On The Density of Binary Matroids Without A Given Minor

by

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A thesis
presented to the University of Waterloo
in fulfilment of the
thesis requirement for the degree of
Master in Mathematics
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2016

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This thesis is motivated by the following question: how many elements can a simple binary matroid with no $\mathrm{PG}(t, 2)$-minor have? This is a natural analogue of questions asked about the density of graphs in minor-closed classes. We will answer this question by finding the eventual growth rate function of the class of matroids with no $\mathrm{PG}(t, 2)$-minor, for any $t \geq 2$. Our main tool will be the matroid minors structure theory of Geelen, Gerards, and Whittle, and much of this thesis will be devoted to frame templates, the notion of structure in that theory.

Acknowledgements

I would first like to thank my supervisor, Dr. Peter Nelson. I have learned a great deal under his guidance thus far, and I look forward to learning much more in the years to come. I would also like to thank my family, who have supported me unfailingly, and are the basis for anything I accomplish.

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1 Introduction

This thesis is concerned with a notion of density in matroids: the relationship between the rank and number of elements. In particular, we are interested in the density of binary matroids without a given projective geometry minor. Using the matroid minors structure theory of Geelen, Gerards, and Whittle, we will prove a result, Theorem 1.12, which has the following two corollaries.

Theorem 1.1. *Let $t \geq 0$ be an integer, and let M be a simple binary matroid with no* $\mathrm{PG}(t+2,2)$*-minor. If $r(M)$ is sufficiently large, then* $|M| \leq 2^t\binom{r(M)-t+1}{2} + 2^t - 1$.

Theorem 1.2. *Let $t \geq 0$ be an integer, and let M be a simple binary matroid with no* $\mathrm{AG}(t+3,2)$*-minor. If $r(M)$ is sufficiently large, then* $|M| \leq 2^t\binom{r(M)-t+1}{2} + 2^t - 1$.

Theorem 1.12 characterizes for each integer $t \geq 0$ the binary matroids N for which the above bound holds for matroids with no N-minor and is best-possible. Before proving this result we will highlight important past results concerning growth rates of classes of binary matroids. We will also introduce 'frame templates', objects defined by Geelen, Gerards, and Whittle in [17] which precisely describe structure of representable matroids in minor-closed classes, and we show how they can be simplified. After proving our main result, we will discuss how the techniques of this thesis may be applied in more general settings. We direct the reader to the appendix for a brief overview of concepts in matroid theory relevant for this thesis.

1.1 Growth Rates

In this thesis we will describe density using the growth rate function, which for a class of matroids gives the maximum number of elements of a simple rank-n matroid in the class for any n. More precisely, the *growth rate function* of a non-empty class of matroids $\mathcal{M}$ is

$$h_{\mathcal{M}}(n) = \max\{\varepsilon(M) : M \in \mathcal{M}, r(M) = n\}.$$

Some classes of matroids have growth rate functions which are easy to calculate. For example, the class of all binary matroids has growth rate function $h(n) = 2^n - 1$ because a binary matrix with n rows has at most $2^n - 1$ nonzero columns, and $\varepsilon(\mathrm{PG}(n-1, 2)) = 2^n - 1$. The class $\mathcal{G}$ of all graphic matroids has growth rate function $h_{\mathcal{G}}(n) = \binom{n+1}{2}$, because a simple graph on $n+1$ vertices has at most $\binom{n+1}{2}$ edges, and $\varepsilon(M(K_{n+1})) = \binom{n+1}{2}$. However, exact growth rate functions are often nontrivial to determine, even for naturally defined classes such as regular matroids [3], dyadic matroids [4, 5], sixth-root-of-unity matroids [6], and near-regular matroids [6].

We can also restrict our attention to classes of graphic matroids and find linear bounds on the growth rate function within $\mathcal{G}$. We rephrase the following result of Mader [7] in the language of matroids.

Theorem 1.3 (Mader 1967)**.** *For any graphic matroid N there is some constant c_N such that any graphic matroid M with no N-minor satisfies $\varepsilon(M) \leq c_N r(M)$.*

This theorem shows that there is a gap between the growth rate function for the class of all graphic matroids, which is quadratic, and the growth rate function within $\mathcal{G}$ after excluding any graphic matroid, which is linear. When $M = M(K_t)$ the constant c_N is determined by Thomason in [8], and again we rephrase the result matroidally.

Theorem 1.4 (Thomason 2001)**.** *For any positive integers t and n, there exists a rank-n graphic matroid M with no $M(K_t)$-minor such that $\varepsilon(M) = c_t n$, where $c_t = (\alpha + o(1))t\sqrt{\log t}$ and $\alpha = .319\ldots$ is an explicit constant.*

Thomason and Myers prove a similar theorem for excluding a general graphic matroid $M(G)$ in [9], only replacing α by $\alpha \cdot \gamma(G)$, where γ is a constant depending on properties of G. Theorem 1.4 implies Theorem 1.3 because any graph is a minor of some complete graph, and $h_{\mathrm{Ex}(M_1)}(n) \leq h_{\mathrm{Ex}(M_2)}(n)$ whenever M_1 is a minor of M_2.

In this thesis we will find a theorem analogous to Theorem 1.4, but for the class of binary matroids. Thomason determined the growth rate function obtained by excluding the densest graphs, so we will determine the growth rate function obtained by excluding the densest binary matroids, which are projective geometries. Let $\mathcal{M}_t$ denote the class of binary matroids with no $\mathrm{PG}(t+2,2)$-minor. The following classical theorem proved independently by Sauer [10] and Shelah [11] gives an upper bound on $h_{\mathcal{M}_t}(n)$ for any $t \geq 0$.

Theorem 1.5. *For any integer $t \geq 0$, $h_{\mathcal{M}_t}(n) \leq \binom{n+1}{t+2}$.*

The bound of $\binom{n+1}{2}$ for $t = 0$ is tight, because $\mathrm{PG}(2,2)$ is not graphic, and thus $\binom{n+1}{2}$ is a lower bound on $h_{\mathcal{M}_0}(n)$. However, for $t > 1$ the bound is at least cubic in n, and the following theorem from [12] shows that there is a quadratic upper bound on $h_{\mathcal{M}_t}(n)$ for any $t \geq 0$. The authors prove a more general theorem, but here we state their result restricted to classes of binary matroids.

Theorem 1.6 (Geelen, Kung, Whittle 2009)**.** *For any minor-closed class $\mathcal{M}$ of binary matroids there is some constant c such that either*

(i) $h_{\mathcal{M}}(n) \leq cn$,

(ii) $\binom{n+1}{2} \leq h_{\mathcal{M}}(n) \leq cn^2$ *and $\mathcal{M}$ contains all graphic matroids, or*

(iii) $h_{\mathcal{M}}(n) = 2^n - 1$, *and $\mathcal{M}$ is the class of all binary matroids.*

This theorem implies Theorem 1.3, and shows that there is a significant gap in the possible growth rate functions of minor-closed classes of binary matroids. If we exclude any non-graphic binary matroid the growth rate function drops from exponential to quadratic, and if we exclude any graphic matroid it drops from exponential to linear. Since $\mathrm{PG}(t+2, 2)$ is not graphic for any $t \geq 0$, $\mathcal{M}_t$ contains all graphic matroids. Thus, Theorem 1.6 tells us that for sufficiently large n, $h_{\mathcal{M}_t}(n)$ is bounded above by some fixed quadratic function. We will find this function and show that the bound is tight, but we will not consider $h_{\mathcal{M}_t}(n)$ for small values of n because there is too much anomalous behavior among low-rank matroids. For example, $h_{\mathcal{M}_t}(n)$ is exponential for $n \leq t+2$ because no matroid of rank less than $t+3$ has a $\mathrm{PG}(t+2, 2)$-minor, but is eventually quadratic. For a class of matroids $\mathcal{M}$, if $h_{\mathcal{M}}(n) = g(n)$ for all n sufficiently large we say that $g(n)$ is an *eventual growth rate function* of $\mathcal{M}$, and we write $h_{\mathcal{M}}(n) \approx g(n)$.

Previous results have established the eventual growth rate function for $\mathcal{M}_t$ only for $t = 0$ and $t = 1$. We discussed previously that Theorem 1.5 implies that $h_{\mathcal{M}_t}(n) = \binom{n+1}{2}$. In [14], Grace and van Zwam prove that $h_{\mathcal{M}_1}(n) \approx 2\binom{n}{2} + 1$. Theorem 1.5 is not proved in the framework of matroid theory, but the proof in [14] uses frame templates and the matroid structure theory of Geelen, Gerards, and Whittle, as we will in this thesis. For excluding $\mathrm{AG}(t+3, 2)$ as a minor, Kung, Mayhew, Pivotto, and Royle [13] prove that the class of matroids with no $\mathrm{AG}(3, 2)$-minor has growth rate function $h(n) = \binom{n+1}{2}$ for $n \geq 5$, but there are no previous results for $t > 0$.

For each integer $t \geq 0$ we will determine the eventual growth rate function of both $\mathcal{M}_t$ and the class of matroids with no $\mathrm{AG}(t+3, 2)$-minor by first finding a lower bound via a class of examples, and then showing that this lower bound is also an upper bound.

1.2 t-Graphic Matroids

To find a lower bound for $h_{\mathcal{M}_t}(n)$ we will exhibit a class of matroids with no $\mathrm{PG}(t+2,2)$-minor. A matroid M is *pinched-graphic* if there is some binary matrix A with columns indexed by $E(M) \cup \{e\}$ such that $A[E]$ is the incidence matrix of a graph, $A[e]$ has support at most 3, and M is isomorphic to $M(A)/e$. Geelen and Nelson show in [18] that the class of pinched-graphic matroids is equivalent to the class of 'even-cycle matroids having a representation with a blocking pair', which has been studied in [15], and elsewhere. Let $\mathcal{P}$ denote the class of pinched-graphic matroids. If $A[E]$ represents $M(K_{n+1})$, then $r(M) = n$ and $\varepsilon(M) = \binom{n+1}{2} + n - 2$, so $h_{\mathcal{P}}(n) = \binom{n+1}{2} + n - 2$. It is straightforward to show that $\mathcal{P}$ is minor-closed.

Lemma 1.7. *The class of pinched-graphic matroids is minor-closed.*

Proof. Let M be a pinched-graphic on ground set E. Then there is some binary matrix A with columns indexed by $E(M) \cup e$ and rows indexed by a set R such that $A[E]$ is the incidence matrix of a graph, $A[e]$ has support at most 3, and M is isomorphic to $M(A)/e$. Then $M \backslash d$ is isomorphic to $M(A[E-d])/e$, so $M \backslash d$ is pinched-graphic. Let $f \in E(M)$ be a non-loop of M. If $A[f]$ is a unit column with nonzero entry in row r, then $A[R-r, E-f]$ is graphic, $A[R-r, e]$ has support at most three, and $M(A[R-r, E \cup e - f])/e$ is isomorphic to M/f. Otherwise, $A[f]$ has nonzero entries in rows $r, r' \in R$. Let A' be obtained from A by adding row r to row r'. Then $A'[R-r, E-f]$ is graphic, $A'[R-r, e]$ has support at most three, and $M(A'[R-r, E \cup e - f])/e$ is isomorphic to M/f. $\square$

The following lemma shows that we can generalize any minor-closed class of $\mathbb{F}$-representable matroids to a larger minor-closed class. For a fixed field $\mathbb{F}$, and any class $\mathcal{M}$ of $\mathbb{F}$-representable matroids and any integer $t \geq 0$, define $\mathcal{M}^t$ to be the class of matroids having a representation K over $\mathbb{F}$ of the form

$$K = \left[\begin{array}{c} K_1 \\ \hline K_2 \end{array}\right],$$

where $M(K_2) \in \mathcal{M}$ and K_1 has at most t rows.

