Ehrhart Theory and Unimodular Decompositions of Lattice Polytopes

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Ehrhart theory studies the behaviour of lattice points contained in dilates of lattice polytopes. We provide an introduction to Ehrhart theory. In particular, we prove Ehrhart's Theorem, Stanley Non-negativity, and Ehrhart-Macdonald Reciprocity via lattice triangulations. We also introduce the algebra $\mathscr{P}(\mathbb{R}^d)$ spanned by indicator functions of polyhedra, and valuations (linear functions) on $\mathscr{P}(\mathbb{R}^d)$. Through this, we derive Brion's Theorem, which gives an alternate proof of Ehrhart's Theorem. The proof of Brion's Theorem makes use of decomposing the lattice polytope in $\mathscr{P}(\mathbb{R}^d)$ into support cones and other polyhedra. More generally, Betke and Kneser proved that every lattice polytope in $\mathscr{P}(\mathbb{R}^d)$ (or the sub-algebra $\mathscr{P}(\mathbb{Z}^d)$, spanned by lattice polytopes) admits a unimodular decomposition; it can be expressed as a formal sum of unimodular simplices. We give a new streamlined proof of this result, as well as some applications and consequences.

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Dedication

Laura Janzen. Amy Ng. Tammy Tam.

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Chapter 1

Introduction

Consider a convex polygon $P \in \mathbb{R}^2$ where the vertices of P are in $\mathbb{Z}^2$. We would like to explore the relationships between P and the integer points contained in P. Pick's Theorem states that

$$A = i + \frac{b}{2} - 1,$$

where A is the area of P, i is the number of integer points in the interior of P, and b is the number of integer points in the boundary of P. We may also like to explore the behaviour of the number of integer points in P as we scale the size of P by some integer and generalize to higher dimensions.

More formally, given a lattice polytope $P \subseteq \mathbb{R}^d$, we would like to explore the behaviour of

$$L_P(t) = |tP \cap \mathbb{Z}^d|.$$

Ehrhart's Theorem states that for each lattice polytope P, $L_P(t)$ is a polynomial where the degree is bounded by the dimension of P. In addition, the Ehrhart-Macdonald reciprocity states a relationship between the number of lattice points of a lattice polytope and the number of lattices points in the interior of the polytope.

The proofs of the above results made use of a technique that is going to become a recurring theme in this paper: we triangulate polytopes out of simplices. In particular, we would like to use only simplices of volume 1; such simplices are called unimodular simplices. The reason we are so eager to triangulate polytopes using unimodular simplices is that studying properties of simplices is in many cases simpler than for a lattice polytope.

Thankfully, some properties of lattice polytopes can be proven as a result of being built up from simplices. However, a slight problem arises: some polytopes can not be built with unimodular simplices alone. It is true that every polytope can be built with simplices, but there are polytopes that require simplices of volume greater than 1. For example, the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 0, 1)$, and $(1, 2, 1)$ has volume 2, but cannot be built from smaller lattice tetrahedra.

In light of this complication, we turn to another way of viewing polytopes. Instead of additively combining simplices, we allow for the subtraction, or deleting of simplices as well.

One result that is discussed in this paper is that any polytope can be built by combining unimodular simplices, and then "trimming off" unwanted pieces to get the desired polytope. More importantly, there is a way to build the polytope so the trimmed off pieces are also unimodular simplices.

More formally, a polytope has an associated indicator function. Many ideas in this paper on this topic was originally due to McMullen. These functions span the polytope algebra. The combining of simplices and the trimming off excess pieces correspond to the addition and subtraction of elements in the polytope algebra. We can study linear functions in the lattice polytope algebra, which we call valuations. For valuations that are invariant under translation and unimodular maps, we can actually determine whether two valuations are equal by determining their values on a finite number of unimodular simplices. These results in valuations see applications in alternate proofs of Ehrhart's Theorem and Ehrhart Reciprocity.

Betke and Kneser proved the existence of unimodular decompositions of lattice polytopes, as well as the equivalence of integer unimodular invariant valuations. This paper provides more streamlined proofs of such results.

For the rest of Chapter 1, we introduce the foundational properties of polytopes and their unbounded relatives, polyhedra.

In Chapter 2, we will introduce polyhedral complexes. We are particularly interested in polyhedral subdivisions and lattice triangulations. The chapter will conclude with a way to subdivide a polyhedron, and a way to triangulate a polytope. We see that any polytope can be triangulated, where each simplex uses only vertices of the polytope. We also see an example where some maximal lattice triangulations do not necessarily have the same number of k-dimensional simplices.

Chapter 3 introduces the Ehrhart series of a polytope. We prove a few fundamental results in Ehrhart theory including Ehrhart's Theorem and Ehrhart-Macdonald Reciprocity. Tying in with the previous chapter, we see that there is a relationship between the number of k-dimensional faces of a unimodular lattice triangulation and the coefficients of the Ehrhart series. If we try to impose and extend this result to polytopes that do not admit a unimodular lattice triangulation, things go awry; we give a brief 3-dimensional case study on this matter.

We view polytopes and Ehrhart theory in a different light in Chapter 4 by introducing indicator functions of polyhedra. These indicator functions span the algebra of polyhedra. Brion's Theorem provides a way to decompose a polyhedron using indicator functions of cones and other polyhedra. By considering valuations (linear functions) on elements of the algebra of rational polyhedra (namely, cones), it is possible to prove Ehrhart's theorem via valuations and Brion's Theorem. We also see a valuation called the Euler valuation, which is closely related to the Euler characteristic. We derive a few Euler-type relations that will be used in Chapter 5.

In Chapter 5, we provide a new streamlined proof of the results by Betke and Kneser. We see that there is a way to decompose the indicator function of any polytope into indicator functions of unimodular simplices. This may seem contradictory to the fact that there are polytopes that do not admit unimodular lattice triangulations. However, indicator functions allow us the freedom to include simplices that would have "protruded" out of the original polytope. We give an algorithm for such a decomposition, and analyze its runtime: the algorithm is not efficient, but does its job in proving that every polytope has a unimodular decomposition.

As a consequence of this result, we formulate conditions under which two integer unimodular invariant valuations are equivalent. We see an application of this result by proving Ehrhart-Macdonald Reciprocity and revisiting the case study in Section 3 in a more general light.

1.1 Background Check

For the remainder of the chapter, we will be laying the foundational properties and terminology of the paper. More details on these topics can be found in [1], [8], [9], and [13].

1.1.1 Convex Sets

Let us consider two sets in $\mathbb{R}^2$, displayed below. For any two points a, b in A, the line segment $\{\lambda a + (1-\lambda)b : 0 \le \lambda \le 1\}$ with endpoints a and b is contained in A, while we cannot say that the statement holds for B.

Figure 1.1. A is a convex set while B is not.

We denote $[a, b]$ as the closed line segment between a and b and sometimes (a, b) as the open line segment. The family of sets that satisfies this distinguishing property is called the family of *convex* sets. More formally, a set $S \subseteq \mathbb{R}^d$ is convex if

$$a, b \in S \implies (1-\lambda)a + \lambda b \in S, \ 0 \le \lambda \le 1. \tag{1.1}$$

The expression $(1-\lambda)a + \lambda b$ is an example of a *convex combination* of two points. More generally, a convex combination of points $a_1, \ldots, a_m$ in S is of the form

$$\sum_{i=1}^{m} \lambda_i a_i \text{ where } \sum_{i=1}^{m} \lambda_i = 1 \text{ and } 0 \le \lambda_1, \ldots, \lambda_m \le 1. \tag{1.2}$$

For convenience of indices, we will denote $[m] := \{1, \ldots, m\}$. An equivalent definition of a convex set is a set that is closed under convex combinations. If a point p in a convex set S cannot be expressed as a convex combination of other points in S, then p is a *extreme point*.

The *convex hull* $\text{conv}(S)$ of a set of points S is the smallest convex set that contains S. In other words, any convex set that contains S must also contain $\text{conv}(S)$. In two dimensions, one can imagine the convex hull of a set of points to be the shape if an elastic band were stretched and wrapped around the "outside" of the points. In any dimension, the convex hull of S is the set of all convex combinations of every subset of points in S.

Figure 1.2. A set of points and its convex hull.

For convenience, we denote $\mathrm{conv}(S_1, \ldots, S_m)$ to be $\mathrm{conv}(S_1 \cup \cdots \cup S_m)$ for sets $S_1, \ldots, S_m$, and whenever a point appears as an argument for conv( ), we really mean the singleton $\{v\}$. For example, $\mathrm{conv}(v, S)$ denotes $\mathrm{conv}(\{v\} \cup S)$. A property of convex hulls is that

$$\mathrm{conv}(v, S_1) \cap \mathrm{conv}(v, S_2) = \mathrm{conv}(v, S_1 \cap S_2).$$

for convex sets S_1 and S_2 such that if $s_1 \in S_1$, $s_2 \in S_2$ and v are co-linear, then $s_1, s_2 \in S_1 \cap S_2$. More generally, the intersection of convex sets is convex.

Another operation that we can perform on a number of sets $A_1, \ldots, A_k$ is the *Minkowski sum*

$$\sum_{i=1}^{k} A_i = A_1 + \cdots + A_k := \{a_1 + \cdots + a_k : a_i \in A_i, i = 1, \ldots, k\}.$$

A specific case of Minkowski sum is a *translation* $A + S$, where A is a singleton $\{a\}$; for convenience we drop the set notation and simply write $a + S$ to mean the Minkowski sum of $\{a\}$ and S. As demonstrated in the following theorem, convex sets and Minkowski sums cooperate rather nicely with each other.

Theorem 1.1.1. *For sets $S_1, \ldots, S_n$,*

$$\mathrm{conv}\left(\sum_{i=1}^{n} S_i\right) = \sum_{i=1}^{n} \mathrm{conv}(S_i). \tag{1.3}$$

For a convex set $S \subseteq \mathbb{R}^d$, the *interior* $\mathrm{int}(S)$ of S is the set of points $s \in S$ such that there is an open ball $B(s, \epsilon) = \{x + s : \langle x, x \rangle < \epsilon\}$ centred at s with $\epsilon > 0$ such that $B(s, \epsilon) \subseteq S$. The *closure* $\mathrm{cl}(S)$ of S is the set of points s such that for any $\epsilon > 0$, $B(s, \epsilon) \cap S \neq \emptyset$. The boundary $\partial(S)$ of S is the set $\partial(S) := \mathrm{cl}(S) \setminus \mathrm{int}(S)$. There do exist non-empty convex sets $S \subseteq \mathbb{R}^d$ that have an empty interior. In order classify such sets, we need the notion of the dimension of a set.

An *affine combination* of a set of points $a_1, \ldots, a_m$ is of the form

$$\sum_{i=1}^{m} \lambda_i a_i \text{ where } \sum_{i=1}^{m} \lambda_i = 1;$$

it is similar to convex combination, but without the non-negativity constraint on the coefficients. The terminology of affine sets are similar to the terminology used for convex sets. A set is *affine* if it is closed under affine combinations. The *affine hull* $\mathrm{aff}(S)$ of a set of points S is the smallest affine set containing S. Equivalently, the affine hull of S is the set of all affine combinations of points of S.

Theorem 1.1.2. *An affine set S in $\mathbb{R}^d$ is a translation of a linear subspace.*

Consider an affine set $S = a + V$ where V is a linear subspace. Since $0 \in V$, we see that $a \in S$. The dimension of $V = S - a$ is the the number of linearly independent points in $S - a$. We can define a set of points $a_0, \ldots, a_k$ to be *affinely independent* if the set $\{a_1 - a_0, \ldots, a_k - a_0\}$ is linearly independent. Then the dimension $\dim(S)$ of the affine set S is the dimension of V or equivalently, the maximum k such that there exists $k + 1$ affinely independent points in S.

We can now define the dimension of any convex set S to be the dimension of $\mathrm{aff}(S)$, or equivalently, one less than the maximum number of affinely independent points in S. A set in $\mathbb{R}^d$ is *full-dimensional* if it has dimension d. In a collection of sets, a set is *top-dimensional* if its dimension is largest among all the sets in that collection. Recalling the motivation of this definition, if a set $S \subseteq \mathbb{R}^d$ is not full-dimensional, then $S^\circ = \emptyset$. However, suppose $\dim(S) = k < d$; we can identify $\mathrm{aff}(S)$ with $\mathbb{R}^k$ and define the *relative interior* S° of S to be the interior of S in $\mathrm{aff}(S)$, represented as points in $\mathbb{R}^d$.

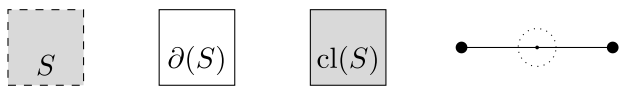

Figure 1.3. The interior of a 2-dimensional open set $S \subset \mathbb{R}^2$ is S. A line segment in $\mathbb{R}^2$ has an empty interior.

1.1.2 Polyhedra

We define an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^d$; for two points $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$ in $\mathbb{R}^d$,

$$\langle a, b \rangle = a_1 b_1 + \cdots + a_d b_d.$$

We define a *hyperplane* of $\mathbb{R}^d$ to be a set of the form

$$H = \{x : \langle c, x \rangle = b\}.$$

If c and b are integral, then H is a *rational hyperplane* and contains integer points. We can see that H "splits" $\mathbb{R}^d$ into two sides. Let *positive and negative closed half-spaces* be

$$H^+ := \{x : \langle c, x \rangle \geq b\}$$
$$H^- := \{x : \langle c, x \rangle \leq b\}$$

and open half-spaces to be similar to closed half-spaces, but with a strict inequality constraint. A *polyhedron* P is the intersection of half-spaces; it is *rational* if all associated hyperplanes of the half-spaces are rational. Since half-spaces are convex, polyhedra are convex. Note that the intersection of half-spaces can correspond to the solution set of a system of linear inequalities $P = \{x : Ax \leq b\}$. By setting $A = 0$ and $b = 0$, $\mathbb{R}^d$ is a polyhedron. The empty set is also a polyhedron, by setting $A = 0$ and $b = -1$.

Example 1.1.3. Consider the triangle below. It can be represented as a convex hull of points

(1,0), (1,1) and (2,0). Alternatively, it is the intersection of half-spaces

$$\{x : \langle (1,0), x\rangle \geq 1\}$$
$$\{x : \langle (0,1), x\rangle \geq 0\}$$
$$\{x : \langle (1,1), x\rangle \leq 2\}.$$

It could also be expressed as the solution set of the system of inequalities $\{x : Ax \leq b\}$, where

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

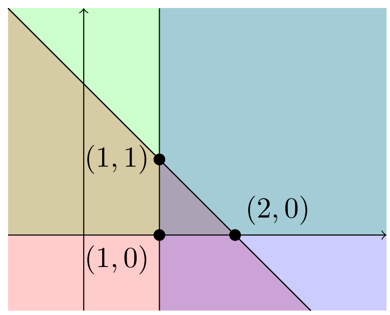

Figure 1.4. A simplex expressed as an intersection of half-spaces can also be expressed as a convex hull of its vertices.

If P is bounded, then P is a *polytope*. We say that a d-polyhedron or d-polytope is a d-dimensional polyhedron or polytope. A polyhedron is unbounded if and only if it contains a ray (or a line, but a line is just a union of two rays). We have the following two results to help us see if a polyhedron is indeed a polytope.

Lemma 1.1.4. *Let $P = \{x \in \mathbb{R}^d : Ax \leq b\}$ be a non-empty polyhedron. P contains a ray if and only if there exists an $x \neq 0$ such that $Ax \leq 0$.*

Corollary 1.1.5. *Let $P = \{x \in \mathbb{R}^d : Ax \leq b\}$ be a non-empty polyhedron. P contains a line if and only if* $\text{Nullity}(A) > 0$.

The Minkowski sum of polyhedra is a polyhedron. If we have two polyhedra $P_1 \in \mathbb{R}^{d_1}$ and $P_2 \in \mathbb{R}^{d_2}$, then their direct product

$$P_1 \times P_2 = \{(p_1, p_2) \in \mathbb{R}^{d_1+d_2} : p_1 \in P_1, p_2 \in P_2\}$$

is also a polyhedron.

1.1.3 Faces of Polyhedra

A hyperplane H or half-space H^+ (similarly with H^-) is said to *cut* a d-polyhedron $P \subseteq \mathbb{R}^d$ is on both sides of H. More formally, if $H = \{x : \langle u, x\rangle = b\}$, then P is on both sides of H if

$$\{x : \langle u, x\rangle < b\} \cap P \text{ and } \{x : \langle u, x\rangle > b\} \cap P$$

are both non-empty. If H does not cut P, then $H^+ \cap P$ is called a *face* of P; $H^- \cap P$ is also called a face of P. Note that faces are also polyhedra, since they are just an intersections of half-spaces. Also, note that P itself and the empty set are always faces of P. These two faces are called *trivial faces*. If H does not cut P but intersects P, and $P \subseteq H^+$, then $H^- \cap P$ is a *proper face* of P. We denote $F \leq P$ to mean that F is a face of P. If $H^+ \cap P$ is a proper face $F \leq P$, then H^+ is an *inward half-space* and H^- is an *outward half-space*. Note that proper faces can also be defined as $P \cap H$, where H is a supporting hyperplane.

A *supporting hyperplane* of a face F is a hyperplane H such that $F = H^- \cap P$. A 0-dimensional face (equivalently, an extreme point of P) is called a *vertex*. A polytope can actually be defined as the convex hull of its vertices. If all its vertices are lattice points (ie. if its vertices have integer coordinates), then the polytope is a *lattice polytope*.

A $(d-1)$-dimensional face is called a *facet*. If $F = H^- \cap P$ is a facet, then H is called the *facet hyperplane* of F. Note that facet hyperplanes are unique for each facet, while supporting hyperplanes for lower dimensional faces are not. Therefore, it is suitable to define the *inward normal* of a facet to be the vector u such that u is the normal of the facet hyperplane pointing into P, and the *outward normal* to be the normal pointing away from P. We will work primarily with rational polyhedra. In this case, the facet normals could all be scaled to be integral, and so we can assume that these normals are integral and primitive (ie. the greatest common divisor of its entries is 1).

Theorem 1.1.6. *Let P be a non-empty polyhedron. Then*

- *The intersection of two faces of a polyhedron is a face*
- *The faces of a face F are exactly the faces of P that are contained in F*
- *Every proper face of P is an intersection of a subset of its facets.*
- *P can be expressed as the intersection its of inward facet half-spaces.*
- *P contains a vertex if and only if P does not contain a line*
- *A face of P is the solution set of the facet constraints of P, where a subset of these constraints are tight.*

1.1.4 Simplices

A *simplex* of dimension d (or a *d-simplex*) is the convex hull of $d+1$ affinely independent points. Note that $d+1$ points is the fewest number of points we need such that their convex hull is d-dimensional. If a simplex contains no lattice points aside from its vertices, the simplex is *empty*.

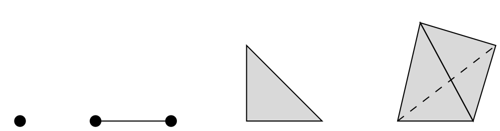

Figure 1.5. $0, 1, 2$ and 3-dimensional simplices.

The set of faces of a simplex S is exactly the collection of convex hulls of every subset of vertices of S. Since vertices of S are affinely independent, the faces of a simplex are themselves simplices. If the vertices of S are integral, the S *lattice simplex*. Let lattice simplex S have vertices $v_1, \ldots, v_{d+1}$. Then the *volume* $\mathrm{Vol}(S)$ of S is

$$\mathrm{Vol}(S) = |\det(V)|$$

where V is a $d \times d$ matrix whose columns correspond to $v_1 - v_{d+1}, \ldots, v_d - v_{d+1}$. Note that we have scaled our definition of volume by $d!$ for convenience. Equivalently, if we append 1 onto all the vertex vectors, we have

$$\mathrm{Vol}(S) = \left| \det \begin{pmatrix} v_1 & \ldots & v_{d+1} \\ 1 & \ldots & 1 \end{pmatrix} \right|.$$

If $\mathrm{Vol}(S) = 1$, then S is said to be *unimodular*. Unimodular simplices are empty. A family of important unimodular simplices are the standard simplices. Let e_i be the the standard basis vector, where the i^{th} entry of e_i is 1, and zero elsewhere. The standard d-simplex is the convex hull $\mathrm{conv}(e_1, \ldots, e_d, 0)$.

For a lattice d-simplex $S \subset \mathbb{R}^d$ with vertices $v_1, \ldots, v_{d+1}$, let $S' \subset \mathbb{R}^{d+1}$ be the d-simplex with vertices $v'_1, \ldots, v'_{d+1}$, where $v'_i = (v_i, 1)$. The *parallelepiped* generated by vertices of S' is the polytope

$$\Pi_{S'} = \left\{ \sum_{i=1}^{d+1} \lambda_i v_i : 0 \leq \lambda_i \leq 1, i = 1, \ldots, d+1 \right\}.$$

The *top-open* parallelepiped $\overline{\Pi}_{S'}$ and *bottom-open* parallelepiped $\underline{\Pi}_{S'}$ generated by vertices of S' have the additional constraint that $\lambda_i < 1$ and $0 < \lambda_i$ respectively. If S is unimodular, then the only lattice points in $\Pi_{S'}$ are its vertices.

1.1.5 Cones

A *ray emanating from v in the direction u* is a set of points

$$R = \{v + \tau u : \tau \geq 0\},$$

where $u, v \in \mathbb{R}^d$ and $u \neq 0$. A *cone* is a polyhedron that can be represented as a union of rays emanating from a point. If this point is unique, then it is the only vertex of the cone, and we call the cone a *pointed cone*. Note that $\emptyset$ and $\mathbb{R}^d$ are also cones. We often "cone over a polyhedron" P, by which we mean constructing the cone that is the collection of rays from the origin passing through points in P:

$$\mathrm{cone}(P) := \{\tau p : p \in P, \tau \geq 0\}.$$

If P is a polytope with vertices $v_1, \ldots, v_n$ then $\mathrm{cone}(P)$ can actually be expressed as the non-negative span of its vertices:

$$\mathrm{cone}(P) = \mathrm{span}_{\geq 0}(v_1, \ldots, v_n).$$

P is a *base* of cone K if K can be expressed as $\operatorname{cone}(P)$; note that bases of K are not unique. A cone has a base if and only if it has a vertex. For any pointed rational cone, there will always be a base that is a lattice polytope.

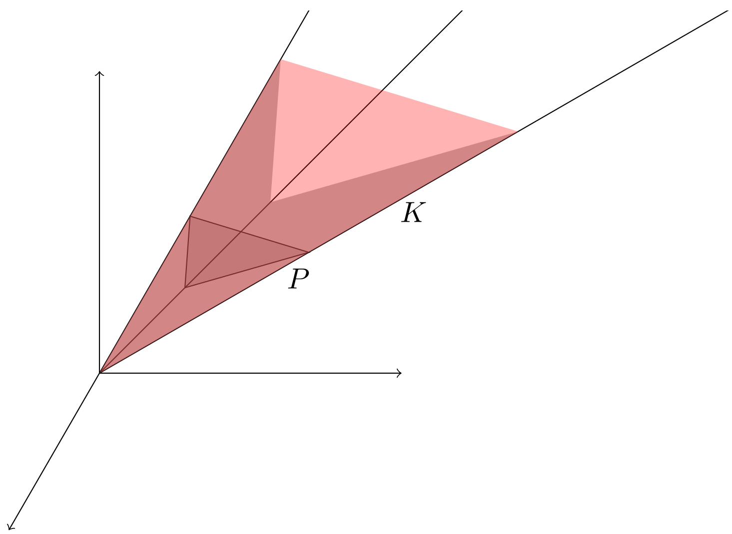

Figure 1.6. Polytope P is the base of cone K.

The *recession cone* of a polyhedron P is a cone K such that for a $p \in P$, the set of rays in P emanating from p is equal to some translate of K. There exists a recession cone for every polyhedron. Since a polytope contains no rays, the recession cone of any polytope is the origin.

Theorem 1.1.7. *Let $P \subset \mathbb{R}^d$ be a polyhedron without straight lines. Then P can be expressed as the Minkowski sum of its recession cone and the convex hull of its vertices.*

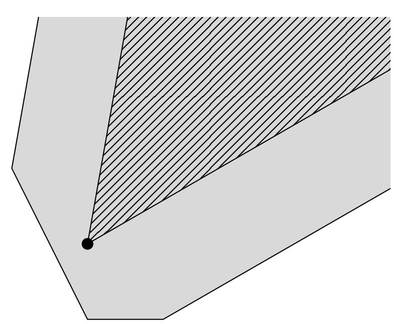

Figure 1.7. A polyhedron and its recession cone. Figure inspired by [1].

Chapter 2

Polyhedral Subdivisions

A *polyhedral complex* $\mathscr{C}$ is a collection of polytopes (called *cells* of $\mathscr{C}$) such that

(CP) If $C \in \mathscr{C}$ and $F \leq C$, then $F \in \mathscr{C}$. (Closure Property)

(IP) If $C \neq C'$ are two cells in $\mathscr{C}$, then $C \cap C'$ is a (potentially empty) face of both C and C'. (Intersection Property)

Given a polytope (or polyhedron) P, the a *polytope (or polyhedral) subdivision* $\mathscr{S}$ is a polyhedral complex that satisfies the additional condition

(UP) The union of all cells in $\mathscr{S}$ is P. (Union Property)

Additionally, if P has a vertex and if the vertices of each cell in $\mathscr{S}$ is a vertex of P, then $\mathscr{S}$ is said to be a *subdivision with no new vertices.* From properties of polyhedra, it is easy to verify that for a lattice polytope P, the collection of all the faces of P is a subdivision of P with no new vertices.

If every cell in $\mathscr{S}$ is a simplex, then $\mathscr{S}$ is called a *triangulation* of P. If every cell of a triangulation is a lattice simplex, then $\mathscr{S}$ is a *lattice triangulation.* Note that only polyhedra with integral vertices have lattice triangulations. Since we are mainly interested in lattice triangulations, let us take "triangulation" to really mean "lattice triangulation". A triangulation is *maximal* if all of its cells are empty simplices. It is possible for some polytopes to have more than one maximal triangulation; these maximal triangulations need not have the same number of cells, as illustrated in Example 2.3.7.

(a) (b)

Figure 2.1. (a) does not describe a subdivision of a square, while (b) does.

Lemma 2.0.8. *The only triangulation of an empty simplex S is the collection of all the faces of S.*

Proof. We assume S is full-dimensional; otherwise, we can just work in the affine subspace that contains S. Let $\mathscr{S}$ be the collection of faces of S. $\mathscr{S}$ is a subdivision of S. Since all faces of a simplex are simplices, this subdivision is indeed a triangulation. If the vertices of S are its only lattice points, then any vertex of a simplex in a triangulation of S must also be a vertex of S, implying that the simplex is a face of S. By (UP), the face $S \leq S$ must be in the triangulation. By (IP), all faces of S must also be in the triangulation. Therefore, $\mathscr{S}$ is the only triangulation of S. $\square$

A triangulation is *unimodular* if all of its simplices are unimodular. Since unimodular simplices are empty, every unimodular triangulation is maximal. However, the converse is not true as demonstrated in the following example. It is also possible for a polytope to have a unimodular triangulation, but also to have another maximal non-unimodular triangulation, as illustrated in Example 2.3.7.

Example 2.0.9. Consider a simplex S with vertices $(0,0,0)$, $(1,0,0)$, $(0,0,1)$, and $(a,b,1)$ where $0 < a < b$ and a and b are coprime. This simplex is empty. Therefore, the only triangulation of S is the collection of its faces. Since there is only one triangulation, this triangulation is maximal. However, S has volume b, and $b \geq 2$, so S is not unimodular, and the triangulation is not unimodular.

It is often useful to subdivide or triangulate P, but we need to know that triangulations do exist for every P. A method of triangulating a polytope P is called a *pushing triangulation.* This method starts with a simplex in P and iteratively introduces a new vertex to build a bigger polytope. In order to outline the method, we need to introduce the concept of visibility.

2.1 Point Visibility

Let $P \subset \mathbb{R}^d$ be a d-polytope, and let v be a point that is not contained in the relative interior of P. A point $p \in P$ is visible to v if the line segment

$$\{\lambda v + (1-\lambda)p : 0 \leq \lambda \leq 1\}$$

intersects P at exactly p. A face $F \leq P$ is visible to v if every point $p \in F$ is visible to v. Note that if v is in P, then no other point in P is visible to v.

Lemma 2.1.1. *Facet F is visible to v if and only if the facet hyperplane separates v and P.*

In other words, if u is the outward normal of F and $H = \{x : \langle u, x\rangle = b\}$ is the facet hyperplane of F, then F is visible to v if and only if $\langle u, v\rangle > b$.

Proof. Let u be the outward normal of F and $H = \{x : \langle u, x\rangle = b\}$ be the supporting hyperplane of F. Suppose $\langle u, v\rangle \leq b$. Let $p \neq v$ be a point in the relative interior of F. There exists a ball B centred at p such that $H^- \cap B$ is a closed half-ball in P, and the other open half-ball is not in P. Any line segment $[v,p]$ must intersect B. Since p is on the hyperplane H, any point on $[v,p)$ is on the same side of H that v is on (if $v \in H$, then $[v,p] \in H$). Therefore, if $\langle u,v\rangle \leq b$, then the line segment $[v,p]$ intersects P at p and other points, and so F is not visible to v.

Now consider $\langle u,v\rangle > b$. Note that for any point p in F, $\langle u,p\rangle = b$. Then for $0 \leq \lambda \leq 1$ and any $p \in F$,

$$\begin{aligned}
\langle u, \lambda v + (1-\lambda)p\rangle &= \lambda\langle u,v\rangle + (1-\lambda)\langle u,p\rangle \\
&= \lambda\langle u,v\rangle + (1-\lambda)b \\
&\geq \lambda b + (1-\lambda)b \\
&= b
\end{aligned}$$

with equality if and only if $\lambda = 0$. Therefore, F is visible to v. □

Let H be a hyperplane that does not cut through P, and let pr be the orthogonal projection of points in P onto H. Then a face $F \leq P$ is visible to the hyperplane H if for every $p \in F$, p is visible to $\mathrm{pr}(p)$.

Lemma 2.1.2. *Let $P \subset \mathbb{R}^d$ be a d-polytope and suppose $H = \{x : \langle c,x\rangle = b\}$ is a hyperplane such that $P \subset H^-$. Then facet $F \leq P$ with outward normal u is visible to H if and only if* $\langle c,u\rangle > 0$.

Proof. Consider a point p in facet F, with outward normal u. Then the projected point $v := \mathrm{pr}(p)$ can be expressed as $p + \tau c$ for some $\tau \geq 0$. If $\tau = 0$, then $v = p$, and p is visible to v. Now, if $\tau > 0$, then

$$\begin{aligned}
\langle u,v\rangle &= \langle u, p + \tau c\rangle \\
&= \langle u,p\rangle + \tau\langle u,c\rangle \\
&= b + \tau\langle u,c\rangle.
\end{aligned}$$

We see that $\langle u,c\rangle > 0$ if and only if $\langle u,v\rangle > b$, and since u is the outward normal of F, it is equivalent to F being visible to v; in particular, p is visible to v. □

Where it is clear, we shall take "visible facet" to mean "facet visible to a point v".

