idea of the significance of this theorem, in Section 7 we are going to look at the crossed product von Neumann algebra $\mathcal{M} \rtimes_{\sigma^\phi} \mathbb{R}$ with respect to modular automorphisms. As it turns out, the relationship between the algebras $\mathcal{M}$ and $\mathcal{M} \rtimes_{\sigma^\phi} \mathbb{R}$ is interesting only when σ^ϕ is not inner.

6 Equivalence of Hilbert Algebras and Weights

In Sections 3 and 4, it was shown that if ϕ is an fns weight on a von Neumann algebra $\mathcal{M}$, then $\mathcal{M}$ can be realized as the von Neumann algebra generated by left multiplication operators on the full Left Hilbert algebra $\mathcal{U}_\phi = q_\phi(n_\phi \cap n_\phi^*)$. We summarize this in the following Theorem.

Theorem 6.1. *Let $\mathcal{M}$ be a von Neumann algebra, let ϕ be an fns weight on $\mathcal{M}$, and set*

$$\mathcal{U}_\phi = q_\phi(n_\phi \cap n_\phi^*).$$

Define an involution and product on $\mathcal{U}_\phi$ by

$$q_\phi(x)^\sharp = q_\phi(x^*), \qquad\qquad q_\phi(x)q_\phi(y) = q_\phi(xy).$$

Then $\mathcal{U}_\phi$ is a full left Hilbert algebra. If we let $(\pi_\phi, \mathcal{H}_\phi)$ be the GNS representation of $\mathcal{M}$ coming from ϕ, then $\pi_\phi(\mathcal{M})$ is unitarily equivalent to $\mathcal{R}_\ell(\mathcal{U})$.

In this Section we complete the picture by showing that full left Hilbert algebras characterize the GNS representation spaces corresponding to fns weights. Let $\mathcal{U}$ be a full left Hilbert algebra with Hilbert space completion $\mathcal{H}$, and let $\mathcal{M} = \mathcal{R}_\ell(\mathcal{U})$. Let $\mathcal{B} \subseteq \mathcal{H}$ to be the set of left bounded elements and let $n_\ell = \pi_\ell(\mathcal{B})$ so that $\pi_\ell(\mathcal{U}) = n_\ell \cap n_\ell^*$. Finally, let $m_\ell = n_\ell^* n_\ell = \{\sum_{i=1}^n x_i^* y_i : x_i, y_i \in n_\ell\}$. Define a function $\phi_\ell : \mathcal{M}_+ \to [0, \infty]$ by

$$\phi_\ell(x) = \begin{cases} \|\xi\|^2 & \text{if } x^{1/2} = \pi_\ell(\xi) \text{ for } \xi \in \mathcal{U}, \\ \infty & \text{otherwise.} \end{cases}$$

In this Section we prove the following Theorem. The proof follows that given in Chapter 7 of [13].

Theorem 6.2. *The function ϕ_ℓ is an fns weight on $\mathcal{M}$ with domain of definition m_ℓ and for which n_ℓ corresponds to the left ideal n_{ϕ_ℓ}. Moreover the GNS representation $(\pi_{\phi_\ell}, \mathcal{H}_{\phi_\ell}, \gamma_{\phi_\ell})$ is unitarily equivalent to $\mathcal{M} = \mathcal{R}_\ell(\mathcal{U})$ via the unitary U satisfying $U\xi = q_{\phi_\ell}(\pi_\ell(\xi))$ for $\xi \in \mathcal{B}$.*

We will prove the Theorem by a series of Lemmas.

Lemma 6.3. *The function ϕ_ℓ is a semifinite weight, and $m_\ell^+ = \{x \in \mathcal{M}^+ : \phi_\ell(x) < \infty\}$.*

Proof.

Claim: If $x \in m_\ell^+$ then $x^{1/2} \in n_\ell$. Hence $m_\ell^+ = \{x \in \mathcal{M}^+ : \phi_\ell(x) < \infty\}$ and m_ℓ^+ is hereditary.

First, assume that x is of the form $\sum_i^n x_i^* x_i$ for $x_i \in n_\ell$. Then by the Generalized Polar Decomposition Theorem there exists $s_i \in \mathcal{M}$ such that $x_i = s_i x^{1/2}$ and $\sum_{i=1}^n s_i$ is the range projection of x. For each i, let $\xi_i \in \mathcal{B}$ be chosen such that $x_i = \pi_\ell(\xi_i)$ and set $\xi = \sum_{i=1}^n s_i^* \xi_i$. Then $\xi \in \mathcal{B}$ and

$$\pi_\ell(\xi) = \sum_{i=1}^n s_i^* \pi_\ell(\xi) = \sum_{i=1}^n s_i^* x_i = \sum_{i=1}^n s_i^* s_i x^{1/2} = x^{1/2}.$$

Now suppose that $x \in m_\ell^+$ is of the form $\sum_{i=1}^n y_i^* z_i$ for $y_i, z_i \in n_\ell$ and that $y \in \mathcal{M}^+$ such that $y \leq x$. Then

$$y \leq x = \frac{1}{2}(x + x^*) = \frac{1}{2}\sum_{i=1}^n (y_i^* z_i + z_i^* y_i) \leq \frac{1}{2}\sum_{i=1}^n (y_i^* y_i + z_i^* z_i).$$

As shown above, the right hand side is of the form $\pi_\ell(\xi)$ for some $\xi \in \mathcal{B}$, and by the Generalized Polar Decomposition Theorem there exists $s \in \mathcal{M}$ such that

$$y^{1/2} = s\pi_\ell(\xi) = \pi_\ell(s\xi),$$

so $y^{1/2} \in n_\ell$. Hence $y \in m_\ell^+$, and the claim follows.

It is clear that ϕ_ℓ is homogeneous for positive scalars. Suppose that $x, y \in m_\ell^+$ so that $z = x + y \in m_\ell^+$. Let $z^{1/2} = \pi_\ell(\xi)$ for $x \in \mathcal{B}$ and let $s, t \in \mathcal{M}$ be such that

$$x^{1/2} = sz^{1/2} = \pi_\ell(s\xi), y^{1/2} = tz^{1/2} = \pi_\ell(t\xi),$$

and $s^*s + t^*t$ is the range projection of z. We then have

$$\phi_\ell(x) + \phi_\ell(y) = \|s\xi\|^2 + \|t\xi\|^2 =< (s^*s + t^*t)\xi, \xi >= \|\xi\|^2 = \phi_\ell(z),$$

where the second last equality follows from the fact that $\xi \in [\pi_\ell(\xi)\mathcal{H}]$. On the other hand, let $x, y \in \mathcal{M}^+$, let $z = x + y$, and suppose that $\phi_\ell(z) < \infty$ so that $z \in m_\ell^+$. Since m_ℓ^+ is hereditary we also have $x, y \in m_\ell^+$ so that $\phi_\ell(x), \phi_\ell(y) < \infty$. By the above work, we have $\phi_\ell(z) = \phi_\ell(x) + \phi_\ell(y)$. Therefore, ϕ_ℓ is a semifinite weight.

□

Let U be the unitary as defined in Theorem 6.2. For $\xi \in \mathcal{B}$, we have

$$\|U\xi\|^2 = \|q_{\phi_\ell}(\pi_\ell(\xi))\|^2 = \phi_\ell(\pi_\ell(\xi)^*\pi_\ell(\xi)) = \|\xi\|^2,$$

so that U extends to a unitary $U : \mathcal{H} \to \mathcal{H}_{\phi_\ell}$. If $\xi, \eta \in \mathcal{U}$, we have

$$\begin{aligned}(U\pi_\ell(\xi)U^*)q_{\phi_\ell}(\pi_\ell(\eta)) &= U\pi_\ell(\xi)\eta \\ &= U\xi\eta \\ &= q_{\phi_\ell}(\pi_\ell(\xi\eta)) \\ &= q_{\phi_\ell}(\pi_\ell(\xi)\pi_\ell(\eta)) \\ &= \pi_{\phi_\ell}(\pi_\ell(\xi))q_{\phi_\ell}(\pi_\ell(\eta)).\end{aligned}$$

Since $q_{\phi_\ell}(\pi_\ell(\mathcal{U}))$ is dense in $\mathcal{H}_{\phi_\ell}$ we have $U\pi_\ell(\xi)U^* = \pi_{\phi_\ell}(\pi_\ell(\xi))$. Since $\pi_\ell(\mathcal{U})$ is SOT-dense, we can conclude that this will hold for all $a \in \mathcal{M}$ if we can show that π_{ϕ_ℓ} is a normal representation. This will then follow by Proposition 3.6 once we show that ϕ_ℓ is normal, which is what we now turn to prove.

For $\eta \in \mathcal{B}'$ define $\omega_\eta \in \mathcal{M}_*^+$ by $\omega_\eta(x) =< x\eta, \eta >$. Define

$$\Phi_{\ell,0} = \{\omega_\eta : \eta \in \mathcal{B}', \|\pi_r(\eta)\| < 1\}.$$

We analogously define set n_r, m_r for the right hilbert algebra $\mathcal{U}'$.

Lemma 6.4. *There exists a positive map $\theta : m_r \to \mathcal{M}_*$ such that*

$$\theta(\pi_\ell(\zeta)^*\pi_\ell(\eta)) = \omega_{\eta,\zeta}, \qquad \textit{for } \eta, \zeta \in \mathcal{B}'$$

and θ maps the open unit ball of m_r^+ onto $\Phi_{\ell,0}$.

Proof. By the dual form of Lemma 6.3, every element of m_r^+ is of the form x^*x for some $x \in n_r$, so consider the map θ given by $x^*x \mapsto \omega_\eta$ where $x = \pi_r(\eta), \eta \in \mathcal{B}'$. We check that this map is well-defined. If $x, y \in n_r$ are such that $x^*x = y^*y$, and if we let $s \in \mathcal{M}'$ be such that $x = sy$ and ker $s \subseteq (\text{ran } y)^\perp$, then by construction s will be a partial isometry such that s^*s is the range projection of y, and ss^* is the range projection of x. If $x = \pi_\ell(\eta), y = \pi_\ell(\zeta), \xi \in \mathcal{U}$, then

$$\begin{aligned}< \pi_\ell(\xi)\eta, \eta > &=< \pi_r(\eta)\xi, \eta > \\ &=< s\pi_r(\zeta)\xi, \eta > \\ &=< \pi_r(\zeta)\xi, s^*\eta > \\ &=< \pi_\ell(\xi)\zeta, \zeta >,\end{aligned}$$

which proves the claim.

We now show that θ can be extended to a linear map on m_r. Since m_r is spanned by its positive elements, is suffices to show that θ is additive and positive scalar homogeneous on m_r^+. We just prove additivity. Let $x, y \in n_r$ and let $z = x^*x + y^*y$. Then there exists

$s, t \in \mathcal{M}'$ such that $x = sz^{1/2}$ and $y = tz^{1/2}$ and $p = s^*s + t^*t$ is the range projection of z. Let $z^{1/2} = \pi_r(\eta)$ for $\eta \in \mathcal{B}'$, so that $x = \pi_r(s\eta), y = \pi_r(t\eta)$. Then for $\xi \in \mathcal{U}$,

$$\begin{aligned} < \pi_\ell(\xi), \theta(x^*x) > + < \pi_\ell(\xi)\theta(y^*y) > &=< \pi_\ell(\xi)s\eta, s\eta > + < \pi_\ell(\xi)t\eta, t\eta > \\ &=< \pi_r(s\eta)\xi, s\eta > + < \pi_r(t\eta)\xi, t\eta > \\ &=< (s^*s + t^*t)\pi_r(\eta)\xi, \eta > \\ &=< \pi_r(\eta)\xi, \eta > \\ &=< \pi_\ell(\xi), \theta(z) >, \end{aligned}$$

so taking limits in the WOT, we have $\theta(z) = \theta(x^*x)+\theta(y^*y)$, and the claim follows. Moreover, by polarizing we find that for $\eta, \zeta \in \mathcal{B}'$,

$$\theta(\pi_\ell(\zeta)^*\pi_\ell(\eta)) = \omega_{\eta,\zeta},$$

and the last claim follows immediately by construction of θ.

□

Lemma 6.5. *For $x \in \mathcal{M}^+$, $\phi_\ell(x) = \sup_{\omega\in\Phi_{\ell,0}} \omega(x)$. Hence ϕ_ℓ is normal.*

Proof. Define $\psi : \mathcal{M}^+ \to [0, \infty]$ by $\phi(x) = \sup_{\omega\in\Phi_{\ell,0}} \omega(x)$. If $a \in m_\ell^+$, then by Lemma 6.3, $a^{1/2} = \pi_\ell(\xi)$ for some $\xi \in \mathcal{U}$, so that $\phi_\ell(a) = \|\xi\|^2$. By definition, we have

$$\begin{aligned} \psi(a) &= \sup_{\omega\in\Phi_{\ell,0}} \omega(a) \\ &= \sup_{\eta\in\mathcal{B}',\|\pi_r(\eta)\|<1} \omega^r_\eta(a) \\ &= \sup_{\eta\in\mathcal{B}',\|\pi_r(\eta)\|<1} \|\pi_r(\eta)\xi\|^2 \\ &\leq \|\xi\|^2. \end{aligned}$$

But $\pi_r(\mathcal{B}') \cap \pi_r(\mathcal{B}')^*$ acts non-degenerately so that there exists a net $\{\pi_{\eta_i}\}$ in the unit ball of $\pi_r(\mathcal{B}')$ which converges to 1 in the SOT, so that $\|\xi\|^2 = \lim_i\|\pi_r(\eta_i)\xi\|^2 \leq \psi(a)$. Hence, $\phi_\ell(a) = \psi(a)$.