Lemma 1.8. *Let $\mathbb{F}$ be a field, and let $\mathcal{M}$ be a minor-closed class of $\mathbb{F}$-representable matroids. For any integer $t \geq 0$, $\mathcal{M}^t$ is minor-closed.*

Proof. Let $M \in \mathcal{M}^t$ with ground set E, and let

$$K = \left[\begin{array}{c} K_1 \\ \hline K_2 \end{array}\right]$$

be a representation of M such that $M(K_2) \in \mathcal{M}$ and K_1 has at most t rows. For any $d \in E$, $K[E-d]$ is a representation of $M \backslash d$, so $M \backslash d \in \mathcal{M}^t$. It suffices to show that if $M \in \mathcal{M}^t$, then $M/e \in \mathcal{M}^t$ for any non-loop e of M. By applying elementary row operations we may assume that either $K_2[e]$ is a unit column and $K_1[e] = 0$, or $K_2[e] = 0$ and $K_1[e]$ is a unit column. In the first case, let K_2' be a representation of M/e, and then

$$\left[\begin{array}{c} K_1 \\ \hline K_2' \end{array}\right]$$

is a representation of M. In the second case, let K_1' be the matrix obtained from K_1 by deleting the row in which $K_1[e] \neq 0$, and then

$$\left[\begin{array}{c} K_1'[E-e] \\ \hline K_2[E-e] \end{array}\right]$$

is a representation of M. $\square$

Another nice feature of $\mathcal{M}^t$ is that we can easily determine the growth rate function of $\mathcal{M}^t$ from the growth rate function for $\mathcal{M}$. We just require that $\mathcal{M}$ is closed under adding loops and *parallel extension*, which for a matroid M defines a matroid $M' = (E(M) \cup \{f\}, r')$ such that $r'(X) = r(X)$ for any $X \subseteq E(M)$, and f is parallel with e for some $e \in E(M)$. Most naturally defined classes of matroids are closed under this operation, including $\mathrm{Ex}(M)$ for any simple matroid M.

Lemma 1.9. *Let $\mathbb{F}$ be a finite field, and let $\mathcal{M}$ be a non-empty class of $\mathbb{F}$-representable matroids which is closed under parallel extension and adding loops. Then $h_{\mathcal{M}^t}(n) = |\mathbb{F}|^t(h_{\mathcal{M}}(n-t)) + |\mathbb{F}|^t - 1$.*

Proof. We will first show that $h_{\mathcal{M}^t}(n) \leq |\mathbb{F}|^t(h_{\mathcal{M}}(n-t)) + |\mathbb{F}|^t - 1$. Let $M \in \mathcal{M}^t$, and let

$$K = \left[\begin{array}{c} K_1 \\ \hline K_2 \end{array}\right]$$

be a representation of M such that $M(K_2) \in \mathcal{M}$ and K_1 has at most t rows. If $r(M) = n$, then $\mathrm{rank}(K_2) \leq n-t$. Then K_2 has at most $h_{\mathcal{M}}(n-t)$ distinct nonzero columns, so there are at most $|\mathbb{F}|^t(h_{\mathcal{M}}(n-t))$ distinct columns e of K such that $K_2[e]$ is nonzero. Since there are at most $|\mathbb{F}|^t-1$ distinct nonzero columns e of K such that $K_2[e]$ is zero, we have $h_{\mathcal{M}^t}(n) \leq |\mathbb{F}|^t(h_{\mathcal{M}}(n-t)) + |\mathbb{F}|^t - 1$.

We will now show that $h_{\mathcal{M}^t}(n) \geq |\mathbb{F}|^t(h_{\mathcal{M}}(n-t)) + |\mathbb{F}|^t - 1$. Let $n \geq 0$ be an integer. For $n \leq t$ the statement holds, so assume $n > t$. Let M be a simple matroid in $\mathcal{M}$ such that $r(M) = n-t$ and $|M| = h_{\mathcal{M}}(n-t)$. Let K_2 be a representation of M. Consider the matrix

$$K = \left[\begin{array}{c} K_1' \\ \hline K_2' \end{array}\right]$$

such that the columns of K are precisely the nonzero vectors of the form

$$\left[\begin{array}{c} v_1 \\ \hline v_2 \end{array}\right]$$

where v_1 has t entries and v_2 is either the zero vector or a column of K_2. Then $M(K_2') \in \mathcal{M}$ since $\mathcal{M}$ is closed under parallel extensions and adding loops, so $M(K) \in \mathcal{M}^t$. Since $M(K)$ is simple and has $|\mathbb{F}|^t(h_{\mathcal{M}}(n-t)) + |\mathbb{F}|^t - 1$ elements, we have $h_{\mathcal{M}^t}(n) \geq |\mathbb{F}|^t(h_{\mathcal{M}}(n-t)) + |\mathbb{F}|^t - 1$. $\square$

We will now define two classes of binary matroids which are extremely important for this thesis. A matroid is *t-graphic* if it is in $\mathcal{G}^t$, and a matroid is *t-pinched-graphic* if it is in $\mathcal{P}^t$. Since $h_{\mathcal{G}}(n) = \binom{n+1}{2}$ and $h_{\mathcal{P}}(n) = \binom{n+1}{2}+n-2$, Lemma 1.9 tells us that

$$h_{\mathcal{G}^t}(n) = 2^t\binom{n-t+1}{2} + 2^t - 1$$

and

$$h_{\mathcal{P}^t}(n) = 2^t\left(\binom{n-t+1}{2} + n - t - 2\right) + 2^t - 1.$$

We will let $f_t(n)$ denote $2^t\binom{n-t+1}{2} + 2^t - 1$ for the remainder of this thesis. These observations give us the following corollaries of Lemma 1.8.

Corollary 1.10. *If N is a binary matroid which is not t-graphic, then $h_{\mathrm{Ex}(N)}(n) \geq f_t(n)$.*

Corollary 1.11. *If N is a binary matroid which is not t-pinched-graphic, then $h_{\mathrm{Ex}(N)}(n) > f_t(n)$ for $n > t-2$.*

Since $\mathrm{PG}(t+2,2)$ has rank $t+3$ and

$$f_t(t+3) = 2^t\binom{4}{2} + 2^t - 1 < 2^{t+3} - 1 = \varepsilon(\mathrm{PG}(t+2,2)),$$

we know $\mathrm{PG}(t+2,2)$ is not t-graphic. Then Corollary 1.10 tells us that $h_{\mathcal{M}_t}(n) \geq f_t(n)$ for any $t \geq 0$. The rest of this thesis will be devoted to proving the following.

Theorem 1.12. *Let $t \geq 0$ be an integer, and let N be a binary matroid. Then $h_{\mathrm{Ex}(N)}(n) \approx f_t(n)$ if and only if N is not t-graphic and is t-pinched-graphic.*

We will show later that $\mathrm{PG}(t+2,2)$ is t-pinched-graphic, and then we have as a corollary that $h_{M_t}(n) = f_t(n)$ for sufficiently large n. In the next section we will introduce frame templates, which are the tools we will use to find upper bounds on the growth rate functions for these classes.

2 Frame Templates & Structure Theory

In this section we will provide an overview of the structure theory of Geelen, Gerards, and Whittle for minor-closed classes of $\mathbb{F}$-representable matroids. This theory generalizes to representable matroids the groundbreaking work of Robertson and Seymour on the Graph Minors Structure Theorem [16]. The matroid structure theory considers perturbation, which is a way to describe the 'distance' between two matroids representable over the same field. The authors work with objects called 'represented matroids', but we will describe their results in terms of matroids as we have defined them. This means that we will give a different definition of perturbations, and a slight weakening of their main theorem.

For $\mathbb{F}$-representable matroids $M_1 = (E, r_1)$ and $M_2 = (E, r_2)$ we say that M_1 is a *rank-*$(\leq t)$ *perturbation* of M_2 if and there are matrices A_1 and A_2 with same row indices such that $M(A_1) = M_1$, $M(A_2) = M_2$, and $\text{rank}(A_1 - A_2) \leq t$. We write $\text{pert}(M_1, M_2)$ for the smallest integer t such that M_1 is a $(\leq t)$-perturbation of M_2. The following theorem from [17] says that any matroid M in a minor-closed class of $\mathbb{F}$-representable matroids, either M or M^* is close to being a frame matroid.

Theorem 2.1. *For any prime field $\mathbb{F}$ and any proper minor-closed class $\mathcal{M}$ of $\mathbb{F}$-representable matroids there exist integers k and t such that each vertically k-connected matroid M in $\mathcal{M}$ satisfies either* $\text{pert}(M, N) \leq t$ *or* $\text{pert}(M^*, N) \leq t$ *for some frame matroid N.*

This is a remarkable result, but the authors go even further by characterizing the perturbations given in the theorem. They do so with objects called frame templates, which precisely describe perturbations of frame matroids. A *frame template* over a finite field $\mathbb{F}$ is a 9-tuple $\Phi = (\Gamma, C, D, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ such that

(i) Γ is a subgroup of the multiplicative group of $\mathbb{F}$,

(ii) C, D, X, Y_0, Y_1 are disjoint finite sets,

(iii) $A_1 \in \mathbb{F}^{(D \cup X) \times (Y_0 \cup Y_1 \cup C)}$,

(iv) Δ is a subgroup of the additive group of $\mathbb{F}^{Y_0 \cup Y_1 \cup C}$ and is closed under scaling by elements of Γ, and

(v) Λ is a subgroup of the additive group of $\mathbb{F}^D$ and is closed under scaling by elements of Γ.

A frame template describes two classes of matrices. A matrix $A' \in \mathbb{F}^{B\times(E-B)}$ *respects* Φ if

(i) $X, D \subseteq B$, and $Y_0, Y_1, C \subseteq E - B$,

(ii) there is some set $Z \subseteq E-(B\cup Y_0\cup Y_1\cup C)$ such that $A'[X\cup D, Z] = 0$ and $A'[B-(X\cup D), Z]$ consists of unit columns,

(iii) $A'[X, E-(B\cup Y_0\cup Y_1\cup C\cup Z)] = 0$, $A'[D, E-(B\cup Y_0\cup Y_1\cup C\cup Z)]$ has columns from Λ, and $A'[B-(X\cup D), E-(B\cup Y_0\cup Y_1\cup C\cup Z)]$ is a Γ-frame matrix,

(iv) $A'[X\cup D, Y_0\cup Y_1\cup C] = A_1$, and $A'[B-(X\cup D), Y_0\cup Y_1\cup C]$ has rows from Δ.

Any matrix respecting Φ is of the form

$$\begin{array}{r|c|c|ccc|}
 & & Z & Y_0 & Y_1 & C \\ \hline
X & 0 & 0 & & & \\ \cline{2-3}
D & \Lambda\text{-columns} & 0 & & A_1 & \\ \hline
 & \Gamma\text{-frame} & \text{unit} & & \Delta\text{-rows} & \\ \hline
\end{array}.$$

A matrix $A \in \mathbb{F}^{B\times(E-B)}$ *conforms* to Φ if there is some matrix A' which respects Φ such that

(i) $A[B, E-(B\cup Z)] = A'[B, E-(B\cup Z)]$,

(ii) for each $z \in Z$ there is some $y \in Y_1$ such that $A[z] = A'[z] + A'[y]$.

Any matrix conforming to Φ is of the form

$$\begin{array}{r|c|c|ccc|}
 & & Z & Y_0 & Y_1 & C \\ \hline
X & 0 & * & & & \\ \cline{2-3}
D & \Lambda\text{-columns} & * & & A_1 & \\ \hline
 & \Gamma\text{-frame} & * & & \Delta\text{-rows} & \\ \hline
\end{array}.$$

An $\mathbb{F}$-representable matroid M *conforms* to Φ if there is some matrix A conforming to Φ such that $M \cong M([I,A])/C\backslash(B-X)\cup Y_1$. An $\mathbb{F}$-representable matroid *co-conforms* to Φ if M^* conforms to Φ. Let $\mathcal{M}(\Phi)$

denote the set of matroids conforming to Φ. The *complexity* of Φ is $|C \cup D \cup Y_0 \cup Y_1|$, and is denoted $c(\Phi)$. This is a natural template parameter because any matroid conforming to Φ is a $(\leq c(\Phi))$-perturbation of a frame matroid, since for any matrix conforming to Φ we can add a matrix of rank at most $c(\Phi)$ to zero out all rows indexed by D and all columns indexed by $Y_0 \cup Y_1 \cup C$. This shows that templates are closely related to perturbations in that they describe specific perturbations of frame matroids. We can now state the main structure theorem from [17].

Theorem 2.2 (Geelen, Gerards, Whittle). *For any finite field $\mathbb{F}$ and any proper minor-closed class of $\mathbb{F}$-representable matroids there exist two finite sets of templates T and T^* and a positive integer k such that*

(i) for any vertically k-connected matroid $M \in \mathcal{M}$ of rank at least $2k$, either M conforms to some template in T, or M^* conforms to some template in T^*, and

(ii) for every template $\Phi \in \mathsf{T}$, $\mathcal{M}(\Phi) \subseteq \mathcal{M}$, and for any template $\Psi \in \mathsf{T}^*$, $\mathcal{M}^*(\Psi) \subseteq \mathcal{M}$.

