Heavy-tail Sensitivity of Stable Portfolios

by

Marko Agatonovic

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

This thesis documents a heavy-tailed analysis of stable portfolios. Stock market crashes occur more often than is predicted by a normal distribution,which provides empirical evidence that asset returns are heavy-tailed. The motivation of this thesis is to study the effects of heavy-tailed distributions of asset returns. It is imperative to know the risk that is incurred for unlikely tail events in order to develop a safer and more accurate portfolio. The heavy-tailed distribution that is used to model asset returns is the stable distribution. The problem of optimally allocating assets between normal and stable distribution portfolios is studied. Furthermore, a heavy-tail sensitivity analysis is performed in order to see how the optimal allocation changes as the heavy-tail coefficient is altered. In order to solve both problems, we use a mean-dispersion risk measure and a probability of loss risk measure. Our analysis is done for two-asset stable portfolios, one of the assets being risk-free, and one risky. The approach used involves changing the heavy-tail parameter of the stable distribution and finding the differences in the optimal asset allocation. The key result is that relatively more wealth is allocated to the risk-free asset when using stable distributions than when using normal distributions. The exception occurs when using a loss probability risk measure with a very high risk tolerance. We conclude that portfolios assuming normal distributions incorrectly calculate the risk in two types of situations. These portfolios do not account for the heavy-tail risk when the risk tolerance is low and they do not account for the higher peak around the mean when the risk tolerance is high.

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Dedication

To my brother, my parents, my grandparents and my lovely Marija.

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Chapter 1

Introduction

In recent history, there has been vast growth in the field of quantitative financial analysis. The rise of powerful computer systems has facilitated the implementation of an immense number of complicated mathematical models. Many of the concepts in theoretical and empirical finance that have been created over the last half century, however, incorrectly rest upon the assumption that asset returns follow a normal distribution. More specifically, one of the most important theories in portfolio construction was designed under the normal assumption. This theory, known as mean-variance portfolio theory, or modern portfolio theory, was developed by Harry Markowitz in the 1950s through the early 1970s, and is still widely used in the financial world.

Historically, many crashes in the capital markets, including the 1987 stock market crash, the 2001 dot-com bubble collapse and the current financial crisis, are empirical evidence that low probability negative returns are more likely to occur than is predicted by the normal distribution. The fundamental work and investigation of Mandelbrot and Fama led them to reject the normal assumption and to propose the stable distribution as a statistical model for asset returns. This sparked considerable interest in the study of empirical distributions of financial assets, and in subsequent years, Mandelbrot and Fama's conjecture was supported by numerous empirical investigations. Since it is well-known that asset returns are not normally distributed, the models should be expanded to use heavy-tailed stable distributions in order to correctly account for potentially large losses in a portfolio.

Unfortunately, the stable distribution poses several key obstacles, and is not entirely efficient and effective for use in portfolio selection. The first difficulty lies in the fact that there is no closed form probability density function (PDF) for a stable distribution. Consequently, it is very difficult to obtain an analytic solution to a stable portfolio, and even though approximation exists, numerical integration must be used for accurate results. The second difficulty is that the stable distribution has an infinite variance, meaning that the variance cannot be used as a measure of risk. Instead, a new measure of risk must be defined. Furthermore, in modern portfolio theory, the set of efficient portfolios is defined as those portfolios which have maximum expected returns for a given set of portfolio variances, however, in the case of heavy tails, the efficient frontier loses its meaning and must be replaced so as not to depend on the variance as a measure of risk.

Infinite variances displayed in the financial markets can be modeled very accurately with the use of a non-Gaussian stable distribution. The practical and theoretical appeals of this approach are given by its attractive properties that are similar to those of the normal distribution approach. For the purpose of portfolio analysis, one of these key stable distribution properties is that of stability. By definition, a stable distribution is any distribution that is invariant under addition. That is, the distribution of sums of independent, identically distributed stable variables is itself stable, and has the same form as the distribution of the individual summands. This property is very appealing, as it broadens the central limit theorem to include distributions of infinite variance. The property that stable distribution the sum of stable distributions is a stable distribution makes them appropriate for use in optimal allocation problems.

In this thesis we address the question of how the addition of heavy-tails affects optimal asset allocation. Furthermore, we aim to obtain insight into how a change in the heaviness of the tails of the stable distribution affects our optimal asset allocation. In other words, we perform a heavy-tail sensitivity analysis of a stable portfolio. It is important to answer these questions to get an idea of how much riskier an optimal asset allocation is without the measurement of risk located in the heavy-tails. We also answer how much better our portfolio is with respect to a risk to reward ratio.

The thesis is divided into two parts and has two objectives. The first part compares the normal and stable assumptions with respect to the optimal portfolio, analyzes heavy tail sensitivity on the portfolio models. We quantify the differences in the allocation of assets when the data is fitted to the stable non-Gaussian distribution, rather than to the normal distribution. The latter part of this thesis consists in determining the effects of changes in the heavy-tail coefficient on the optimal asset allocation.

This thesis is conducted within the context of several portfolio optimization models that use stable distributions (and normal distribution in the limiting case) for asset returns. In order to create the necessary framework for the analysis, we define two new portfolio risk measures. The first asset allocation model considers the expected value of some power of the absolute deviation of the portfolio from its mean as the measure of risk. In the case of the power absolute deviation (also termed mean-dispersion) risk measure, when the power is equal to two, we are left with the classical quadratic utility as is seen in modern portfolio theory. The second asset allocation model's risk measure assumes that the probability that the portfolio return is less than the value at risk. This is known as a loss probability model. With the aid of these two models, analysis is done for a two-asset portfolio and a multi-asset portfolio. Empirical analysis for major US indices is used to show the results from the developed theory.

This thesis consists of 5 chapters. Chapter 2 is a background chapter in which the basic framework and problem definition for our portfolio models is stated. Stable portfolio models in the form of Fama's two separation model are outlined and introduced. The effects of diversification within the framework of a stable portfolio are described. The stable distribution is defined, and the numerical methods used for stable distributions are explained. Data fitting and sampling methods for these distributions are explained, as are modifications to the maximum likelihood and quantile based parameter estimation technique which allow us to fit the data to any amount of tail heaviness.

In the next two chapters of this thesis, two two-asset portfolio models are described, and empirical results and analysis of the developed models are shown. These are the core chapters of this thesis. The mean-dispersion model and the loss probability portfolio model are given in Chapter 3 and Chapter 4, respectively. Finally, the conclusions are stated in Chapter 5.

Chapter 2

Background

In this chapter we describe the framework and formulate the problem that is addressed in the thesis. All of the assumptions and constraints are stated and the formulations of the stable distribution that are used in the two subsequent chapters are shown. We give a brief introduction to stable portfolios in the form of Fama's two separation model which is used to describe the diversification effects within these portfolios. We build a foundation for the two models that are numerically analyzed in the following two chapters.

2.1 Basic Framework

Our portfolios are optimized and tested in a single investment period where money is invested at an initial time and payoff is attained at the end of the period. We assume that the investor optimizing the portfolio is risk-averse and therefore wants to minimize risk. The two main constraints are that short selling is not permitted and that the problem is analyzed in discrete time using daily returns of the stocks as the data points. Once the optimal portfolio is determined, no reallocation of capital is allowed during the investment period for testing purposes.

2.2 Problem Formulation

We now define some terms and variables that are used to develop and analyze our portfolio model. Although in general, portfolios consist of a large number of securities we focus on two-asset portfolio models (one risky asset and one risk-free asset) for mathematical efficiency and to be able to obtain useful insight into the effects of the addition of heavy-tails on the optimal allocation. The following variables are used to construct our portfolios:

- W - portfolio return
- λ - percent of wealth allocated to the risk-free asset ($\lambda \in [0, 1]$ for no short selling)
- z_0 - risk-free asset return
- $z \sim S_\alpha(\beta_z, \sigma_z, \mu_z)$ - stable distributed risky asset return

In general, we aim to solve the following problem

$$\begin{array}{ll} \max & \mathrm{E}[W] \\ \text{subject to} & \mathrm{f}(W) \leq q \end{array} \tag{2.1}$$

where $\mathrm{f}(W)$ is the function that specifies the risk measure for the portfolio, and $W = \lambda z_0 + (1 - \lambda)z$ is a linear combination of the risk-free asset and the risky asset. In other words, the objective of this portfolio is to maximize the expected value of W subject to a constraint of the risk measure. In the following two chapters, two different risk measure functions are specified which constrain the amount of risk an investor wants to allow in the portfolio. The next section describes the definition of the stable distribution, $S_\alpha(\beta_z, \sigma_z, \mu_z)$, used to model the daily returns of the risky asset.

2.3 Stable Distribution

It is well documented in financial literature that asset returns are leptokurtotic. Stable distributions are leptokurtotic distributions that are four-parameter functions with infinite variance for which the generalized Central Limit Theorem holds and determines the domain of attraction. That is, the importance of stable probability distributions is that they are attractors for sums of independent, identically distributed random variables. The normal distribution is a limiting case of the general stable distribution and is the only special case stable distribution that is not heavy-tailed. The probability theory of stable distributions applicable to our portfolio model is described in the following section.

2.3.1 Definition & Properties

By definition, the distribution of any random variable X with distribution function $F(x)$ is said to have a heavy right tail if

$$\lim_{x \to \infty} e^{\alpha x} \mathrm{P}(X > x) = \infty \quad \text{for all} \quad 0 < \alpha < 2. \tag{2.2}$$

In particular, for empirical equity data we observe that $1 < \alpha < 2$. In this case our random variable X is heavy-tailed having finite mean and infinite variance. The tail condition in Equation 2.2 also implies that this random variable X is in the domain of attraction of the $\alpha-$stable law. In other words, given t i.i.d. observations of stable random variables X_i, there exist constants a_t and b_t such that

$$\sum_{i=1}^{t} \frac{X_i}{a_t} + b_t \xrightarrow{d} X \quad \text{as} \quad t \to \infty, \tag{2.3}$$

where $X \sim S_\alpha(\sigma, \beta, \mu)$ is a stable random variable and is heavy-tailed. This convergence result is a consequence of the stationarity of returns and of the generalized Central Limit Theorem for normalized sums of i.i.d. random variables.

The key property of stable distributions is that of stability. A random variable X is stable if for any independent copies of X, X_1 and X_2, and any $a, b > 0$ we have

$$aX_1 + bX_2 \stackrel{d}{=} cX + d \tag{2.4}$$

for some $c > 0$ and $d \in \Re$. The distribution of sums of i.i.d. stable random variables is stable and has the same form as the distribution of the individual summands. This property tells us that stable distributions are the only possible limiting distributions for sums of i.i.d distributed random variables with infinite variances.

The general form of the univariate stable distribution is normally defined in most recent literature as: $X \sim S_\alpha(\sigma, \beta, \mu)$ if the characteristic function of X is given by

$$\phi(t) = \mathrm{E}[e^{itX}] = \begin{cases} \exp\{-\sigma^\alpha |t|^\alpha \left[1 - i\beta (\tan \frac{\pi\alpha}{2})(\operatorname{sign} t)\right] + i\mu t\} & \alpha \neq 1 \\ \exp\{-\sigma |t| \left[1 + i\beta \frac{2}{\pi} (\operatorname{sign} t) \ln |t|\right] + i\mu t\} & \alpha = 1 \end{cases} \tag{2.5}$$

The above stable distribution has four parameters, α, β, σ, μ. The first parameter $0 < \alpha \leq 2$ determines the probability contained in the extreme tails of the distribution and is called the characteristic exponent of the distribution. For $\alpha = 2$, the stable distribution becomes the normal distribution and for $\alpha < 2$, the stable distribution is heavy-tailed with the probability in the tails increasing as α decreases. When $\alpha \leq 2$ the variance is infinite. The second parameter $-1 \leq \beta \leq 1$ is an index of skewness. For $\beta = 0$, the distribution is symmetric. For positive and negative β the distribution is skewed right and left, respectively. When $\beta = 1$ the left tail disappears completely and vice versa when $\beta = -1$ making the distribution completely asymmetric. The third parameter σ determines the scale of a stable distribution which determines the spread or dispersion. For $\alpha = 2$, σ is one half the variance and for $\alpha < 2$, the variance is infinite and σ defines the scale but in a different manner than the variance. The final parameter μ is the location parameter of the distribution. When $\alpha > 1$, μ is the expected value of the distribution, however, when $\alpha \leq 1$ the mean is infinite and μ represents the location of the distribution differently.

From the above characteristic function, Equation 2.5, there are three special cases where the formula simplifies to another distribution. For $\alpha = 2$, the stable distribution becomes the normal distribution. For $\alpha = 1$ and $\beta = 0$, it becomes the Cauchy distribution and finally for $\alpha = 1/2$ and $\beta = 1$, it becomes the Lévy distribution.

In order to state the theoretical formulas which are used for numerical calculations required in both data fitting and stimulation, we define a different parametrization of the characteristic function. The standardized stable random variable has the following characteristic function

$$\phi(t) = \mathrm{E}[e^{itX}] = \begin{cases} \exp\{-|t|^\alpha \left[1 + i\beta (\tan \frac{\pi\alpha}{2})(\operatorname{sign} t)(|t|^{1-\alpha} - 1)\right]\} & \alpha \neq 1 \\ \exp\{-|t| \left[1 + i\beta \frac{2}{\pi} (\operatorname{sign} t) \ln |t|\right]\} & \alpha = 1 \end{cases} \tag{2.6}$$

In order to define the computational formulas for the cumulative distribution function (CDF), $\mathrm{F}(x; \alpha, \beta)$, and the probability density function (PDF), $\mathrm{f}(x; \alpha, \beta)$, for the above parametrization, we must define the following functions

$$\zeta = \zeta(\alpha, \beta) = \begin{cases} -\beta \tan \frac{\pi\alpha}{2} & \alpha \neq 1 \\ 0 & \alpha = 1, \end{cases} \tag{2.7}$$

$$\xi = \xi(\alpha, \beta) = \begin{cases} \frac{1}{\alpha} \arctan(\beta \tan \frac{\pi\alpha}{2}) & \alpha \neq 1 \\ \frac{\pi}{2} & \alpha = 1, \end{cases} \tag{2.8}$$

$$c_1(\alpha, \beta) = \begin{cases} \frac{1}{\pi}(\frac{\pi}{2} - \xi) & \alpha < 1 \\ 0 & \alpha = 1 \\ 1 & \alpha > 1, \end{cases} \tag{2.9}$$

$$V(\theta; \alpha, \beta) = \begin{cases} \cos(\alpha\xi)^{\frac{1}{\alpha-1}} \left(\frac{\cos\theta}{\sin\alpha(\xi+\theta)}\right)^{\frac{\alpha}{\alpha-1}} \frac{\cos(\alpha\xi+(\alpha-1)\theta)}{\cos\theta} & \alpha \neq 1 \\ \frac{2}{\pi}\left(\frac{\frac{\pi}{2}+\beta\theta}{\cos\theta}\right) \exp\left(\frac{1}{\beta}\left(\frac{\pi}{2} + \beta\theta\right) \tan\theta\right) & \alpha = 1, \beta \neq 0. \end{cases} \tag{2.10}$$

For a stable random variable, X, that has the characteristic function in Equation 2.6, the PDF and CDF of X [11] are given by

(a) When $\alpha \neq 1$ and $x > \zeta$,

$$f(x; \alpha, \beta) = \frac{\alpha(x-\zeta)^{\frac{1}{\alpha-1}}}{\pi|\alpha - 1|} \int_{-\xi}^{\frac{\pi}{2}} V(\theta; \alpha, \beta) \exp\left(-(x-\zeta)^{\frac{\alpha}{\alpha-1}} V(\theta; \alpha, \beta)\right) d\theta \tag{2.11}$$

$$F(x; \alpha, \beta) = c_1(\alpha, \beta) + \frac{\operatorname{sign}(1-\alpha)}{\pi} \int_{-\xi}^{\frac{\pi}{2}} \exp\left(-(x-\zeta)^{\frac{\alpha}{\alpha-1}} V(\theta; \alpha, \beta)\right) d\theta \tag{2.12}$$

(b) When $\alpha \neq 1$ and $x = \zeta$,

$$f(\zeta; \alpha, \beta) = \frac{\Gamma(1 + \frac{1}{\alpha}) \cos(\xi)}{\pi(1+\zeta^2)^{1/2\alpha}} \tag{2.13}$$

$$F(\zeta; \alpha, \beta) = \frac{1}{\pi}\left(\frac{\pi}{2} - \xi\right) \tag{2.14}$$

(c) When $\alpha \neq 1$ and $x < \zeta$,

$$f(x; \alpha, \beta) = f(-x; \alpha, -\beta) \tag{2.15}$$

$$F(x; \alpha, \beta) = 1 - F(-x; \alpha, -\beta) \tag{2.16}$$

(d) When $\alpha = 1$,

$$f(x;1,\beta) = \begin{cases} \frac{1}{2|\beta|} e^{-\frac{\pi x}{2\beta}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} V(\theta;1,\beta) \exp\big(- e^{-\frac{\pi x}{2\beta}} V(\theta;1,\beta) \big) d\theta & \beta \neq 0 \\ \frac{1}{\pi(1+x^2)} & \beta = 0 \end{cases} \tag{2.17}$$

$$F(x;1,\beta) = \begin{cases} \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\big(- e^{-\frac{\pi x}{2\beta}} V(\theta;1,\beta) \big) d\theta & \beta > 0 \\ \frac{1}{2} + \frac{1}{\pi} \arctan x & \beta = 0 \\ 1 - F(x;\alpha,-\beta) & \beta < 0. \end{cases} \tag{2.18}$$

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Given the above formulation of the univariate stable distribution, it is important that accurate and efficient parameter estimation and simulation of stable random variables is possible in order to find the optimal distribution of wealth within the portfolio. The next two sections describe the methods used to estimate the parameter of the stable distribution and simulate random variables, respectively.