2.2 Pushing Triangulations

Now, we are ready to outline the construction of a pushing triangulation. We will prove our claims after outlining the whole method. Let P have n vertices. Order the vertices of P such that the first $d+1$ vertices are affinely independent; by relabelling, and for ease of notation, we can assume that the vertices are ordered and labelled as $v_{-d}, \ldots, v_{n-d-1}$. Starting at the zeroeth iteration, let $\mathscr{T}$ be the collection containing $\mathrm{conv}(v_{-d}, \ldots, v_0)$ and all of its faces. At the beginning of each iteration $k \geq 1$, $\mathscr{T}$ is a triangulation of $P_{k-1} := \mathrm{conv}(v_{-d}, \ldots, v_{k-1})$. Then for any $(d-1)$-dimensional cell C of $\mathscr{T}$ that is visible to v_k, add $\mathrm{conv}(C, v_k)$ and all its faces to $\mathscr{T}$. We continue until $P_k = P$.

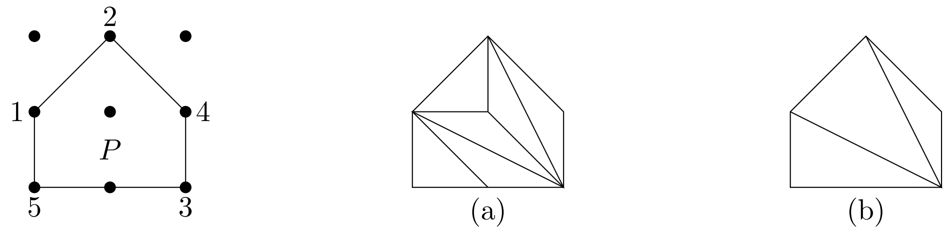

Figure 2.2. (a) is a unimodular triangulation of P; (b) is the pushing triangulation of P, taking its vertices in the order listed. Note that (b) uses no new vertices.

Lemma 2.2.1. *By our construction, $\mathscr{T}$ is a triangulation with no new vertices.*

Again, we need to verify the three triangulation conditions. Let us break the proof down into several parts. We proceed by induction on n. Since P is full-dimensional, our base case is $n = d + 1$. Our algorithm terminates after the zeroeth iteration; $\mathscr{T}$ is a subdivision. Since P is a simplex, $\mathscr{T}$ is a triangulation.

Taking the inductive step, $\mathscr{T}'$ is the triangulation of $P_{k-1} := \operatorname{conv}(v_{-d}, \ldots, v_{k-1})$ that we get at the beginning of the k^{th} iteration.

Let $\mathcal{F}$ be the set of facets of P_{k-1} visible to v_k. We can define

$$\mathscr{T}'|_F := \{S : S \in \mathscr{T}', S \subseteq F\}$$

to be the triangulation restricted to the facet $F \in \mathcal{F}$.

Note that since $\mathscr{T}'$ does not use new vertices, neither do any of the $\mathscr{T}'|_F$. Let $\mathscr{T}_{\text{vis}}$ be the collection of convex hulls of v_k with each $(d-1)$-simplex of $\mathscr{T}'|_F$ for all visible faces $F \in \mathcal{F}$. We claim that

$$\mathscr{T} = \mathscr{T}_{\text{vis}} \cup \mathscr{T}'$$

is a triangulation for $P_k := \operatorname{conv}(v_{-d}, \ldots, v_k)$.

Lemma 2.2.2. *$\mathscr{T}$ satisfies (CP).*

Proof. Since $\mathscr{T}'$ already satisfies the closure property by the induction hypothesis, we just have to consider simplices in $\mathscr{T}_{\text{vis}}$. Suppose that $S \in \mathscr{T}_{\text{vis}}$ is a simplex of the form $\operatorname{conv}(C, v_k)$, where $C \in \mathscr{T}'$ is contained in a face visible to v_k. Without loss of generality, let C be the convex hull of $v_1, \ldots, v_\ell$. A face of S is of the form $C_I := \operatorname{conv}(v_i : i \in I)$ where $I \subseteq \{1, \ldots, \ell, k\}$. If $k \notin I$, then $C_I \leq C \in \mathscr{T}'$ and (CP) holds. If $k \in I$, then note that $C_{I\setminus\{k\}}$ is a face of C, and therefore, is a cell of $\mathscr{T}'$, contained in a face visible to v_k. By construction, C_I is then a cell in $\mathscr{T}_{\text{vis}}$ and therefore in $\mathscr{T}$. □

Lemma 2.2.3. *$\mathscr{T}$ satisfies (IP).*

Proof. Given $S_1, S_2 \in \mathscr{T}$, there are three cases:

1. $S_1, S_2 \in \mathscr{T}'$

2. $S_1, S_2 \in \mathscr{T}_{\text{vis}} \backslash \mathscr{T}'$

3. $S_1 \in \mathscr{T}_{\text{vis}} \backslash \mathscr{T}'$ and $S_2 \in \mathscr{T}'$

1. The result is immediate from the induction hypothesis.

2. Let $S_1 = \text{conv}(C_1, v_k)$ and $S_2 = \text{conv}(C_2, v_k)$, where C_1 and C_2 are cells of $\mathscr{T}'$ that are contained in faces visible to v_k. Then by convexity,

$$\begin{aligned} S_1 \cap S_2 &= \text{conv}(C_1, v_k) \cap \text{conv}(C_2, v_k) \\ &= \text{conv}((C_1 \cap C_2), v_k). \end{aligned}$$

   Since C_1 and C_2 are in $\mathscr{T}'$, induction hypothesis implies that $C_1 \cap C_2$ is a face of both C_1 and C_2. Since both are simplices, $C_1 \cap C_2$ is the convex hull of the common vertices of C_1 and C_2. Therefore, $S_1 \cap S_2$ is the convex hull of their common vertices, and hence is a face of both S_1 and S_2.

3. Let $S_1 = \text{conv}(C_1, v_k)$, where $C_1 \in \mathscr{T}'$ is in a visible facet. Note that

$$\begin{aligned} S_1 \cap S_2 &= S_1 \cap (S_2 \cap P_{k-1}) \\ &= (S_1 \cap P_{k-1}) \cap S_2 \\ &= C_1 \cap S_2. \end{aligned}$$

   Since C_1 and S_2 are both in cells $\mathscr{T}'$, induction hypothesis implies that $S_1 \cap S_2$ is a face of both C_1 and S_2. Now, since C_1 is a face S_1, $S_1 \cap S_2$ is a face of S_1. $\square$

Lemma 2.2.4. *$\mathscr{T}$ satisfies (UP).*

Proof. Any point in $\bigcup_{C \in \mathscr{T}} C$ is in a simplex whose vertices are also the vertices of P_k. Therefore, $\bigcup_{C \in \mathscr{T}} C \subseteq P_k$. Now consider some $p \in P_k$. If $p \in P_{k-1}$, induction hypothesis states that $p \in \bigcup_{C' \in \mathscr{T}'} C' \subseteq \bigcup_{C \in \mathscr{T}} C$, so suppose $p \notin P_{k-1}$. Since v_k is a cell in $\mathscr{T}$, suppose also that $p \neq v_k$. Consider the line through v_k and p. Because P_k is convex, this line meets P_{k-1}. Let $x \in P_{k-1}$ be the first point we meet in P_{k-1} when traveling on the line from v_k towards P_{k-1}. The point x is on a facet of P_{k-1} that is visible to v_k, and so $p \in S$ for some $S \in \mathscr{T}_{\text{vis}}$. Therefore, $\bigcup_{C \in \mathscr{T}} C \supseteq P_k$. $\square$

Lastly, note that we have used no new vertices, since each simplex is by construction a convex hull of vertices of P.

2.3 Regular Subdivisions

Another useful type of polytope subdivision requires viewing the vertices of the polytope in a higher dimension. Let $P \subset \mathbb{R}^d$ be a d-polytope with vertex set $V = \{v_1, \ldots, v_n\}$. We assign a height function $\omega : \mathbb{R}^d \to \mathbb{Z}_+$, and for each $v \in V$, we denote the point $v' := (v, \omega(v)) \in \mathbb{R}^{d+1}$. If we project the polytope $P_\omega := \text{conv}(v_1', \ldots, v_n')$ back to the first d coordinates, we get P. In other words, we identify $\mathbb{R}^d$ with some hyperplane $H_m = \{x \in \mathbb{R}^{d+1} : x_{d+1} = m\}$ and apply the orthogonal projection $\mathrm{pr}_m : (x_1, \ldots, x_{d+1}) \mapsto (x_1, \ldots, x_d, m)$ on P_ω to get P.

If we set $m := \max\{\omega(v) : v \in V\}$, then P_ω is contained in the closed region bounded by $H_0 = \{x : x_{d+1} = 0\}$ and H_m. A facet of P_ω is a lower facet if it is visible to H_0 and an upper facet if it is visible to H_m. Note that there may exist facets that are neither lower nor upper facets.

Example 2.3.1. Consider the polytope $P \subset \mathbb{R}^3$ with vertices

$$v_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}.$$

We can see that $\mathrm{conv}(v_0, \ldots, v_3)$ is a facet of P with outward normal $(0, 0, -1)$. Assign a height function and let $P_\omega \subset \mathbb{R}^4$ be the polytope with vertices

$$v'_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v'_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, v'_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, v'_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, v'_4 = \begin{pmatrix} 0 \\ 0 \\ 4 \\ 0 \end{pmatrix}.$$

Again, we can see that $F' = \mathrm{conv}(v'_0, \ldots, v'_3)$ is a facet of P_ω with outward normal $(0, 0, -1, 0)$. A normal of H_0 is $(0, 0, 0, -1)$. By Lemma 2.1.2, since $\langle (0, 0, 0, -1), (0, 0, -1, 0) \rangle = 0$, F' is neither a lower or upper facet.

Lemma 2.3.2. *Lower faces of P form a polyhedral complex.*

Proof. The faces of P form a polyhedral complex, so the intersection property is preserved when taking a subset of faces of P. To prove the closure property, we need to show that faces of lower faces are lower faces. Let $F \leq P$ be a lower face. For every point $p \in F$, let $\mathrm{pr}(p)$ be the orthogonal projection of p onto H_0. By definition, the closed line segment $[p, \mathrm{pr}(p)]$ intersects P at exactly p. Since any face of $F' \leq F$ is a subset of F, this property is preserved in F' and therefore, by definition of lower faces and visibility, F' must be a lower face. $\square$

We would like to project the lower faces of P onto H_0. Suppose there are two points p_1 and p_2 that are contained in lower faces such that $\mathrm{pr}(p_1) = \mathrm{pr}(p_2)$. Then p_1 and p_2 are on the line $\{\mathrm{pr}(p_1) + te_{d+1} : t \in \mathbb{R}$. Equivalently, p_1, p_2 and $\mathrm{pr}(p_1)$ are co-linear. Without loss of generality, let $[\mathrm{pr}(p_1), p_1]$ be a closed line segment containing p_2. Since p_1 is visible to H_0, p_1 is the only point in $[\mathrm{pr}(p_1), p_1]$ that is also in P. Since p_2 is also in P, we must have $p_1 = p_2$. Therefore, points in the lower faces are in $1-1$ correspondence with the projection (onto H_0) of points in the lower faces.

Lemma 2.3.3. *Let F be a lower face. A set of points $M = \{p_1, \ldots, p_k\} \subseteq F$ is affinely independent if and only if $\mathrm{pr}(M)$ is affinely independent.*

Proof. M is affinely dependent if and only if there exists a set of α_i, such that

$$\sum_{i=1}^{k} \alpha_i p_i = 0$$

with $\sum_{i=1}^{k} \alpha_i = 0$ and not all α_i are zero. Then

$$\begin{aligned}\sum_{i=1}^{k} \alpha_i \operatorname{pr}(p_i) &= \operatorname{pr}\left(\sum_{i=1}^{k} \alpha_i p_i\right) \\ &= \operatorname{pr}(0) \\ &= 0.\end{aligned}$$

Therefore $\operatorname{pr}(M)$ is affinely dependent. Since there is a $1-1$ correspondence with H_0 and the lower faces of P, the inverse of pr is well-defined. Tracing the above argument backwards would prove the other direction of the statement. □

In other words, the number of affinely independent points in lower faces are preserved in the projection pr. Since the dimension of a polytope is equivalent to the number of affinely independent points, we have the following corollary.

Corollary 2.3.4. *A lower face F with dimension k projects (onto H_0) to a k-polytope.*

Lemma 2.3.5. *Let $\mathcal{F}$ be the collection of lower faces of P_ω. Then the collection*

$$\{\operatorname{pr}(F) : F \in \mathcal{F}\}$$

is a polytope subdivision of P.

Proof. Since $\mathcal{F}$ is a polyhedral complex, Corollary 2.3.4 implies that (CP) and (IP) are preserved from $\mathcal{F}$.

For a point $p \in P$, the intersection of line $p \times \mathbb{R}$ with P_ω is a closed line segment $L := \{(p, \tau) : s \leq \tau \leq t\}$; L could possibly contain just a point. To prove the union property, suppose for a contradiction that no point in L is in a lower face of P_ω. This will lead to the conclusion that L is not a closed line segment.

Let us look at P as an intersection of facet half-spaces; $P = \{x : \langle u_i, x\rangle \leq b_i\}$. Let $\mathcal{H}_{low}$ be the set of facet half-spaces associated with lower facets. In other words, the half-space $\{x : \langle c, x\rangle \leq b\}$ is in $\mathcal{H}_{low}$ if and only if $c_{d+1} < 0$. Since no point in L are on lower facets, any point in L satisfies the half-space constraints of P_ω, with strict inequality on constraints in $\mathcal{H}_{low}$. Consider (p, s') with $s' < s$; note that $(p, s') \notin P_\omega$. For any facet half-space not in $\mathcal{H}_{low}$, we have

$$\begin{aligned}\langle c, (p, s')\rangle &= c_1 p_1 + \cdots + c_d p_d + c_{d+1} s' \\ &\leq c_1 p_1 + \cdots + c_d p_d + c_{d+1} s \\ &= \langle c, (p, s)\rangle \\ &\leq b\end{aligned}$$

since $c_{d+1} \geq 0$. For any facet half-space in $\mathcal{H}_{low}$, we have

$$\begin{aligned}\langle c, (p, s')\rangle &= c_1 p_1 + \cdots + c_d p_d + c_{d+1} s' \\ &> c_1 p_1 + \cdots + c_d p_d + c_{d+1} s \\ &= \langle c, (p, s)\rangle.\end{aligned}$$

Since the inner product is continuous, we can choose our s' to be close enough to s such that

$$\langle c, (p, s)\rangle < \langle c, (p, s')\rangle < b.$$

However this means that (p, s') in is P_ω and therefore in L, which leads to a contradiction. $\square$

Note that as long as P_ω is on the same side of H_0, any translations of P_ω and H_0 preserve whether a facet is upper, lower or neither. Let P'_ω be a polytope we obtain by reflecting P_ω across H_0 and translating it to be on the same side of H_0 as P_ω. Then the lower and upper faces of P_ω correspond to the upper and lower faces (respectively) of P'_ω. Therefore, upper faces of P_ω also form a polytope subdivision. By a change of basis, we can extend this result for any hyperplane.

Theorem 2.3.6. *Let $P \subset \mathbb{R}^d$ be a polytope, H be a hyperplane that does not cut P, and $\mathcal{F}$ be the set of faces of P visible to H. Then the orthogonal projection of faces in $\mathcal{F}$ onto H form a polytope subdivision of the orthogonal projection of P onto H.*

Example 2.3.7. Let P be the 3-dimensional polytope with vertices

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \; v_4 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \; v_5 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Consider the height function that results in the following lifted vertices:

$$v'_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; v'_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; v'_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; v'_4 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \; v'_5 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

The upper and lower faces induce two different triangulations for P, as shown in Figure 2.3. Note that these vertices are affinely independent, so the convex hull of any four of these vertices is a simplex. Let Δ_i be the simplex with vertices v'_j such that $1 \leq j \leq 5, j \neq i$. Each projection $\mathrm{pr}(\Delta_i)$ onto the hyperplane $\{x \in \mathbb{R}^4 : x_4 = 0\}$, when viewed in $\mathbb{R}^3$, would be a cell in exactly one of the two triangulations described above.

The simplex $\mathrm{pr}(\Delta_5)$ has volume 2, while each of the other $\mathrm{pr}(\Delta_i)$ has volume 1. From Example 2.0.9, we see that $\mathrm{pr}(\Delta_5)$ cannot be subdivided into any more simplices. Since the sum of the volumes of the cells in each triangulation of P must be equal, P has a maximal triangulation containing cells $\mathrm{pr}(\Delta_5)$ and $\mathrm{pr}(\Delta_2)$, and another maximal triangulation containing $\mathrm{pr}(\Delta_1)$, $\mathrm{pr}(\Delta_3)$ and $\mathrm{pr}(\Delta_4)$. Note that P admits a unimodular triangulation, yet also has another maximal triangulation that is not unimodular.

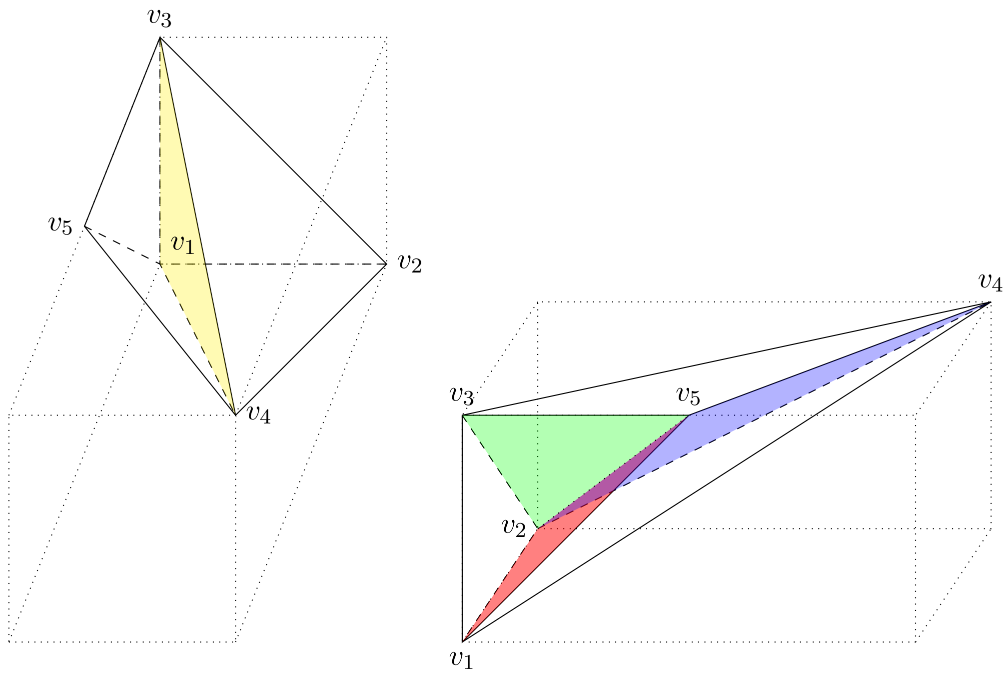

Figure 2.3. Two maximal triangulations of P.

Chapter 3

Ehrhart Theory

Ehrhart theory is in some way the "skeleton" of this paper. The different ideas that we will cover in the paper are connected together by Ehrhart theory, whether by finding applications in Ehrhart theory, or using concepts for the proofs of this chapter, or inspired by Ehrhart theory. In this chapter, we will see the development of Ehrhart's Theorem, Stanley Non-negativity, Ehrhart-Macdonald Reciprocity, and relationships between a polytope's Ehrhart series and triangulations.

3.1 Ehrhart's Theorem

Given a lattice d-polytope $P \subset \mathbb{R}^d$, we define the *lattice point enumerator* $L_P : \mathbb{Z}_+ \to \mathbb{Z}_+$ of P to be

$$L_P(t) = |tP \cap \mathbb{Z}^d|.$$

In words, $L_P(t)$ is the number of lattice points in the t-dilate of P. The Ehrhart series is the generating function

$$\mathrm{Ehr}_P(z) := \sum_{t \geq 0} L_P(t) z^t.$$

Note that $L_P(0)$ is equal to 0 if $P = \emptyset$ and 1 otherwise. We will prove later in this section that for lattice polytopes, the lattice point enumerator can actually be represented as polynomial in t. Hence, $L_P(t)$ is referred to as an Ehrhart polynomial.

Theorem 3.1.1 (Ehrhart's Theorem)**.** *Let P be a lattice polytope in $\mathbb{R}^d$. Then $L_P(t)$ is a polynomial in t of degree d.*

We take *lifting P up one dimension to P'* to mean the following:

- Let $v_1, \ldots, v_n \in \mathbb{R}^d$ be the vertices of P
- For $i \in [n]$, let $v_i' := (v_i, 1) \in \mathbb{R}^{d+1}$
- Let $P' := \mathrm{conv}\{v_1', \ldots, v_n'\}$.

In words, we embed P in the hyperplane $\{x \in \mathbb{R}^{d+1} : x_{d+1} = 1\}$ and call it P'. Recall the since P' is a polytope, $\mathrm{cone}(P')$ is the cone generated by the vertices of P'. We say that the point $(x_1, \ldots, x_{d+1}) \in \mathrm{cone}(P')$ is at level or height x_{d+1} and denote it as $\mathrm{level}(x)$.

Lemma 3.1.2. *Let $P \subset \mathbb{R}^d$ be a lattice polytope. For $t \in \mathbb{Z}_+$, the intersection of* $\mathrm{cone}(P')$ *with the hyperplane $H = \{x \in \mathbb{R}^{d+1} : x_{d+1} = t\}$ is (after identifying $\mathbb{R}^d$ with H) tP, the t-dilate of P.*

Proof. The result is immediate for $t = 0$. Let $v_1, \ldots, v_n$ be the vertices of P, and define v_i' as above. Fix a $t > 0$ and let $H = \{x \in \mathbb{R}^{d+1} : x_{d+1} = t\}$. Firstly, note that $\mathrm{cone}(P') \cap H$ is the set of points of $\mathrm{cone}(P')$ at height t. Secondly, note that $\mathrm{cone}(P')$ is a convex cone. Let $x \in \mathbb{R}^{d+1}$ be a point in $\mathrm{cone}(P')$ at height t, and let $y := \frac{1}{t}x$. The point y is in $\mathrm{cone}(P')$ and is at height 1; therefore, $y \in P'$, which implies that $x \in tP'$. Identifying $\mathbb{R}^d$ with H, we see that $x \in tP$. $\square$

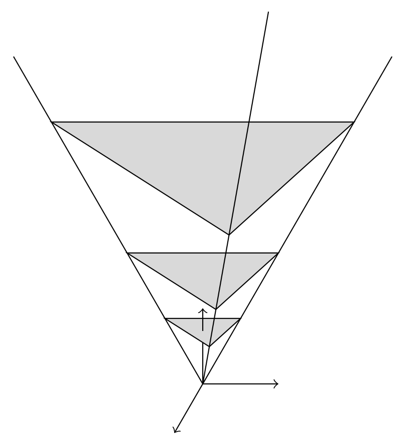

Figure 3.1. Height cross sections of a cone over a lifted 2-simplex.

We want to encode the information of lattice points in a set into a generating function. The *Hilbert series* $\sigma_S(\mathbf{x})$ of a set $S \subseteq \mathbb{R}^n$ to be

$$\sigma_S(\mathbf{z}) := \sum_{\mathbf{p} \in (S \cap \mathbb{Z}^n)} \mathbf{z}^{\mathbf{p}}$$

where, given $\mathbf{z} = (z_1, \ldots, z_n)$ and $\mathbf{p} = (p_1, \ldots, p_n)$,

$$\mathbf{z}^{\mathbf{p}} := z_1^{p_1} z_2^{p_2} \ldots z_n^{p_n}.$$

Let us set $S = \mathrm{cone}(P') \subseteq \mathbb{R}^{d+1}$ and evaluate $\sigma_S(\mathbf{z})$ at $(1, \ldots, 1, z)$. Applying Lemma 3.1.2, we have

$$\begin{aligned}
\sigma_{\mathrm{cone}(P')}(\mathbf{z}) &= \sum_{\mathbf{p'}\in(\mathrm{cone}(P')\cap\mathbb{Z}^{d+1})} \mathbf{z}^{\mathbf{p}} \\
&= \sum_{t\geq 0}\sum_{\mathbf{p}\in(tP\cap\mathbb{Z}^d)} (z_1,\ldots,z_d)^{\mathbf{p}} z_{d+1}^t \\
\sigma_{\mathrm{cone}(P')}(1,\ldots,1,z) &= \sum_{t\geq 0}\sum_{\mathbf{p}\in(tP\cap\mathbb{Z}^d)} (1,\ldots,1)^{\mathbf{p}} z^t \\
&= \sum_{t\geq 0}\sum_{\mathbf{p}\in(tP\cap\mathbb{Z}^d)} z^t \\
&= \sum_{t\geq 0} L_P(t) z^t.
\end{aligned}$$

Therefore, we have the following result.

Lemma 3.1.3. *Let $P \subset \mathbb{R}^d$ be a lattice polytope, and let $P' \subset \mathbb{R}^{d+1}$ be the associated lifted polytope. Then*

$$\mathrm{Ehr}_P(z) = \sigma_{\mathrm{cone}(P')}(1,\ldots,1,z).$$

We now represent $\mathrm{cone}(P')$ using a technique we refer to as *tiling*.

Lemma 3.1.4. *Let $K \subset \mathbb{R}^n$ be a closed pointed cone with vertex at the origin, generated by linearly independent vectors $v_1,\ldots,v_n \in \mathbb{Z}^n$, and $\overline{\Pi}_K$ be the top-open parallelepiped generated by $v_1,\ldots,v_n$. Then for any $u \in \mathbb{R}^n$, every lattice point p in $u+K$ can be uniquely represented as*

$$p = x + \sum_{i\in[n]} \tau_i v_i$$

where $\tau_i \in \mathbb{Z}_+$ and x is a lattice point in $u + \overline{\Pi}_K$.

Proof. Since the generators are linearly independent, each point $p \in u + K$ is uniquely represented as a non-negative linear combination of the generators, translated by u:

$$p = u + \sum_{i\in[n]} \alpha_i v_i.$$

Let $\lfloor \alpha_i \rfloor$ be the largest integer such that $\lfloor \alpha_i \rfloor \leq \alpha_i$, and $\{\alpha_i\} := \alpha_i - \lfloor \alpha_i \rfloor$. We say that $\lfloor \alpha_i \rfloor$ and $\{\alpha_i\}$ are the *integer part* and *fractional part* of α_i, respectively. By definition, $0 \leq \{\alpha_i\} < 1$. We then have

$$p = u + \sum_{i\in[n]} \{\alpha_i\} v_i + \sum_{i\in[n]} \lfloor \alpha_i \rfloor v_i.$$

Note that since $\lfloor \alpha_i \rfloor \in \mathbb{Z}_+$ and $p, v_i \in \mathbb{Z}^n$,

$$x := u + \sum_{i\in[n]} \{\alpha_i\} v_i = p - \sum_{i\in[n]} \lfloor \alpha_i \rfloor v_i$$

is integral. Since $0 \leq \{\alpha_i\} < 1$, x is a lattice point in $u + \overline{\Pi}_K$. To show uniqueness, suppose that

$$\begin{aligned}
\sum_{i\in[n]} \{\beta_i\} v_i + \sum_{i\in[n]} \lfloor \beta_i \rfloor v_i =& p - u = \sum_{i\in[n]} \{\alpha_i\} v_i + \sum_{i\in[n]} \lfloor \alpha_i \rfloor v_i \\
\sum_{i\in[n]} \{\beta_i\} v_i - \sum_{i\in[n]} \{\alpha_i\} v_i =& \sum_{i\in[n]} \lfloor \alpha_i \rfloor - v_i \sum_{i\in[n]} \lfloor \beta_i \rfloor v_i \\
\sum_{i\in[n]} (\{\beta_i\} - \{\alpha_i\}) v_i =& \sum_{i\in[n]} (\lfloor \alpha_i \rfloor - \lfloor \beta_i \rfloor) v_i.
\end{aligned}$$

Note that $\lfloor \alpha \rfloor - \lfloor \beta \rfloor \in \mathbb{Z}$, and that linear combinations of the generators are unique. Therefore, $\{\beta_i\} - \{\alpha_i\} = \lfloor \alpha_i \rfloor - \lfloor \beta_i \rfloor$ for all $i \in [n]$. In particular, this implies that $\{\beta_i\} - \{\alpha_i\}$ is an integer. Since $0 \leq \{\beta_i\}, \{\alpha_i\} < 1$, we must have

$$\begin{aligned}
\{\beta_i\} - \{\alpha_i\} &= \lfloor \alpha_i \rfloor - \lfloor \beta_i \rfloor = 0 \\
\beta_i &= \lfloor \beta_i \rfloor + \{\beta_i\} = \lfloor \alpha_i \rfloor + \{\alpha_i\} = \alpha_i.
\end{aligned}$$

□

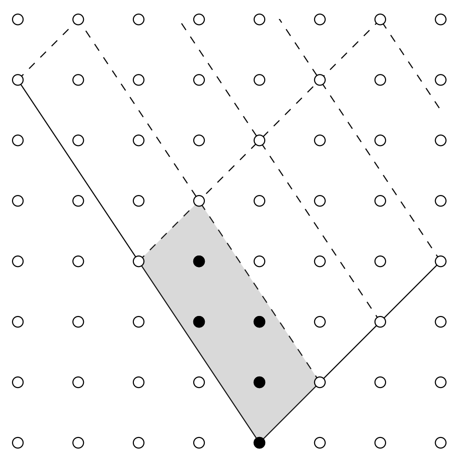

Figure 3.2. Tiling of a 2-dimensional cone using a top-open parallelepiped. The bottom-open parallelepiped would tile the interior of the cone. Figure inspired by [5].

Using the same ideas as the above proof, the use of the bottom-open parallelepiped yields a similar result.

Corollary 3.1.5. *Let $K \subset \mathbb{R}^n$ be a closed pointed cone with vertex at the origin, generated by linearly independent vectors $v_1, \ldots, v_n \in \mathbb{Z}^n$, and $\underline{\Pi}_K$ be the bottom-open parallelepiped generated by $v_1, \ldots, v_n$. Then for any $u \in \mathbb{R}^n$, every interior lattice point p in $u + K$ can be uniquely represented as*

$$p = x + \sum_{i\in[n]} \tau_i v_i$$

where $\tau_i \in \mathbb{Z}_+$ and x is a lattice point in $u + \overline{\Pi}_K$.