This result is stated in [17] but not proved. It says that for any proper minor-closed class of $\mathbb{F}$-representable matroids there are two finite sets of templates which precisely describe the ways in which the high rank, highly vertically connected matroids in the class are perturbations of frame matroids. Frame templates are somewhat difficult to work with, so in the next section we will show how they can be simplified.

2.1 Standardized Templates

We now define a standardized frame template, and show that for every frame template there is an equivalent standardized frame template. A frame template $\Phi = (\Gamma, C, D, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ over a prime field $\mathbb{F}$ is *standardized* if $X = \emptyset$, and there is some $R \subseteq D$ such that

(i) $\Lambda = \mathbb{F}^{D-R} \times \{0\}^R$,

(ii) $A_1[R, C] = 0$,

(iii) $A_1[D-R, C]$ and $A_1[R, Y_1]$ are in reduced-row echelon form,

(iv) $A_1[R', Y_0]$ is in reduced-row echelon form where $R' \subseteq R$ denotes zero rows of $A_1[R, Y_1]$,

(v) $A_1[D - R, y] = 0$ for each $y \in Y_0 \cup Y_1$ for which $A_1[Y_0]$ or $A_1[Y_1]$ has a leading 1, and

(vi) $\Delta[y] = 0$ for each $y \in Y_0 \cup Y_1$ for which $A_1[Y_0]$ or $A_1[Y_1]$ has a leading 1.

Every conforming matrix of a standardized template is of the form

$$
\begin{array}{c|c|c|cc|cc|cc|}
 & & Z & \multicolumn{2}{c}{Y_0} & \multicolumn{2}{c}{Y_1} & \multicolumn{2}{c}{C} \\
\cline{2-10}
\multirow{2}{*}{R} & 0 & * & * & * & I & * & 0 & 0 \\
\cline{2-10}
 & 0 & 0 & I & * & 0 & 0 & 0 & 0 \\
\cline{2-10}
\multirow{2}{*}{$D-R$} & \multirow{2}{*}{$\Lambda[D-R]\text{-columns}$} & * & 0 & * & 0 & * & I & * \\
\cline{3-10}
 & & * & 0 & * & 0 & * & 0 & 0 \\
\cline{2-10}
 & \Gamma\text{-frame} & * & \multicolumn{6}{c|}{\Delta\text{-rows}} \\
\cline{2-10}
\end{array}.
$$

In this thesis we will use '$*$' to help visualize the structure of matrices. For example, in the above matrix each '$*$' in a set of columns contained in $Y_0 \cup Y_1 \cup C$ represents a fixed submatrix of A_1, and each '$*$' in Z represents a submatrix of A chosen according to (ii) of the definition of a conforming matrix.

Standardized templates are convenient to work with because we can easily contract a basis of C, Y_0, or Y_1. We say templates Φ and Φ' are *equivalent* if $\mathcal{M}(\Phi) = \mathcal{M}(\Phi')$, and in the remainder of this section we will show that for every frame template there is an equivalent standardized frame template. We will first show that X is unnecessary. The following lemma appears in [14], but we include a proof for completeness.

Lemma 2.3. *For every frame template* $\Phi = (\Gamma, C, D, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ *over a finite field* $\mathbb{F}$ *there is an equivalent frame template* $\Phi' = (\Gamma, C, D', \varnothing, Y_0', Y_1, A_1', \Delta', \Lambda)$.

Proof. Define $A_1' \in \mathbb{F}^{(D \cup X) \times (X \cup Y_0 \cup Y_1 \cup C)}$ such that $A_1'[Y_0 \cup Y_1 \cup C] = A_1$, and $A_1'[D, X] = 0$ and $A_1'[X, X] = I_X$. Define $\Delta' = \Delta \times \{0\}^X$, and define $\Phi' = (\Gamma, C, D \cup X, \emptyset, Y_0 \cup X, Y_1, A_1', \Delta', \Lambda)$. Clearly Δ' is a subgroup of the additive group of $\mathbb{F}^{X \cup Y_0 \cup Y_1 \cup C}$.

We will first show that $\mathcal{M}(\Phi) \subseteq \mathcal{M}(\Phi')$. Consider the matrix

$$A = \begin{array}{c|c|c|c|}
 \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{Z} & \multicolumn{1}{c}{Y_0 \quad Y_1 \quad C} \\ \cline{2-4}
 \begin{matrix} X \\ D \end{matrix} & A_\Lambda & * & A_1 \\ \cline{2-4}
 & A_\Gamma & * & A_\Delta \\ \cline{2-4}
\end{array}$$

which conforms to Φ. Here A_Γ is a Γ-frame matrix, A_Λ is a matrix with columns in Λ, and A_Δ is a matrix with rows in Δ. Consider the matrix

$$A' = \begin{array}{c|c|c|c|}
 \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{Z} & \multicolumn{1}{c}{Y_0 \cup X \quad Y_1 \quad C} \\ \cline{2-4}
 D \cup X & A_\Lambda & * & A_1' \\ \cline{2-4}
 & A_\Gamma & * & A_{\Delta'} \\ \cline{2-4}
\end{array}$$

which conforms to Φ', where $A_{\Delta'}[Y_0 \cup Y_1 \cup C] = A_\Delta$ and $A_{\Delta'}[X] = 0$. Clearly $M([I, A]) \backslash (B - X) = M([I, A']) \backslash B$ by the definition of A_1'. Then since $M(A[C]) = M(A'[C])$ and $M(A[Y_1]) = M(A'[Y_1])$, we know that $M([I, A])/C \setminus Y_1 \cup (B - X)$ is isomorphic to $M([I, A'])/C \setminus B \cup Y_1$, so $\mathcal{M}(\Phi') \subseteq \mathcal{M}(\Phi')$.

We will now show that $\mathcal{M}(\Phi') \subseteq \mathcal{M}(\Phi)$. Consider the matrix

$$A' = \begin{array}{c|c|c|c|}
 \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{Z} & \multicolumn{1}{c}{Y_0 \cup X \quad Y_1 \quad C} \\ \cline{2-4}
 \begin{matrix} X \\ D \end{matrix} & A_\Lambda & * & A_1' \\ \cline{2-4}
 & A_\Gamma & * & A_{\Delta'} \\ \cline{2-4}
\end{array}$$

which conforms to Φ'. Consider the matrix

$$A = \begin{array}{c|c|c|c|}
 \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{Z} & \multicolumn{1}{c}{Y_0 \quad Y_1 \quad C} \\ \cline{2-4}
 \begin{matrix} X \\ D \end{matrix} & A_\Lambda & * & A_1 \\ \cline{2-4}
 & A_\Gamma & * & A_\Delta \\ \cline{2-4}
\end{array}$$

which conforms to Φ, where $A_\Delta = A_{\Delta'}[Y_0 \cup Y_1 \cup C]$. Then by the same reasoning as before, $M([I, A])/C \backslash (B - X) \cup Y_1$ is isomorphic to $M([I, A'])/C \backslash B \cup Y_1$, so $\mathcal{M}(\Phi) \subseteq \mathcal{M}(\Phi)$. $\square$

In light of this lemma, for the remainder of this thesis we specify templates in terms of just eight parameters, where $X = \emptyset$ is implicit. We also say that an $\mathbb{F}$-representable matroid M *conforms* to a template Φ if there is some matrix A conforming to Φ such that $M \cong M(A)/C\backslash Y_1$, because $M(A)/C\backslash Y_1 = M([I,A])/C\backslash Y_1 \cup B$. The following lemma is an important tool for simplifying templates with $X = \emptyset$. For any subgroup Λ of the additive group of $\mathbb{F}^D$ and any matrix $U \in \mathbb{F}^{D\times D}$ we define $U\Lambda = \{Uw\colon w \in \Lambda\}$.

Lemma 2.4. *Let $\Phi = (\Gamma, C, D, Y_0, Y_1, A_1, \Delta, \Lambda)$ be a frame template over a finite field $\mathbb{F}$, and let $U \in \mathbb{F}^{D\times D}$ be a nonsingular matrix. Then $\Phi' = (\Gamma, C, D, Y_0, Y_1, UA_1, \Delta, U\Lambda)$ is equivalent to Φ.*

Proof. By linearity, $U\Lambda$ is a subgroup of the additive group of $\mathbb{F}^D$. By symmetry it suffices to show that $\mathcal{M}(\Phi) \subseteq \mathcal{M}(\Phi')$. Consider the matrix

$$A = \begin{array}{c} \\ D \\ \\ \end{array}\begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{Z} & \multicolumn{1}{c}{Y_0 \quad Y_1 \quad C} \\ \hline A_\Lambda & * & A_1 \\ \hline A_\Gamma & * & A_\Delta \\ \hline \end{array},$$

which conforms to Φ. Consider the matrix

$$A' = \begin{array}{c} \\ D \\ \\ \end{array}\begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{Z} & \multicolumn{1}{c}{Y_0 \quad Y_1 \quad C} \\ \hline UA_\Lambda & * & UA_1 \\ \hline A_\Gamma & * & A_\Delta \\ \hline \end{array},$$

conforming to Φ'. Since $M(A) = M(A')$, $M(A[Y_1]) = M(A'[Y_1])$, and $M(A[C]) = M(A'[C])$, we have $M(A)/C\backslash Y_1 \cong M(A')/C\backslash Y_1$, so $\mathcal{M}(\Phi) \subseteq \mathcal{M}(\Phi')$. $\square$

For templates over prime fields we can apply Lemma 2.4 to impose structure on Λ and A_1.

Lemma 2.5. *Let $\Phi = (\Gamma, C, D, Y_0, Y_1, A_1, \Delta, \Lambda)$ be a frame template over a prime field $\mathbb{F}$. Then there exists an equivalent frame template $\Phi' = (\Gamma, C', D', Y_0, Y_1, A_1', \Delta', \Lambda')$ such that $\Lambda' = \mathbb{F}^{D'-R} \times \{0\}^R$ and $A_1'[R, C'] = 0$ for some $R \subseteq D'$.*

Proof. Since Λ is a subgroup of the additive group of $\mathbb{F}^D$ and $\mathbb{F}$ is a prime field, Λ is a subspace of $\mathbb{F}^D$. This means that there is some nonsingular matrix $U_1 \in \mathbb{F}^{D\times D}$ such that the map $x \mapsto U_1 x$ is an isomorphism from Λ to $\Lambda' = \mathbb{F}^{D-R} \times \{0\}^R$, where $|D - R| = \dim(\Lambda)$. There is a nonsingular matrix $U_2 \in \mathbb{F}^{D\times D}$ such that

(i) $U_2[R, D - R] = 0$,

(ii) $(U_2U_1A_1)[R, C]$ is in reduced-row echelon form with basis C_1, and

(iii) $(U_2U_1A_1)[D - R, C_1] = 0$.

Since $U_2[R, D - R] = 0$ we know that $U_2\Lambda' = \Lambda'$. Let $A_1'' = U_2U_1A_1$. By Lemma 2.4 we know that $\Phi'' = (\Gamma, C, D, Y_0, Y_1, A_1'', \Delta, \Lambda')$ is equivalent to Φ.

Let $P \subseteq R$ denote the nonzero rows of $A_1''[R, C]$. For each vector $v \in \Delta$ there is a unique $x_v \in \mathbb{F}^P$ such that $x_v^T A_1''[P, C_1] = -v[C_1]$. Define $\Delta'' = \{v + x_v^T A_1''[P] : v \in \Delta\}$, and define a function f from Δ to Δ'' by $f(v) = v + x_v^T A_1''[P]$. Then f is a group homomorphism and Δ'' is a subgroup of the additive group of $\mathbb{F}^{Y_0 \cup Y_1 \cup C}$. Then $\Delta' = \Delta''[Y_0 \cup Y_1 \cup (C - C_1)]$ is a subgroup of the additive group of $\mathbb{F}^{Y_0 \cup Y_1 \cup (C - C_1)}$. There is a homomorphism h from Δ to Δ' defined by $h(v) = f(v)[Y_0 \cup Y_1 \cup (C - C_1)]$.