2.3.2 Parameter Estimation

There are many different methods for fitting data to stable random variables. Two of these methods are commonly used and we employ them in this thesis. One of the easiest methods to implement, and the least computationally intensive method for data fitting of stable distributions, is a quantile based estimation method (QBE) proposed by McCulloch (1986) [4]. It estimates all 4 parameters (for $\alpha > 0.5$) using 5 quantiles. They are the 5^{th}, 25^{th}, 50^{th}, 75^{th}, 95^{th} percentiles of the data. This method involves calculation of the five sample quantiles and a simple linear interpolation of tabulated index numbers, and a simple continuity correction for the β parameter. Most of the computation time is spent ordering the sample observations to obtain the desired quantiles. Quantile based estimation is reliable and works well for large sample sets and when the data is not too saturated. Some rounding errors in the estimation of the quantiles occurs due to the discretization yielding inaccuracies in the parameter estimates.

In our analysis, we use the QBE method for the heavy-tail sensitivity analysis due to its speed since many computations have to be made to fit the stable distribution to the data for preset α values. In order to fit the data for a specified α, a simple modification to the algorithm developed by McCulloch must be implemented. Instead of obtaining the α parameter based on interpolation of the tabulated index numbers which are obtained from the calculated quantiles, we simply use the chosen value of α. All of the subsequent calculations are then carried out with this α value exactly the same way as if α were fitted correctly. This method is fairly accurate and computationally fast, however, it does not perform quite as well as the maximum likelihood estimation method (MLE) which is the most accurate estimation method [6].

The MLE method is used for the normal distribution versus the stable distribution analysis due to its accuracy and the fact that only one data fitting is required. This method yields the highest probability of being correct and works well for any values of the parameters. The parametrization used for this method of estimation, $X \sim S(\mu, \beta, \gamma, \delta_1; 1)$, has the characteristic function given in Equation 2.5. Given a family $\mathcal{D}_{\boldsymbol{\theta}}$ of stable distributions for unknown parameters, $\boldsymbol{\theta}$, let the stable density be denoted by f$(x; \alpha, \beta)$ where the parameter vector $\boldsymbol{\theta}$ = $(\alpha,\beta,\gamma,\delta)$. For the purpose of maximum likelihood estimation for a set of sample values, $x_1, x_2, \ldots, x_n$, taken from the empirical data since the sample is i.i.d and since the maximum is unaffected by the logarithm as it is a monotone transformation, it is more convenient to use a trick and define the likelihood function as follows

$$\mathcal{L}(\boldsymbol{\theta}) = \log \mathcal{L}(\boldsymbol{\theta}) = \log \prod_{i=1}^{n} f(x_i|\boldsymbol{\theta}) = \sum_{i=1}^{n} \log f(x_i|\boldsymbol{\theta}). \tag{2.19}$$

Maximum likelihood estimation requires choosing a value for $\boldsymbol{\theta}$ such that $\mathcal{L}(\boldsymbol{\theta})$ is maximized over the parameter space Θ = (0,2] $\times$ [-1,1] $\times$ (0,∞) $\times$ (∞, ∞) [6]. The difficulty in finding the optimum $\boldsymbol{\theta}$, denoted by $\hat{\boldsymbol{\theta}}$, lies in the fact that no closed formulas exist for stable densities. We employ the numerical methods for calculating the density of the stable distribution described by Nolan (1997) [5]. The fitted parameters of the quantile based estimation method are used to give the starting point for finding the maximum of the log-likelihood function as they give an initial value which is a close approximation to the solution.

2.3.3 Random Sampling

The complexity of the problem of simulating α-stable random variables stems from the fact that there are no analytic expressions for the cumulative distribution function or the inverse. Therefore, all standard approaches like the rejection or inversion methods would require tedious computations. Chambers, Mallows, and Stuck (1976) [1] devised an elegant and superior method for efficiently stimulating stable distributions. The first step in the algorithm is to generate a uniformly distributed random variable U on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and an independent, exponential random variable W with unit mean. The following equation then yields a random variable $X \sim S_\alpha(1, \beta, 0)$

$$X = \begin{cases} (1+\zeta^2)^{\frac{1}{2\alpha}} \frac{\sin\{\alpha(U+\xi)\}}{\{\cos(U)\}^{1/\alpha}} \left[\frac{\cos\{U-\alpha(U+\xi)\}}{W}\right]^{\frac{1-\alpha}{\alpha}} & \alpha \neq 1 \\ \frac{1}{\xi}\left\{\left(\frac{\pi}{2}+\beta U\right)\tan U - \beta \log\left(\frac{\frac{\pi}{2} W \cos U}{\frac{\pi}{2}+\beta U}\right)\right\} & \alpha = 1 \end{cases} \tag{2.20}$$

where ξ is given by Equation 2.8 and ζ is given by Equation 2.7 [1]. Given the formulas for simulation of a standard α-stable random variable, we are able to easily simulate a stable random variable for all admissible values of the parameters α, β, σ, and μ using the following property:

$$Y = \begin{cases} \sigma X + \mu & \alpha \neq 1 \\ \sigma X + \frac{2}{\pi}\beta\sigma \log \sigma + \mu & \alpha = 1 \end{cases} \tag{2.21}$$

where $X \sim S_\alpha(1, \beta, 0)$ and our resulting random variable $Y \sim S_\alpha(\sigma, \beta, \mu)$ [1]. This method of simulating stable random variables is regarded as the fastest and most accurate. This method is used for all of the required random sampling needed for the numerical results of this thesis.

2.4 Related Work

This section of the thesis gives a general overview of stable distributions in the form of Fama's Two Fund Separation Model and uses this model to describe the diversification effects for different cases within stable portfolios. This gives an overview of stable portfolios and insight into how risk will be adjusted to be within a certain risk tolerance in the models given in the following two chapters.

2.4.1 Two Fund Separation Model

Given the equations, data fitting and stimulation methods shown for stable random variables, this section uses their properties and formulations to outline Fama's two fund portfolio model for a stable market [2]. The next chapters then build on this general portfolio model to develop optimal allocation models for different cases of stable assets. The main concept demonstrated here is diversification within stable portfolios.

Let Z_j, $j = 1, ..., n$, denote the daily returns of n securities where the return on each security is a random variable with its own probability distribution. Assume that the returns on the different securities, Z_j, are related to each other by a common underlying factor which is the market index number, a random variable M. Therefore, the daily return is $Z_j = a_j + b_j M + \epsilon_j$ where the coefficient a_j is a measure of the return on the i^{th}asset. It is the equivalent of an asset's α as defined in modern portfolio theory and is written as a_j to avoid confusion with the stable parameter α. The variable b_j is a measure of the relationship between the return Z_j and the market index M. This parameter is the equivalent of the asset's β as defined in modern portfolio theory and again is written as b_j to avoid confusion with the stable parameter β. The random variable ϵ_j is the error or noise in the relationship and has an expected value of zero.

In this model, if the distributions of all the random variables are normal then we simply have the Markowitz security characteristic line [3]. We now assume that M and ϵ_j, $j = 1, ..., n$ are independent, stable random variables. Then $Y_j = a_j + \epsilon_j$ is also a stable random variable. Let the term $b_j M$ be the market component of the return on security j, then Y_j is the individualistic component which is the portion of the return due to factors affecting component j alone. We have $Z_j = Y_j + b_j M$ and since Z_j is a linear combination of the stable variables Y_j and M, the logarithm of its characteristic function can be expressed as

$$\log f_{Z_j}(t) = i(a_j + b_j \mu)t - (\sigma_{\epsilon_j} + \sigma_M |b_j|^{\alpha})|t|^{\alpha}. \tag{2.22}$$

The location and scale parameters of the distribution of Z_j are $\mu_{Z_j} = a_j + b_j \mu$ and $\gamma_{Z_j} = \sigma_{\epsilon_j} + \sigma_M |b_j|^{\alpha}$, respectively. The return on a portfolio of securities, Z_p, can be expressed as $Z_p = \sum_{j=1}^{n} \lambda_j Z_j$ and so the logarithm of its characteristic function of Z_p will be

$$\log f_{Z_p}(t) = i \sum_{j=1}^{n} \lambda_j (a_j + b_j \mu)t - [\sum_{j=1}^{n} |\lambda_j|^{\alpha} \sigma_{\epsilon_j} + \sigma_M |\bar{b}_n|^{\alpha}]|t|^{\alpha}. \tag{2.23}$$

The location and scale parameters of the distribution of Z_p are therefore $\mu_{Z_p} = \sum_{j=1}^{n} \lambda_j A_j + \bar{b}_n \mu$ and $\sigma_{Z_p} = \sum_{j=1}^{n} |\lambda_j|^{\alpha} \sigma_{\epsilon_j} + \sigma_M |\bar{b}_n|^{\alpha}$ [2]. It is clear that the scale parameter ,σ_{Z_p}, is a sum of two components. This type of portfolio model is most naturally thought of as a series of investments in individual securities plus an investment in the market. It is analogous to the mean-variance portfolio model. The model just described, however, can be applied to $0 < \alpha \leq 2$ where as the mean-variance portfolio model requires $\alpha = 2$. We now demonstrate the effects of diversification within this simple stable portfolio framework.

2.4.2 Diversification within Stable Portfolios

Given the above model construction, we will show the effects of diversification on stable portfolios. Instead of diversifying the portfolio to reduce the variance, we will diversify the portfolio to reduce the dispersion, σ. However, diversification is not always effective and does not reduce the dispersion for all α.

Assume that all securities have the same proportion of the total value of the portfolio, $\lambda_j = \frac{1}{n}$, $j = 1, ..., n$. We then have

$$\sigma_{Z_p} = \left(\frac{1}{n}\right)^{\alpha} \sum_{j=1}^{n} \sigma_{\epsilon_j} + \sigma_M |\bar{b}_n|^{\alpha} \tag{2.24}$$

In Equation 2.24 as n is increased the behavior of σ_{Z_p} depends very definitely on the value of α. The dispersion parameter σ_{Z_p} is made up of two terms. The first term depends on the scale parameters of the distributions of the individual components of security returns, and the second term depends on the scale parameter of the market index. Diversification must affect the individual components of the portfolio since the market component of σ_{Z_p} is independent of the number of securities.

There are three cases for which the effects of diversification need to be determined: 1) $\alpha > 1$, 2) $\alpha = 1$, and 3) $\alpha < 1$. Note that for $\alpha < 1$ we have $(1/n)^{\alpha} \sum_{j=1}^{n} \sigma_{\epsilon_j} > \bar{\sigma}_{\epsilon_{j_n}}$ and for $\alpha > 1$ we have $(1/n)^{\alpha} \sum_{j=1}^{n} \sigma_{\epsilon_j} < \bar{\sigma}_{\epsilon_{j_n}}$. Now from Equation 2.24, taking the limit as n goes to infinity we have

$$\lim_{n \to \infty} \sigma_{Z_p} = \begin{cases} \infty & \alpha < 1 \\ \bar{\sigma}_{\epsilon_{j_n}} + \sigma_M |\bar{b}_n| & \alpha = 1 \\ \sigma_M |\bar{b}|^{\alpha} & \alpha > 1 \end{cases} . \tag{2.25}$$

From Equation 2.25, for $\alpha < 1$, diversification has a negative effect on the portfolio, increases the dispersion and creates more risk. For $\alpha = 1$ diversification is in general ineffective in reducing the dispersion of the distribution of the returns of the portfolio, unless there is very substantial uncertainty about the values of σ_{ϵ_j} and b_j for the different securities. Finally, diversification is effective only when the characteristic exponent $\alpha > 1$. Moreover, when $\alpha > 1$ the rate of approach of σ_{Z_p} to $\sigma_M|\bar{b}|^\alpha$ will be greater the larger the value of α.

Since it is well known that $\alpha > 1$ for all equities and indices, we are able to efficiently diversify a portfolio and decrease the scale or dispersion parameter. The larger α is in this range, the more diversification will decrease the dispersion of the distribution of returns on the portfolio. The limiting case is where diversification has the greatest effect and returns follow the Gaussian case, $\alpha = 2$. However, $\alpha = 2$ does not realistically account for the fact that heavy-tails have been empirically observed in equities. The following two chapters create a detailed portfolio optimization solution for two-asset models for both the mean-dispersion risk measure and the loss probability risk measure.

Chapter 3

Mean-Dispersion Portfolio Model

In this chapter, we analyze and present the solution to the problem of finding the optimal asset allocation for a two-asset portfolio model with a mean-dispersion risk measure.

3.1 Model Formulation

The portfolio we analyze consists of two positions: one risky asset and one risk-free asset. The returns of the risky asset follow a stable distribution, while the returns of the risk-free asset are constant. In this analysis, we assume that no short sales are allowed, but similar arguments can be made when short sales are permitted. A second assumption is that any risk-averse investor chooses an optimal asset allocation that maximizes returns given a specific risk tolerance. For this model, our risk measure of portfolio loss is given by [9]

$$\mathrm{E}[|W - \mathrm{E}[W]|^r]. \tag{3.1}$$

where $W = \lambda z_0 + (1-\lambda)z$ is the portfolio return. Given the risk measure, we attempt to maximize the expected value of wealth, $\mathrm{E}[W]$, and minimize the defined risk measure. In other words, we wish to solve the following constrained optimization problem

$$\begin{array}{ll} \max & \mathrm{E}[W] \\ \text{subject to} & \mathrm{E}\left[|W - \mathrm{E}[W]|^r\right] \leq q \end{array} \tag{3.2}$$

where W is the portfolio return, $1 \leq r < \alpha$ is the power of the mean-dispersion risk measure, and the real number $q > 0$ is the maximum acceptable tolerance for the risk measure. The coefficient α is the heavy tail parameter of the non-Gaussian stable distributed risky asset returns.

In our problem definition, we attempt to maximize our returns given that our risk level is within a certain maximum allowed tolerance, however, the inverse problem can also be solved. We could attempt to minimize our risk level given that our mean attains a specific requirement that an investor chooses at his discretion. Mathematically, the opposite problem to the one we are solving is

$$\begin{aligned} &\min && \mathrm{E}\left[|W - \mathrm{E}[W]|^r\right] \\ &\text{subject to} && \mathrm{E}[W] = w \end{aligned} \tag{3.3}$$

Note that our risk averse investor always wants to maximize returns given a certain risk level. In order to show that this investor will always choose a portfolio that maximizes the above utility function, or solve the constrained optimization problem, we must apply the principle of second-order stochastic dominance. By definition, W_1 dominates W_2 in the second-order stochastic sense if and only if for a concave utility function we have $\mathrm{E}[u(W_1)] \geq \mathrm{E}[u(W_2)]$, or alternatively, in terms of cumulative distribution functions $\mathrm{F}[W_1]$ and $\mathrm{F}[W_2]$, if and only if

$$\int_{-\infty}^{t} \mathrm{F}_{W_1}(u)du \leq \int_{-\infty}^{t} \mathrm{F}_{W_2}(u)du \tag{3.4}$$

where strict inequality holds at some t. Assume that we have two portfolios, W_1 and W_2, and suppose that W_1 dominates W_2 in the second-order stochastic sense. Because $-c|W - \mathrm{E}[W]|^r$ is a concave utility function, then for $r \in [1, \alpha)$ it follows that

$$\mathrm{U}[W_1] = \mathrm{E}[W_1] - c\,\mathrm{E}\left[|W_1 - \mathrm{E}(W_1)|^r\right] \geq \mathrm{U}[W_2]. \tag{3.5}$$

The above inequality shows that $\mathrm{U}[W_1] \geq \mathrm{U}[W_2]$ if and only if W_1 stochastically dominates W_2 [9]. We know that all expected utility maximizers, who are risk averse, prefer a second-order stochastically dominant portfolio to a dominated one. Therefore, the above inequality implies that every risk-averse investor with the given utility function should choose a portfolio that solves our optimization problem.