Lemma 3.1.6. *For a pointed cone $K \subset \mathbb{R}^n$ generated by linearly independent vectors $v_1, \ldots, v_n \in \mathbb{Z}^n$,*

$$\sigma_K(\mathbf{z}) = \frac{\sigma_{\overline{\Pi}_K}(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{v}_1})\ldots(1-\mathbf{z}^{\mathbf{v}_n})}$$

Proof. This proof simply follows from the definition of $\sigma_K(\mathbf{z})$ and Lemma 3.1.4:

$$\begin{aligned}
\sigma_K(\mathbf{z}) &= \sum_{\mathbf{p}\in(K\cap\mathbb{Z}^n)} \mathbf{z}^{\mathbf{p}} \\
&= \sum_{\mathbf{x}\in(\overline{\Pi}_K\cap\mathbb{Z}^n)} \sum_{\substack{\lambda_i\in\mathbb{Z}_+ \\ i\in[n]}} \mathbf{z}^{\mathbf{x}+\lambda_1\mathbf{v}_1+\cdots+\lambda_n\mathbf{v}_n} \\
&= \left(\sum_{\lambda_1\geq 0} (\mathbf{z}^{\mathbf{v}_1})^{\lambda_1}\right)\cdots\left(\sum_{\lambda_n\geq 0} (\mathbf{z}^{\mathbf{v}_n})^{\lambda_n}\right)\left(\sum_{\mathbf{x}\in\overline{\Pi}_K} \mathbf{z}^{\mathbf{x}}\right) \\
&= \left(\frac{1}{1-\mathbf{z}^{\mathbf{v}_1}}\right)\cdots\left(\frac{1}{1-\mathbf{z}^{\mathbf{v}_n}}\right)\left(\sum_{\mathbf{x}\in(\overline{\Pi}_K\cap\mathbb{Z}^n)} \mathbf{z}^{\mathbf{x}}\right).
\end{aligned}$$

□

We are now ready prove Ehrhart's Theorem.

Proof of Ehrhart's Theorem. First, note that $L_\emptyset(t) = 0$, and is a polynomial, so suppose P is non-empty. Let $\mathscr{T}$ be a triangulation of P. Note that P can be expressed (via the inclusion-exclusion principle) as the union and differences of simplices in $\mathscr{T}$. Then

$$L_P(t) = \sum_{S_i\in\mathscr{T}} \gamma_i L_{S_i}(t)$$

where $\gamma_i \in \{-1, 1\}$. Since the sum of polynomials is still a polynomial and the degree of the sum is the maximum degree among the summands, it suffices to prove the theorem for simplices. Let $\Delta \subset \mathbb{R}^d$ be a lattice simplex. Without loss of generality, we may assume Δ is full dimensional; otherwise we can work in the affine hull of Δ. Let $v_1, \ldots, v_{d+1}$ be the vertices of Δ, and let Δ' be the associated lifted simplex.

$$\begin{aligned}
\mathrm{Ehr}_\Delta(z) &= \sigma_{\mathrm{cone}(\Delta')}(1,\ldots,1,z) \\
&= \left.\frac{\sum_{\mathbf{x}\in(\overline{\Pi}_{\mathrm{cone}(\Delta')}\cap\mathbb{Z}^{d+1})} \mathbf{z}^{\mathbf{x}}}{(1-\mathbf{z}^{\mathbf{v}'_1})\ldots(1-\mathbf{z}^{\mathbf{v}'_{d+1}})}\right|_{\mathbf{z}=(1,\ldots,1,z)} \\
&= \frac{\sum_{\mathbf{x}\in(\overline{\Pi}_{\mathrm{cone}(\Delta')}\cap\mathbb{Z}^{d+1})} z^{\mathrm{level}(\mathbf{x})}}{(1-(1,\ldots,1)^{\mathbf{v}_1}z^1)\ldots(1-(1,\ldots,1)^{\mathbf{v}_{d+1}}z^1)} \\
&= \frac{h^*(z)}{(1-z)^{d+1}}
\end{aligned}$$

where $h^*(z) := \sum_{\mathbf{x}\in(\overline{\Pi}_{\mathrm{cone}(\Delta')}\cap\mathbb{Z}^{d+1})} z^{\mathrm{level}(\mathbf{x})}$. Recall that for all $d+1$ generators $v'_i = (v_i, 1)$ of $\mathrm{cone}(\Delta')$, we have $\mathrm{level}(v'_i) = 1$. Therefore, for any lattice point $\mathbf{x}$ in the top-open parallelepiped $\overline{\Pi}_{\mathrm{cone}(\Delta')}$, $\mathrm{level}(\mathbf{x}) < d+1$, which implies that the degree of $f(z)$ is at most d. Let $h^*(z) =$

$h_0 + h_1 z + \cdots + h_d z^d$. Note that the origin is the only lattice point in $\overline{\Pi}_{\mathrm{cone}(\Delta')}$ that is at level zero, which implies that $h_0 = 1$.

$$\begin{aligned}\frac{h^*(z)}{(1-z)^{d+1}} &= \sum_{i=0}^{d} h_i \frac{z^i}{(1-z)^{d+1}} \\ &= \sum_{i=0}^{d} h_i \sum_{j\geq 0} \binom{d+j}{d} z^{j+i} \\ &= \sum_{t\geq 0} \left(\sum_{i=0}^{d} h_i \binom{d+t-i}{d} \right) z^t.\end{aligned}$$

In other words,

$$L_\Delta(t) = h_0 \binom{t+d}{d} + h_1 \binom{t+d-1}{d} + \cdots + h_{d-1} \binom{t+1}{d} + h_d \binom{t}{d}, \tag{3.1}$$

and therefore, $L_P(t)$ is a polynomial. It remains to show that $L_P(t)$ has degree d.

Each $\binom{t+i}{d}$ for $i = 0, \dots, d$ is a polynomial of degree d with positive leading coefficient. Note that the coefficients of $h^*(z)$ are non-negative; they count the number of lattice points at a certain level of $\overline{\Pi}_{\mathrm{cone}(\Delta')}$. In particular, note that $h_0 > 0$. Since there is at least one positive coefficient in $h^*(z)$, $[t^d]L_\Delta(t) \neq 0$; the degree of $L_\Delta(t)$ is $\dim(\Delta) = d$. The coefficient $[t^d]L_P(t)$ is then the sum of all the leading coefficients of $L_\Delta(t)$, indexing over all top dimensional cells $\Delta \in \mathscr{T}$. All such leading terms are positive, and therefore, the degree of $L_P(t)$ is $\dim(P) = d$. $\square$

Example 3.1.7. Let Δ be the standard d-simplex with vertices $e_0, e_1 \dots, e_d$, where $e_0 = 0$ and let $e_i' := (e_i, 1)$ and $\Delta' := \mathrm{conv}(e_0', \dots, e_d')$. There is only one lattice point in the associated top-open parallelepiped $\overline{\Pi}_{\Delta'}$, and that point is the origin. Therefore

$$\mathrm{Ehr}_\Delta(z) = \frac{z^0}{(1-z)^{d+1}} = \frac{1}{(1-z)^{d+1}}$$

and

$$L_\Delta(t) = \binom{t+d}{d}.$$

3.2 Stanley's Non-negativity Theorem

For a simplex, the coefficients of $h^*(z)$ are non-negative, since they are counting something. Does this property hold when the polytope is not a simplex? Even though we may not be counting something for the general polytope, the answer is still *yes*. The following result was proved by Beck, Haase, and Sottile.

Theorem 3.2.1 (Stanley's Non-negativity Theorem)**.** *Suppose $P \subset \mathbb{R}^d$ is an lattice d-polytope with Ehrhart series*

$$\mathrm{Ehr}_P(z) = \frac{h^*(z)}{(1-z)^{d+1}}$$

where $h^*(z)$ is a polynomial in z. Then the coefficients of $h^*(z)$ are non-negative.

We defer the proof to the end of the section. If P is a lattice polytope, then P' would also have integer vertices, which implies that each facet-hyperplane H of cone(P') is rational; they can be expressed as

$$H = \{x \in \mathbb{R}^d : \langle c, x \rangle = 0\}$$

for some $c \in \mathbb{Z}^{d+1}$. The boundary of cone(P') would then contain lattice points. Triangulate cone(P') into simplicial cones $K_1, \ldots, K_\ell$. We aim to "perturb" cone(P') by a small amount, say vector $v \in \mathbb{R}^d$ so that

1. Any lattice point is in at most one $K_i + v$
2. The set of lattice points contained in $v +$ cone(P') is set of lattice points contained in cone(P').

For any $c \in \mathbb{Z}^n \backslash \{0\}$, the hyperplane $H = \{x : \langle c, x \rangle = 0\}$ is a linear subspace of dimension $n-1$, and therefore $\mathbb{R}^n/H$ is a 1-dimensional subspace. We can use the line L_c through the origin and c to represent $\mathbb{R}^n/H$.

Lemma 3.2.2. *For any primitive integer vector $c \in \mathbb{Z}^n \backslash \{0\}$ the orthogonal projection of $\mathbb{Z}^n$ onto L_c is equal to the lattice generated by $\frac{c}{||c||^2}$.*

Proof. Let $x = (x_1, \ldots, x_n)$ be a lattice point. The projection of x onto L_c is

$$\mathrm{pr}_{L_c}(x) = \frac{\langle c, x \rangle}{||c||^2} c = \frac{c}{||c||^2} c_1 x_1 + \cdots + c_n x_d.$$

Since $x, c \in \mathbb{Z}^n$, the expression $c_1 x_1 + \cdots + c_n x_n$ is an integer. Now we must prove that $c_1 x_1 + \cdots + c_n x_n$ attains every integer value. Without loss of generality, suppose c_1 and c_2 are coprime. Then there is an integer solution $(\bar{x}_1, \bar{x}_2)$ to the Diophantine equation

$$c_1 \bar{x}_1 + c_2 \bar{x}_2 = 1.$$

Therefore, the projection of $(k\bar{x}_1, k\bar{x}_2, 0, \ldots, 0)$ onto L_c is $\frac{k}{||c||^2} c$ for any integer k, which completes our proof. □

By identifying L_c with $\mathbb{R}$, we can represent $\mathbb{R}^n/H$ as $\mathbb{R}$. We observe that

- $v + H$ is represented by $\langle c, v \rangle$
- $v + H^+$ is represented by ray $\{x \in \mathbb{R} : x \geq \langle c, v \rangle\}$
- $v + H$ contains a lattice point if and only if $\langle c, v \rangle$ is an integer

For a pointed cone $K \subset \mathbb{R}^n$ with vertex at the origin, facet hyperplanes $H_1, \ldots, H_k$ and primitive inward normals $c_1, \ldots, c_k \in \mathbb{Z}^n$, point v satisfies the *perturbing vector condition* for K if $-1 < \langle c_i, v \rangle < 0$ for all $i = 1, \ldots, k$.

Lemma 3.2.3. *For any v that satisfies the perturbing vector condition for K, the set of lattice points in $v + K$ is equal to the set of lattice points in K.*

Proof. Since K is the intersection of its facet half-spaces, it suffices to show that for each $i = 1, \ldots, k$, the set of lattice points in $v + H_i^+$ is equal to the set of lattice points in H_i^+. Let us look at $\mathbb{R}^n/H$, treated as $\mathbb{R}$.

We view H^+ and $v + H^+$ as the rays $R_1 := \{x \in \mathbb{R} : x \geq 0\}$ and $R_2 := \{x \in \mathbb{R} : x \geq \langle c, v\rangle\}$ respectively. We see that $R_1 \cap \mathbb{Z} = \mathbb{Z}_+$ and $R_2 \cap \mathbb{Z} = \{x \in \mathbb{Z} : x \geq \lceil\langle c, v\rangle\rceil\}$. By our choice of v, we see that $\lceil\langle c, v\rangle\rceil = 0$. Therefore, $R_1 \cap \mathbb{Z} = R_2 \cap \mathbb{Z}$. $\square$

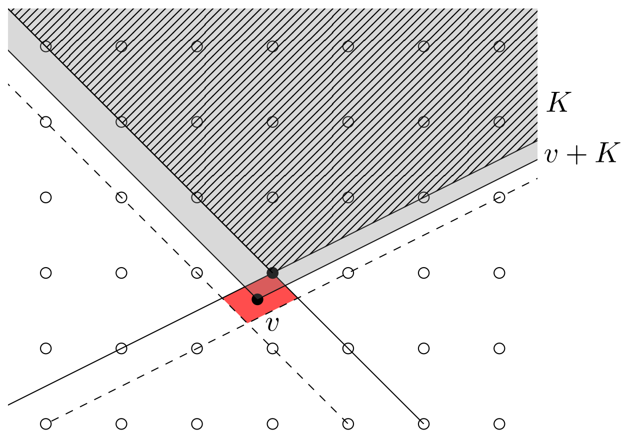

Figure 3.3. Perturbing cone K. Any point in the interior of the red region satisfies the perturbing vector condition.

Proof of Stanley's Non-negativity Theorem. Triangulate $\mathrm{cone}(P')$ into simplicial cones $K_1, \ldots, K_\ell$. Let $\mathcal{N}$ be the set of normals of the facet hyperplanes of $K_1, \ldots, K_\ell$. Pick a $v \in \mathbb{R}^d$ such that

- $\langle c, v\rangle$ is not an integer for $c \in \mathcal{N}$
- v satisfies the perturbing vector condition with respect to $\mathrm{cone}(P')$

There is such a perturbing vector v because there are only finitely many facet hyperplanes. Since $\langle c_i, v\rangle$ is not an integer, each $v + H_i$ does not contain any lattice points. There are no lattice points in the facet hyperplanes of simplicial cones $v + K_1, \ldots, v + K_\ell$. Therefore, any lattice point is in at most one K_i. Applying Lemma 3.2.3 , we see that

$$\mathrm{cone}(P') \cap \mathbb{Z}^d = (v + \mathrm{cone}(P')) \cap \mathbb{Z}^d = \bigsqcup_{i=1}^{\ell} ((v + K_i) \cap \mathbb{Z}^d).$$

We then have

$$\begin{aligned}\frac{h^*(z)}{(1-z)^{d+1}} &= \sigma_{\mathrm{cone}(P')}(1, \ldots, 1, z) \\ &= \sigma_{v+\mathrm{cone}(P')}(1, \ldots, 1, z) \\ &= \sum_{i=1}^{m} \sigma_{v+K_i}(1, \ldots, 1, z).\end{aligned}$$

Since the denominator of each $\sigma_{v+K_i}(1, \ldots, 1, z)$ is $(1-z)^{d+1}$, and their numerators count the number of integer points in a parallelepiped, the coefficients of $h^*(z)$ are non-negative. $\square$

3.3 Unimodular Triangulations and Ehrhart Polynomials

We now apply some of the tools we used in the previous section to see a relationship between a triangulation of a polytope and its Ehrhart polynomial. We define the f-vector of a triangulation $\mathscr{T}$ of a lattice d-polytope to be

$$f(\mathscr{T}) = (f_{-1}(\mathscr{T}), f_0(\mathscr{T}), \dots, f_d(\mathscr{T}))$$

where $f_{-1}(\mathscr{T}) = 1$ and each other $f_k(\mathscr{T})$ is the number of k-dimensional faces in the triangulation. Where there is no ambiguity, we will use f_k to denote $f_k(\mathscr{T})$.

Lemma 3.3.1. *If a lattice d-polytope $P \subset \mathbb{R}^d$ has a unimodular triangulation $\mathscr{T}$ with f-vector $(f_{-1}, \dots, f_d)$, then*

$$L_P(t) = \sum_{k=0}^{d} \binom{t-1}{k} f_k$$

Proof. This proof uses similar steps to the proof of Ehrhart's theorem. First we let $\mathscr{T}$ be a unimodular triangulation of P. By noting that P is the disjoint union of all the relative interiors of the cells of $\mathscr{T}$, we have

$$L_P(t) = \sum_{C \in \mathscr{T}} L_{C^\circ}(t).$$

In light of this realization, let us take a k-dimensional unimodular simplex $S \in \mathscr{T}$ and view it in $\mathbb{R}^k$. Let S have vertices $v_1, \dots, v_{k+1} \in \mathbb{Z}^k$. Consider the $(k+1)$-dimensional cone of S'

$$K_{S'} := \operatorname{cone}(S') = \operatorname{cone}\{v'_1, \dots, v'_{k+1}\}$$

where $v'_i = (v_i, 1)$. Now, we consider the bottom-open parallelepiped $\underline{\Pi}_{S'}$ generated by the vertices of S'. Recall that any lattice point p in the interior of K_S can be written as a unique combination

$$p = \sum_{i=1}^{k+1} \tau_i v'_i + x \tag{3.2}$$

where $\tau_i \in \mathbb{Z}_+$ and x is a lattice point in $\underline{\Pi}_{S'}$. Since $\operatorname{level}(v'_i) = 1$ for all $i = 1, \dots, k+1$, we see that

$$\begin{aligned} \operatorname{level}(p) &= \tau_1 + \cdots + \tau_{k+1} + \operatorname{level}(x) \\ \operatorname{level}(p) - \operatorname{level}(x) &= \tau_1 + \cdots + \tau_{k+1} \end{aligned} \tag{3.3}$$

Fix a lattice point $x \in \underline{\Pi}_{S'}$ and $t \geq \operatorname{level}(x)$. We are interested in the number of points p of the form in (3.2) such that $\operatorname{level}(p) = t$. From (3.3), we see that this number is equal to the number of un-ordered partitions of $t - \operatorname{level}(x)$ of size at most $k+1$, which is

$$\binom{t - \operatorname{level}(x) + k}{k}.$$

Therefore, we can express $L_{S^\circ}(t)$ as

$$L_{S^\circ}(t) = \sum_{x \in \underline{\Pi}_S} \binom{t - \operatorname{level}(x) + k}{k} = \sum_{i=1}^{k+1} \delta_i \binom{t - i + k}{k} \tag{3.4}$$

where δ_i is the number of lattice points $x \in \underline{\Pi}_{S'}$ such that $\text{level}(x) = i$. Since S is unimodular, $\delta_{k+1} = 1$ and for all other i, $\delta_i = 0$. Thus, we have

$$L_{S^\circ}(t) = \binom{t-1}{k}.$$

Going back to P, we have

$$L_P(t) = \sum_{C \in \mathscr{T}} L_{C^\circ}(t) = \sum_{k=0}^{d} \binom{t-1}{k} f_k. \qquad \square$$

Instead of determining $L_P(t)$, let us try to to determine the Ehrhart series $\text{Ehr}_P(z)$ directly. We know that the $(k+1)$-dimensional bottom-open parallelepiped associated with a k-dimensional unimodular simplex contains only one lattice point, and that this lattice point is at level $k+1$. Therefore, the Ehrhart series of the interior of a k-dimensional unimodular simplex is

$$\text{Ehr}_{\Delta^\circ} = \frac{z^{k+1}}{(1-z)^{k+1}}.$$

Note that tiling with the bottom-open parallelepiped would omit the origin, or the zero dilation of P. Taking into account the zero dilation of P, we have

$$\begin{aligned} \text{Ehr}_P(z) &= 1 + f_0 \frac{z}{1-z} + \cdots + f_d \frac{z^{d+1}}{(1-z)^{d+1}} \\ &= \frac{h^*(z)}{(1-z)^{d+1}} \end{aligned}$$

where $h^*(z)$ is a polynomial in z. From the above expression, it may seem like the degree of $h^*(z)$ may be $d+1$, but the proof of Ehrhart's Theorem implies that the degree of $h^*(z)$ is in fact at most d.. We encode the coefficients of $h^*(z)$ into the h^*-vector of P:

$$(h_0, \dots, h_{d+1})$$

where $h_\ell = [z^\ell]h^*(z)$. Let us determine the h^*-vector of P in terms of the entries of the f-vector of a triangulation of P.

Theorem 3.3.2. *Let $P \subset \mathbb{R}^d$ be a lattice d-polytope. If P has h^*-vector $(h_0, \dots, h_{d+1})$ and a unimodular triangulation with f-vector $(f_{-1}, \dots, f_d)$, then*

$$h_\ell = \sum_{k=0}^{\ell} f_{k-1} \binom{d+1-k}{\ell-k} (-1)^{\ell-k}$$

and

$$f_{\ell-1} = \sum_{k=0}^{\ell} h_k \binom{d+1-k}{\ell-k}.$$

Proof. The first result follows from looking at the coefficients of $h^*(z)$.

$$\begin{aligned}
h^*(z) &= (1-z)^{d+1}\operatorname{Ehr}(z) \\
&= (1-z)^{d+1}\sum_{k=0}^{d+1} f_{k-1}(1-z)^{-k}z^k \\
&= \sum_{k=0}^{d+1} f_{k-1}(1-z)^{d+1-k}z^k \\
&= \sum_{k=0}^{d+1} f_{k-1}\sum_{i=0}^{d+1-k}\binom{d+1-k}{i}(-1)^i z^{k+i} \\
[z^\ell]h^*(z) &= \sum_{k=0}^{\ell} f_{k-1}\binom{d+1-k}{\ell-k}(-1)^{\ell-k}.
\end{aligned}$$

For the second result, let us define two polynomial functions

$$f(z) := f_{-1}z^{d+1} + f_0 z^d + \cdots + f_{d-1}z + f_d \tag{3.5}$$

$$h(z) := h_0 z^{d+1} + h_1 z^d + \cdots + h_d z + h_{d+1}. \tag{3.6}$$

First, we show that $h(z) = f(z-1)$ by comparing coefficients. For $0 \le \ell \le d+1$,

$$\begin{aligned}
[z^{d+1-\ell}]f(z-1) &= [z^{d+1-\ell}]\sum_{k=0}^{d+1} f_{k-1}(z-1)^{d+1-k} \\
&= [z^{d+1-\ell}]\sum_{k=0}^{d+1} f_{k-1}\sum_{i=0}^{d+1-k}\binom{d+1-k}{i}(-1)^i z^{d+1-k-i} \\
&= \sum_{k=0}^{\ell} f_{k-1}\binom{d+1-k}{\ell-k}(-1)^{\ell-k} \\
&= h_\ell \\
&= [z^{d+1-\ell}]h(z).
\end{aligned}$$

Now, we compute the coefficients of $f(z) = h(z+1)$:

$$\begin{aligned}
f_{\ell-1} &= [z^{d+1-\ell}]f(z) \\
&= [z^{d+1-\ell}]h(z+1) \\
&= [z^{d+1-\ell}]\sum_{k=0}^{d+1} h_k(z+1)^{d+1-k} \\
&= [z^{d+1-\ell}]\sum_{k=0}^{d+1} h_k\sum_{i=0}^{d+1-k}\binom{d+1-k}{i} z^{d+1-k-i} \\
&= \sum_{k=0}^{\ell} h_k\binom{d+1-k}{\ell-k}.
\end{aligned}$$

□

Theorem 3.3.2 tells us that the f-vector of a unimodular triangulation (if one exists) of a lattice polytope P is uniquely expressed in terms of the polytope's h^*-vector. Since $h^*(z)$ is unique, all unimodular triangulations of P have the same f-vector. However, if P does not admit a unimodular triangulation then this theorem might not be able to give us any f-vector of any triangulation of P at all. Example 3.4.3 demonstrates what happens when we misuse this theorem. We will see in section later that the f-vector gives appropriate information on decompositions rather than triangulations.

3.4 Examples of Ehrhart Polynomials

Example 3.4.1. Let us revisit the Ehrhart series of a standard d-simplex Δ. Since Δ is already a unimodular simplex, it has a unimodular triangulation: the collection of all faces of Δ. Let the f-vector of Δ be $(f_{-1}, f_0, \dots, f_d)$. In this particular case, we know that

$$f_i = \binom{d+1}{i+1}.$$

The Ehrhart series of Δ is then

$$\begin{aligned} 1 + \sum_{k=0}^{d} f_k \frac{z^{k+1}}{(1-z)^{k+1}} &= 1 + \sum_{k=0}^{d} \binom{d+1}{k+1} \frac{z^{k+1}}{(1-z)^{k+1}} \\ &= \frac{h^*(z)}{(1-z)^{d+1}}. \end{aligned}$$

By defining $f(z)$ and $h(z)$ as defined in (3.5) and (3.6), we have

$$\begin{aligned} f(z) &= \binom{d+1}{0} z^{d+1} + \binom{d+1}{1} z^d + \cdots + \binom{d+1}{d} z + \binom{d+1}{d+1} \\ &= (1+z)^{d+1} \\ h(z) &= f(z-1) \\ &= z^{d+1}. \end{aligned}$$

By the definition of $h(z)$, we see that $h_0 = 1$ and $h_\ell = 0$ for $\ell = 1, \dots, d+1$, which agrees with Example 3.1.7. For a more specific example, if Δ is 3-dimensional, then the Ehrhart series is

$$1 + \frac{4z}{(1-z)} + \frac{6z^2}{(1-z)^2} + \frac{4z^3}{(1-z)^3} + \frac{z^4}{(1-z)^4}.$$

Notice that since the associated bottom-open parallelepiped of a unimodular simplex has only one lattice point, if lattice polytope admits a unimodular triangulation, then it is very easy to determine its Ehrhart series. However, using the f-vector method can quickly get tedious for a polytope without a unimodular triangulation, since we would need to look at each specific face of the triangulation to determine relative interior lattice points of its associated bottom-open parallelepiped.

Example 3.4.2. Let us look at a polytope that does not have a unimodular triangulation. Let S be the simplex with vertices

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \; v_4 = \begin{pmatrix} a \\ b \\ 1 \end{pmatrix}$$

with $0 < a < b$, coprime. This simplex has volume b and is an empty simplex. Additionally, any lattice 3-simplex with volume b could be mapped to S via a unimodular transformation. Thus, the only triangulation of S is the collection of its faces. Since S has volume b, S does not have a unimodular triangulation. Let us look at the faces of the triangulation in order of dimension. Note that since S is an empty simplex, the vertices, edges and 2-dimensional faces of S have no relative interior lattice points. In these dimensions, this implies that these faces are unimodular. Therefore, their Ehrhart series are going to take on the form of the Ehrhart series in Example 3.4.1. The resulting Ehrhart series would be

$$\mathrm{Ehr}_\Delta(z) = 1 + \frac{4z}{(1-z)} + \frac{6z^2}{(1-z)^2} + \frac{4z^3}{(1-z)^3} + \frac{p(z) + z^4}{(1-z)^4}$$

for some polynomial $p(z)$ of degree at most 3. We determine $p(z)$ by looking at the interior lattice points of the associated parallelepiped. In other words, we need to count the number of integer points $x = (x_1, x_2, x_3, x_4)$ of the form

$$x = \begin{pmatrix} 0 & 1 & 0 & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix}$$

where $0 < \lambda_1, \lambda_2, \lambda_3, \lambda_4 < 1$. We see that $x_2 = b\lambda_4$ must be integer, so let

$$\lambda_4 = \frac{k}{b}$$

for some integer $0 < k < b$. Also, $x_3 = \lambda_3 + \lambda_4$ must be integer, but we see that $0 < \lambda_3, \lambda_4 < 1$, which implies that

$$\begin{aligned} 0 <& \lambda_3 + \lambda_4 < 2 \\ \implies& \lambda_3 + \lambda_4 = 1. \end{aligned}$$

Since

$$\begin{aligned} \mathrm{level}(x) &= x_4 \\ &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \\ &= \lambda_1 + \lambda_2 + 1 \end{aligned}$$

must be an integer, we see that $\lambda_1 + \lambda_2$ must also be integer. Using the same argument that $0 < \lambda_1, \lambda_2 < 2$, we see that

$$\lambda_1 + \lambda_2 = 1.$$

Therefore, the only interior lattice points of the parallelepiped are at level 2. Let us count how many such lattice points there are. Again,

$$\begin{aligned} x_1 &= \lambda_2 + a\lambda_4 \\ &= \lambda_2 + \frac{ak}{b} \end{aligned}$$

must be integer, and $0 < \lambda_2 < 1$, so

$$\lambda_2 = 1 - \left\{ \frac{ak}{b} \right\}$$

where $\{n\}$ denotes the fractional part of n. However, since $\lambda_2 < 1$, we need $\left\{\frac{ak}{b}\right\} \neq 0$, which happens if and only if $b \nmid ak$. Since a and b are coprime, for any $0 < k < b$, we have $b \nmid ak$. Therefore, there are $b-1$ such lattice points;

$$p(z) = (b-1)z^2.$$

Note that $p(z)$ does not depend of a at all! Now suppose we force our idea and claim that we can use Theorem 3.3.2 to determine the f-vector of some triangulation of P, even though P does not admit a unimodular triangulation.

Example 3.4.3. Let us take the simplex Δ with vertices $(0,0,0)$, $(1,0,0)$, $(0,0,1)$ and $(a,b,1)$ where $0 < a < b$ are coprime integers. We know that the Ehrhart series is

$$\mathrm{Ehr}_\Delta(z) = 1 + \frac{4z}{(1-z)} + \frac{6z^2}{(1-z)^2} + \frac{4z^3}{(1-z)^3} + \frac{(b-1)^2 + z^4}{(1-z)^4}.$$

Using partial fraction decomposition, the Ehrhart series can be written as

$$\mathrm{Ehr}_\Delta(z) = 1 + \frac{4z}{(1-z)} + \frac{(b+5)z^2}{(1-z)^2} + \frac{(2b+2)z^3}{(1-z)^3} + \frac{bz^4}{(1-z)^4}.$$

Now, we suppose (by way of contradiction) that $(1,4,b+5,2b+2,b)$ is the f-vector for a triangulation for $b \geq 2$. This is obviously false, since there are 4 vertices in the triangulation, which means that there should be at most $\binom{4}{2} = 6$ edges in the triangulation. With $b \geq 2$ our supposed f-vector has at least 7 edges in its triangulation, which leads to a contradiction.

3.5 Ehrhart-Macdonald Reciprocity

We now explore the relationship between the number of lattice points in P and the number of lattice points in the interior of P. More specifically, we see the development of the following reciprocity theorem. We defer the proof of this theorem to the end of the section.

Theorem 3.5.1 (Ehrhart-Macdonald Reciprocity)**.** *Suppose P is a lattice d-poytope. Then the evaluation of L_P at negative integers yields*

$$L_P(-t) = (-1)^d L_{P^\circ}(t).$$

Example 3.5.2. Recall that for a unimodular d-simplex Δ, we have $L_\Delta(t) = \binom{d+t}{d}$ and $L_{\Delta^\circ}(t) = \binom{t-1}{d}$. Since

$$\begin{aligned} L_\Delta(-t) &= \binom{d-t}{d} \\ &= \frac{(d-t)(d-t-1)\dots(-t+1)}{t!} \\ &= (-1)^d \frac{(t-d)(t-d+1)\dots(t-1)}{t!} \\ &= (-1)^d \binom{t-1}{d} \\ &= (-1)^d L_{\Delta^\circ}(t), \end{aligned}$$

we see that the reciprocity theorem holds for unimodular simplices.

Like the proof of Stanley's non-negativity theorem, we would like to triangulate cone(P') into top-dimensional simpicial cones $K_1, \dots, K_\ell$ and translate cone(P') by $v \in \mathbb{R}^d$ such that

1. Any lattice point is in at most one K_i
2. The set of lattice points contained in $v + \text{cone}(P')$ is the set of lattice points contained in cone(P').