Now suppose that $\psi(a) < \infty$ and define a linear functional ω_a on m_r by $\omega_a(x) =< a, \theta(x) >$. Since θ is positive we have that ω_a is positive, and since $\theta(x) \in \Phi_{\ell,0}$, if $\|x\| \leq 1$ we have $\omega_a(x) \leq \psi(a) < \infty$. Let $x \in m_r$ be self-adjoint, and let $\mathcal{A}_x$ be the von Neumann algebra generated by x. Then $m_r \cap \mathcal{A}_x$ sits as an ideal in $\mathcal{A}_x$ so that in particular, if we write $x = x_+ - x_-$, the positive and negative parts of x, with range projections $p_+, p_- \in \mathcal{A}_x$, then $x_+ = xp_+, x_- = xp_-$ both belong to m_r. If $x \in m_r$ is arbitrary we can write $x = h + ik$, where $h = \frac{1}{2}(x + x^*), k = \frac{1}{2}(ix - ix^*) \in m_r$ are self-adjoint. Then

$$\begin{aligned} |\omega_a(x)| &\leq \omega_a(h_+) + \omega_a(h_-) + \omega_a(k_+) + \omega_a(k_+) \\ &\leq \psi(a)(\|h_+\| + \|h_-\| + \|k_+\| + \|k_+\|) \\ &\leq 4\psi(a)\|x\|. \end{aligned}$$

Hence ω_a is bounded, so extends to a bounded positive linear functional, also denoted ω_a, on the norm closure A_r of m_r such that $\|\omega_a\| \leq 4\psi(a)$. For $x \in A_r$, using an approximate identity and the Cauchy-Schwartz inequality, we see that $|\omega_a(x)|^2 \leq \|\omega_a\|\omega_a(x^*x)$. Then for any $\eta \in \mathcal{U}'$, we have

$$\begin{aligned} |\omega_a(\pi_r(\eta))| &\leq 2\psi(a)^{1/2}\omega_a(\pi_r(\eta))^*\pi_r(\eta))^{1/2} \\ &= 2\psi(a)^{1/2} < a\eta, \eta >^{1/2} \\ &= 2\psi(a)^{1/2}\|a^{1/2}\eta\|. \end{aligned}$$

Therefore the linear functional $a^{1/2}\eta \to \omega_a(\pi_r(\eta))$ is bounded, so there exists $\xi \in [a^{1/2}\mathcal{H}]$ such that for all $\eta \in \mathcal{U}'$, $\omega_a(\pi_r(\eta)) =< a^{1/2}\eta, \xi >$. If $\zeta, \eta \in \mathcal{U}'$,

$$\begin{aligned} < a^{1/2}\eta, \pi_r(\zeta)\xi > &=< \pi_r(\zeta^\flat)a^{1/2}\eta, \xi > \\ &=< a^{1/2}\pi_r(\zeta^\flat)\eta, \xi > \\ &=< a^{1/2}\eta\zeta^\flat, \xi > \\ &= \omega_a(\pi_r(\eta\zeta^\flat)) \\ &= \omega_a(\pi_r(\zeta)^*\pi_r(\eta)) \\ &=< a\eta, \zeta > \\ &=< a^{1/2}\eta, a^{1/2}\zeta > . \end{aligned}$$

If we know that $\pi_r(\zeta)\xi \in [a^{1/2}\mathcal{H}]$, then the above calculation shows that $\pi_r(\zeta)\xi = a^{1/2}\zeta$, so that ξ is left bounded and $\pi_\ell(\xi) = a^{1/2}$. But $\xi \in [a^{1/2}\mathcal{H}]$ and $a^{1/2}$ commutes with $\pi_r(\zeta)$ so that by continuity $\pi_r(\zeta)\xi \in [\pi_r(\zeta)a^{1/2}\mathcal{H}] = [a^{1/2}\pi_r(\zeta)\mathcal{H}] \subseteq [a^{1/2}\mathcal{H}]$ and the claim follows. Hence $a^{1/2} \in n_\ell$ which implies that $a \in m_\ell$. Then as shown above, this implies that $\phi_\ell(a) = \psi(a)$. Therefore, $\phi = \psi$.

With the above identification, suppose $\{x_i\}$ is an increasing net in $\mathcal{M}^+$ with $x = \sup_i x_i$, and let $\epsilon > 0$. Let $\omega \in \Phi_{\ell,0}$ such that $\omega(x) > \phi(x) - \epsilon$. Since ω is normal, there exists i such that $j \geq i$ implies $\omega(x_j) > \omega(x) - \epsilon$. Then $\phi(x_j) \geq \omega(x_j) > \omega(x) - \epsilon > \phi(x) - 2\epsilon$, and so $\sup_i \phi(x_i) \geq \phi(x)$. Since the reverse inequality is trivial, ϕ is normal.

$\square$

7 Crossed Products and Dual Weights

In this Section we introduce the von Neumann crossed product as a tool to construct new von Neumann algebras. Given an action $\alpha : G \to Aut(\mathcal{M})$ where G is a locally compact group G, $\mathcal{M}$ is a von Neumann algebra, we would like to generalize the notion of the semidirect product of groups to construct a von Neumann algebra which is an extension of $\mathcal{M}$ by the subgroup $\alpha(G) \leq Aut(\mathcal{M})$. The material of this Section is from Chapter 10 of [13].

We require that all G-actions are continuous in the sense that if $x \in \mathcal{M}$, then the function $t \in G \mapsto \alpha_t(x)$ is SOT-continuous, and we say $(\mathcal{M}, G, \alpha)$ is a **covariant system**. Recall that if ϕ is an fns weight on a von Neumann algebra $\mathcal{M}$ then the modular automorphism group is of the form $\sigma_t^\phi = Ad(\Delta^{it})$ where Δ is a positive self-adjoint operator. In particular, the one-parameter unitary group $t \mapsto \Delta^{it}$ is SOT-continuous, and hence for fixed $x \in \mathcal{M}$, the map $t \mapsto \sigma_t^\phi(x)$ is also SOT-continuous. Hence $(\mathcal{M}, \mathbb{R}, \sigma^\phi)$ is a covariant system.

Definition 7.1. *Let $(\mathcal{M}, G, \alpha)$ be a covariant system. Then a normal representation $\rho : \mathcal{M} \to B(\mathcal{K})$ of $\mathcal{M}$ together with a SOT-continuous unitary representation $U : G \to B(\mathcal{K})$ are said to be* ***covariant*** *if $\rho \circ \alpha_t = Ad(U_t) \circ \rho$.*

Let (ρ, U) be a covariant representation of a covariant system $(\mathcal{M}, G, \alpha)$ and define $\mathcal{M}_\rho[G]$ to be the set of all operators of the form

$$\sum_{s \in G} \rho(x_s) U_s, \qquad x_s \in \mathcal{M}, x_s = 0 \text{ for all but finitely many } s.$$

Then $\mathcal{M}_\rho[G]$ is a *-subalgebra of $B(\mathcal{K})$ since we have

$$\begin{aligned}\left(\sum_{s \in G} \rho(x_s) U_s\right)\left(\sum_{s \in G} \rho(y_s) U_s\right) &= \sum_{s,t \in G} \rho(x_s) U_s \rho(y_t) U_t \\ &= \sum_{s,t \in G} \rho(x_s) \rho(\alpha_s(y_t)) U_{st} \\ &= \sum_{r \in G} \rho(\sum_{s \in G} x_s \alpha_s(y_{s^{-1}r})) U_r,\end{aligned}$$

and

$$\left(\sum_{s\in G}\rho(x_s)U_s\right)^* = \sum_{s\in G}U_s^*\rho(x_s^*)$$
$$= \sum_{s\in G}U_s\rho(x_{s^{-1}}^*)$$
$$= \sum_{s\in G}\rho(\alpha_s(x_{s^{-1}})^*)U_s.$$

Therefore it is natural to think of the von Neumann algebra generated by $\mathcal{M}_\rho[G]$ as an extension of $\mathcal{M}$ by $\alpha(G)$. We will now show that this can always be done in a canonical fashion.

Let $\pi : \mathcal{M} \to B(\mathcal{H})$ be a normal representation, and let $C_c(G,\mathcal{H})$ be the space of all compactly supported continuous function $\xi : G \to \mathcal{H}$. For $\xi, \eta \in C_c(G,\mathcal{H})$ define the inner-product

$$< \xi, \eta >= \int < \xi(t), \eta(t) > dt.$$

Let $L^2(G,\mathcal{H})$ be the Hilbert space completion of $C_c(G,\mathcal{H})$ with respect to this inner product. For $\xi \in C_c(G,\mathcal{H}), x \in \mathcal{M}$ consider the function $s \in G \mapsto \pi(\alpha_s^{-1}(x))\xi(s)$. This will again belong to $C_c(G,\mathcal{H})$ by the continuity assumption on α, and moreover,

$$\int \|\pi(\alpha_s^{-1}(x))\xi(s)\|^2 ds \leq \|x\|^2\|\xi\|^2,$$

so that we obtain an operator $\pi_\alpha(x) \in B(L^2(G,\mathcal{H}))$. Suppose that π is a faithful representation. For non-zero $x \in \mathcal{M}$, we choose $\xi \in \mathcal{H}$ such that $\pi(\alpha_1(x))\xi \neq 0$. If we let $f \in C_c(G)$ be chosen such that $f(1) \neq 0$, we can define the function $f_\xi \in C_c(G,\mathcal{H})$ by $f_\xi(s) = f(s)\xi$. Then we have $(\pi_\alpha(x)f_\xi)(1) = \pi(\alpha_1(x))f(1)\xi \neq 0$, so that $\pi_\alpha(x)f_\xi \neq 0$. Therefore π_α is also a faithful represenentation. We now show that π_α is normal. Let $\{x_i\}$ be an increasing net in $\mathcal{M}^+$ with $x = \sup_i x_i$. Let $\xi \in \mathcal{H}, f \in C_c(G)$. Then

$$< \pi_\alpha(x_i)f_\xi, f_\xi >= \int |f(s)|^2 < \alpha_{s^{-1}}(x_i)\xi, \xi > ds.$$

The functions $s \mapsto < \alpha_{s^{-1}}(x_i)\xi, \xi >$ are continuous and monotone increasing, so by Dini's theorem converge uniformly on the support of f. Hence $< \pi_\alpha(x_i)f_\xi, f_\xi > \to < \pi_\alpha(x)f_\xi, f_\xi >$, and since the elements f_ξ span a dense subspace of $L^2(G,\mathcal{H})$, it follows that $\pi_\alpha(x) = \sup_i \pi_\alpha(x_i)$, and therefore, π_α is normal.

Now for $\xi \in C_c(G,\mathcal{H})$, $t \in G$, define

$$(\lambda_{\mathcal{H}}(t)\xi)(s) = \xi(t^{-1}s).$$

Then it is easy to check that $\lambda_{\mathcal{H}}$ extends to an SOT-continuous unitary representation of G in $L^2(G,\mathcal{H})$. Moreover, for $x \in \mathcal{M}$, we have

$$\begin{aligned}(\lambda_{\mathcal{H}}(t)\pi_\alpha(x)\lambda_{\mathcal{H}}(t)^*)\xi)(s) &= (\pi_\alpha(x)\lambda_{\mathcal{H}}(t)^*\xi)(t^{-1}s)\\ &= \alpha_{s^{-1}t}(x)(\lambda_{\mathcal{H}}(t)^*\xi)(t^{-1}s)\\ &= (\pi_\alpha(\alpha_t(x))\xi)(s),\end{aligned}$$

so $(\pi_\alpha, \lambda_{\mathcal{H}})$ is a covariant representation. We have the following uniqueness result.

Lemma 7.2. *Let $(\pi,\mathcal{H}), (\rho,\mathcal{K})$ be faithful representations of $\mathcal{M}$. If we let $(\pi_\alpha,\lambda_{\mathcal{H}}), (\rho_\alpha,\lambda_{\mathcal{K}})$ be covariant representations as above, then there exists a unique isomorphism*

$$\Phi : (\pi_\alpha(\mathcal{M}) \cup \lambda_{\mathcal{H}}(G))'' \to (\rho_\alpha(\mathcal{M}) \cup \lambda_{\mathcal{K}}(G))''$$

such that $\Phi \circ \pi_\alpha = \rho_\alpha$ and $\Phi \circ \lambda_{\mathcal{H}} = \lambda_{\mathcal{K}}$.