From Equation 3.2 it is clear that we attempt to maximize returns while keeping the risk measure less than or equal to q. In this case the risk measure is some power, $r < \alpha$, of the absolute deviation from the mean. The variable r must be less than α since the centralized moment of r greater than α do not exist and tends to infinity. This is shown by the following equation

$$\mathrm{E}[|X|^r] = \int_0^\infty \mathrm{P}(|X|^r > x)dx. \tag{3.6}$$

where X is a stable random variable. From the equation above it follows that if $r < \alpha$ then

$$\mathrm{E}[|X - \mathrm{E}(X)|^r] < \infty, \tag{3.7}$$

and if $r \geq \alpha$ then we have the following result

$$\mathrm{E}[|X - \mathrm{E}(X)|^r] = \infty. \tag{3.8}$$

From this statement it is clear that r must be such that $r < \alpha$, otherwise, the risk measure will be meaningless [9].

The main reason for choosing to solve Equation 3.2 as the allocation problem is that when $r = 2$ and a normal distribution is assumed for the risky asset, the solution to the optimization problem reduces down to the mean variance approach for optimal asset allocation. For $r = 2$ and normal returns are assumed, we obtain the following optimization problem

$$\begin{array}{ll} \max & \mathrm{E}[W] \\ \text{subject to} & \mathrm{E}\left[|W - \mathrm{E}[W]|^2\right] = \mathrm{Var}[W] \leq q \end{array} \tag{3.9}$$

From this equation we know that the mean and the variance of the portfolio would be the following

$$\mathrm{E}[W] = \lambda z_0 + (1-\lambda)\mu_z \tag{3.10}$$

$$\mathrm{Var}[W] = (1-\lambda)^2\sigma^2 \tag{3.11}$$

where μ_z is the risky asset return, z_0 is the risk-free asset return, and λ is the percentage allocated to the risk-free asset. We assume that $\mu_z > z_0$ otherwise trivially $\lambda = 1$ is the optimal solution. The above equation holds since if $X \sim \mathcal{N}(\mu, \sigma^2)$, a linear transformation $aX + b$ follows the normal distribution $aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$. Due to the short selling restriction, we know that $\lambda \in [0, 1]$. Since the variance of W increases as the expected value of W increases our inequality constraint in Equation 3.9 is in fact an equality constraint. The new constraint $\mathrm{Var}[W] = q$ allows for no addition of a complementary slackness variable for this particular constraint function. Using the method of Lagrange multipliers, the Lagrangian becomes

$$\mathrm{L}(\lambda, k_1, k_2, k_3) = \lambda z_0 + (1-\lambda)\mu_z - k_1\left((1-\lambda)^2\sigma^2 - q\right) + k_2\lambda + k_3(\lambda - 1). \tag{3.12}$$

After derivation and the addition of the two complementary slackness variables our system of equations becomes

$$\begin{gathered}
z_0 - \mu_z + 2k_1(1-\lambda)\sigma^2 + k_2 + k_3 = 0 \\
(1-\lambda)^2\sigma^2 - q = 0. \\
k_2\lambda = 0. \\
k_3(\lambda - 1) = 0. \\
\lambda \geq 0. \\
\lambda \leq 1. \\
k_2, k_3 \geq 0.
\end{gathered}$$

Given the first order conditions and the two complementary slackness conditions we have the following cases:

$$\lambda = \begin{cases} 0 & k_2 = 2k_1\sigma^2 + \mu_z - z_0, k_3 = 0 \\ 1 - \left(\frac{q}{\sigma^2}\right)^{\frac{1}{2}} & k_1 = \frac{\mu_z - z_0}{2\sigma^2}, k_2 = 0, k_3 = 0 \\ 1 & k_2 = 0, k_3 = \mu_z - z_0 \end{cases}$$

Summing up the previous equations, the solution to the optimization problem is

$$\lambda_{\text{opt}} = \max\left(0, 1 - \left(\frac{q}{\sigma^2}\right)^{\frac{1}{2}}\right) \tag{3.13}$$

for a portfolio with $r = 2$ and normally distributed returns. Given these two assumption our optimal portfolio is the same as one achieved by the mean-variance portfolio theory [13]. Modern portfolio theory is based on the quadratic utility for normal distributions with finite variances [3]. Given the normal distribution assumption, finite expected returns and finite variances, it is sufficient to use quadratic utility for asset choice to be completely described in terms of a preference relation. However, an undesirable property of the quadratic utility function is the implication that only the first two moments of a distribution are considered to make the decision of asset allocation. For this reason, when $r = 2$, even if our optimal allocation problem motivates the mean variance analysis in terms of preference relations, it is more realistic to consider models which are motivated by accounting for the distribution of returns.

Going back to the original constrained optimization problem given by Equation 3.2, the return of the portfolio, W, is given by

$$W = \lambda z_0 + (1 - \lambda)z. \tag{3.14}$$

where μ_z is the risky asset return, z_0 is the risk-free asset return and λ is the percentage of allocation in the risk-free asset. We assume that the distribution of the risky return, z, is an $\alpha-$stable distributed random variable with $\alpha > 1$ as defined in the previous chapter. That is,

$$z \sim S_\alpha(\sigma_z, \beta_z, \mu_z)$$

where α is the index of stability, σ_z is the scale parameter, β_z is the skewness parameter, and μ_z is the parameter representing the mean of z when $\alpha = 2$.

Since portfolio returns are given by $W = \lambda z_0 + (1 - \lambda)z$, when $\lambda = 1$ we obtain that $W = z_0$ and when $\lambda \neq 1$ all the portfolio returns admit the stable distribution where the parameters of the distribution are dependent on λ. Since $\lambda \in [0, 1]$, we obtain

$$W \stackrel{d}{=} \begin{cases} S_\alpha((1-\lambda)\sigma_z, \beta_z, \lambda z_0 + (1-\lambda)\mu_z) & \lambda \in [0,1) \\ z_0 & \lambda = 1 \end{cases} \tag{3.15}$$

From the above equation, the portfolio scale parameter is given by $\sigma_W = (1-\lambda)\sigma_z$ and the portfolio mean is given by $\mu_W = \lambda z_0 + (1-\lambda)\mu_z$. Since the portfolio skewness parameter is fixed, all the solutions can be represented in the mean-dispersion plane by

$$\mu_W = z_0 + \frac{\mu_z - z_0}{\sigma_z}\sigma_W \tag{3.16}$$

This equation is the efficient frontier for our optimization problem. Every risk averse investor with the utility function given by Equation 3.2 should choose a portfolio that maximizes the expected return, $\mu_W = \lambda z_0 + (1-\lambda)\mu_z$, while keeping the risk measure less than or equal to q for some $r \in [1, \alpha)$. The following optimization problem

$$\begin{aligned} \max_{\lambda} \quad & \mathrm{E}[W] \\ \text{subject to} \quad & \mathrm{E}\left[|W - \mathrm{E}[W]|^r\right] \leq q, \end{aligned} \tag{3.17}$$

can be simplified to contain an equality constraint instead of an inequality constraint since $\mathrm{E}[|W - \mathrm{E}[W]|^r]$ increases with respect to $\mathrm{E}[W]$ for all $r > 1$ and therefore our optimizing problem can be rewritten as

$$\begin{aligned} \max_{\lambda} \quad & \lambda z_0 + (1-\lambda)\mu_z \\ \text{subject to} \quad & \sigma_z^r \, \mathrm{V}(\alpha, \beta, r)^r (1-\lambda)^r = q. \end{aligned} \tag{3.18}$$

In the above equation [10],

$$\mathrm{V}(\alpha, \beta, r) = \begin{cases} (\mathrm{H}(\alpha, \beta, r))^r & 1 < \alpha < 2 \quad \text{(stable distribution)} \\ \frac{2^{r/2}\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{\pi}} & \alpha = 2 \quad \text{(normal distribution).} \end{cases} \tag{3.19}$$

The function $\mathrm{H}(\alpha, \beta, r))^r$ [10] is represented by

$$\mathrm{H}(\alpha, \beta, r)^r = \mathrm{G}(r)\left(1 + \beta^2\left(\tan^2\left(\frac{\alpha\pi}{2}\right)\right)\right)^{r/2\alpha} \cos\left(\frac{r}{\alpha}\arctan\left(\beta\tan\left(\frac{\alpha\pi}{2}\right)\right)\right),$$

where

$$\mathrm{G}(r) = \frac{2^{r-1}\Gamma\left(1-\frac{r}{\alpha}\right)}{r\displaystyle\int_0^\infty u^{-r-1}\sin^2 u du}.$$

The optimal allocation problem is considered for $r \in [1, \alpha)$ and we solve Equation 3.18 using Lagrange multipliers. We make the assumption that $\mu_z > z_0$, otherwise all of the wealth should be trivially allocated to the risk-free asset. The Lagrangian for Equation 3.18 is

$$\mathrm{L}(\lambda, k_1, k_2, k_3) = \lambda z_0 - (1-\lambda)\mu_z - k_1\left(\sigma_z^r \mathrm{V}(\alpha, \beta, r)^r (1-\lambda)^r - q\right) + k_2\lambda + k_3(\lambda - 1), \quad (3.20)$$

where k_1, k_2, k_3 are the Lagrange multipliers. Therefore, the first order conditions of the problem along with the complementary slackness conditions are

$$z_0 - \mu_z + rk_1\sigma_z^r \mathrm{V}(\alpha, \beta, r)^r (1-\lambda)^{r-1} + k_2 + k_3 = 0,$$
$$\sigma_z^r \mathrm{V}(\alpha, \beta, r)^r (1-\lambda)^r - q = 0.$$
$$k_2\lambda = 0.$$
$$k_3(\lambda - 1) = 0.$$
$$\lambda \geq 0.$$
$$\lambda \leq 1.$$
$$k_2, k_3 \geq 0.$$

From the above equations, we obtain the following solution by cases

$$\lambda = \begin{cases} 0 & k_2 = rk_1\sigma_z^r \mathrm{V}(\alpha, \beta, r)^r (1-\lambda)^{r-1} + \mu_z - z_0, k_3 = 0 \\ 1 - \left(\frac{q}{\sigma^2}\right)^{\frac{1}{2}} & k_1 = \frac{\mu_z - z_0}{2\sigma^2}, k_2 = 0, k_3 = 0 \\ 1 & k_2 = 0, k_3 = \mu_z - z_0 \end{cases}$$

Therefore, our final solution to this asset allocation problem is the following

$$\lambda_{opt} = \begin{cases} 1 & z_0 > \mu_z \\ \max\left(0, 1 - \left(\frac{q}{V(\alpha,\beta,r)^r \sigma_z^r}\right)^{\frac{1}{r}}\right) & \mu_z > z_0. \end{cases} \tag{3.21}$$

The method for finding an optimal portfolio is simple for this two-asset case. The first step is to choose a value for the risk tolerance, q, which the risk averse investor is comfortable with. The second step is to use a data fitting technique to find the stable parameters given the historical data for the risky asset. The third and final step is to use all of the obtained variables and solve Equation 3.21 to find the optimal asset allocation.

The next two sections show empirical results for optimal asset allocations using this model. The first section shows the differences between a portfolio based on the normal distribution of returns versus the stable distribution of returns, while the second section uses a series of different test cases to analyze the effects of altering the heavy tail coefficient on the optimal asset allocation.

3.2 Numerical Results

The following two subsections show the numerical results for the mean-dispersion stable portfolio model. We perform a comparison between the normal and stable distribution optimal allocations as well as a heavy-tail sensitivity analysis of the mean-dispersion stable portfolio model.

3.2.1 Normal Distribution and Stable Distribution Comparison

In this section, we compare portfolios optimized with the assumption of normal returns and those optimized with the assumption of stable non-Gaussian returns. The portfolio analysis is performed for three of the most common indices in North America. The S&P 500, Dow Jones, and NASDAQ indices are used to show the differences between the two portfolios. Table 3.1 and Table 3.2 show the estimated parameters of the risky asset, z, fitted to the normal distribution and the stable distribution for the three different indices, respectively. For this analysis daily close returns are used. The stable distribution are fitted to 1000 data points. The maximum likelihood estimation technique is used for the stable parameter data fitting as is shown in the previous chapter.

Table 3.1: Estimated Normal Daily Index Parameters

Index	Normal Parameters	
	μ	σ
S&P 500	0.0001494	0.01075
Dow Jones	0.0001038	0.01050
NASDAQ	0.0001518	0.01738

Table 3.2: Estimated Stable Daily Index Parameters

Index	Stable Parameters			
	α	β	σ	μ
S&P 500	1.6119	-0.06149	0.006119	0.0001309
Dow Jones	1.6301	0.005661	0.005897	0.0001351
NASDAQ	1.4864	-0.1749	0.008236	0.0001770

Using the above estimated parameters, the optimal asset allocation is calculated for the two-asset mean-dispersion portfolio model and analysis of the differences in optimal allocations is discussed and mathematically justified. Assuming a 2.5% risk-free rate, it is clear that risk-free asset return is less than the risky asset return over the time period we are testing, and thus the non-trivial optimal allocation holds. The risky asset parameter estimates in the Table 3.1 and Table 3.2 are used to compute the optimal allocation, λ_{opt}. Consider the optimal allocation when $\alpha > r \leq 1$ given by

$$\begin{aligned} \text{max} \quad & \mathrm{E}[W] \\ \text{subject to} \quad & \mathrm{E}\left[|W - \mathrm{E}[W]|^r\right] \leq q \end{aligned} \tag{3.22}$$

.

Table 3.3 lists the optimal allocation, λ_{opt}, given by 3.21, for both the normal and the stable distribution portfolios. The choices for the values of r are $r = 1, 1.05, 1.1, 1.15, 1.2$. Various risk tolerances, q, are used to show the effect on the differences between the normal and the stable portfolio as the allocation changes from all wealth in the risk-free investment to all wealth in the risky investment.

Table 3.3: Optimal Allocation for the Two-asset Mean-dispersion Model

Index	q	Normal Optimal Allocation					Stable Optimal Allocation				
		r=1	r=1.05	r=1.1	r=1.15	r=1.2	r=1	r=1.05	r=1.1	r=1.15	r=1.2
S&P 500	0.00001	0.9988	0.9980	0.9968	0.9950	0.9925	0.9989	0.9982	0.9971	0.9956	0.9937
	0.0001	0.9883	0.9822	0.9738	0.9628	0.9487	0.9890	0.9836	0.9765	0.9676	0.9569
	0.001	0.8334	0.8402	0.7875	0.7244	0.6506	0.8903	0.8531	0.8095	0.7602	0.7062
	0.002	0.7668	0.6909	0.6009	0.4965	0.3775	0.7807	0.7157	0.6423	0.5618	0.4765
	0.003	0.6502	0.5451	0.4230	0.2837	0.1272	0.6710	0.5817	0.4828	0.3766	0.2660
	0.004	0.5336	0.4018	0.2505	0.0800	0.0000	0.5613	0.4499	0.3283	0.1994	0.0672
	0.005	0.4170	0.2601	0.0820	0.0000	0.0000	0.4517	0.3196	0.1772	0.0279	0.0000
	0.006	0.3004	0.1198	0.0000	0.0000	0.0000	0.3420	0.1906	0.0289	0.0000	0.0000
	0.007	0.1838	0.0000	0.0000	0.0000	0.0000	0.2323	0.0626	0.0000	0.0000	0.0000
	0.008	0.0672	0.0000	0.0000	0.0000	0.0000	0.1227	0.0000	0.0000	0.0000	0.0000
	0.009	0.0000	0.0000	0.0000	0.0000	0.0000	0.0130	0.0000	0.0000	0.0000	0.0000
	0.01	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Dow Jones	0.00001	0.9988	0.9980	0.9967	0.9949	0.9923	0.9988	0.9981	0.9969	0.9953	0.9932
	0.0001	0.9881	0.9817	0.9732	0.9619	0.9475	0.9884	0.9826	0.9751	0.9656	0.9540
	0.001	0.8806	0.8364	0.7824	0.7179	0.6423	0.8841	0.8444	0.7979	0.7449	0.6865
	0.002	0.7613	0.6835	0.5914	0.4845	0.3627	0.7682	0.6990	0.6204	0.5339	0.4414
	0.003	0.6419	0.5343	0.4093	0.2666	0.1065	0.6522	0.5571	0.4512	0.3368	0.2169
	0.004	0.5225	0.3876	0.2327	0.0582	0.0000	0.5363	0.4175	0.2872	0.1483	0.0047
	0.005	0.4032	0.2426	0.0602	0.0000	0.0000	0.4204	0.2796	0.1269	0.0000	0.0000
	0.006	0.2838	0.0989	0.0000	0.0000	0.0000	0.3045	0.1430	0.0000	0.0000	0.0000
	0.007	0.1644	0.0000	0.0000	0.0000	0.0000	0.1885	0.0074	0.0000	0.0000	0.0000
	0.008	0.0451	0.0000	0.0000	0.0000	0.0000	0.0726	0.0000	0.0000	0.0000	0.0000
	0.009	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.01	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Nasdaq	0.00001	0.9993	0.9988	0.9980	0.9969	0.9953	0.9993	0.9989	0.9982	0.9974	0.9964
	0.0001	0.9928	0.9890	0.9838	0.9770	0.9683	0.9931	0.9898	0.9857	0.9807	0.9752
	0.001	0.9279	0.9012	0.8685	0.8295	0.7839	0.9307	0.9086	0.8839	0.8574	0.8308
	0.002	0.8558	0.8088	0.7531	0.6886	0.6149	0.8613	0.8232	0.7919	0.7394	0.6984
	0.003	0.7836	0.7186	0.6431	0.5569	0.4601	0.7920	0.7399	0.6847	0.6293	0.5772
	0.004	0.7115	0.6300	0.5364	0.4309	0.3139	0.7226	0.6579	0.5905	0.5239	0.4627
	0.005	0.6394	0.5423	0.4321	0.3091	0.1736	0.6533	0.5769	0.4984	0.4220	0.3529
	0.006	0.5673	0.4556	0.3298	0.1904	0.0380	0.5839	0.4966	0.4080	0.3227	0.2467
	0.007	0.4951	0.3695	0.2290	0.0743	0.0000	0.5146	0.4170	0.3189	0.2255	0.1434
	0.008	0.4230	0.2840	0.1294	0.0000	0.0000	0.4452	0.3380	0.2310	0.1301	0.0426
	0.009	0.3509	0.1990	0.0311	0.0000	0.0000	0.3759	0.2594	0.1441	0.0363	0.0000
	0.01	0.2788	0.1144	0.0000	0.0000	0.0000	0.3065	0.1812	0.0581	0.0000	0.0000