We've already seen that any v satisfying the perturbing vector condition would suffice. We now prove a counterpart of Lemma 3.2.3.

Lemma 3.5.3. *Let $K \subset \mathbb{R}^n$ be a pointed cone with vertex at the origin. Then for any vector v satisfying the perturbing vector condition, the set of lattice points in $-v + K$ is the set of interior lattice points in K.*

Proof. Mimicking the proof of Lemma 3.2.3, we aim to show that the set of interior lattice points in each inward facet half-space H^+ is equal to the set of lattice points in $-v + H^+$. Again, we consider R^n/H and view H^+ and $-v + H^+$ as rays $R_1 := \{x \in \mathbb{R} : x \geq 0\}$ and $R_2 := \{x \in \mathbb{R} : x \geq \langle c, -v\rangle\}$ respectively, where $c \in \mathbb{Z}^n$ is the inward normal of H^+. The set of interior lattice points of R_1 is $\mathbb{Z}_{>0}$. The set of lattice points in $-v + H^+$ is and $R_2 \cap \mathbb{Z} = \{x \in \mathbb{Z} : x \geq \lceil\langle c, -v\rangle\rceil\}$. Recalling that v satisfies the perturbing vector condition, implying that $0 < \langle c, -v\rangle < 1$ completes our proof. □

In summary, for such a v and cone K, we have the following:

1. $K^\circ \cap \mathbb{Z}^d = (-v + K) \cap \mathbb{Z}^d$
2. $\partial(-v + K_i) \cap \mathbb{Z}^d = \emptyset$ for all $i = 1, \dots, \ell$
3. $\partial(v + K_i) \cap \mathbb{Z}^d = \emptyset$ for all $i = 1, \dots, \ell$
4. $K \cap \mathbb{Z}^d = (v + K) \cap \mathbb{Z}^d$.

Now, we will need to prove a chain of results that will lead us to the proof of the Ehrhart-Macdonald Reciprocity theorem.

Theorem 3.5.4. *Suppose K is a (simplicial) cone generated by linearly independent integer vectors $w_1, \dots, w_n$, and that Π° is the open parallelepiped*

$$\Pi^\circ := \{\lambda_1 w_1 + \cdots + \lambda_n w_n : 0 < \lambda_1, \dots, \lambda_n < 1\}.$$

Then for any v that satisfies the perturbing vector conditions with respect to K,

(a) $-v + \Pi^\circ = -(v + \Pi^\circ) + w_1 + \cdots + w_n$

(b) $\sigma_{-v+K}\left(\frac{1}{z_1}, \dots, \frac{1}{z_n}\right) = (-1)^n \sigma_{v+K}(z_1, \dots, z_n)$

(c) $\sigma_K\left(\frac{1}{\mathbf{z}}\right) = (-1)^d \sigma_{K^\circ}(\mathbf{z})$ *(Stanley Reciprocity)*

Proof. Using the fact that $0 < \lambda_i < 1 \implies 0 < 1 - \lambda_i < 1$, we have

$$\begin{aligned}
-(v + \Pi^\circ) + w_1 \cdots + w_n &= -v - \{\lambda_1 w_1 + \cdots + \lambda_n w_n : 0 < \lambda_i < 1\} + w_1 + \cdots + w_n \\
&= -v + \{(1-\lambda_1)w_1 + \cdots + (1-\lambda_n)w_n : 0 < \lambda_i < 1\} \\
&= -v + \{\lambda_1 w_1 + \cdots + \lambda_n w_n : 0 < \lambda_i < 1\} \\
&= -v + \Pi^\circ,
\end{aligned}$$

thus completing the proof of (a). And so,

$$\begin{aligned}
\sigma_{-v+\Pi^\circ}(\mathbf{z}) &= \sigma_{-(v+\Pi^\circ)}(\mathbf{z})\mathbf{z}^{w_1} \dots \mathbf{z}^{w_d} \\
&= \mathbf{z}^{w_1} \dots \mathbf{z}^{w_d} \sum_{\mathbf{x} \in -(v+\Pi^\circ) \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{x}} \\
&= \mathbf{z}^{w_1} \dots \mathbf{z}^{w_d} \sum_{\mathbf{x} \in (v+\Pi^\circ) \cap \mathbb{Z}^d} \mathbf{z}^{-\mathbf{x}} \\
&= \mathbf{z}^{w_1} \dots \mathbf{z}^{w_d} \sum_{\mathbf{x} \in (v+\Pi^\circ) \cap \mathbb{Z}^d} \left(\frac{1}{\mathbf{z}}\right)^{\mathbf{x}} \\
&= \sigma_{v+\Pi^\circ}\left(\frac{1}{\mathbf{z}}\right) \mathbf{z}^{w_1} \cdots \mathbf{z}^{w_d} \\
\sigma_{-v+\Pi^\circ}\left(\frac{1}{\mathbf{z}}\right) &= \sigma_{v+\Pi^\circ}(\mathbf{z})\mathbf{z}^{-w_1} \cdots \mathbf{z}^{-w_d}
\end{aligned}$$

where we denote $\frac{1}{\mathbf{z}} := \left(\frac{1}{z_1}, \dots, \frac{1}{z_d}\right)$. Since there are no lattice points on the boundary of $-v+K$,

$$\sigma_{-v+K}(\mathbf{z}) = \frac{\sigma_{-v+\Pi^\circ}(\mathbf{z})}{(1 - \mathbf{z}^{w_1}) \dots (1 - \mathbf{z}^{w_n})}.$$

Reciprocating $\mathbf{z}$, we then have

$$\begin{aligned}
\sigma_{-v+K}\left(\frac{1}{\mathbf{z}}\right) &= \frac{\sigma_{-v+\Pi^\circ}(1/\mathbf{z})}{(1-\mathbf{z}^{-w_1})\cdots(1-\mathbf{z}^{-w_d})} \\
&= \frac{\sigma_{v+\Pi^\circ}(\mathbf{z})\mathbf{z}^{-w_1}\cdots\mathbf{z}^{-w_d}}{(1-\mathbf{z}^{-w_1})\cdots(1-\mathbf{z}^{-w_d})} \\
&= \frac{\sigma_{v+\Pi^\circ}(\mathbf{z})}{(\mathbf{z}^{-w_1}-1)\cdots(\mathbf{z}^{-w_d}-1)} \\
&= (-1)^d\frac{\sigma_{v+\Pi^\circ}(\mathbf{z})}{(1-\mathbf{z}^{w_1})\cdots(1-\mathbf{z}^{w_d})} \\
&= (-1)^d\sigma_{v+K}(\mathbf{z})
\end{aligned}$$

thus completing the proof of (b). Since none of the K_i's share lattice points, we have

$$\begin{aligned}
\sigma_K\left(\frac{1}{\mathbf{z}}\right) &= \sigma_{v+K}\left(\frac{1}{\mathbf{z}}\right) \\
&= \sum_{i=1}^{\ell}\sigma_{v+K_i}\left(\frac{1}{\mathbf{z}}\right) \\
&= \sum_{i=1}^{\ell}(-1)^d\sigma_{-v+K_i}(\mathbf{z}) \\
&= (-1)^d\sigma_{-v+K}(\mathbf{z}) \\
&= (-1)^d\sigma_{K^\circ}(\mathbf{z})
\end{aligned}$$

thus completing the proof of (c). $\square$

Now, we define the Ehrhart series for the relative interior of a rational polytope P to be

$$\mathrm{Ehr}_{P^\circ}(z) := \sum_{t\geq 1} L_{P^\circ}(t)z^t.$$

We have the analogue of Lemma 3.1.3:

$$\mathrm{Ehr}_{P^\circ}(z) = \sigma_{(\mathrm{cone}(P'))^\circ}(1,\ldots,1,z).$$

Lemma 3.5.5. *Suppose P is a lattice d-polytope. Then the evaluation of the rational function* Ehr_P *at* $\frac{1}{\mathbf{z}}$ *yields*

$$\mathrm{Ehr}_P\left(\frac{1}{z}\right) = (-1)^{d+1}\,\mathrm{Ehr}_{P^\circ}(z).$$

Proof. Let $K := \mathrm{cone}(P')$. We have

$$\begin{aligned}
\mathrm{Ehr}_P\left(\frac{1}{z}\right) &= \sigma_K\left(\mathbf{1},\frac{1}{z}\right) \\
&= (-1)^{d+1}\sigma_{K^\circ}(\mathbf{1},z) \\
&= (-1)^{d+1}\,\mathrm{Ehr}_{P^\circ}(z)
\end{aligned}$$
$\square$

Corollary 3.5.6. *Let Δ_k be a unimodular k-simplex. Then*

$$\sum_{t\leq 0} L_{\Delta_k}(-t)z^t = -\sum_{t\geq 1} L_{\Delta_k}(-t)z^t$$

Proof. We make use of the result of Example 3.5.2 and see that

$$\begin{aligned}
\sum_{t\leq 0} L_{\Delta_k}(-t)z^t &= \sum_{t\geq 0} L_{\Delta_k}(t)\left(\frac{1}{z}\right)^t = \operatorname{Ehr}_{\Delta_k}\left(\frac{1}{z}\right)\\
\sum_{t\geq 1} L_{\Delta_k}(-t)z^t &= \sum_{t\geq 1}(-1)^k L_{\Delta_k^\circ}(t)z^t = (-1)^k \operatorname{Ehr}_{\Delta_k^\circ}(z)
\end{aligned}$$

Applying Lemma 3.5.5 completes this proof. □

Proof of Ehrhart-Macdonald Reciprocity. We note that $L_{\Delta_k}(t)$ for $k \geq 0$ forms a basis for polynomials in t, so we can express $L_P(t)$ as

$$L_P(t) = \sum_{k=0}^{d} \alpha_k L_{\Delta_k}(t)$$

for some $\alpha_k \in \mathbb{R}$. Then we have

$$\begin{aligned}
\sum_{t\geq 1} L_{P^\circ}(t)z^t &= \operatorname{Ehr}_{P^\circ}(z)\\
&= (-1)^{d+1}\operatorname{Ehr}_P\left(\frac{1}{z}\right)\\
&= (-1)^{d+1}\sum_{t\leq 0} L_P(-t)z^t\\
&= (-1)^{d+1}\sum_{t\leq 0}\sum_{k=1}^{d} \alpha_k L_{\Delta_k}(-t)z^t\\
&= (-1)^{d+1}\sum_{k=1}^{d} \alpha_k \sum_{t\leq 0} L_{\Delta_k}(-t)z^t\\
&= (-1)^{d+1}(-1)\sum_{k=1}^{d} \alpha_k \sum_{t\geq 1} L_{\Delta_k}(-t)z^t\\
&= (-1)^{d}\sum_{t\geq 1}\sum_{k=1}^{d} \alpha_k L_{\Delta_k}(-t)z^t\\
&= (-1)^{d}\sum_{t\geq 1} L_P(-t)z^t.
\end{aligned}$$

Comparing coefficients completes the proof. □

Chapter 4

Polytope Algebra

The polytope algebra becomes our main interest, as it is the key to unlocking the way to "express any polytope in terms of" unimodular simplices; we will see this in Chapter 5. In this chapter, we introduce the lattice polytope algebra and show a specific way of decomposing of a lattice polytope. A consequence of this algorithm is Brion's Theorem, which we will use to provide any alternate proof of Ehrhart's Theorem.

4.1 Valuations

A *valuation* is a map that maps sets in a vector space to elements in an abelian group. Throughout this section, let $\mathscr{S}$ denote some collection of sets in $\mathbb{R}^d$. For example, $\mathscr{S}$ could be the set of lines passing through the origin; or it could be the set of all balls centred at some fixed point, including the ball with zero radius.

We say that $\mathscr{S}$ is *intersectional* if it is closed under intersection. In other words, $\mathscr{S}$ is intersectional if, for any finite non-empty index set I,

$$A_i \in \mathscr{S} \ \forall\ i \in I \implies \bigcap_{i \in I} A_i \in \mathscr{S}. \tag{4.1}$$

In the above two examples, we see that the latter is intersectional, since for any two distinct balls centred at the same point, one must contain the other. However, the former is not intersectional, since the intersection of two elements in this family is the singleton $\{0\}$, not a line.

A *valuation* is a function $\varphi : \mathscr{S} \to G$ (where G is an abelian group) satisfying what we will refer to as the *"inclusion-exclusion relation"*:

$$\varphi(A \cup B) + \varphi(A \cap B) = \varphi(A) + \varphi(B) \tag{4.2}$$

whenever $A, B, A \cup B,$ and $A \cap B$ are in $\mathscr{S}$. We take $\varphi(\emptyset)$ to be 0. If $\mathscr{S}$ is intersectional, then we let $U(\mathscr{S})$ denote the set of finite unions of elements in $\mathscr{S}$, and $\overline{U}(\mathscr{S}) := \{A \backslash B : A, B \in U(\mathscr{S})\}$. Later on, we will see an example of a valuation on $\mathscr{S}$ extending to $U(\mathscr{S})$ and $\overline{U}(\mathscr{S})$; but first, we need to develop more tools.

For a set $A \in \mathscr{S}$, let $[A] : \mathbb{R}^d \to \{0, 1\}$ be the *indicator function* of A, defined by

$$[A](x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \tag{4.3}$$

Note that $[\mathbb{R}^d] = 1$ and $[\emptyset] = 0$. Here are a few more properties of indicator functions:

Lemma 4.1.1. *For $A, B \in \mathscr{S}$*

(a) $[A][B] = [A \cap B]$

(b) $\left[\overline{A}\right] = 1 - [A]$

(c) $[A \cup B] + [A \cap B] = [A] + [B]$

Proof.

(a) $[A][B]$ evaluated at x is $[A](x) \cdot [B](x)$.

$$\begin{aligned} [A](x) \cdot [B](x) = 1 &\iff [A](x) = 1 = [B](x) \\ &\iff x \in A \text{ and } x \in B \\ &\iff x \in (A \cap B) \\ &\iff [A \cap B](x) = 1 \end{aligned}$$

(b) We perform a simple case analysis. If $x \in A$, then $x \notin \overline{A}$; $1 - [A](x) = 1 - 1 = 0$, which is what we wanted $\left[\overline{A}\right](x)$ to be, in this case. If $x \notin S$, then $x \in \overline{A}$; $1 - [A](x) = 1 - 0 = 1$, which is also what we wanted $\left[\overline{A}\right](x)$ to be, in this case.

(c) We use the fact that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

$$\begin{aligned} [A \cup B] &= 1 - \left[\overline{A \cup B}\right] \\ &= 1 - \left[\overline{A} \cap \overline{B}\right] \\ &= 1 - \left[\overline{A}\right]\left[\overline{B}\right] \\ &= 1 - (1 - [A])(1 - [B]) \\ &= [A] + [B] - [A][B] \\ [A \cup B] + [A \cap B] &= [A] + [B] \end{aligned}$$

□

Note that part c) is an example of the inclusion-exclusion principle. We can generalize this result to more than two sets.

Theorem 4.1.2 (Inclusion-Exclusion)**.** *Let $A_1, \dots, A_m \subset \mathbb{R}^d$ be sets. Then*

$$\begin{aligned} [A_1 \cup \dots \cup A_m] &= 1 - (1 - [A_1]) \dots (1 - [A_m]) \\ &= \sum_{\substack{K \subseteq [m] \\ K \neq \emptyset}} (-1)^{|K|-1} \left[\bigcap_{k \in K} A_k\right]. \end{aligned}$$

Proof. The second equality in the statement of the theorem is a result of expanding the product before it. Now, to prove the first equality.

$$\begin{aligned}
[A_1 \cup \cdots \cup A_m] &= 1 - \left[\ \overline{A_1 \cup \cdots \cup A_m}\ \right] \\
&= 1 - \left[\ \overline{A_1} \cap \cdots \cap \overline{A_m}\ \right] \\
&= 1 - \left[\ \overline{A_1}\ \right] \ldots \left[\ \overline{A_m}\ \right] \\
&= 1 - (1 - [A_1]) \ldots (1 - [A_m])
\end{aligned}$$

□

Now, let us define $V(\mathscr{S})$ to be the free abelian group generated by the indicator functions of elements of $\mathscr{S}$. An element $f \in V(\mathscr{S})$ would be of the form

$$f = \sum_i \alpha_i [A_i],$$

where $A_i \in \mathscr{S}$ and $\alpha_i \in \mathbb{R}$, for which finitely many α_i are non-zero. We can equip $V(\mathscr{S})$ with addition (for two elements $\sum_i \alpha_i [A_i]$ and $\sum_j \beta_j [A_j] \in V(\mathscr{S})$)

$$\sum_i \alpha_i [A_i] + \sum_j \beta_j [A_j] = \sum_i (\alpha_i + \beta_i)[A_i], \tag{4.4}$$

multiplication

$$\left(\sum_i \alpha_i [A_i] \right) \left(\sum_j \beta_j [A_j] \right) = \sum_{i,j} \alpha_i \beta_j [A_i][A_j] = \sum_{i,j} \alpha_i \beta_j [A_i \cap A_j] \tag{4.5}$$

and scalar multiplication (for any $c \in \mathbb{R}$)

$$c \left(\sum_i \alpha_i [A_i] \right) = \sum_i c \alpha_i [A_i]$$

Lemma 4.1.3. *Let $\mathscr{S}$ be a non-empty collection of subsets of $\mathbb{R}^d$. Then $V(\mathscr{S})$ is a commutative algebra over $\mathbb{R}$.*

Proof. Note that $V(\mathscr{S})$ is a subset of $\mathscr{F}(\mathbb{R}^d, \mathbb{R})$, the set of functions from $\mathbb{R}^d$ to $\mathbb{R}$. Conveniently, $\mathscr{F}(\mathbb{R}^d, \mathbb{R})$ is a commutative algebra. Thus, showing that $V(\mathscr{S})$ contains the additive identity and is closed under addition and multiplication would suffice to prove that $V(\mathscr{S})$ is a commutative algebra.

We set the additive identity to be $\sum_i 0[A_i]$. Consider (4.4); since finitely many α_i and β_i are non-zero, finitely many $\alpha_i + \beta_i$ are non-zero. Therefore, we have closure under addition. Next, consider (4.5); again, finitely many non-zero α_i and β_i implies finitely many non-zero $\alpha_i \beta_i$. Therefore, we have closure under multiplication. □

Now, we will see an extension on a valuation on $\mathscr{S}$. Let $\varphi : \mathscr{S} \to V(\mathscr{S})$ be the function that takes $A \in \mathscr{S}$ to $[A]$.

Lemma 4.1.4. *φ is a valuation on $\mathscr{S}$.*

Proof. To show that φ is a valuation, we need to prove that

1. $V(\mathscr{S})$ is an abelian group
2. Whenever $A, B, A \cup B,$ and $A \cap B$ are in $\mathscr{S}$, we get

$$\varphi(A \cup B) + \varphi(A \cap B) = \varphi(A) + \varphi(B)$$

To prove part 1, observe by Lemma 4.1.3 that $\mathscr{S}$ is a commutative algebra, and therefore, an abelian group.

To prove part 2, observe by Lemma 4.1.1c that for any $A, B \subseteq \mathbb{R}^d$,

$$[A \cup B] + [A \cap B] = [A] + [B]$$

Therefore, if $A, B, A \cup B,$ and $A \cap B$ are in $\mathscr{S}$, then part 2 must hold. $\square$

Suppose that $\mathscr{S}$ is intersectional. For $A_i \in \mathscr{S}, i = 1, \ldots, m$, we can use inclusion-exclusion to express $[\bigcup_{i=1}^m A_i] \in V(\mathscr{S})$ as a finite linear combination of $\left[\bigcap_{k \in K} A_k\right]$'s. Since $\mathscr{S}$ is intersectional, these intersections are also in $\mathscr{S}$. Therefore, we can try to extend φ to $U(\mathscr{S})$ by defining

$$\begin{aligned}
\varphi\left(\bigcup_{i=1}^m A_i\right) &:= \sum_{K \vdash [m]} (-1)^{|K|-1} \ \varphi\left(\bigcap_{k \in K} A_k\right) \\
&= \sum_{K \vdash [m]} (-1)^{|K|-1} \left[\bigcap_{k \in K} A_k\right] \\
&= \left[\bigcup_{i=1}^m A_i\right]
\end{aligned}$$

Lemma 4.1.5. *The extension $\varphi : U(\mathscr{S}) \to V(\mathscr{S})$ is a valuation.*

Proof. We have already shown that $V(\mathscr{S})$ is an abelian group. Suppose A and B are sets in $U(\mathscr{S})$. We must show that

$$\varphi(A) + \varphi(B) = \varphi(A \cup B) + \varphi(A \cap B).$$

However, this follows immediately from our definition $\varphi(A) = [A]$ and Lemma 4.1.1c. $\square$

Finally, let us try to extend φ to $\overline{U}(\mathscr{S})$. For, $A, B \in U(\mathscr{S})$, note that

$$\begin{aligned}
[A \backslash B] &= [A \cap \overline{B}] \\
&= [A][\overline{B}] \\
&= [A](1 - [B]) \\
&= [A] - [A \cap B]
\end{aligned}$$

Let $A := \bigcup_{i\in I} A_i$ and $B := \bigcup_{j\in J} B_j$ where $A_i, B_i \in \mathscr{S}$, for all i and j in finite index sets I and J, respectively. Note that

$$\begin{aligned}
A \cap B &= \left(\bigcup_{i\in I} A_i\right) \cap B \\
&= \bigcup_{i\in I} (A_i \cap B) \\
&= \bigcup_{i\in I} \left(A_i \cap \left(\bigcup_{j\in J} B_j\right)\right) \\
&= \bigcup_{i\in I, j\in J} (A_i \cap B_j)
\end{aligned}$$

Since $\mathscr{S}$ is intersectional, $A \cap B$ is also in $U(\mathscr{S})$. We try to extend φ to $\overline{U}(\mathscr{S})$ by defining (for any $A, B \in U(\mathscr{S})$)

$$\begin{aligned}
\varphi(A \backslash B) :=& \varphi(A) - \varphi(A \cap B) \\
=& [A] - [A \cap B] \\
=& [A \backslash B]
\end{aligned}$$

Lemma 4.1.6. *The extension $\varphi : \overline{U}(\mathscr{S}) \to V(\mathscr{S})$ is a valuation.*

Proof. The proof is almost identical to the proof of the previous lemma. We just need to change A and B to be in $\overline{U}(\mathscr{S})$ instead of $U(\mathscr{S})$. □

A pattern emerges from the similarities of the previous proofs. Given any set $A \subseteq \mathbb{R}^d$, let us set $\varphi(A) := [A]$. Then for any $A, B \in \mathbb{R}^d$, we can use Lemma 4.1.1c to show

$$\begin{aligned}
\varphi(A) + \varphi(B) &= [A] + [B] \\
&= [A \cup B] + [A \cap B] \\
&= \varphi(A \cup B) + \varphi(A \cap B)
\end{aligned}$$

Now suppose we have a linear map $\bar{\varphi}$ from $V(\mathscr{S})$ into an abelian group. In other words, for any $A, B \in \mathscr{S}$ and $\alpha, \beta \in \mathbb{R}$,

$$\bar{\varphi}(\alpha[A] + \beta[B]) = \alpha\bar{\varphi}([A]) + \beta\bar{\varphi}([B]). \tag{4.6}$$

By defining $\varphi(A) := \bar{\varphi}([A])$ we see that

$$\begin{aligned}
\varphi(A) + \varphi(B) &= \bar{\varphi}([A]) + \bar{\varphi}([B]) \\
&= \bar{\varphi}([A] + [B]) \\
&= \bar{\varphi}([A \cup B] + [A \cap B]) \\
&= \bar{\varphi}([A \cup B]) + \bar{\varphi}([A \cap B]) \\
&= \varphi(A \cup B) + \varphi(A \cap B)
\end{aligned}$$

Therefore, for every linear map $\bar{\varphi} : V(\mathscr{S}) \to G$ (where G is an abelian group), $\varphi(A) := \bar{\varphi}([A)]$ is a valuation on $\mathscr{S}$. In light of this discovery, let us say that a *valuation on* $V(\mathscr{S})$ is a linear map from $V(\mathscr{S})$ to an abelian group. For convenience of notation, given a valuation $\bar{\varphi}$ on $V(\mathscr{S})$, let us denote $\bar{\varphi}(P)$ to mean $\bar{\varphi}([P])$.

4.2 Algebra of Polyhedra

We have seen that for a collection $\mathscr{S}$ of sets in $\mathbb{R}^d$, $V(\mathscr{S})$ is an algebra (if we choose to equip it with addition, multiplication and scalar multiplication). If we take $\mathscr{S}$ to be the set of polyhedra, then we define the *algebra of polyhedra*, denoted $\mathscr{P}(\mathbb{R}^d)$, to be $V(\mathscr{S})$, equipped with addition and scalar multiplication. In other words, the algebra of polyhedra is the real vector space generated by indicator functions $[P]$, where $P \subset \mathbb{R}^d$ is a polyhedron. If instead, we take $\mathscr{S}$ to be the set of rational polyhedra, we would have the *algebra of rational polyhedra*, denoted $\mathscr{P}(\mathbb{Q}^d)$. Similarly, let us denote $\mathcal{K}(\mathbb{R}^d)$ to be the *algebra of compact convex sets* in $\mathbb{R}^d$.

Lemma 4.2.1. *$\mathscr{P}(\mathbb{Q}^d)$ is spanned by indicator functions $[P]$ where P is a rational polyhedron without straight lines.*

Proof. We prove the lemma by showing that for any non-empty rational polyhedron $Q = \{x \in \mathbb{R}^d : Ax \leq b\}$, $[Q]$ can be expressed as a linear combination of $[P]$'s, where each P is a rational polyhedron without straight lines. To do so, we use induction on the nullity of A. If $\text{Nullity}(A) = 0$, then by Corollary 1.1.5, Q does not have straight lines, and we're done. Taking the inductive step, assume that $\text{Nullity}(A) > 0$. Let $z \neq 0$ be in the null-space of A. Since A is an integer matrix, we can assume without loss of generality that z is an integer vector. Let P_1 and P_2 be

$$P_1 = Q \cap \{x : \langle x, z\rangle \leq 0\}$$
$$P_2 = Q \cap \{x : \langle x, z\rangle \geq 0\}.$$

P_1 and P_2 are rational polyhedra, since z is integer. Since $Q = P_1 \cup P_2$, by inclusion-exclusion, we have

$$[Q] = [P_1] + [P_2] - [P_1 \cap P_2].$$

It remains to show that $[P_1], [P_2]$ and $[P_1 \cap P_2]$ can be expressed as linear combinations of indicator functions of polytopes without straight lines. Note that the constraint matrices associated to P_1, P_2 and $P_1 \cap P_2$ are

$$\begin{pmatrix} A \\ z^T \end{pmatrix}, \begin{pmatrix} A \\ -z^T \end{pmatrix}, \begin{pmatrix} A \\ z^T \\ -z^T \end{pmatrix}$$

respectively. Since z is in the null-space of A, the nullities of the above three matrices are less than $\text{Nullity}(A)$. Applying the induction hypothesis completes the proof. $\square$

4.2.1 Euler Valuation

We would like to define a unique valuation $\chi : \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}$ such that $\chi(P) = 1$ for every non-empty polyhedron in $\mathbb{R}^d$. We call this valuation the *Euler valuation*. The following approach to constructing the Euler valuation can be found in Barvinok [1].

Consider an element $f = \sum_i \alpha_i [P_i]$ in $\mathscr{P}(\mathbb{R}^d)$. If $\chi(f)$ exists, it must take on the value $\sum_{i:P_i \neq \emptyset} \alpha_i$. Since finitely many α_i are non-zero, this sum has a finite value and so $\chi(f)$ would be well-defined. Therefore, if $\chi(f)$ exists, then it is unique.

Before defining χ for $\mathscr{P}(\mathbb{R}^d)$, let us define the Euler valuation $\chi_{\mathcal{K}}$ on elements $f \in \mathcal{K}(\mathbb{R}^d)$.

Lemma 4.2.2. *The Euler valuation $\chi_{\mathcal{K}} : \mathcal{K}(\mathbb{R}^d) \to \mathbb{R}$ exists.*

Proof. We use induction on d. For the base case, we consider $d = 0$. Elements in $\mathcal{K}(\mathbb{R}^0)$ will be of the form $\alpha[0]$ for some $\alpha \in \mathbb{R}$, so $\chi_{\mathcal{K}}(\alpha[0]) = \alpha$. Now suppose $d > 0$.

