Taub-NUT Spacetime in the (A)dS/CFT and M-Theory

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Physics

Waterloo, Ontario, Canada, 2005

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In the following thesis, I will conduct a thermodynamic analysis of the Taub-NUT spacetime in various dimensions, as well as show uses for Taub-NUT and other Hyper-Kahler spacetimes.

Thermodynamic analysis (by which I mean the calculation of the entropy and other thermodynamic quantities, and the analysis of these quantities) has in the past been done by use of background subtraction. The recent derivation of the (A)dS/CFT correspondences from String theory has allowed for easier and quicker analysis. I will use Taub-NUT space as a template to test these correspondences against the standard thermodynamic calculations (via the Nöether method), with (in the Taub-NUT-dS case especially) some very interesting results.

There is also interest in obtaining metrics in eleven dimensions that can be reduced down to ten dimensional string theory metrics. Taub-NUT and other Hyper-Kahler metrics already possess the form to easily facilitate the Kaluza-Klein reduction, and embedding such metrics into eleven dimensional metrics containing M2 or M5 branes produces metrics with interesting Dp-brane results.

Acknowledgements

I would like to first thank my parents Tony and Anne, and my brother Ed, for their support and encouragement during my scholastic endeavors.

I am grateful to my supervisor Dr. Robert Mann for his support and assistance, both during my Ph.D. studies and in the writing and editing of this thesis. I would also like to thank my Master's supervisor(s), Dr. D.G.C. McKeon and (unfortunately posthumously) Dr. R. Migneron, for their encouragement and teaching.

Also, I would like to thank the two other collaborators who co-authored the papers around which this thesis is based, Dr. L. Fatibene and Dr. A.M. Ghezelbash. Specifically the Taub-NUT-AdS thermodynamics in chapter 4 are based on the results [16] in collaboration with Dr. Fatibene and Dr. Mann. The Taub-NUT-dS thermodynamics in the same chapter are from the collaboration [20] with Dr. Ghezelbash and Dr. Mann. Finally, the results in chapter 5 are from the results in [28, 29] with Dr. Ghezelbash and Dr. Mann.

Finally, I would like to thank Cristian Stelea for his assistance with the Kaluza-Klein mathematics and Taub-NUT space, and Dr. J.P. Gauntlett for his aid in understanding M-theory.

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Chapter 1

Introduction

1.1 The Standard Model and Gravity

Our current view of the physics of the universe, developed over the last century, divides the forces of nature into four distinct groups: the Strong Nuclear forces, which govern the interactions of the quarks inside atoms; Electromagnetism, which describes the interaction between charged particles; the Weak Nuclear forces, governing radiation; and Gravity, which is the interaction between mass and energy.

One of the overall goals of physics has always been an attempt to unify these four forces into one complete theory. This complete theory would allow one to predict events and phenomena without having to resort to different theories and equations, depending on the phenomena one was studying. Such a complete theory would also provide new predictions concerning the nature of the universe we live in, on both the atomic and large scales. There is even suggestive evidence that such a theory exists; modern Grand Unified Theories (that are a unification of the Strong, Weak, and Electromagnetic forces) suggest that the three gauge coupling constants unify at a scale of at least $10^{15} GeV$, with gravity seeming to unify at an energy scale slightly higher than this. Finally, there is the beauty argument: one feels that in a universe that is working properly, there should be one theory to describe everything, instead of a collection of separate theories that describe parts of the whole.

The name given to the collective Strong, Weak and Electromagnetic interactions is the *Standard Model*. This model includes the matter particles, *force carriers* and the Higgs boson. The force carriers are particles that mediate the force they are associated with; for example, the electromagnetic force carrier is the photon. The Higgs boson, whose existence was postulated by Peter Higgs, gives masses to particles that need mass, while not giving masses to those that don't.

