# Horizon Thermodynamics from Einstein's Equation of State

by

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**Statement of Contributions:** – Some of the work in chapters III, IV, and V was co-authored by David Kubiznak and Robert Mann.

**Abstract:** The striking resemblance between geometric theories of gravity and thermodynamics suggests a more fundamental relationship between the two seemingly distinct theories. It has been postulated that the radial Einstein equations projected onto the event horizon of a black hole are in fact an equation of state of an ensemble whose temperature is given by the Hawking temperature of the spacetime and whose pressure is the stress-energy on the horizon. We study the consequences of this postulate in detail and find several interesting results. First, we demonstrate how a cohomogeneity-2 first law can be derived from the equation of state, and find that in the process we obtain an independent definition of black hole entropy which matches the Wald entropy in several nontrivial cases. Second, we find that the spectrum of thermodynamic behaviour in this paradigm is rich and includes the standard Hawking-Page transition as well as re-entrant phase transitions, Van der-Waals transitions, and triple points similarly to what has been observed in the Black Hole Chemistry approach. Finally, we find that horizon thermodynamics does not easily extend to spacetimes whose symmetries do not sufficiently reduce the number of independent field equations. Though we do not yet have a quantitative limit to horizon thermodynamics, we find that conceptual sacrifices must be made in order to keep ignorant of conserved charges in more general spacetime backgrounds.

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## I. INTRODUCTION AND BACKGROUND

Since seminal papers of Bekenstein, Hawking, Bardeen, and Carter [1–4], there has been a strong belief that Einstein's gravitational field equations have a deep connection with the first law of thermodynamics, e.g. [5–8]. For example, Jacobson [6] showed that the Einstein equations were in fact equivalent to the Clausius relation imposed for the local Rindler horizon of an accelerated observer at every spacetime point. Hayward [7] showed that the equations projected on a null surface could be written as a first law. Similarly, it was explicitly shown that the Einstein equations on the horizon of a spherically symmetric black hole spacetime can be interpreted as a thermodynamic identity. This was the origin of *horizon thermodynamics* [9].

Horizon thermodynamics is an extension of the standard first law of black hole thermodynamics, which states that variations in the mass of a black hole are proportional to variations in the horizon area. An analogy is drawn to classical thermodynamics by identifying the horizon area as an entropy for the black hole, and proportionality term as the black hole temperature. Horizon thermodynamics is one proposed method to extend this picture by including an effective "work term" in the first law. The key feature of horizon thermodynamics is that black hole pressure is defined as the stress-energy of the spacetime, and volume is assumed to be the geometric volume enclosed by the horizon. Given this assumption, horizon thermodynamics is a procedure for rewriting a component of the Einstein equations as an equation of state, $P = P(V, T)$.

The original observation for spherically symmetric black holes in Einstein's gravity [9] noticed that by identifying thermodynamic pressure as the radial component of the stress-energy tensor (along with any cosmological constant), the radial field equations were simply an equation of state $P = P(V, T)$ for Hawking temperature $T$ and volume $V$. It was also noted [9] that by simple algebraic manipulation this equation of state could be re-cast as a first law of thermodynamics of the form $\delta E = T\delta S + P\delta V$ for entropy given by the area law $S = A/4$ and $E$ related to the black hole's mass.

Horizon thermodynamics has since been extended to a number of other interesting cases, many of which have been highlighted in recent reviews [10, 11]. For example, the horizon thermodynamics has been extended to spherically symmetric black holes in Lovelock and Quasi-topological gravities [12–15], $f(R)$ gravity [16], and Horava–Lifshitz gravity [17], to time evolving and axisymmetric stationary black hole horizons [18, 19], to horizons in FRW spacetime [20–22] and braneworld scenarios [23, 24]. More recently the general thermodynamic properties of null surfaces have been investigated e.g. in [25]. In this thesis, I concentrate on horizon thermodynamics of black holes.

While these results are rather suggestive, there are several issues in this procedure that arise upon further inspection. First, all relevant thermodynamic quantities must already be known in order to identify them in the field equations. Namely, $S, T$ and $V$ have to be *independently specified* and the only derived quantity is the quasilocal energy $E$. Consequently this procedure cannot be used as a way to derive any thermodynamic properties of a spacetime; instead, it serves purely as means to identify a peculiar relationship between the field equations and thermal systems provided the thermodynamic properties of the solution are already known. The focus was previously on the provocative relation hidden within the Einstein Equations when the appropriate identifications were made. Consequently this procedure provides no direct algorithmic method to derive thermodynamic properties of a spacetime where appropriate identifications are yet unknown, and has instead been used as means of highlighting the presence of known thermodynamics in the gravitational field equations.

The second issue concerns the restriction to virtual displacements $\delta r_+$ of the horizon radius. This renders the first law to be of '*cohomogeneity-one*', since both $S$ and $V$ are functions only of $r_+$. Indeed the first law could just as well be written as $\delta E = (TS' + PV')\delta r_+$, with primes denoting differentiation with respect to $r_+$. This yields an *ambiguity* between 'heat' and 'work' terms and leads to a 'vacuum interpretation' of the first law [26].

These issues are not inherent to horizon thermodynamics but in fact can be avoided completely by treating the initial equation of state with more care. By directly varying the radial field equations on the horizon, we show that one can obtain a cohomogeneity-2 first law directly from the equation of state. Interestingly, in the process we find an independent definition for horizon entropy which seems to be consistent with the Wald entropy in several nontrivial cases despite the fact that the notion of conserved charges was never used.

Horizon thermodynamics also provides a rich spectrum of thermodynamic behavior to study. We find the Hawking-Page phase transition is present in spherically-symmetric solutions of the Einstein equations, and more exotic phase transitions can be obtained by moving to higher curvature theories.

We also explore the interpretation of the equations of state and corresponding first laws of thermodynamics which result from this approach. We find the energy in the first law corresponds to a "horizon curvature energy". If one includes the cosmological constant as well as the stress-energy term in the pressure, the corresponding energy is the Misner-Sharp mass, which has been shown previously in [27, 28]. We show that in spherical symmetry, one need not make any additional assumptions to generate an equation of state and a first law.

The idea of pressure and volume as well as that of the equation of state have in recent years been the subject of much attention, see e.g. [29, 30] for recent short reviews. In an alternative framework one identifies the cosmological constant as a thermodynamic variable analogous to pressure [31–34]. Its conjugate thermodynamic volume can be obtained via geometric means by generalizing the first law of black hole mechanics in spacetimes that have a cosmological constant [32, 35]. This in turn implies that the mass of an AdS black hole is the enthalpy of spacetime. This approach emerged from geometric derivations of the Smarr formula for AdS black holes [32] and led to a *reverse* isoperimetric inequality conjecture [34], which states that for fixed thermodynamic volume, the entropy of an AdS black hole is maximized for Schwarzschild AdS. In this thesis, I refer to this alternate approach as the *variable lambda approach.* A very rich and interesting array of thermodynamic behaviour for both AdS and dS black holes has been shown to emerge from extended phase space thermodynamics. Examples of the so-called $P-V$ *criticality* include a complete analogy between 4-dimensional Reissner–Nördstrom AdS black holes and the Van der Waals liquid-gas system [36], the existence of reentrant phase transitions in rotating [37] and Born–Infeld [38] black holes, tricritical points in rotating black holes analogous to the triple point of water [39], and isolated critical points in Lovelock gravities [40, 41]. These phenomena continue to be subject to intensive study in a broad variety of contexts e.g. [29, 38, 42–71][12–14, 72].

Despite the enhancements given to horizon thermodynamics, limitations are still present when extending to general metric ansatz and modified gravity theories. For example, when applied to axially symmetric solutions in asymptotically flat Einstein gravity we find there are several ambiguities in the procedure. The first is that the stress-energy is not constant over the horizon. This can be mediated by re-defining pressure as some form of average of the stress-energy over the horizon. From there, there is further ambiguity as to how to separate the resulting equation into total differentials. We discuss several possible resolutions to this problem, including introducing a "surface tension" term, however there is no clear unambiguously preferred choice. This suggests that an extension of horizon thermodynamics to more complex metric ansatz would be difficult in cases where thorough understanding of the black hole's thermodynamics were not present to guide the procedure.

The primary goal of this thesis is to rigorously explore the consequences of the conjecture that the radial field equations projected onto a black hole horizon are an equation of state in the context of specific spacetime backgrounds. The remainder of the thesis is broken up as follows: Sec. II discusses the basics of black hole horizon thermodynamics and demonstrates the procedure for the Schwarzschild metric. Sec. III introduces the enhanced, full-cohomogeneity thermodynamics and new entropy definition and provides several examples of its use. Sec. IV looks at $P-V$ criticality in horizon thermodynamics and identifies several interesting phase transitions. Sec. V looks at the consequences of generalizing horizon thermodynamics to various nontrivial metrics and highlights the ambiguities that arise in each case. Finally, Sec. VI provides a summary and closing remarks for the thesis. Appendix A gives a review of Wald's formalism for black hole horizon entropy and Appendix B provides explicit calculations for Einstein and Lovelock gravities. This thesis is based on the following three papers [26, 73, 74]. For the remainder of this paper we use units where $\hbar = G = c = 1$.

## II. FUNDAMENTALS OF HORIZON THERMODYNAMICS

### A. The Standard Laws of Black Hole Thermodynamics

A discussion of horizon thermodynamics must first begin with a review of the standard laws of black hole thermodynamics first proposed by Bardeen, Carter and Hawking in 1973 [4]. Black hole solutions to Einsteins gravitational equations appear to have little to do with thermodynamics at first glance. Classically, the so called "no hair" theorem [75] states that classical black holes are without internal structure and can be fully described by three values: the mass of the black hole $M$, the angular momentum $J$ and the electrostatic charge $Q$. As a classical ensemble, a black hole has zero temperature, absorbs all incoming radiation and emits nothing.

Hawking [3] showed that in the presence of quantum fields, black holes do radiate with a characteristic temperature proportionate to the *surface gravity*,

$$\xi^\alpha \nabla_\alpha \xi^\beta = \kappa \xi^\beta \tag{0}$$

for Killing vector $\xi$. The *Hawking temperature* is defined

$$T \equiv \frac{\kappa}{2\pi}\,, \tag{0}$$

with a thermodynamic conjugate entropy proportional to the horizon area, which has since been shown to be the integral of a Noether charge for diffiomorphism covariant Lagrangians (see Appendix A). The discovery of these charges of black holes lead to the study of black hole thermodynamics.

The rules govern the dynamics of black holes and their horizons in terms of the thermodynamic quantities defined above bear a striking resemblance to the classical laws of thermodynamics [75]. They read as follows:

- *0th Law:* The surface gravity of a black hole is a constant across its horizon. Since the surface gravity $\kappa$ is directly proportional to the Hawking temperature by the relation $T = \kappa/2\pi$, this is equivalent to saying the temperature is a constant across the horizon.

- *1st Law:* Perturbations to the energy of a stationary black holes in Einstein gravity can be written as

$$\delta E = \frac{\kappa}{2\pi}\frac{\delta A}{4} + \Omega\delta J + \Phi\delta Q\,, \tag{0}$$

  where $A$ is the horizon area, $\Omega$ and $J$ are the angular velocities and momenta respectively, $\Phi$ is the electrostatic potential and $Q$ is the charge of the black hole. By identifying the temperature as $\kappa/2\pi$ and the entropy as $A/4$ this can be related to the standard first law of thermodynamics,

$$\delta E = T\delta S + \Omega\delta J + \Phi\delta Q\,. \tag{0}$$

- *2nd Law:* Classical perturbations will never decrease the area of the black hole horizon. Relating the horizon area to the entropy gives the analogy to the second law of thermodynamics that states that entropy is never decreasing. This only holds for classical perturbations however, as quantum processes cause the horizon area to shrink due to Hawking radiation.

- *3rd Law:* The surface gravity of a black hole cannot be reduced to zero in a finite number of steps.

The striking resemblance between these laws for black hole dynamics and classical thermodynamics have caused much speculation as to a fundamental connection between black hole horizons and thermal ensembles. It is of note, however, that the first law (II A) does not have a "work" term $PdV$ as is generally seen in classical thermodynamics. Allong with the classical charges of a black hole $M$, $Q$ and $J$, temperature and entropy are given physical meaning for black hole spacetimes from Hawking and Wald respectively. However, a thermodynamic pressure and volume may not be obviously present in black hole systems.