For a $\tau \in \mathbb{R}$ define

$$H_\tau = \{x \in \mathbb{R}^d : x_d = \tau\}$$

to be the hyperplane at height τ. Let $\mathcal{K}(H_\tau)$ be the algebra of compact convex sets in H_τ. By identifying H_τ with $\mathbb{R}^{d-1}$, induction hypothesis tells us that there exists Euler valuation $\chi_\tau : \mathcal{K}(H_\tau) \to \mathbb{R}$. For a function $f = \sum_i \alpha_i[A_i]$ in $\mathcal{K}(\mathbb{R}^d)$, let

$$f_\tau := \sum_i \alpha_i[A_i \cap H_\tau]$$

We claim that $A_i \cap H_\tau$ is compact and convex. Indeed, since A_i is compact (closed and bounded) and convex, and H_τ is closed, $A_i \cap H_\tau$ is closed and bounded (compact) as well. Since A_i and H_τ are both convex, $A_i \cap H_\tau$ is convex as well. Therefore, $f_\tau \in \mathcal{K}(H_\tau)$, and $\chi_\tau(f_\tau)$ is well defined:

$$\chi_\tau(f_\tau) = \sum_{i : A_i \cap H_\tau \neq \emptyset} \alpha_i.$$

Now let us compare the limit

$$\lim_{\epsilon \to 0^+} \chi_{\tau-\epsilon}(f_{\tau-\epsilon}).$$

to $\chi_\tau(f_\tau)$. Let A be a compact convex set. By convexity, for some sufficiently small $\epsilon' > 0$,

$$\text{the value of } \chi_t(f_t) \text{ will be constant for } \tau - \epsilon' \leq t < \tau.$$

Therefore, the value of $\lim_{\epsilon \to 0^+} \chi_{\tau-\epsilon}(f_{\tau-\epsilon})$ equals the value of $\chi_t(f_t)$ on the half-open interval $[\tau - \epsilon', \tau)$. Let us refer to the above result as the *constant neighbourhood property.* We perform some case analysis. Let $f = [A]$.

a) Suppose $\chi_\tau(f_\tau) = 1$ (i.e. $H_\tau \cap A \neq \emptyset$) and $\lim_{\epsilon \to 0^+} \chi_{\tau-\epsilon}(f_{\tau-\epsilon}) = 1$. Then by the constant neighbourhood property, $\chi_{\tau-\epsilon'}(f_{\tau-\epsilon'}) = 1$, which implies that $A \cap H_{\tau-\epsilon'} \neq \emptyset$.

b) Suppose $\chi_\tau(f_\tau) = 1$ and $\lim_{\epsilon \to 0^+} \chi_{\tau-\epsilon}(f_{\tau-\epsilon}) = 0$. Then, by the constant neighbourhood property, $\chi_t(f_t) = 0$ (i.e. $H_t \cap A = \emptyset$) on $[\tau - \epsilon', \tau)$. By convexity, we have that $H_t \cap A = \emptyset$ for $t < \tau$.

c) Suppose $\chi_\tau(f_\tau) = 0$ (i.e. $H_\tau \cap A = \emptyset$). Since A in is closed, there is an open interval $(\tau - \epsilon, \tau)$ such that for $t \in (\tau - \epsilon, \tau)$, we have $A \cap H_t = \emptyset$. Therefore,

$$\lim_{\epsilon \to 0^+} \chi_{\tau-\epsilon}(f_{\tau-\epsilon}) = 0$$

From these cases, we conclude that

$$\chi_\tau([A]_\tau) - \lim_{\epsilon \to 0^+} \chi_{\tau-\epsilon}([A]_{\tau-\epsilon}) = \begin{cases} 1 & \text{if } \min_{x \in A} x_d = \tau \\ 0 & \text{otherwise} \end{cases} \tag{4.7}$$

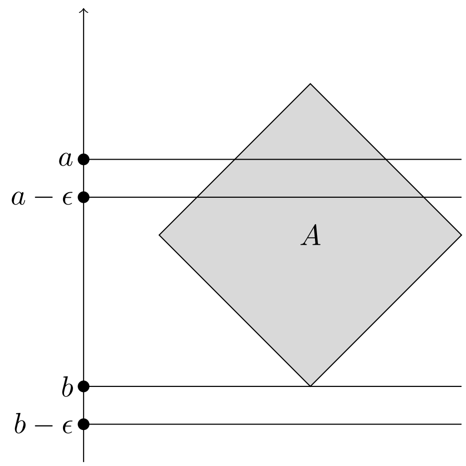

Figure 4.1. We see that $\lim_{\epsilon\to 0^+}\chi_{a-\epsilon} = \chi_a(f_a) = 1$, but $0 = \lim_{\epsilon\to 0^+}\chi_{b-\epsilon}(f_{b-\epsilon}) \neq \chi_b(f_b) = 1$. Figure inspired by [1]

Now we define (for $f = \sum_i \alpha_i[A_i]$)

$$\chi_{\mathcal{K}}(f) := \sum_{\tau\in\mathbb{R}}\left(\chi_\tau(f_\tau) - \lim_{\epsilon\to 0^+}\chi_{\tau-\epsilon}(f_{\tau-\epsilon})\right)$$

By (4.7), A is compact (closed and bounded) and so for any non-empty A, $\min_{x\in A} x_d$ exists and is unique. Therefore, $A \neq \emptyset$, $\chi_{\mathcal{K}}([A]) = 1$. Since $\min_{x\in\emptyset} x_d$ does not exist, by (4.7), $\chi_{\mathcal{K}}([\emptyset]) = 0$. Also, since finitely many α_i are non-zero, $\chi_{\mathcal{K}}$ is well defined.

It remains to show linearity. Let f and g be two functions in $\mathcal{K}(\mathbb{R}^d)$ and note

$$\begin{aligned}
\chi_{\mathcal{K}}(f+g) &= \sum_{\tau\in\mathbb{R}}\left(\chi_\tau(f_\tau + g_\tau) - \lim_{\epsilon\to 0^+}\chi_{\tau-\epsilon}(f_{\tau-\epsilon} + g_{\tau-\epsilon})\right)\\
&= \sum_{\tau\in\mathbb{R}}\left(\chi_\tau(f_\tau) + \chi_\tau(g_\tau) - \lim_{\epsilon\to 0^+}\chi_{\tau-\epsilon}(f_{\tau-\epsilon}) - \lim_{\epsilon\to 0^+}\chi_{\tau-\epsilon}(g_{\tau-\epsilon})\right)\\
&= \sum_{\tau\in\mathbb{R}}\left(\chi_\tau(f_\tau) - \lim_{\epsilon\to 0^+}\chi_{\tau-\epsilon}(f_{\tau-\epsilon}) + \chi_\tau(g_\tau) - \lim_{\epsilon\to 0^+}\chi_{\tau-\epsilon}(g_{\tau-\epsilon})\right)\\
&= \chi_{\mathcal{K}}(f) + \chi_{\mathcal{K}}(g)
\end{aligned}$$

and (for $c \in \mathbb{R}$)

$$\begin{aligned}
\chi_{\mathcal{K}}(cf) &= \sum_{\tau\in\mathbb{R}}\chi_\tau(cf_\tau) - \lim_{\epsilon\to 0^+}\chi_{\tau-\epsilon}(cf_{\tau-\epsilon})\\
&= \sum_{\tau\in\mathbb{R}}c\chi_\tau(f_\tau) - c\lim_{\epsilon\to 0^+}\chi_{\tau-\epsilon}(f_{\tau-\epsilon})\\
&= c\left(\sum_{\tau\in\mathbb{R}}\chi_\tau(f_\tau) - \lim_{\epsilon\to 0^+}\chi_{\tau-\epsilon}(f_{\tau-\epsilon})\right)\\
&= c\chi_{\mathcal{K}}(f)
\end{aligned}$$

Therefore, $\chi_{\mathcal{K}}$ is the Euler valuation of $\mathcal{K}(\mathbb{R}^d)$. $\square$

Finally, we move on to the unbounded case of $\mathscr{P}(\mathbb{R}^d)$. Let $B(r)$ be the ball of radius r, centred at the origin. we define (for $f \in \mathscr{P}(\mathbb{R}^d)$)

$$\chi(f) := \lim_{r\to\infty} \chi_{\mathcal{K}}(f \cdot [B(r)]). \tag{4.8}$$

First, note that

$$\chi([\emptyset]) = \lim_{r\to\infty} \chi_{\mathcal{K}}([\emptyset \cap B(r)]) = \lim_{r\to\infty} \chi_{\mathcal{K}}([\emptyset]) = 0.$$

If $A \neq \emptyset$, then $\lim_{r\to\infty}(A \cap B(r)) \neq \emptyset$, so $\chi([A]) = 1$.

To check for linearity, we have (for $f, g \in \mathscr{P}(\mathbb{R}^d)$ and $a, b \in \mathbb{R}$)

$$\begin{aligned}
\chi(af + bg) &= \lim_{r\to\infty} \chi_{\mathcal{K}}((af + bg) \cdot [B(r)]) \\
&= \lim_{r\to\infty} \chi_{\mathcal{K}}(af \cdot [B(r)] + bg \cdot [B(r)]) \\
&= \lim_{r\to\infty} a\chi_{\mathcal{K}}(f \cdot [B(r)]) + b\chi_{\mathcal{K}}(g \cdot [B(r)]) \\
&= a \lim_{r\to\infty} \chi_{\mathcal{K}}(f \cdot [B(r)]) + b \lim_{r\to\infty} \chi_{\mathcal{K}}(g \cdot [B(r)]) \\
&= a\chi(f) + b\chi(g)
\end{aligned}$$

Therefore, our definition of χ is the Euler valuation of $\mathscr{P}(\mathbb{R}^d)$.

4.2.2 Linear Transformations of Polyhedra

We can use the Euler valuation to prove the following nice result of linear transformations on polyhedra, which we will prove at the end of the section.

Theorem 4.2.3. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a linear transformation $\mathcal{T} : \mathscr{P}(\mathbb{R}^n) \to \mathscr{P}(\mathbb{R}^m)$ such that $\mathcal{T}([P]) = [T(P)]$ for any polyhedron $A \subset \mathbb{R}^n$.*

Lemma 4.2.4. *Let $\mathrm{pr} : \mathbb{R}^d \to \mathbb{R}^{d-1}$ be the projection $\mathrm{pr}(x_1, \ldots, x_d) = (x_1, \ldots, x_{d-1})$. If $P \subset \mathbb{R}^d$ is a polyhedron then $\mathrm{pr}(P)$ is a polyhedron in $\mathbb{R}^{d-1}$.*

Proof. Let $P = \{x : Ax \leq b\}$ for some matrix $A \in \mathbb{R}^{n\times d}$ and vector $b \in \mathbb{R}^n$. Let us look at the last column of A and define

$$\begin{aligned}
I_0 &:= \{i : A_{id} = 0\} \\
I_+ &:= \{i : A_{id} > 0\} \\
I_- &:= \{i : A_{id} < 0\}
\end{aligned}$$

Denote a_i to be the i^{th} row of A. A point $x = (x_1, \ldots, x_{d-1})$ is in $pr(P)$ if and only if there exists a number q such that

$$\langle a_i, (x, q)\rangle = a_{id}q + \sum_{j=1}^{d-1} a_{ij}x_j \leq b_i \tag{4.9}$$

for all $i = 1, \dots, n$. If $i \in I_0$, the term $a_{id}q$ would disappear. If i was an index in I_+ or I_-, then we could isolate q to see its restrictions:

$$q \leq \frac{1}{a_{id}} \left(b_i - \sum_{j=1}^{d-1} a_{ij} x_j \right) \text{ for } i \in I_+ \tag{4.10}$$

$$q \geq \frac{1}{a_{id}} \left(b_i - \sum_{j=1}^{d-1} a_{ij} x_j \right) \text{ for } i \in I_- \tag{4.11}$$

In order for q to exist, we see that if we take one index each from I_+ and I_-,the right side of (4.10) must be equal or greater than the right side of (4.11). Therefore, the projection $pr(P)$ is a polyhedron in $\mathbb{R}^{d-1}$ with the following linear inequalities:

$$\sum_{j=1}^{d-1} a_{ij} x_j \leq b_i \text{ for all } i \in I_0$$

$$\frac{1}{a_{kd}} \left(b_k - \sum_{j=1}^{d-1} a_{kj} x_j \right) \leq \frac{1}{a_{id}} \left(b_i - \sum_{j=1}^{d-1} a_{ij} x_j \right) \text{ for all pairs } i \in I_+, k \in I_-$$

If I_0 is empty, then there will be no inequalities of the first kind; a similar statement can be said for $I_+ \cup I_-$ as well. $\square$

We have just seen that the projection of a polyhedron is a polyhedron. Now, let us generalize this statement to invertible linear transformations.

Lemma 4.2.5. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Then for any polyhedron P in $\mathbb{R}^n$, $T(P)$ is also a polyhedron.*

Proof. Let $P = \{x : Ax \leq b\}$ for some matrix $A \in \mathbb{R}^{m \times n}$ and vector $b \in \mathbb{R}^m$. By veiwing T as a matrix in $\mathbb{R}^{n \times n}$, we see that $T(P)$ would be of the form

$$\begin{aligned} T(P) &= \{T(x) : Ax \leq b\} \\ &= \{y : AT^{-1}y \leq b\} \\ &= \{y : By \leq b\} \end{aligned}$$

for some matrix $B := AT^{-1} \in \mathbb{R}^{m \times n}$. The second line is possible since we said T is invertible. Therefore, $T(P)$ is a polyhedron. $\square$

Note that if T is injective, and not necessarily invertible, we could “force” T to be invertible by restricting the co-domain to $T : \mathbb{R}^n \to \text{Im}(T)$, and the above proof would still hold. Now, we generalize once again to see that any linear transformation of a polyhedron is a polyhedron.

Theorem 4.2.6. *Let $P \in \mathbb{R}^n$ be a polyhedron and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $T(P)$ is a polyhedron in $\mathbb{R}^m$.*

Proof. We have seen this property when T is invertible and one-to-one functions. Now suppose T is not necessarily one-to-one. We define $\widehat{T} : \mathbb{R}^n \to \mathbb{R}^{m+n}$ to be the linear transformation

$$\widehat{T}(x) = (T(x), x)$$

Note that $\ker(\widehat{T}) = \{0\}$, which implies that $\widehat{T}$ is one-to-one, and therefore, $\widehat{T}(P)$ is a polyhedron. Now, we apply Lemma 4.2.4 m times to the last m coordinates to complete the proof. $\square$

Tracing through the proof with integer matrix A, rational matrix T and integer vector b, we have the following.

Corollary 4.2.7. *Given rational polyhedron $P \in \mathbb{R}^n$ and linear transformation $T : \mathbb{R}^n \to \mathbb{R}^d$ with a rational matrix, the image $T(P)$ is a rational polyhedron in $\mathbb{R}^d$.*

With these results ready for use, we proceed to proving Theorem 4.2.3.

Proof of Theorem 4.2.3. We have seen that $T(P)$ is a polyhedron. Let us define a function $G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, where

$$G(x, y) = \begin{cases} 1 & \text{if } T(x) = y \\ 0 & \text{if } T(x) \neq y \end{cases}$$

For a function $f = \sum_i \alpha_i [P_i] \in \mathscr{P}(\mathbb{R}^n)$, let f'_y be the function where

$$\begin{aligned} f'_y(x) &:= G(x, y) f(x) \\ &= \sum_i \alpha_i G(x, y)[P_i](x) \\ &= \sum_i \alpha_i [P_i \cap T^{-1}(y)](x) \end{aligned}$$

where $T^{-1}(y)$ is the preimage of y. For any number of elements $z_1, \ldots, z_k$ in $T^{-1}(y)$, let $z = \sum_{i=1}^k \lambda_i z_i$ be an affine combination of such points (so $\lambda_1 + \cdots + \lambda_k = 1$). We have

$$\begin{aligned} T(z) &= \sum_{i=1}^k \lambda_i T(z_i) \\ &= \sum_{i=1}^k \lambda_i y \\ &= y \end{aligned}$$

$T^{-1}(y)$ is the affine hull of points in $T^{-1}(y)$, and therefore is an affine space. Since affine spaces can be represented by a set of points satisfying a system of linear inequalities, $T^{-1}(y)$ is in fact a polyhedron. Therefore, $f'_y(x)$ is a function in $\mathscr{P}(\mathbb{R}^n)$. This implies that the Euler valuation χ for $\mathscr{P}(\mathbb{R}^n)$ acting on f'_y is well-defined. Keeping in mind that the $\mathcal{T}$ maps functions to functions, let us write $\mathcal{T}_f$ to be the same as $\mathcal{T}(f)$. We now define $\mathcal{T}$ to be

$$\begin{aligned} \mathcal{T}_f(y) &:= \chi(f'_y) \\ &= \sum_{i \in I} \alpha_i \end{aligned}$$

where $I = \{i : P_i \cap T^{-1}(y) \neq \emptyset\}$. However,

$$\begin{aligned} P_i \cap T^{-1}(y) \neq \emptyset &\iff T(P_i) \cap T(T^{-1}(y)) \neq \emptyset \\ &\iff T(P_i) \cap \{y\} \neq \emptyset \\ &\iff y \in T(P_i), \end{aligned}$$

so

$$\mathcal{T}(f) = \sum_i \alpha_i [T(P_i)].$$

Since $T(P_i)$ is a polyhedron in $\mathbb{R}^m$, $\mathcal{T}$ is well-defined. Let us show linearity of $\mathcal{T}$. Let $a, b \in \mathbb{R}$ and $f = \sum_i \alpha_i P_i$ and $g = \sum_i \beta_i P_i$ be functions in $\mathscr{P}(\mathbb{R}^n)$;

$$\begin{aligned} \mathcal{T}(af + bg) &= \mathcal{T}\left(\sum_i (a\alpha_i + b\beta_i)[P_i]\right) \\ &= \sum_{i \in I} (a\alpha_i + b\beta_i)[T(P_i)] \\ &= a \sum_{i \in I} \alpha_i [T(P_i)] + b \sum_{i \in I} \beta_i [T(P_i)] \\ &= a\mathcal{T}(f) + b\mathcal{T}(g). \end{aligned}$$

For some fixed i, set $a = 1, b = 0$ and $\alpha_i = 1$ and $\alpha_j = 0,\ \forall\, j \neq i$. Then, the second line of the above chain of equailities would yield $\mathcal{T}[P] = [T(P)]$ for any polyhedron in $\mathbb{R}^n$. $\square$

4.3 Euler Type Relations

The Euler valuation paves the way to a number of relations between polytopes and cells of its subdivision. We have already made use of the following theorem in our proof of Lemma 3.3.1.

Theorem 4.3.1. *If $\mathscr{S}$ is a subdivision of polytope P, then*

$$[P] = \sum_{C \leq \mathscr{S}} [C^\circ].$$

Proof. Suppose x is in the relative interior of two cells C_1 and C_2 of $\mathscr{S}$. Since x is the relative interior of C_1, it cannot be in a proper face of C_1; similarly, x cannot be in a proper face of C_2. However C_1 and C_2 intersect, and so by definition of a subdivision, they must share a face in common and that face contains x. The only face of C_1 that contains x is C_1 itself, and likewise with C_2. Therefore, $C_1 = C_2$; the relative interiors of cells of $\mathscr{S}$ are disjoint. Since $P = \bigcup_{C \in \mathscr{S}} C$, for any point $x \in \mathbb{R}^d$,

$$[P](x) \geq \sum_{C \in \mathscr{S}} [C^\circ](x).$$

To complete the proof, it suffices to show that every x in P is in the relative interior of a cell of $\mathscr{S}$. Let $x \in P$. Since $P = \bigcup_{C \in \mathscr{S}} C$, there is at least one $C \in \mathscr{S}$ that contains x. Let C' be

$$C' := \bigcap_{\substack{C \in \mathscr{S} \\ x \in C}} C.$$

By definition of a subdivision, C' must be itself a cell of $\mathscr{S}$. Moreover, we claim that $x \in C^\circ$; otherwise, x would be in a proper face G of C', which leads to a contradiction, since C' would then be defined as the intersection of G with some other faces of $\mathscr{S}$, implying that C' is a face of G. $\square$

Consider the subdivision $\mathscr{S}$ of polytope P. Let us restrict $\mathscr{S}$ to the boundary and interior of P:

$$\begin{aligned} \mathscr{S}|_{\partial(P)} &:= \{C \in \mathscr{S} : C \in \partial(P)\} \\ \mathscr{S}|_{P^\circ} &:= \{C \in \mathscr{S} : C^\circ \in P^\circ\} \\ &= \mathscr{S} \backslash (\mathscr{S}|_{\partial(P)}). \end{aligned}$$

Using the same idea as the above proof, we see that

$$[\partial(P)] = \sum_{C \in \mathscr{S}|_{\partial(P)}} [C^\circ].$$

Corollary 4.3.2. *Let $\mathscr{S}$ be a polytope subdivision of P. Then*

$$[P^\circ] = [P] - [\partial(P)] = \sum_{C \in \mathscr{S}} [C^\circ] - \sum_{D \in \mathscr{S}|_{\partial(P)}} [D^\circ] = \sum_{C \in \mathscr{S}|_{P^\circ}} [C^\circ].$$

A closely related idea to the Euler valuation is the Euler characteristic, which we will define as

$$\bar{\chi}(P) := \sum_{F \leq P} (-1)^{\dim(F)}.$$

Lemma 4.3.3. *For any non-empty polytope P,*

$$\chi(P^\circ) = (-1)^{\dim(P)}.$$

Proof. By induction on $\dim(P)$, for the base case of $\dim(P) = 0$, a point, we have

$$\begin{aligned} [P^\circ] &= [P] \\ \chi(P^\circ) &= \chi(P) \\ (-1)^0 &= 1. \end{aligned}$$

Taking the inductive step, fix a facet F. Then we have

$$\begin{aligned} \chi(P^\circ) &= \chi(P) - \chi(\partial P) \\ &= \chi(P) - \chi(F) - \chi(\partial P \backslash F) \\ &= 1 - 1 - \chi(\partial P \backslash F) \\ &= -\chi(\partial P \backslash F) \\ &= -\chi\left(\sum_{C \leq \partial P \backslash F} [C^\circ]\right). \end{aligned}$$

Consider the Schlegel diagram $\mathcal{D}(P, F)$. By [13] , we can view $\mathcal{D}(P, F)$ as a subdivision of F or as $\mathscr{S}|_{\partial(P)}\backslash\{F\}$. Let us take $\mathscr{S}$ to be the subdivision of F induced by $\mathcal{D}(P, F)$. Note that the subdivision of faces of $\partial P\backslash F$ is isomorphic to $\mathscr{S}|_{F^\circ}$. Therefore,

$$\begin{aligned}
\chi(P^\circ) &= -\chi\left(\sum_{C\in\mathscr{S}|_{F^\circ}} [C^\circ]\right) \\
&= -\chi(F^\circ) \\
&= -(-1)^{\dim(F)} \\
&= (-1)^{\dim(P)}.
\end{aligned}$$

□

Then the Euler characteristic of a non-empty closed polytope P can be expressed as

$$\begin{aligned}
\bar{\chi}(P) &= \sum_{F\leq P} (-1)^{\dim(F)} \\
&= \sum_{F\leq P} \chi(F^\circ) \\
&= \chi\left(\sum_{F\leq P} F^\circ\right) \\
&= \chi(P) \\
&= 1.
\end{aligned}$$

Therefore, we have the following two results.

Corollary 4.3.4. *For any closed polytope,* $\bar{\chi}(P) = 1$.

Corollary 4.3.5. *For any polytope* P*,* $\chi(\partial P) = 1 - (-1)^{\dim(P)}$.

4.3.1 Vertex Figure

Now, we consider that a polyhedral subdivision can form a partial ordered set (a *poset*), ordered by inclusion. Let R to a polytope. Let V be the vertex set of R and let $v \in V$. There is a hyperplane H that separates v from all other vertices of R; for example, we can start with a supporting hyperplane of v and move the hyperplane into R by a small distance. The *vertex figure* of R at v, denoted $R\backslash v$ is

$$R\backslash v := H \cap R.$$

Clearly, $R\backslash v$ is a polytope.

Lemma 4.3.6. *The poset (ordered by inclusion) of faces of vertex figure* $R\backslash v = H \cap R$ *is isomorphic to the poset (ordered by inclusion) of faces of* R *that contain* v*.* k*-dimensional faces of* $R\backslash v$ *correspond to* $(k+1)$*-dimensional faces of* R *that contain* v*.*

Proof. Recall that a face of R is a convex hull of a particular subset of V. Since H separates v from all other vertices, the faces of R that intersect H are the faces that contain v and at least one other vertex. Alternatively, $R\backslash v$ can be expressed as

$$\begin{aligned} R\backslash v &= H \cap (H_1^+ \cap \cdots \cap H_n^+) \\ &= (H \cap H_1^+) \cap \cdots \cap (H \cap H_n^+) \end{aligned}$$

where $H_1^+, \ldots, H_n^+$ are the facet half-spaces of facets containing v. Let $F \leq R$ be a face that intersects H and let $H_F = \{x : \langle c, x \rangle \leq b\}$ be a supporting hyperplane of F. By definition, the inequality $\langle c, x \rangle \leq b$ is tight on F and strict on $R\backslash F$. In particular, the inequality is tight on $F \cap H$ and strict on $H \cap (R\backslash F)$. Therefore, $H_F \cap H$ is a supporting hyperplane of $R\backslash v$ and $F \cap H$ is a face of $R\backslash v$.

We now know that faces of R that contain v correspond to the faces of $R\backslash v$. By looking at a face as the intersection of facet half-spaces, we see that inclusion is preserved. Noting that edges of R that contain v pass through H at one point, and therefore correspond to vertices of $R\backslash v$, we can conclude that the k-dimensional faces of $R\backslash v$ correspond to $(k+1)$-dimensional faces of R containing v. Note that v would correspond to the empty face. □

4.3.2 Möbius function on Polytope Faces and Subdivisions

Given a poset S, the Möbius function $\mu(Q, R)$ of S satisfies

1. If $Q = R$, $\mu(Q, R) = 1$
2. If $Q < R$, $\sum_{Q \leq T \leq R} \mu(Q, T) = 0$
3. Otherwise, $\mu(Q, R) = 0$

An equivalent statement for condition (2) is

$$\sum_{Q \leq T \leq R} \mu(T, R) = 0.$$

Let P be a polytope. Consider the poset of faces of P ordered by inclusion. Let $\emptyset$ be the empty face; we define $\dim(\emptyset) = -1$. For the rest of this section, we aim to prove the following theorem.

Theorem 4.3.7. *For $Q \leq R$, $\mu(Q, R) = (-1)^{\dim(R) - \dim(Q)}$.*

Proof. Let us proceed by induction on $\dim(Q)$. For the base case, we show that $\mu(\emptyset, R) = (-1)^{\dim(R)+1}$. We have two cases: $R = \emptyset$ must satisfy condition (1) and $R > \emptyset$ must satisfy condition (2). If $R = \emptyset$, then

$$\mu(\emptyset, R) = \mu(\emptyset, \emptyset) = (-1)^{-1+1} = 1.$$

Now suppose $R > \emptyset$. We want to show that

$$\sum_{\emptyset \leq T \leq R} (-1)^{\dim(T)+1} = 0.$$

Re-writing the left hand side, we have

$$\begin{aligned}
\sum_{\emptyset \le T \le R} (-1)^{\dim(T)+1} &= \mu(\emptyset, \emptyset) + \sum_{\emptyset < T \le R} (-1)^{\dim(T)+1} \\
&= 1 - \sum_{\emptyset < T \le R} (-1)^{\dim(T)} \\
&= 1 - \bar{\chi}(R) \\
&= 1 - 1 \\
&= 0.
\end{aligned}$$

Now, take the inductive step and consider $\emptyset < Q \le R$ and try to verify that $\mu(Q, R) = (-1)^{\dim(R)-\dim(Q)}$. For $Q = R$, we see that

$$\mu(Q, R) = \mu(R, R) = (-1)^0 = 1,$$

which satisfies condition (1). Now suppose $\emptyset < Q < R$. Consider the vertex figure $R \backslash v$, where v is a vertex of Q. We have seen that the poset of faces of the polytope $R \backslash v$ is isomorphic to the poset of faces of R that contain v and that the dimension of each corresponding face differ by one. Let Q' and R' be the corresponding faces in $R \backslash v$. Then $\dim(Q') = \dim(Q) - 1$ and $\dim(R') = \dim(R) - 1$. By induction hypothesis,

$$\mu(Q, R) = \mu(Q', R') = (-1)^{(\dim(R)-1)-(\dim(Q)-1)} = (-1)^{\dim(R)-\dim(Q)}.$$

Let us now consider a polytope subdivision $\mathscr{S}$ of P, and the poset of cells of $\mathscr{S}$, ordered by inclusion. Once again, we claim that for $Q, R \in \mathscr{S}$, $\mu(Q, R) = (-1)^{\dim(R)-\dim(Q)}$. For $Q = R$, the result is immediate. For $Q < R$ a rearranging of terms of the second Möbius function condition would give

$$\mu(Q, R) = -\sum_{Q < T \le R} \mu(T, R).$$

However, the only cells $T \in \mathscr{S}$ that contribute to $\mu(Q, R)$ are the faces of R. Therefore, we can omit all the other cells of $\mathscr{S}$ and compute $\mu(Q, R)$ as if it were in the context of the face poset of R (ordered by inclusion). Therefore, $\mu(Q, R) = (-1)^{\dim(R)-\dim(Q)}$. $\square$

4.3.3 More Euler-type Relations

Using Möbius functions, we are able to derive the following two results.

Theorem 4.3.8. *For any polytope P,*

$$[P^\circ] = \sum_{F \le P} (-1)^{\dim(P)-\dim(F)} [F]$$

Proof. If we take our poset to be the faces of P ordered by inclusion, we would have

$$\begin{aligned}
\sum_{F\leq P}(-1)^{\dim(P)-\dim(F)}[F] &= \sum_{F\leq P}\mu(F,P)[F] \\
&= \sum_{F\leq P}\mu(F,P)\left(\sum_{T\leq F}[T^\circ]\right) \\
&= \sum_{T\leq P}[T^\circ]\left(\sum_{T\leq F\leq P}\mu(F,P)\right) \\
&= [P^\circ]\mu(P,P)+\sum_{T<P}[T^\circ]\left(\sum_{T\leq F\leq P}\mu(F,P)\right) \\
&= [P^\circ](1)+\sum_{T<P}[T^\circ](0) \\
&= [P^\circ].
\end{aligned}$$

□

Theorem 4.3.9. *For any polyhedral subdivision of polytope P,*

$$[P]=\sum_{C\in\mathscr{S}|_{P^\circ}}(-1)^{\dim(P)-\dim(C)}[C].$$

Proof. We proceed by induction on $\dim(P)=d$. If $d=0$, then P is a point p, and $\mathscr{S}=\{p,\emptyset\}$. The result is then immediate. Taking the inductive step, we let $d\geq 1$ and pick a facet F of P and consider the Schlegel diagram $\mathcal{D}(P,F)$. By [13], we can view $\mathcal{D}(P,F)$ as a subdivision of F or as $\mathscr{S}|_{\partial(P)}\backslash\{F\}$. By the former interpretation, we can apply Theorem 4.3.1 to get

$$\sum_{C\in\mathcal{D}(P,F)}(-1)^{\dim(F)-\dim(C)}[C]=[F^\circ]$$

Translating this result to the language of the latter interpretation, we have

$$\sum_{C\in\mathscr{S}|_{\partial(P)}\backslash\{F\}}(-1)^{\dim(F)-\dim(C)}[C]=[\partial(P)\backslash F].$$

Let us now look at $\mathscr{S}':=\mathscr{S}|_F$, the subdivision $\mathscr{S}$ restricted to the cells contained in F. Note that the intersection

$$\left(\mathscr{S}|_{\partial(P)}\backslash\{F\}\right)\cap\mathscr{S}'$$

is the set of cells in the boundary of F, and so

$$\mathscr{S}|_{\partial(P)}=\mathscr{S}|_{\partial(P)}\backslash\{F\}\sqcup\mathscr{S}'|_{F^\circ}.$$

By induction,

$$\sum_{C\in\mathscr{S}|_{F^\circ}}(-1)^{\dim(F)-\dim(C)}[C]=[F].$$

Recalling the interpretations of the Schlegel diagram, we have

$$\sum_{C\in\mathscr{S}|_{\partial(P)}} (-1)^{\dim(P)-\dim(C)}[C] = \sum_{C\in\mathscr{S}|_{\partial(P)}\setminus\{F\}} (-1)^{\dim(P)-\dim(C)}[C] + \sum_{C\in\mathscr{S}|_{F^\circ}} (-1)^{\dim(P)-\dim(C)}[C].$$

Noting that $\dim(P) = \dim(F) + 1$ yields

$$\sum_{C\in\mathscr{S}|_{\partial(P)}} (-1)^{\dim(P)-\dim(C)}[C] = -[\partial(P)\setminus F] - [F] = -[\partial(P)].$$

Lastly, we conclude that

$$\begin{aligned}\sum_{C\in\mathscr{S}|_{P^\circ}} (-1)^{\dim(P)-\dim(C)}[C] &= \sum_{C\in\mathscr{S}} (-1)^{\dim(P)-\dim(C)}[C] - \sum_{C\in\mathscr{S}|_{\partial(P)}} (-1)^{\dim(P)-\dim(C)}[C] \\ &= [P^\circ] - (-[\partial(P)]) \\ &= [P].\end{aligned}$$

□

4.4 Generating Functions and Convergence

As we will be dealing with generating functions and infinite sums, let us a look at some of the generating functions and the issue of convergence. We denote $\mathbb{Z}_+^d$ and $\mathbb{R}_+^d$ the non-negative orthant of $\mathbb{Z}^d$ and $\mathbb{R}^d$ respectively. We start with the basic geometric series

$$f(n,z) = \sum_{\substack{p\in\mathbb{Z}_+\\ p\leq n}} z^p = \frac{1-z^{n+1}}{1-z} \tag{4.12}$$

and recall the convergence

$$\lim_{n\to\infty} f(n,z) = \sum_{p\in\mathbb{Z}_+} x^p = \frac{1}{1-z} \tag{4.13}$$

for z such that $|z| < 1$.