Proof. By Theorem 5.5 on page 222 of [12], there exists a Hilbert space $\mathcal{H}_0$ and a unitary $U : \mathcal{H}_0 \otimes \mathcal{K} \to \mathcal{H}_0 \otimes \mathcal{H}$ such that for $x \in \mathcal{M}$,

$$U(1_{\mathcal{H}_0} \otimes \rho(x))U^* = 1_{\mathcal{H}_0} \otimes \pi(x).$$

Then $\tilde{U} = U \otimes 1_{L^2(G)} : L^2(G,\mathcal{H}_0 \otimes \mathcal{K}) \to L^2(G,\mathcal{H}_0 \otimes \mathcal{H})$ is a unitary such that

$$\begin{aligned}\tilde{U}(1_{\mathcal{H}_0} \otimes \rho_\alpha(x))\tilde{U}^* &= 1_{\mathcal{H}_0} \otimes \pi_\alpha(x),\\ \tilde{U}(1_{\mathcal{H}_0} \otimes \lambda_{\mathcal{K}}(s))\tilde{U}^* &= 1_{\mathcal{H}_0} \otimes \lambda_{\mathcal{H}}(s),\end{aligned}$$

and the proof easily follows.

□

Now let ϕ and ψ be fns weights defined on $\mathcal{M}$. This yields covariant systems $(\mathcal{M},\mathbb{R},\sigma^\phi)$ and $(\mathcal{M},\mathbb{R},\sigma^\psi)$. The Connes Cocycle Derivative Theorem says that there exists an SOT-continuous family of unitaries $\{u_t\}$ such that $u_{s+t} = u_s\sigma^\psi_s(u_t)$ and $\sigma^\phi_t = Ad(u_t)\sigma^\psi_t$. We now consider the implications of this result to the study of covariant representations.

Definition 7.3. *A SOT-continuous family $\{u_t\}_{t\in G}$ of unitaries is called an α-**cocycle** if $u_{st} = u_s\alpha_s(u_t)$. We denote by $Z^1(\mathcal{M},\alpha)$ the set of all α-cocycles.*

If $u \in Z^1(\mathcal{M},\alpha)$, we define a new action $\beta : G \to \mathrm{Aut}(\mathcal{M})$ by $\beta_t(x) = u_t\alpha_t(x)u_t^*$. We say that β is **cocycle conjugate** to α and write $\beta =_u \alpha$. This defines an equivalence relation of G-actions on $\mathcal{M}$, called **cocycle classes**, for if $\beta =_u \alpha$ the family $u^* = \{u_t^*\}_{t\in G}$ is a β-cocycle and $\alpha =_{u^*} \beta$. In particular, the Connes Cocycle Derivative Theorem together, with its converse, says that the cocycle class of the modular action σ^ϕ of an fns weight ϕ is precisely the actions of the form σ^ψ where ψ is an fns weight.

Lemma 7.4. *For $u \in Z^1(\mathcal{M}, \alpha)m$, define a unitary U on $L^2(G, \mathcal{H})$ by $(U\xi)(s) = u_{s^{-1}}\xi(s)$. Then*

$$U(\pi_\alpha(\mathcal{M}) \cup \lambda_{\mathcal{H}}(G))''U^* = (\pi_{{}_u\alpha}(\mathcal{M}) \cup \lambda_{\mathcal{H}}(G))''.$$

The proof is a simple computation. With Lemmas 7.2 and 7.4 the following definition makes sense.

Definition 7.5. *Let $(\mathcal{M}, G, \alpha)$ be a covariant system and assume that $\mathcal{M}$ is standard in the sense that it acts on the Hilbert space $\mathcal{H} = \mathcal{H}_\phi$ via the normal representation π_ϕ for some fns weight ϕ. Let (π_α, λ) be the associated covariant representation on $L^2(G, \mathcal{H}_\phi)$. Then the von Neumann algebra* ***crossed product*** *$\mathcal{M} \rtimes_\alpha G$, defined up to unitary equivalence, is the von Neumann algebra generated by $\pi_\alpha(\mathcal{M}) \cup \lambda(G)$.*

We now shall fix an fns weight ϕ on $\mathcal{M}$ and normal representation $(\pi_\phi, \mathcal{H}_\phi)$, with left Hilbert algebra $\mathcal{U}_\phi$ and modular automorphism group $\{\sigma_t^\phi\}_{t\in\mathbb{R}}$. For convenience we will supress the representation π_ϕ and write $\mathcal{H}$ for $\mathcal{H}_\phi$. We now want to find a weight on $\mathcal{M} \rtimes_\alpha G$ in terms of the weight ϕ and for that we need another way to construct the crossed product. As a motivating example, note that we have two ways to view the left group von Neumann algebra $\mathcal{R}_\ell(G)$. The first way is as the crossed product $\mathbb{C} \rtimes_{\mathrm{id}} G$, where id denotes the trivial action of G. The other way is as the algebra generated by the left multiplication operators $\pi_\ell(\xi)$ for $\xi \in C_c(G)$. But this is really just the restriction to $C_c(G)$ of the representation $\lambda : L^1(G) \to B(L^2(G))$ obtained from the left regular representation $\lambda : G \to B(L^2(G))$. The advantage is that we can introduce the theory of left Hilbert algebras to obtain a weight on $\mathcal{R}_\ell(G)$ by

$$\phi_\ell(\pi_\ell(\xi)^*\pi_\ell(\xi)) = \|\xi\|^2 = \int_G |\xi(t)|^2 dt.$$

Moreover, in this case, the weight ϕ is just $\phi(x) = |x|$, so we have

$$\phi_\ell(\pi_\ell(\xi)^*\pi_\ell(\xi)) = \int_G \phi(\xi(t)^*\xi(t))dt.$$

In fact, the above can be generalized.

Define $C_c(G, \mathcal{M})$ to be the space of all bounded, compactly supported, strong*-continuous, functions $x : G \to \mathcal{M}$. We write $\|x\|_\infty = \sup_{s\in G}\|x(s)\|$. Let μ be a left Haar measure on G and write ds in place of $d\mu(s)$ for integration with respect to μ. Let δ_G be the modular function of μ. For $x \in C_c(G, \mathcal{M})$ consider the function $s \mapsto \alpha_s(x(s))$. If $\xi \in \mathcal{H}$, then

$$\begin{aligned}\|(\alpha_s(x(s)) - \alpha_{s_0}(x(s_0)))\xi\| &\le \|\alpha_s(x(s) - x(s_0))\xi\| + \|(\alpha_s - \alpha_{s_0})(x(s_0))\xi\| \\ &\le \|(x(s) - x(s_0))\xi\| + \|(\alpha_s - \alpha_{s_0})(x(s_0))\xi\|,\end{aligned}$$

so as $s \to s_0$, both terms on the right converge to 0 by the continuity assumptions on x and α. Since each α_s is *-preserving, we also have $\lim_{s\to s_0} \|(\alpha_s(x(s)) - \alpha_{s_0}(x(s_0)))^*\xi\| = 0$, so that $s \mapsto \alpha_s(x(s))$ is strong*-continuous, and hence belongs to $C_c(G, \mathcal{M})$. Moreover, if $x, y \in C_c(G, \mathcal{M})$, then x, y are bounded so it follows that $s \mapsto x(s)y(s)$ also belongs to $C_c(G, \mathcal{M})$. Then for $t \in G$, the integral

$$x * y(t) = \int_G \alpha_s(x(ts))y(s^{-1})ds$$

belongs to $\mathcal{M}$, and for $\xi, \eta \in \mathcal{H}$,

$$| < x * y(t)\xi, \eta > | = \left| \int_G < \alpha_s(x(ts))y(s^{-1})\xi, \eta > ds \right| \leq \|x\|_\infty \|y\|_\infty \|\xi\| \|\eta\| \mu(\text{supp } y^{-1}),$$

where y^{-1} is the compactly supported function $s \mapsto y(s^{-1})$. It follows that that $x * y$ is bounded. We note the following inequality for $t, t_0 \in G$,

$$\begin{aligned} \|x * y(t - t_0)\xi\| &= \| \left(\int_G \alpha_s(x(ts) - x(t_0 s))y(s^{-1})ds \right) \xi \| \\ &= \sup_{\eta \in \mathcal{H}, \|\eta\| \leq 1} \left| \int_G < \alpha_s(x(ts) - x(t_0 s))y(s^{-1})\xi, \eta > ds \right| \\ &\leq \int_G \|\alpha_s(x(ts) - x(t_0 s))y(s^{-1})\xi\| ds. \end{aligned}$$

Since for each $s \in G$ we have $\lim_{t \to t_0} \|\alpha_s(x(ts) - x(t_0 s))y(s^{-1})\xi\| = 0$, and since

$$\|\alpha_s(x(ts) - x(t_0 s))y(s^{-1})\xi\| \leq 2\|x\|_\infty \|y\|_\infty \|\xi\|,$$

and vanishes off supp y^{-1}, by the Lebesgue Dominated Convergence Theorem, it follows that the above integrals converge to 0, so that $x * y$ is continuous in the SOT. A similar argument show that $x * y$ is Strong*-continuous. We can therefore give $C_c(G, \mathcal{M})$ a product and involution as follows.

$$x * y(t) = \int_G \alpha_s(x(ts))y(s^{-1})ds, \qquad x^\sharp(t) = \delta_G(t)^{-1} \alpha_{t^{-1}}(x(t^{-1})^*)$$

We make $C_c(G, \mathcal{M})$ into an $\mathcal{M}$-bimodule by defining for $a \in \mathcal{M}, x \in C_c(G, \mathcal{M})$,

$$(x \cdot a)(t) = x(t)a, \qquad (a \cdot x)(t) = \alpha_{t^{-1}}(a)x(t).$$

We note the following properties.

$$\begin{aligned} a \cdot (x * y) = (a \cdot x) * y, &\qquad (x * y) \cdot a = x * (y \cdot a); \\ (a \cdot x)^\sharp = x^\sharp \cdot a^*, &\qquad (x \cdot a)^\sharp = a^* \cdot x^\sharp. \end{aligned}$$

Lemma 7.6. *We obtain a *-representation $\tilde{\pi}$ of $C_c(G, \mathcal{M})$ by*

$$\tilde{\pi}(x) = \int \lambda(s)\pi_\alpha(x(s))ds$$

and we have that $\mathcal{M} \rtimes_\alpha G = \tilde{\pi}(C_c(G, \mathcal{M}))''$. Moreover, for $a \in \mathcal{M}, x \in C_c(G, \mathcal{M})$, we have $\tilde{\pi}(a \cdot x) = \pi_\alpha(a)\tilde{\pi}(x)$ and $\tilde{\pi}(x \cdot a) = \tilde{\pi}(x)\pi_\alpha(a)$.

Proof. The proof that $\tilde{\pi}$ is a *-representation and satisfies the module properties is routine, so we will just show that the von Neumann algebra generated by the image is $\mathcal{M} \rtimes_\alpha G$. We can view $C_c(G)$ as a subalgebra of $C_c(G, \mathcal{M})$ by identifying the function $f \in C_c(G)$ with the function $s \mapsto f(s)1$. Then by definition, $\tilde{\pi}(f)$ is just $\lambda(f)$ where λ is the representation of $L^1(G)$ obtained from the representation λ of G so that $\lambda(G)'' = \tilde{\pi}(C_c(G))''$. Therefore $\lambda(G) \subseteq \tilde{\pi}(C_c(G, \mathcal{M}))''$. Now let $\{f_i\}$ be a bounded approximate identity in $C_c(G)$. Then for $a \in C_c(G, \mathcal{M})$ we have $\tilde{\pi}(f_i)\pi_\alpha(a) = \tilde{\pi}(f_i \cdot a)$, the left side of which converges in the SOT to $\pi_\alpha(a)$ so that $\pi_\alpha(\mathcal{M}) \subseteq \tilde{\pi}(C_c(G, \mathcal{M}))''$. Hence, $\mathcal{M} \rtimes_\alpha G \subseteq \tilde{\pi}(C_c(G, \mathcal{M}))''$. On the other hand, if $y \in (\pi_\alpha(\mathcal{M}) \cup \lambda(G))'$ and $x \in C_c(G, \mathcal{M})$, then for $\xi, \eta \in C_c(G, \mathcal{H})$, we have

$$\begin{aligned}
< \tilde{\pi}(x)y\xi, \eta > &= \int < \lambda(s)\pi_\alpha(x(s))y\xi, \eta > ds \\
&= \int < y\lambda(s)\pi_\alpha(x(s))\xi, \eta > ds \\
&= \int < \lambda(s)\pi_\alpha(x(s))\xi, y^*\eta > ds \\
&=< \tilde{\pi}(x)\xi, y^*\eta > \\
&=< y\tilde{\pi}(x)\xi, \eta >,
\end{aligned}$$

so that $\tilde{\pi}(x) \in \mathcal{M} \rtimes_\alpha G$, completing the proof.