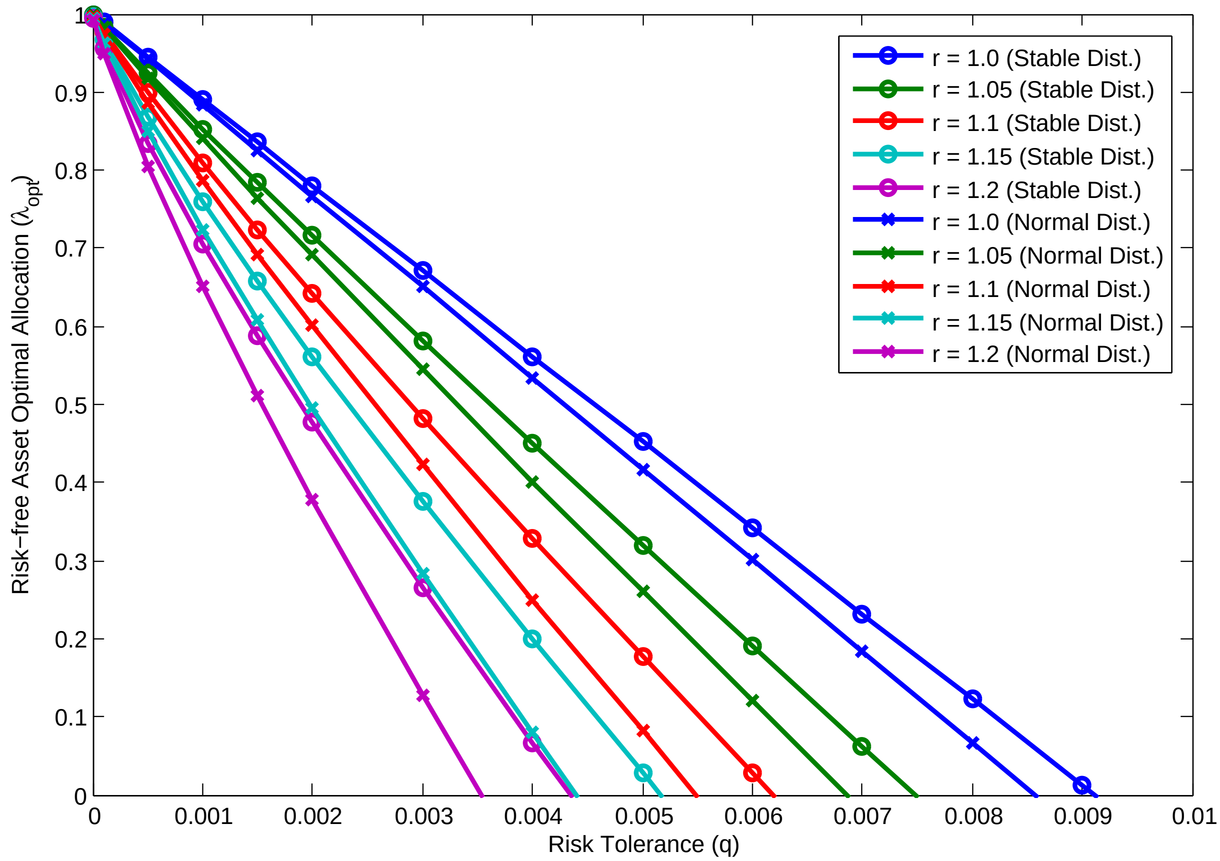

Figure 3.1: Optimal Asset Allocations of S&P 500 Two-asset Mean-dispersion Model

From Table 3.3 we observe that more capital is allocated to the risky asset in the normal case in comparison to the stable case. These results are displayed visually in Figure 3.1, Figure 3.2 and Figure 3.3 for the S&P 500, Dow Jones and Nasdaq indices, respectively.

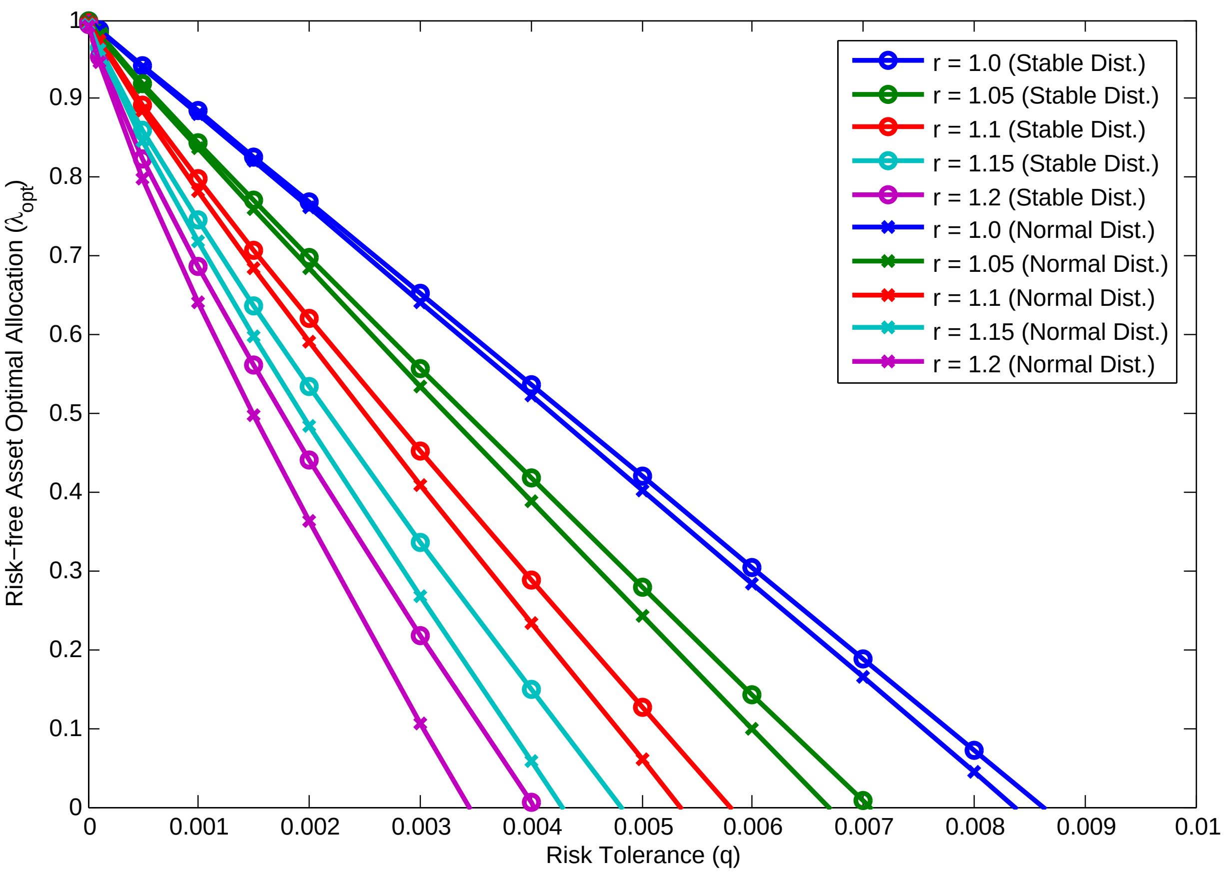

Figure 3.2: Optimal Asset Allocations of Dow Jones Two-asset Mean-dispersion Model

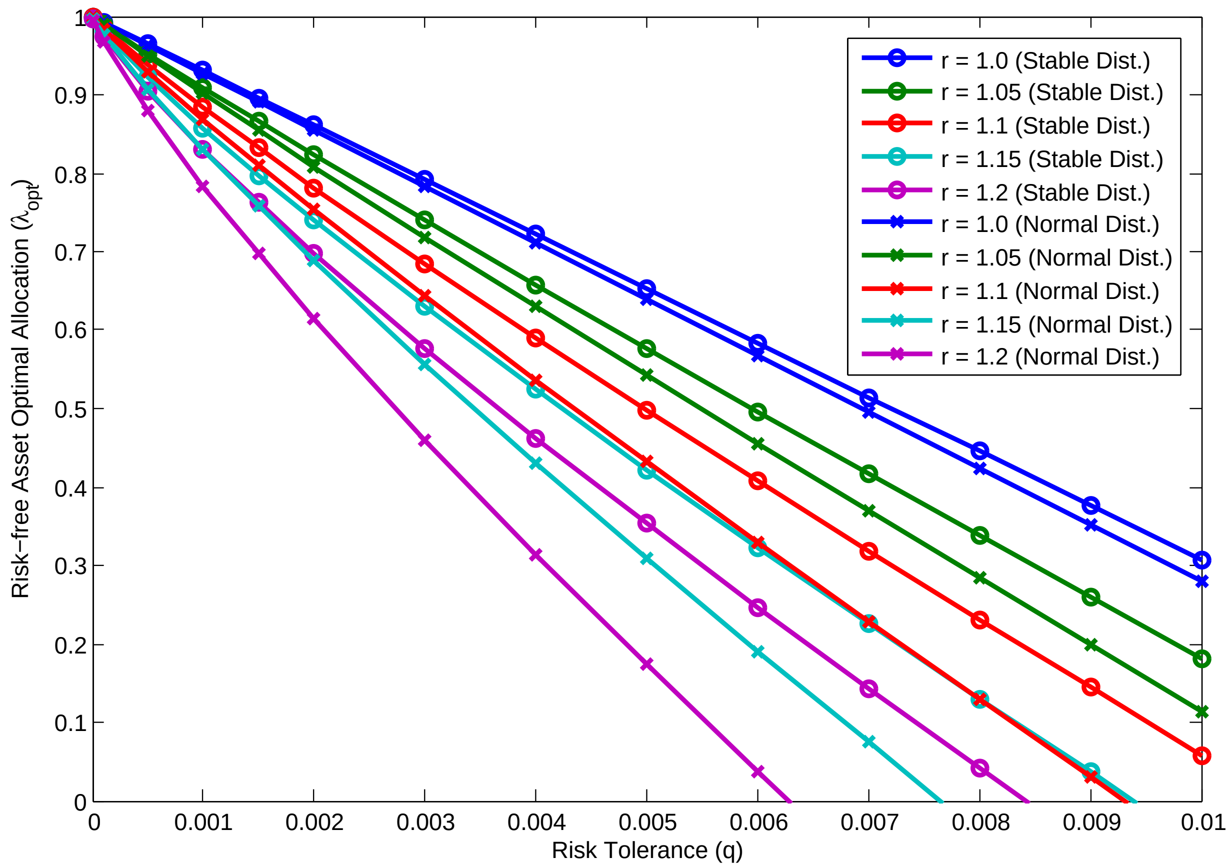

Figure 3.3: Optimal Asset Allocations of Nasdaq Two-asset Mean-dispersion Model

These three figures clearly show that in the stable non-Gaussian case, the wealth allocated to the riskless asset is always greater than that allocated to the riskless asset in the normal case. This amount varies significantly depending on several factors including r and α. A key observation is that the greater the distance between r and α, the lower the maximum difference in the allocation, and the closer r is to α, the larger the difference between the portfolios. Moreover, the further α is from the normal case, $\alpha = 2$, the larger the difference between the normal and stable distribution portfolios. This occurs due to the heavy tails of the stable distribution and due to the fact that their effects are exaggerated with smaller α and larger r. This can clearly be seen from the equations in the previous section. From Equation 3.21 it is clear that there is a positive relationship between r and λ_{opt}. As r increases we note that λ_{opt} also increases. This occurs because the fraction containing r in Equation 3.21 becomes smaller with increasing $r > 1$ and after applying the exponent is reduced even further since the exponent is a positive number less than one. In a similar fashion from the same equation we see that as α increases that $V(\alpha, \beta, r)^r$ increases as well. The relationship is easily seen from Equation 3.19. The effects of α on the allocation are analyzed in detail in the following section.

3.2.2 Heavy Tail Sensitivity Analysis

In this section we study the effects of the α parameter on the optimal asset allocation. For this analysis, we also use the three primary North American indices: the S&P 500, Dow Jones, and Nasdaq. In order to fit the data for different values of the α parameter with the stable distribution, we employ the modified quantile based estimation technique for fitting data for different α. The other technique for fitting the data to different α coefficients is the modified maximum likelihood technique. This maximum likelihood estimation technique is more accurate, however, it takes significantly longer to fit the data and requires a lot more computing power. For these reasons we will use the modified quantile based estimation technique to fit the data. As in the previous section for this analysis daily close returns are used and 1000 data points are used to fit each of the distributions.

The two-asset mean-dispersion optimal allocation problem is analyzed for various values of r. The parameter r ranges from 1 to 1.2 in 0.05 step increments. The value of r is always chosen such that $\alpha > r$ in order for the maximization of the utility function to be consistent. The second parameter that changes throughout this test is the stable parameter, α. The value of α ranges from 1.50 to 1.95 with a step size of 0.05. For this two-asset mean-dispersion

Table 3.4: Fitted Stable Parameters with Fixed α Parameter

Index	Stable Parameters			
	α	β	σ	μ
S&P 500	1.50	-0.0354	0.0059	0.0003
	1.55	-0.0403	0.0059	0.0003
	1.60	-0.0468	0.0060	0.0003
	1.65	-0.0551	0.0060	0.0003
	1.70	-0.0669	0.0060	0.0004
	1.75	-0.0826	0.0060	0.0004
	1.80	-0.1077	0.0060	0.0004
	1.85	-0.1476	0.0060	0.0004
	1.90	-0.2341	0.0060	0.0004
	1.95	-0.4695	0.0060	0.0004
Dow Jones	1.50	0.0047	0.0057	0.0003
	1.55	0.0053	0.0057	0.0003
	1.60	0.0062	0.0057	0.0002
	1.65	0.0073	0.0057	0.0002
	1.70	0.0089	0.0057	0.0002
	1.75	0.0109	0.0057	0.0002
	1.80	0.0143	0.0057	0.0002
	1.85	0.0195	0.0057	0.0002
	1.90	0.0310	0.0058	0.0002
	1.95	0.0619	0.0058	0.0002
Nasdaq	1.50	-0.1108	0.0079	0.0002
	1.55	-0.1262	0.0079	0.0002
	1.60	-0.1465	0.0079	0.0003
	1.65	-0.1724	0.0080	0.0003
	1.70	-0.2094	0.0080	0.0003
	1.75	-0.2587	0.0080	0.0003
	1.80	-0.3401	0.0080	0.0003
	1.85	-0.4670	0.0080	0.0003
	1.90	-0.7378	0.0080	0.0003
	1.95	-1.0000	0.0080	0.0005

model, the optimal asset allocation does not depend on the mean of the data, but instead depends only on the α, β, and σ coefficients. The two stable parameters, β and σ, are both determined by the corresponding changing value of α. Table 3.4 shows the fitted parameters for the stable distribution for α ranging from 1.5 to 1.95.