Now, let us take a look at

$$g(n,\mathbf{z}) := \sum_{\substack{\mathbf{p}\in\mathbb{Z}_+^d\\ p_i\leq n}} \mathbf{z}^{\mathbf{p}}. \tag{4.14}$$

Note that $g(n,\mathbf{z})$ can be written as

$$\sum_{p_d\in\mathbb{Z}_+} x_d^{p_d} \left(\sum_{\substack{\mathbf{p}'\in\mathbb{Z}_+^{d-1}\\ p_i\leq n}} \mathbf{x}^{\mathbf{p}'} \right). \tag{4.15}$$

By factoring each index out recursively, we would get

$$\sum_{\substack{p_d \in \mathbb{Z}_+ \\ p_d \le n}} x_d^{p_d} \left(\sum_{\substack{p_{d-1} \in \mathbb{Z}_+ \\ p_{d-1} \le n}} z_{d-1}^{p_{d-1}} \left(\cdots \left(\sum_{\substack{p_2 \in \mathbb{Z}_+ \\ p_2 \le n}} z_2^{p_2} \left(\sum_{\substack{p_1 \in \mathbb{Z}_+ \\ p_1 \le n}} z_1^{p_1} \right) \right) \cdots \right) \right). \tag{4.16}$$

Therefore, starting from the inner most summation, we factor summations out to get

$$\left(\sum_{\substack{p_d \in \mathbb{Z}_+ \\ p_d \le n}} z_d^{p_d} \right) \left(\sum_{\substack{p_{d-1} \in \mathbb{Z}_+ \\ p_{d-1} \le n}} z_{d-1}^{p_{d-1}} \right) \cdots \left(\sum_{\substack{p_2 \in \mathbb{Z}_+ \\ p_2 \le n}} z_2^{p_2} \right) \left(\sum_{\substack{p_1 \in \mathbb{Z}_+ \\ p_1 \le n}} z_1^{p_1} \right). \tag{4.17}$$

For $z \in \mathbb{R}^d$ such that $|z_i| < 1$, and taking n to infinity, we have

$$\lim_{n \to \infty} g(n, \mathbf{z}) = \sum_{\mathbf{p} \in \mathbb{Z}_+^d} \mathbf{z}^{\mathbf{p}} = \prod_{i=1}^{d} \frac{1}{1 - z_i}. \tag{4.18}$$

Now suppose we fix $\mathbf{v} \in \mathbb{Z}^d$ and let $U = \{\mathbf{z} \in \mathbb{R}^d : |\mathbf{z}^{\mathbf{v}}| < 1\}$. Let

$$h(n, \mathbf{z}) := \sum_{\substack{p \in \mathbb{Z}_+ \\ p \le n}} \mathbf{z}^{p\mathbf{v}}. \tag{4.19}$$

Then by (4.13), we see that

$$\lim_{n \to \infty} h(n, \mathbf{z}) = \sum_{p \in \mathbb{Z}_+} \mathbf{z}^{p\mathbf{v}} = \frac{1}{1 - \mathbf{z}^{\mathbf{v}}} \tag{4.20}$$

for every $\mathbf{z} \in U$. Additionally, we have the following lemma.

Lemma 4.4.1. *The above convergence is absolute and uniform on compact subsets of U.*

Proof. Since $0 \le |\mathbf{z}^{\mathbf{v}}| < 1$, the absolute convergence follows from (4.13). Now, by (4.12), we see that for any $\mathbf{z} \in U$

$$\begin{aligned} \left| \frac{1}{1 - \mathbf{z}^{\mathbf{v}}} - h(n, \mathbf{z}) \right| &= \left| \frac{1}{1 - \mathbf{z}^{\mathbf{v}}} - \frac{1 - (\mathbf{z}^{\mathbf{v}})^{n+1}}{1 - \mathbf{z}^{\mathbf{v}}} \right| \\ &= \frac{(\mathbf{z}^{\mathbf{v}})^{n+1}}{1 - \mathbf{z}^{\mathbf{v}}} \end{aligned}$$

Taking n to infinity, we see that the difference goes to zero. Therefore, the convergence is uniform on compact subsets of U. $\square$

Now, let us consider convergence on the tiling of a cone using half-open parallelepipeds.

Lemma 4.4.2. *Let K be the cone generated by linearly independent vectors $\mathbf{v}_1, \dots, \mathbf{v}_d \in \mathbb{Z}^d$ and $\overline{\Pi}$ be the top-open parallelepiped generated by these vectors. If*

$$U = \{\mathbf{z} \in \mathbb{R}^d : |\mathbf{z}^{\mathbf{v}_i}| < 1, i = 1, \dots, d\}, \tag{4.21}$$

then for all $\mathbf{z} \in U$, the series

$$\sum_{\mathbf{p} \in K \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{p}} \tag{4.22}$$

converges absolutely and uniformly on compact subsets of U to the rational function

$$f(K, \mathbf{z}) = \left(\sum_{\mathbf{x} \in \overline{\Pi} \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{x}} \right) \prod_{i=1}^{d} \frac{1}{1 - \mathbf{z}^{\mathbf{v}_i}} \tag{4.23}$$

Proof. We have seen that every lattice point in $\mathbb{R}^d$ can be uniquely represented as

$$x + \sum_{i=1}^{d} \tau_i v_i$$

where $x \in \overline{\Pi}$ and $\tau_i \in \mathbb{Z}_+$. By Lemma 4.4.1, for any $\mathbf{z} \in U$, we have absolute and uniform convergence

$$\begin{aligned} \sum_{\mathbf{p} \in K \cap Z^d} \mathbf{z}^{\mathbf{p}} &= \left(\sum_{\mathbf{x} \in \overline{\Pi} \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{x}} \right) \left(\sum_{\mathbf{v}_1, \dots, \mathbf{v}_d \in \mathbb{Z}^d_+} \mathbf{z}^{\tau_1 \mathbf{v}_1 + \dots + \tau_k \mathbf{v}_k} \right) \\ &= \left(\sum_{\mathbf{x} \in \overline{\Pi} \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{x}} \right) \prod_{i=1}^{d} \frac{1}{1 - \mathbf{z}^{\mathbf{v}_i}}. \end{aligned}$$

□

Let us introduce the *polar set* $S^{\triangle}$ of $S \subseteq \mathbb{R}^n$ to be

$$S^{\triangle} := \{a : \langle a, s \rangle \leq 1 \ \forall \ s \in S\}.$$

It is common to denote the polar to be S°, but we reserve S° to denote the relative interior of S instead. Now suppose S is a non-empty convex cone, and that the polar cone $S^{\triangle}$ is non-empty. For any $a \in S^{\triangle}$, we have $\langle a, s \rangle \leq 1$. Since S is a convex cone, τs is also in S for any $\tau \geq 0$, so

$$\langle a, \tau s \rangle = \tau \langle a, s \rangle \leq 1.$$

Therefore, for any convex cone S, we can tighten the constraint "$\langle a, s \rangle \leq 1$" in the definition of $S^{\triangle}$ to "$\langle a, s \rangle \leq 0$".

Lemma 4.4.3. *Let $K \subset \mathbb{R}^d$ be a rational cone without straight lines and $U \subset \mathbb{C}^d$ be the set*

$$W = \left\{ \mathbf{e}^{a+ib} : a \in (K^{\triangle})^{\circ} \text{ and } b \in \mathbb{R}^d \right\}$$

where $\mathbf{e}^{a+ib} = (e^{a_1+ib_1}, \dots, e^{a_d+ib_d})$. Then W is a non-empty open set where for every $z \in W$ the series

$$\sum_{\mathbf{p} \in K \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{p}}$$

converges absolutely and uniformly on compact subsets of U to a rational function

$$f(K,\mathbf{z}) = \sum_{i=1}^{n} \frac{p_i(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{u}_{i1}})\dots(1-\mathbf{z}^{\mathbf{u}_{id}})}$$

where $p_i(\mathbf{z})$ are Laurent polynomials and $\mathbf{u}_{ij} \in \mathbb{Z}^d$ are integer vectors for $i = 1,\dots,n$ and $j = 1,\dots,d$.

Proof. Without loss of generality, assume that $K \neq \{0\}$. Since K is a rational cone, there exists an integer polytope Q that is a base of K. Thus, we can triangulate K into rational simplicial cones K_i, $i \in I$ (where I is some index set). Using indicator functions and inclusion-exclusion, we see that

$$[K] = \sum_{i \in I} \gamma_i [K_i]$$

where $\gamma_i \in \{-1,1\}$. Translating to the language of generating functions, we get

$$\sum_{\mathbf{p} \in K \cap \mathbb{Z}^d} \mathbf{x}^{\mathbf{p}} = \sum_{i \in I} \gamma_i \left(\sum_{\mathbf{p} \in K_i \cap \mathbb{Z}^d} \mathbf{x}^{\mathbf{p}} \right).$$

Note that for any $a \in K^{\triangle}$ and $p \in K$,

$$|(\mathbf{e}^{a+ib})^p| = |e^{\langle a,p \rangle}| \leq e^0 = 1$$

with equality if and only if a is on the boundary of $K^{\triangle}$. Therefore, for any $\mathbf{z} \in W$, $|\mathbf{z}^m| < 1$. By Lemma (4.4.2), the right hand side is a sum of rational functions with denominators $(1-\mathbf{z}^{\mathbf{u}})$ where $\mathbf{u}$ is a generator ray for some cone K_i. By multiplying numerator and denominator by some binomials $(1-\mathbf{z}^{\mathbf{u}})$, we can ensure that each denominator is the product of exactly d binomials, which is the desired form.

To show that W is open, we need $(K^{\triangle})^{\circ}$ to be open and full-dimensional. It is easy to see that $(K^{\triangle})^{\circ} = \{a : \langle a,p \rangle < 0 \ \forall \ p \in K\}$ is open. To show that $\dim(K^{\triangle}) = d$, suppose $\dim(K^{\triangle}) < d$. Then $K^{\triangle}$ is contained in hyperplane, implying that $(K^{\triangle})^{\triangle}$ contains a straight line. However, recalling that $(K^{\triangle})^{\triangle} = K$, we have a contradiction. □

Now we generalize the result from rational cones to rational polyhedra.

Lemma 4.4.4. *Let $P \subset \mathbb{R}^d$ be a rational polyhedron without straight lines. Then there exists a non-empty open set $U \subset \mathbb{C}^d$ such that for all $\mathbf{x} \in U$ the series*

$$\sum_{m \in P \cap \mathbb{Z}^d} \mathbf{x}^m$$

converges absolutely and uniformly on compact subsets of U to a rational function $f(P,\mathbf{x})$ of $\mathbf{x}$.

Proof. Again, we lift P into $P' \subset \mathbb{R}^{d+1}$ and let $K = \operatorname{cone}(P')$. We apply Lemma (4.4.3) to see that there exists a non-empty open set U' such that for all $y = (\mathbf{z}, z_{d+1}) \in U'$,

$$\sum_{\mathbf{p}' \in K \cap \mathbb{Z}^{d+1}} \mathbf{y}^{\mathbf{p}'} = \sum_{(\mathbf{p},\mu) \in K \cap \mathbb{Z}^{d+1}} \mathbf{z}^{\mathbf{p}} z_{d+1}^{\mu}$$

converges absolutely and uniformly on compact subsets of U' to $f(K, (\mathbf{z}, z_{d+1}))$, a rational function. Note that for any $(\mathbf{z}, z_{d+1}) \in U' = (U')^\circ$,

$$\begin{aligned}\frac{\partial f}{\partial z_{d+1}} &= \frac{\partial}{\partial z_{d+1}} \sum_{(\mathbf{p},\mu)\in K\cap\mathbb{Z}^{d+1}} \mathbf{z}^{\mathbf{p}} z_{d+1}^{\mu} \\ &= \sum_{(\mathbf{p},1)\in K\cap\mathbb{Z}^{d+1}} \mathbf{z}^{\mathbf{p}} + \sum_{i>1} (i+1) z_{d+1}^{i} \sum_{(\mathbf{p},i)\in K\cap\mathbb{Z}^{d+1}} \mathbf{z}^{\mathbf{p}}\end{aligned}$$

converges absolutely and uniformly on compact sets in U' to a rational function. Since this series keeps track of the number of lattice points in a certain set, and we know it converges to a rational function, setting z_{d+1} to 0 will only decrease the number of points that we are keeping track of. Therefore, setting z_{d+1} to 0 will not cause any convergence issues:

$$\left.\frac{\partial f}{\partial z_{d+1}}\right|_{z_{d+1}=0} = \sum_{(\mathbf{p},1)\in K\cap\mathbb{Z}^{d+1}} \mathbf{z}^{\mathbf{p}} = \sum_{\mathbf{p}\in P\cap\mathbb{Z}^{d}} \mathbf{z}^{\mathbf{p}}.$$

Let U be the projection of U'; $(\mathbf{z}, z_{d+1}) \mapsto \mathbf{z}$. Then for any $\mathbf{z} \in U$, the series

$$\sum_{\mathbf{p}\in P\cap\mathbb{Z}^{d}} \mathbf{z}^{\mathbf{p}}$$

converges absolutely and uniformly on compact subsets of U to the rational function

$$f(P, \mathbf{z}) = \left.\frac{\partial}{\partial z_{d+1}} f(K, (\mathbf{z}, z_{d+1}))\right|_{z_{d+1}=0}. \qquad \square$$

Now, we get to a nice theorem tying together some notions of the Ehrhart series and the algebra of polyhedra.

4.4.1 A special valuation

We would like to develop a valuation

$$\mathcal{F} : \mathscr{P}(\mathbb{Q}^d) \to \mathbb{C}(x_1, \ldots, x_d)$$

such that the following hold:

($\mathcal{F}1$) If $P \subset \mathbb{R}^d$ is a rational polyhedron without straight lines, then $\mathcal{F}[P] = f(P, \mathbf{x})$ is the rational function

$$f(P, \mathbf{z}) = \sum_{\mathbf{p}\in P\cap\mathbb{Z}^{d}} \mathbf{z}^{\mathbf{p}},$$

provided it converges absolutely.

($\mathcal{F}2$) For a function $g \in \mathscr{P}(\mathbb{Q}^d)$ and an integer vector $u \in \mathbb{Z}^d$, let $h^*(z) = g(z - u)$ be the shift of g. Then $\mathcal{F}(h) = \mathbf{z}^{\mathbf{u}} \mathcal{F}(g)$.

($\mathcal{F}3$) If $P \subset \mathbb{R}^d$ is a rational polyhedron containing a straight line, then $\mathcal{F}[P] \equiv 0$; the rational function is identically zero.

Note that the series mentioned in ($\mathcal{F}1$) is the Hilbert series of a polytope. For this reason, we shall refer to the valuation $\mathcal{F}$ as the "Hilbert Valuation". We already know how to define $\mathcal{F}[P]$ for rational polyhedra P without straight lines. By Lemma 4.4.4, we see that there is a non-empty open set $U \subset \mathbb{C}^d$ such that $\sum_{\mathbf{p} \in P \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{p}}$ converges absolutely and uniformly to a rational function $f(P, \mathbf{z})$ on compact subsets of U. Let us define

$$\mathcal{F}[P] := f(P, \mathbf{z}) \tag{4.24}$$

for rational polytopes P without straight lines. Recall that the algebra of rational polyhedra is spanned by the indicator functions $[P]$ where each P is a rational polyhedron without straight lines (Lemma 4.2.1). Then, every function $f \in \mathscr{P}(\mathbb{Q}^d)$ can be written as

$$f = \sum_i \alpha_i [P_i]$$

where each P_i does not contain straight lines. Let us define $\mathcal{F}(f)$ as

$$\mathcal{F}(f) := \sum_i \alpha_i \mathcal{F}[P_i].$$

Consider another function $g = \sum_i \beta_i [P_i]$ in $\mathscr{P}(\mathbb{Q}^d)$ and $a, b \in \mathbb{R}$. Let us verify that our definition of $\mathcal{F}$ is a valuation; in other words, we need to verify linearity.

$$\begin{aligned}
\mathcal{F}(af + bg) &= \sum_i (a\alpha_i + b\beta_i)\mathcal{F}[P_i] \\
&= a\sum_i \alpha_i \mathcal{F}[P_i] + b\sum_i \beta_i \mathcal{F}[P_i] \\
&= a\mathcal{F}(f) + b\mathcal{F}(g).
\end{aligned}$$

We also need to prove that $\mathcal{F}$ is well-defined. For $f, g \in \mathscr{P}(\mathbb{Q}^d)$, we want

$$f = g \implies \mathcal{F}(f) = \mathcal{F}(g).$$

Equivalently, by moving all terms to one side, we would like to prove

$$\sum_{i=1}^n \alpha_i [P_i] = 0 \implies \sum_{i=1}^n \alpha_i f(P_i, x) = 0$$

for rational polyhedra P_i without straight lines and real numbers α_i. Suppose $\sum_{i=1}^n \alpha_i [P_i] = 0$. We denote for some subset $I \subseteq [n]$

$$P_I := \bigcap_{i \in I} P_i.$$

We fix some i and use inclusion-exclusion to get

$$\left[\bigcup_{j=1}^{n} P_j\right] = \sum_{\substack{I\subset[n]\\ I\neq\emptyset}} (-1)^{|I|-1}[P_I]$$

$$[P_i]\left[\bigcup_{j=1}^{n} P_j\right] = [P_i]\sum_{\substack{I\subset[n]\\ I\neq\emptyset}} (-1)^{|I|-1}[P_I]$$

$$[P_i] = \sum_{\substack{I\subset[n]\\ I\neq\emptyset}} (-1)^{|I|-1}[P_{I\cup\{i\}}].$$

Taking lattice points, we have the following identity of power series:

$$\sum_{\mathbf{p}\in P_i\cap\mathbb{Z}^d} \mathbf{z}^{\mathbf{p}} = \sum_{\substack{I\subset[n]\\ I\neq\emptyset}} (-1)^{|I|-1} \sum_{\mathbf{p}\in P_{I\cup\{i\}}\cap\mathbb{Z}^d} \mathbf{z}^{\mathbf{p}}.$$

We would like to use Lemma 4.4.4 to help us in this proof. We already know that for any non-empty I, P_I is a rational polyhedron without straight lines. However, we also need to make sure that our one choice of open set U is good enough that all $\sum_{\mathbf{p}\in P_{I\cup\{i\}}\cap\mathbb{Z}^d} \mathbf{z}^{\mathbf{p}}$ and $\sum_{\mathbf{p}\in P_i\cap\mathbb{Z}^d} \mathbf{z}^{\mathbf{p}}$ would converge to rational functions. In the proof of Lemma 4.4.4, our choice of U comes from the projection of $(K^{\triangle})^{\circ}$ onto $\mathbb{R}^d$, where K is the cone over the lifted polytope P' of P. Note that

$$P_{I\cup\{i\}} \subseteq P_i,$$

and that this containment implies

$$\mathrm{pr}((\mathrm{cone}(P'_{I\cup\{i\}})^{\triangle})^{\circ}) \supseteq \mathrm{pr}((\mathrm{cone}(P'_i)^{\triangle})^{\circ}).$$

In light of the above result, we can use Lemma 4.4.4 on P_i and the choice of U for P_i (as described in the lemma's proof) is good enough that each $\sum_{\mathbf{p}\in P\cap\mathbb{Z}^d} \mathbf{z}^{\mathbf{p}}$ converges to a rational function. Therefore,

$$f(P_i,\mathbf{z}) = \sum_{\substack{I\subset[n]\\ I\neq\emptyset}} (-1)^{|I|-1} f(P_{I\cup\{i\}},\mathbf{z}). \tag{4.25}$$

Let $I\subset\{1,\ldots,n\}$ be a non-empty index set. We have

$$\sum_{i=1}^{n} \alpha_i[P_i] = 0$$

$$[P_I]\sum_{i=1}^{n} \alpha_i[P_i] = [P_I]0$$

$$\sum_{i=1}^{n} \alpha_i[P_{I\cup\{i\}}] = 0$$

$$\sum_{i=1}^{n} \alpha_i \sum_{\mathbf{p}\in P_{I\cup\{i\}}} \mathbf{z}^{\mathbf{p}} = 0,$$

where the left side of the last line is a formal power series. Again, since P_I is a rational polyhedron without straight lines and $P_{I\cup\{i\}} \subseteq P_I$, there is a non-empty open set U such that all $\sum_{\mathbf{p}\in P\cap\mathbb{Z}^d} \mathbf{z}^{\mathbf{p}}$ converge to a rational function. Therefore, for non-empty $I \subset [n]$,

$$\sum_{i=1}^{n} \alpha_i f(P_{I\cup\{i\}}, \mathbf{z}) \equiv 0. \tag{4.26}$$

Combining (4.25) and (4.26) proves property ($\mathcal{F}1$):

$$\begin{aligned}
\sum_{i=1}^{n} \alpha_i f(P_i, \mathbf{z}) &= \sum_{i=1}^{n} \alpha_i \sum_{\substack{I\subset[n]\\ I\neq\emptyset}} (-1)^{|I|-1} f(P_{I\cup\{i\}}, \mathbf{z}) \\
&= \sum_{\substack{I\subset[n]\\ I\neq\emptyset}} (-1)^{|I|-1} \left(\sum_{i=1}^{n} \alpha_i f(P_{I\cup\{i\}}, \mathbf{z}) \right) \\
&\equiv \sum_{\substack{I\subset[n]\\ I\neq\emptyset}} (-1)^{|I|-1} (0) \\
&= 0.
\end{aligned}$$

It remains to prove properties ($\mathcal{F}2$) and ($\mathcal{F}3$).

Lemma 4.4.5. *For a function $g \in \mathscr{P}(\mathbb{Q}^d)$ and an integer vector $u \in \mathbb{Z}^d$, let $h(z) = g(z-u)$ be the shift of g. Then $\mathcal{F}(h) = \mathbf{z}^{\mathbf{u}}\mathcal{F}(g)$.*

Proof. Suppose $g = [P]$ for some rational polyhedron P without straight lines. Then

$$g(z-u) = 1 \iff z - u \in P \iff z \in u + P.$$

Therefore, if $g = \sum_i \alpha_i [P_i]$ where each P_i is a rational polyhedron without straight lines, then $h = \sum_i \alpha[u + P_i]$. Since u is and integer point, we have

$$\begin{aligned}
\mathcal{F}(h) &= \sum_i \alpha_i \mathcal{F}[u + P_i] \\
&= \sum_i \alpha_i f(u + P_i, \mathbf{z}) \\
&= \sum_i \alpha_i \sum_{\mathbf{p}\in(u+P_i)\cap\mathbb{Z}^d} \mathbf{z}^{\mathbf{p}} \\
&= \sum_i \alpha_i \sum_{\mathbf{p}\in P_i\cap\mathbb{Z}^d} \mathbf{z}^{\mathbf{p}} \mathbf{z}^{\mathbf{u}} \\
&= \mathbf{z}^{\mathbf{u}} \sum_i \alpha_i f(P_i, \mathbf{x}) \\
&= \mathbf{z}^{\mathbf{u}} \sum_i \alpha_i \mathcal{F}[P_i] \\
&= \mathbf{z}^{\mathbf{u}} \mathcal{F}(g)
\end{aligned}$$

□

Lemma 4.4.6. *If $P \subset \mathbb{R}^d$ is a rational polyhedron containing a straight line, then $\mathcal{F}[P] \equiv 0$; the rational function is identically zero.*

Proof. Suppose rational polyhedron $P = \{x : Ax \leq b\}$ contains a straight line. By Corollary 1.1.5, Nullity$(A) > 0$; let u be in the null-space of A. Since

$$A(u + x) = Au + Ax = 0 + Ax = Ax,$$

$x \in P$ if and only if $u + x \in P$. An immediate result from the forward and reverse implications is that $P \subseteq u + P$ and $P \supseteq u + P$, so $u + P = P$. Applying Lemma 4.4.5,

$$\mathcal{F}[P] = \mathcal{F}[u + P] = \mathbf{z}^{\mathbf{u}}\mathcal{F}[P].$$

Since $\mathcal{F}[P]$ is a rational function and $u \neq 0$, $\mathcal{F}[P]$ must be identically zero. □

4.5 Brion's Theorem

By exploring the structure of the algebra of polyhedra, we will develop a formula for the valuation $\mathcal{F}$ and a polyhedron's support cones.

4.5.1 Support Cones

Given a polyhedron $P \subseteq \mathbb{R}^d$ and point $v \in P$, the *support cone* of P at v is defined as

$$\operatorname{cone}(P, v) = \left\{x \in \mathbb{R}^d : \lambda x + (1 - \lambda)v \in P \text{ for some } 0 < \lambda < 1\right\}.$$

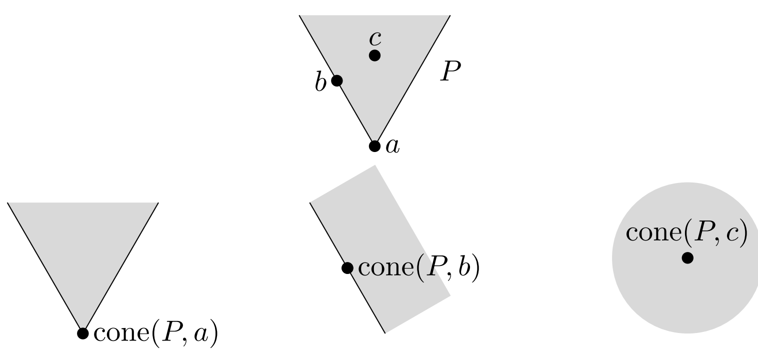

Figure 4.2. A polyhedron P and its support cones at points a, b and c. Figure inspired by [1].

Note that the vertex of the cone (if a vertex exists) is at v and not necessarily at the origin. Another way of viewing cone(P, v) is as follows:

Lemma 4.5.1. *The support cone of P at v is the cone generated by all the rays emanating from v that intersect P.*

Proof. The condition "$\lambda x + (1-\lambda)v \in P$ for some $0 < \lambda < 1$" can be viewed as "the open line segment $L = [x, v]$ joining x and v intersects P." We can then state that

$$x \text{ is in } \operatorname{cone}(P, v) \text{ if and only if } L \text{ intersects } P,$$

which is equivalent to saying L contains a line segment or a point that intersects P. Let p be a point in $L \cap P$. By convexity the open line segment L' joining v and p is contained in P. For any $\tau > 0$, the open line segment joining v and τx must contain part of L', which implies that $\tau x \in \operatorname{cone}(P, v)$. □

If v is a vertex, then $\operatorname{cone}(P, v)$ is the cone generated by the rays emanating from v along the edges of P that contain v. Alternatively, recall that a pointed cone generated by a set R of rays can be expressed as the intersection of facet half-spaces, where each facet hyperplane is the affine subspace containing linearly independent subsets of R.

Corollary 4.5.2. *Suppose $P \subseteq \mathbb{R}^d$ is a d-polyhedron. If P contains a vertex v, then* $\operatorname{cone}(P, v)$ *is the intersection of facet half-spaces H^+ of P such that $v \in H$.*

Under linear transformations, we have the following lemma.

Lemma 4.5.3. *Let $P \subset \mathbb{R}^n$ be a polyhedron and let $T : \mathbb{R}^n \to \mathbb{R}^d$ be a linear transformation. For any $v \in P$,*

$$T(\operatorname{cone}(P, v)) = \operatorname{cone}(T(P), T(v))$$

Proof. Suppose $z \in \operatorname{cone}(P, v)$. For some fixed $0 < \lambda < 1$, the point $x := \lambda z + (1-\lambda)v$ is in P. Note that by definition of $T(P)$ and $T(\operatorname{cone}(P, v))$, $T(x) \in T(P)$ and $T(z) \in T(\operatorname{cone}(P, v))$. Since T is linear, we have

$$\begin{aligned} T(x) &= T(\lambda z + (1-\lambda)v) \\ &= \lambda T(z) + (1-\lambda)T(v) \end{aligned}$$

Therefore $T(z) \in \operatorname{cone}(T(P), T(v))$, which implies that $T(\operatorname{cone}(P, v)) \subseteq \operatorname{cone}(T(P), T(v))$.

Now suppose $y \in \operatorname{cone}(T(P), T(v))$, which implies for some fixed $0 < \lambda < 1$, the point $w = \lambda y + (1-\lambda)T(v)$ is in $T(P)$. Let $x \in P$ be a point such that $T(x) = w$. By noting that $\lambda \neq 0$ and that T is linear, we have

$$\begin{aligned} w &= \lambda y + (1-\lambda)T(v) \\ y &= \frac{1}{\lambda}(w - (1-\lambda)T(v)) \\ y &= \frac{1}{\lambda}(T(x) - (1-\lambda)T(v)) \\ y &= T(\frac{1}{\lambda}(x - (1-\lambda)v)) \end{aligned}$$

Let $z = \frac{1}{\lambda}(x - (1-\lambda)v)$ be the point such that $T(z) = y$. After rearranging, we see that $x = \lambda z + (1-\lambda)v$. Since $x \in P$, we have that $z \in \operatorname{cone}(P, v)$, which means that $y = T(z) = T(\operatorname{cone}(P, v))$. Therefore, $T(\operatorname{cone}(P, v)) \supseteq \operatorname{cone}(T(P), T(v))$. □

Lemma 4.5.4. *Let $P_1, P_2 \subset \mathbb{R}^d$ be polyhedra. For any $v_1 \in P_1$ and $v_2 \in P_2$,*

$$\operatorname{cone}(P_1 \times P_2, (v_1, v_2)) = \operatorname{cone}(P_1, v_1) \times \operatorname{cone}(P_2, v_2).$$

Proof. Let the P_1 and P_2 be defined by a set of linear inequalities (facet half-spaces) $\langle c_i, x\rangle \leq \alpha_i$ and $\langle d_j, y\rangle \leq \beta_j$ for $i \in [m], j \in [n]$. Therefore, if $(x, y) \in P_1 \times P_2$, then (x, y) would need to satisfy

$$\langle (c_i, d_j), (x, y)\rangle \leq (\alpha_i, \beta_j) \ \ \forall \ \ i \in [m], \ j \in [n] \tag{4.27}$$

Therefore, $P_1 \times P_2$ is contained in the polyhedron P, where P is defined by the set of linear inequalities described in (4.27). However, for any $(x, y) \in P$, x and y satisfies all inequalities defining P_1 and P_2 respectively. Therefore, P is contained in $P_1 \times P_2$. Therefore, we see that $P_1 \times P_2 = P$ is a polyhedron.