The Standard Model is consistent with every experiment to date, but also has predictions that have not yet been seen. For instance there is no direct evidence for the Higgs boson. The Standard Model, and most GUT's, are described by Quantum Field Theory (QFT), which is what we are led to when we combine Special Relativity (SR) and Quantum Mechanics (QM). It is in QFT that each force is associated with a force carrier. One application of QFT, Quantum Electrodynamics (QED), is the study of the interaction between electrons and photons. QED is the most successful theory to date, in that it has exceptionally close agreement with experimental results. For example the *g-factor*, which is a measure of the magnetism of the electron, is given in QED by[1]

$$\frac{g}{2} = 1.001159652190$$

This is very close to the experimental value of

$$\frac{g}{2} = 1.001159652193$$

The success of QED has led scientists to attempt to describe the other forces though the use of force carriers. For example, the Strong force is described by Quantum Chromo-dynamics (QCD), and is the study of the interaction between coloured quarks and gluons, though the existence of the gluon has only been established indirectly (1978). Also, the collective study of the interactions between quarks and leptons with photons and the W and Z bosons is called *Electroweak theory.* For example, the radioactive decay of the neutron

$$n \to p + e^- + \overline{\nu}$$

is mediated by an exchange of a W-boson.

One final particle needed in field theory is the Higgs boson. The Higgs boson itself, however, is neither a force carrier nor a matter particle. It does its job by spontaneously breaking the gauge symmetries; this also has the benefit of keeping the theory renormalizable.

[1] Values from [1]

The final force, not included in the Standard Model (or not completely successfully included at any rate) is Gravity. The force carrier for Gravity is the *graviton*, which is only theoretical at the moment. The theory we have to describe gravity, found by Einstein in 1915, is the General Theory of Relativity (GR). GR provides an understanding of the gravitational force. The idea of GR is that all of the laws of physics should be the same for *all* observers. According to the picture provided by GR, the force of gravity has a geometric description that curves spacetime. This means that observers in free fall through this curved spacetime will move along an extremal path, called a *geodesic*. Most often, this extremal path is the shortest path.

The Standard Model, when combined with GR, is consistent with almost all physics known today, down to scales probed by modern day particle accelerators. However, there still remains the problem of correctly *combining* gravity along with the other three forces, into one complete theory, so that we don't have to switch between the Standard Model and GR.

Thus the Standard Model, despite its successes, is not complete. Many of the parameters in the Standard Model must be input by hand, in order to get out the correct predictive results. Also, gravity is generally neglected, when dealing with atoms or elementary particles, in the Standard Model and in GUT's. This is due to the weakness of gravity when compared to the other forces. Gravity becomes important, however, when we remember that it is a long range force, and that it is always attractive. This means its effects will add up, and so for a collection of a large number of particles, gravity can be the dominant force.

There are difficulties in forming a theory that includes gravity in with the other three forces, however. One difficulty with gravity is that the theory we have to describe it is a theory of *spacetime*, and so gravity is a part *of* spacetime, unlike the forces of the Standard Model, which are *in* spacetime. To put it another way, the gravitational field in GR is manifested as a curvature of spacetime, and so is a part of spacetime itself, as opposed to the electromagnetic field travelling in a spacetime. So, if we quantize gravity, we are in a sense quantizing spacetime itself.

Gravity also presents problems mathematically. Einstein's equations are non-linear, for example. Thus the superposition principle, which only applies to linear equations, cannot be used. Another difficulty is that gravity is a classical theory, in that it does not depend on or incorporate the uncertainty principle. The Standard Model and other GUT's depend on quantum mechanics and the uncertainty principle, however. A first step would seem to be to form a quantum theory of gravity, that includes the uncertainty principle as part of its framework. This is conceptually difficult, though, in that this would seem to indicate we would be quantizing spacetime itself.

Such a theory of quantum gravity has already been shown to have many strange predictions. For example, as Hawking showed [2], quantum effects mean that black holes aren't really "black" - they can emit black body radiation. There have been a few attempts at a quantum theory of gravity, one of which is the path integral approach.

1.2 Quantizing Gravity

One approach to quantizing gravity involves the path integral, to be briefly described here, and more fully introduced in section 2.3. The quantization of gravity through the path integral method involves partition functions of the form

$$Z = \int D[g]D[\phi]e^{\mathrm{i}I[g,\phi]} \tag{1.2.1}$$

Here, $D[g]$ is a measure on the space of metrics g, $D[\phi]$ a measure on the space of matter fields ϕ, and $I[g, \phi]$ is the action in terms of the metrics and matter fields. Although the action is in general divergent, one could calculate (1.2.1) by first calculating the action, and then use the results to calculate the gravitational entropy of spacetimes containing horizons, as was done by Gibbons and Hawking in 1977 [3]. It is important to note that the entropy can be calculated for spacetimes containing horizons, and not just black holes, as a cosmological horizon has many of the same thermodynamic properties as the event horizon of a black hole.