□

We can now begin to construct a left Hilbert algebra for $\mathcal{M} \rtimes_\alpha G$. We define b_ϕ to be the (non self-adjoint) algebra generated by the set

$$L = \{x \cdot a : x \in C_c(G, \mathcal{M}), a \in n_\phi\}.$$

Note that for each $t, (x \cdot a)(t) = x(t)a \in n_\phi$, and since n_ϕ is a left ideal, it follows that b_ϕ consists of functions $G \to n_\phi$. Then for $x \in b_\phi$, we define the function $\tilde{q_\phi}(x) : G \to \mathcal{H}_\phi$ by $(\tilde{q_\phi}(x))(s) = q_\phi(x(s))$. In fact, we have that $\tilde{q}_\phi \in C_c(G, \mathcal{H})$. To see this, we note that if $x \in C_c(G, \mathcal{H})$ and $a \in n_\phi$, then for $s \in G$, we have $\tilde{q}_\phi(x \cdot a)(s) = q_\phi(x(s)a) = x(s)q_\phi(a)$, so that $\tilde{q}_\phi(x \cdot a) \in C_c(G, \mathcal{H})$ by the continuity assumption on x. Moreover, if we have $x, y \in b_\phi$ such that $\tilde{q}_\phi(y) \in C_c(G, \mathcal{M})$, then by the following lemma we have $\tilde{q}_\phi(x * y) = \tilde{\pi}(x)\tilde{q}_\phi(y)$, which also belongs to $C_c(G, \mathcal{H})$, proving the assertion.

Lemma 7.7. *With the notation above, set $\tilde{\mathcal{U}_\phi} = \tilde{q_\phi}(b_\phi \cap b_\phi^\sharp)$, and define multiplication in $\tilde{\mathcal{U}_\phi}$ by*

$$\tilde{q_\phi}(x)\tilde{q_\phi}(y) = \tilde{q_\phi}(x * y).$$

Then $\tilde{\mathcal{U}_\phi}$ is dense in $L^2(G, \mathcal{H}_\phi)$, and for $x, y \in b_\phi$ we have $\tilde{\pi}(x)\tilde{q_\phi}(y) = \tilde{q_\phi}(x * y)$. Therefore, left multiplication is bounded in $\tilde{\mathcal{U}_\phi}$ and the left multiplication operator for x coincides with $\tilde{\pi}(x)$. Moreover, $\tilde{\pi}(b_\phi \cap b_\phi^\sharp)$ generates $\mathcal{M} \rtimes_\alpha G$.

Proof. Let $x, z \in C_c(G, \mathcal{M}), a \in n_\phi, y = z \cdot a$. Then for $s \in G$, we have

$$\begin{aligned}\tilde{q_\phi}(x * y)(s) &= q_\phi(x * y(s)) \\ &= q_\phi(x * z(s)a) \\ &= x * z(s) q_\phi(a) \\ &= \int \alpha_t(x(st)) z(t^{-1}) q_\phi(a) dt.\end{aligned}$$

But if we set $\xi(s) = z(s) q_\phi(a)$, then

$$\begin{aligned}(\tilde{\pi}(x)\xi)(s) &= \int \lambda_t \pi(x(t))\xi(s) dt \\ &= \int (\pi(x(t))\xi)(t^{-1}s) dt \\ &= \int \alpha_{s^{-1}t}(x(t))\xi(t^{-1}s) dt \\ &= \int \alpha_t(x(st))\xi(t^{-1}) dt \\ &= \int \alpha_t(x(st)) z(t^{-1}) q_\phi(a) dt,\end{aligned}$$

so that $\tilde{\pi}(x)\tilde{q_\phi}(y) = \tilde{q_\phi}(x * y)$.

To see that $b_\phi \cap b_\phi^*$ generates $\mathcal{M}$ we note that for $a, b \in n_\phi, x \in C_c(G, \mathcal{M})$ we have $\tilde{\pi}_\alpha(a^* x b) = \pi_\alpha(a)^* \tilde{\pi}(x) \pi_\alpha(b) \in \tilde{\pi}(b_\phi \cap b_\phi^\sharp)$, and that $n_\phi, C_c(G, \mathcal{M})$ respectively generate $\mathcal{M}, \mathcal{M} \rtimes_\alpha G$. Lastly, for $a, b \in n_\phi, f \in C_c(G)$ consider the element $a^* \cdot f \cdot b \in b_\phi \cap b_\phi^*$. We have for $s \in G$,

$$\tilde{q}_\phi(a^* \cdot f \cdot b)(s) = q_\phi(f(s)\alpha_{s^{-1}}(a)^* b) = f(s)\alpha_{s^{-1}}(a)^* q_\phi(a) = [\pi_\alpha(a^*)(q_\phi(a) \otimes f)](s).$$

Since n_ϕ generates $\mathcal{M}$ and since $q_\phi(n_\phi) \otimes C_c(G)$ is dense in $\mathcal{H}_\phi \otimes L^2(G) = L^2(G, \mathcal{H}_\phi)$, it follows that $\tilde{q}_\phi(b_\phi \cap b_\phi^*)$ is dense in $L^2(G, \mathcal{H}_\phi)$.

□

Now let $\tilde{\mathcal{U}_\phi}$ be as in Lemma 7.7, and define an involution $^\sharp$ by

$$\tilde{q}_\phi(x)^\sharp = \tilde{q}_\phi(x^\sharp).$$

Then by Lemmas 7.6 and 7.7, $\tilde{\mathcal{U}_\phi}$ with the involution $^\sharp$ and product as in Lemma 7.7 satisfies 1), 2), and 4) of Definition 4.1, and the left multiplication operators generate $\mathcal{M} \rtimes_\alpha G$. To show that $\tilde{\mathcal{U}_\phi}$ is a left Hilbert algebra we just have to show that the involution is closable. First we define potential candidates for a modular operator and modular conjugation. Using the notation of Section 5, for $t \in \mathbb{R}$, we define a unitary u_t on $L^2(G, \mathcal{H})$ by

$$u_t \xi(s) = \delta_G(s)^{it} \Delta^{it}_{\phi\circ\alpha_s, \phi} \xi(s),$$

where $\Delta^{it}_{\phi\circ\alpha_s,\phi} = (D_{\phi\circ\alpha_s,\phi} : D_\phi)_t\Delta^{it}$. By Lemma 2.10 on page 245 of [13], the map $(s,t) \in G \times \mathbb{R} \mapsto (D_{\phi\circ\alpha_s} : D_\phi)_t$ is continuous in the SOT, so it follows that the family $\{u_t\}$ is a one-parameter group of unitaries continuous in the SOT. By Stone's Theorem, there exists a positive self-adjoint operator $\tilde{\Delta}$ such that $u_t = \tilde{\Delta}^{it}$ for all $t \in \mathbb{R}$.

Lemma 7.8. *For $x \in C_c(G, \mathcal{M})$ define*

$$[\rho^\phi_t(x)](s) = \delta_G(s)^{it}\sigma_t^{\phi\circ\alpha_s,\phi}(x(s)).$$

Then $\{\rho^\phi_t\}$ defines a one-parameter group of automorphisms for $C_c(G, \mathcal{M})$ which leaves $b_\phi \cap b^*_\phi$ invariant. Moreover, for $x \in b_\phi \cap b^*_\phi$, we have

$$\tilde{\Delta}^{it}\tilde{q}_\phi(x) = \tilde{q}_\phi(\rho^\phi_t(x)).$$

Proof. Writing $\sigma_t^{\phi\circ\alpha_s,\phi} = (D_{\phi\circ\alpha_s} : D_\phi)_t\sigma^\phi_t$, we see that $\rho^\phi_t(x) \in C_c(G, \mathcal{M})$ for $x \in C_c(G, \mathcal{M})$, and moreover, that $\{\rho^\phi_t\}$ is a one-parameter group of tranformations. Let $x, y \in C_c(G, \mathcal{M})$. By Lemma 5.7, we have

$$\begin{aligned}
&[\rho^\phi_t(x) * \rho^\phi_t(y)](s)\\
&\quad= \int_G \alpha_r(\rho^\phi_t(x)(sr))\rho^\phi_t(y)(r^{-1})dr\\
&\quad= \int_G \alpha_r(\delta_G(sr)^{it}\sigma_t^{\phi\circ\alpha_{sr},\phi}(x(sr)))\delta_G(r^{-1})^{it}\sigma_t^{\phi\circ\alpha_{r^{-1}},\phi}(y(r^{-1}))dr\\
&\quad= \delta_G(s)^{it}\int_G \alpha_r((D_{\phi\circ\alpha_{sr}} : D_\phi)_t\sigma^\phi_t(x(sr)))(D_{\phi\circ\alpha_{r^{-1}}} : D_\phi)_t\sigma^\phi_t(y(r^{-1}))dr\\
&\quad= \delta_G(s)^{it}\int_G (D_{\phi\circ\alpha_s} : D_{\phi\circ\alpha_{r^{-1}}})_t\alpha_r \circ \sigma^\phi_t(x(sr))(D_{\phi\circ\alpha_{r^{-1}}} : D_\phi)_t\sigma^\phi_t(y(r^{-1}))dr\\
&\quad= \delta_G(s)^{it}(D_{\phi\circ\alpha_s} : D_\phi)_t\int_G (D_\phi : D_{\phi\circ\alpha_{r^{-1}}})_t\sigma_t^{\phi\circ\alpha_{r^{-1}}}\alpha_r(x(sr))(D_\phi : D_{\phi\circ\alpha_{r^{-1}}})^*_t\sigma^\phi_t(y(r^{-1}))dr\\
&\quad= \delta_G(s)^{it}(D_{\phi\circ\alpha_s} : D_\phi)_t\int_G \sigma^\phi_t\alpha_r(x(sr))\sigma^\phi_t(y(r^{-1}))dr\\
&\quad= \delta_G(s)^{it}(D_{\phi\circ\alpha_s} : D_\phi)_t\sigma^\phi_t\left(\int_G \alpha_r(x(sr))y(r^{-1})dr\right)\\
&\quad= \delta_G(s)^{it}(D_{\phi\circ\alpha_s} : D_\phi)_t\sigma^\phi_t(x * y(t))\\
&\quad= [\rho^\phi_t(x * y)](s),
\end{aligned}$$

Also,

$$\begin{aligned}
[\rho_t^\phi(x)^\sharp](s) &= \delta_G(s)^{-1}\alpha_{s^{-1}}([\rho_t^\phi(x)](s^{-1})^*) \\
&= \delta_G(s)^{-1-it}\alpha_{s^{-1}}(\sigma_t^\phi(x^*)(D_{\phi\circ\alpha_{s^{-1}}:D_\phi})_t^*) \\
&= \delta_G(s)^{-1-it}\alpha_{s^{-1}}((D_{\phi\circ\alpha_{s^{-1}}:D_\phi})_t^*\sigma_t^{\phi\circ\alpha_{s^{-1}}}(x^*)) \\
&= \delta_G(s)^{-1-it}(D_{\phi\circ\alpha_s} : D_\phi)_t\alpha_{s^{-1}}(\sigma_t^{\phi\circ\alpha_{s^{-1}}}(x^*)) \\
&= \delta_G(s)^{-1-it}(D_{\phi\circ\alpha_s} : D_\phi)_t\sigma_t^\phi(\alpha_{s^{-1}}(x^*)) \\
&= [\rho_t^\phi(x)]^\sharp(t),
\end{aligned}$$

so that ρ_t^ϕ is a *-automorphism of $C_c(G, \mathcal{M})$.

Let $x \in C_c(G, \mathcal{M}), a \in n_\phi$. Then

$$\begin{aligned}
[\rho_t^\phi(x \cdot a)](s) &= \delta_G(s)^{it}(D_{\phi\circ\alpha_s} : D_\phi)_t\sigma_t^\phi(x(s)a) \\
&= \delta_G(s)^{it}(D_{\phi\circ\alpha_s} : D_\phi)_t\sigma_t^\phi(x(s))\sigma_t^\phi(a) \\
&= [\rho_t^\phi(x)](s)\sigma_t^\phi(a),
\end{aligned}$$

so that $\rho_t^\phi(x \cdot a) = \rho_t^\phi(x) \cdot \sigma_t^\phi(a)$. Since n_ϕ is σ_t^ϕ-invariant, it follows that $\rho_t^\phi(x \cdot a) \in b_\phi$. Moreover, if we choose $x, y \in b_\phi$ such that $\rho_t^\phi(x), \rho_t^\phi(y) \in b_\phi$, then

$$\rho_t^\phi(x * y) = \rho_t^\phi(x) * \rho_t^\phi(y),$$

which is also in b_ϕ. Since the set $\{x \cdot a : x \in C_c(G, \mathcal{M}), a \in n_\phi\}$ generates b_ϕ, it follows that $\rho_t^\phi(b_\phi) \subseteq b_\phi$. The reverse inclusion holds replacing t with $-t$, so that $\rho_t^\phi(b_\phi) = b_\phi$. Since ρ_t^ϕ is a *-automorphism, this also shows that $\rho_t^\phi(b_\phi \cap b_\phi^\sharp) = b_\phi \cap b_\phi^\sharp$. Lastly, if $x \in b_\phi \cap b_\phi^\sharp$,

$$\begin{aligned}
\tilde{q}_\phi(\rho_t^\phi(x))(s) &= q_\phi([\rho_t^\phi(x)](s)) \\
&= \delta_G(s)^{it}(D_{\phi\circ\alpha_s} : D_\phi)_t q_\phi(\sigma_t^\phi(x(s))) \\
&= \delta_G(s)^{it}(D_{\phi\circ\alpha_s} : D_\phi)_t \Delta^{it} q_\phi(x(s)) \\
&= [\tilde{\Delta}^{it}\tilde{q}_\phi(x)](s).
\end{aligned}$$