Define $A_1' = A_1''[D - P, (Y_0 \cup Y_1 \cup C) - C_1]$, and define $\Phi' = (\Gamma, C - C_1, D - P, Y_0, Y_1, A_1', \Delta', \mathbb{F}^{D-R} \times \{0\}^{R-P})$. We will first show that $\mathcal{M}(\Phi'') \subseteq \mathcal{M}(\Phi')$. Consider the matrix

$$A = \begin{array}{r|c|c|c|c|}
 & & Z \cup Y_0 \cup Y_1 & C_1 & C - C_1 \\
\cline{2-5}
P & 0 & * & I & * \\
\cline{2-5}
R - P & 0 & * & 0 & 0 \\
\cline{2-5}
D - R & A_\Lambda & * & 0 & * \\
\cline{2-5}
 & A_\Gamma & \multicolumn{3}{c|}{A_\Delta} \\
\cline{2-5}
\end{array},$$

which conforms to Φ'', where A_Λ is a matrix with columns from $\mathbb{F}^{D-R}$, A_Γ is a Γ-frame matrix, and A_Δ is a matrix with rows in Δ. Consider the matrix

$$A' = \begin{array}{r|c|c|c|}
 & & Z \cup Y_0 \cup Y_1 & C - C_1 \\
\cline{2-4}
R - P & 0 & * & 0 \\
\cline{2-4}
D - R & A_\Lambda & * & * \\
\cline{2-4}
 & A_\Gamma & \multicolumn{2}{c|}{A_{\Delta'}} \\
\cline{2-4}
\end{array},$$

which conforms to Φ', where $A_{\Delta'}[b] = h(A_\Delta[b])$ for each row b of A_Δ. We see that $M(A)/C \backslash Y_1$ is isomorphic to $M(A')/(C - C_1) \backslash Y_1$, so $\mathcal{M}(\Phi'') \subseteq \mathcal{M}(\Phi')$.

We will now show that $\mathcal{M}(\Phi') \subseteq \mathcal{M}(\Phi'')$. Consider the matrix

$$A' = \begin{array}{r|c|c|c|}
 & & Z \cup Y_0 \cup Y_1 & C - C_1 \\
\cline{2-4}
R - P & 0 & * & 0 \\
\cline{2-4}
D - R & A_\Lambda & * & * \\
\cline{2-4}
 & A_\Gamma & \multicolumn{2}{c|}{A_{\Delta'}} \\
\cline{2-4}
\end{array},$$

which conforms to Φ', where A_Λ is a matrix with columns from $\mathbb{F}^{D-R}$, A_Γ is a Γ-frame matrix, and $A_{\Delta'}$ is a matrix with rows in Δ'. Consider the matrix

$$A = \begin{array}{r|c|c|c|c|}
 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{Z \cup Y_0 \cup Y_1} & \multicolumn{1}{c}{C_1} & \multicolumn{1}{c}{C - C_1} \\ \cline{2-5}
P & 0 & * & I & * \\ \cline{2-5}
R - P & 0 & * & 0 & 0 \\ \cline{2-5}
D - R & A_\Lambda & * & 0 & * \\ \cline{2-5}
 & A_\Gamma & \multicolumn{3}{c|}{A_\Delta} \\ \cline{2-5}
\end{array},$$

which conforms to Φ'', where $A_\Delta[b] \in h^{-1}(A_{\Delta'}[b])$ for each row b of $A_{\Delta'}$. We again see that $M(A)/C \setminus Y_1$ is isomorphic to $M(A')/(C - C_1) \setminus Y_1$, so $M \in \mathcal{M}(\Phi'')$. Thus, $\mathcal{M}(\Phi'') \subseteq \mathcal{M}(\Phi')$ and $\mathcal{M}(\Phi') \subseteq \mathcal{M}(\Phi'')$, so Φ'' and Φ' are equivalent. By construction, Φ' has the desired properties, and is equivalent to Φ. $\square$

Now we can use Lemma 2.4 to put Y_1, Y_0, and Δ in the desired form.

Lemma 2.6. *Let $\Phi = (\Gamma, C, D, Y_0, Y_1, A_1, \Delta, \Lambda)$ be a frame template over a prime field $\mathbb{F}$ such that $\Lambda = \mathbb{F}^{D-R} \times \{0\}^R$ and $A_1[R, C] = 0$ for some $R \subseteq D$. Then there exists an equivalent standardized frame template.*

Proof. There is a nonsingular matrix $U_1 \in \mathbb{F}^{D \times D}$ such that

(i) $U_1[R, D - R] = 0$,

(ii) $(U_1 A_1)[R, Y_1]$ is in reduced-row echelon form with basis columns Y_1',

(iii) $(U_1 A_1)[D - R, Y_1'] = 0$, and

(iv) $(U_1 A_1)[D - R, C]$ is in reduced-row echelon form.

Since $U_1[R, D - R] = 0$ we know that $U_1 \Lambda = \Lambda$. Let $S \subseteq R$ denote the zero rows of $(U_1 A_1)[R, Y_1]$. There is a nonsingular matrix $U_2 \in \mathbb{F}^{D \times D}$ such that

(i) $U_2 U_1[S, D - S] = 0$ and $U_2 U_1[D - S, D - S] = I$,

(ii) $(U_2 U_1 A_1)[S, Y_0]$ is in reduced-row echelon form with basis columns Y_0', and

(iii) $(U_2 U_1 A_1)[D - R, Y_0'] = 0$.

Since $U_2U_1[S, D-S] = 0$ and $U_2U_1[D-S, D-S] = I$ and $S \subseteq R$ we know that $U_2\Lambda = \Lambda$. Let $A_1' = U_2U_1A_1$, and define $\Phi' = (\Gamma, C, D, Y_0, Y_1, A_1', \Delta, \Lambda)$. By Lemma 2.4 we know that Φ' is equivalent to Φ, and by construction we know that Φ' satisfies (i)-(v) of the definition of standardized frame template.

For each vector $v \in \Delta$ there is a unique $x_v \in \mathbb{F}^R$ such that $x_v^T A_1[Y_0' \cup Y_1'] = -v[Y_0' \cup Y_1']$. Define $\Delta' = \{v + x_v^T A_1[R] : v \in \Delta\}$, and define a function f from Δ to Δ' by $f(v) = v + x_v^T A_1[R]$. Then f is a group homomorphism and Δ' is a subgroup of the additive group of $\mathbb{F}^{Y_0 \cup Y_1 \cup C}$ such that $\Delta'[Y_0' \cup Y_1'] = 0$.

Define $\Phi'' = (\Gamma, C, D, Y_0, Y_1, A_1', \Delta', \mathbb{F}^{D-R} \times \{0\}^R)$. Since Φ' and Φ'' have equal finite sets and for any matrix conforming to Φ' there is a row-equivalent matrix conforming to Φ'', we have $\mathcal{M}(\Phi') \subseteq \mathcal{M}(\Phi'')$. Similarly, $\mathcal{M}(\Phi'') \subseteq \mathcal{M}(\Phi')$, so Φ' and Φ'' are equivalent. Then Φ'' is equivalent to Φ, and by construction, Φ'' is standardized. $\square$

The following proposition follows from Lemma 2.3, Lemma 2.5, and Lemma 2.6.

Proposition 2.7. *For any frame template over a prime field $\mathbb{F}$ there is an equivalent standardized template.*

For the rest of this thesis we will assume that the templates given by the structure theorem are standardized frame templates. We now define an important property of standardized frame templates. Note that over a prime field $\mathbb{F}$, Λ is a subspace of $\mathbb{F}^D$ and Δ is a subspace of $\mathbb{F}^{Y_0 \cup Y_1 \cup C}$. The *dimension* of Φ is

$$\dim(\Lambda) - \dim(\Lambda \cap \operatorname{col}(A_1[C])) + \dim(\Delta[C] \cap \operatorname{row}(A_1[C])),$$

and is denoted $\dim(\Phi)$. This parameter loosely describes which perturbations of frame matroids conform to Φ. In the next section we will bound the density of matroids conforming and co-conforming to Φ in terms of $c(\Phi)$ and $\dim(\Phi)$, and the purpose of this parameter will become clear.

2.2 Templates and Density

We will use perturbations to bound the density of matroids conforming or co-conforming to a template. In order to bound the density of matroids co-conforming to a template we need the following lemma, which shows that perturbations behave well under duality.

Lemma 2.8. *Let $\mathbb{F}$ be a field, and let M_1 and M_2 be $\mathbb{F}$-representable matroids with $E(M_1) = E(M_2)$. Then* $\text{pert}(M_1, M_2) = \text{pert}(M_1^*, M_2^*)$.

Proof. Let $n = |E(M_1)|$. There exist matrices A_1 and A_2 with the same set of row indices such that $M_1 = M(A_1)$ and $M_2 = M(A_2)$ and $\text{rank}(A_1 - A_2) \leq t$. Let $r_1 = \text{rank}(A_1)$ and let $r_2 = \text{rank}(A_2)$ and assume $r_1 \geq r_2$. We may assume A_1 and A_2 are of the form

$$A_1 = \left[\begin{array}{c|c} I_{r_1} & D_1 \end{array}\right], A_2 = \left[\begin{array}{c|c} I_{r_2} & D_2 \\ \hline 0 & 0 \end{array}\right].$$

We know that $M_1^* = M(A_1^*)$ and $M_2^* = M(A_2^*)$ where

$$A_1^* = \left[\begin{array}{c|c} -D_1^T & I_{n-r_1} \\ \hline 0 & 0 \end{array}\right], A_2^* = \left[\begin{array}{c|c} -D_2^T & I_{n-r_2} \end{array}\right].$$

We see that $-(A_1^* - A_2^*)^T$ is equal to $A_1 - A_2$ up to swapping columns, so $\text{rank}(A_1^* - A_2^*) \leq t$. This shows that $\text{pert}(M_1^*, M_2^*) \leq \text{pert}(M_1, M_2)$, and by duality we have $\text{pert}(M_1, M_2) \leq \text{pert}(M_1^*, M_2^*)$. $\square$

We also need a lemma which relates the number of points of a matroid and a low-rank perturbation of that matroid.

Lemma 2.9. *Let $\mathbb{F}$ be a field, and let M_1 and M_2 be $\mathbb{F}$-representable matroids with $E(M_1) = E(M_2)$. If* $\text{pert}(M_1, M_2) \leq t$*, then* $\varepsilon(M_2) \leq |\mathbb{F}|^t \varepsilon(M_1) + |\mathbb{F}|^t - 1$.

Proof. There are matrices A_1 and A_2 with the same set of row indices such that $M_1 = M(A_1)$, $M_2 = M(A_1)$, and $\text{rank}(A_2 - A_1) = t$. Each column of A_1 is a nonzero multiple of some vector in $\{v_1, v_2, \ldots, v_{\varepsilon(M_1)}\}$. Since $\text{rank}(A_2 - A_1) \leq t$, there are vectors $w_1, w_2, \ldots, w_t$ such that each column of $A_2 - A_1$ is in the span of $\{w_1, w_2, \ldots, w_t\}$. Since $A_2 = A_1 + (A_2 - A_1)$, each column of A_2 is of the form $a_0 v_i + a_1 w_1 + a_2 w_2 + \cdots + a_t w_t$ for constants $a_i \in \mathbb{F}$. For each choice of v_i, there are at most $|\mathbb{F}|^t$ parallel classes of columns of A_2 for which a_0 is nonzero. There are at most $|\mathbb{F}|^t - 1$ for which $a_0 = 0$. Thus, there are at most $\varepsilon(M_1)|\mathbb{F}|^t + |\mathbb{F}|^t - 1$ parallel classes of columns of A_2, so $\varepsilon(M_2) \leq \varepsilon(M_1)|\mathbb{F}|^t + |\mathbb{F}|^t - 1$. $\square$

We can now find an upper bound for the density of a matroid co-conforming to a template.

Lemma 2.10. *Let Φ be a frame template over a finite field $\mathbb{F}$. If $M^* \in \mathcal{M}(\Phi)$ then $\varepsilon(M) \leq |\mathbb{F}|^{c(\Phi)+2}(r(M) + c(\Phi)) + |\mathbb{F}|^{c(\Phi)}$.*

Proof. Let $c = c(\Phi)$, and let $n = |M_1| = |M_2|$. If M conforms to Φ, then there is a frame matroid M_1 such that $\mathrm{pert}(M, M_1) \le c$. Then $\mathrm{pert}(M^*, M_1^*) \le c$ by Lemma 2.8, so $\varepsilon(M^*) \le |\mathbb{F}|^c \varepsilon(M_1^*) + |\mathbb{F}|^c - 1$, by Lemma 2.9. We will show that $\varepsilon(M_1^*) \le 3r(M_1^*)$. It suffices to show that $|M_1^*| \le 3r(M_1^*)$ if M_1^* is simple. Since M_1 is a frame matroid, it has a representation with at most $2n$ nonzero entries. If M_1^* is simple, then each row of the representation has at least three nonzero entries, or else there is a loop or a parallel pair in M_1^*. This means that $3r(M_1) \le 2n$, and since $r(M_1) + r(M_1^*) = n$ we find that $|M_1^*| \le 3r(M_1^*)$. Then we have $\varepsilon(M^*) \le |\mathbb{F}|^c 3r(M_1^*) + |\mathbb{F}|^c - 1 \le |\mathbb{F}|^{c+2} r(M_1^*) + |\mathbb{F}|^c - 1$. Since $r(M_1^*) \le r(M^*) + c$, this gives $\varepsilon(M^*) \le |\mathbb{F}|^{c+2}(r(M^*) + c) + |\mathbb{F}|^c$. □

Finding an upper bound on the density of conforming matroids is somewhat harder.