Table 3.5: Heavy Tail Sensitivity Analysis of the Two-asset Mean-dispersion Model

Index	Stable Parameter α	Optimal Allocation				
		r=1	r=1.05	r=1.1	r=1.15	r=1.2
S&P 500	1.50	0.8250	0.7477	0.6881	0.6261	0.5653
	1.55	0.7896	0.7292	0.6622	0.5905	0.5170
	1.60	0.7775	0.7120	0.6382	0.5576	0.4726
	1.65	0.7664	0.6961	0.6159	0.5271	0.4316
	1.70	0.7559	0.6812	0.5953	0.4989	0.3937
	1.75	0.7462	0.6674	0.5760	0.4727	0.3585
	1.80	0.7372	0.6545	0.5581	0.4483	0.3259
	1.85	0.7287	0.6425	0.5415	0.4257	0.2956
	1.90	0.7208	0.6312	0.5259	0.4045	0.2673
	1.95	0.7134	0.6207	0.5113	0.3847	0.2410
Dow Jones	1.50	0.7935	0.7363	0.6739	0.6091	0.5456
	1.55	0.7800	0.7169	0.6468	0.5718	0.4951
	1.60	0.7674	0.6989	0.6217	0.5374	0.4485
	1.65	0.7557	0.6822	0.5984	0.5055	0.4056
	1.70	0.7447	0.6666	0.5767	0.4759	0.3659
	1.75	0.7346	0.6521	0.5565	0.4484	0.3290
	1.80	0.7251	0.6386	0.5377	0.4229	0.2948
	1.85	0.7162	0.6260	0.5203	0.3992	0.2632
	1.90	0.7080	0.6143	0.5041	0.3771	0.2337
	1.95	0.7002	0.6033	0.4888	0.3564	0.2061
Nasdaq	1.50	0.8520	0.8109	0.7662	0.7196	0.6740
	1.55	0.8423	0.7971	0.7468	0.6930	0.6379
	1.60	0.8333	0.7842	0.7289	0.6685	0.6047
	1.65	0.8250	0.7723	0.7123	0.6457	0.5741
	1.70	0.8173	0.7613	0.6969	0.6247	0.5458
	1.75	0.8100	0.7510	0.6825	0.6051	0.5196
	1.80	0.8032	0.7412	0.6690	0.5867	0.4950
	1.85	0.7968	0.7322	0.6564	0.5696	0.4721
	1.90	0.7909	0.7237	0.6447	0.5538	0.4510
	1.95	0.7852	0.7157	0.6337	0.5388	0.4310

Finally, the tolerance level of risk for the portfolio, q, is set to a value of 0.002. This value is chosen because the optimal allocations calculated are spread out well across the range of all possible allocations. Table 3.5 shows the optimal allocation for the three indices for a varying stable parameter, α, and different powers of the absolute deviation risk measure, r.

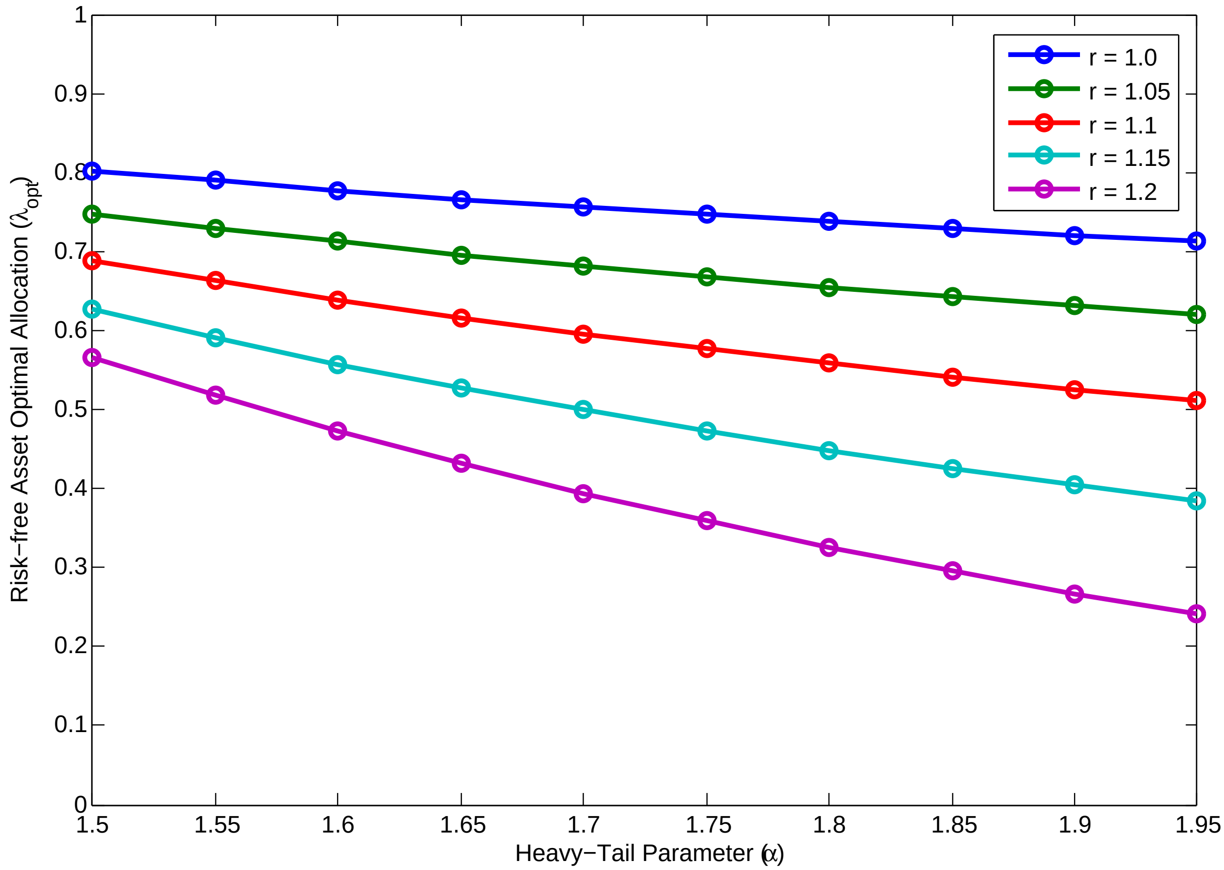

Figure 3.4: Heavy-tail Sensitivity Analysis of S&P 500 Two-asset Mean-dispersion Model

From Table 3.5 we observe that as the stable distribution inputted into the model is more heavy-tailed. The percentage of the risky asset in the optimal portfolio is smaller. This relationship is consistent for all three of the indices where the stable parameter α ranges from 1.5 to 1.95. For $r = 1$, the change in the asset allocation differs by $< 1\%$ in all three cases, while for $r = 1.2$, the change in the asset allocation differs by $> 2\%$ in all three cases. In summary, as r becomes further away from the coefficient α the changes in the optimal asset allocation are smaller. Analyzing Table 3.5 more closely, we note that the relationship is almost linear, with some curvature. This is seen more clearly for larger values of r. There is a more significant change in allocation for changes in α when α is further away from the normal distribution case, $\alpha = 2$. Figure 3.4, Figure 3.5, and Figure 3.6 below graphically depict the change in the optimal allocation, λ_{opt}, against the change of the stable parameter α for the test cases previously conducted.

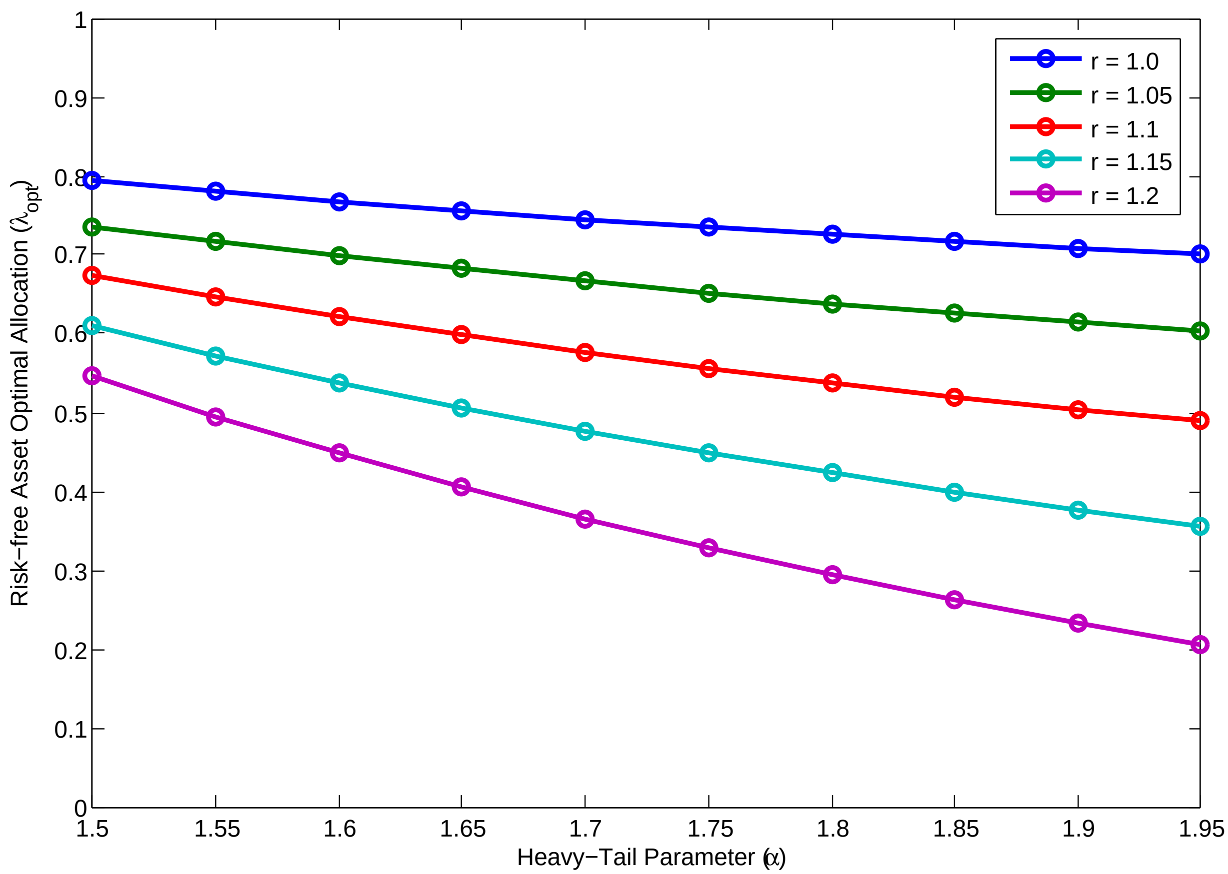

Figure 3.5: Heavy-tail Sensitivity Analysis of Dow Jones Two-asset Mean-dispersion Model

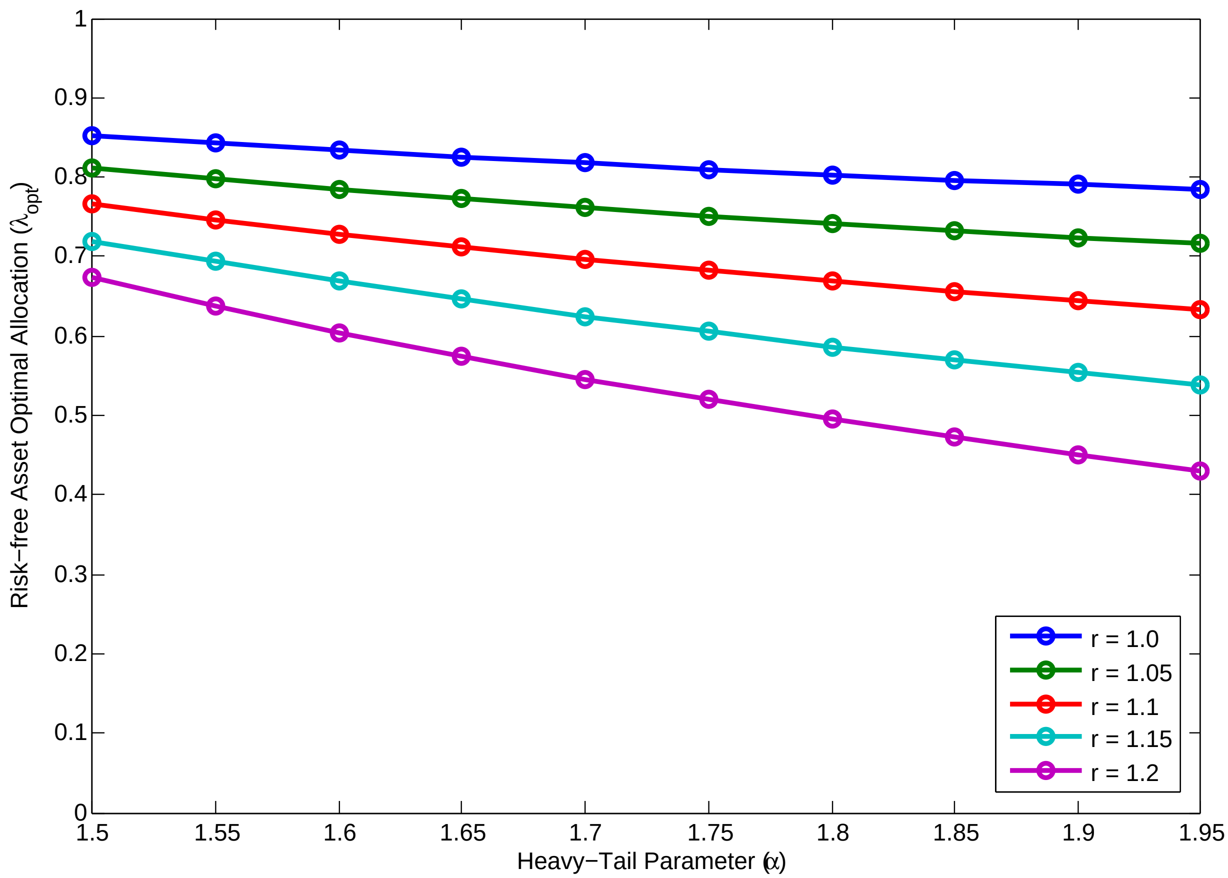

Figure 3.6: Heavy-tail Sensitivity Analysis of Nasdaq Two-asset Mean-dispersion Model

We clearly see from the figures above that the relationship between the change in the stable parameter α and the optimal allocation, λ_{opt}, is concave up. A change in the optimal allocation is much larger the closer the stable parameter α is to the normal distribution case. Overall, this sensitivity analysis shows that the stable parameter α plays a very significant role in the optimal allocation and has a profound effect on the asset weights of a portfolio. Analyzing the tail behavior of the stable non-Gaussian distribution can show the effect of the heavy-tails on the portfolio. The tail behavior of every stable non-Gaussian distribution $X \sim S_\alpha(\beta, \sigma, \mu)$, with $1 < \alpha < 2$ is given by

$$\lim_{x \to \infty} x^\alpha \, \mathrm{P}(X > x) = \sigma^\alpha C_\alpha \frac{1+\beta}{2} \tag{3.23}$$

where

$$C_\alpha = \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\frac{\pi\alpha}{2})}. \tag{3.24}$$

Figure 3.7 below shows the tail behavior for $1.5 \leq \alpha \leq 1.95$, $\beta = 0$ and various σ values.

From Figure 3.7 it is clear that as α increases, the limiting tail behavior is lower in all cases. The shape of the curves are concave up and they follow an inverse law for $\sigma < 1$ as seen in the figure. As the value of σ becomes closer to being unitary, the curve becomes straighter. It is linear for $\sigma = 1$. For $\sigma > 1$, the curve is concave down. No matter what the value of σ is as α increases, going closer to the normal case, the size of the tail is smaller. It is very important that accurate estimates of the stable parameters are made so that an accurate portfolio is achieved because, as is shown by the tail behavior, a portfolio is very sensitive to these parameters.

This heavy-tail analysis confirms that the weight of the risk measure for $r \in [1, \alpha)$, is greater for investors who use the stable distribution for asset returns. In general, the effects of the heavy-tails are very significant. It is imperative to account for their effects in order to have a correct estimate of the risk of a large loss in the portfolio. An implication of the above statement is that models that assume a normal distribution miss a component of risk. On the contrary, models that assume stable returns account for and attempt to approximate the component of risk in the heavy-tails of the distributions. For this reason, the stability parameter plays a strategic role in the optimal portfolio selection. The empirical analysis shows that the component of risk in the heavy-tails of the stable distribution has a large importance and a significant effect on the optimal asset allocation of a portfolio.