### B. Variable $\Lambda$ Thermodynamics & Black Hole Chemistry

More recently, the nature of spacetimes with a non-zero cosmological constant have been used to introduce the concept of thermodynamic pressure into black hole mechanics. By treating the (negative) cosmological constant of asymptotically AdS black hole spacetimes as a pressure experienced by the black hole [29], and introducing a thermodynamic conjugate volume, one is able to right a full first law for black holes $dE = TdS - PdV$. From here a more full spectra of thermodynamic behavior was observed in black hole mechanics. In particular, it was seen that black holes in asymptotically AdS spaces were remarkably analogous to Van der Waals fluids [36, 39]. Re-entrant and Van der Waals like phase transitions were discovered[39, 76], as well as a triple point analogous to that of water is seen under the assumption that $\Lambda$ serves as a thermodynamic pressure. Specifically, the black hole ensemble is treated as if it were submerged in a bath of fluid

$$P_\Lambda = -\frac{\Lambda}{8\pi}\,, \tag{0}$$

(where the subscript $\Lambda$ is used to differentiate from the horizon thermodynamic pressure).

Notably, in this picture, the black hole mass is not associated with the internal energy, but rather the black hole's enthalpy, defined by the first law

$$\delta M = T\delta S + V_\Lambda \delta P_\Lambda + \Omega\delta J + \Phi\delta Q.\,. \tag{0}$$

The notion of thermodynamic volume as a conjugate to pressure then arises naturally in this picture as

$$V_\Lambda \equiv \left(\frac{\partial M}{\partial P}\right)_{S,Q,J}\,. \tag{0}$$

Note that this quantity need not be a geometric volume associated with the black hole horizon, but is defined purely as a dimensionally appropriate conjugate to pressure.

In this setting, one treats variations in the newly defined pressure along with the Hawking temperature and the classically defined charges of a black hole (mass, charge and angular momentum) together to form a complete picture of the *extended thermodynamic phase space* of the black hole. By studying variations in the extended phase space one is able to explore the thermodynamic possibilities of the ensemble similarly to classical thermodynamic systems. Note that by construction, slices in the extended thermodynamic phase space fix conserved charges in the spacetime (for example, one can study changes in free energy with respect to temperature at fixed pressure, charge, and angular momentum).

Black hole chemistry can be generalized beyond standard Einstein gravity in a straightforward manner. In D-dimensional spacetime, a black hole with charges $Q_i$ and angular momenta $J_i$ obeys the generalized first law

$$\delta M = T\delta S + V_\Lambda \delta P_\Lambda + \Omega^i \delta J_i + \Phi^i \delta Q_i\,, \tag{0}$$

and, by dimensional analysis, satisfies the Smarr relation

$$\frac{D-3}{D-2}M = TS - \frac{2}{D-2}P_\Lambda V_\Lambda + \Omega^i J_i + \frac{D-3}{D-2}\Phi^i Q_i\,. \tag{0}$$

### C. Horizon Thermodynamics – Spherically Symmetric Ansatz in Einstein Gravity

Horizon thermodynamics takes a different approach to the problem of pressure in black hole thermodynamics. Instead of defining pressure as the cosmological constant term in an asymptotically AdS spacetime, horizon thermodynamics assumes that the the radial field equations themselves serve as an *equation of state*

$$\frac{1}{8\pi}G^r_r|_r+ = T^r_r|r_+ - \Lambda \equiv P(V,T)\,, \tag{0}$$

where $r_+$ is the radius of the black hole horizon and the thermodynamic pressure is identified as the entire right hand side of the field equations

$$P \equiv T^r_r|_r + -\Lambda\,. \tag{0}$$

Note that $P = P_\Lambda$ in the case of a vacuum spacetime where $T^r_r = 0$. Also note that black hole volume does not naturally arise as it does in the black hole chemistry approach. Rather, horizon thermodynamics assumes the thermodynamic volume to be the geometric volume bounded by the event horizon.

A first law of horizon thermodynamics can then be derived algebraically from this equation of state (II C). In what follows we sketch this for a static, spherically symmetric spacetime in 4-D Einstein gravity.

Consider a static spherically symmetric black hole spacetime described by the geometry

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2\,, \tag{1}$$

with a non-degenerate horizon located at $r = r_+$, determined as the largest positive root of $f(r_+) = 0$. Assuming minimal coupling to the matter, with the stress energy tensor $T_{ab}$, the radial Einstein equation evaluated on the horizon reads

$$8\pi T^r{}_r|_{r_+} = G^r{}_r|_{r_+} = \frac{f'(r_+)}{r_+} - \frac{1-f(r_+)}{r_+^2}\,, \tag{1}$$

where primes denote differentiation with respect to $r$. Horizon thermodynamics postulates that

$$P = T^r{}_r|_{r_+}\,, \tag{1}$$

and the temperature to be the Hawking temperature,

$$T = \frac{f'(r+)}{4\pi}\,. \tag{1}$$

These identifications yield

$$P(V,T) = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2}\,, \tag{1}$$

which is the horizon equation of state for $V$ a geometric volume $V = 4/3\pi r_+^3$. Multiplying this by $4\pi r_+^2 \delta r_+$ then gives

$$\frac{\delta r_+}{2} = T\delta S - P\delta V\,, \tag{1}$$

which is the horizon first law, provided we either identify

$$S = \frac{A}{4} = \pi r_+^2\,, \quad E = \frac{r_+}{2}\,, \tag{1}$$

entropy and energy respectively.

If we require the horizon first law to be of the form

$$\delta E = T\delta S - P\delta V\,, \tag{1}$$

then identifying the volume as a geometric volume and the entropy as the Wald entropy can be interpreted as a definition of the internal energy of the black hole. Similarly an identification of the volume and the requirement that the energy be purely a function of $r_+$ can be used to obtain the horizon entropy up to a total derivative.

The approach presented here is not restricted to Einstein gravity alone, but also extends naturally to spherically symmetric black holes in Lovelock gravity theories [77]. However, as with the Einstein case, the many of the thermodynamic quantities must be given *a-priori*.

While it is striking that a first law can be algebraically obtained as a consequence of interpreting the radial field equations as an equation of state, the first law obtained here has several worrying features. Firstly, the first law (II C) is cohomogeneity one, as its every term varies solely with $r_+$, and suffers from the ambiguity of defining independent heat and work terms. While in principal, $T$, $P$, and $V$ can all vary independently of one another, in this case volume and entropy are degenerate with one another. The second troublesome point is the sheer number of a-priori inputs required to arrive at a first law. Entropy, temperature and mass are obtained through independent means, and the volume is assumed without any further justification. This suggests that horizon thermodynamics may not be able to be predictive in cases where the thermodynamic properties of a spacetime are not already well understood, but rather is purely prescriptive to spacetimes once the relevant quantities are derived through independent means. However some of these issues are not inherent features of horizon thermodynamics and instead are consequences of the traditional approach to horizon thermodynamics. In the following section, we provide an enhanced version of horizon thermodynamics which is able to provide a full-cohomogeneity first law as a direct consequence of the equation of state.

## III. ENHANCED HORIZON THERMODYNAMICS

In the previous section we demonstrated the traditional method for arriving at a first law of horizon thermodynamics, which is explicitly degenerate. In this section we demonstrate our new construction of a full cohomogeneity first law directly from Einstein's equation of state, as well as demonstrate its applicability in more general spacetimes and higher curvature Lovelock gravities. We also find an independent definition of black hole entropy which is separate from Wald's prescription of Noether charges [74].

### A. Full Cohomogeneity Thermodynamics – Spherical Symmetry

Again consider the metric ansatz (1). The identification of the temperature $T$ as in (II C) is via standard arguments in thermal quantum field theory; it does not require any gravitational field equations. By definition the pressure is identified with the matter stress-energy as in (II C). With this information the radial Einstein equation can be rewritten as

$$P = B(r_+) + C(r_+)T\,, \tag{1}$$

where $B$ and $C$ are some known functions of $r_+$ that in general depend on the theory of gravity under consideration, as does the linearity of the equation of state in the temperature $T$. Formally varying the generalized equation of state (III A), we obtain

$$V\delta P = V\left(B' + C'T\right)\delta r_+ + VC\delta T\,, \tag{1}$$

upon multiplication by a function $V(r_+)$ that we shall identify as the volume, assuming all other parameters are fixed. It is now straightforward to rewrite this equation as

$$V\delta P = S\delta T + \delta G\,, \tag{1}$$

where

$$\begin{aligned} G &= \int^{r_+} dx V(x)B'(x) + T\int^{r_+} dx V(x)C'(x) \\ &= PV - ST - \int^{r_+} dx V'(x)B(x)\,, \\ S &= \int^{r_+} dx V'(x)C(x)\,, \end{aligned} \tag{1}$$

using the integration by parts. Since (by postulate) we have identified $T$ with temperature, $P$ with pressure, and $V$ with volume, we therefore conclude that $S$ is the *entropy* and $G$ is the *Gibbs free energy* of the black hole. Note that these are *derived* quantities from the premises (II C), and the field equations that yield (III A), along with the assumption that the volume (whose explicit form (II C) was not really required up to now) does not depend on $T$.

The relation (III A) for the Gibbs free energy $G = G(P,T)$ is the cohomogeneity-two horizon first law, where $P$ and $T$ are independent quantities. It is valid for any gravitational theory whose field equations yield a linear relation between pressure and temperature. Note that since $G$ depends on the matter content only implicitly (via $P$ and $T$) it characterizes the gravitational theory. This is the origin of recently observed 'universality' of the corresponding phase behavior [73].

We can define the *horizon enthalpy* by the associated Legendre transformation $H = H(S,P) = G + TS$, and recover

$$\delta H = T\delta S + V\delta P\,, \tag{0}$$

which is another non-degenerate horizon first law. Likewise we can employ the Euler scaling argument, e.g. [78], to obtain

$$H = 2TS - 2VP\,, \tag{0}$$

which is the accompanying (four-dimensional) Smarr relation.

We can also make the degenerate Legendre transformation, whose degeneracy originates in the fact that $S$ and $V$ can both be expressed purely as function of the horizon radius $r_+$ and thus are not independent quantities, and obtain so the 'old' cohomogeneity-one horizon first law (II C).

Specifying to Einstein gravity in four dimensions, it is straightforward to identify $B(r_+) = -(8\pi r_+^2)^{-1}$ and $C(r_+) = 1/(2r_+)$ from (II C), yielding from (1)

$$S = \pi r_+^2 \,, \quad G = \frac{r_+}{3}(1 - \pi r_+ T) \,, \tag{0}$$

using the geometric definition (II C) of the volume. This Gibbs free energy was previously derived and its phase diagrams studied in [72, 73]; it is understood as $G = G(P, T)$ through the equation of state $r_+ = r_+(P, T)$, (II C). Performing the degenerate Legendre transformation, one finds $E = \frac{r_+}{2}$, in accordance with the previous approach.

### B. Full Cohomogeneity Thermodynamics – Other Examples

Enhanced horizon thermodynamics is applicable to spherically symmetric solutions beyond standard Einstein gravity. Similarly to the traditional approach, the enhanced approach extends to Lovelock gravities, where enhanced thermodynamics is able to provide the nontrivial expressions for the horizon entropy and the Lovelock potentials. Enhanced horizon thermodynamics is also able to extend to theories of gravity with equations of state which are non-linear in $T$ such as cubic gravity [79]. This is something which could not be unambiguously done in standard horizon thermodynamics.

#### *1. Lovelock gravity and horizon equation of state*

Lovelock gravity [80] is a geometric higher curvature theory of gravity that can be considered as a natural generalization of Einstein's theory to higher dimensions—it is the unique higher-derivative theory that gives rise to second-order field equations for all metric components. In $d$ spacetime dimensions, the Lagrangian reads

$$\mathcal{L} = \frac{1}{16\pi G_N} \sum_{k=0}^{K} \alpha_k \mathcal{L}^{(k)} + \mathcal{L}_m \,. \tag{0}$$

Here, $K = \lfloor \frac{d-1}{2} \rfloor$ is the largest integer less than or equal to $\frac{d-1}{2}$, $\mathcal{L}^{(k)}$ are the $2k$-dimensional Euler densities, given by

$$\mathcal{L}^{(k)} = \frac{1}{2^k} \delta^{a_1 b_1 \ldots a_k b_k}_{c_1 d_1 \ldots c_k d_k} R_{a_1 b_1}{}^{c_1 d_1} \ldots R_{a_k b_k}{}^{c_k d_k} \,, \tag{0}$$

with the 'generalized Kronecker delta function' $\delta^{a_1 b_1 \ldots a_k b_k}_{c_1 d_1 \ldots c_k d_k}$ totally antisymmetric in both sets of indicies, $R_{a_k b_k}{}^{c_k d_k}$ is the Riemann tensor, and the $\alpha_{(k)}$ are the Lovelock coupling constants. In what follows we identify the (negative) cosmological constant $\Lambda = -\alpha_0/2$, and set $\alpha_1 = 1$ to remain consistent with general relativity. We also assume minimal coupling to the matter, described by the matter Lagrangian $\mathcal{L}_m$. The Lovelock equations of motion that follow from the variation of (III B 1) are

$$\sum_{k=0}^{K} \alpha_k G^{(k)}_{\mu\nu} = 8\pi T_{\mu\nu} \,, \tag{0}$$

where $G^{(k)}_{\mu\nu}$ are the $k$th-order Einstein–Lovelock tensors [28, 80].