□

To find the modular conjugation we need some background on automorphisms. Let $\theta \in Aut(\mathcal{M})$, and for $x \in n_\phi$, define

$$V_\theta q_\phi(x) = q_{\phi\circ\theta^{-1}}(\theta(x)).$$

This is well-defined since $n_{\phi\circ\theta^{-1}} = \theta(n_\phi)$, and so extends to a unitary $V_\theta : \mathcal{H}_\phi \to \mathcal{H}_{\phi\circ\theta^{-1}}$. Identifying the representations $\pi_\phi, \pi_{\phi\circ\theta}, \pi_{\phi\circ\theta^{-1}}$, for $a \in \mathcal{M}, x \in n_\phi$, we have

$$\begin{aligned} V_\theta a V_\theta^*(x) &= V_\theta a q_{\phi\circ\theta}(\theta^{-1}(x)) \\ &= V_\theta q_{\phi\circ\theta}(a\theta^{-1}(x)) \\ &= q_\phi(\theta(a)x) \\ &= \theta(a)q_\phi(x), \end{aligned}$$

so that $\theta = \mathrm{Ad}(V_\theta)$. Moreover, for $x \in n_\phi \cap n_\phi^*$ we have

$$\begin{aligned} V_\theta S_\phi q_\phi(x) &= V_\theta q_\phi(x^*) \\ &= q_{\phi\circ\theta^{-1}}(\theta(x)^*) \\ &= S_{\phi\circ\theta^{-1}} q_{\phi\circ\theta^{-1}}(\theta(x)) \\ &= S_{\phi\circ\theta^{-1}} V_\theta q_\phi(x)), \end{aligned}$$

so $V_\theta S_\phi = S_{\phi\circ\theta^{-1}} V_\theta$. Then

$$S_{\phi\circ\theta^{-1}} = V_\theta S_\phi V_\theta^* = (V_\theta J_\phi V_\theta^*)(V_\theta \Delta_\phi^{1/2} V_\theta^*),$$

so by uniqueness of the polar decomposition, we have $J_{\phi\circ\theta^{-1}} = V_\theta J_\phi V_\theta^*$ and $\Delta_{\phi\circ\theta^{-1}} = V_\theta \Delta_\phi V_\theta^*$. Identifying the conjugations $J_\phi, J_{\phi\circ\theta^{-1}}$ with the operator J under the unitary equivalence of the representations $\pi_\phi, \pi_{\phi\circ\theta^{-1}}$ (see Chapter 9, Section 1, of [13]), we can write $JV_\theta = V_\theta J$. Setting $V_{\alpha_s} = U(s)$ we define a conjugate linear operator $\tilde{J}$ on $L^2(G, \mathcal{H})$ by

$$(\tilde{J}\xi)(s) = \delta_G(s)^{-1/2} U(s^{-1}) J \xi(s^{-1}).$$

Lemma 7.9. *The operator $\tilde{J}$ satisfies $\tilde{J} = \tilde{J}^* = \tilde{J}^{-1}$, and the involution $^\sharp$ has closure $\tilde{S}_\phi$ which admits polar decomposition $\tilde{S}_\phi = \tilde{J}\tilde{\Delta}_\phi^{1/2}$.*

Proof. For $s \in G$, we have

$$\begin{aligned} \tilde{J}^2\xi(s) &= \delta_G(s)^{-1/2} U(s^{-1}) J (\tilde{J}\xi)(s^{-1}) \\ &= U(s^{-1}) J U(s) J \xi(s) \\ &= \xi(s), \end{aligned}$$

so that $\tilde{J}^2 = 1$. For $x \in b_\phi \cap b_\phi^\sharp$, we have

$$\begin{aligned} (\tilde{J}\tilde{q}_\phi(x^\sharp))(s) &= \delta_G(s)^{-1/2} U(s^{-1}) J q_\phi(x^\sharp(s^{-1})) \\ &= \delta_G(s)^{-1/2} J U(s^{-1}) q_\phi(\delta_G(s)\alpha_s(x(s)^*)) \\ &= \delta_G(s)^{1/2} J S_{\phi\circ\alpha_s} q_{\phi\circ\alpha_s}(x(s)) \\ &= \delta_G(s)^{1/2} J S_{\phi\circ\alpha_s,\phi} q_\phi(x(s)) \\ &= \delta_G(s)^{1/2} \Delta_{\phi\circ\alpha_s,\phi}^{1/2} q_\phi(x(s)) \\ &= \tilde{\Delta}^{1/2}\tilde{q}_\phi(x)(s), \end{aligned}$$

so $\tilde{q}_\phi(x)^\sharp = \tilde{J}\tilde{\Delta}^{1/2}\tilde{q}_\phi(x)$. Hence, $\sharp$ has closure $\tilde{S}_\phi$ such that $\tilde{S}_\phi\xi = \tilde{J}\tilde{\Delta}^{1/2}\xi$ for $\xi \in \mathcal{D}(\tilde{S}_\phi) \subseteq \mathcal{D}(\tilde{\Delta}^{1/2})$. But by Lemma 7.8, $\tilde{q}_\phi(b_\phi \cap b_\phi^*)$ is invariant under each $\tilde{\Delta}^{it}$ so that by Appendix 4 of [13], $\tilde{q}_\phi(b_\phi \cap b_\phi^*)$ is a core for $\tilde{\Delta}^{1/2}$, completing the proof.

□

Theorem 7.10. *The algebra $\tilde{\mathcal{U}}_\phi$ is a left Hilbert algebra such that $\mathcal{R}_\ell(\tilde{\mathcal{U}}_\phi) = \mathcal{M} \rtimes_\alpha G$. There exists a unique fns weight $\tilde{\phi}$ on $\mathcal{M} \rtimes_\alpha G$ satisfying the following:*

1) for $x \in b_\phi$,

$$\tilde{\phi}(\tilde{\pi}(x)^*\tilde{\pi}(x)) = \int_G \phi(x(t)^*x(t))dt;$$

2) for $x \in \mathcal{M}$, $\sigma_t^{\tilde{\phi}}(\pi_\alpha(x)) = \pi_\alpha(\sigma_t(x))$;
3) $\sigma_t^{\tilde{\phi}}(\lambda(s)) = \delta_G(s)^{it}\lambda(s)\pi_\alpha((D_{\phi\circ\alpha_s} : D_\phi)_t)$.

We say that $\tilde{\phi}$ is the weight on $\mathcal{M} \rtimes_\alpha G$ dual to ϕ.

Verification of 1), 2), and 3) are routine, and the proof of uniqueness can be found in [13]. We finish this Section with an application to the case that α is the modular automorphism group of ϕ.

Corollary 7.11. *Let ϕ be an fns weight on a von Neumann algebra $\mathcal{M}$, and let $\tilde{\phi}$ be the weight on $\mathcal{M} \rtimes_{\sigma^\phi} \mathbb{R}$ dual to ϕ. Then $\sigma_t^{\tilde{\phi}} = Ad(\lambda(t))$.*

We conclude by Theorem 5.22 that $\mathcal{M} \rtimes_{\sigma^\phi} \mathbb{R}$ is semifinite. In Section 8 we expand upon this by constructing a new action on the crossed product algebra.

8 Duality of Crossed Products

Let $(\mathcal{M}, G, \alpha)$ be a covariant system where $\mathcal{M}$ acts on a Hilbert space $\mathcal{H}$. We assume throughout that G is a locally compact abelian group, with group operation written additively, and with dual group $\hat{G}$. In this Section we generalize the notion of Pontryagin duality to von Neumann algebras. The material follows the exposition given in Part 1, Section 4 of [4].

Let λ and ρ respectively denote the left and right regular representations of G on $L^2(G)$. Define a unitary representation $\nu : \hat{G} \to B(L^2(G))$ by $(\nu_\gamma \xi)(s) = \overline{\gamma(s)}\xi(s)$.

Lemma 8.1. *The von Neumann algebra generated by $\lambda(G) \cup \nu(\hat{G})$ on $L^2(G)$ is all of $B(L^2(G))$.*

Proof.

Claim: The von Neumann algebra generated by $\nu(\hat{G})$ is $L^\infty(G)$.

Let $\nu : L^1(\hat{G}) \to B(L^2(G))$ be the representation given by

$$\nu(f) = \int_{\hat{G}} f(\gamma)\nu_\gamma d\gamma.$$

Then the von Neumann algebra generated by $\nu(\hat{G})$ coincides with that generated by $\nu(L^1(\hat{G}))$. But for $f \in L^1(\hat{G}), \xi \in L^2(G)$, we have

$$\begin{aligned} \nu(f)\xi &= \int f(\gamma)\nu_\gamma \xi d\gamma \\ &= \int f(\gamma)\overline{\gamma}\xi d\gamma \\ &= \hat{f}\xi. \end{aligned}$$

That is, $\nu(f)$ is just multiplication by $\hat{f}$. Since the set $\{\hat{f} : L^1(\hat{G})\}$ is dense in $C_0(G)$ and $C_0(G)$ is WOT-dense in $L^\infty(G)$, the claim follows.

The algebra $L^\infty(G)$, and the group von Neumann algebra $\mathcal{R}_\ell(G)$ are maximal abelian, so that

$$[L^\infty(G) \cup \mathcal{R}_\ell(G)]' \subseteq L^\infty(G)' \cap \mathcal{R}_\ell(G)' = L^\infty(G) \cap \mathcal{R}_\ell(G).$$

But if $f \in L^\infty(G)$ acts by the multiplication operator $\nu(f) \in B(L^2(G))$, then for $\xi \in L^2(G)$, $x \in G$,

$$(\lambda_s v(f)\lambda_s^*\xi)(t) = (\nu(f)\lambda_s^*\xi)(s^{-1}t) = f(s^{-1}t)(\lambda_s^*\xi)(s^{-1}t) = f(s^{-1}t)\xi(t).$$

Hence if $\nu(f)\lambda_s = \lambda_s v(f)$, then we have $f(t) = f(s^{-1}t)$ almost everywhere. If $\nu(f) \in \lambda(G)'$, f must be constant almost everywhere. Therefore $L^\infty(G) \cap \mathcal{R}_\ell(G) = \mathbb{C}$ so that $[L^\infty(G) \cup \mathcal{R}_\ell(G)]'' = B(L^2(G))$.

□

It was shown in the proof of Lemma 8.1 that for $s \in G, \gamma \in \hat{G}$ we have the relation

$$\lambda_s \nu_\gamma \lambda_s^* = \gamma(s)\nu_\gamma.$$

We say that the pair (λ, ν) satisfy the **Heisenberg-Weyl commutation** relation. We have the following uniqueness result, which will be useful in a later Section, from page 257 of [13].

Lemma 8.2. *Let U, V respectively be unitary representation of $G, \hat{G}$ on a Hilbert space $\mathcal{H}$ which satisfies the Heisenberg-Weyl commutation relation. Then there exists a Hilbert space $\mathcal{H}_0$ and an isomorphorphism $\Omega : L^2(G) \otimes \mathcal{H}_0 \to \mathcal{H}$ such that*

$$U = \lambda \otimes id, \qquad V = \nu \otimes id.$$

Returning to the covariant system $(\mathcal{M}, G, \alpha)$, we define unitary representations $\lambda_\mathcal{H}, \nu_\mathcal{H}$ of $G, \hat{G}$ on $L^2(G, \mathcal{H})$ as follows:

$$(\lambda_\mathcal{H}(r)\xi)(s) = \xi(s - r), \qquad (\nu_\mathcal{H}(\gamma)\xi)(s) = \overline{\gamma(s)}\xi(s)$$

so that $\lambda_\mathcal{H}(r) = 1 \otimes \lambda_r$ and $\nu_\mathcal{H}(\gamma) = 1 \otimes \nu_\gamma$. We then define an action $\hat{\alpha}$ of $\hat{G}$ on $L^2(G, \mathcal{H})$ by $\hat{\alpha}_\gamma = \mathrm{Ad}(\nu_\mathcal{H}(\gamma))$. In fact, $\hat{\alpha}$ restricts to an action on $\mathcal{M} \rtimes_\alpha G$, which we will call the action **dual** to α. Unless the context requires clarification, we will just refer to it as the **dual action**.