To prove the identity for the support cones, consider a point $(u_1, u_2) \in \mathbb{R}^{2d}$.

$$\begin{aligned}
&(u_1, u_2) \in \operatorname{cone}(P, (v_1, v_2)) \\
\iff &\exists\ 0 < \lambda < 1 : \lambda(u_1, u_2) + (1-\lambda)(v_1, v_2) \in P \\
\iff &(\lambda u_1 + (1-\lambda)v_1, \lambda u_2 + (1-\lambda)v_2) \in P \\
\iff &\lambda u_1 + (1-\lambda)v_1 \in P_1 \text{ and } \lambda u_2 + (1-\lambda)v_2 \in P_2 \\
\iff &u_1 \in \operatorname{cone}(P_1, v_1) \text{ and } u_2 \in \operatorname{cone}(P_2, v_2) \\
\iff &(u_1, u_2) \in \operatorname{cone}(P_1, v_1) \times \operatorname{cone}(P_2, v_2).
\end{aligned}$$

□

4.5.2 Brion's Theorem

Let $\mathscr{P}_0(\mathbb{Q}^n)$ denote the subspace generated by indicator functions of rational polyhedra in $\mathbb{R}^n$ that contain straight lines. Consider the lattice d-simplex

$$\Delta := \operatorname{conv}(e_i : i = 1, \ldots, d+1) \subset \mathbb{R}^{d+1}.$$

Let us define a few hyperplanes and half-spaces:

- $H := \{x : \langle \mathbf{1}, x\rangle = 1\} = \operatorname{aff}(e_i : i = 1, \ldots, d+1)$
- $H_i^+ := \{x : \langle e_i, x\rangle \geq 0\}$
- $\overline{H_i^+} := H_i^+ \cap H$

By identifying $\mathbb{R}^d$ with H, $\overline{H_i^+}$ can be thought of as a half-space in $\mathbb{R}^d$. Let H_i and $\overline{H_i}$ be hyperplanes associated with H_i^+ and $\overline{H_i^+}$ respectively. Note that

$$\begin{aligned}
\Delta &= H \cap H_1^+ \cap \cdots \cap H_{d+1}^+ \\
&= (H \cap H_1^+) \cap \cdots \cap (H \cap H_{d+1}^+) \\
&= \bigcap_{i=1}^{d+1} \overline{H_i^+}
\end{aligned}$$

By noting that $\overline{H_j}$ is the only facet hyperplane that does not contain vertex e_j, we can apply Corollary 4.5.2 to see that

$$\operatorname{cone}(\Delta, e_j) = \bigcap_{i \neq j} \overline{H_i^+}.$$

Lemma 4.5.5. *There exist rational polyhedra $Q_1, \ldots, Q_N \subseteq \mathbb{R}^d$ such that*

(a) *each polyhedron Q_k contains a straight line parallel to $e_i - e_j$ for some pair $1 \leq i < j \leq d+1$;*

(b) *we have*

$$[\Delta] = \sum_{i=1}^{d+1} [\operatorname{cone}(\Delta, e_i)] + \sum_{k=1}^{N} \gamma_k [Q_k] \textit{ for some } \gamma_k \in \{-1, 1\}.$$

In particular, modulo $\mathscr{P}_0(\mathbb{Q}^d)$, the indicator function of the standard simplex is the sum of the indicator functions of the support cones at its vertices.

Proof. Let $P_I := \bigcap_{i \in I} \overline{H_i^+}$. Using inclusion-exclusion, we have

$$[\mathbb{R}^d] = \left[\bigcup_{i=1}^{d+1} \overline{H_i^+} \right] = \sum_{\substack{I \subseteq [d+1] \\ I \neq \emptyset}} (-1)^{|I|-1} [P_I]. \tag{4.28}$$

Note that

- $I = \{1, \ldots, d+1\}$ implies $P_I = \Delta$
- $I = \{1, \ldots, d+1\} \backslash \{i\}$ implies $P_I = \operatorname{cone}(\Delta, i)$
- If there are two distinct i, j that are not in I, then P_I contains a straight a line in the direction of $e_i - e_j$.

To show the third point, suppose 1 and 2 are not in I. Then a point x in P_I has the restrictions $\sum_{i=1}^{d+1} x_i = 1$ and $x_j \geq 0, j \in I$. Then we see that $(a, 1-a, 0, \ldots, 0)$ is in P_I for any $a \in \mathbb{R}$. Rearranging (4.28) completes the proof. $\square$

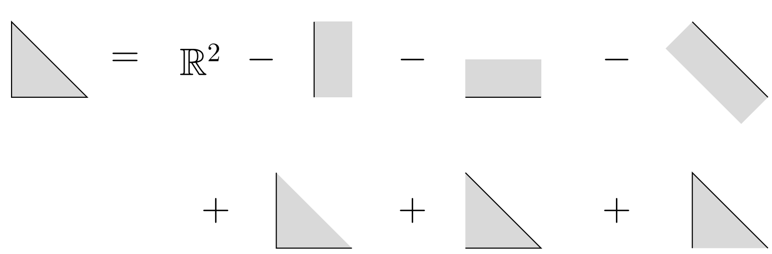

Figure 4.3. Viewing the above pictures as indicator functions, we can express the standard 2-simplex in terms of the intersection of its facet half-spaces.

Now we can generalize from simplices to polytopes by mapping each vertex of a simplex to a distinct vertex of the polytope.

Lemma 4.5.6. *Let $P \subset \mathbb{R}^d$ be a polytope (resp. rational polytope) with vertices $v_1, \ldots, v_n$. Then we can write*

$$[P] = g + \sum_{i=1}^{n} [\text{cone}(P, v_i)]$$

for some function $g \in \mathscr{P}_0(\mathbb{R}^d)$ (resp $\mathscr{P}_0(\mathbb{Q}^d)$).

Proof. Let $\Delta \subset \mathbb{R}^n$ be the standard $(n-1)$-dimensional simplex, and let $T : \mathbb{R}^n \to \mathbb{R}^d$ be a linear transformation such that $T(e_i) = v_i$. Then $T(\Delta) = P$. By Theorem 4.2.3, there exists a linear transformation $\mathcal{T}$ such that $\mathcal{T}([\Delta]) = [T(\Delta)]$. Applying in combination with Lemma 4.5.5, we have

$$\begin{aligned}
[P] &= [T(\Delta)] \\
&= \mathcal{T}([\Delta]) \\
&= \mathcal{T}\left(\sum_{i=1}^{n} [\text{cone}(\Delta, e_i)] + \sum_{k=1}^{N} \gamma_k [Q_k]\right) \\
&= \sum_{i=1}^{n} \mathcal{T}[\text{cone}(\Delta, e_i)] + \sum_{k=1}^{N} \gamma_k \mathcal{T}([Q_k]) \\
&= \sum_{i=1}^{n} [T(\text{cone}(\Delta, e_i))] + \sum_{k=1}^{N} \gamma_k [T(Q_k)] \\
&= \sum_{i=1}^{n} [\text{cone}(P, v_i)] + \sum_{k=1}^{N} \gamma_k [T(Q_k)]
\end{aligned}$$

where each Q_k is a (rational) polyhedron that contains a straight line. Since T is linear, $T(Q_k)$ is also a (rational) polyhedron. Linear functions map straight lines to straight lines, so each $T(Q_k)$ has a straight line. □

Finally, we can generalize from polytopes to polyhedra.

Theorem 4.5.7. *Let $P \subset \mathbb{R}^d$ be a polyhedron (resp. rational polyhedron). Then*

$$[P] = g + \sum_{v \text{ vertex of } P} [\text{cone}(P, v)]$$

for some function $g \in \mathscr{P}_0(\mathbb{R}^d)$ (resp. $\mathscr{P}_0(\mathbb{Q}^d)$) .

In other words, if we mod out indicator functions of polyhedra that contain straight lines, then the indicator function of polyhedron P is sum of the indicator functions of the support cones of P at its vertices.

Proof. Let us prove the rational version. Suppose

$$P := \{x : \langle c_i, x\rangle \le \beta_i, i = 1, \dots, n\}$$

for $c_i \in \mathbb{Z}^d$ and $\beta_i \in \mathbb{Z}$. If P does not contain any vertices, then by Theorem 1.1.6, P is either empty or contains a straight line. In either case, the result is immediate. Therefore, suppose that P has a vertex set V. Since P does not contain straight lines, we can write P as

$$P = Q + K$$

where K is the recession cone of P and Q is the convex hull of the vertices of P. Since P is rational, the vertices of P are rational, and therefore Q is a rational polytope and K is a rational cone. Consider the rational polyhedron $Q \times K$. By Lemma 4.5.6, there exist polyhedra $Q_i, i \in I$ containing straight lines and $\gamma \in \{-1, 1\}$ such that

$$\begin{aligned}
[Q] &= \sum_{v\in V}[\mathrm{cone}(Q, v)] + \sum_{i\in I}\gamma[Q_i] \\
[Q \times K] &= \sum_{v\in V}[\mathrm{cone}(Q, v) \times K] + \sum_{i\in I}\gamma[Q_i \times K] \\
&= \sum_{v\in V}[\mathrm{cone}(Q \times K, \bar{v})] + \sum_{i\in I}\gamma[Q_i \times K]
\end{aligned}$$

where $\bar{v} = (v, 0)$. Let $T : \mathbb{R}^{2d} \to \mathbb{R}^d$ be the linear transformation $T(x, y) = x + y$. By Lemma 4.5.3, we get

$$\begin{aligned}
[T(Q \times K)] &= \sum_{v\in V}[T(\mathrm{cone}(Q \times K, \bar{v}))] + \sum_{i\in I}\gamma_i[T(Q_i \times K)] \\
[Q + K] &= \sum_{v\in V}[\mathrm{cone}(T(Q \times K), T(\bar{v}))] + \sum_{i\in I}\gamma_i[Q_i + K] \\
[P] &= \sum_{v\in V}[\mathrm{cone}(Q + K, v)] + \sum_{i\in I}\gamma_i[Q_i + K].
\end{aligned}$$

To complete the proof, note that $Q_i + K$ is a rational polyhedron containing straight lines. $\square$

Combining the above theorem with the Hilbert valuation, we get Brion's Theorem.

Corollary 4.5.8 (Brion's Theorem)**.** *For every rational polyhedron $P \subset \mathbb{R}^d$,*

$$\mathcal{F}[P] = \sum_{v \text{ vertex of } P} \mathcal{F}[\mathrm{cone}(P, v)],$$

where $\mathcal{F} : \mathscr{P}(\mathbb{Q}^d) \to \mathbb{C}(x_1, \dots, x_d)$ is Hilbert valuation.

4.5.3 Proving Ehrhart's Theorem via Brion's Theorem

We now see another way to prove Ehrhart's Theorem. Let $P \subset \mathbb{R}^d$ be a lattice d-polytope with vertices $v_1, \dots, v_n$. Also, let

$$K_i = \mathrm{cone}(P, v_i) - v_i$$

be the support cone of P at v_i, translated so that the vertex is the origin. Dilating coordinates by a positive factor t is a linear function; denoting this dilation as a linear transformation (as a matrix) T, we can explicitly state that $T = tI$, where I is the identity matrix. Since the vertex of K_i is the origin, K_i is invariant under T; $T(K_i) = K_i$. Recalling Lemma 4.5.3,

$$\begin{aligned} T(K_i) &= T(\text{cone}(P, v_i) - v_i) \\ K_i &= T(\text{cone}(P, v_i)) - T(v_i) \\ &= \text{cone}(T(P), T(v_i)) - T(v_i) \\ &= \text{cone}(tP, tv_i) - tv_i \\ \text{cone}(tP, tv_i) &= K_i + tv_i. \end{aligned}$$

Applying Brion's Theorem yields

$$\sum_{\mathbf{p} \in tP \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{p}} = \mathcal{F}[tP] \tag{4.29}$$

$$= \mathcal{F}\left(\sum_{i \in [n]} [\text{cone}(tP, tv_i)] \right) \tag{4.30}$$

$$= \sum_{i \in [n]} \mathcal{F}[\text{cone}(tP, tv_i)] \tag{4.31}$$

$$= \sum_{i \in [n]} \mathcal{F}[K_i + tv_i] \tag{4.32}$$

$$= \sum_{i \in [n]} \mathbf{z}^{tv_i} \mathcal{F}[K_i] \tag{4.33}$$

$$= \sum_{i \in [N]} \frac{\mathbf{z}^{t\mathbf{v}_i} p_i(\mathbf{z})}{(1 - \mathbf{z}^{\mathbf{u}_{i1}}) \dots (1 - \mathbf{z}^{\mathbf{u}_{id}})} \tag{4.34}$$

where $u_{i1}, \dots, u_{id}$ are integer vectors and $p_i(\mathbf{z})$ is a Laurent polynomial on $\mathbf{z} = (z_1, \dots, z_d)$. Note that the indices for the summations in (4.33) and (4.34) may not be equal, since the support cones K_i may not be simplicial cones, and hence we would have to subdivide the K_i into simplicial cones. We would like $\mathbf{z}$ to approach $(1, \dots, 1)$. Choose a $c \in \mathbb{R}^d$ such that $\langle c, u_{ij} \rangle \neq 0$ for all i and j. Now we set $\mathbf{z} = \mathbf{e}^{\tau c}$ for $\tau \in \mathbb{R}^d$, and let $\tau \to 0$. By doing this, we have

$$\sum_{\mathbf{p} \in tP \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{p}} = \sum_{p \in tP \cap \mathbb{Z}^d} \exp(\tau \langle c, p \rangle).$$

Expanding this analytic function around $\tau = 0$, we see that the constant term is the number of lattice points in tP. By performing the same substitution on each fraction of (4.34), we get

$$\frac{e^{\tau \langle c, tv_i \rangle} \; p_i(\mathbf{e}^{\tau c})}{(1 - e^{\tau \langle c, u_{i1} \rangle}) \dots (1 - e^{\tau \langle c, u_{id} \rangle})} = \tau^{-d} e^{\tau \langle c, tv_i \rangle} \; p_i(\mathbf{e}^{\tau c}) \prod_{j=1}^{d} \frac{\tau}{1 - e^{\tau \langle c, u_{ij} \rangle}}. \tag{4.35}$$

Note that each

$$\frac{\tau}{1 - e^{\tau \langle c, u_{ij} \rangle}}$$

does not depend on t and converges to the function

$$\frac{\tau}{1-(1+\tau\langle c,u_{ij}\rangle+\frac{\tau^2\langle c,u_{ij}\rangle^2}{2!}+\dots)}=\frac{1}{-\langle c,u_{ij}\rangle-\frac{\tau\langle c,u_{ij}\rangle^2}{2!}-\dots}.$$

By recalling that $p_i(\mathbf{e}^{\tau c})$ is a Laurent polynomial, we have the following power series representation:

$$p_i(\mathbf{e}^{\tau c})\prod_{j=1}^{d}\frac{\tau}{1-e^{\tau\langle c,u_{ij}\rangle}}=\sum_{\ell\geq 0}\alpha_{i\ell}\tau^{\ell}$$

for some $\alpha_{i\ell}\in\mathbb{R}$. Also,

$$\tau^{-d}e^{\tau\langle c,tv_i\rangle}=\tau^{-d}\sum_{\ell\geq 0}t^{\ell}\frac{\langle c,v_i\rangle^{\ell}}{\ell!}\tau^{\ell}.$$

Thus, the constant term of (4.35) is

$$\sum_{\ell=0}^{d}t^{\ell}\frac{\langle c,v_i\rangle^{\ell}}{\ell!}\alpha_{i,d-\ell}.$$

Comparing constant terms, we conclude that

$$|tP\cap\mathbb{Z}^d|=\sum_{i=1}^{N}\sum_{\ell=0}^{d}t^{\ell}\frac{\langle c,v_i\rangle^{\ell}}{\ell!}\alpha_{i,d-1}$$

is a polynomial in t with degree at most d. Note that this proof approach provides an algorithm for finding $h^*_P(z)$. Barvinok proved that there is an algorithm that can compute the Ehrhart polynomial of a lattice polytope P in time polynomial in the size of P, formulated as a system of linearly inequalities. We shall refer to this algorithm as Barvinok's Algorithm.

Chapter 5

Unimodular Decomposition of Lattice Polytopes

As we have seen in Chapter 2, some polytopes do not admit unimodular triangulations. However, if we view polytopes as their indicator functions, we are allowed the freedom to "subtract off" polytopes. A consequence is that polytopes can be decomposed with unimodular simplices. In proving this consequence, we will detail an algorithm to unimodularly decompose polytopes. We detail a number of applications that stem from this decomposition. Among them, we will show a few nice results for valuations that are constant over integer translations and unimodular maps of polytope. Among these results is a theorem of Betke and Kneser. The proof of the Betke-Kneser theorem can be found in Gruber [8]. We will give a new streamlined proof of this theorem and the theorem on the existence of unimodular decompositions.

5.1 Unimodular Decompositions

Let $\mathscr{P}(\mathbb{Z}^d)$ be the algebra of lattice polytopes in $\mathbb{R}^d$. Let $\mathcal{U}$ be the sub-algebra of $\mathscr{P}(\mathbb{Z}^d)$ generated by

$$[P] - [UP + u]$$

where $u \in \mathbb{Z}^d$, $U \in GL_d(\mathbb{Z})$, and P is a lattice polytope in $\mathbb{R}^d$. A lattice polytope P has a *unimodular decomposition* if $[P]$ can be written as a linear combination of $[\Delta_0], \ldots, [\Delta_d]$ in $\mathscr{P}(\mathbb{Z}^d)/\mathcal{U}$. In other words, we can write $[P]$ as a finite linear combination

$$[P] = \sum_{i=1}^{m} \alpha_i [U_i S_i + u_i]$$

where $\alpha_i \in \mathbb{R}$, $U_i \in GL_d(\mathbb{Z})$, $u_i \in \mathbb{Z}^d$, and each S_i is some standard simplex. We will prove that any lattice polytope has a unimodular decomposition.

Lemma 5.1.1. *Let $0, v_1, \ldots, v_d \in \mathbb{Z}^d$ form a non-unimodular lattice d-simplex. There exists a lattice point p contained in the parallelepiped generated by $v_1, \ldots, v_d$ such that p is not a vertex of the parallelepiped.*

Proof. Let V be the $d \times d$ matrix with v_i as its i^{th} column. The parallelepiped generated by $v_1, \ldots, v_d$ is

$$\Pi_V := \left\{ \sum_{i=1}^{d} \lambda_i v_i : 0 \leq \lambda_i \leq 1 \right\}.$$

Since the vertices of Π_V are $0, v_1, \ldots, v_d$, and $\sum_{i=1}^{d} v_i$, it suffices to prove that there is a lattice point p contained in $\overline{\Pi}_V \backslash \{0\}$. Consider V^{-1}. Since $|\det(V)| = \mathrm{Vol}(S) > 1$, we see that $|\det(V^{-1})| = \frac{1}{|\det(V)|}$ is not an integer. Since integer matrices have integer determinants, V^{-1} is not an integer matrix. Let $w = (w_1, \ldots, w_d)$ be some k^{th} column of V^{-1} such that w has a non-integer entry.

$$\begin{aligned} V^{-1} e_k &= w \\ e_k &= Vw \\ &= w_1 v_1 + \cdots + w_d v_d \\ &= \sum_{i=1}^{d} (\lfloor w_i \rfloor + \{w_i\}) v_i \\ e_k - \lfloor w_1 \rfloor v_1 - \cdots - \lfloor w_d \rfloor v_d &= \sum_{i=1}^{d} \{w_i\} v_i \end{aligned}$$

Let $p := e_k - \lfloor w_1 \rfloor v_1 - \cdots - \lfloor w_d \rfloor v_d$. Since all v_i are integer vectors, it is clear that p is a lattice point. Also, since $0 \leq \{w_i\} < 1$, it is easy to see from the right side of the above equation that $p \in \overline{\Pi}_V$. By our choice, w has a non-integer entry, so at least one $\{w_i\}$ is non-zero. Recalling that $v_1, \ldots, v_d$ are vertices of a simplex, and therefore linearly independent, we see that $p \neq 0$ and so p is a lattice point in $\overline{\Pi}_V \backslash \{0\}$. □

Let $S \subset \mathbb{R}^d$ be a non-unimodular lattice d-simplex. We would like to find a lattice point p such that

$$\mathrm{Vol}(\mathrm{conv}(F, p)) < \mathrm{Vol}(S) \tag{5.1}$$

for all facets $F \leq S$. We will refer to this condition as the *decreasing volume condition* for S.

We can translate S by $u \in \mathbb{Z}^d$ such that the origin is a vertex of the translated simplex. If we can find a p that satisfies the decreasing volume condition for $S + u$, then we know that $p - u$ would satisfy the decreasing volume condition for S. In other words, we are translating S so that a vertex is at the origin, finding p and then translating p and S back.

Therefore we can assume without loss of generality that S has vertices $v_1, \ldots, v_d$ and $v_0 = 0$. Let V be the $d \times d$ matrix with v_i as its i^{th} column. We denote facet F_i as $\mathrm{conv}(v_j : j \neq i)$. The volume of $\mathrm{conv}(F_i, x)$ can be written as

$$\mathrm{Vol}(\mathrm{conv}(F_i, x)) = h_i \, \mathrm{Vol}(F_i)$$

where h_i is the "height" of x from F_i, or the lattice distance x is away from F_i. In other words, h_i is the lattice length of x viewed in $\mathbb{Z}^d / (\mathrm{aff}(F_i) \cap \mathbb{Z}^{d-1})$.

We want

$$\begin{aligned}\operatorname{Vol}(\operatorname{conv}(F_i,p)) &< \operatorname{Vol}(S) = \operatorname{Vol}(\operatorname{conv}(F_i,v_i))\\ h_p \operatorname{Vol}(F_i) &< h_{v_i} \operatorname{Vol}(F_i)\\ h_p &< h_{v_i}\end{aligned}$$

for all i. We see that p cannot be past a certain distance (h_{v_i}) away from either side of hyperplane F_i. Note that if $p \in (\operatorname{aff}(F_i) + (z - v_i))$ for any arbitrary $z \in \operatorname{aff}(F_i)$, then $h_i = h_{v_i}$.

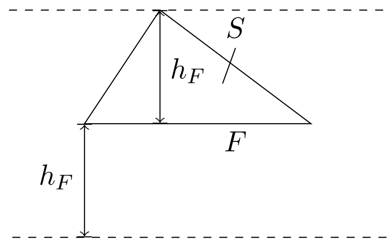

Figure 5.1. For any point p in between the dashed lines, we have $\operatorname{Vol}(S) \geq \operatorname{Vol}(\operatorname{conv}(F,p))$, with equality if and only if p is on one of the dashed lines.

For $i \in [d]$, since $0 \in \operatorname{aff}(F_i)$, we take $z = 0$ to get

$$p \in \Omega_i := \{x : x \in \operatorname{aff}(F_i) + \lambda_i v_i : -1 < \lambda_i < 1\}. \tag{5.2}$$

Since $0 \in \operatorname{aff}(F_i)$ for $i \in [d]$, we can actually see that

$$\begin{aligned}\operatorname{aff}(F_i) &= \operatorname{span}\{v_j : j \neq i\}\\ &= \left\{\sum_{j=1}^{d} \lambda_j v_j : \lambda_i = 0\right\}\\ \Omega_i &= \left\{\sum_{j=1}^{d} \lambda_j v_j : -1 < \lambda_i < 1\right\}.\end{aligned}$$

Therefore, intersecting for all $i \in [d]$, we have

$$\bigcap_{i=1}^{d} \Omega_i = \left\{\sum_{i=1}^{d} \lambda_i v_i : -1 < \lambda_i < 1\right\}.$$

Also, note that $v_i \in \operatorname{aff}(F_0)$ for any $i \in [d]$, so we can take $z = \sum_{i=1}^{d} \alpha_i v_i$ where $\sum_{i=1}^{d} \alpha_i = 1$ (and recalling that $v_0 = 0$) to get

$$p \in \Omega_0 := \{\operatorname{aff}(F_0) + \lambda_0 z : -1 < \lambda_0 < 1\}. \tag{5.3}$$

Any point in Ω_0 can be written as

$$\begin{aligned}\lambda_0 z + \sum_{i=1}^{d} \beta_i v_i &= \lambda_0 \left(\sum_{i=1}^{d} \alpha_i v_i\right) + \sum_{i=1}^{d} \beta_i v_i\\ &= \sum_{i=1}^{d} (\lambda_0 \alpha_i + \beta_i) v_i\end{aligned}$$

such that $\sum_{i=1}^d \beta_i = \sum_{i=1}^d \alpha_i = 1$ and $-1 < \lambda_0 < 1$. By noting that

$$\begin{aligned}\sum_{i=1}^d (\lambda_0\alpha_i + \beta_i) &= \lambda_0 \sum_{i=1}^d \alpha_i + \sum_{i=1}^d \beta_i \\ &= \lambda_0 + 1,\end{aligned}$$

we can write any point in Ω_0 as $\sum_{i=1}^d \gamma_i v_i$, where $0 < \sum_{i=1}^d \gamma_i < 2$, and so

$$\Omega_0 = \left\{ \sum_{i=1}^d \gamma_i v_i : 0 < \sum_{i=1}^d \gamma_i < 2 \right\}.$$

Intersecting all the Ω_i's, we have

$$\bigcap_{i=0}^d \Omega_i = \left\{ \sum_{i=1}^d \lambda_i v_i : -1 < \lambda_i < 1, 0 < \sum_{i=1}^d \lambda_i < 2 \right\} =: \Omega_V.$$

By Lemma 5.1.1, there is a non-zero lattice point $p = \sum_{i=1}^d \lambda_i v_i$ where $0 \leq \lambda_i < 1$ in the top-open parallelepiped $\overline{\Pi}_V$. Since $p \neq 0$, we have $0 < \sum_{i=1}^d \lambda_i$. Without loss of generality, let $\lambda_1, \ldots, \lambda_k$ be non-zero and $\lambda_{k+1} = \cdots = \lambda_d = 0$ (if any). Let

$$m := \lfloor \sum_{i=1}^d \lambda_i \rfloor = \lfloor \sum_{i=1}^k \lambda_i \rfloor$$

and note that $k = \sum_{i=1}^k 1 > \sum_{i=1}^k \lambda_i \geq m$. If $m < 2$, then $p \in \Omega_V$, and therefore satisfies the decreasing volume condition. If $m \geq 2$, then we redefine

$$\begin{aligned}\lambda_1 &\to \lambda_1 - 1 \\ &\vdots \\ \lambda_{m-1} &\to \lambda_{m-1} - 1.\end{aligned}$$

Notice that these redefined values interpret to a translation of p:

$$p \to p - v_1 - \cdots - v_{m-1},$$

which is obviously still a lattice point. Finally, with these redefined values of λ_i, we have

$$\lfloor \sum_{i=1}^d \lambda_i \rfloor = m - (m-1) = 1.$$

Theorem 5.1.2. *If lattice simplex S is not unimodular, then there exists a lattice point p such that*

$$\mathrm{Vol}(\mathrm{conv}(F, p)) < \mathrm{Vol}(S)$$

for all facets $F \leq S$.

Theorem 5.1.3. *Any lattice polytope has a unimodular decomposition.*

Proof. Since every polytope admits a triangulation, we can write $[P]$ as a finite linear combination of simplices $[S]$. Therefore, it suffices to prove the theorem for simplices. We proceed by induction on d and on the volume of the simplex. Let S be a lattice simplex with vertices $v_1, \ldots, v_{d+1}$ and facets $F_i = \mathrm{conv}(v_j : j \neq i)$. If $\mathrm{Vol}(S) = 1$, then S is unimodular already, and so there is a unimodular map U that maps S to Δ_d, which implies that $[S] = [U^{-1}\Delta_d]$. If $d = 1$, then S is a closed line segment $[a, b]$ where $a < b$ are integers. Noting that Δ_1 is the line segment $[0, 1]$, we have $[S] = \sum_{i=a}^{b-1}[\Delta_1 + i]$.

Taking the inductive step, consider $d > 1$ and $\mathrm{Vol}(S) > 1$. We choose a lattice point p be such that $\mathrm{Vol}(\mathrm{conv}(F_i, p)) < \mathrm{Vol}(S)$ for all i. We then assign a height function $\omega : \{v_1, \ldots, v_{d+1}, p\} \to \mathbb{Z}$ such that

$$\omega(x) = \begin{cases} 1 & \text{if } x = p \\ 0 & o.w. \end{cases}$$

Let R be the convex hull of the v_i's and p, and let R' be the convex hull of the $(v_i, 0)$'s and $(p, 1)$. By defining $\mathrm{pr} : \mathbb{R}^{d+1} \to \mathbb{R}^d$ to be the projection $(x_1, \ldots, x_{d+1}) \mapsto (x_1, \ldots, x_d)$ back to the first d coordinates, we see that $\mathrm{pr}(R') = R$. Note that $F_0 := \mathrm{conv}\{(v_1, 0), \ldots, (v_{d+1}, 0)\}$ is a lower facet of R' and that $\mathrm{pr}(F_0) = S$. For every other facet F of R', we have $\mathrm{pr}(F) = \mathrm{conv}(F_i, p)$ for some i. The upper and lower faces of R' induce two triangulations of R. By inclusion-exclusion, we can write $[\mathrm{pr}(R')]$ as

$$[\mathrm{pr}(F_0)] + \sum_{\substack{F \text{ lower facet of } R' \\ F \neq F_0}} [\mathrm{pr}(F)] + A = \sum_{F \text{ upper facet of } R'} [\mathrm{pr}(F)] + B \tag{5.4}$$

where A and B are finite linear combinations of $[\mathrm{pr}(f)]$'s and each f is a lower dimensional (less than d) face of R'. By induction hypothesis, A and B can be written as finite linear combinations of $[U\Delta + u]$'s, where $U \in GL_d(\mathbb{Z})$, $u \in \mathbb{Z}^d$ and Δ is a standard simplex. Our choice of p is such that $\mathrm{Vol}(\mathrm{pr}(F)) < \mathrm{Vol}(S)$ for all facets $F \neq F_0$ of R'. By induction hypothesis, each $\mathrm{pr}(F)$ can be written as a finite linear combination of $[U\Delta + u]$'s, where $U \in GL_d(\mathbb{Z})$, $u \in \mathbb{Z}^d$ and Δ is a standard simplex. Isolating $[\mathrm{pr}(F_0)] = [S]$ in (5.4) completes our proof. $\square$

5.2 3-Dimensional Example of Unimodular Decomposition

Let us illustrate the above theorem on the simplex $\Delta \subset \mathbb{R}^3$, with vertices $v_0 = (0, 0, 0), v_1 = (1, 0, 0), v_2 = (0, 0, 1)$ and $v_3 = (1, 2, 1)$. Since Δ has a volume of 2, this decomposition is not trivial. We define the matrix V with columns comprised of the non-zero vectors of Δ:

$$V := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}.$$

To find a non-zero lattice point in the top-open parallelepiped $\overline{\Pi}_V$, we invert V to get

$$V^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

Focusing on the second column of V^{-1}, we have

$$
\begin{aligned}
\begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \\
\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 + \frac{1}{2} \\ -1 + \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \\
\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}
\end{aligned}
$$

and so $(1,1,1) =: p$ is a non-zero lattice point in $\overline{\Pi}_V$. For each facet F_i of Δ, we want $\mathrm{Vol}(\Delta) > \mathrm{Vol}(\mathrm{conv}(F_i, p))$. This happens if $p = \sum_{i=1}^{3} \lambda_i v_i$ such that $-1 < \lambda_i < 1$ and $0 < \sum_{i=1}^{3} \lambda_i < 2$. Note that $(1,1,1) = \frac{1}{2}v_1 + \frac{1}{2}v_2 + \frac{1}{2}v_3$, so p satisfies the decreasing volume condition. Indeed, one can check that the volumes of $\mathrm{Vol}(\mathrm{conv}(F_i, p))$ are all 1.