Lemma 2.11. *Let $\Phi = (\Gamma, C, D, Y_0, Y_1, A_1, \Delta, \Lambda)$ be a standardized frame template over a prime field $\mathbb{F} = \mathrm{GF}(p)$, and let $t = \dim(\Phi)$. If $M \in \mathcal{M}(\Phi)$, then*

$$\varepsilon(M) \le |\Gamma(\Phi)| p^t \binom{r(M) + c(\Phi) + 1}{2} + c(\Phi) r(M) + c(\Phi) + p^t.$$

Proof. Let $s = \dim(\Delta[C] \cap \mathrm{row}(A_1[C]))$. Consider a matrix A conforming to Φ of the form

$$A = \begin{array}{c|c|c|c|c|c|c|}
 \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{Z} & \multicolumn{1}{c}{Y_0} & \multicolumn{1}{c}{Y_1} & \multicolumn{1}{c}{C_1} & \multicolumn{1}{c}{C_2} \\ \cline{2-7}
R & 0 & * & * & * & 0 & 0 \\ \cline{2-7}
F & \multirow{2}{*}{A_Λ} & * & * & * & I & * \\ \cline{3-7}
D' & & * & * & * & 0 & 0 \\ \cline{2-7}
 & A_\Gamma & * & \multicolumn{4}{c|}{A_\Delta} \\ \cline{2-7}
\end{array}.$$

Then $M(A)/C_1$ has a representation of the form

$$\begin{array}{c|c|c|c|c|c|}
 \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{Z} & \multicolumn{1}{c}{Y_0} & \multicolumn{1}{c}{Y_1} & \multicolumn{1}{c}{C_2} \\ \cline{2-6}
R & 0 & * & * & * & 0 \\ \cline{2-6}
D' & A_\Lambda[D'] & * & * & * & 0 \\ \cline{2-6}
 & A_\Gamma + B & * & \multicolumn{3}{c|}{P} \\ \cline{2-6}
\end{array},$$

where B and P are matrices such that $\mathrm{rank}(B) \le s$ and $\mathrm{rank}(P[C_2]) \le |C_2|$. Let U be a nonsingular matrix such that $UP[C_2]$ is in reduced-row echelon form. Let V' denote the set of nonzero rows of $UP[C_2]$, and let V denote the set of zero rows. Then $M(A)/C\backslash Y_1$ has a representation of the form

$$\begin{array}{c|c|c|c|}
 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{Z} & \multicolumn{1}{c}{Y_0} \\ \cline{2-4}
R & 0 & * & * \\ \cline{2-4}
D' & A_\Lambda[D'] & * & * \\ \cline{2-4}
V & UA_\Gamma[V] + UB[V] & * & * \\ \cline{2-4}
\end{array}.$$

We know $\text{rank}(UB[V]) \leq s$ since $\text{rank}(B) \leq s$. Also, $UA_\Gamma[V]$ has at most as many distinct columns as A_Γ. Let $M_1 = M(A_\Gamma)$, $M_2 = M(UA_\Gamma[V])$, $M_3 = M(UA_\Gamma[V] + UB[V])$, and $M_4 = M\backslash Z\backslash Y_0$. Since A_Γ is a Γ-frame matrix we know $\varepsilon(M_1) \leq |\Gamma|\binom{r(M_1)+1}{2}$. Note that M_2 is a $(\leq |C_2|)$-perturbation of M_1, since $\text{rank}(P[C_2]) \leq |C_2|$. This means that $r(M_1) \leq r(M_2) + |C_2|$, so we have

$$\varepsilon(M_2) \leq \varepsilon(M_1) \leq |\Gamma|\binom{r(M_1)+1}{2} \leq |\Gamma|\binom{r(M_2)+|C_2|+1}{2}.$$

Then since $\text{pert}(M_2, M_3) \leq s$ since $\text{rank}(UB[V]) \leq s$ by Lemma 2.9, we have

$$\varepsilon(M_3) \leq p^s\varepsilon(M_2) + p^s - 1 \leq p^s|\Gamma|\binom{r(M_3)+s+|C_2|+1}{2} + p^s - 1.$$

Then since $|D'| = t-s$ we have $\text{pert}(M_3, M_4) \leq t-s$ and $r(M_3) \leq r(M_4)$, we have

$$\varepsilon(M_4) \leq p^t|\Gamma|\binom{r(M_4)+s+|C_2|+1}{2} + p^t - 1.$$

Then by definition of M_4 we have

$$\begin{aligned}
\varepsilon(M) &\leq \varepsilon(M_4) + |Z| + |Y_0| \\
&\leq p^t|\Gamma|\binom{r(M)+s+|C_2|+1}{2} + p^t - 1 + |Y_1|r(M) + |Y_0| \\
&\leq |\Gamma|p^t\binom{r(M)+c(\Phi)+1}{2} + c(\Phi)r(M) + c(\Phi) + p^t,
\end{aligned}$$

as required. □

We will use these lemmas to show that matroids conforming to a template of dimension less than t or co-conforming to a template are not dense enough to affect the growth rate function of $\mathcal{M}_t$. We will use the next lemma to show that no template for $\mathcal{M}_t$ can have dimension greater than t.

Lemma 2.12. *Let $t \geq 0$ be an integer, and let N be a t-graphic matroid. Let $\Phi = (\Gamma, C, D, Y_0, Y_1, A_1, \Delta, \Lambda)$ be a standardized binary frame template. If $\dim(\Phi) \geq t$, then there is some matroid M conforming to Φ such that N is a minor of M.*

Proof. Since any t-graphic matroid is also m-graphic for any $m > t$, we may assume that $\dim(\Phi) = t$. Let $s = \dim(\Delta[C] \cap \mathrm{row}(A_1[C]))$, and $s' = \dim(\Lambda) - \dim(\Lambda \cap \mathrm{col}(A_1[C])$, so $s + s' = t$. Let

$$K = \begin{bmatrix} K_1 \\ \hline K_2 \\ \hline K_3 \end{bmatrix}$$

be a representation of N with columns indexed by L such that K_1 has s' rows, K_2 has s rows, and $M(K_3)$ is graphic.

Let $F \subseteq D$ denote the set of rows for which $A_1[C]$ has a leading 1. Since $\dim(\Delta[C] \cap \mathrm{row}(A_1[C])) = s$, there are vectors $v_1, v_2, \ldots, v_s$ in Δ such that $\{v_i[C] : i \in [s]\}$ is a basis of $\Delta[C] \cap \mathrm{row}(A_1[C])$. For each v_i there is some $x_i \in \mathrm{GF}(2)^F$ such that $x_i^T A_1[C] = v_i[C]$. We see that $\{x_i : i \in [s]\}$ is linearly independent, since $\{v_i[C] : i \in [s]\}$ is linearly independent. Let $X \in \mathrm{GF}(2)^{F \times [s]}$ denote the matrix such that $X[i] = x_i$. Since $\{x_i : i \in [s]\}$ is linearly independent X^T is left-invertible, and thus there is some $B \in \mathrm{GF}(2)^{F \times L}$ such that $X^T B = K_2$. Consider the following matrix

$$A = \begin{array}{c|c|c|c|c|}
 & L & Y_0 \cup Y_1 & C_1 & C_2 \\ \hline
R & 0 & * & 0 & 0 \\ \hline
F & B & * & I & * \\ \hline
D' & K_1 & * & 0 & 0 \\ \hline
 & 0 & \multicolumn{3}{c|}{v_1} \\
 & 0 & \multicolumn{3}{c|}{v_2} \\
{[s]} & \vdots & \multicolumn{3}{c|}{\vdots} \\
 & 0 & \multicolumn{3}{c|}{v_s} \\ \hline
 & & \multicolumn{3}{c|}{0} \\
 & & \multicolumn{3}{c|}{0} \\
 & & \multicolumn{3}{c|}{0} \\
 & K_3 & \multicolumn{3}{c|}{0} \\
 & & \multicolumn{3}{c|}{\vdots} \\
 & & \multicolumn{3}{c|}{0} \\ \hline
\end{array},$$

which conforms to Φ. Then there is a representation A' of $M(A)/C\backslash Y_0 \cup Y_1$ of the form

$$\begin{array}{c|c|}
 & L \\ \cline{2-2}
D' & K_1 \\ \cline{2-2}
[s] & K_2 \\ \cline{2-2}
 & K_3 \\ \cline{2-2}
\end{array},$$

and clearly $M(A') = N$. □

2.3 The Main Lemma

All lemmas in the previous section deal with general properties of standardized templates, but we need one lemma about the density of templates of dimension t which do not have some fixed matroid as a minor of a conforming matroid.

Lemma 2.13. *Let $t \geq 0$, let N be a t-pinched-graphic matroid, and let $\Phi = (\Gamma, C, D, Y_0, Y_1, A_1, \Delta, \Lambda)$ be a standardized binary frame template of dimension t. Then either*

- *N is a minor of a matroid conforming to Φ, or*
- *every vertically $c(\Phi)$-connected matroid M of rank greater than $c(\Phi)$ conforming to Φ satisfies $\varepsilon(M) \leq f_t(r(M))$.*

Proof. Since Φ is standardized, every matrix respecting Φ is of the form

		Z	Y_0'	$Y_0 - Y_0'$	Y_1'	$Y_1 - Y_1'$	C_1	C_2
$R - S$	0	0	$*$	$*$	I	$*$	0	0
S	0	0	I	$*$	0	0	0	0
F	$\Lambda[D-R]$-columns	0	0	$*$	0	$*$	I	$*$
D'		0	0	$*$	0	$*$	0	0
	frame	unit	Δ-rows					

Recall that each '$*$' represents a fixed submatrix of A_1. Let $C_1 \subseteq C$ denote the set of columns for which $A_1[C]$ has a leading 1, and let $C_2 = C - C_1$. Let $Y_1' \subseteq Y_1$ denote the set of columns for which $A_1[R, Y_1]$ has a leading 1, and let $S \subseteq R$ denote the set of zero rows of $A_1[R, Y_1]$. Let $Y_0' \subseteq Y_0$ denote the set of columns for which $A_1[S, Y_0]$ has a leading 1. Let $F \subseteq D - R$ denote the set of rows for which $A_1[C]$ has a leading 1, and let $D' = D - (R \cup F)$.

Assume that N is not a minor of any matroid conforming to Φ. Let $L = E(N)$. Let $s = \dim(\Delta[C] \cap \mathrm{row}(A_1[C]))$ and $s' = \dim(\Lambda) - \dim(\Lambda \cap \mathrm{col}(A_1[C]))$, and note that $s + s' = t$. Since N is t-pinched-graphic, it has a representation

$$K = \begin{bmatrix} K_1 \\ \hline K_2 \\ \hline A'_N \end{bmatrix}$$

such that K_1 has s' rows, K_2 has s rows, and $M(A'_N)$ is pinched-graphic. Since $M(A'_N)$ is pinched-graphic, there is a matrix

$$A''_N = \left[\begin{array}{c|c} & 1 \\ & 1 \\ & 1 \\ A_N & 0 \\ & \vdots \\ & 0 \end{array} \right],$$

such that A_N is graphic and $M(A''_N)/e = M(A'_N)$, where e is the label of the support-3 column. Fix some ordering of the rows of A_N. Let w denote the top row of A_N in this ordering, and let $\tilde{A}_N$ denote the matrix obtained from A_N by removing the top row.

Since $\dim(\Delta[C] \cap \mathrm{row}(A_1[C])) = s$, there are vectors $v_1, v_2, \ldots, v_s$ in Δ such that each v_i satisfies $v_i[C] \in \mathrm{row}(A_1[C])$, and $\{v_i[C] : i \in [s]\}$ is a basis of $\Delta[C] \cap \mathrm{row}(A_1[C])$. For each v_i there is a unique $x_i \in \mathrm{GF}(2)^F$ such that $x_i^T A_1[C] = v_i[C]$. Since $\{v_i[C] : i \in [s]\}$ is a linearly independent set, so is $\{x_i : i \in [s]\}$. Let $X \in \mathrm{GF}(2)^{F \times [s]}$ denote the matrix such that $X[i] = x_i$. Then X^T is left-invertible because $\{x_i : i \in [s]\}$ is linearly independent, so there is some $B \in \mathrm{GF}(2)^{F \times L}$ such that $X^T B = K_2$.