We shall restrict our attention to spherically symmetric AdS Lovelock black holes, employing the ansatz [28]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \gamma_{ab}(r)dx^a dx^b + r^2 h_{ij}dx^i dx^j\,, \tag{0}$$

where the non-trivial part of the metric is described by a 2-dimensional metric $\gamma_{ab}$ $(a, b = 0, 1)$, while $h_{ij}$ $(i, j = 2, \ldots, d-1)$ stands for the line element of a $(d-2)$-dimensional space of constant curvature $\sigma(d-2)(d-3)$, with $\sigma = +1, 0, -1$ for spherical, flat, and hyperbolic geometries respectively of finite volume $\Sigma_{d-2}$, the latter two cases being compact via identification [81–83]. The $(a, b)$-components of the $k$th Lovelock–Einstein tensor then are [28]

$$\begin{aligned} G^{(k)}_{ab} = &\frac{k(d-2)!}{(d-2k-1)!}\frac{(D^2r)\gamma_{ab} - D_aD_br}{r}\left(\frac{\sigma-(Dr)^2}{r^2}\right)^{k-1} \\ &-\frac{(d-2)!(d-2k-1)}{2(d-2k-1)!}\gamma_{ab}\left(\frac{\sigma-(Dr)^2}{r^2}\right)^{k}\,, \end{aligned} \tag{0}$$

where $(Dr)^2 = \gamma^{ab}(D_ar)(D_br)$ and $D^2r = D^aD_ar$. The remaining $(i, j)$ components can be found in [28]. As long as at least one $\alpha_k \neq 0$ for $k > 1$ all possible values of $\sigma$ yield solutions, even if $\Lambda \propto \alpha_0 = 0$.

Consider a black hole for which

$$\gamma = \gamma_{ab}(r)dx^a dx^b = -f(r)dt^2 + \frac{dr^2}{g(r)}\,, \tag{0}$$

with the outer black hole horizon located at $r = r_+$, determined from $f(r_+) = 0$. Employing (III B 1), we have

$$\begin{aligned} D^2r = \frac{1}{2}\frac{(fg)'}{f}\,, \quad (Dr)^2 = g\,, \\ D^tD_tr = \frac{1}{2}\frac{gf'}{f}\,, \quad D^rD_rr = \frac{1}{2}g'\,. \end{aligned} \tag{0}$$

The Einstein–Lovelock equations (1) then read

$$8\pi T^t{}_t = \frac{g'}{2r}\sum_{k=1}^{K}\alpha_k\frac{k(d-2)!}{(d-2k-1)!}\left(\frac{\sigma-g}{r^2}\right)^{k-1} - \sum_{k=0}^{K}\alpha_k\frac{(d-2)!(d-2k-1)}{2(d-2k-1)!}\left(\frac{\sigma-g}{r^2}\right)^{k}, \tag{1}$$

$$,8\pi T^r{}_r = \frac{f'g}{2rf}\sum_{k=1}^{K}\alpha_k\frac{k(d-2)!}{(d-2k-1)!}\left(\frac{\sigma-g}{r^2}\right)^{k-1} - \sum_{k=0}^{K}\alpha_k\frac{(d-2)!(d-2k-1)}{2(d-2k-1)!}\left(\frac{\sigma-g}{r^2}\right)^{k}. \quad . \tag{2}$$

We concentrate on the case where

$$f(r) = g(r)\,, \tag{2}$$

for simplicity. The more general case will be discussed in Sec. (V). Then, identifying temperature with surface gravity yields

$$T = \frac{\kappa}{2\pi} = \frac{f'(r_+)}{4\pi}\,. \tag{2}$$

Horizon thermodynamics is based on the proposal that the energy–momentum tensor on the horizon is interpreted as

$$P_m \equiv T^r{}_r|_{r=r_+}\,. \tag{2}$$

with the assumption that

$$V = \frac{\Sigma_{d-2} r_+^{d-1}}{d-1}\,, \tag{2}$$

is the conjugate black hole volume. Note that $T^t{}_t|_{r=r_+} = T^r{}_r|_{r=r_+}$ due to our assumption that $f(r) = g(r)$. On the horizon, equation (2) (or equivalently (1)) thus reduces to

$$8\pi P_m = \frac{2\pi T}{r_+} \sum_{k=1}^{K} \alpha_k \frac{k(d-2)!}{(d-2k-1)!} \left(\frac{\sigma}{r_+^2}\right)^{k-1} - \sum_{k=0}^{K} \alpha_k \frac{(d-2)!(d-2k-1)}{2(d-2k-1)!} \left(\frac{\sigma}{r_+^2}\right)^{k}\,, \tag{3}$$

upon using the regularity conditions (III B 1) and (III B 1) and the definition (III B 1) of temperature $T$.

Let us further identify

$$P_\Lambda = -\frac{\Lambda}{8\pi} = \frac{\alpha_0}{16\pi}\,, \tag{3}$$

as the pressure associated with the the cosmological constant, and

$$P = P_m + P_\Lambda\,, \tag{3}$$

as the *total pressure* of all the matter fields. Note that such $P$ is determined from the matter content and is not necessarily positive. We therefore arrive at

$$P = \sum_{k=1}^{K} \frac{\alpha_k}{4r_+} \frac{(d-2)!}{(d-2k-1)!} \left(\frac{\sigma}{r_+^2}\right)^{k-1} \left[kT - \frac{\sigma(d-2k-1)}{4\pi r_+}\right], \tag{4}$$

which, together with the identification (III B 1), gives the HES for Lovelock gravity, $P = P(V,T)$. Note that to write down this equation of state one does not need to know the explicit form of $f$. Furthermore, equation (III B 1) is an ansatz in this approach that has to be justified (similar to the prescription for temperature $T$) by some other means, e.g. [34, 84–86].

### 2. *Full-Cohomogeneity Lovelock Gravity*

Lovelock gravity is a particularly interesting class of theories to consider, as the Wald entropy for spherically symmetric black holes is not a simple area law (see Appendix A). The entropy here is

$$S = \sum_{k=1}^{[D/2]} \frac{k}{4} \alpha_k \oint \sqrt{\tilde{h}} \mathcal{L}_{k-1}(\tilde{h}) d^{d-2}x\,, \tag{4}$$

where $\tilde{h}$ is the induced metric on the horizon. We find that the procedure detailed above is able to reproduce the correct entropy in spherical symmetry without the use of Eq. (III B 2).

We continue to consider the static spherically-symmetric metric ansatz (III B 1). Following [73], let us include the contribution of the cosmological constant (if present) to the matter part, replacing the definitions in (II C) by

$$P = T^r{}_r|_{r_+} - \frac{\Lambda}{8\pi}\,, \qquad T = \frac{f'(r+)}{4\pi}\,. \tag{4}$$

The radial Lovelock equation evaluated on the horizon rewrites as the horizon equation of state, which again assumes the form (III A), where now [73]

$$B(r_+) = \sum_{k=1}^{K} \alpha_k B_k(r_+)\,, \quad C(r_+) = \sum_{k=1}^{K} \alpha_k C_k(r_+)\,, \tag{4}$$

and

$$B_k(r_+) = -\frac{(d-2k-1)(d-2)!}{16\pi(d-2k-1)!}\left(\frac{\sigma}{r_+^2}\right)^k ,$$
$$C_k(r_+) = \frac{1}{4r_+}\frac{k(d-2)!}{(d-2k-1)!}\left(\frac{\sigma}{r_+^2}\right)^{k-1} . \tag{4}$$

Identifying the volume $V$ with the black hole geometric volume [76]

$$V = \frac{\Sigma_{d-2}}{d-1} r_+^{d-1} , \tag{4}$$

the formulae (1) imply

$$S = \frac{\Sigma_{d-2}}{4}\sum_{k=1}^{K}\alpha_k \frac{(d-2)!}{(d-2k-1)!}\frac{k\sigma^{k-1}}{d-2k} r_+^{d-2k} , \tag{5}$$

$$G = \frac{\Sigma_{d-2}}{d-1}\sum_{k=1}^{K}\frac{k\alpha_k(d-2)!}{4(d-2k-1)!} r_+^{d-2}\left(\frac{\sigma}{r_+^2}\right)^{k-1}\Big[\frac{\sigma(1-\delta_{d,2k+1})}{2\pi r_+} - \frac{(2k-1)}{d-2k}T\Big] , \tag{6}$$

and we recover the horizon first law. Note that the derived $S$ is a non-trivial generalization of entropy for the Lovelock black holes [87]. By performing the degenerate Legendre transformation, we obtain

$$E = \frac{\Sigma_{d-2}}{16\pi}\sum_{k=1}^{K}\alpha_k \frac{\sigma^k(d-2)!}{(d-2k-1)!} r_+^{d-2k-1} , \tag{6}$$

which is the generalized Misner–Sharp energy [28]. The degenerate horizon first law (II C) can therefore be understood as a special case of the 'unified first law' [28].

As argued in [88], to obtain a consistent Smarr relation, the first law has to be extended to contain variations of the Lovelock coupling constants. This is easily achieved in our approach. Namely, starting again from the horizon equation of state (III A) with (III B 2), we can also vary the Lovelock couplings $\alpha_k$ $(k = 2, \ldots, K)$, thereby obtaining a generalized horizon first law

$$\delta G = -S\delta T + V\delta P + \sum_{k=2}^{K}\Psi^k\delta\alpha_k , \tag{6}$$

where $G$ and $S$, (1), are given above and

$$\begin{aligned}\Psi^k &= (\mathcal{C}_k - VC_k)T + \mathcal{B}_k - VB_k \\ &= \frac{\Sigma_{d-2}(d-2)!\sigma^{k-1}}{16\pi(d-2k-1)!} r_+^{d-2k}\Big[\frac{\sigma(1-\delta_{d,2k+1})}{r_+} - \frac{4\pi kT}{d-2k}\Big] ,\end{aligned} \tag{7}$$

are the conjugate potentials to variable $\alpha_k$; quantities $\mathcal{C}_k$ were determined from $\mathcal{C} = \sum_{k=1}^{K}\alpha_k\mathcal{C}_k$ and similar for quantities $\mathcal{B}_k$. Note that by construction the potentials $\Psi^k$ depend on matter only implicitly through the temperature $T$. Because of this, Eq. (7) corresponds to the vacuum values of the potentials, c.f. Eq. (2.23) in [40]. The obtained horizon first law (III B 2) is obviously of cohomogeneity-$(K+1)$. It is now easy to verify that one obtains the following Smarr relation:

$$(d-3)H = (d-2)TS - 2VP + \sum_{k=2}^{K}2(k-1)\Psi^k\alpha_k , \tag{6}$$

for the enthalpy, completing the horizon thermodynamic description of Lovelock black holes.

Another interesting example where horizon thermodynamics seems to be applicable is in the cubic gravity theory studied recently in [79]. The Lagrangian for Einsteinian cubic gravity reads:

$$\mathcal{L} = \frac{1}{16\pi}\left(-2\Lambda_R\right) + 8\pi\lambda\mathcal{P}\,, \tag{6}$$

with dimensionful coupling constant $\lambda$ and

$$\mathcal{P} \equiv 12 R_{\mu\nu}^{\rho\sigma} R_{\rho\sigma}^{\gamma\delta} R_{\gamma\delta}^{\mu\nu} + R_{\mu\nu}^{\rho\sigma} R_{\rho\sigma}^{\gamma\delta} R_{\gamma\delta}^{\mu\nu} - 12 R_{\mu\nu\rho\sigma} R^{\mu\nu} R^{\nu\sigma} + 8 R_{\mu}^{\nu} R_{\nu}^{\sigma} R_{\rho}^{\mu}\,. \tag{6}$$

This theory is interesting in that it introduces cubic terms which are active even in four dimensions remains ghost free.