We now check that $\hat{\alpha}$ defines an action on $\mathcal{M} \rtimes_\alpha G$. If $x \in \mathcal{M}, \gamma \in \hat{G}, \xi \in L^2(G, \mathcal{H})$, then we have

$$\begin{aligned}
(\nu_\mathcal{H}(\gamma)\pi_\alpha(x)\nu_\mathcal{H}(\gamma)^*\xi)(s) &= \overline{\gamma(s)}(\pi_\alpha(x)\nu_\mathcal{H}(\gamma)^*\xi)(s) \\
&= \overline{\gamma(s)}\alpha_{s^{-1}}(x)(\nu_\mathcal{H}(\gamma)^*\xi)(s) \\
&= \overline{\gamma(s)}\alpha_{s^{-1}}(x)\overline{\gamma^{-1}(s)}\xi(s) \\
&= \alpha_{s^{-1}}(x)\xi(s) \\
&= (\pi_\alpha(x)\xi)(s),
\end{aligned}$$

so $\hat{\alpha}$ fixes $\pi_\alpha(\mathcal{M})$. Now if $t \in G$,

$$\begin{aligned}(\nu_{\mathcal{H}}(\gamma)\lambda_{\mathcal{H}}(t)\mu_\gamma^*\xi)(s) &= \overline{\gamma(s)}(\lambda_{\mathcal{H}}(t)\mu_\gamma^*\xi)(s) \\ &= \overline{\gamma(s)}(\nu_{\mathcal{H}}(\gamma)^*\xi)(s-t) \\ &= \overline{\gamma(s)\gamma^{-1}(s-t)}\xi(s-t) \\ &= \overline{\gamma(t)}(\lambda_{\mathcal{H}}(t)\xi)(s)\end{aligned}$$

so $\hat{\alpha}_\gamma(\lambda_{\mathcal{H}}(t)) = \overline{\gamma(t)}\lambda_{\mathcal{H}}(t)$, which belongs to $\mathcal{M} \rtimes_\alpha G$. Hence $\hat{\alpha}_\gamma$ maps the generators of $\mathcal{M} \rtimes_\alpha G$ into $\mathcal{M} \rtimes_\alpha G$, so the claim follows. We now state the main Theorem of this Section.

Theorem 8.3. *There exists an isomorphism $\pi : (\mathcal{M} \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G} \to \mathcal{M}\overline{\otimes}B(L^2(G))$ which transforms the bidual action $\hat{\hat{\alpha}}_t$ to $\{\alpha_s \otimes Ad(\rho_s)\}_{s\in G}$. Moreover, the fixed point algebra $(\mathcal{M} \rtimes_\alpha G)^{\hat{\alpha}}$ is precisely $\pi_\alpha(\mathcal{M})$.*

We prove the Theorem with a series of Lemmas. First we consider the following definition.

Definition 8.4. *We say that the covariant systems $(\mathcal{M}, G, \alpha), (\mathcal{N}, G, \beta)$ are* ***conjugate*** *if there exists an isomorphism $\pi : \mathcal{M} \to \mathcal{N}$ such that for $x \in \mathcal{M}, t \in G$, we have*

$$\pi(\alpha_t(x)) = \beta_t(\pi(x)),$$

and we say that π is a ***conjugating*** *isomorphism.*

The following is proven in [4].

Lemma 8.5. *Let $(\mathcal{M}, G, \alpha), (\mathcal{N}, G, \beta)$ be conjugate covariant systems with conjugating isomorphism $\pi : \mathcal{M} \to \mathcal{N}$. Then $\tilde{\pi} = \pi \otimes 1$ is a conjugating isomorphism for the covariant systems $(\mathcal{M} \rtimes_\alpha G, \hat{G}, \hat{\alpha})$ and $(\mathcal{N} \rtimes_\beta G, \hat{G}, \hat{\beta})$.*

Returning to the covariant system $(\mathcal{M}, G, \alpha)$, let $(\pi_\alpha, \lambda_{\mathcal{H}})$ be the covariant representation on $L^2(G, \mathcal{H})$ constructed in Section 7. Then we obtain a new covariant system $(\pi(\mathcal{M}), G, \alpha')$ where $\alpha'_t = \pi_\alpha \circ \alpha_t \circ \pi_\alpha^{-1}$. Note that by the covariance condition, for $x \in \mathcal{M}$ we have $\alpha'_t(\pi_\alpha(x)) = \pi_\alpha \circ \alpha_t(x) = \lambda_t \pi_\alpha(x)\lambda_t^*$ so that α' is unitarily implemented. Moreover, these systems are conjugate. To prove Theorem 8.3 it suffices to work with the covariant system $(\pi(\mathcal{M}), G, \alpha')$ in place of $(\mathcal{M}, G, \alpha)$. Hence, for the remainder we will assume that α is implemented by a one-parameter unitary group $\{u_s\}_{s\in G}$ which is SOT-continuous.

Define a unitary W on $L^2(G, \mathcal{H})$ by $(W\xi)(s) = u_s\xi(s)$. We have

$$\begin{aligned}(W\pi_\alpha(x)W^*\xi)(s) &= u_s(\pi_\alpha(x)W^*\xi)(s) \\ &= u_s(u_s^* x u_s)(W^*\xi)(s) \\ &= x u_s u_s^* \xi(s) \\ &= x\xi(s),\end{aligned}$$

and

$$\begin{aligned}(W\lambda_t W^*\xi)(s) &= u_s(\lambda_{\mathcal{H}}(t)W^*\xi)(s)\\ &= u_s(W^*\xi)(s-t)\\ &= u_s u_{s-t}^*\xi(s-t)\\ &= u_t\xi(s-t),\end{aligned}$$

so that $W\pi_\alpha(x)W^* = x\otimes 1$ and $W\lambda_t W^* = u_s \otimes \lambda_t$. In particular,

$$W(\mathcal{M} \rtimes_\alpha G)W^* = \{x\otimes 1, u_s\otimes\lambda_s\}''_{x\in\mathcal{M}, s\in G}.$$

Lemma 8.6. *The algebra $(\mathcal{M}\rtimes_\alpha G)\rtimes_{\hat{\alpha}}\hat{G}$ is spatially isomorphic to the von Neumann algebra $\mathcal{M}_1$ acting on $\mathcal{H}\otimes L^2(G)\otimes L^2(\hat{G})$ with generating set*

$$\{x\otimes 1\otimes 1, u_s\otimes\lambda_s\otimes 1, 1\otimes\nu_\gamma\otimes\lambda_\gamma\}_{x\in\mathcal{M}, s\in G, \gamma\in\hat{G}},$$

where λ_γ comes from the left regular representation of $\hat{G}$. Moreover, this isomorphism leaves the bidual action $\hat{\hat{\alpha}}$ unchanged.

Proof. We define the unitary $\tilde{W}$ on $L^2(\hat{G}, L^2(G,\mathcal{H}))$ by $(\tilde{W}\xi)(\gamma) = \nu_{\mathcal{H}}(\gamma)\xi(\gamma)$ so that for $\tilde{x}\in\mathcal{M}\rtimes_\alpha G$,

$$\tilde{W}\pi_{\hat{\alpha}}(\tilde{x})\tilde{W}^* = \tilde{x}\otimes 1$$

and for $\gamma\in\hat{G}$,

$$\tilde{W}\lambda_{L^2(G,\mathcal{H})}(\gamma)\tilde{W}^* = \nu_{\mathcal{H}}(\gamma)\otimes\lambda_\gamma.$$

We then define the unitary W on $L^2(G,\mathcal{H})$ by $(W\xi)(s) = u_s\xi(s)$ so that for $x\in\mathcal{M}$,

$$W\pi_\alpha(x)W^* = x\otimes 1,$$

and for $s\in G$,

$$W\lambda_{\mathcal{H}}(s)W^* = u_s\otimes\lambda_s.$$

Then,

$$\begin{aligned}(W\otimes 1)(\tilde{W}\pi_{\hat{\alpha}}(\pi_\alpha(x)))\tilde{W})(W^*\otimes 1) &= W\otimes 1(\pi_\alpha(x)\otimes 1)W^*\otimes 1\\ &= x\otimes 1\otimes 1,\end{aligned}$$

and

$$\begin{aligned}(W\otimes 1)(\tilde{W}\pi_{\hat{\alpha}}(\lambda_{\mathcal{H}}(s))\tilde{W})(W^*\otimes 1) &= W\otimes 1(\lambda_{\mathcal{H}}(s)\otimes 1)W^*\otimes 1\\ &= u_s\otimes\lambda_s\otimes 1.\end{aligned}$$

Moreove, since

$$\begin{aligned}(W\nu_{\mathcal{H}}W^*\xi)(s) &= u_s(\nu_{\mathcal{H}}W^*\xi)(s)\\ &= \overline{\gamma(s)}u_s(W^*\xi)(s)\\ &= \overline{\gamma(s)}\xi(s)\\ &= (\nu_{\mathcal{H}}\xi)(s),\end{aligned}$$

we have that

$$\begin{aligned}(W\otimes 1)(\tilde{W}\lambda_{L^2(G,\mathcal{H})}\tilde{W})(W^*\otimes 1) &= W\otimes 1(\nu_{\mathcal{H}}(\gamma)\otimes\lambda_\gamma)W^*\otimes 1\\ &= \nu_{\mathcal{H}}(\gamma)\otimes\lambda_\gamma\\ &= 1\otimes\nu_\gamma\otimes\lambda_\gamma.\end{aligned}$$

Lastly, we note that the unitary $(W\otimes 1)\tilde{W}$ commutes with the unitaries $\{\nu_{L^2(G,\mathcal{M})}(s)\}_{s\in G}$ implementing the bidual action, which completes the proof.

$\square$

Lemma 8.7. *The algebra $\mathcal{M}_1$ of Lemma 8.6 is spatially isomorphic to the von Neumann algebra $\mathcal{M}_2$ acting on $\mathcal{H}\otimes L^2(G)\otimes L^2(G)$ with generating set*

$$\{x\otimes 1\otimes 1, u_s\otimes\lambda_s\otimes 1, 1\otimes\nu_\gamma\otimes\nu_\gamma\}_{x\in\mathcal{M},s\in G,\gamma\in\hat{G}}.$$

The isomorphism transforms the bidual action $\hat{\hat{\alpha}}$ to the action β implemented by the one-parameter unitary group $\{1\otimes 1\otimes\rho_s\}_{s\in G}$.

Proof. Let $\mathfrak{F} : L^2(\hat{G})\to L^2(G)$ be the isomorphism extending the Fourier transform on $L^1(\hat{G})\cap L^2(\hat{G})$. If $\gamma\in\hat{G}, f\in C_c(G)$, we have

$$\begin{aligned}(\mathfrak{F}\lambda_\gamma\mathfrak{F}^*f)(t) &= \int_{\hat{G}}\overline{\gamma'(t)}(\lambda_\gamma\mathfrak{F}^*f)(\gamma')d\gamma'\\ &= \int_{\hat{G}}\overline{\gamma'(t)}(\mathfrak{F}^*f)(\gamma^{-1}\gamma')d\gamma'\\ &= \overline{\gamma(t)}\int_{\hat{G}}\overline{\gamma'(t)}(\mathfrak{F}^*f)(\gamma')d\gamma'\\ &= \overline{\gamma(t)}\mathfrak{F}(\mathfrak{F}^*f)(t)\\ &= \nu_\gamma f(t),\end{aligned}$$

so that the desired isomorphism is given by the unitary $1\otimes 1\otimes\mathfrak{F}$. A simple calculation yields for $s\in G$, $\mathfrak{F}\nu_\gamma\mathfrak{F}^* = \rho_s$, completing the proof.

$\square$

Lemma 8.8. *The algebra $\mathcal{M}_2$ of Lemma 8.7 is isomorphic to the von Neumann algebra $\mathcal{M}_3$ acting on $\mathcal{H} \otimes L^2(G)$ with generating set*

$$\{x \otimes 1, u_s \otimes \lambda_s, 1 \otimes \nu_\gamma\}_{x \in \mathcal{M}, s \in G, \gamma \in \hat{G}}.$$

The isomorphism transforms the action β of Lemma 8.7 to the action μ implemented by the one parameter unitary group $\{1 \otimes \rho_s\}_{s \in G}$.

Proof. Identify $L^2(G \times G)$ with $L^2(G) \otimes L^2(G)$ by extending the map which identifies the element $f \otimes g$ with the function $(s,t) \mapsto f(s)g(t)$, and define a unitary U on $L^2(G \times G)$ by $Uf(s,t) = f(st,t)$. Then,

$$\begin{aligned} U^*(\nu_\gamma \otimes \nu_\gamma)Uf(s,t) &= (\nu_\gamma \otimes \nu_\gamma)Uf(st^{-1},t) \\ &= \overline{\gamma(st^{-1})\gamma(t)}Uf(st^{-1},t) \\ &= \overline{\gamma(s)}f(s,t) \\ &= (\nu_\gamma \otimes 1)f(s,t), \end{aligned}$$

and

$$\begin{aligned} U^*(\lambda_r \otimes 1)Uf(s,t) &= (\lambda_r \otimes 1)Uf(st^{-1},t) \\ &= Uf(r^{-1}st^{-1},t) \\ &= f(r^{-1}s,t) \\ &= (\lambda_r \otimes 1)f(s,t). \end{aligned}$$

Hence,

$$\begin{aligned} (1 \otimes U^*)(x \otimes 1 \otimes 1)(1 \otimes U) &= x \otimes 1 \otimes 1, \\ (1 \otimes U^*)(u_s \otimes \lambda_s \otimes 1)(1 \otimes U) &= u_s \otimes \lambda_s \otimes 1, \\ (1 \otimes U^*)(1 \otimes \nu_\gamma \otimes \nu_\gamma)(1 \otimes U) &= 1 \otimes \nu_\gamma \otimes 1. \end{aligned}$$

Therefore, $1 \otimes U^*$ gives the desired isomorphism. Lastly, we note that for $s \in G$,

$$(1 \otimes U^*)(1 \otimes 1 \otimes \rho_s)(1 \otimes U) = 1 \otimes \rho_s \otimes \rho_s,$$

completing the proof.