In the first part of this proof we will show that $\Delta = \{0\}$. In the second part we will show that $A_1[R, Y_1]$ consists of unit columns and $A_1[R, Y_0]$ is graphic.

Claim 2.13.1. *$s = 0$.*

Proof. Assume for a contradiction that $s \geq 1$. Consider the matrix

$$A = \begin{array}{c|c|c|c|c|}
 & \multicolumn{1}{c}{L} & \multicolumn{1}{c}{Y_0 \cup Y_1} & \multicolumn{1}{c}{C_1} & \multicolumn{1}{c}{C_2} \\ \cline{2-5}
R & 0 & * & 0 & 0 \\ \cline{2-5}
F & B & * & I & * \\ \cline{2-5}
D' & K_1 & * & 0 & 0 \\ \cline{2-5}
 & 0 & \multicolumn{3}{c|}{v_1} \\
 & 0 & \multicolumn{3}{c|}{v_2} \\
 & \vdots & \multicolumn{3}{c|}{\vdots} \\
 & 0 & \multicolumn{3}{c|}{v_{s-1}} \\
 & w & \multicolumn{3}{c|}{v_s} \\ \cline{2-5}
 & & \multicolumn{3}{c|}{v_s} \\
 & & \multicolumn{3}{c|}{v_s} \\
 & \tilde{A}_N & \multicolumn{3}{c|}{0} \\
 & & \multicolumn{3}{c|}{0} \\
 & & \multicolumn{3}{c|}{\vdots} \\
 & & \multicolumn{3}{c|}{0} \\ \cline{2-5}
\end{array},$$

which conforms to Φ. This matrix is row-equivalent to

$$A = \begin{array}{c|c|c|c|c|}
 & \multicolumn{1}{c}{L} & \multicolumn{1}{c}{Y_0 \cup Y_1} & \multicolumn{1}{c}{C_1} & \multicolumn{1}{c}{C_2} \\ \cline{2-5}
R & 0 & * & 0 & 0 \\ \cline{2-5}
F & B & * & I & * \\ \cline{2-5}
D' & K_1 & * & 0 & 0 \\ \cline{2-5}
 & 0 & \multicolumn{3}{c|}{v_1} \\
 & 0 & \multicolumn{3}{c|}{v_2} \\
 & \vdots & \multicolumn{3}{c|}{\vdots} \\
 & 0 & \multicolumn{3}{c|}{v_{s-1}} \\
 & w & \multicolumn{3}{c|}{v_s} \\ \cline{2-5}
 & & \multicolumn{3}{c|}{0} \\
 & & \multicolumn{3}{c|}{0} \\
 & & \multicolumn{3}{c|}{0} \\
 & A'_N & \multicolumn{3}{c|}{0} \\
 & & \multicolumn{3}{c|}{\vdots} \\
 & & \multicolumn{3}{c|}{0} \\ \cline{2-5}
\end{array}.$$

Then $M(A)/C\backslash Y_0 \cup Y_1$ has a representation of the form

$$\begin{array}{c|c|}
 & L \\ \cline{2-2}
D' & K_1 \\ \cline{2-2}
[s] & K_2 \\ \cline{2-2}
 & A'_N \\ \cline{2-2}
\end{array},$$

which is K, a representation of N. This contradicts that no matroid conforming to Φ has N as a minor. $\square$

This means that $s' = t = |D'|$, and

$$K = \left[\begin{array}{c} K_1 \\ \hline A'_N \end{array}\right].$$

Claim 2.13.2. $\Delta[C] = \{0\}$.

Proof. Assume for a contradiction that there is some $v \in \Delta$ such that $v[C] \neq 0$. By the previous claim, $v[C] \notin \text{row}(A_1[C])$. Let $x_v \in \text{GF}(2)^F$ be the unique vector such that $v[C_1] + x_v^T A_1[F, C_1] = 0$. Let $v' = v + x_v^T A_1[F]$, and note that $v'[C_2] \neq 0$. Consider the matrix

$$A = \begin{array}{c|c|c|c|c|}
 & L & Y_0 \cup Y_1 & C_1 & C_2 \\ \hline
R & 0 & * & 0 & 0 \\ \hline
F & 0 & * & I & * \\ \hline
D' & K_1 & * & 0 & 0 \\ \hline
 & & \multicolumn{3}{c|}{v} \\
 & & \multicolumn{3}{c|}{v} \\
 & & \multicolumn{3}{c|}{v} \\
 & A_N & \multicolumn{3}{c|}{0} \\
 & & \multicolumn{3}{c|}{\vdots} \\
 & & \multicolumn{3}{c|}{0} \\ \hline
\end{array}.$$

which conforms to Φ. Then $M(A)/C_1\backslash Y_0 \cup Y_1$ has a representation of the form

$$\begin{array}{r|c|c|}
\multicolumn{1}{r}{} & \multicolumn{1}{c}{L} & \multicolumn{1}{c}{C_2} \\
\cline{2-3}
D' & K_1 & 0 \\
\cline{2-3}
 & & v' \\
 & & v' \\
 & & v' \\
 & A_N & 0 \\
 & & \vdots \\
 & & 0 \\
\cline{2-3}
\end{array},$$

and we see that $M(A)/C\backslash Y_0 \cup Y_1$ is isomorphic to N, a contradiction. $\square$

We can now show that Δ is trivial.

Claim 2.13.3. $\Delta = \{0\}$.

Proof. Assume that there is some $v \in \Delta - \{0\}$. Then by condition (iv) in the definition of standard form there is some $y \in (Y_0 - Y_0') \cup (Y_1 - Y_1')$ such that $v[y] \neq 0$. Let $U = \{d \in D' \colon y[d] \neq 0\}$, and define $Q \in \mathrm{GF}(2)^{D' \times L}$ such that $Q[D' - U] = K_1[D' - U]$ and $Q[u] = K_1[u] + w$ for each $u \in U$, where w is the top row of A_N. Consider the matrix

$$A = \begin{array}{r|c|c|c|c|c|c|c|c|}
\multicolumn{1}{r}{} & \multicolumn{1}{c}{L} & \multicolumn{1}{c}{Y_0'} & \multicolumn{1}{c}{Y_0 - Y_0'} & \multicolumn{1}{c}{Y_1'} & \multicolumn{1}{c}{Y_1 - Y_1'} & \multicolumn{1}{c}{C_1} & \multicolumn{1}{c}{C_2} & \multicolumn{1}{c}{Z} \\
\cline{2-9}
 & 0 & * & * & I & * & 0 & 0 & \\
\cline{2-8}
R & 0 & I & * & 0 & * & 0 & 0 & A_1[Y_1] \\
\cline{2-8}
F & 0 & 0 & * & 0 & * & I & * & \\
\cline{2-8}
D' & Q & 0 & * & 0 & * & 0 & 0 & \\
\cline{2-9}
 & & \multicolumn{6}{c|}{v} & v[Y_1] \\
 & & \multicolumn{6}{c|}{v} & v[Y_1] \\
 & & \multicolumn{6}{c|}{v} & v[Y_1] \\
 & A_N & \multicolumn{6}{c|}{0} & 0 \\
 & & \multicolumn{6}{c|}{\vdots} & \vdots \\
 & & \multicolumn{6}{c|}{0} & 0 \\
\cline{2-9}
\end{array}$$

which conforms to Φ. Let $Z' \subseteq Z$ denote the set of columns corresponding to Y_1'. There is a representation of $M(A)/C \cup Y_0' \cup Z' \backslash Y_1 \cup (((Y_0 - Y_0') \cup (Z -$

$Z')) - y)$ of the form

$$
\begin{array}{c|c|c|}
 & L & y \\
\cline{2-3}
D' & Q & A_1[D', y] \\
\cline{2-3}
 & & 1 \\
 & & 1 \\
 & & 1 \\
 & A_N & 0 \\
 & & \vdots \\
 & & 0 \\
\cline{2-3}
\end{array},
$$

By our choice of Q, contracting y gives a matroid isomorphic to N, a contradiction. $\square$

The next three claims will impose structure on A_1.

Claim 2.13.4. *Each column of $A_1[R, Y_1]$ is a unit column.*

Proof. Assume for a contradiction that there is some column y_1 of $A_1[R, Y_1 - Y_1']$ with two or more nonzero entries. Then there are columns y_2 and y_3 in Y_1' with nonzero entries in two of the rows for which $A_1[R, y_1]$ is nonzero. Let $P = \{d \in D' \colon z_1[d] \neq 0\}$, and define $Q[D' - P] = K_1[D' - P]$ and $Q[p] = K_1[p] + w$ for each $p \in P$, where w is top row of A_N. Consider the matrix

$$
A = \begin{array}{c|c|c|ccc|c|}
 & L & Z - \{z_1, z_2, z_3\} & & \{z_1, z_2, z_3\} & & Y_0 \cup Y_1 \cup C \\
\cline{2-7}
R & 0 & & & & & \\
\cline{2-2}
F & 0 & A_1[Y_1' - \{y_2, y_3\}] & & A_1[\{y_1, y_2, y_3\}] & & A_1 \\
\cline{2-2}
D' & Q & & & & & \\
\cline{2-7}
 & & 0 & 0 & 0 & 1 & 0 \\
 & & 0 & 0 & 1 & 0 & 0 \\
 & & 0 & 1 & 0 & 0 & 0 \\
 & A_N & 0 & 0 & 0 & 0 & 0 \\
 & & \vdots & & \vdots & & \vdots \\
 & & 0 & 0 & 0 & 0 & 0 \\
\cline{2-7}
\end{array}
$$

which conforms to Φ. There exits a representation of the matroid

$M(A)/C \cup (Z - \{z_1\}) \backslash Y_0 \cup Y_1$ of the form

$$
\begin{array}{c|c|c|}
 & L & z_1 \\
\cline{2-3}
D' & Q & A_1[D', z_1] \\
\cline{2-3}
 & & 1 \\
 & & 1 \\
 & & 1 \\
 & A_N & 0 \\
 & & \vdots \\
 & & 0 \\
\cline{2-3}
\end{array}.
$$

Due to our choice of Q this matroid has N as a minor. This is a contradiction, so each column of $A_1[R, Y_1 - Y_1']$ is a unit column. Since each column of $A_1[R, Y_1']$ is a unit column by definition of Y_1', the claim holds. □

The next claim shows that either no high rank matroid conforming to Φ is highly connected, or $A_1[R, Y_1]$ spans $A_1[R, Y_0]$.