It has recently been shown [79] that this theory admits spherically symmetric metrics of the form (1). Interestingly, however, the equation of state is not linear in $T$ but rather quadratic. While such a theory could not be described by classical horizon thermodynamics, enhanced thermodynamics is perfectly able to process it. In 4 spacetime dimensions concentrating on spherical horizon topology, the on-horizon radial field equation reads [79]

$$G_r^r = \frac{1}{16\pi}\left(\frac{r_+ f' - 1}{r_+^2}\right) - \lambda 48\pi \frac{(f')^2}{r_+^4}\,. \tag{6}$$

For consistency with [79], in this section we take $\kappa = 8\pi G$, not the surface gravity.

Identifying the temperature as $T = f'/4\pi$, this rewrites as the equation of state

$$P(V,T) = \frac{1}{2}\left(\frac{T}{r_+} - \frac{1}{8\pi r_+^2}\right) - \lambda \frac{3072\pi^3}{r_+^4} T^2\,. \tag{6}$$

Varying and multiplying by geometric volume yields the first law:

$$V\delta P = \left(\pi r_+^2 + \frac{3072\pi^3\lambda}{r_+} T\right)\delta T + \delta\left(\frac{1}{3} r_+ - \frac{\pi}{3} T r_+^2 - \frac{2048\pi^3\lambda T^2}{r_+}\right)\,. \tag{6}$$

By taking advantage of our total derivative freedom, we can add and subtract the total derivative $\delta(4\pi\lambda\kappa T^3) = 12\pi\lambda\kappa T^2 \delta T$ to the right hand side to obtain

$$V\delta P = \left(\pi r_+^2 + \frac{3072\pi^3\lambda}{r_+} T + 768\pi^3\lambda T^2\right)\delta T + \delta\left(\frac{1}{3} r_+ - \frac{\pi}{3} T r_+^2 - \frac{2048\pi^3\lambda T^2}{r_+} - 256\pi^3\lambda T^3\right)\,, \tag{6}$$

to correctly identify the entropy as

$$S = \pi r_+^2 + \frac{3072\pi^3\lambda}{r_+} T + 768\pi^3\lambda T^2\,, \tag{6}$$

with a corresponding Gibbs energy of

$$G = \frac{1}{3} r_+ - \frac{\pi}{3} T r_+^2 - \frac{2048\pi^3\lambda T^2}{r_+} - 256\pi^3\lambda T^3\,. \tag{6}$$

in order to arrive at the desired first law,

$$\delta G = -S\delta T + V\delta P\,. \tag{6}$$

Enhanced horizon thermodynamics is able to capture the temperature dependant entropy for an equation of state which is non-linear in temperature. In fact, horizon thermodynamics *predicts* temperature dependance in the entropy for any radial field equation which has terms $(f')^n$, $n \geq 2$ on the horizon. Note that the algebraic approach to standard horizon thermodynamics would be unable to unambiguously find a temperature dependant entropy and energy, as the procedure only identifies entropy as the coefficient of the temperature term in the equation of state.

By allowing the coupling constant $\lambda$ to vary, enhanced horizon thermodynamics can calculate the potentials conjugate to the coupling, $\Phi_\lambda$. With variable $\lambda$, the first law reads

$$V\delta P = \left(\pi r_+^2 + \frac{3072\pi^3\lambda}{r_+}T + 768\pi^3\lambda T^2\right)\delta T + \delta\left(\frac{1}{3}r_+ - \frac{\pi}{3}Tr_+^2 - \frac{2048\pi^3\lambda T^2}{r_+} - 256\pi^3\lambda T^3\right) + \left(\frac{1536\pi^3 T^2}{r_+} + 256\pi^3 T^3\right)\delta\lambda\,.$$

The first law can then be written as

$$\delta G = -S\delta T + V\delta P - \Phi_\lambda\delta\lambda\,, \tag{5}$$

where in agreement with [79] we find

$$\Phi_\lambda = \frac{1536\pi^3}{r_+}T^2 + 256\pi^3 T^3\,, \tag{5}$$

for the thermodynamic potential conjugate to $\lambda$.

## IV. $P-V$ CRITICALITY IN HORIZON THERMODYNAMICS

As with the variable-$\Lambda$ approach, a rich spectrum of thermodynamics is observed in the extended phase space diagrams in horizon thermodynamics. Similarly to variable-$\Lambda$, we find a similarity between black hole horizon thermodynamics and Van der Waals fluids. Similar phase transitions, including triple points, are seen in horizon thermodynamics, though in higher curvature theories than in the black hole chemistry counterparts. Notably, however, horizon thermodynamics is blind to the contents of the stress-energy of the spacetime, taking only the whole of the right hand side of the field equations into consideration. This means that slices in extended phase space are not taken along fixed values of the conserved charges of the spacetime.

### A. Phase Transitions in Horizon Thermodynamics

We illustrate the possible behaviour of the horizon Gibbs free energy and the associated variety of interesting phase transitions that occur in the horizon thermodynamics of spherically symmetric black holes in first few lower-order Lovelock gravities (small values of $K$), generalizing recent results for the Gauss-Bonnet case [72].

#### *1. Einstein gravity*

We start with an example from Einstein gravity ($K = 1$) in $d = 4$ dimensions (similar results hold in higher $d$). Irrespective of the matter content, the equation of state (4) reads

$$P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2}\,, \quad V = \frac{\Sigma_2 r_+^3}{3}\,, \tag{5}$$

while the other thermodynamic quantities take the following explicit form:

$$S = \frac{\Sigma_2 r_+^2}{4}\,,\ E = \frac{\Sigma_2 r_+}{8\pi}\,,\ G = \frac{\Sigma_2 r_+}{6}\Big(\frac{3}{8\pi} - r_+^2 P\Big), \tag{5}$$

and satisfy the horizon first law (III A).

The behaviour of the horizon Gibbs free energy is for $\sigma = 1$ displayed in Fig. 1. Whereas for $P > 0$ we observe a shape characteristic for the Hawking–Page transition of Schwarzschild-AdS black holes [89] (illustrated in Fig. 4), for $P = 0$ and $P < 0$ we see that $G$ is relatively simple and respectively reminiscent of what happens for asymptotically dS and asymptotically flat (uncharged) black holes [43, 44]. However, this similarity is only superficial and the actual physical interpretation depends on the matter content of the theory, as we shall demonstrate below.

#### *2. Gauss–Bonnet gravity*

Carrying out the same analysis in Gauss–Bonnet gravity ($K = 2$) in $d = 5$ dimensions, the equation of state reads

$$P = \frac{3T}{4r_+} - \frac{3}{8\pi r_+^2} + \frac{3\alpha_2 T}{r_+^3}\,, \quad V = \frac{\Sigma_3 r_+^4}{4}\,, \tag{5}$$

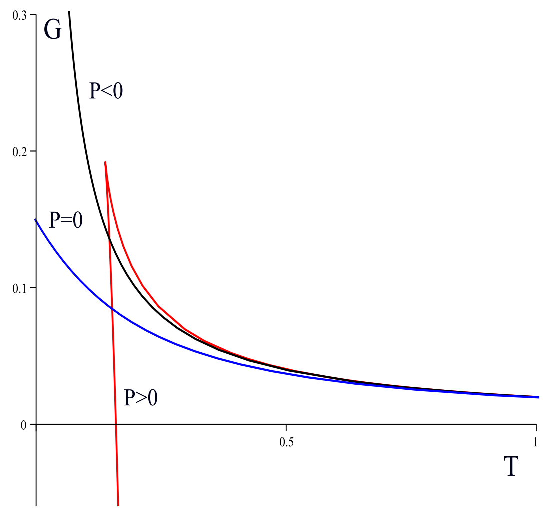


FIG. 1. **Horizon thermodynamics:** $d = 4$ **spherical Einstein black holes.** The $G-T$ diagram is displayed for $P = 0.03$ (red curve), $P = 0$ (black curve) and $P = -0.2$ (blue curve). For positive pressures we observe a characteristic shape reminiscent of the Hawking–Page behavior.

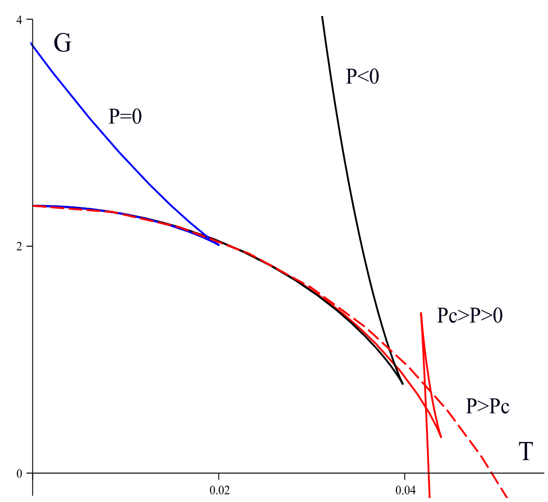


FIG. 2. **Horizon thermodynamics:** $d = 5$ **spherical Gauss–Bonnet black holes.** The $G-T$ diagram is displayed for $P = 0.01$ (red dash curve), $P = 0.0025$ (red solid curve), $P = 0$ (black curve), and $P = -0.05$ (blue curve) and $\alpha_2 = 1$. For small positive pressures we observe a characteristic swallow tail reminiscent of the Van der Waals-like phase transition.

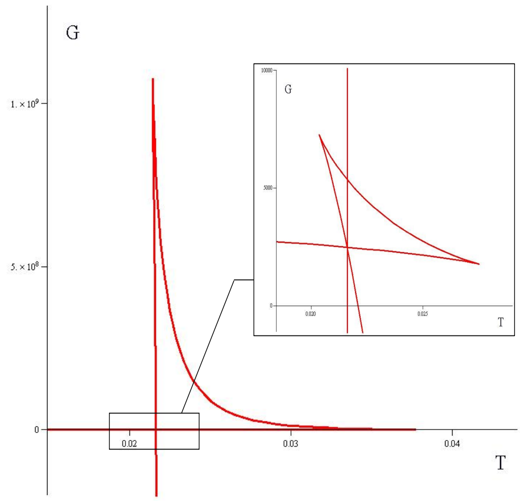


FIG. 3. **Horizon thermodynamics: triple point.** The $G-T$ diagram is displayed for a spherical black hole in the 4-th order Lovelock gravity for the following choice of parameters: $\alpha_2=0.2, \alpha_3=2.8, \alpha_4=1, P=0.000425$. We observe two swallow tails merging together, characterizing an existence of a triple point.

while the other quantities are

$$\begin{aligned} S &= \frac{\Sigma_3 r_+^3}{4}\left(1+\frac{12\alpha_2}{r_+^2}\right), \quad E=\frac{3\Sigma_3 r_+^2}{16\pi}\left(1+\frac{2\alpha_2}{r_+^2}\right), \\ G &= \frac{\Sigma_3[72\alpha_2^2-18r_+^2(1+8\pi r_+^2 P)\alpha_2+3r_+^4-4\pi P r_+^6]}{48\pi(r_+^2+4\alpha_2)}, \end{aligned} \tag{4}$$

and satisfy the horizon first law (III A).

The corresponding $G-T$ diagram for spherical ($\sigma=1$) black holes is displayed in Fig. 2. In contrast to the $K=1$ case, we now see that the additional gravitational non-linearity can yield more interesting phase behaviour. Namely, for sufficiently small positive pressures [40, 72]

$$0<P<P_c=\frac{1}{96\pi\alpha_2}, \tag{4}$$

we observe a characteristic swallow tail reminiscent of the Van der Waals-like phase transition for $d=4$ charged black holes in extended phase space [36], illustrated in Fig. 5. For $P>P_c$ the swallow tail disappears and the Gibbs free energy becomes smooth. On the other hand for $P=0$ and $P<0$ we observe a cusp (corresponding to a divergent specific heat) and the shape of $G=G(T)$ reminds that of the charged asymptotically dS and asymptotically flat black holes, c.f. [43, 44].