One way of calculating the entropy of a black hole spacetime (and other spacetimes containing a horizon) involves a background subtraction - an example of which is the Nöether method, to be reviewed in section 2.2. Here, one matches the asymptotic boundary geometries of the spacetime of interest with a suitably chosen background metric in order to get finite surface contributions. For example, one would use flat space as a background for Schwarzschild, or Taub-NUT-AdS as a background for Taub-Bolt-AdS [4]. This method suffers from the fact that, for a lot of spacetimes one cannot choose a suitable background metric.

There has recently been proposed a theory that connects any asymptotically Anti-de Sitter (AdS) spacetime in $(d+1)$ dimensions with a holographic dual conformal field theory (CFT) on its boundary in d dimensions. The proposed AdS/CFT conjecture (to be more fully reviewed in section 2.5) depends fundamentally on the use of the path-integral formalism, in that the partition function of the bulk AdS theory is equated with the partition function of the boundary CFT theory. The first suggestion of this AdS/CFT correspondence was made by Strominger and Vafa [5]. Here, the authors were able to connect the entropy of certain black holes to a system of D-branes. In hindsight, their findings can be viewed as the first hint of the AdS/CFT conjecture.

The AdS/CFT conjecture offers an alternative method of calculation of the action, through the use of counterterms instead of background subtraction. These counterterms, derived from the Gauss-Codazzi equations, are added as extra terms in the action that cancel out the divergent parts in a manner analogous to the counterterms applied in field theory renormalization. The counterterm action does not depend on the metric, but rather on the boundary metric, and so also leaves the equations of motion invariant.

1.3 M-Theory

One of the most recent attempts to unify the four forces is string theory, from which the AdS/CFT is derived. String theory, not to be discussed here in any detail, is now known to be a part of a much larger, overall theory known as M-theory.

M-theory arises as our next attempt to unify the four forces of nature into one complete theory. It relies on spacetime supersymmetry (susy), in which bosons and fermions are interchangeable - every boson has a supersymmetric fermionic partner, and vice versa, under variation of the Lagrangian. Supersymmetry is clearly a broken symmetry, as unbroken susy would mean that every elementary particle would have a super-partner having the same mass, but opposite statistics. For example, spin 1/2 quarks would have spin 0 squark super-partners, the photon would have a spin 1/2 photino, etc.. However, clearly no such equal mass partners exist in our world, hence susy must be broken. At sufficiently high energies, supersymmetry may be restored. One major advantage of susy is that local[2] susy predicts gravity. This is because the supersymmetry algebra contains the generator of translations, P_μ.

[2]Symmetries are *global* if changes are the same throughout the spacetime, and *local* if they differ from point to point.

In local (super)symmetry, the group parameters should be functions of the points of spacetime x^μ. This means we should consider translations P_μ that vary from point to point, and so local supersymmetry should be a theory of general coordinate transformations of spacetime, or in other words a theory of gravity. Thus, we are forced into a supergravity - i.e. if Einstein hadn't invented GR, local susy would have demanded it. In supergravity, the spin 2 graviton is partnered with the spin 3/2 gravitino. Supergravity confronts the problem from which GR and Grand Unified Theories shy away from: neither takes the others symmetries into account.

M-theory is the overall theory that contains as a low energy limit all five of the known string theories[3] and eleven dimensional supergravity. It was Witten [6] in 1995 who put forward the idea that the distinction between the five consistent string theories was due to our approximations, and that there would only be one theory if we could look at it exactly. Moreover, this theory had to be supersymmetric, and eleven dimensional. In M-theory, the Electromagnetic duality in $D = 4$ is a consequence of the M2-M5 brane duality in $D = 11$.

The idea that the elementary particles might correspond to modes of a vibrating membrane was put forward by Dirac (1960). The idea is as follows; regular field theory uses the idea that the elementary particles are point-like objects that move in spacetime, with different particles having different properties. A particle starting at point A and ending at point B will travel a path between the two - i.e. will sweep out a *world-line*. String theory takes this a step further; in String theory, the elementary particles are taken to be strings, with different particles now corresponding to different vibrational modes of the string. A particle string will sweep out a *world-sheet* as it moves from A to B. But, once you have strings, you can move up to two dimensional membranes and higher dimensional objects, that sweep out three or more dimensional world-volumes as they move from A to B.