Claim 2.13.5. *If $A_1[R, Y_1]$ does not span $A_1[R, Y_0]$, then no matroid of rank greater than $c(\Phi)$ conforming to Φ is vertically $c(\Phi)$-connected.*

Proof. Any matrix A conforming to Φ is of the form

$$
\begin{array}{c|c|c|c|c|c|c|c|}
 & & Z & Y_0 & Y_1' & Y_1 - Y_1' & C_1 & C_2 \\
\cline{2-8}
\multirow{2}{*}{R} & 0 & * & * & I & * & 0 & 0 \\
\cline{2-8}
 & 0 & 0 & * & 0 & 0 & 0 & 0 \\
\cline{2-8}
F & \multirow{2}{*}{A_Λ} & * & * & 0 & * & I & * \\
\cline{3-8}
D' & & * & * & 0 & * & 0 & 0 \\
\cline{2-8}
 & A_\Gamma & * & \multicolumn{5}{c|}{0} \\
\cline{2-8}
\end{array},
$$

by Claim 2.13.3. If $A_1[R, Y_1]$ does not span $A_1[R, Y_0]$, then Y_0 is a separation in $M(A)/C \backslash Y_1$ whenever $r(M(A)) > |D|$. Since $|Y_0| \leq c(\Phi)$ and $|D| \leq c(\Phi)$, no matroid of rank greater than $c(\Phi)$ conforming to Φ is vertically $c(\Phi)$-connected. □

Claim 2.13.6. *If $A_1[R, Y_1]$ spans $A_1[R, Y_0]$, then each column of $A_1[R, Y_0]$ has at most two nonzero entries.*

Proof. Assume for a contradiction that there is a column $y \in Y_0$ such that $A_1[R, y]$ has three or more nonzero entries. Let $P = \{d \in D' : y[d] \neq 0\}$, and define $Q[D' - P] = K_1[D' - P]$ and $Q[p] = K_1[p] + w$ for each $p \in P$, where w

is the top row of A_N. There are columns $y_1, y_2, y_3 \in Y_1'$ such that they have a leading 1 in three rows for which $A_1[R, y]$ is nonzero. Consider the matrix

$$A = \begin{array}{c|c|c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{L} & \multicolumn{1}{c}{Z-\{z_1,z_2,z_3\}} & \multicolumn{1}{c}{\{z_1,z_2,z_3\}} & \multicolumn{1}{c}{Y_0\cup Y_1\cup C} \\ \cline{2-5}
R & 0 & & & \\ \cline{2-2}
F & 0 & A_1[Y_1'-\{y_1,y_2,y_3\}] & A_1[\{y_1,y_2,y_3\}] & A_1 \\ \cline{2-2}
D' & Q & & & \\ \cline{2-5}
 & A_N & 0 & \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ & \vdots & \\ 0 & 0 & 0 \end{array} & 0 \\ \cline{2-5}
\end{array}$$

which conforms to Φ. There is a representation of $M(A)/C \cup (Z - \{z_1, z_2, z_3\})\backslash Y_1 \cup (Y_0 - \{y\})$ of the form

$$\begin{array}{c|c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{L} & \multicolumn{1}{c}{\{z_1,z_2,z_3\}} & \multicolumn{1}{c}{y} \\ \cline{2-4}
 & 0 & \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} & \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \\ \cline{2-4}
D' & Q & 0 & A_1[D', y] \\ \cline{2-4}
 & A_N & \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ & \vdots & \\ 0 & 0 & 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{array} \\ \cline{2-4}
\end{array}.$$

Then there is a representation of $M(A)/C \cup Z\backslash Y_1 \cup (Y_0 - \{y\})$ of the form

$$\begin{array}{c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{L} & \multicolumn{1}{c}{y} \\ \cline{2-3}
D' & Q & A_1[D', y] \\ \cline{2-3}
 & A_N & \begin{array}{c} 1 \\ 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{array} \\ \cline{2-3}
\end{array}.$$

Due to our choice of Q this matroid N as a minor. This is a contradiction, so each column of $A_1[R, Y_0]$ has at most two nonzero entries. □

Now we know that if $A_1[R, Y_1]$ spans $A_1[R, Y_0]$, then any matrix conforming to Φ is of the form

<table>
<tr><th></th><th></th><th>Z</th><th>Y_0</th><th>Y_1</th><th colspan="2">C</th></tr>
<tr><td>R</td><td>0</td><td>unit</td><td>frame</td><td>unit</td><td>0</td><td>0</td></tr>
<tr><td rowspan="2">$D - R$</td><td rowspan="2">$\Lambda[D - R]$-columns</td><td>$*$</td><td>$*$</td><td>$*$</td><td>I</td><td>$*$</td></tr>
<tr><td>$*$</td><td>$*$</td><td>$*$</td><td>0</td><td>0</td></tr>
<tr><td></td><td>frame</td><td>unit</td><td colspan="4">0</td></tr>
</table>

.

This means that any vertically $c(\Phi)$-connected matroid of rank greater than $c(\Phi)$ conforming to Φ has a representation of the form

<table>
<tr><th></th><th></th><th>Z</th><th>Y_0</th></tr>
<tr><td>D'</td><td>$*$</td><td>$*$</td><td>$*$</td></tr>
<tr><td>R</td><td>0</td><td>unit</td><td>frame</td></tr>
<tr><td></td><td>frame</td><td>unit</td><td>0</td></tr>
</table>

,

and thus has at most $f_t(n)$ distinct elements. □

3 The Main Proof

We now combine the technical material from the previous chapter to prove Theorem 1.12, which we restate here as a combination of Theorems 3.2 and 3.3. The following theorem from [18] allows us to restrict our attention to high rank, highly vertically connected matroids in $\mathcal{M}_t$.

Theorem 3.1 (Geelen, Nelson 2015)**.** *Let $\mathcal{M}$ be a quadratically dense minor-closed class of $\mathbb{F}$-representable matroids, and let f be a quadratic polynomial with positive leading coefficient. For any positive integers k and m, if there are infinitely many positive integers n such that $h_{\mathcal{M}}(n) > f(n)$, then there is some vertically k-connected matroid $M \in \mathcal{M}$ such that $r(M) \geq m$ and $h_{\mathcal{M}}(r(M)) > f(r(M))$.*

This theorem says that if f is a candidate for an upper bound on the growth rate function of $\mathcal{M}$ but there are counterexamples for infinitely many positive integers, then there are counterexamples which are arbitrarily highly vertically connected and have arbitrarily high rank. We can now prove the main result. Recall that $f_t(n) = 2^t\binom{n-t+1}{2}+2^t-1$, and we write $h_{\mathcal{M}}(n) \approx g(n)$ for a class of matroids $\mathcal{M}$ if $h_{\mathcal{M}}(n) = g(n)$ for n sufficiently large.

Theorem 3.2. *For any integer $t \geq 0$ and any binary matroid N which is t-pinched-graphic and not t-graphic, $h_{\mathrm{Ex}(N)}(n) \approx f_t(n)$.*

Proof. Since $\mathrm{Ex}(N)$ is minor-closed, by Theorem 2.2 and Proposition 2.7 there are finite sets of standardized templates T and T^* and a positive integer k such that any vertically k-connected matroid of rank at least $2k$ in $\mathrm{Ex}(N)$ either conforms to some template in T or co-conforms to some template in T^*. Let $c = \max\{c(\Phi) : \Phi \in \mathsf{T}\}$, and let $k_0 = \max\{k, c\}$.

By only considering matroids of high rank, we may restrict our attention to matroids conforming to templates of dimension t in T, as follows. Choose m_1 such that $f_t(n) > 2^{t-1}\binom{n+c+1}{2} + cn + c + p^t$ for any $n \geq m_1$. Then by Lemma 2.11 any matroid of rank at least m_1 conforming to a template in T of dimension less than t satisfies $\varepsilon(M) < f_t(r(M))$. Choose m_2 such that $f_t(n) > 2^{c+2}(n+c)+2^c$ for any $n \geq m_2$. Then by Lemma 2.10 any matroid of rank at least m_2 co-conforming to a template in T^* satisfies $\varepsilon(M) < f_t(r(M))$. Let $m_0 = \max\{2k, m_1, m_2, c\}$.

Assume for a contradiction that $h_{\mathrm{Ex}(N)}(n) > f_t(n)$ for infinitely many positive integers n. Then since $\mathrm{Ex}(N)$ is quadratically dense because N is not graphic, Theorem 3.1 tells us that there is some vertically k_0-connected matroid M of rank at least m_0 such that $\varepsilon(M) > f_t(n)$. Since $k_0 \geq k$ and $m_0 \geq 2k$, either M conforms to some template in T or M^* conforms to some template in T^*. But since $r(M) \geq m_2$, M^* cannot conform to a template in T^* or else $\varepsilon(M) < f_t(r(M))$. And since $r(M) \geq m_1$, M cannot conform to a template in T of dimension less than t. So by Lemma 2.12, M conforms to a template $\Phi \in \mathsf{T}$ of dimension t. But M does not have an N-minor since $\mathcal{M}(\Phi) \subseteq \mathrm{Ex}(N)$ by the Structure Theorem. Then since $k_0 \geq c(\Phi)$ and $m_0 \geq c(\Phi)$, by Lemma 2.13 we have $\varepsilon(M) \leq f_t(n)$, which contradicts $\varepsilon(M) > f_t(n)$. So we have $h_{\mathrm{Ex}(N)}(n) \leq f_t(n)$ for sufficiently large n, and since N is not t-graphic we have $h_{\mathrm{Ex}(N)}(n) \geq f_t(n)$, giving us the desired result. □

The converse of Theorem 3.2 is also true.

Theorem 3.3. *For any integer $t \geq 0$ and any binary matroid N, if $h_{\mathrm{Ex}(N)}(n) \approx f_t(n)$, then N is t-pinched-graphic and not t-graphic.*

Proof. First note that if N is graphic, then $h_{\mathrm{Ex}(N)}(n) < f_t(n)$ for large n because $h_{\mathrm{Ex}(N)}(n)$ is linear, so we will assume that N is not graphic. If N is not t-pinched-graphic, then $h_{\mathrm{Ex}(N)}(n) > f_t(n)$ by Corollary 1.11. So it suffices to show that if N is t-graphic but not graphic, then $h_{\mathrm{Ex}(N)}(n) < f_t(n)$ for n sufficiently large.

By Theorem 2.2 and Proposition 2.7 there are finite sets of standardized templates T and T^* and a positive integer k such that any vertically k-connected matroid of rank at least $2k$ in $\mathrm{Ex}(N)$ either conforms to some template in T or co-conforms to some template in T^*. Let $c = \max\{c(\Phi) : \Phi \in \mathsf{T}\}$, and let $k_0 = \max\{k, c\}$.

Choose m_1 such that $f_t(n) > 2^{t-1}\binom{n+c+1}{2} + cn + c + p^t$ for all $n \geq m_1$. Then by Lemma 2.11 any matroid of rank at least m_1 conforming to a template in T of dimension less than t satisfies $\varepsilon(M) < f_t(r(M))$. Choose m_2 such that $f_t(n) > 2^{c+2}(n+c) + 2^c$ for any $n \geq m_2$. Then by Lemma 2.10 any matroid of rank at least m_2 co-conforming to a template in T^* satisfies $\varepsilon(M) < f_t(r(M))$. Let $m_0 = \max\{2k, m_1, m_2\}$.

Assume for a contradiction that $h_{\mathrm{Ex}(N)}(n) > f_t(n) - 1$ for infinitely many positive integers n. Then since $\mathrm{Ex}(N)$ is quadratically dense, Theorem 3.1 tells us that there is some vertically k_0-connected matroid M of rank at least m_0 such that $\varepsilon(M) > f_t(r(M)) - 1$. Since $k_0 \geq k$ and $m_0 \geq 2k$, either M conforms to some template in T or M^* conforms to some template in

T^*. But since $r(M) \geq m_2$, M^* cannot conform to a template in T^* or else $\varepsilon(M) < f_t(r(M))$. And since $r(M) \geq m_1$, M cannot conform to a template in T of dimension less than t. But by Lemma 2.12, any template in T has dimension less than t since N is t-graphic. Thus, M does not conform to any template in T or co-conform to any template in T^*, which is a contradiction. Therefore $h_{\mathrm{Ex}(N)}(n) \leq f_t(n) - 1 < f_t(n)$ for n sufficiently large. □

Our two main corollaries follow from Theorem 3.2.

Corollary 3.4. *Let $t \geq 0$ be an integer, and let $\mathcal{M}_t$ denote the class of binary matroids with no* $\mathrm{PG}(t+2,2)$*-minor. Then $h_{\mathcal{M}_t}(n) \approx f_t(n)$.*

Proof. We need only show that $\mathrm{PG}(t+2,2)$ is t-pinched-graphic but not t-graphic. Since $f_t(t+3) = 2^t\binom{4}{2} + 2^t - 1 < 2^{t+3} - 1 = \varepsilon(\mathrm{PG}(t+2,2))$, it is not t-graphic. Since any rank-3 matroid is pinched-graphic, any rank-$(t+3)$ matroid is t-pinched-graphic, so $\mathrm{PG}(t+2,2)$ is t-pinched-graphic. □

Corollary 3.5. *Let $t \geq 0$ be an integer, and let $\mathcal{N}_t$ denote the class of binary matroids with no* $\mathrm{AG}(t+3,2)$*-minor. Then $h_{\mathcal{N}_t}(n) \approx f_t(n)$.*

Proof. We need only show that $\mathrm{AG}(t+3,2)$ is t-pinched-graphic but not t-graphic. Since $f_t(t+3) = 2^t\binom{4}{2} + 2^t - 1 < 2^{t+3} = \varepsilon(\mathrm{AG}(t+3,2))$, it is not t-graphic. We can take a $(t+1)$-pinched-graphic representation of $\mathrm{PG}(t+3,2)$ and delete a hyperplane to obtain a $(t+1)$-pinched-graphic representation of $\mathrm{AG}(t+3,2)$ with $t+4$ rows. Since $\mathrm{AG}(t+3,2)$ has rank $t+3$, we can delete a row of this representation to find a t-pinched-graphic matrix which represents $\mathrm{AG}(t+3,2)$. □

4 Generalizations

4.1 Other Prime Fields

It is very likely possible to generalize the results of this thesis to prime fields other than GF(2). Let $\mathcal{M}_{t,p}$ denote the class of matroids representable over GF(p) with no PG($t+2, p$)-minor, where p is an odd prime and $t \geq 0$ is an integer. Over GF(2) we extended graphic matroids to t-graphic matroids. Over a general finite field $\mathbb{F}$ and a subgroup Γ of $\mathbb{F}^\times$ we can extend Γ-frame matroids to (t, Γ)-*frame matroids*, matroids which have a representation of the form

$$\left[\frac{K_1}{K_2}\right],$$

where K_2 is a Γ-frame matrix and K_1 has at most t rows. By Lemma 1.8, the class of (t, Γ)-frame matroids is minor-closed, and has growth rate function $h(n) = p^t|\Gamma|\binom{n-t+1}{2} + p^t - 1$, where $p = |\mathbb{F}|$. Note that PG($t+2, p$) is not a $(t+1, \Gamma)$-frame matroid for any Γ, because $|\Gamma| \leq \frac{p-1}{2}$ and

$$p^{t+1}\left(\frac{p-1}{2}\right)\binom{3}{2} + p^{t+1} - 1 < \frac{p^{t+3}-1}{p-1} = \varepsilon(\mathrm{PG}(t+2, p))$$

for any $p \geq 3$. This tells us that $h_{\mathcal{M}_{t,p}}(n) \geq p^{t+1}(\frac{p-1}{2})\binom{n-t}{2} + p^{t+1} - 1 = f_{t,p}(n)$. If we can then extend Lemma 2.12 and Lemma 2.13, we would prove the following result when p is prime.