### 3. *General Lovelock gravities*

For $K>2$ we find further interesting phase behaviour. At each additional order in the Lovelock expansion, we gain an additional degree of freedom corresponding to the additional Lovelock coupling $\alpha_K$, allowing for more complex structures to arise. We find phenomena similar to those seen previously in extended phase space thermodynamics for $K=1$, such as reentrant phase transitions [37], double swallow tails and a corresponding triple point [39],and even (for $K>2$) isolated critical points [40, 41, 66]. However in contrast to the extended phase space approach, such behaviour in horizon thermodynamics is entirely due to the non-linearity of gravity (the larger values of $K$), fully independent of the matter distribution. We depict a triple point in 4-th order Lovelock gravity in Fig. 3.

It remains an interesting open question whether the horizon thermodynamics of higher-order Lovelock theories can bring some additional qualitatively new phase transitions to those described in this section. In particular, can one find '$n$-tuple swallow tails' and the corresponding $n$-tuple critical points? This question is left for future work.

### B. Comparison to extended thermodynamics with variable $\Lambda$

In this section we shall compare horizon thermodynamics to the recently studied (canonical ensemble) extended phase space thermodynamics of asymptotically AdS black holes. The latter, sometimes referred to as black hole chemistry [29], is essentially 'standard black hole thermodynamics' with the additional feature that the (negative) cosmological constant is treated as an additional thermodynamic variable, which is interpreted as a thermodynamic pressure $P_\Lambda$ according to Eq. (III B 1) and allowed to vary in the corresponding first law. The first law for spherically symmetric Lovelock black holes then takes the following form [88]:

$$\delta M = T\delta S + \sum_i \Phi_i \delta Q_i + V_{\text{TD}} \delta P_\Lambda + \sum_{k=2} \Psi^{(k)} \delta\alpha_k \,, \tag{4}$$

and implies the associated Smarr formula

$$\begin{aligned}(d-3)M = (d-2)TS + (d-3)\sum_i \Phi_i Q_i - 2V_{\text{TD}} P_\Lambda \\ + \sum_{k=2} 2(k-1)\Psi^{(k)}\alpha_k\end{aligned} \tag{4}$$

through the Euler scaling argument. Here $M$ stands for the black hole mass, now interpreted as a *gravitational enthalpy*. We have also included the possibility that the black holes are multiply-charged with several $U(1)$ charges $Q_i$ and corresponding electric potentials $\Phi_i$ (see Sec. V for an example with an STU black hole). The horizon temperature $T$ and associated entropy $S$ are the same as in the horizon thermodynamics approach.

Let us now study some differences between the HFL (III B 2) and the extended first law (IV B). The most obvious distinction is the appearance of extra work terms, $\sum_i \Phi_i \delta Q_i$, in (III B 2). These terms in the horizon case (III B 2) are instead interpreted as contributions to the pressure, which is associated with all matter fields. In the extended case (IV B) one only has a completely isotropic pressure due to the cosmological constant.

A more important difference between (III B 2) and (IV B) is the nature of the black hole volume. In the horizon approach $V$ is assumed to be given by (III B 1); it is associated with the 'Euclidean geometric volume' of the black hole and is independent of the matter content, c.f. [34, 84–86]. In contrast to this the volume in extended thermodynamics

$$V_{\text{TD}} = \Big(\frac{\partial M}{\partial P_\Lambda}\Big)_{S,Q_1,\dots} \tag{4}$$

is a *thermodynamic volume* [34], a quantity conjugate to the pressure $P_\Lambda$. Hence $V_{\text{TD}}$ is not an independent input but directly follows from the identification of the black hole mass. It can also depend on the matter content of the theory; for example the thermodynamic volumes of supergravity black holes have this feature [34].

Another important difference is the nature and distinction between the quantities $E$, $H$, and $M$. Whereas the latter is the black hole mass and can be calculated by standard methods, e.g. the method of conformal completion [90, 91], the physical meaning of $E$ is distinct. It evidently plays the role of energy in (III B 2), but this quantity is not the mass of black hole; indeed its properties are quite different. It vanishes for planar/toroidal black holes and can be negative for higher-genus topological/hyperbolic black holes. It has been noted that it is associated with the transverse geometry of the horizon [12].

Since it is a function only of the horizon curvature and the horizon radius $r_+$, we propose that it is the *horizon curvature energy*: the energy required to warp space time so that it embeds an horizon. This definition is analogous to that of the spatial curvature density in cosmology, which depends only on the curvature of spatial slices at constant time in an FRW cosmology. Likewise, the horizon enthalpy $H$ then can be interpreted as the energy required to both warp spacetime and displace its matter content so that a black hole can be created.

This physical interpretation is contingent upon the fact that the energy of horizon thermodynamics corresponds to the generalized Misner–Sharp mass $m_{\text{MS}} = m_{\text{MS}}(r)$ [27, 28]

$$m_{\text{MS}}(r_+) = P_\Lambda V + E = M \tag{4}$$

evaluated on the black hole horizon [19] and whose properties in Einstein gravity have been previously elaborated upon [7]. In this sense it is a quasi-local quantity that can be associated with the horizon itself without referral to asymptotics and can be independently defined. This indeed is a primary motivation of horizon thermodynamics. Furthermore, we note that the last equality in (IV B) follows from (IV B) (which holds for $P_m = 0$), and so we see that the mass of a Schwarzschild AdS black hole is the Misner–Sharp mass on the horizon. Setting $P_m \neq 0$, it has been shown that $m_{\text{MS}}(r_+)$ satisfies the HFL [7, 19].

In particular, using the Smarr relation (5), we find the following relation between $M$ and $H$:

$$M = H + \sum_i Q_i \Phi_i + \frac{2}{d-3}\big(VP - V_{\text{TD}} P_\Lambda\big) \tag{4}$$

valid for the charged AdS Lovelock black holes. For singly charged Lovelock black holes, $V = V_{\text{TD}}$ [40, 88] yielding

$$M = H + Q\Phi + \frac{2}{d-3} V P_m\,. \tag{4}$$

as the relationship between mass and horizon enthalpy $H$.

If no matter apart from a cosmological constant is present $P_m = 0$. $H$ and $M$ then represent the same quantities, and so

$$H = M = E + P_\Lambda V \tag{4}$$

which is the sum of the energy $E$ needed for warping the spacetime to embed the black hole horizon plus the energy $P_\Lambda V$ needed to place the black hole into a cosmological environment ('to displace the vacuum energy'). Note that for *planar black holes* $E$ vanishes and the mass is entirely given by the $P_\Lambda V$ term.

Criticality and possible phase transitions in the framework of extended phase space are governed by the associated Gibbs free energy

$$G_\Lambda = M - TS\,, \tag{4}$$

in comparison to the horizon Gibbs free energy $G$ (III B 2).

In particular, and obvious from the above discussion, in the vacuum with negative cosmological constant case we have the same expressions

$$G = G_\Lambda\,, \quad P = P_\Lambda\,, \tag{4}$$

for the Gibbs free energy and equation of state. Only in this case and for *positive* $P$ do the two approaches yield the same kind of thermodynamic behaviour and phase transitions (Van der Waals behaviour, reentrant transitions, triple points, isolated critical points) in any Lovelock theory. These phenomena will only take place for sufficiently large $K$ (sufficient gravitational non-linearity).

The two approaches differ significantly once matter is introduced. Generically they give rise to very distinct phase diagrams with completely different physical interpretations. The difference is rooted in the inherent degeneracy in horizon thermodynamics: it is described by only two parameters $T$ and $P$ (together with their conjugates). This degeneracy is removed in extended phase space thermodynamics, with each matter field having its own contribution to the free-energy, leading to a description in a different (often incompatible) thermodynamic ensemble. Furthermore, in horizon thermodynamics negative pressures are possible even if $\Lambda < 0$, whereas in the extended case negative pressure requires $\Lambda > 0$.

We shall now illustrate these distinctions for a spherical ($\sigma = 1$) charged-AdS black hole in $d = 4$ dimensions ($K = 1$)

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2 \,,$$
$$F = dA \,, \quad A = -\frac{Q}{r} dt \,, \tag{4}$$

where $d\Omega^2 = r^2(d\theta^2 + \sin^2\theta d\varphi^2)$,

$$f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{l^2} \,, \tag{4}$$

and $\Lambda = -\frac{3}{l^2}$ is the cosmological constant. This simple example will allow us to discuss all important differences without the need for complicated expressions; generalization to 'arbitrary' charged Lovelock black holes is straightforward [40].

The HES (4) now reads

$$P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2} \,, \quad V = \frac{4}{3}\pi r_+^3 \,, \tag{4}$$

upon setting $\sigma = 1$ in (IV A 1). Interestingly, using the expression for the energy-momentum tensor,

$$P_m = T^r{}_r = -\frac{Q^2}{8\pi r_+^4} \,, \tag{4}$$

the HES (IV B) can be rewritten as

$$P_\Lambda = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2} - P_m = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2} + \frac{Q^2}{8\pi r_+^4} \,, \tag{4}$$

which is the extended phase space equation of state in the canonical ensemble [36] upon setting $Q$ constant and identifying $P_\Lambda = -\Lambda/(8\pi)$. Note that $V_\Lambda = V$ and so the thermodynamic and geometric volumes are the same and

$$P_\Lambda = P + \frac{Q^2}{8\pi r_+^4} \,, \tag{4}$$

since $P = P_m + P_\Lambda$.

Note that in the extended phase space approach there is no need to 'invoke the Einstein equations' to derive this equation of state since we are using a concrete solution. In fact (IV B) simply follows from the 'definition' of the temperature

$$T = \frac{f'}{4\pi} \,, \tag{4}$$

upon using the explicit form of $f$ from (IV B). The horizon enthalpy

$$H = \frac{r_+(1 + 2\pi T r_+)}{3} \,, \tag{5}$$

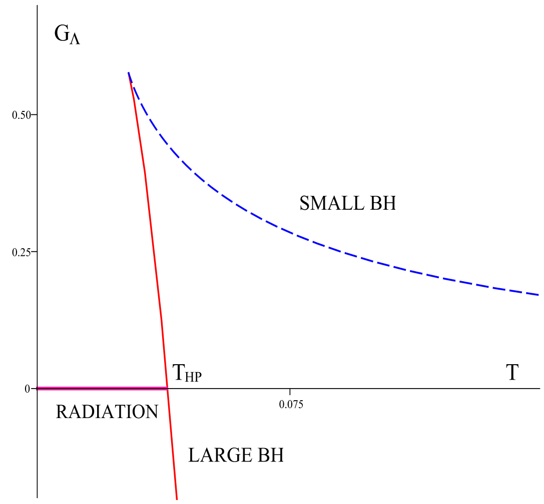


FIG. 4. **Hawking–Page transition.** The characteristic $G_\Lambda - T$ diagram is displayed for the uncharged ($Q = 0$) AdS spherical black hole in $d = 4$. The black hole Gibbs free energy admits two branches of black holes: small black holes (displayed by blue dashed curve) have negative specific heat and are thermodynamically unstable while large black holes (solid red curve) have positive specific heat and thermodynamically dominate for large temperatures, $T > T_{\rm HP}$, over the radiation phase displayed by horizontal magenta line. Note that (being in the framework of extended phase space thermodynamics) each point on the black hole curve corresponds to different black holes (of increasing horizon radius $r_+$ from right on the dashed blue curve to bottom left) in the same environment of fixed $\Lambda$ and fixed $Q = 0$.

and mass (gravitational enthalpy)

$$M = \frac{r_+^2 l^2 + Q^2 l^2 + r_+^4}{2 l^2 r_+}\,, \tag{5}$$

of the black hole are related via (IV B), $M = H + \Phi Q + 2 V P_m$ , where $\Phi = Q/r_+$, and $P_m$ and $V$ are given by (IV B) and (IV B). This then implies the following relation:

$$\begin{aligned} G_\Lambda &= G + \Phi Q + 2 V P_m = G + \frac{2}{3}\frac{Q^2}{r_+}\,, \\ P_\Lambda &= P + \frac{Q^2}{8\pi r_+^4}\,, \end{aligned} \tag{5}$$

between the horizon and extended Gibbs free energies.[1]

These relations imply fundamentally different thermodynamic behaviour in the two approaches. Even after removing the degeneracy in (IV B) by imposing a constant $Q$ constraint, the $P = const$ and $P_\Lambda = const$ slices of thermodynamic phase space are incompatible, and yield different behaviour of the Gibbs free energies $G(T)$ and $G_\Lambda(T)$. We shall illustrate this point by comparing the positive pressure curve in Fig. 1 describing the behaviour of $G$ in horizon thermodynamics to that of $G_\Lambda$ displaying the Hawking–Page transition for $Q = 0$ and the Van der Waals like behavior for $Q \neq 0$ in the extended phase space thermodynamics, Fig. 4 and Fig. 5.