Let $R := \mathrm{conv}(\Delta, p)$. Assigning heights $\varphi(v) = 0$ for all vertices of Δ and $\varphi(p) = 1$, the lower facets of this lifted configuration correspond to Δ and $\mathrm{conv}(v_1, v_2, v_3, p)$, and the upper facets correspond to $\mathrm{conv}(0, v_2, v_3, p)$, $\mathrm{conv}(v_1, 0, v_3, p)$ and $\mathrm{conv}(v_1, v_2, 0, p)$. By inclusion-exclusion, the triangulation of R using lower faces is

$$[R] = [\Delta] + [\mathrm{conv}(v_1, v_2, v_3, p)] - [\mathrm{conv}(v_1, v_2, v_3)] \tag{5.5}$$

and the triangulation of R using upper faces is

$$
\begin{aligned}
[R] = {} & [\mathrm{conv}(0, v_2, v_3, p)] + [\mathrm{conv}(v_1, 0, v_3, p)] + [\mathrm{conv}(v_1, v_2, 0, p)] \\
& - [\mathrm{conv}(v_3, p, 0)] - [\mathrm{conv}(v_2, p, 0)] - [\mathrm{conv}(v_1, p, 0)] \\
& + [\mathrm{conv}(p, 0)].
\end{aligned} \tag{5.6}
$$

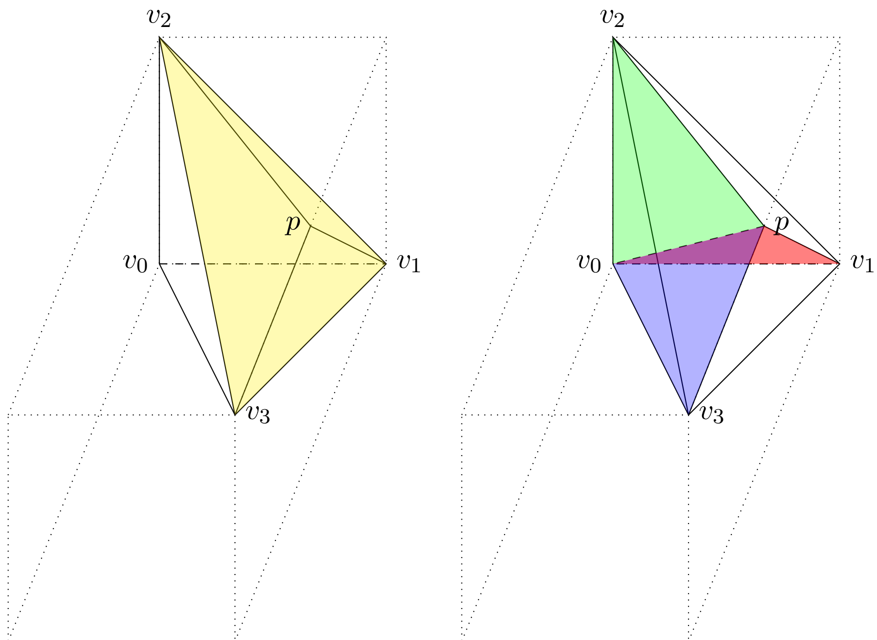

Figure 5.2. Two triangulations of $\operatorname{conv}(v_0, v_1, v_2, v_3, p)$.

We have already claimed that some of the above simplices are unimodular. Instead of explicitly checking the volumes of each simplex, we will verify their unimodularity by representing them as a unimodular transformation of a unimodular simplex, up to translation. We define three different dimensional simplices $\Delta_1 = \operatorname{conv}(0, e_1)$, $\Delta_2 = \operatorname{conv}(0, e_1, e_2)$ and $\Delta_3 = \operatorname{conv}(0, e_1, e_2, e_3)$. Note that they are all unimodular. Finally, we will represent each of the above simplices as a translation of a unimodular map of one of the $\Delta_1, \Delta_2, \Delta_3$. We use the fact that for any linear map U, vertices of a polytope P get mapped to vertices of UP. Therefore, it suffices to check that the vertices of Δ_1, Δ_2 and Δ_3 get mapped (up to translation) to the simplices that they are to represent.

$$\operatorname{conv}(v_1, v_2, v_3, p) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \Delta_3 + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =: S_1$$

$$\operatorname{conv}(v_1, v_2, v_3) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \Delta_2 + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =: S_2$$

$$\operatorname{conv}(0, v_2, v_3, p) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \Delta_3 =: S_3$$

$$\operatorname{conv}(v_1, 0, v_3, p) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \Delta_3 =: S_4$$

$$\operatorname{conv}(v_1, v_2, 0, p) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \Delta_3 =: S_5$$

$$\operatorname{conv}(v_1, p, 0) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \Delta_2 =: S_6$$

$$\operatorname{conv}(v_2, p, 0) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \Delta_2 =: S_7$$

$$\operatorname{conv}(v_3, p, 0) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \Delta_2 =: S_8$$

$$\operatorname{conv}(p, 0) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \Delta_1 =: S_9$$

Therefore, by equating (5.5) with (5.6) and isolating $[\Delta]$, we have

$$[\Delta] = -[S_1] + [S_2] + [S_3] + [S_4] + [S_5] - [S_6] - [S_7] - [S_8] + [S_9].$$

5.2.1 Runtime of Unimodular Decomposition

Note that the above method of unimodular decomposition recurses on the volume of each simplex $\operatorname{conv}(F, p)$. The algorithm above for finding a point suitable p is in fact not very efficient. Let us illustrate the run-time on the simplex $\Delta \subset \mathbb{R}^3$, with vertices $v_0 = (0,0,0), v_1 = (1,0,0), v_2 = (0,0,1)$ and $v_3 = (a,b,1)$, where $0 < a < b$ and a and b are co-prime. As before, we define

$$V := \begin{pmatrix} 1 & 0 & a \\ 0 & 0 & b \\ 0 & 1 & 1 \end{pmatrix}$$

Again, we invert V to get

$$V^{-1} = \begin{pmatrix} 1 & -a/b & 0 \\ 0 & -1/b & 1 \\ 0 & 1/b & 0 \end{pmatrix}.$$

Focusing on the second column, we would always end up with

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a \\ 0 & 0 & b \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} (b-a)/b \\ (b-1)/b \\ 1/b \end{pmatrix}$$

and so $(1,1,1) =: p$ is always the point that we would compute. The simplices $\operatorname{conv}(F, p)$ for each facet $F \leq S$ would then have volumes $b-1$, and two simplices of volume $b-a$. We now recurse on the three non-unimodular simplices.

Pick one of the three simplices and call it Δ_W. By translating the simplex, let us assume that Δ_W has vertices $0, w_1, w_2$ and w_3. Let W be the matrix with columns w_1, w_2 and w_3, and note that Δ_W has volume $|\det(W)|$. Now, we can use a change of basis map (by way of multiplying by a unimodular matrix) so that the inputs w_1, w_2 and w_3 are in a suitable form. Let U be the unimodular map; we want

$$UW = \begin{pmatrix} 1 & 0 & r \\ 0 & 0 & s \\ 0 & 1 & 1 \end{pmatrix} =: X$$
$$U = XW^{-1},$$

where we choose co-prime r and s such that $0 < r < s$. We can then run our algorithm on the columns of X. Note that all we did was apply a unimodular map, so volume is preserved: $|\det(W)| = |\det(X)| = s$. The algorithm would then produce 4 more simplices with volumes $1, s-1, s-r$ and $s-r$. For all iterations, let us set $r = s-1$. In doing so, each iteration would yield three unimodular simplices plus a simplex with volume one less than that of the non-unimodular simplex of the previous iteration. Running the algorithm on Δ_W would then take $|\det(W)| - 1$ steps.

Focusing back on Δ, running our algorithm takes

$$1 + ((a-1)-1) + ((b-a)-1) + ((b-a)-1) = 3a - 2b - 3$$

iterations on the top-dimensional simplices alone. From this example, we see that it is beneficial to choose a point p to minimize the maximum volume among $\operatorname{conv}(F, p)$. This optimization problem takes the form

$$\begin{array}{lll} \min & t & \\ & \operatorname{Vol}(\operatorname{conv}(F,p)) \quad \leq t & \text{facets } F \leq S \\ & t \quad \in \mathbb{R} & \\ & p \quad \in \mathbb{Z}^d & \end{array}$$

The volume of a simplex is the the absolute value of the determinant of the matrix with column set $\{(v,1) : v \text{ a vertex of } S\}$. Therefore, for a simplex S with $v_0, \ldots, v_d$, we define

$$V := \begin{pmatrix} v_0 & \ldots & v_d \\ 1 & \ldots & 1 \end{pmatrix}$$

Letting c_i be the i^{th} column of the co-factor matrix $\operatorname{co}(V)$, we can re-formulate the facet constraints by linear inequalities:

$$\langle c_i, (p,1) \rangle \leq t$$
$$\langle c_i, (p,1) \rangle \geq -t$$

for $i = 0, \ldots, d$ and see that this formulation is a mixed integer linear program.

5.3 Applications of Unimodular Decomposition

5.3.1 Equivalence of Valuations

A linear map F is *translation-invariant* if for any $u \in \mathbb{Z}^d$, $F(u + P) = F(P)$. F is $GL_d(\mathbb{Z})$-*invariant* if for any unimodular map $U \in GL_d(\mathbb{Z})$, $F(UP) = F(P)$. A valuation that demonstrates both of these invariances is called *integer unimodular invariant*.

Lemma 5.3.1. *$F_t : [P] \mapsto L_P(t)$ is a valuation.*

Proof. For $t = 0$, we have $tP = \{0\}$ and so $F_t(P_i) = 1$ for any non-empty P_i; in other words, F_t is the Euler valuation, and we've already shown that it is a valuation. Again, it suffices to prove that this map is independent of how it is expressed; in other words, we need to prove

$$\sum_{i=1}^{n} \alpha_i [P_i] = 0 \implies \sum_{i=1}^{n} \alpha_i F_t(P_i) = 0,$$

so let us assume that $\sum_{i=1}^{n} \alpha_i [P_i] = 0$. Now, consider $t \neq 0$. Noting $x \in tP \iff \frac{x}{t} \in P$,

$$\begin{aligned}
0 = \sum_{i=1}^{n} \alpha_i [P_i] \left(\frac{x}{t}\right) &= \sum_{i=1}^{n} \alpha_i [tP_i](x) \\
0 = \sum_{x \in \mathbb{Z}^d} \sum_{i=1}^{n} \alpha_i [P_i] \left(\frac{x}{t}\right) &= \sum_{x \in \mathbb{Z}^d} \sum_{i=1}^{n} \alpha_i [tP_i](x) \\
&= \sum_{i=1}^{n} \alpha_i \sum_{x \in \mathbb{Z}^d} [tP_i](x) \\
&= \sum_{i=1}^{n} \alpha_i L_{P_i}(t) \\
0 &= \sum_{i=1}^{n} \alpha_i F_t(P_i).
\end{aligned}$$

□

Remark 5.3.2. This proof does not use any property of closed polytopes. F_t could be extended to indicator functions of any set in $\mathbb{R}^d$.

Lemma 5.3.3. *F_t is integer unimodular invariant.*

Proof. Let us prove the two properties separately.

(a) Indeed, for any $u \in \mathbb{Z}^d$ and $t \in \mathbb{Z}$, we have a bijection $T_{u,t} : \mathbb{Z}^d \to \mathbb{Z}^d$ where $x \mapsto tu + x$, and so the number of lattices points in tP is the same as the number of lattice points in $t(u + P) = tu + tP$.

(b) Note that for any $U \in GL_d(\mathbb{Z})$, U^{-1} exists, and both U and U^{-1} are integer matrices. The columns $u_1, \ldots, u_d$ of U^{-1} are linearly independent; define $B := \{u_1, \ldots, u_d\}$. Let $\mathcal{L}_B$, be the lattice generated by the vectors in B. Let $\mathcal{L}_I$ be the standard lattice, generated by $e_1, \ldots, e_d$.

We will show that U is a bijection between $\mathcal{L}_I$ and $\mathcal{L}_B$. For any $x \in \mathbb{Z}^d$, note that $x = U^{-1}(Ax)$, and so we can interpret Ux as x represented in the B basis. Since U is integer and x is integer, x can be represented as a lattice point in $\mathcal{L}_B$. Conversely, for any $x \in \mathcal{L}_B$, we can represent x in the standard basis by $U^{-1}x$. Since U^{-1} is integer and x is integer, x can be represented as a lattice point in $\mathcal{L}_I$.

Therefore, the number of lattices points in $t(UP)$ is equal to the number of lattice points of tP. □

Since $F_t(P)$ is a polynomial, and polynomials are equal if their coefficients are equal, Lemmas 5.3.1, and 5.3.3 imply the following.

Corollary 5.3.4. *$L_n : [P] \mapsto [t^n]L_P(t)$ is an integer unimodular invariant valuation.*

Suppose we have a valuation $F : \mathscr{P}(\mathbb{Z}^d) \to K$ that is integer unimodular invariant. Recall that $\mathcal{U}$ is defined to be the the sub-algebra generated by $[P] - [uP]$. For any element in $\mathcal{U}$, we have

$$F([P] - [UP + u]) = F(P) - F(UP + u) = F(P) - F(P) = 0.$$

For any lattice polytope P, we can consider $P+\mathcal{U}$; we see that the valuation of any $P+h \in P+\mathcal{U}$ is

$$F(P + h) = F(P) + F(h) = F(P).$$

Since F is constant over all elements in a coset, we could view F as a valuation from $\mathscr{P}(\mathbb{Z}^d)/\mathcal{U}$ to K.

Theorem 5.3.5. *Let $F, G : \mathscr{P}(\mathbb{Z}^d) \to K$ be two valuations that are integer unimodular invariant. If $F(\Delta_i) = G(\Delta_i)$ for all standard simplices Δ_i, then $F(P) = G(P)$ for all lattice polytopes in $\mathbb{R}^d$.*

Proof. By an application of Theorem 5.1.3, $[P]$ can be written as

$$[P] = \sum_{i=1}^{m} \alpha_i [U_i \Delta_i + u_i]$$

for some finite m, where $\alpha_i \in \mathbb{R}$, $U_i \in GL_d(\mathbb{Z})$, $u_i \mathbb{Z}^d$, and each S_i is some standard simplex. Then

$$\begin{aligned} F(P) &= \sum_{i=1}^{m} \alpha_i F(U_i \Delta_i + u_i) \\ &= \sum_{i=1}^{m} \alpha_i F(\Delta_i) \\ G(P) &= \sum_{i=1}^{m} \alpha_i G(U_i \Delta_i + u_i) \\ &= \sum_{i=1}^{m} \alpha_i G(\Delta_i). \end{aligned}$$

Since $F(\Delta_i) = G(\Delta_i)$, we have $F(P) = G(P)$. □

Let us write the unimodular decomposition of a lattice d-polytope $P \subset \mathbb{R}^d$ to keep track of the dimension of the simplices used. Let n_k be the number of k-dimensional simplices used in a unimodular decomposition of P. Then our decomposition has the form

$$\begin{aligned} [P] &= \sum_{i=1}^{m} \gamma_i [U_i S_i + u_i] \\ &= \sum_{k=0}^{d} \sum_{i=1}^{n_k} \alpha_{ki} [U_{ki} \Delta_k + u_{ki}] \end{aligned}$$

where $\gamma_i, \alpha_{ki} \in \{-1, 1\}$, all U_i and U_{ki} are unimodular maps, all u_i and u_{ki} are integral vectors, and Δ_k is a unimodular k-simplex. In $\mathscr{P}(\mathbb{Z}^d)/\mathcal{U}$, we then have the formulation

$$[P] = \sum_{k=0}^{d} \alpha_k [\Delta_k]$$

where $\alpha_k = \sum_{i=1}^{n_k} \alpha_{ki}$. Note that for any integer unimodular invariant valuation F,

$$F(P) = \alpha_0 F(\Delta_0) + \alpha_1 F(\Delta_1) + \cdots + \alpha_d F(\Delta_d).$$

Lemma 5.3.6. *For any lattice d-polytope, the vector $(\alpha_0, \ldots, \alpha_d)$ is unique.*

Proof. Recall that

$$F_t(\Delta_k) = L_{\Delta_k}(t) = \binom{t+k}{k}.$$

Therefore, $L_{\Delta_0}(t), \ldots, L_{\Delta_d}(t)$ form a basis for polynomials of degree at most d. Given a lattice d-polytope $P \subset \mathbb{R}^d$, suppose we have two unimodular decompositions. In $\mathscr{P}(\mathbb{Z}^d)/\mathcal{U}$, suppose

$$\sum_{k=0}^{d} \alpha_k [\Delta_k] = [P] = \sum_{k=0}^{d} \beta_k [\Delta_k].$$

By applying the integer unimodular invariant F_t, we have

$$\sum_{k=0}^{d} \alpha_k F_t(\Delta_k) = \sum_{k+0}^{d} \beta_k F_t(\Delta_k)$$
$$\sum_{k=0}^{d} \alpha_k \binom{t+k}{k} = \sum_{k=0}^{d} \beta_k \binom{t+k}{k}.$$

Since the set of $\binom{t+k}{k}$ for $k = 0, \ldots, d$ is linearly independent, we must have that $\alpha_k = \beta_k$. $\square$

Since $[P]$ can be represented in $\mathscr{P}(\mathbb{Z}^d)/\mathcal{U}$ as a linear combination of $[\Delta_0], \ldots, [\Delta_d]$, we can view the set of integer unimodular invariant valuations as a $(d+1)$-dimensional vector space. The integer unimodular invariant valuation F corresponds to the point $(F(\Delta_0), \ldots, F(\Delta_d))$. Theorem 5.3.5 then tells us that two integer unimodular invariant valuations are equivalent if and only if they are the same point in this vector space.

Theorem 5.3.7 (Betke-Kneser)**.** *The valuations $L_0(P), \ldots, L_d(P)$ form a basis for the space of integer unimodular invariant valuations of $\mathscr{P}(\mathbb{Z}^d)$.*

Proof. Recall that L_n corresponds to the vector

$$b_n := (L_n(\Delta_0), \ldots, L_n(\Delta_d)).$$

Since the space of integer unimodular invariant valuations of $\mathscr{P}(\mathbb{Z}^d)$ has dimension $d+1$, it suffices to show that the vectors b_n are linearly independent for $n = 0, \ldots, d$. Consider the matrix B where the n^{th} column is b_n. Note that since $L_{\Delta_k}(t)$ is a polynomial in t with degree k, $L_n(\Delta_n) \neq 0$ and $L_n(\Delta_k) = 0$ for $k > n$. Since B is upper triangular and has non-zero diagonal, it has full rank, and therefore its columns are linearly independent. $\square$

5.3.2 Proof of Ehrhart Reciprocity via Valuation

We would like to prove that $F_t(P) = L_P(t)$ and $G_t(P) = (-1)^{\dim(P)} L_{P^\circ}(-t)$ are integer unimodular invariant valuations. By recalling from Example 3.5.2 that

$$F_t(\Delta) = G_t(\Delta)$$

for all standard simplices, we see that Ehrhart Reciprocity is an immediate consequence of Theorem 5.3.5:

$$L_P(t) = F_t(P) = G_t(P) = (-1)^{\dim(P)} L_{P^\circ}(-t).$$

Now let $G_t : \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}$ be the function $[P] \mapsto (-1)^{\dim(P)} L_P(-t)$. After establishing that G_t is a valuation, proving that G_t is integer unimodular invariant uses the same ideas as proving that F_t is integer unimodular invariant. For this reason, we will not explicitly provide the proof that G_t is integer unimodular invariant. We would like to prove that G_t also a valuation, but it is a bit more involved than the proof of F_t.

Before proving G_t is a valuation, we first need the following results.

Theorem 5.3.8. *Let $C_1, \ldots, C_n$ be cells of a polyhedral complex. Then*

$$\sum_{i=1}^{n} \alpha_i [C_i] = 0 \implies \sum_{i=1}^{n} \alpha_i (-1)^{\dim(C_i)} [C_i^\circ] = 0.$$

Proof. Suppose $\sum_{i=1}^{n} \alpha_i [C_i] = 0$. Noting that the collection of all non-empty faces of C_i is a subdivision of C_i, we have

$$[C_i^\circ] = \sum_{F \leq C_i} (-1)^{\dim(C_i) - \dim(F)} [F],$$

so

$$\begin{aligned}
\sum_{i=1}^{n} \alpha_i (-1)^{\dim(C_i)} [C_i^\circ] &= \sum_{i=1}^{n} \alpha_i (-1)^{\dim(C_i)} \sum_{F \leq C_i} (-1)^{\dim(C_i) - \dim(F)} [F] \\
&= \sum_{\substack{F : F \leq C_i \\ \text{for some } i}} (-1)^{\dim(F)} \left(\sum_{i : F \leq C_i} \alpha_i \right) [F].
\end{aligned}$$

If x is not in any F, then $[F](x) = 0$. Now suppose $x \in F$. Since all C_i's are faces of a polyhedral complex, if F is a face of some C_i and is also contained in some other C_j, then F must also be a face of C_j. In other words, $x \in C_i$ if and only if $F \leq C$. Then, the sum $\sum_{i : F \leq C_i} \alpha_i$ would be equal to

$$\sum_{i=1}^{n} \alpha_i [C_i](x) = 0.$$

In either case, $(\sum_{i : F \leq C_i} \alpha_i)[F] = 0$. □

Theorem 5.3.9. *Let $P_1, \ldots, P_n$ be lattice polytopes. Then*

$$\sum_{i=1}^{n} \alpha_i [P_i] = 0 \implies \sum_{i=1}^{n} \alpha_i (-1)^{\dim(P_i)} [P_i^\circ] = 0.$$

Proof. Let $\mathcal{H}$ be the set of facet hyperplanes of all of $P_1, \ldots, P_n$. Each of these hyperplanes $H \in \mathcal{H}$ partitions $\mathbb{R}^d$ into three parts: $(H^-)^\circ$, H, and $(H^+)^\circ$. The collection $\mathcal{H}$ then partitions $\mathbb{R}^d$ into cells, where each cell is of the form

$$\bigcap_{H \in \mathcal{H}} J_H$$

where $J_H \in \{H^-, H, H^+\}$. Note that these cells are polyhedra; let $\mathscr{S}$ be the collection of these cells. We claim that $\mathscr{S}$ is a polyhedral complex. The faces of a cell $C = \bigcap_{H \in \mathcal{H}} J_H$ can be represented as the intersection of C with a subset of its facet hyperplanes. By definition, any facet hyperplane of C is a hyperplane in $\mathcal{H}$. Therefore, faces of C are in $\mathscr{S}$. Consider another cell $C' = \bigcap_{H \in \mathcal{H}} J'_H$. For any fixed $H \in \mathcal{H}$, $J_H \cap J'_H$ is either H or a half-space of H; this half-space would be $J_H = J'_H$. Therefore,

$$C \cap C' = \bigcap_{H \in \mathcal{H}} (J_H \cap J'_H)$$

is a cell in $\mathscr{S}$. Again, since $J_H \cap J'_H$ is either H or half-space $J_H = J'_H$, $C \cap C'$ is a face of both C and C'. For any P_i, let $\mathscr{S}_i \subset \mathscr{S}$ be the set of cells $\bigcup_{H \in \mathcal{H}} J_H$ where for each facet hyperplane $\overline{H}$ of P_i, $J_{\overline{H}}$ is an inward facet half-space of P_i. Then

$$P_i = \bigcup_{C \in \mathscr{S}_i} C,$$

and so $\mathscr{S}_i$ is a subdivision of P_i. Corollary 4.3.2 and Theorem 4.3.9 and then give us two identities:

$$[P_i^\circ] = \sum_{C \in \mathscr{S}_i|_{P_i^\circ}} [C^\circ] \tag{5.7}$$

$$[P_i] = \sum_{C \in \mathscr{S}_i|_{P_i^\circ}} (-1)^{\dim(P_i) - \dim(C)} [C]. \tag{5.8}$$

Suppose

$$\sum_{i=1}^{n} \alpha_i [P_i] = \sum_{i=1}^{n} \alpha_i \sum_{C \in \mathscr{S}_i|_{P_i^\circ}} (-1)^{\dim(P_i) - \dim(C)} [C] = 0.$$

We treat $\alpha_i (-1)^{\dim(P_i) - \dim(C)} =: \beta_C$ as the coefficient of $[C]$ and note that $\mathscr{S}_1 \cup \cdots \cup \mathscr{S}_n$ is a polyhedral complex. Theorem 5.3.8 implies that $\sum_{i=1}^{n} \sum_{C \in \mathscr{S}_i|_{P_i}} \beta_C [C] = 0$. Equation (5.7) then implies

$$\begin{aligned}
0 &= \sum_{i=1}^{n} \sum_{C \in \mathscr{S}_i|_{P_i}} \beta_C (-1)^{\dim(C_{ij})} [C^\circ] \\
&= \sum_{i=1}^{n} \alpha_i \sum_{C \in \mathscr{S}_i|_{P_i}} (-1)^{\dim(P_i)-\dim(C)} (-1)^{\dim(C)} [C^\circ] \\
&= \sum_{i=1}^{n} \alpha_i (-1)^{\dim(P_i)} \sum_{C \in \mathscr{S}_i|_{P_i}} [C^\circ] \\
&= \sum_{i=1}^{n} \alpha_i (-1)^{\dim(P_i)} [P_i^\circ]. \qquad \square
\end{aligned}$$

Lemma 5.3.10. *G_t is a valuation.*

Proof. It suffices to prove that

$$\sum_{i=1}^{n} \alpha_i [P_i] = 0 \implies \sum_{i=1}^{n} \alpha_i G_t(P_i) = 0,$$

so suppose $\sum_{i=1}^{n} \alpha_i [P_i] = 0$. Note that $G_t(P) = (-1)^{\dim(P)} F_{-t}(P)$. By Theorem 5.3.9 , we have $\sum_{i=1}^{n} \alpha_i (-1)^{\dim(P_i)} [P_i^\circ] = 0$. By Remark 5.3.2 , we can apply the valuation F_{-t} to get

$$0 = \sum_{i=1}^{n} \alpha_i (-1)^{\dim(P_i)} F_{-t}(P_i^\circ) = \sum_{i=1}^{n} \alpha_i G_t(P_i). \qquad \square$$

5.3.3 Obtaining f-vector via Barvinok's Algorithm

As we have seen, the run-time of our approach is not very efficient in finding a unimodular decomposition. However, if we were just interested in computing the number of simplices of each dimension in a unimodular triangulation, we can use Barvinok's algorithm.

Let us define φ_{Ehr} to be the the map that takes $[P]$ to $\mathrm{Ehr}_P(z)$. For any lattice polytope,

$$\varphi_{\mathrm{Ehr}}(P) = \sum_{t \geq 0} L_P(t) z^t.$$

Note that since $L_P(t)$ is an integer unimodular invariant valuation (by Lemmas 5.3.1, and 5.3.3), the map that takes $[P]$ to $L_P(t)z^t$ is an integer unimodular invariant valuation, and therefore φ_{Ehr} is an integer unimodular invariant valuation. Now consider a unimodular decomposition of P in $\mathscr{P}(\mathbb{Z}^d)/\mathcal{U}$.

$$\begin{aligned}
[P] &= \sum_{k=0}^{d} a_k \Delta_d \\
\varphi_{\mathrm{Ehr}}(P) &= \sum_{k=0}^{d} a_k \varphi_{\mathrm{Ehr}}(\Delta_k) \\
\mathrm{Ehr}_P(z) &= \sum_{k=0}^{d} a_k \,\mathrm{Ehr}_{\Delta_k}(z) \\
\frac{h_P^*(z)}{(1-z)^{d+1}} &= \sum_{k=0}^{d} a_k \frac{h_{\Delta_k}^*(z)}{(1-z)^{k+1}}.
\end{aligned}$$

Multiplying both sides by $(1-z)^{d+1}$, we get

$$h_P^*(z) = a_d + (1-z)a_{d-1} + \cdots + (1-z)^d a_0.$$

Then the coefficients of $h_P^*(1-z)$,

$$h_P^*(1-z) = a_d + a_{d-1}z + a_{d-2}z^2 + \cdots + a_0 z^d$$

will give us the number of k-dimensional unimodular simplices in a unimodular decomposition of P. In addition, if any of these coefficients is negative, then we know that P does not admit a unimodular triangulation, since some part of the unimodular decomposition requires us to "take away" a simplex. Therefore we can use Barvinok's algorithm to obtain $h_P^*(z)$ and efficiently gain some information about a unimodular decomposition of P.

Bibliography

[1] A. Barvinok, *A Course in Convexity*, Amer. Math. Soc., 2002.

[2] A. Barvinok, Computing the Ehrhart Polynomial of a Convex Lattice Polytope, *Disc. Comp. Geom.* **12** (1994), 35–48.

[3] A. Barvinok, K. Woods, Short Rational Generating Functions For Lattice Point Problems, *J. Amer. Math. Soc.* **16** (2003) (4), 957–979.

[4] M. Beck, C. Haase, Irrational proofs for three theorems of Stanley, *European J. Combin.* **28** (2007) (1), 403–409.

[5] M. Beck, S. Robins, *Computing the Continuous Discretely: Integer-point Enumeration in Polyhedra*, Springer, 2007.

[6] U. Betke, M. Kneser, Zerlegungen und Bewertungen von Gitterpolytopen, *J. Reine Ang. Math.* **358** (1985), 202-208.

[7] J. De Loera, J. Rambau, F. Santos, *Triangulations: Structures for Algorithms and Applications*, Springer, 2010.

[8] P. Gruber, *Convex and Discrete Geometry*, Springer, 2007.

[9] B. Grünbaum, *Convex Polytopes*, Springer, 2003.

[10] P. McMullen, Triangulations Of Simplicial Polytopes, *Beitr. Algebra Geom.* **45** (2004) (1), 37–46.

[11] P. McMullen, Valuations and Euler-Type Relations on Certain Classes of Convex Polytopes, *Proc. London Math. Soc.* **s3-35** (1977) (1), 113–135.

[12] P. McMullen, Valuations on lattice polytopes, *Adv. Math.* **220** (2009), 303–323.

[13] G. Ziegler, *Lectures on Polytopes* Springer, 1995.