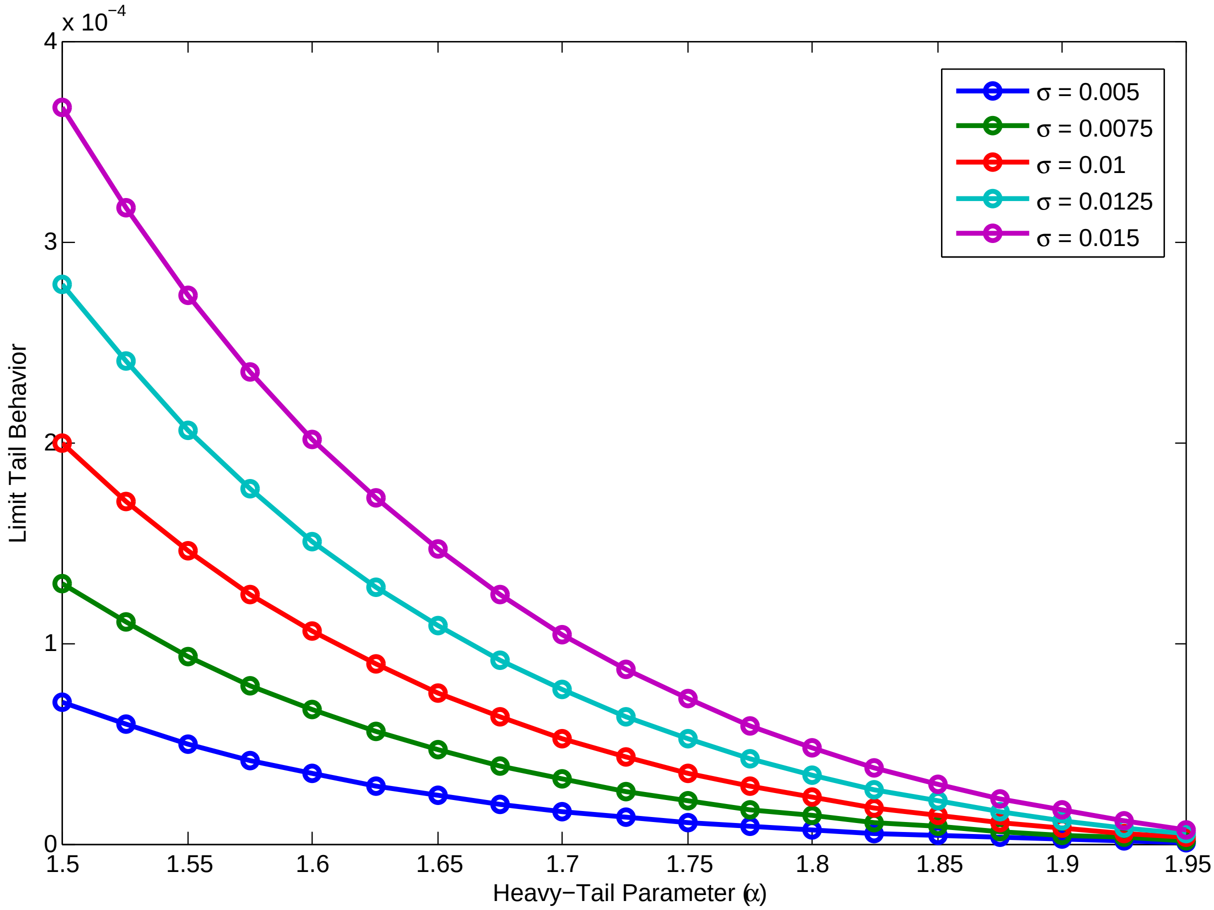

Figure 3.7: Asymptotic Heavy-tail Behavior of Stable Distributions

In the next chapter, we perform a heavy-tail analysis of the two-asset loss probability optimal portfolio model. This model uses the probability of incurring a portfolio loss equal to the empirical value at risk as the risk measure for the portfolio. The empirical analysis of this model is shown in the following as well.

Chapter 4

Loss Probability Portfolio Model

In a similar format to the previous chapter, we perform a heavy-tail analysis and compare the normal distribution and stable distribution portfolios. The difference is that the analysis is done for a two-asset portfolio model where the risk measure is the probability of incurring a certain loss.

4.1 Model Formulation

The portfolio that is analyzed is the same as that of the previous chapter. It consists of two positions: one risky asset and one risk-free asset. The returns of the risky asset follow a stable distribution while the returns of the risk-free asset are constant. Again for this analysis, we assume that no short sales are allowed and that any risk averse investor would like to choose an optimal asset allocation to maximize returns given a specific risk level. For this portfolio, the risk measure of portfolio loss is [7]

$$\mathrm{P}(W \leq -VaR). \tag{4.1}$$

where W is the portfolio return as defined previously. We attempt to maximize wealth, W, and minimize risk, $\mathrm{P}(W \leq -VaR)$. The variable, VaR, is the empirical value at risk of the portfolio. This is a commonly used measure of the risk of loss which is defined as a threshold value such that the probability that the loss of the portfolio over the given time horizon exceeds this value is the given probability level. We use the 1% and 5% daily VaR for our calculations. Mathematically, we wish to solve the following constrained optimization problem

$$\begin{aligned} \max \quad & \mathrm{E}[W] \\ \text{subject to} \quad & \mathrm{P}(W \leq -VaR) \leq q \end{aligned} \tag{4.2}$$

where W is the portfolio return, VaR is the estimated empirical value at risk, and the real number $q > 0$, as in the previous chapter, is the maximum acceptable tolerance for the risk measure. Our optimization problem requires that returns are maximized given that the risk measure is within a chosen tolerance, however, the opposite problem where we attempt to minimize our risk measure given that our mean is equal to a specific requirement can also be solved. This other optimization problem is given by

$$\begin{aligned} \min \quad & \mathrm{P}(W \leq -VaR) \\ \text{subject to} \quad & \mathrm{E}[W] = w \end{aligned} \tag{4.3}$$

Even though this problem can also be solved, we solve the optimization problem that was originally defined and perform the analysis based on this problem for the remainder of this chapter. Our optimization problem can be considered the inverse of typical VaR analysis. This is the case since in VaR analysis, we analyze the VaR of a given portfolio for a fixed probability of loss, while in our case we choose a portfolio that minimizes the probability of loss.

In order to show that any risk averse investor who maximizes the above utility function is maximizing returns while minimizing risk, we must show that for our utility function, $\mathrm{U}(W_1) \geq \mathrm{U}(W_1)$ if and only if W_1 dominates W_2 in the second-order stochastic sense. Using a similar argument to the one in the previous chapter, this can be easily shown.

There are many reasons for studying this particular allocation problem for stable distributions along with the allocation problem described in the previous chapter. One important reason for using this allocation problem is can be considered a safety first optimal allocation problem [8]. In fact, every non-satiable investor with increasing utility function

$$u(x) = x - cI_{[x \leq -VaR]}(x) \tag{4.4}$$

tends to choose portfolios that maximize the given utility function. At the same time, the investor, who uses the increasing utility function, Equation 4.4, maximizes the expected value and the probability of survival of his portfolio, as is postulated in the safety first principles [12].

Recent studies in portfolio selection have shown that there are many reasons for the consideration of the safety first approach as an alternative to the classic mean-volatility approach. Some of the main motivations leading to loss probabliity portfolio choices are that we can consider a portfolio selection for returns with unknown distributions and that we can develop a multi-parameter loss probability analysis of optimal choices in the market. Furthermore, this analysis provides a representation of the efficient frontier in terms of the threshold VaR which is very beneficial. There are also efficient programming methods to approximate loss probability optimal portfolios.

The same formulation for the portfolio returns is made as in the previous chapter. That is,

$$W = \lambda z_0 + (1-\lambda)z, \tag{4.5}$$

where z_0 is the risk-free asset return and z is the risky asset return. The portfolio is $\alpha-$stable distributed, with $\alpha > 1$, where

$$z \sim S_\alpha(\sigma_z, \beta_z, \mu_z). \tag{4.6}$$

Here, α, σ_z, β_z, and μ_z are defined as in the previous chapter. Since portfolio returns are given by $W = \lambda z_0 + (1-\lambda)z$, we have

$$W \stackrel{d}{=} \begin{cases} S_\alpha((1-\lambda)\sigma_z, \beta_z, \lambda z_0 + (1-\lambda)\mu_z) & \lambda \in [0,1) \\ z_0 & \lambda = 1 \end{cases} \tag{4.7}$$

The parameters of the portfolio are dependent on the value of λ. The portfolio scale parameter is given by $\sigma_W = (1-\lambda)\sigma_z$ and the portfolio mean is given by $\mu_W = \lambda z_0 + (1-\lambda)\mu_z$. Going back to the optimization problem we have

$$\begin{aligned} \max_{\lambda} \quad & \lambda z_0 + (1-\lambda)\mu_z \\ \text{subject to} \quad & \mathrm{P}(W \le -VaR) \le q. \end{aligned} \tag{4.8}$$

The distribution of the risk measure in the above optimization problem is the stable distribution with the parameters defined above. In order to solve the allocation problem we use the method of Lagrange multipliers as we did in the previous chapter. However, since our risk measure is a monotonically increasing function with respect to $\mathrm{E}[W]$ it can be seen that the constraint in the optimization problem becomes an equality constraint rather than an inequality constraint. Therefore, no complementary slack condition is necessary given that the new constraint is $\mathrm{P}(W \leq -VaR) \leq q$. The Lagrangian for Equation 4.3 is

$$\mathrm{L}(\lambda, k_1, k_2, k_3) = \lambda z_0 + (1-\lambda)\mu_z - k_1\left(\mathrm{F}_W(-VaR) - q\right) + k_2\lambda + k_3(\lambda - 1). \qquad (4.9)$$

where k_1, k_2, k_3 are the Lagrange multipliers and $\mathrm{F}_W(-VaR)$ is the cumulative distribution function of W. We assume that the inequality, $\mu_z > z_0$, always holds, otherwise we get the trivial solution, $\lambda_{opt} = 1$. The first order conditions and additional slackness equations are the following

$$\begin{gathered}
z_0 - \mu_z - k_1\left((z_0 - \mu_z)\,\mathrm{f}_W(-VaR)\right) + k_2 + k_3. \\
\mathrm{F}_W(-VaR) - q = 0. \\
k_2\lambda = 0. \\
k_3(\lambda - 1) = 0. \\
\lambda \geq 0. \\
\lambda \leq 1. \\
k_2, k_3 \geq 0.
\end{gathered}$$

From the above equations, we know that the following solutions corresponding to the complementary slackness conditions when $\lambda = 0, 1$ are

$$\lambda = \begin{cases} 0 & k_2 = k_1\left((z_0 - \mu_z)\,\mathrm{f}_W(-VaR)\right) + \mu_z - z_0, k_3 = 0 \\ 1 & k_2 = 0, k_3 = \mu_z - z_0. \end{cases}$$

Two cases must now be considered for this asset allocation problem, the normal distribution case and the stable distribution case. The trivial solutions above are the same for both of the cases. The first case is the normal distribution case, $\alpha = 2$. In this case, the solution when $\lambda \in (0, 1)$ is

$$\int_{-\infty}^{\frac{-VaR-\lambda z_0-(1-\lambda)\mu_z}{(1-\lambda)\sigma_z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = q. \tag{4.10}$$

Here $W \sim \mathcal{N}(\lambda z_0 + (1-\lambda)\mu_z, (1-\lambda)\sigma_z)$ and therefore from the above equation. The optimal solution, λ_{opt} is found numerically. The second case that must be solved occurs when $\alpha \in (1,2)$. In this case, $W \sim S_\alpha((1-\lambda)\sigma_z, \beta_z, \lambda z_0 + (1-\lambda)\mu_z)$. The solution is then given by

$$\mathrm{F}_{\alpha,\beta}\left(\frac{-VaR - \lambda z_0 - (1-\lambda)\mu_z - (1-\lambda)\beta_z\sigma_z \tan(\frac{\pi\alpha}{2})}{(1-\lambda)\sigma_z}\right) = q. \tag{4.11}$$

The cumulative distribution function (CDF), $\mathrm{F}_{\alpha,\beta}(x)$, of the stable random variable, $S_\alpha(1, \beta_z, -\beta_z \tan(\pi\alpha/2))$, is defined as

$$\mathrm{F}_{\alpha,\beta}(x) = \begin{cases} 1 - \frac{1}{\pi}\displaystyle\int_{-\theta_0}^{\pi/2} \exp\left(-(x-\zeta)^{\alpha/(\alpha-1)}\,\mathrm{V}(\theta;\alpha,\theta_0)\right) d\theta & x > \zeta \\ \frac{1}{\pi}\left(\frac{\pi}{2} - \theta_0\right) & x = \zeta \\ 1 - \mathrm{F}_{\alpha,-\beta}(-x) & x < \zeta \end{cases} \tag{4.12}$$

where

$$\begin{aligned} \zeta &= \zeta(\alpha,\beta) = -\beta\tan\left(\frac{\pi\alpha}{2}\right), \\ \theta_0 &= \theta_0(\alpha,\beta) = \frac{\arctan\left(\beta\tan\left(\frac{\pi\alpha}{2}\right)\right)}{\alpha}, \\ \mathrm{V}(\theta;\alpha,\theta_0) &= \cos(\alpha\theta_0)^{1/(\alpha-1)}\left(\frac{\cos\theta}{\sin(\alpha(\theta+\theta_0))}\right)^{\frac{\alpha}{\alpha-1}} \frac{\cos(\alpha\theta_0 + (\alpha-1)\theta)}{\cos\theta}. \end{aligned}$$

The above definition of the CDF of the α-stable distribution is a simplified formula of the CDF of a stable distribution described in the previous chapter. This simplification occurs because we are only considering $\alpha > 1$. In order to the find the solution, we must numerically search for the λ_{opt} that satisfies the above equations. Solutions for both non-trivial cases of the loss probability allocation problem can be found numerically for both the normal and stable non-Gaussian optimal allocation using Equation 4.10 and 4.11. The final solution is $\lambda_{opt} = \min(\max(0, \lambda^*), 1)$ where λ^* is the numerical solution to Equation 4.10 and 4.11 for the normal distribution case and the stable distribution case, respectively. The next two

Table 4.1: Estimated Daily Value at Risk

Index	Value at Risk	
	95%	99%
S&P 500	0.0164	0.0283
Dow Jones	0.0160	0.0274
NASDAQ	0.0267	0.0504

sections describe the empirical results obtained by using the safety-first optimal allocation portfolio model.

4.2 Numerical Results

The following two subsections show the numerical results for the loss probability stable portfolio model. We perform a comparison between the normal and stable distribution optimal allocations as well as a heavy-tail sensitivity analysis of the loss probability stable portfolio model.

4.2.1 Normal Distribution and Stable Distribution Comparison

In order to compare the differences in allocation between the normal distribution and the stable distribution, we use the same three indices as in the previous chapter, the S&P 500, Dow Jones, and Nasdaq. Table 3.1 and Table 3.2 display the fitted normal and stable parameters for the indices, respectively. However, given this probability of loss model, we must find one more variable that is the empirical value at risk of the indices. All of our fitted parameters are obtained from 1000 data points of daily close returns. For the analysis, 5% and 1% daily VaR values are calculated from the most recent 1000 data points consisting of the closing prices of the indices. We choose the 5% and 1% daily VaR since it is most commonly used in VaR analysis. Table 4.1 shows the results.

From Table 4.1 we observe that the S&P 500 and the Dow Jones indices have very similar VaR values of 1.6% and 2.8% for the 95% and 99% intervals, respectively. The Nasdaq index has an empirical VaR that is approximately double that of the other two indices showing that it is likely to incur twice as large a loss as the other two indices given a bad trading day in the markets.

Using the above empirical VaR value, the optimal asset allocation is calculated for the two-asset loss probability portfolio model. We analyze the differences in optimal allocations when the investor chooses the normal distribution assumption and the stable non-Gaussian distribution as a model for the asset returns in the portfolio. For the risky asset, the parameter estimates in Table 3.1 and Table 3.2 are used to compute the optimal allocation, λ_{opt}. The mean of the risk-free asset returns is less than the mean of the risky asset returns over the same time period and therefore the non-trivial allocation must occur. Consider the optimal allocation for the 5% VaR and the 1% VaR for the problem

$$\begin{array}{ll} \max & \mathrm{E}[W] \\ \text{subject to} & \mathrm{P}(W \le -VaR) \le q \end{array} \tag{4.13}$$

.

Table 4.2, Table 4.3 and Table 4.4 list the optimal allocation, λ_{opt}, for the normal and the stable distribution assumptions for the S&P 500, Dow Jones and Nasdaq indices, respectively. The VaR values are calculated using the same data set for which the parameters are fitted. Various risk tolerances, q, are used to show the differences as the allocation changes from all assets in the risk free investment to all assets in the risky investment. The yearly risk-free rate in our calculations is 2.5%.