Conjecture 4.1. *Let $t \geq 0$ be an integer, let p be an odd prime, and let Γ be a subgroup of the multiplicative group of* GF(p) *of order $\frac{p-1}{2}$. If N is a $(t+2, \{1\})$-frame matroid which is not a $(t+1, \Gamma)$-matroid, then $h_{\mathrm{Ex}(N)}(n) \approx f_{t,p}(n)$.*

Any matroid of rank at most $t+3$ representable over GF(p) is a $(t+2, \{1\})$-frame matroid, so PG($t+2, p$) is a $(t+2, \{1\})$-frame matroid. Thus, if the conjecture holds we would find that the growth rate function for the class of matroids with no PG($t+2, p$)-minor is $f_{t,p}(n)$ for any odd prime p.

4.2 (t, k)-Graphic Matroids

We can generalize the notion of a pinched-graphic matroid to a $(0, k)$-*graphic matroid*, in which we extend a graphic matrix by up to k columns of support-3 and then contract them. For any integer $k \geq 0$ let $\mathcal{P}_k$ denote the class of

$(0,k)$-graphic matroids. A similar argument to Lemma 1.7 shows that $\mathcal{P}_k$ is minor-closed for any $k \geq 0$. If the graphic matrix is the incidence matrix of K_{n+k-1} and no two support-3 columns have nonzero entries in the same row, then we get a simple matroid M in $\mathcal{P}_k$ with $|M| = \binom{n+k+1}{2}$ and $r(M) = n$. Since this is the arrangement of support-3 columns which gives the most distinct elements, we have

$$h_{\mathcal{P}_k}(n) = \binom{n+k+1}{2}.$$

We say a matroid is (t,k)-graphic matroid if it is in $\mathcal{P}_k^t$. Note that $\mathcal{P}_1^t$ is the class of t-pinched-graphic matroids. By Lemma 1.8 we know that the class of (t,k)-graphic matroids is minor-closed for any $t, k \geq 0$, and by Lemma 1.9 we have

$$h_{\mathcal{P}_k^t}(n) = 2^t\binom{n-t+k+1}{2} + 2^t - 1.$$

This gives a family of functions $\{f_{t,k} : t, k \geq 0\}$ where $f_{t,k} = h_{\mathcal{P}_k^t}(n)$.

For any binary matroid N which is not graphic, we can find the maximum integer t such that N is not t-graphic, and then find the maximum k such that N is not (t,k)-graphic. This tells us that $f_{t,k}(n) \leq h_{\mathrm{Ex}(N)}(n)$ for all integers n, and that N is $(t+1,0)$-graphic and $(t,k+1)$-graphic. It would be nice to believe that $h_{\mathrm{Ex}(N)}(n) \approx f_{t,k}(n)$, but this is not always true because when t is small relative to k we can use the columns indexed by Y_1 of a matrix conforming to a template to find minor-closed classes which do not contain N and have an extremal growth rate function greater than $f_{t,k}(n)$. However, it may be possible to extend the techniques of this thesis to prove the following conjecture.

Conjecture 4.2. *For any integer $k \geq 1$ there is some integer t_k such that for any $t \geq t_k$, any matroid N which is (t,k)-graphic and $(t+1,0)$-graphic but not $(t,k-1)$-graphic satisfies $h_{\mathrm{Ex}(N)}(n) = f_{t,k-1}$ for n sufficiently large.*

This conjecture says that if N is a matroid which is sufficiently far from graphic, then we can determine the eventual growth rate function of $\mathrm{Ex}(N)$ based on properties of N.

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Appendices

A Matroids

A matroid is an object which generalizes dependence from linear algebra and graph theory. A set of vectors is dependent if there is a nontrivial linear combination which sums to the zero vector. A set of edges in a graph is dependent if it contains a cycle. We will define a matroid, and show how a matroid can be constructed from either a set of vectors and a graph. The next two sections are short versions of discussions found in Oxley [1] and Zaslavsky [2]. We follow the notation in [1], unless stated otherwise.

A *matroid* is a pair $M = (E, r)$ where E is a finite set and r is a function from the set of subsets of E to the integers such that

(1) $r(\varnothing) = 0$,

(2) $r(X) \le r(Y)$ whenever $X \subseteq Y$, and

(3) $r(X \cap Y) + r(X \cup Y) \le r(X) + r(Y)$ for all $X, Y \subseteq E$.

We say that r is the *rank function* of M and E is the *ground set* of M. We write $|M|$ for $|E|$ and $r(M)$ for $r(E)$ and say $r(M)$ is the *rank* of M. A set $X \subseteq E$ is *independent* if $r(X) = |X|$, and is a *basis* if $r(X) = r(E) = |X|$. An element $e \in E$ is a *loop* of M if $r(\{e\}) = 0$. Two elements $e_1, e_2 \in E$ are *parallel* in M if both are non-loops and $r(\{e_1, e_2\}) = 1$. We say M is *simple* if it has no loops and no two elements are parallel.

For any matrix A over a field $\mathbb{F}$ with columns indexed by E we obtain a matroid $M = (E, r)$ with $r(X) = \text{rank}(A[X])$. We say that M is the *vector matroid* of A and that A *represents* M. We write $M(A)$ for the vector matroid of A. A matroid M is $\mathbb{F}$-*representable* if there is some matrix A over $\mathbb{F}$ such that $M = M(A)$. In particular, a matroid is *binary* if it is GF(2)-representable. Note that an $\mathbb{F}$-representable matroid may have many different representations. If A represents a matroid M, then any matrix obtained from A by elementary row operations and column scaling also represents M. The matroid represented by the matrix over GF(q) with n rows and all possible nonzero columns with first nonzero entry equal to one is called the *rank-n projective geometry* over GF(q), and is denoted PG$(n-1, q)$. Any simple GF(q)-representable matroid has a representation which is a submatrix of a representation of $\mathrm{PG}(n-1,q)$. In this regard $\mathrm{PG}(n-1,q)$ plays a similar role for $\mathrm{GF}(q)$-representable matroids to the role that K_n does for graphs, because any simple graph on n vertices is a subgraph of K_n.

For any graph $G=(V,E)$, allowing loops and parallel edges, there is an associated matroid $M(G)=(E,r)$ defined by $r(X)=|V|-\kappa(X)$, where $\kappa(X)$ denotes the number of connected components of (V,X). A matroid M is *graphic* if there is some graph G such that $M=M(G)$. Equivalently, a matroid M is graphic if there is some matrix A over $\mathrm{GF}(2)$ such that $M=M(A)$ and A is the incidence matrix of a graph.

We can generalize graphic matroids to matroids which encode graphs with edges labelled by elements of a group, as Zaslavsky does in [2]. For any field $\mathbb{F}$, a *frame matrix* is a matrix over $\mathbb{F}$ with at most two nonzero entries in each column. A *frame matroid* is a matroid with a representation as a frame matrix. Let $\mathbb{F}^\times$ denote the multiplicative group of $\mathbb{F}$, and let Γ be a subgroup of $\mathbb{F}^\times$. A Γ-*frame matrix* is a matrix over $\mathbb{F}$ such that each nonzero column is either a unit column or has two nonzero entries and they are 1 and $-\alpha$ for some $\alpha\in\Gamma$. A matroid is a Γ-*frame matroid* if it has a representation as a Γ-frame matrix.

B Matroid Properties

Many notions in matroid theory are motivated by graph theory, including deletion, contraction, duality, and connectivity. For any matroid $M=(E,r)$ we can define operations deletion and contraction using the rank function of M. For any $D\subseteq E$ we define a matroid $M\setminus D=(E\setminus D, r_{M\setminus D})$, where $r_{M\setminus D}(X)=r(X)$ for any $X\subseteq E\setminus D$, and we say that $M\setminus D$ is obtained by *deleting* D. For any $C\subseteq E$ we define a matroid $M/C=(E\setminus C, r_{M/C})$ where $r_{M/C}(X)=r(X\cup C)-r(C)$, and we say that M/C is obtained by *contracting* C. Let $M_1=(E_1,r_1)$ and $M_2=(E_2,r_2)$ be matroids. We say that M_1 is a *minor* of M_2 if $M_1=M_2/C\setminus D$ for some disjoint $C,D\subseteq E_2$. We say M_1 and M_2 are *isomorphic* if there is a bijection ϕ from E_1 to E_2 such that $r_1(X)=r_2(\phi(X))$ for each $X\subseteq E_1$, and we write $M_1\cong M_2$. A class $\mathcal{M}$ of matroids is *minor-closed* if it is closed under taking minors and isomorphism. For any matroid M we will denote the class of matroids with no M-minor by $\mathrm{Ex}(M)$.

Given matrix A over a field $\mathbb{F}$ we can obtain representations of $M(A)\setminus D$ and $M(A)/C$ in terms of A for all $D,C\in E(M(A))$. Let $M=M(A)$

and let $E = E(M)$. Certainly $A[E - D]$ is a representation of $M \backslash D$. For any $C \subseteq E$, there is some invertible matrix U over $\mathbb{F}$ such that $(UA)[C]$ is in reduced-row echelon form. Note that UA represents M because row-equivalent matrices have the same vector matroid. Let A' denote the matrix obtained from UA by removing $(UA)[C]$ and any row in which $(UA)[C]$ has a nonzero entry. Then A' represents M/C, because for any $X \subseteq E - C$ we have $\text{rank}(A'[X]) + \text{rank}(A[C]) = \text{rank}(A[X \cup C])$. In this thesis we will frequently obtain representations of minors of representable matroids in this way.

There are other ways we can construct a matroid from another matroid. The *dual* of a matroid $M = (E, r)$ is $M^* = (E, r^*)$ where r^* is defined by $r^*(X) = r(E - X) + |X| - r(E)$. This agrees with planar duality in graphs, so for any planar embedding G with dual embedding G^* we have $M(G)^* = M(G^*)$. A *simplification* of a matroid $M = (E, r)$ is any matroid $\text{si}(M)$ obtained by deleting any loop of M and all but one element of each maximal $X \subseteq E$ such that $r(X) = 1$. Note that $\text{si}(M)$ is a simple matroid for any matroid M. We denote $|\,\text{si}(M)|$ by $\varepsilon(M)$.

We define the *connectivity function* of a matroid $M = (E, r)$ to be $\lambda_M(X) = r(X) + r(E - X) - r(M)$ for any $X \subseteq E$. We say a matroid $M = (E, r)$ is *vertically k-connected* if for every $X \subseteq E$ such that $\lambda_M(X) < k - 1$, either $r(X) = r(M)$ or $r(E - X) = r(M)$. This deviates slightly from the definition found in [1], but it is an equivalent definition. If M is not vertically k-connected, then there is some $X \subseteq E$ such that $\lambda_M(X) < k - 1$ and $r(X) < r(M)$ and $r(E - X) < r(M)$, and we say that X is a vertical *k-separation*.