M-theory is the current best candidate for the "theory of everything", the GUT's that have been the holy grail of physics for the past 100 years. It is currently the only theory that seems to give hope for unifying Einstein's General Relativity with Quantum Mechanics, the two major theories that are at the core of our current understanding of physics, and yet appear to be mutually incompatible. Though there is no well defined theory as yet,

[3]Five known string theories: Type I, Type IIA, Type IIB, $SO(32)$ Heterotic and $E_8 \times E_8$ Heterotic.

the low energy limit of M-theory is generally understood to be eleven dimensional supergravity. Indeed, local supersymmetry predicts the existence of supergravity - one is forced through local supersymmetry to have the graviton (spin 2 boson) partnered with the gravitino (spin 3/2 fermion), along with all of the other particles from quantum mechanics. Hence, supergravity (sugra) would seem to be succeeding where all other unifying attempts have failed. The interest in eleven dimensional sugra is that eleven dimensions seems to be the maximum number of dimensions allowable by supersymmetry. In fact, $N = 8$ supersymmetry is only fully realized in eleven dimensions, and $N = 8$ susy is the only truly unified theory - in it, gravity and all other lower spin particles appear in the same multiplet.

The theory that M-theory supersedes is of course the idea of superstrings, a ten dimensional theory (nine spatial and one time dimension (9+1)) that involves one-dimensional relativistic string-like objects that vibrate, with each vibratory mode associated with a specific particle. M-theory contains both two dimensional (M2) and five dimensional (M5) branes, that now live in eleven dimensions (10+1). Both of these theories live in dimensions greater than the four we are used to in our everyday lives, however. Thus, the extra six dimensions from string theory, or the extra seven from M-theory, must be extremely small, or invisible in some way, if such theories are to explain our (3+1) dimensional world. The idea of a hidden dimension was used by Theodor Kaluza and Oskar Klein when they tried to unify gravity with electromagnetism by adding a fifth, hidden dimension (for a good review of this, see either [7] or for a non-technical overview, [1]).

1.4 (A)dS and Taub-NUT Spacetimes

1.4.1 (A)dS Spacetimes

Anti-de Sitter (AdS) and de Sitter (dS) spacetimes, in $(3 + 1)$ dimensions, can be written in static coordinates as

$$ds^2 = -\left(1 \pm \frac{r^2}{\ell^2}\right) dt^2 + \frac{dr^2}{\left(1 \pm \frac{r^2}{\ell^2}\right)} + r^2(d\theta^2 + \sin^2(\theta) d\phi^2) \qquad (1.4.1)$$

where ℓ is the characteristic length. AdS is negatively curved, with a negative cosmological constant $\Lambda = -3/\ell^2$, and dS is positively curved, with positive cosmological constant $\Lambda = 3/\ell^2$. Both spacetimes solve the classical Einstein equations with appropriate cosmological term;

$$0 = R_{ab} - \frac{1}{2}g_{ab}R + g_{ab}\Lambda \tag{1.4.2}$$

Both spacetimes are maximally symmetric; specifically, the Riemann curvature tensor can be written as

$$R_{abcd} = \mp\frac{1}{\ell^2}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right) \tag{1.4.3}$$

This ensures the geometry of a spacetime is asymptotically (A)dS - any spacetime whose Riemann curvature tensor can be written in the form (1.4.3) is asymptotically (A)dS. The condition for a spacetime to be asymptotically locally AdS (aLAdS) is that the Riemann tensor for the spacetime approaches (1.4.3) asymptotically to $O\left(r^{-3}\right)$ [9]. This means, near the boundary, conformally compact manifolds have a curvature tensor that looks like the AdS curvature tensor. For example, the Riemann tensor of the Taub-NUT-(A)dS spacetimes discussed below can be written asymptotically in the form (1.4.3), but the Eguchi-Hanson-(A)dS spacetimes [10], even though they solve the Einstein equations (1.4.2), cannot.