Table 4.2: Optimal Allocation of S&P 500 Two-asset Loss Probability Model

Risk Tolerance q	Normal Optimal Allocation		Stable Optimal Allocation	
	5% $VaR = 0.0164$	1% $VaR = 0.0283$	5% $VaR = 0.0164$	1% $VaR = 0.0283$
1^{-12}	0.7913	0.6405	1.0000	1.0000
1^{-5}	0.6401	0.3801	0.9954	0.9920
0.0001	0.5872	0.2889	0.9739	0.9557
0.0002	0.5663	0.2529	0.9600	0.9311
0.0004	0.5420	0.2111	0.9382	0.8936
0.0006	0.5258	0.1833	0.9205	0.8630
0.0008	0.5133	0.1617	0.9053	0.8368
0.001	0.5030	0.1439	0.8915	0.8131
0.0025	0.4527	0.0573	0.8116	0.6754
0.005	0.4034	0.0000	0.7125	0.5048
0.0075	0.3681	0.0000	0.6368	0.3744
0.01	0.3393	0.0000	0.5734	0.2652
0.015	0.2915	0.0000	0.4694	0.0860
0.02	0.2512	0.0000	0.3851	0.0000
0.025	0.2153	0.0000	0.3131	0.0000
0.03	0.1821	0.0000	0.2499	0.0000
0.035	0.1509	0.0000	0.1928	0.0000
0.04	0.1211	0.0000	0.1399	0.0000
0.045	0.0923	0.0000	0.0904	0.0000
0.05	0.0643	0.0000	0.0434	0.0000
0.055	0.0368	0.0000	0.0000	0.0000
0.06	0.0098	0.0000	0.0000	0.0000
0.065	0.0000	0.0000	0.0000	0.0000
0.07	0.0000	0.0000	0.0000	0.0000

Table 4.3: Optimal Allocation of Dow Jones Two-asset Loss Probability Model

Risk Tolerance q	Normal Optimal Allocation		Stable Optimal Allocation	
	5% $VaR = 0.0160$	1% $VaR = 0.0274$	5% $VaR = 0.0160$	1% $VaR = 0.0274$
1^{-12}	0.7916	0.6438	1.0000	1.0000
1^{-5}	0.6409	0.3861	0.9947	0.9910
0.0001	0.5881	0.2959	0.9704	0.9494
0.0002	0.5673	0.2603	0.9548	0.9228
0.0004	0.5431	0.2189	0.9305	0.8812
0.0006	0.5270	0.1914	0.9108	0.8476
0.0008	0.5145	0.1702	0.8940	0.8188
0.001	0.5042	0.1525	0.8789	0.7930
0.0025	0.4541	0.0669	0.7909	0.6425
0.005	0.4051	0.0000	0.6845	0.4606
0.0075	0.3699	0.0000	0.6039	0.3230
0.01	0.3412	0.0000	0.5374	0.2092
0.015	0.2937	0.0000	0.4300	0.0256
0.02	0.2536	0.0000	0.3447	0.0000
0.025	0.2178	0.0000	0.2730	0.0000
0.03	0.1849	0.0000	0.2102	0.0000
0.035	0.1538	0.0000	0.1539	0.0000
0.04	0.1242	0.0000	0.1019	0.0000
0.045	0.0956	0.0000	0.0533	0.0000
0.05	0.0677	0.0000	0.0074	0.0000
0.055	0.0404	0.0000	0.0000	0.0000
0.06	0.0136	0.0000	0.0000	0.0000
0.065	0.0000	0.0000	0.0000	0.0000
0.07	0.0000	0.0000	0.0000	0.0000

Table 4.4: Optimal Allocation of Nasdaq Two-asset Loss Probability Model

Risk Tolerance q	Normal Optimal Allocation		Stable Optimal Allocation	
	5% $VaR = 0.0267$	1% $VaR = 0.0504$	5% $VaR = 0.0267$	1% $VaR = 0.0504$
1^{-12}	0.7903	0.6046	1.0000	1.0000
1^{-5}	0.6386	0.3186	0.9965	0.9946
0.0001	0.5855	0.2185	0.9832	0.9682
0.0002	0.5645	0.1790	0.9727	0.9485
0.0004	0.5402	0.1331	0.9563	0.9176
0.0006	0.5240	0.1026	0.9426	0.8917
0.0008	0.5115	0.0789	0.9304	0.8687
0.001	0.5011	0.0594	0.9192	0.8476
0.0025	0.4507	0.0000	0.8503	0.7178
0.005	0.4013	0.0000	0.7621	0.5515
0.0075	0.3660	0.0000	0.6892	0.4140
0.01	0.3371	0.0000	0.6249	0.2927
0.015	0.2893	0.0000	0.5125	0.0809
0.02	0.2490	0.0000	0.4144	0.0000
0.025	0.2130	0.0000	0.3259	0.0000
0.03	0.1798	0.0000	0.2445	0.0000
0.035	0.1486	0.0000	0.1685	0.0000
0.04	0.1187	0.0000	0.0973	0.0000
0.045	0.0900	0.0000	0.0276	0.0000
0.05	0.0620	0.0000	0.0000	0.0000
0.055	0.0345	0.0000	0.0000	0.0000
0.06	0.0075	0.0000	0.0000	0.0000
0.065	0.0000	0.0000	0.0000	0.0000
0.07	0.0000	0.0000	0.0000	0.0000

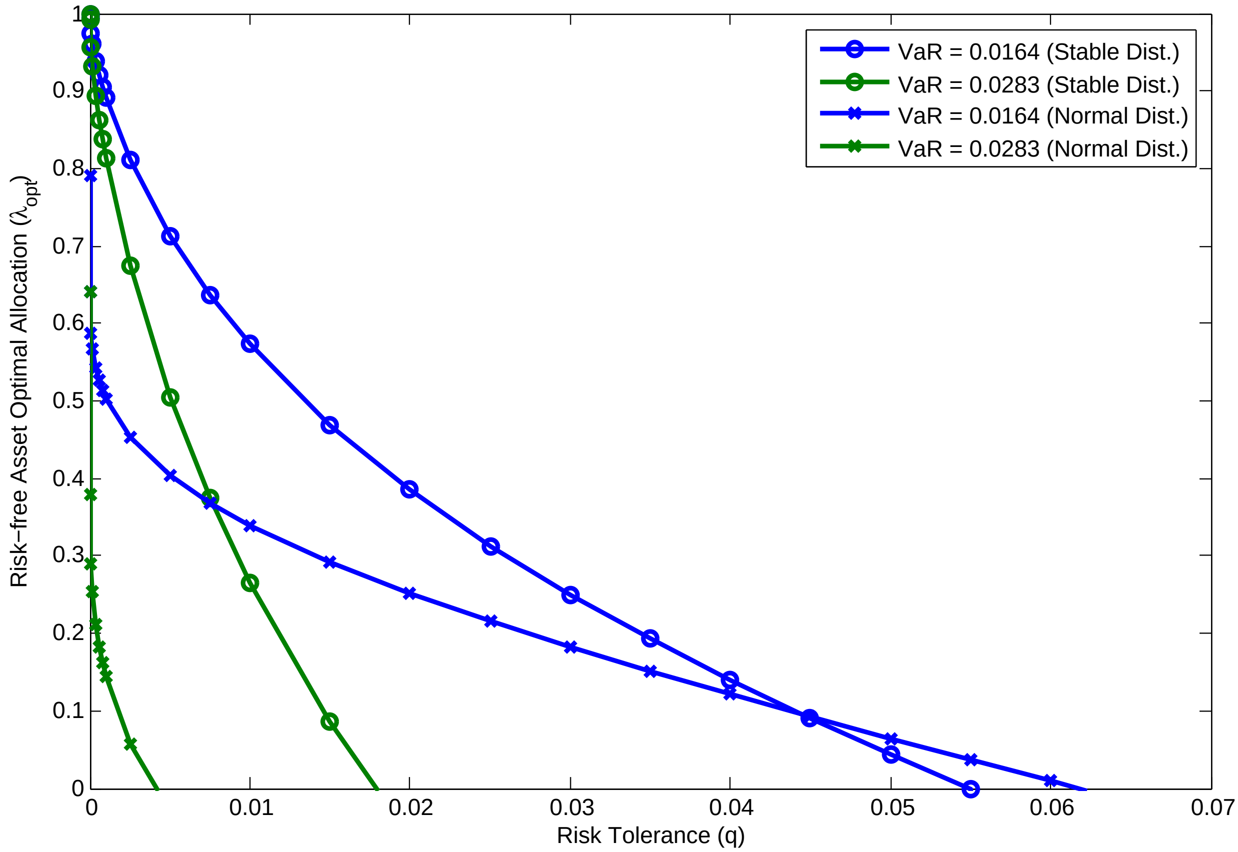

Figure 4.1: Optimal Asset Allocations S&P 500 Two-asset Loss Probability Model

Analyzing the three tables, we note that the optimal allocation of wealth in the risky asset is usually greater in the normal case than in the stable case. The only case where this is not true is for large tolerances (larger q). These results are displayed visually in Figure 4.1, Figure 4.2 and Figure 4.3 for the S&P 500, Dow Jones and Nasdaq indices, respectively.

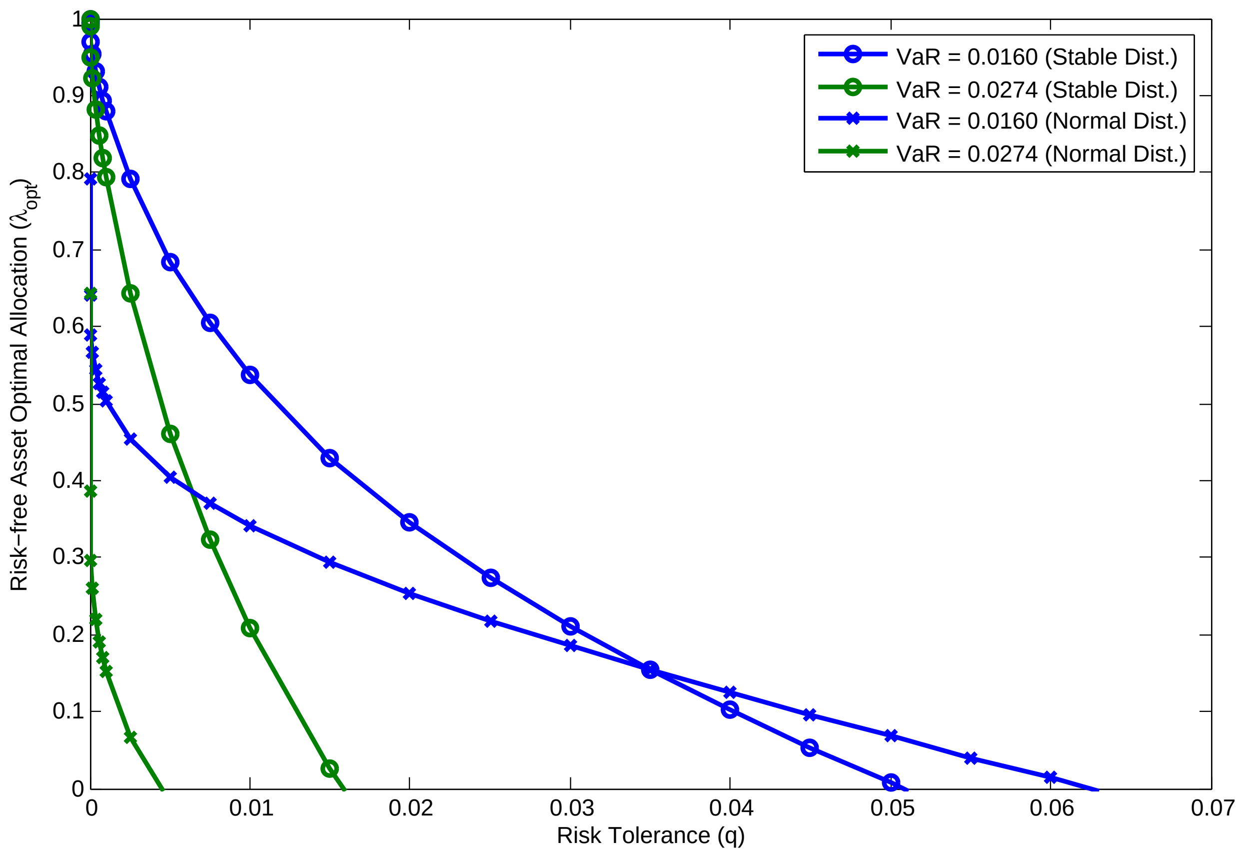

Figure 4.2: Optimal Asset Allocations of Dow Jones Two-asset Loss Probability Model

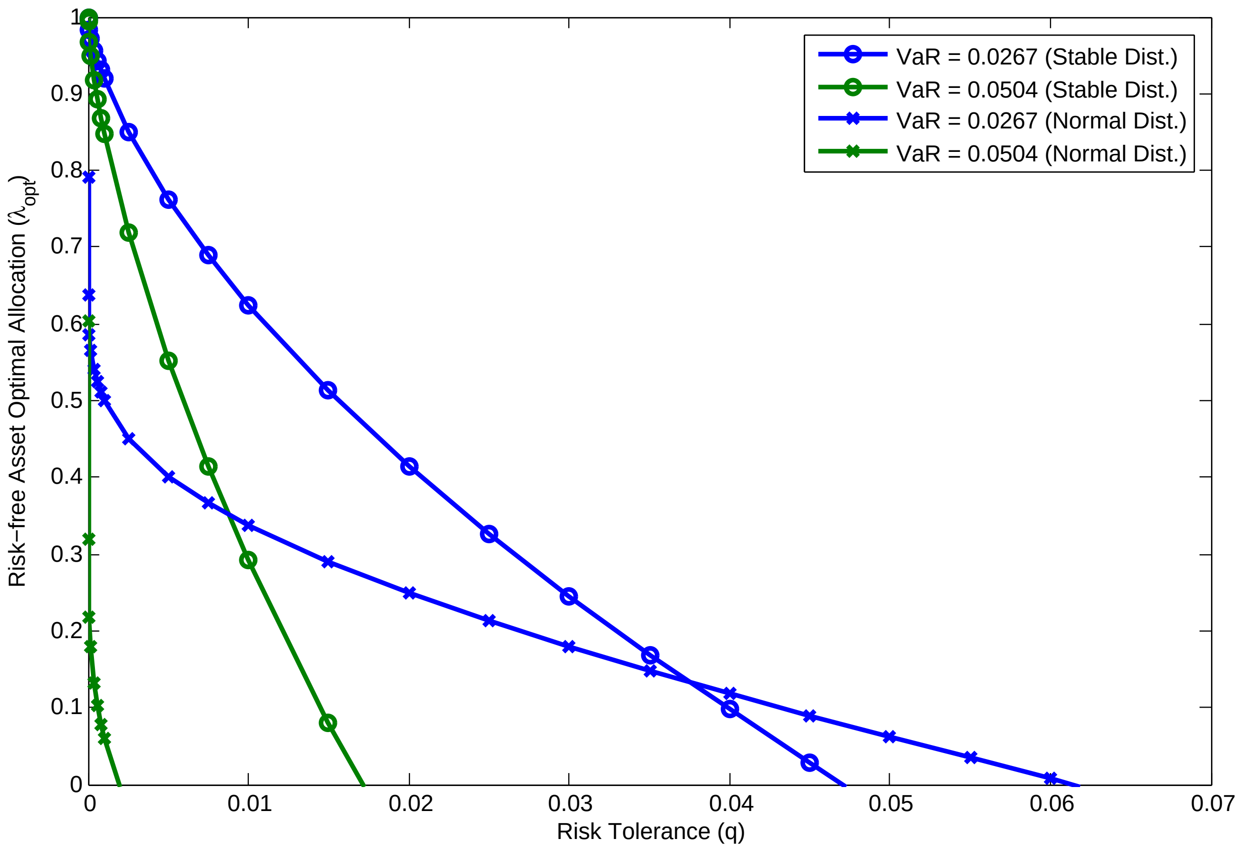

Figure 4.3: Optimal Asset Allocations of Nasdaq Two-asset Loss Probability Model

These three figures show that the difference between the optimal allocation in the normal case and in the stable case can be as much as 60%. In fact, for all except very large q, in the optimal case, more capital is allocated to the riskless asset for the stable portfolio. The opposite occurs for relaxed risk tolerances where the normal distribution allocates more capital to the risk-free asset. There is a breaking point where a specific risk tolerance yields the same portfolio for both the stable and normal distribution assumptions. There are two explanations for why we notice such differences in allocation.

In the case of a high-risk tolerance, large q, less money is invested in the riskless asset in the stable portfolio. This difference can be up to 20% in certain cases. This particular difference in allocation occurs due to the kurtosis effect. That is, the stable distribution is significantly more peaked around the mean than the normal distribution. Furthermore, the stable distribution has a much narrower peak than the normal distribution. Consequently, an investor with a very low risk aversion coefficient (i.e. high risk tolerance) who uses a stable portfolio will put more importance on the mean than an investor with that same low risk aversion coefficient who uses a normal portfolio. The latter investor prefers lower returns for a lower potential drawdown. In this case, the normal investor loses some potential returns due to the ill-shaped distribution and lower peak.