□

Lemma 8.9. *The algebra $\mathcal{M}_3$ of Lemma 8.8 is spatially isomorphic to $\mathcal{M} \overline{\otimes} B(L^2(G))$, and transforms the action μ into $\alpha \otimes \rho$.*

Proof. Let W be the unitary on $\mathcal{H} \otimes L^2(G)$ as before satisfying

$$W(\mathcal{M} \rtimes_\alpha G)W^* = \{x \otimes 1, u_s \otimes \lambda_s\}''_{x \in \mathcal{M}, s \in G}.$$

We will show that

$$W(\mathcal{M}\overline{\otimes}L^\infty(G))W^* = \mathcal{M}\overline{\otimes}L^\infty(G).$$

For $x \in \mathcal{M}, \xi \in \mathcal{H} \otimes L^2(G), t \in G$, we have

$$\begin{aligned}
(W(x \otimes 1)W^*\xi)(t) &= (W^2\pi_\alpha(x)W^{*2}\xi)(t) \\
&= u_t^2(\pi_\alpha(x)W^{*2}\xi)(t) \\
&= u_t^2\alpha_{t^{-1}}(x)(W^{*2}\xi)(t) \\
&= u_t^2\alpha_{t^{-1}}(x)u_t^{*2}\xi(t) \\
&= \alpha_t(x)\xi(t).
\end{aligned}$$

If $y \in \mathcal{M}', \gamma \in \hat{G}$, then

$$\begin{aligned}
[(W(x \otimes 1)W^*)(y \otimes \nu_\gamma)\xi](t) &= \alpha_t(x)[(y \otimes \nu_\gamma)\xi](t) \\
&= \alpha_t(x)y\overline{\gamma(t)}\xi(t) \\
&= (y \otimes \nu_\gamma)[\alpha_t(x)\xi](t) \\
&= [(y \otimes \nu_\gamma)(W(x \otimes 1)W^*)\xi](t),
\end{aligned}$$

so $W(x \otimes 1)W^* \in [\mathcal{M}'\overline{\otimes}L^\infty(G)]' = \mathcal{M}\overline{\otimes}L^\infty(G)$. On the other hand, we have that $W^*(x \otimes 1)W = \pi_\alpha(x)$, and if $y \in \mathcal{M}', \gamma \in \hat{G}, \xi \in \mathcal{H} \otimes L^2(G)$, then

$$\begin{aligned}
[\pi_\alpha(x)(y \otimes \nu_\gamma)\xi](t) &= \alpha_{t^{-1}}[(y \otimes \nu_\gamma)\xi](t) \\
&= \alpha_{t^{-1}}y\overline{\gamma(t)}\xi(t) \\
&= (y \otimes \nu_\gamma)\alpha_{t^{-1}}\xi(t) \\
&= [(y \otimes \nu_\gamma)\pi_\alpha(x)\xi](t),
\end{aligned}$$

so $W^*(x \otimes 1)W \in \mathcal{M}\overline{\otimes}L^\infty(G)$. Lastly, since $W^*(1 \otimes \nu_\gamma)W = 1 \otimes \nu_\gamma$, the claim follows.

Now $W^*(u_s \otimes \lambda_s)W = 1 \otimes \lambda_s$, so that $W^*\{u_s \otimes \lambda_s\}''_{s \in G}W = \mathbb{C}\overline{\otimes}\mathcal{R}_\ell(G)$. Therefore, by Lemma 8.1,

$$W^*\mathcal{M}_3W = [\mathcal{M}\overline{\otimes}L^\infty(G) \cup \mathbb{C} \otimes \mathcal{R}_\ell(G)]'' = \mathcal{M}\overline{\otimes}B(L^2(G)).$$

Lastly, we note that for $s \in G$,

$$W^*(1 \otimes \rho_s)W = u_s \otimes \lambda_s,$$

completing the proof.

□

Combining Lemmas 8.6, 8.7, 8.8, 8.9 and the discussion preceding Lemma 8.6 we have that $(\mathcal{M} \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}$ is isomorphic to $\mathcal{M}\overline{\otimes}B(L^2(G))$, and that the bidual action $\hat{\hat{\alpha}}$ is transformed into $\{\alpha_s \otimes \mathrm{Ad}(\rho_s)\}_{s\in G}$. To finish the proof of Theorem 8.3 we need to show that the fixed point algebra $(\mathcal{M} \rtimes_\alpha G)^{\hat{\alpha}}$ is $\pi_\alpha(\mathcal{M})$.

We already know that $\pi_\alpha(\mathcal{M}) \subseteq (\mathcal{M} \rtimes_\alpha G)^{\hat{\alpha}}$. Note that $\pi_\alpha(\mathcal{M} \rtimes_\alpha G)$ is a subalgebra of $\{u_s \otimes \rho_s\}'_{s\in G}$. Hence if $\tilde{x} \in (\mathcal{M} \rtimes_\alpha G)^{\hat{\alpha}}$, we have that

$$\tilde{x} \in \mathcal{M}\overline{\otimes}B(L^2(G)) \cap \{u_s \otimes \rho_s\}'_{s\in G} \cap \{1 \otimes \nu_\gamma\}'_{\gamma\in\hat{G}}.$$

Let W be the unitary on $\mathcal{H} \otimes L^2(G)$ as before such that

$$W(\mathcal{M} \rtimes_\alpha G)W^* = \{x \otimes 1, u_s \otimes \lambda_s\}''_{x\in\mathcal{M}, s\in G}.$$

Since $W(u_s \otimes \lambda_s)W^* = 1 \otimes \lambda_s$ and $W(1 \otimes \nu_s)W^* = 1 \otimes \nu_s$, we have that by Lemma 8.1,

$$\tilde{x} \in \mathcal{M}\overline{\otimes}B(L^2(G)) \cap W^*(B(\mathcal{H})\overline{\otimes}\mathbb{C})W.$$

Therefore, $\tilde{x} = W^*(x \otimes 1)W$ for some $x \in B(\mathcal{H})$. If we can show that $x \in \mathcal{M}$, then $\tilde{x} = W^*(x \otimes 1)W = \pi_\alpha(x)$ and we are done. But if $y \in \mathcal{M}'$,

$$y \otimes 1 \in \mathcal{M}'\overline{\otimes}\mathbb{C} = (\mathcal{M}\overline{\otimes}B(L^2(G)))',$$

so that $x \otimes 1, W^*(x \otimes 1)W$ commute. Thus, for $\xi \in C_c(G, \mathcal{H})$, we have

$$(u_s^* x u_s y - y u_s^* x u_s)\xi(s) = 0,$$

and if we choose ξ such that $\xi(e) \neq 0$, this gives $xy = yx$, finishing the proof.

We have the following application to modular automorphism groups.

Corollary 8.10. *Let ϕ be an fns weight on a von Neumann algebra $\mathcal{M}$, let $\tilde{\phi}$ be the weight on $\mathcal{M} \rtimes_{\sigma^\phi} \mathbb{R}$ dual to ϕ, and let $\hat{\sigma}^\phi$ be the action on $\mathcal{M} \rtimes_{\sigma^\phi} \mathbb{R}$ dual to σ^ϕ. If we let h be the positive self-adjoint injective operator affiliated with $\mathcal{M} \rtimes_{\sigma^\phi} \mathbb{R}$ such that $\lambda(t) = h^{-it}$, then $\tau = \tilde{\phi}_h$ is a trace such that $\tau \circ \hat{\sigma}^\phi_s = e^{-s}\tau$.*

Proof. By Corollary 7.11, we have that $\sigma_t^{\tilde{\phi}} = \mathrm{Ad}(\lambda_{\mathcal{H}}(t))$. Let h be a positive self-adjoint operator affiliated with $\mathcal{M} \rtimes_{\sigma^\phi} \mathbb{R}$ such that $\lambda(t) = h^{-it}$. Then the modular automorphism group for $\tau = \tilde{\phi}_h$ is trivial so that τ is a trace. Since θ is dual to σ^ϕ we have that $\theta_s(h^{it}) = e^{-ist}h^{it}$. Hence, for $x \in \mathcal{M} \rtimes_{\sigma^\phi} \mathbb{R}^+$,

$$\begin{aligned}
\tau \circ \theta_s(x) &= \lim_{\epsilon\to 0} \hat{\phi}(h(1 + \epsilon h)^{-1}\theta_s(x)) \\
&= \lim_{\epsilon\to 0} e^{-s}(1 + \epsilon \cdot e^{-s})^{-1}\hat{\phi} \circ \theta_s(h(1 + \epsilon h)^{-1}x) \\
&= e^{-s}\tau(x).
\end{aligned}$$

□

In Section 9 we will apply the results of Sections 7 and 8 to properly infinite von Neumann algebras.

9 Structure of Properly Infinite von Neumann Algebras

We now derive some structural implications for a von Neumann algebra $\mathcal{M}$ which comes from the existence of an fns weight ϕ and associated modular automorphism group σ^ϕ. To this end we will apply the crossed product construction as developed in earlier Sections. We begin by proving that there is a relatively simple relation between $\mathcal{M}$ and $\mathcal{M} \rtimes_{\sigma^\phi} \mathbb{R}$ in the case that $\mathcal{M}$ is semifinite. The material for this section is from Chapter 12 of [13], except for Lemmas 9.1 and 9.4 which are from part 2, Section 4 and Appendix C of [4].

Lemma 9.1. *Let $\mathcal{M}$ be semifinite, and let ϕ be an fns weight on $\mathcal{M}$. Then there exists an isomorphism of $\mathcal{M} \rtimes_{\sigma^\phi} \mathbb{R}$ onto $\mathcal{M}\overline{\otimes}L^\infty(\mathbb{R})$ which transforms the dual action $\hat{\sigma}^\phi$ onto $id \otimes \mu$, where μ is the action of $\mathbb{R}$ on $L^\infty(\mathbb{R})$ by translation.*

Proof. Retaining the notation in Sections 7 and 8, let W be the unitary on $\mathcal{H} \otimes L^2(G)$ such that $W\pi_\alpha(x)W^* = x \otimes 1$ and $W\lambda_{\mathcal{H}}(t)W^* = \Delta^{it} \otimes \lambda_t$. Since $\mathcal{M}$ is semifinite, $\Delta^{it} \in \mathcal{M}$, so that

$$\Delta^{it} \otimes \lambda_t \in \{x \otimes 1, 1 \otimes \lambda_s\}''_{x\in\mathcal{M}, s\in\mathbb{R}},$$

and

$$1 \otimes \lambda_t \in \{x \otimes 1, \Delta^{is} \otimes \lambda_s\}''_{x\in\mathcal{M}, s\in\mathbb{R}}.$$

Consequently,

$$W(\mathcal{M} \rtimes_{\sigma^\phi} \mathbb{R})W^* = \{x \otimes 1, 1 \otimes \lambda_s\}''_{x\in\mathcal{M}, s\in\mathbb{R}}.$$

The dual action on $\mathcal{M} \rtimes_{\sigma^\phi} \mathbb{R}$ is implemented by the unitaries $\{\nu_{\mathcal{H}}(s)\}_{s\in G}$, which commute with W so that the dual action remains unchanged. Lastly, if we let $\mathfrak{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the isomorphism coming from the Fourier transform, then by Lemma 8.7, we have

$$(1 \otimes \mathfrak{F})\lambda_{\mathcal{H}}(s)(1 \otimes \mathfrak{F})* = \nu_{\mathcal{H}}(s).$$

By Lemma 8.1, $\{\nu_{\mathcal{H}}(s)\}_{s\in\mathbb{R}}$ generates $\mathbb{C}\overline{\otimes}L^\infty(\mathbb{R})$, so that

$$(1 \otimes \mathfrak{F})W(\mathcal{M} \rtimes_{\sigma^\phi} \mathbb{R})W^*(1 \otimes \mathfrak{F})^* = \mathcal{M}\overline{\otimes}L^\infty(\mathbb{R}).$$

Also, by Lemma 8.7, we have

$$(1 \otimes \mathfrak{F})\lambda_{\mathcal{H}}(s)(1 \otimes \mathfrak{F})* = 1 \otimes \rho_s,$$

which is clearly conjugate to the desired system.

□

Recall that a von Neumann algebra $\mathcal{M}$ has a canonical decomposition of the form

$$\mathcal{M} = \mathcal{M}_s \oplus \mathcal{M}_{III},$$

where $\mathcal{M}_s$ is semifinite and $\mathcal{M}_{III}$ is type III. Using this fact we will obtain a characterization of Type III algebras given in Theorem 9.3. First we start with a definition.