In horizon thermodynamics the description is in terms only of $\{T, P\}$, and only 'Hawking–Page-like behavior' of the horizon Gibbs free energy $G = G(P, T)$ can be observed, as shown in Fig. 1.

[1] Note that the extended phase space equation of state (IV B) was directly derived from the horizon equation of state (IV A 1) by splitting $P = P_m + P_\Lambda$,. This is not true for the Gibbs free energy $G_\Lambda$.

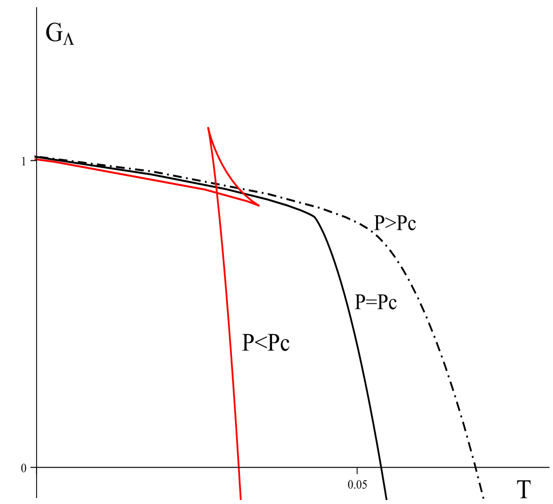


FIG. 5. **Van der Waals-like phase transition.** The characteristic $G_\Lambda - T$ diagram is displayed for the charged ($Q = 1$) AdS spherical black hole in $d = 4$. For sufficiently small pressures, $P < P_c = 1/[96\pi Q^2]$, the $G_\Lambda - T$ diagram displays the characteristic swallow tail behaviour indicating a small to large black hole phase transition ala Van der Waals. As with Fig. 4 , each point on the curve corresponds to different black holes (of increasing horizon radius $r_+$ from left to bottom right) in the same environment of fixed $\Lambda$ and $Q$.

Furthermore, as $T$ changes, moving along a constant-$P$ curve entails modifying some combination of $Q$, $r_+$, and $\Lambda$: different points on the curve are comparing different black holes in *different environments*[2]. The expected transition at $G = 0$ to pure radiation (which has $Q = 0$) can only occur if there is a reservoir of charge, so that $Q$ can appropriately vanish as this transition takes place.

In other words, the physical interpretation of Fig. 1 in horizon thermodynamics depends crucially on the matter content. In contrast to this, the extended phase-space picture breaks this degeneracy, allowing for imposition of independent constraints on $Q$ and the pressure $P_\Lambda$. If $Q = 0$ (Fig. 4) the standard Hawking–Page phase transition is recovered [29], whereas for fixed $Q \neq 0$ (Fig. 5), Van der Waals-like behaviour is observed [36], with the Gibbs free energy $G_\Lambda = G_\Lambda(P_\Lambda, T, Q)$ exhibiting a swallowtail structure. In either case, each point on the curve in a $G_\Lambda$ vs. $T$ diagram corresponds to different black holes in the *same environment* (the same $\Lambda$ and $Q$).

We see that the distinction between the two approaches in this example is reminiscent of the canonical vs. grand-canonical description of charged AdS black holes. For a charged AdS black hole we observe Van der Waals phase transitions only in a canonical (fixed $Q$) ensemble (as in the extended phase space approach), whereas in the grand canonical (fixed $\Phi$) ensemble behaviour similar to Fig. 1 is observed (as in horizon thermodynamics).

In summary, horizon thermodynamics describes a system from the viewpoint of an ensemble described by only two variables $P$ and $T$. The Gibbs free energy therefore only depends on the type of gravity considered. Such a description is 'universal' and 'formally independent' of the matter content. However, the actual interpretation of the thermodynamic behaviour is matter dependent. In general it is not unique due to the degeneracy of the description, in contrast to the non-degenerate description in extended phase space thermodynamics. Consequently in horizon thermodynamics the ensemble is very different from traditional ensembles in standard thermodynamics. The distinguishing feature is that the total pressure $P$ is held fixed. All pressures are summed over to yield this total pressure, and

[2] Since constant-$P$ undetermined condition, its realization can be always achieved by setting $Q = 0$ and tuning $\Lambda$ accordingly. For this reason it is not that surprising that the horizon Gibbs free energy mimics the $Q = 0$ behavior of the extended phase space Gibbs free energy.

in general this renders the ensemble different from both the canonical and grand-canonical ensembles that are usually considered in black hole thermodynamics.

## V. BEYOND HORIZON THERMODYNAMICS

In this section we provide some open topics for consideration in horizon thermodynamics. Namely, we concentrate on extending the principals of horizon thermodynamics to more general spacetimes with fewer spacetime symmetries. As we can see, the procedure for generating first laws becomes more ambiguous and the interpretation of the equation of state can be more questionable.

### A. Thermodynamics of General Spherically Symmetric Black Holes

Consider the most general spherically symmetric metric in 4 dimensions,

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + g(r)^2 d\Omega^2\,, \tag{5}$$

an ansatz that is able to accommodate black holes with more exotic matter content. The radial Einstein equation evaluated on the horizon reads

$$8\pi T^r{}_r|_{r_+} = G^r{}_r|_{r_+} = \frac{f'(r_+)g'(r_+)}{g(r_+)} - \frac{1-f(r_+)(g'(r_+))^2}{(g(r_+))^2}\,. \tag{5}$$

which, since we define the horizon to be at $f(r_+) = 0$ yields

$$P = \frac{T}{2}\frac{g'(r_+)}{g(r_+)} - \frac{1}{8\pi(g(r_+))^2}\,. \tag{5}$$

Examples of such a metric include the STU black hole, where the metric of the STU black hole has the form (V A), where [92]

$$f(r) = \frac{1 - \frac{2m}{r} + \frac{r^2}{\ell^2}H(r)}{\sqrt{H(r)}}\,, \qquad H(r) = \prod_{j=1}^{4}(1 + q_j/r)\,. \tag{5}$$

This new equation of state is now of the form

$$P(T, g, g') = B(g) + g'C(g)T\,. \tag{5}$$

Varying this new equation of state with no a-priori restrictions on $g'$, yields

$$\delta P = \left[\left(\frac{dB}{dg} + g'T\left(\frac{dC}{\delta g}\right)\right)\right]\delta g + g'C\delta T + CT\delta g'\,. \tag{5}$$

Assuming a volume $V(g)$, we can re-write this as

$$V\delta P = g'S\delta T + dG + TS\delta g'\,, \tag{5}$$

where we define

$$S \equiv \int^g C(g)\left(\frac{dV}{dg}\right)dg\,, \tag{5}$$

as derived in [74]. We enforce this relation in part because it provides the correct entropy [34] even for the general STU black hole.

There are two serious problems with the proposed first law (V A); firstly the final anomalous term going as the variation of $g'$ has no clear thermodynamic interpretation. With this in mind, it may be hard to even justify calling $G$ a Gibbs free energy unless there is some feasible thermodynamic quantity that this term can be related to.

The second issue is that the first law (V A) does not obey a Smarr relation.

Let us instead define a new temperature, shifted from the Hawking temperature by a dimensionless factor which depends on the matter content of the spacetime,

$$T_S = g' T \,. \tag{5}$$

In terms of this new “temperature”, our equation of state then takes the form

$$P(T_S, g) = \frac{T_S}{2g(r_+)} - \frac{1}{8\pi (g(r_+))^2} \,. \tag{5}$$

Varying the equation of state and taking volume to be a geometric volume

$$V(g) = \int 4\pi g^2 dg \,, \tag{5}$$

we reach a first law of the form

$$\delta G = V \delta P - S \delta T_S \tag{5}$$

with

$$S = \int^g \frac{\left(\frac{dV}{dg}\right)}{2g} dg = \pi g^2 \,. \tag{5}$$

Note that this enforces the degeneracy between entropy and volume observed in [73]. We find the free energy to be

$$G = \int^g \frac{V(g)}{4\pi g^3} dg - T_S \int^g \frac{V(g)}{2g^2} dg \,. \tag{5}$$

From here, it is straightforward to see that in 4D Einstein gravity, for the equation of state (V A) these definitions satisfy the Smarr relation,

$$G = ST - 2VP \,. \tag{5}$$

For the STU black hole, the shifted temperature evaluates to

$$T_S = \left[ \prod_{i=1}^{4} (1 + \frac{q_i}{r})^{1/4} - \frac{1}{4r} \sum_{i=1}^{4} \frac{q_i \prod_{j \neq i} (1 + \frac{q_j}{r})^{1/4}}{\left(1 + \frac{q_i}{r}\right)^{3/4}} \right] T \,. \tag{5}$$

More interestingly, the entropy evaluates to

$$S = \pi g^2 = \pi r^2 \prod_{i=1}^{4} (1 + \frac{q_i}{r})^{1/4} \,, \tag{5}$$

which is exactly the expected value for the horizon entropy of an STU black hole [34].

### B. Rotating horizon thermodynamics

Another interesting example where complications arise in horizon thermodynamics is black holes with non-zero angular momentum. We work here with the following ansatz for a rotating black hole geometry

$$ds^2 = -\frac{\Delta}{\rho^2} \left(dt - a \sin^2\theta d\varphi\right)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2} \left[a dt - (r^2 + a^2) d\varphi\right]^2 \,, \tag{6}$$

generalizing the Kerr metric, where

$$\rho^2 = r^2 + a^2 \cos^2\theta \,, \tag{6}$$

and we assume that the metric function $\Delta = \Delta(r)$ determines the position of the (non-extremal) black hole horizon located at the largest root of $\Delta(r_+) = 0$.

We begin by deriving the modified HFL and HES (II C) and (III A), assuming Einstein gravity minimally coupled to matter. From the geometry we can immediately identify the black hole horizon area

$$A = 4\pi(r_+^2 + a^2) = 4S \,, \tag{6}$$

in terms of the entropy $S$. The horizon angular velocity

$$\Omega = -\frac{g_{t\varphi}}{g_{\varphi\varphi}}\bigg|_{r_+} = \frac{a}{r_+^2 + a^2} \tag{6}$$

can likewise be identified, as can the black hole temperature

$$T = \frac{\Delta'(r_+)}{4\pi(r_+^2 + a^2)} \,, \tag{6}$$

via standard Wick-rotation arguments. Note that no field equations are required up to this point, though the latter relation in (V B) employs the assumption of Einstein gravity.

Let us next consider the radial Einstein equation, evaluated on the black hole horizon

$$8\pi T^r{}_r|_{r_+} = G^r{}_r|_{r_+} = \frac{a^2 - r_+^2 + r_+\Delta'(r_+)}{\rho_+^4} \,, \tag{6}$$

where $\rho_+^2 = r_+^2 + a^2\cos^2\theta$. Using (V B) we obtain

$$T = \frac{8\pi\rho_+^4 T^r{}_r|_{r_+} + r_+^2 - a^2}{4\pi r_+(r_+^2 + a^2)} \,, \tag{6}$$

which yields

$$T\delta S = \frac{2\rho_+^4 T^r{}_r|_{r_+}}{r_+(r_+^2 + a^2)}\delta S + \frac{r_+^2 - a^2}{4\pi r_+(r_+^2 + a^2)}\delta S \tag{6}$$

upon multiplication by $\delta S = 2\pi(r_+\delta r_+ + a\delta a)$. Note that the first term on the right-hand-side of (V B) depends on the matter content, whereas the second term is universal and completely fixed in terms of $r_+$ and $a$.

Now we make the following interesting observation. This latter term in (V B) can be written as

$$\frac{r_+^2 - a^2}{4\pi r_+(r_+^2 + a^2)}\delta S = \delta E - \Omega\delta J \,, \tag{6}$$

upon defining

$$E = \frac{r_+^2 + a^2}{2r_+} \,, \quad J = Ea \,, \tag{6}$$

the former quantity being defined only up to a total variation. We see that the expressions for $E$ and $J$ are formally identical to those for the mass and angular momentum of vacuum Kerr black hole, respectively. Furthermore, in the absence of rotation, $a \to 0$, $E$ reduces to the Misner–Sharp energy of a spherically symmetric spacetime evaluated on the black hole horizon. We therefore identify $E$ as the *horizon energy* $E$ and $J$ as the *horizon angular momentum* of the black hole described by (6).