In the case of a low risk tolerance, small q, the opposite of the above situation occurs. More capital is invested in the riskless asset in the stable portfolio than in the normal portfolio. This difference occurs due to the heavy-tail effect. For the normal portfolio, the risk measure, $\mathrm{P}(W \leq -VaR)$, tends to zero exponentially and for larger VaR, the probability is essentially zero. On the contrary, for the stable portfolio, the drop off at the tails is significantly slower, in fact sub-exponential, and for higher values the probability is not almost zero. The optimal allocation of the normal investor only improves as the risk tolerance becomes very large. The capital invested by the normal investor in the risky asset is very dangerous since this investor does not consider that returns have heavy tails. The normal investor prefers a larger mean but he does not correctly estimate the potential loss since $\mathrm{P}(W \leq -VaR)$ is almost null in his model. He does not account for a portion of the returns that are away from the mean. At the middle point between a high risk tolerance and a low risk tolerance, the effect of the heavy tails and the difference in peaks offset themselves. Consequently, both the normal portfolio and the stable portfolio have the same optimal allocation for a small set of moderate risk tolerances.

A key observation is as VaR increases, the differences in the portfolio also become larger. Moreover, the further α is away from the normal case, $\alpha = 2$, the larger the difference between the normal and stable distribution portfolio for both the low and high risk tolerances. This occurs due to the heavy-tails of the stable distribution and the fact that their effects are exaggerated with α. This can clearly be seen from the equations in the previous section. The effects of α on the allocation are analyzed in further detail in the following section.

4.2.2 Heavy Tail Sensitivity Analysis

In this section, we study the effects of the α parameter on the optimal asset allocation. As in the previous section, we use the three primary North American indices: the S&P 500, Dow Jones, and Nasdaq. For this analysis, the index data must be fitted to various stable distributions with ranging α. The values of the parameters fitted for varying α are obtained from the previous chapter.

The two-asset loss probability optimal allocation problem is analyzed for the 1% and 5% empirical VaR. The second parameter that is changed for this test is the stable parameter α. The value of α ranges from 1.50 to 1.95 with a step size of 0.05. Given a two-asset loss probability model, the optimal asset allocation depends on the z_0, VaR and all of the parameters of the stable distribution. The values of all of the parameters except α is kept constant and is fitted accordingly for the data. Again 1000 data points of daily close returns are used for this empirical analysis. The parameters β, σ and μ vary slightly for each fitted distribution with different α but this effect should be negligible on the optimal allocation. The values of the fitted parameters are shown in Table 3.4 in the previous chapter. Finally, the tolerance level of risk for the portfolio, q, is set to a value of 0.002 due to the fact that this value will yield a nice variation between the portfolios. Table 4.5, Table 4.6 and Table 4.7 show the optimal allocation for the three indices.

Table 4.5: Heavy-tail Sensitivity Analysis of S&P 500 Two-asset Loss Probability Model

Stable Parameter α	Optimal Allocation	
	5% $VaR = 0.0283$	1% $VaR = 0.0162$
1.50	0.8766	0.7848
1.55	0.8573	0.7511
1.60	0.8355	0.7131
1.65	0.8109	0.6703
1.70	0.7821	0.6201
1.75	0.7479	0.5604
1.80	0.7066	0.4883
1.85	0.6536	0.3959
1.90	0.5825	0.2720
1.95	0.4814	0.0956

Table 4.6: Heavy-tail Sensitivity Analysis of Dow Jones Two-asset Loss Probability Model

Stable Parameter α	Optimal Allocation	
	5% $VaR = 0.0283$	1% $VaR = 0.0162$
1.50	0.8694	0.7767
1.55	0.8485	0.7411
1.60	0.8248	0.7005
1.65	0.7979	0.6546
1.70	0.7658	0.5997
1.75	0.7271	0.5336
1.80	0.6789	0.4511
1.85	0.6152	0.3422
1.90	0.5239	0.1862
1.95	0.4046	0.0000

Table 4.7: Heavy-tail Sensitivity Analysis of Nasdaq Two-asset Loss Probability Model

Stable Parameter α	Optimal Allocation	
	5% $VaR = 0.0283$	1% $VaR = 0.0162$
1.50	0.8546	0.7259
1.55	0.8325	0.6842
1.60	0.8085	0.6389
1.65	0.7810	0.5870
1.70	0.7510	0.5306
1.75	0.7176	0.4677
1.80	0.6780	0.3929
1.85	0.6289	0.3003
1.90	0.5649	0.1796
1.95	0.4249	0.0000

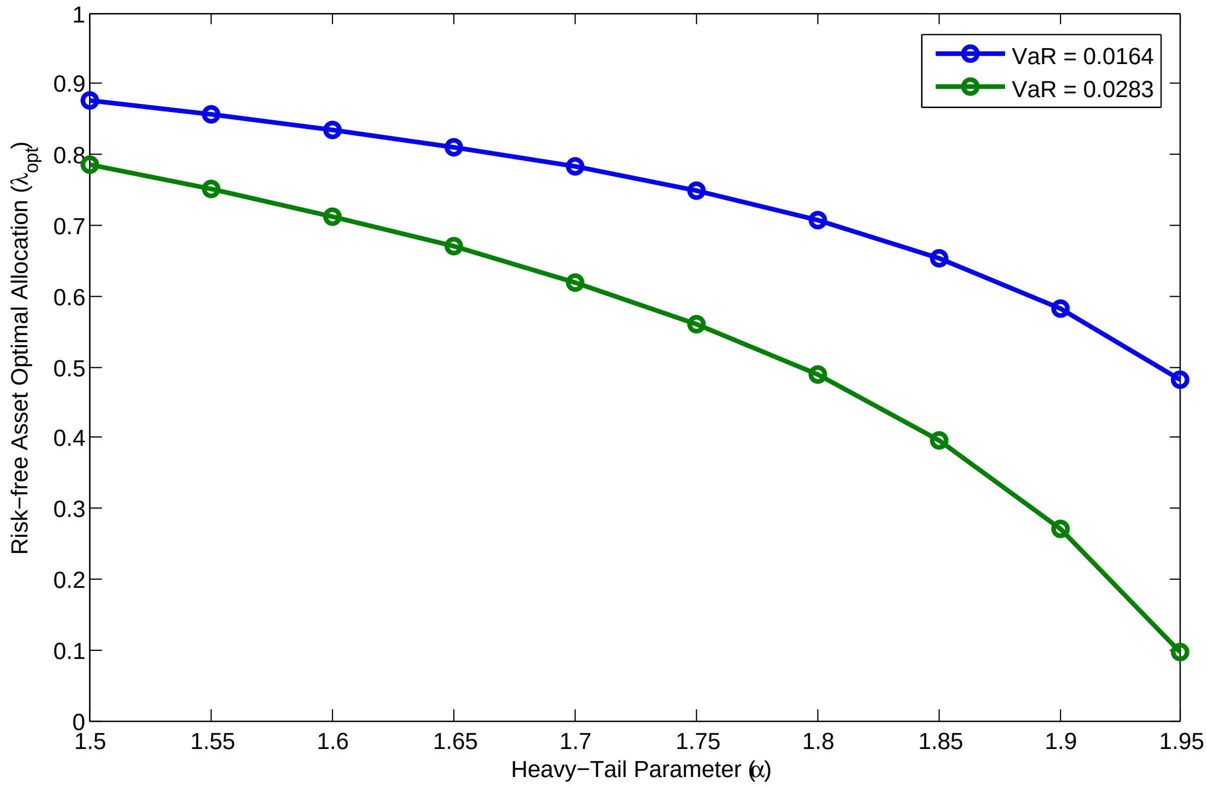

Figure 4.4: Heavy-tail Sensitivity Analysis of S&P 500 Two-asset Loss Probability Model

From the three tables, we observe that as the stable distribution inputted into the model is more heavy-tailed, the percentage of the risky asset in the optimal portfolio is smaller. This relationship is consistent for all three indices where the stable parameter α ranges from 1.5 to 1.95. Figure 3.4, Figure 3.5, and Figure 3.6 below graphically show the change in the optimal allocation λ_{opt} against the change of the stable parameter α for the test cases described.

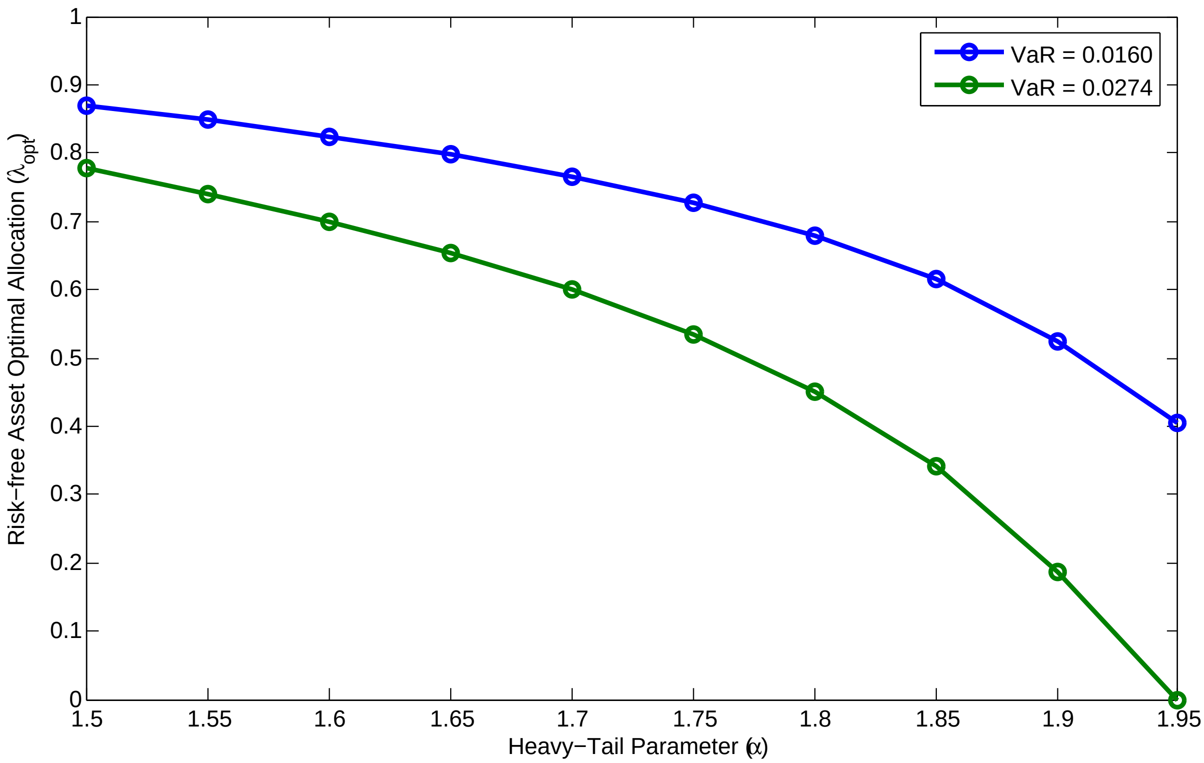

Figure 4.5: Heavy-tail Sensitivity Analysis of Dow Jones Two-asset Loss Probability Model

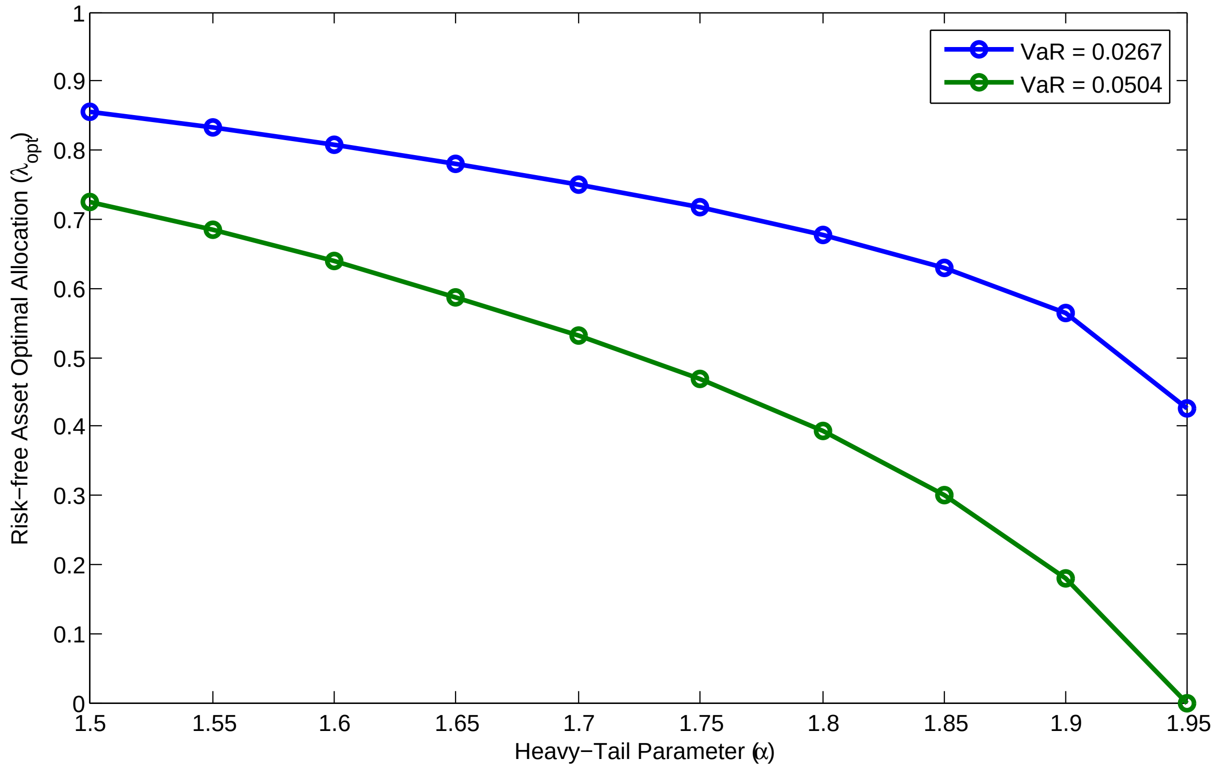

Figure 4.6: Heavy-tail Sensitivity Analysis of Nasdaq Two-asset Loss Probability Model

From the three figures, we note that for the 1% VaR, the change in the asset allocation differs by up to 80% for all three cases, while for the 5% VaR, the change in the asset allocation differs by up to 50% for all three cases. In summary, as the value of VaR becomes smaller, the changes in the optimal asset allocation are smaller for varying α. Examining the figures more closely, we see that the relationship is non-linear and concave down for $q = 0.02$. However, as q becomes larger, the curve becomes more linear. This occurs up until the point described in the previous section, where the kurtosis effect and heavy-tail effect cancel each other out. Once q becomes larger than this median point, the curve is concave up. Therefore, given a low risk tolerance, the changes in the optimal portfolio are larger as α increases. For a high risk tolerance, the changes in the optimal portfolio are smaller as α increases. For the middle point, the relationship is linear. This shows us that there is a more significant change in wealth allocation for changes in the value of α for a risk tolerance that is further away from the point where the allocation for the normal and stable case is the same.

Finally, we state the conclusions obtained from the results in this chapter and the previous chapter.

Chapter 5

Conclusion

The sensitivity analysis and comparison of normal and stable distributions of returns of two portfolios clearly prove that a lot of risk lies in the occurrence of unlikely events. The aim of this dissertation is to quantify the effects of a change in α on asset allocation and to give a quantitative analysis of the difference in asset allocation given different levels of tail heaviness. In order to perform the analysis required, two risk measures are defined. The mean-dispersion risk measure and the loss probability risk measure are used for the two portfolios. In order to perform the sensisitivity analysis, empirical data for 3 major US indices are fitted with different heavy-tail coefficient values and then the optimal asset allocation is found.

The most important result of this thesis is that in most cases, more capital is placed in the risk-free asset. For the first portfolio, which contains a mean-dispersion risk measure, more wealth is always allocated to the risk-free asset as α increases. On the contrary, in the second portfolio which uses a loss probability risk measure, two different effects are noticed which act in opposite directions, placing more or less capital in the risk-free aset as α increases. When the investor assumes a low risk tolerance, the heavy-tail effect dominates and once again more money is placed into the risk-free asset with an increase in the heavy-tail coefficient due to the significance of the heavy-tails on portfolio risk. However, when the investor assumes a high risk tolerance, the kurtosis effect dominates over the heavy-tail effect and the fact that the probability close to the mean of a stable distribution is much higher than it is in a normal distribution makes it more optimal to place more capital in the risky asset as α increases. Overall, it should be noted that in both cases, the portfolio fitted to the stable distribution correctly estimates and accounts for the risk in the tails as well as the gain in equity from the higher peak around the mean.

The main challenge of using stable distributions is that it is computationally difficult to make accurate calculations since there exists no closed form probability density function. This is overcome by the use of maximum likelihood estimation combined with quantile based estimation for large analysis that required fast computation.

Overall, this thesis is successful in outlining the main point that risk is not correctly accounted for when assuming normal asset returns. It also gives a very clear insight into the relationship between α and the optimal allocation for two different types of risk measures. The extension of the thesis is to perform a similar heavy-tail sensitivity analysis for the multi-variate stable portfolios.

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