Definition 9.2. *Let $(\mathcal{N}, \mathbb{R}, \theta)$ be a covariant system and let τ be an fns trace on $\mathcal{N}$. Then we say θ* ***scales*** *τ if for $s \in \mathbb{R}$, we have*

$$\tau \circ \theta_s = e^{-s}\tau.$$

Recall that if $\mathcal{M}$ is a von Neumann algebra then $C_{\mathcal{M}}$ denotes the center of $\mathcal{M}$.

Theorem 9.3. *Let $\mathcal{M}$ be a properly infinite von Neumann algebra. Then there exists a covariant system $(\mathcal{N}, \mathbb{R}, \theta)$ such that $\mathcal{N}$ admits a trace τ scaled by θ and $\mathcal{M}$ is isomorphic to $\mathcal{N} \rtimes_\theta \mathbb{R}$. Furthermore, $\mathcal{M}$ is type III if and only if the central covariant system $(C_{\mathcal{N}}, \mathbb{R}, \theta)$ does not admit an invariant subsystem conjugate to $(L^\infty(\mathbb{R}), \mathbb{R}, \mu)$, where as before μ is the translation action.*

To prove the Theorem we need the following Lemma. First we recall that by Corollary 8.10, if ϕ is an fns weight on $\mathcal{M}$, and θ is the action on $\mathcal{M} \rtimes_{\sigma^\phi} \mathbb{R}$ dual to σ^ϕ, then there exists an fns trace τ on $\mathcal{M} \rtimes_{\sigma^\phi} \mathbb{R}$ which is scaled by θ. The following Lemma allows us to say a bit more.

Lemma 9.4. *If $\mathcal{M}$ is a properly infinite von Neumann algebra and $\mathcal{K}$ is a separable Hilbert space then $\mathcal{M} \cong \mathcal{M} \overline{\otimes} B(\mathcal{K})$.*

Proof.

Claim: There exists a sequence of pairwise othogonal projections $\{e_n\}_{n=1}^\infty$ such that $e_n \sim 1$ for all n, where $\sim$ denotes Murray-von Neumann equivalence of projections. If we let $e = \sum_{n=1}^\infty e_n$, then $\mathcal{M}$ is isomorphic to $e\mathcal{M}e = \mathcal{M}_e$.

Since $\mathcal{M}$ is properly infinite there exists a projection $f \in \mathcal{M}$ such that $f \sim (1-f) \sim 1$. Let $u, v \in \mathcal{M}$ such that

$$u^*u = 1, uu^* = f,$$

and

$$v^*v = f, vv^* = 1-f.$$

Then for $n \geq 1$ if we set $t_n = v^n u$, we have that $t_n^* t_n = f$, and $\{t_n t_n^*\}_{n=1}^\infty$ is a family of non-zero, pairwise orthogonal projections such that $t_n t_n^* \leq f$ for all n. Set $e_n = f_n$ and $e = \sum_{n=1}^\infty e_n$, then $e \preceq 1$ and $1 \preceq e$ so that $e \sim 1$. Finally, if we let $w \in \mathcal{M}$ such that $w^*w = 1, ww^* = e$, then $x \in \mathcal{M} \mapsto wxw^* \in \mathcal{M}_e$ is the desired isomorphism.

By the claim, we can assume that $\sum_{n=1}^{\infty} e_n = 1$. Let $\{f_{i,j}\}_{i,j=1}^{\infty}$ be matrix units for $B(\mathcal{K})$, and let $\{v_i\}_{i=1}^{\infty}$ be elements in $\mathcal{M}$ such that $v_i^* v_i = 1$ and $v_i v_i^* = e_i$. Then, if we set

$$u_k = \sum_{i=1}^{k} v_i \otimes f_{i,1},$$

we have that u_k converges in the SOT to an element $u \in \mathcal{M}\overline{\otimes}B(\mathcal{K})$, that u_k^* converges in the SOT to u^*, and that $u^*u = 1 \otimes f_{1,1}, uu^* = 1 \otimes 1$. Then $x \in \mathcal{M}\overline{\otimes}B(\mathcal{K}) \mapsto u^*xu \in \mathcal{M}\overline{\otimes}(\mathbb{C}\cdot f_{1,1})$ is an isomorphism, and the result follows.

□

proof of Theorem 9.3. The existence of such a system follows immediately from Lemma 9.4 and the discussion preceding it.

Let $(\mathcal{N}, \mathbb{R}, \theta)$ be a covariant sytem with trace τ on $\mathcal{N}$ scaled by θ and such that $\mathcal{M} \cong \mathcal{N} \rtimes_\theta \mathbb{R}$. Let ϕ be the weight on $\mathcal{M}$ dual to τ. Let (π_θ, λ) be the covariant system generating the crossed product as in Section 7. By Theorem 7.10 we have that for $x \in \mathcal{N}$,

$$\sigma_t^\phi(\pi_\theta(x)) = \pi(\sigma^\tau(x)) = \pi_\theta(x),$$

and for $s \in \mathbb{R}$,

$$\sigma_t^\phi(\lambda(s)) = \lambda(s)(D_{\tau\circ\theta_s} : D_\tau)_t = e^{-ist}\lambda(s).$$

Therefore σ^ϕ is precisely the dual action $\hat{\theta}$, and it follows by Theorem 8.3 that $\pi_\theta(\mathcal{N}) = \mathcal{M}_\phi$.

Claim: Identifying $\mathcal{N}$ with $\pi_\theta(\mathcal{N}) \subseteq \mathcal{M}$ and the action θ with the action $\pi_\theta \circ \theta \circ \pi_\theta^{-1}$, the center $C_\mathcal{M}$ coincides with the fixed point algebra $C_\mathcal{N}^\theta$.

We have that $C_\mathcal{M} \subseteq \mathcal{M}_\phi \cap C_\mathcal{N}$, and since $\lambda(s) \in \mathcal{M}$ we have that $C_\mathcal{M}$ commutes with each $\lambda(s)$ so that $C_\mathcal{M} \subseteq C_\mathcal{N}^\theta$. On the other hand, if $x \in C_\mathcal{N}^\theta$, this says precisely that x commutes with $\mathcal{N}$ and each $\lambda(s)$, so that x commutes with the generators of $\mathcal{M}$. Therefore $x \in C_\mathcal{M}$.

Now suppose that there exists an injective *-homomorphism $\pi : L^\infty(\mathbb{R}) \to C_\mathcal{N}$ such that for $t \in \mathbb{R}, f \in L^\infty(\mathbb{R})$, we have

$$\pi(\mu_t f) = \theta_t \pi(f).$$

Let $e = \pi(1)$ so that e is a non-zero projection in $C_\mathcal{N}^\theta$. We will show that the subalgebra $\mathcal{M}_e = \mathcal{N}_e \rtimes_\theta \mathbb{R}$ is semifinite, which shows that $\mathcal{M}$ is not type III.

Define a one-parameter unitary group $\{w_t\}_{t\in\mathbb{R}}$ in $L^\infty(\mathbb{R})$ by

$$w_t(s) = e^{ist},$$

and set $v_t = \pi(w_t)$. By assumption we have $\theta_s(v_t) = e^{ist}v_t$. If $x \in \mathcal{N}_e$, then since $v_t \in C_\mathcal{N}$, we have

$$v_t x v_t^* = x = \sigma_t^\phi(x),$$

and for $s \in \mathbb{R}$, we have

$$v_t \lambda(s) v_t^* = e^{-ist} \lambda(s).$$

Therefore, $\mathrm{Ad}(v_t)$ and $\hat{\theta}_t = \sigma_t^\phi$ agree on $\mathcal{M}_e$, and since $e \in C_{\mathcal{M}}$, we have that $\phi|_{\mathcal{M}_e}$ is semifinite and that $\sigma^{\phi|_{\mathcal{M}_e}} = \sigma^\phi|_{\mathcal{M}_e}$. It follows by Theorem 5.22 that $\mathcal{M}_e$ is semifinite.

On the other hand, write $\mathcal{M} = \mathcal{M}_s \oplus \mathcal{M}_{III}$ as in the paragraph following Lemma 9.1, and let e be the projection corresponding to the identity in $\mathcal{M}_s$. Then e is central, so belongs to $C_{\mathcal{N}}^\theta$. Therefore $\mathcal{M}_s = \mathcal{N}_e \rtimes_\theta \mathbb{R}$, so without loss of generality we will assume $e = 1$. Since $\mathcal{M}$ is semifinite, by Theorem 5.22 there exists a one-parameter SOT-continuous unitary group $\{v_t\}_{t \in \mathbb{R}}$ in $\mathcal{M}_\phi = \mathcal{N}$ such that $\sigma_t^\phi = \mathrm{Ad}(v_t)$. Then we have for $s \in \mathbb{R}$,

$$v_t \lambda(s) v_t^* = \sigma^\phi(u_s) = e^{-ist} \lambda(s),$$

so that (λ, v) satisfies the Heisenberg-Weyl condition. By Lemma 8.2 the von Neumann subalgebra generated in $\mathcal{N}$ by $\{v_t\}_{t \in \mathbb{R}}$ is isomorphic to $L^\infty(\mathbb{R})$ and the isomorphism translates the action $\mathrm{Ad}(\lambda)$, which we have identified with θ, to the translation action μ.

□

We finish this Section with the remark that in the case of a type III algebra, the covariant system $(\mathcal{N}, \mathbb{R}, \theta)$ is actually canonical. We state the following Theorem [13] (See Chapter 12, Section 1 of [13]).

Theorem 9.5. *Let $\mathcal{M}$ be a type III von Neumann algebra. Then there exists a covariant system $(\mathcal{N}, \mathbb{R}, \theta)$, unique up to conjugacy, such that $\mathcal{N}$ admits an fns trace τ scaled by θ, and such that $\mathcal{M}$ is isomorphic to $\mathcal{N} \rtimes_\theta \mathbb{R}$. Moreover, $\mathcal{N}$ is type II_∞.*

It is at this point that the theory becomes interesting. For it turns out that in the case that $\mathcal{M}$ is a type III factor, the covariant system $(C_{\mathcal{N}}, \mathbb{R}, \theta)$ is an isomorphism invariant, called the **flow of weights** associated with $\mathcal{M}$. For this system, we let $T > 0$ be the period of the action (where $T = 0$ if the action is trivial, and $T = \infty$ if the action has no period), and we say that $\mathcal{M}$ is a factor of type III_λ where $\lambda = e^{-T}$ (where $\lambda = 0$ if $T = \infty$). Lastly, we state the following particularly nice decomposition of type III_λ factors for $0 < \lambda < 1$ (see Chapter 12, Section 2 of [13]).

Theorem 9.6. *Let $\mathcal{M}$ be a type III factor. Then $\mathcal{M}$ is of type III_λ for $0 < \lambda < 1$ if and only if there exists a type II_∞ factor $\mathcal{N}$, an automorphism $\theta \in Aut(\mathcal{N})$, and an fns trace τ on $\mathcal{N}$ such that $\tau \circ \theta = \lambda \tau$ and such that $\mathcal{M}$ is isomorphic to $\mathcal{N} \rtimes_\theta \mathbb{Z}$.*

This theorem is particularly interesting because in order to construct non-isomorphic type III factors, we just need to find automorphisms which scale a fixed trace by different degrees.

10 Concluding Remarks

In Section 9 it was noted that a type III factor can be classified by the kernel of the action of the flow of weights. We mention here another method to obtain this same classification. In [1], Arveson discusses a notion of spectrum for an action of a locally compact abelian group G on von Neumann algebra $\mathcal{M}$. This is achieved by taking the associated representation of $L^1(G)$, and defining the spectrum to be the closed subset in $\hat{G}$ (the hull) associated with the kernel. In the case of the modular automorphism group σ^ϕ of an fns weight ϕ, the spectrum coincides with the set $\sigma(\Delta_\phi) \cap \mathbb{R}^+$ (see Section 3.4 of [11]). In [3], Connes noted that the spectrum decreases when we restrict the action to the corner algebras $\mathcal{M}_e$, where e is a non-zero projection fixed by the action. The intersection of the spectra of the corner algebras, called the Connes spectrum, is invariant under cocycle conjugacy (see Section 3.3 of [11]). Therefore, by the Connes Cocycle Derivative Theorem, we can define the spectrum of a von Neumann algebra $\mathcal{M}$, denoted $\Gamma(\mathcal{M})$, to be the Connes spectrum of the action σ^ϕ, where ϕ is any fns weight on $\mathcal{M}$. It turns out that $\Gamma(\mathcal{M})$ coincides with the kernel of the dual action $\hat{\sigma}^\phi$ restricted to $C_{\mathcal{M} \rtimes_{\sigma^\phi} \mathbb{R}}$ (see Chapter 11, Section 2 of [13]). In particular, if $\mathcal{M}$ is a type III factor, then the spectrum of $\mathcal{M}$ is the kernel of the action of the flow of weights (see Chapter 12, Section 1 of [13]). The advantage of this viewpoint is that by Theorem 5.9 we obtain the same classification of type III factors in terms of the spectra of the modular operators. In Section 3.4 of [11], factors of type III_λ are constructed by measure theoretic methods and using the the above picture of the invariant. As an area for further study, it would be interesting to explore in detail the connection between this picture and that given in Section 9 of the invariant for type III factors.

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