We have thus found the following relation

$$\delta E = T\delta S + \Omega\delta J - \frac{2\rho_+^4 T^r{}_r|_{r_+}}{r_+(r_+^2+a^2)}\delta S\,. \tag{6}$$

Since $T$ must be constant on the horizon [93], the last equation is consistent only when $\rho_+^4 T^r{}_r|_{r_+}$ is independent of $\theta$. We therefore introduce the surface tension

$$\tau = \tau(r_+, a) = \frac{\rho_+^4 T^r{}_r|_{r_+}}{2r_+(r_+^2+a^2)}\,, \tag{6}$$

and so obtain (III A) for the modified HFL[3]

$$\delta E = T\delta S + \Omega\delta J - \tau\delta A\,. \tag{8}$$

Such a law is cohomogeneity-2 as both the horizon radius $r_+$ and the rotation parameter $a$ can vary independently. Moreover, Eq. (V B) together with (V B) yields

$$\tau = \tau(A, J, T) = \frac{T}{4} + \frac{a^2 - r_+^2}{16\pi r_+(r_+^2+a^2)} \tag{8}$$

which is the surface tension HES (II C). Here $r_+$ and $a$ are implicitly given in terms of $J$ and $A$ through relations (V B) and (V B).

Equations (V B) and (V B) are together with the definition of the surface tension (V B) the most important results of this section. Note that in order to write these equations down, no new quantities, apart from $E$ and $J$, had to be defined and the expressions are entirely given in terms of geometric horizon properties such as the area $A$, temperature $T$, and angular velocity $\Omega$.

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#### 1. *Surface tension criticality*

Let us now study the possible critical behavior associated with the generalized horizon thermodynamics derived in the previous subsection. For concreteness, we do this in a canonical (fixed $J$) ensemble.

Since according to the HFL (V B), the quantity $E$ in (V B) plays the role of thermodynamic energy (that is a thermodynamic potential expressed in terms of extensive thermodynamic variables $S, J$ and $A$), we define

$$G_\tau = G_\tau(T, \tau, J) = E - TS + \tau A\,, \tag{8}$$

which is the corresponding surface tension Gibbs free energy $G_\tau$. This quantity formally satisfies

$$\delta G = -S\delta T + \Omega\delta J + A\delta\tau\,. \tag{8}$$

The behavior of $G = G(T, \tau, J)$ is displayed in Fig. 6 for fixed $J = 1$ and three representative values of $\tau$. For any $\tau$ we observe two branches of black holes, meeting at a characteristic cusp. For negative

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[3] Note that in vacuum $\tau = 0$ and we recover the standard 1st law of black hole thermodynamics

$$\delta E = T\delta S + \Omega\delta J\,. \tag{7}$$

In the electrovacuum (Kerr–Newman) case, we have

$$T^r{}_r|_{r_+} = -\frac{Q^2}{8\pi\rho_+^4} \quad\Rightarrow\quad \tau = -\frac{Q^2}{16\pi r_+(r_+^2+a^2)}\,. \tag{8}$$

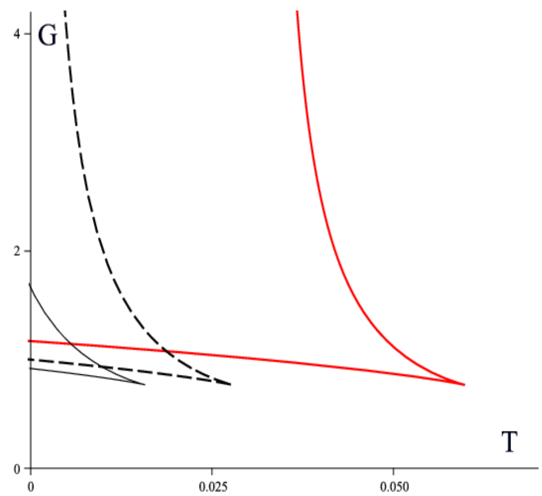


FIG. 6. $\tau - A$ **criticality.** The $G_\tau - T$ diagram is displayed for $J = 1$ and various $\tau$. The red curve corresponds to positive tension $\tau = 0.008$, the dashed black curve to $\tau = 0$, and the thin black curve to negative tension $\tau = -0.003$. We observe a characteristic cusp whose position depends on $\tau$. For positive $\tau$, the corresponding upper branch terminates at finite temperature $T$.

$\tau$ both branches on the other end terminate at finite $G$ and $T = 0$, whereas for positive $\tau$ the upper branch eventually asymptotes to $G \to \infty$ at $T = 4\tau$, with a divergence at $T = 0$ occurring for $\tau = 0$. Apart from the presence of a cusp, no interesting thermodynamic behaviour is observed for any values of $J$.

As with the spherically symmetric case [73], an interpretation of the concrete thermodynamic behaviour depends on the actual matter content. For example, in vacuum, $\tau = 0$ and only the black dashed curve applies. Similarly, for the electrovacuum case with nontrivial charge $\tau < 0$ and behavior similar to the thin black curve in Fig. 6 is realized. We expect that our ansatz could be suitably generalized to accommodate rotating black hole with some type of a scalar hair [94], with free-energy plots similar to the the positive $\tau$ curve.

### 2. *Effective temperature*

The modified HFL (V B) has three terms on its right-hand-side but inherently is only cohomogeneity-2. Furthermore, variation of $S$ is not independent of the variation of $A$. This suggests that we introduce an *effective temperature*

$$T_{\text{eff}} = T - 4\tau = \frac{1}{4\pi}\frac{r_+^2 - a^2}{r_+(r_+^2 + a^2)}\,, \tag{8}$$

which is easily obtained by grouping the $T\delta S$ and $-\tau\delta A$ terms together. Note that this quantity has no explicit dependence on matter, is constant on the horizon, and is positive for $r_+ > a$. With this identification, the modified HFL (V B) becomes manifestly of cohomogeneity two and reads

$$\delta E = T_{\text{eff}}\delta S + \Omega\delta J\,, \tag{8}$$

which is equivalent to Eq. (V B). In fact, since $E$ and $J$ coincide with the mass and angular momentum of the vacuum Kerr black hole, the effective temperature $T_{\text{eff}}$ is nothing other than the temperature of the Kerr solution and (V B 2) is the corresponding first law.

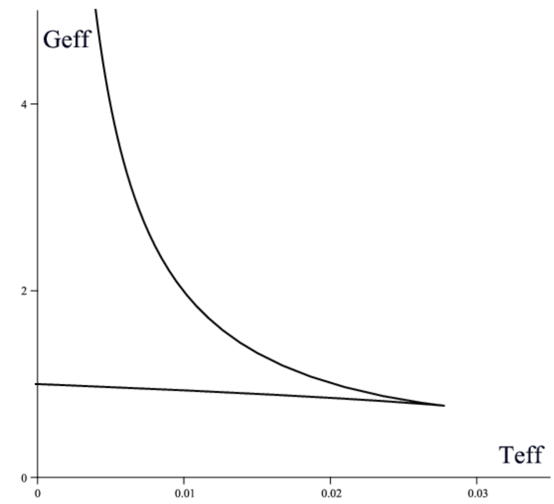


FIG. 7. **Universal criticality.** The $G_{T_{\rm eff}} - T_{\rm eff}$ phase diagram is displayed for $J = 1$. We observe a characteristic cusp that is completely independent of the matter content of the theory.

Stated this way, horizon thermodynamics is recast in universal form that is completely independent of the matter content and represented by the thermodynamics of a vacuum solution. Note that the same is true in the case of spherical symmetry upon absorbing the $-PdV$ term into $TdS$ in (V B), which then simply reads $\delta E = T_{\rm eff}\delta S$, with $T_{\rm eff}$ being the temperature of the Schwarzschild black hole. This interpretation of horizon thermodynamics also opens a new way of deriving the horizon equations (V B) and (V B), as we demonstrate in App. A.

In the light of previous discussion, it is obvious that the criticality of the HFL (V B 2) coincides with that of the Kerr solution. Namely, the associated Gibbs free energy reads

$$G_{T_{\rm eff}} = G_{T_{\rm eff}}(T_{\rm eff}, J) = E - T_{\rm eff}S = \frac{r_+^2 + 3a^2}{4r_+}\,, \tag{8}$$

and obeys

$$\delta G_{T_{\rm eff}} = -S\delta T_{\rm eff} + \Omega\delta J\,. \tag{8}$$

The corresponding $G_{T_{\rm eff}} = G_{T_{\rm eff}}(T_{\rm eff}, J)$ diagram is displayed in Fig. 7. For non-trivial angular momentum $J$, we observe a characteristic cusp, completely independent of the matter content of the theory.

To summarize this section, we stress that both the surface tension and the effective temperature approaches are very natural in the horizon thermodynamics of rotating black holes. Both permit study of cohomogeneity-2 HFLs since variations of both $\delta a$ and $\delta r_+$ are allowed. Furthermore, there is no need to identify any extra structure beyond the horizon energy $E$ and angular momentum $J$ in (V B). We shall now consider an alternate approach in which an additional structure, the black hole volume $V$, is defined.

### C. Tautological derivation of horizon equations

Although precisely in the spirit of horizon thermodynamics [9] generalized to the rotating case, the derivation of the horizon equations (V B) and (V B) in the main text suffers from non-uniqueness of the definition of horizon energy $E$ and horizon angular momentum $J$, (V B). Although their definition is motivated by (V B), the possibility of redefining $E$ by a total derivative (accompanied by a proper modification of $J$) remains. For this reason in this appendix we give an alternate derivation of these equations, turning around the logic of the reasoning. Namely, we start again with the ansatz (6) but consider the vacuum solution first.[4] This allows us to identify $E$ and $J$. We then carry the analysis to the non-vacuum case, keeping the same $E$ and $J$ to rederive Eqs. (V B) and (V B) in a different fashion

Let us start again with the ansatz (6) and specify to the vacuum case, setting $\Delta = r^2 - 2mr + a^2$. The thermodynamic quantities then read

$$\begin{aligned} E &= m = \frac{r_+^2 + a^2}{2r_+}\,, \quad J = Ea\,, \\ \Omega &= \frac{a}{a^2 + r_+^2}\,, \quad S = \frac{A}{4} = \pi(r_+^2 + a^2)\,, \\ T_0 &= T_{\text{eff}} = \frac{r_+^2 - a^2}{4\pi r_+(r_+^2 + a^2)}\,, \end{aligned} \tag{7}$$

and obey the standard first law

$$\delta E = T_0 \delta S + \Omega \delta J\,, \tag{7}$$

which is of course identical to the effective first law (V B 2).

We next consider the spacetime with matter, keeping the same ansatz (6) and general $\Delta = \Delta(r)$ that determines the position of the horizon. The derivation of the horizon equations (V B) and (V B) then consists of the following 4 steps:

- We insist that even in the presence of mater the horizon energy $E$ and the horizon angular momentum $J$ are given by the vacuum expressions (9). (This in some sense directly generalizes the idea of Misner–Sharp quantities to the case with rotation.)
- We employ the Euclidean trick to identify the actual temperature of the black hole horizon according to

$$T = \frac{\Delta'(r_+)}{4\pi(r_+^2 + a^2)}\,. \tag{7}$$

- We impose the radial Einstein equation evaluated on the horizon, to relate $T$ and $T_0$,

$$8\pi T^r{}_r|_{r_+} = G^r{}_r|_{r_+} = \frac{a^2 - r_+^2 + r_+\Delta'(r_+)}{\rho_+^4} \tag{7}$$

  which rewrites, upon using (9), as

$$T = T_0 + 4\tau\,, \quad \tau \equiv \frac{8\pi\rho_+^4 T^r{}_r|_{r=r_+}}{4\pi r_+(r_+^2 + a^2)}\,. \tag{7}$$

  So we identified the matter contribution to the temperature called surface tension $\tau$ in the main text, and recovered the HES (V B).

- The final step is to rewrite the standard first law (V C) in terms of the actual temperature in the presence of matter,

$$\delta E = T_0 \delta S + \Omega \delta J = T \delta S + \Omega \delta J - \tau \delta A\,, \tag{7}$$

  which is the HFL (III A).

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[4] This goes directly against the spirit of horizon thermodynamics that essentially tries to avoid working with concrete solutions of field equations.