AdS spacetime, in global coordinates, can be written as a hyperboloid metric $S^1 \otimes \mathbb{R}^3$, where since the time coordinate is the S^1, there are closed timelike curves. However, the S^1 can be unwrapped (take $-\infty < t < \infty$) to obtain a causal spacetime without closed timelike curves. Half of the spacetime can be represented by the metric [11]

$$ds^2 = -dt^2 + \ell^2\cos^2\left(\frac{t}{\ell}\right)\left[d\chi^2 + \sinh^2(\chi)\left(d\theta^2 + \sin^2(\theta)d\phi^2\right)\right] \tag{1.4.4}$$

and contains apparent singularities at $t = \pm\pi/2$. The whole space can be covered by the static metric (1.4.1). There are no Cauchy surfaces in AdS spacetime - in other words, there are null geodesics that never intersect any given surface. This means, in the above metric, that given the surface $t = 0$, one can only predict events in the region covered by coordinates t, χ, θ, ϕ. Predictions beyond this region are prevented by new information coming in from timelike infinity.

AdS spacetime can be characterized by its geodesics. Future timelike geodesics (or timelike observers), starting at a point p (for example the origin $r = 0$), cannot reach $r = \infty$ in a finite amount of time. However, future null geodesics starting at a point p will reach $r = \infty$ in a finite time, and hence form the boundary of the future of p - i.e. in the units of (1.4.1), a photon starting at $r = 0$ will reach $r = \infty$ in a finite time $\pi\ell/2$.

dS spacetime can also be represented as a hyperboloid [11], however here t is not periodic. Introducing coordinates (t, χ, θ, ϕ) on the hyperboloid, dS spacetime can be written as the metric

$$ds^2 = -dt^2 + \ell^2 \cosh^2\left(\frac{t}{\ell}\right)\left[d\chi^2 + \sinh^2(\chi)\left(d\theta^2 + \sin^2(\theta)d\phi^2\right)\right] \tag{1.4.5}$$

where the coordinates cover the whole space, $-\infty < t < \infty$, $0 \leq \chi \leq \pi$, $0 \leq \theta \leq \pi$, and $0 \leq \phi \leq 2\pi$. Constant t spatial sections are S^3 spheres of positive curvature, and are Cauchy surfaces.

In dS spacetime (see [11] or the appendix in the second paper of reference [19], for example), each observer is surrounded by a cosmological horizon, at $r = \ell$ (best seen in the static coordinates (1.4.1)). An object that is held at a fixed distance from the horizon will be redshifted, with the redshift diverging near the horizon. If released, the object will then accelerate towards the horizon, and once it crosses the horizon cannot be retrieved. Thus, the cosmological horizon acts like the horizon of a black hole, "surrounding" observer. Note that the symmetry of dS spacetimes implies that the location of the cosmological horizon is observer dependent. Semi-classically, then, because matter/entropy can be lost crossing the cosmological horizon, it *must* be assigned a Bekenstein-Hawking entropy, and the horizon must also emit Hawking radiation, with $T = 1/(2\pi\ell)$. (Note that throughout this thesis, except for formal definitions, I will be using units such that $\hbar = c = G = 1$).

1.4.2 Taub-NUT Spacetimes

The metric now known as the Taub-NUT metric (reviewed in more mathematical detail in chapter 3) was originally discovered by Taub [8] in 1951, in his search for metrics with high symmetry. It was later rediscovered by Newman, Tamburino and Unti [12]. The metric, given by (in $(3+1)$ dimensions)

$$\begin{aligned} ds^2 &= -F^2(r)\left[dt + 4n\sin^2\left(\frac{\theta}{2}\right)d\phi\right]^2 + \frac{dr^2}{F^2(r)} + (r^2+n^2)d\Omega_2^2 && (1.4.6)\\ F^2(r) &= \frac{r^2 - n^2 - 2mr}{r^2+n^2} && (1.4.7)\end{aligned}$$

and with the two-sphere metric $d\Omega_2^2$ given by

$$d\Omega_2^2 = d\theta^2 + \sin^2(\theta)d\phi^2$$

can be considered a generalization of the empty-space Schwarzschild metric, as can be seen by taking $n \to 0$. The Taub-NUT metric possesses many unusual properties that make it an interesting metric for study. For example, the "NUT" charge n can be considered a magnetic type of mass, and its presence causes spurious singularities known as Misner strings to crop up, that can be dealt with in a manner similar to the Dirac string singularities that occur around a magnetic monopole (both to be discussed in chapter 3). This metric is also not asymptotically flat (aF); it is only asymptotically locally flat (aLF)[4].