We believe that this derivation in some sense reveals the true nature of horizon thermodynamics. It describes the standard vacuum first law from a perspective of an observer who measures the actual black hole temperature $T$ and the surface tension $\tau$ associated with matter fields present in the spacetime. This is the origin of universality of horizon thermodynamics: all black holes satisfy 'an equivalence class' of first laws (V C) irrespective of the matter content of the theory. Specific features of a given black hole emerge only after the actual matter content and associated conserved charges are identified, along with their respective contributions to the first law.

Of course, exactly the same derivation would apply to the spherically symmetric case.

## VI. SUMMARY

This thesis has taken the assumption that the radial gravitational field equations can be interpreted as an equation of state $P \equiv T^r_r|_{r_+} = P(V,T)$ and explored the thermodynamic consequences. It is important to note that nowhere do we evaluate the legitimacy for this assumption, but instead simply explore what must follows if this analogy does in fact hold. Through the studies presented here, we found that while perhaps not as robust as could be desired, horizon thermodynamics does produce a rich spectrum of thermodynamics with perhaps more predictive capacity than would naively be assumed.

Firstly, we have found that this assumption is more potent than previously assumed and have formulated *enhanced horizon thermodynamics*, a full cohomogeneity first law directly from the equation of state which provides an independent definition of horizon entropy and Gibbs free energy up to a total derivative. This is a significant improvement over traditional horizon thermodynamics, where entropy must be identified with the Wald entropy (discussed in Appendix A) and the standard first law is degenerate. The horizon entropy is independent from Wald's definition of entropy as a Noether charge, and follows purely from the assumption of field equation as an equation of state. Surprisingly however, this entropy is consistent with the Wald value in several non-trivial cases. The enhanced horizon thermodynamic entropy matches the Wald entropy for spherically symmetric black holes in Einstein gravity, higher curvature Lovelock gravities, and even cubic gravity theories where the entropy has intrinsic temperature dependence. The later of these is a unique feature of enhanced horizon thermodynamics, with the degenerate standard horizon thermodynamic prescription unable to unambiguously identify the entropy and energy. In the case of Lovelock gravity, enhanced horizon thermodynamics also provides the Lovelock potentials as a natural consequence of the equation of state. In any case, we found that horizon thermodynamics results in surprisingly similar pictures to the seemingly distinct conserved charge approach. The precise reason for these correspondences may perhaps be an interesting topic for further study.

We found that horizon thermodynamics also provides a full spectrum of thermodynamic behaviour analogous to a Van der Waals fluid when examined in an extended phase space picture. While there are some appealing features of thermodynamics from this approach, namely the simplicity of calculation and the elegance of being able to interpret the field equations as an equation of state, this comes at the cost of abandoning the concept of conserved charges being fundamental to thermodynamics. However, by construction, holding fixed thermodynamic variables in this approach does not fix the conserved charges of the spacetime, resulting in possibly questionable interpretations of phase diagrams.

Finally, we examined the consequences to extending these ideas to more general spacetimes. We found that new ambiguities arise which can be difficult to reconcile without a-priori knowledge of the thermodynamics of the spacetime. Perhaps there is a more general form of horizon thermodynamics, which reduces to the form studied here in the case of sufficient spacetime symmetry. Indeed, it seems reasonable to expect that the pressure would no longer simply be the radial component of the stress-energy in the absence of spherical symmetry. Or perhaps the relations found here are purely a consequence of the limited degrees of freedom in the spacetimes studied; it must be noted that for all of the cases which horizon thermodynamics naturally applied the radial field equation was sufficient to fully determine the metric. We leave these questions as a topic for future research.

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## Appendix A: Entropy as a Noether Charge

The notion of horizon entropy as a conserved charge has been referenced several times throughout this thesis. The calculation by Wald [95] gives entropy as the integral of the Noether charge associated diffeomorphism covariance. We sketch the argument here, while a more complete derivation can be found in [87, 95, 96] for example.

Jacobson and Myers similarly found the entropy of black holes in Lovelock gravity to be the integral of the Noether charge associated with diffeomorphism covariance [97].

This derivation relies on the calculus of forms. For a review, see [98] . For a $D$-dimensional space, we define the $n$-codimensional volume form

$$\epsilon_{\ \mu_1\ldots\mu_n,\nu_{n+1}} \equiv \frac{\sqrt{g}}{(D-n)!}\epsilon_{\mu_1\ldots\mu_n,\nu_{n+1}\ldots\nu_D} dx^{\nu_{n+1}}\wedge\ldots\wedge dx^{\nu_D}\,. \tag{A0}$$

for a spacetime with metric $g^{\mu\nu}$ described by a theory whos Lagrangian is described as a fuction of the metric and the Riemann tensor $R^{\mu\nu\rho\sigma}$.

We can write the Lagrangian as a $D$-form as

$$\mathbf{L} = \mathcal{L}\epsilon\,. \tag{A0}$$

We vary this action with respect to the metric to find

$$\delta\mathbf{L} = \epsilon\mathcal{E}^{\mu\nu}\delta g_{\mu\nu} + d\mathbf{\Theta}(g,\delta g)\,, \tag{A0}$$

where $\mathbf{\Theta}$ is a boundary term arising from integration by parts so that the Lagrangian is extremized for the field equations given by $\mathcal{E}^{\mu\nu}=0$.

The symplectic current form is defined as a function of the symplectic potential $\mathbf{\Theta}$ as

$$\omega(g,\delta_1 g,\delta_2 g) = \delta_1\mathbf{\Theta}(g,\delta_2 g) - \delta_2\mathbf{\Theta}(g,\delta_1 g)\,. \tag{A0}$$

From Eq. (A) we see that

$$d\omega = -\delta_1(\epsilon\mathcal{E}^{\mu\nu})\delta_2 g_{\mu\nu} + (1\leftrightarrow 2)\,. \tag{A0}$$

Recalling that the equations of motion for the theory require $\mathcal{E}^{\mu\nu}=0$, this implies that for systems which satisfy the equations of motion, this current form is closed, i.e. $d\omega=0$. This allows us to invoke Stokes theorem and assert that the integral of $\omega$ over a compact Cauchy surface is a conserved quantity,

$$\Omega(g,\delta_1 g,\delta_2 g) \equiv \int_C \omega(g,\delta_1 g,\delta_2 g)\,. \tag{A0}$$

Now we may invoke the symmetry under diffeomorphisms of the theory to define a Noether current for a vector field $\xi$

$$\mathbf{J}_\xi = \mathbf{\Theta}(g,\mathcal{L}_\xi g) - \xi\cdot\mathbf{L}\,. \tag{A0}$$

The Lagrangian of a diffeomorphism invariant theory varries as

$$\delta_\xi\mathbf{L} = \mathcal{L}_\xi\mathbf{L} = \xi\cdot d\mathbf{L} + d(\xi\cdot\mathbf{L})\,, \tag{A0}$$

by Cartan's formula. Then, again using Eq. (A), we can write the exterior derivative of $\mathbf{J}_\xi$,

$$d\mathbf{J}_\xi = -\epsilon\mathcal{E}^{\mu\nu}\mathcal{L}_\xi g_{\mu\nu}\,, \tag{A0}$$

which we note vanishes when the field equations of the theory are satisfied. It has been shown that we write this current as the derivative of a Noether charge $(D-2)$-form,

$$\mathbf{J}_\xi = d\mathbf{Q}_\xi + \xi^\nu\mathbf{C}_\nu\,, \tag{A0}$$

where $\mathbf{C}_\nu = 2\epsilon_\mu \mathcal{E}^\mu_\nu$. Note that when the field equations are satisfied, the current can be expressed as an exterior derivative $\mathbf{J} = d\mathbf{Q}$.

Then, if there exists infinitesimal time translations $t^\alpha$ and rotations $\varphi^\alpha$ in the spacetime, the cannonical energies and angular momenta are defined

$$\delta E = \int_\infty (\delta \mathbf{Q}[t] - t \cdot \mathbf{\Theta}) \,, \tag{A0}$$

and

$$\delta J = \int_\infty \delta \mathbf{Q}[\varphi] \,. \tag{A0}$$

If one takes $\xi^\alpha$ to be a Killing field which vanishes on some (D-2) Killing surface $\Sigma$, normalized such that

$$\xi^\alpha = t^\alpha + \Omega \varphi^\alpha \,, \tag{A0}$$

where $\Omega$ is the angular velocity of the surface $\Sigma$, then $\mathcal{L}_\xi g = 0$, $d(\delta \mathbf{Q}) = d(\xi \cdot \mathbf{\Theta})$ and as a result

$$\delta \int_\Sigma \mathbf{Q} = \delta E - \Omega dJ \,, \tag{A0}$$

which can be interpreted as a first law of thermodynamics.

This can be improved upon by identifying

$$\delta \mathbf{Q} = \kappa \delta \tilde{\mathbf{Q}} \,, \tag{A0}$$

for surface gravity $\kappa$ defined

$$\xi^\alpha \nabla_\alpha \xi^\beta = \kappa \xi^\beta \tag{A0}$$

via the algorithm presented in [99].

Then, defining the conserved charge

$$S = 2\pi \int_\Sigma \tilde{\mathbf{Q}} \,, \tag{A0}$$

the first law reads

$$\frac{\kappa}{2\pi} \delta S = \delta E - \Omega \delta J \,, \tag{A0}$$

as desired.

## Appendix B: Specific Examples

Now that we have reviewed the formalism behind the Wald entropy, we demonstrate it explicitly in a couple of cases of particular relevance to this thesis–first in General Relativity then in Lovelock Gravity (see [95], [97] for more detail).

### 1. General Relativity

For calculational convenience, define

$$P^{\mu\nu\rho\sigma} = \frac{\partial L}{\partial R_{\mu\nu\rho\sigma}} \,, \tag{B0}$$

for a theory of gravity with Lagrangian $L$. In Einstein gravity where $L(R) = R$, $P^{\mu\nu\rho\sigma}$ evaluates to

$$P^{\mu\nu\rho\sigma} = \frac{1}{2}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) . \tag{B0}$$

Then, under a diffeomorphism

$$x^{\mu} = x^{\mu} + \xi^{\mu} , \tag{B0}$$

the associated Noether current is given explicitly by

$$J^{\mu\nu} = -2P^{\mu\nu\rho\sigma}\nabla_{\rho}\xi_{\sigma} + 4\xi_{\sigma}\left(\nabla_{\rho}P^{\mu\nu\rho\sigma}\right) . \tag{B0}$$

Taking $\xi^{\mu}$ to be a Killing vector vanishing on a Killing horizon $\Sigma_{\mu\nu}$, then the Wald entropy is given by

$$S = \frac{1}{8\kappa}\int_{\Sigma} J^{\mu\nu} d\Sigma_{\mu\nu} . \tag{B0}$$

For Einstein gravity, we can evaluate the integrand

$$S = \frac{1}{8\kappa}\int_{\Sigma}(2\kappa)dA , \tag{B0}$$

which yields the expected area law

$$S = \frac{A}{4} . \tag{B0}$$

### 2. Lovelock Gravity

The entropy of black holes in Lovelock gravity theories was given as a Noether charge in the work of Jacobson and Myers [97].

Following the same procedure, we define

$$P_n^{\mu\nu\rho\theta} = \frac{\partial L_n}{\partial R_{\mu\nu\rho\theta}} , \tag{B0}$$

where $L_n$ is the $n$th Euler characteristic, with $n$ ranging from 1 to a maximum $[D/2]$ in $D$ dimensional gravity.

Defining the Noether current

$$\mathcal{J}^{\mu\nu} = \sum_n \alpha_n J_n^{\mu\nu} = \sum_n^{n_{max}} \big[ -2\alpha_n P_n^{\mu\nu\rho\sigma}\nabla_{\rho}\xi_{\sigma} + 4\xi_{\sigma}\alpha_n\left(\nabla_{\rho}P_n^{\mu\nu\rho\sigma}\right)\big] , \tag{B0}$$

where $\alpha_n$ are the Lovelock coupling constants. Then the Wald entropy is defined as the integral over Killing surface $\Sigma$

$$S = \frac{1}{8\kappa}\int_{\Sigma} \mathcal{J}^{\mu\nu} d\Sigma_{\mu\nu} . \tag{B0}$$

With the Lovelock Lagrangian then, this reduces to

$$S = \sum_{n=1}^{n_{max}} 4\pi n\alpha_n \int_{\Sigma} L_{n-1} d^{D-2}x , \tag{B0}$$

which is the entropy expected from [97].