The Euclideanized version of the metric (1.4.6), (1.4.7)[5]

$$ds^2 = F^2(r)\left[d\tau + 4N\sin^2\left(\frac{\theta}{2}\right)d\phi\right]^2 + \frac{dr^2}{F^2(r)} + (r^2 - N^2)d\Omega_2^2 \quad (1.4.8)$$

$$F^2(r) = \frac{r^2 + N^2 - 2mr}{r^2 - N^2} \quad (1.4.9)$$

admits two separate kinds of solutions. One of these, called the "NUT" solution, occurs when the fixed point set of ∂_τ is zero dimensional. A second solution, called a "Bolt" solution, occurs when the fixed point set is two dimensional. It is these two solutions that are used to evaluate the entropy in the background subtraction method - the entropy of the Bolt solution is calculated relative to the NUT solution.

There are also closed timelike curves (CTC's) in the regions where $r < r_-$ or $r > r_+$, where $r_\pm$ are the zeroes of the metric function (1.4.7). Along with the CTC's, the metric has quasi-regular singularities (again discussed in chapter 3) which are the mildest form of singularity in that, although they are formed by incomplete geodesics spiralling infinitely around a topologically closed spatial dimension, the Riemann tensor is completely finite near the

[4]aF (in four dimensions, and for a Euclideanized metric) means the metric has a boundary at $r \to \infty$ that is $S^1 \otimes S^2$ (for example - Schwarzschild), with the radius of S^1 being asymptotically constant, and the radius of the S^2 being r. aLF means the boundary is an S^1 bundle over an S^2, (or a squashed S^3), with the bundles labelled by their Chern number, $\propto n$ here. (For a Lorentzian metric, aF means the boundary is $\mathbb{R}^1 \otimes S^2$.)

[5]Found by Wick rotating $t \to \mathrm{i}\tau$, $n \to \mathrm{i}N$, and important because the path-integral approach involves the Euclideanized metric to get a converging partition function (1.2.1).

singularity and no observer near the singularity - including those that fall in - feel unbounded tidal forces.

The Taub-NUT metric can be further generalized to include either a positive or negative cosmological constant. Such Taub-NUT-AdS/Taub-NUT-dS metrics have all of the unusual properties of the aLF Taub-NUT solution, but are not asymptotically AdS/dS (aAdS/adS). They are, however, asymptotically locally (Anti-)de Sitter (aLAdS/aLdS); hence, they still contain a cosmological horizon, whose entropy and other thermodynamic properties can be calculated.

1.5 Current Research

However, why are these metrics of use when studying the AdS/CFT conjecture or M-Theory mentioned above? First, the AdS/CFT conjecture of course holds for metrics that are asymptotically AdS (aAdS). It is thus a natural next step to consider whether the conjecture holds for spacetimes with (Euclidean) toplogy different from that of aAdS spacetimes - for example, asymptotically locally AdS (aLAdS) spacetimes. A first test of the conjecture in four dimensions for metrics beyond aAdS metrics, with metrics that are only aLAdS, involved the use of the Taub-NUT-AdS (TNAdS) metric [4, 13, 14]. These metrics describe spacetimes whose topology is aLAdS, i.e. metrics that are aAdS, but with identifications. It is not *a-priori* obvious that the AdS/CFT conjecture (or the counterterms derived from it) can be applied to this case. Indeed, historically the Taub-NUT spacetime has been a "counterexample to almost anything" [15] and, in keeping with this, the TNAdS metric provides a rigorous test of the AdS/CFT correspondence, and specifically of the counterterm approach to the calculation of the action and conserved mass.

Motivated by this, I will examine the validity of the AdS/CFT correspondence for higher dimensional TNAdS metrics [16], whose generalizations of the four dimensional TNAdS metric were found by Awad and Chamblin [17]. As I will show below, the conjecture holds in these higher dimensions - and indeed, I was able to show (see chapter 4) that the full counterterm action is not needed, as the finite contributions from the counterterm action come from the first few terms of this action. Also, the calculation of the thermodynamic properties through the Nöether approach, using Taub-NUT-AdS as a background for Taub-Bolt-AdS, can also be done in higher dimensions, and a comparison can be made. The entropy found from the Nöether method can be written - up to an integration constant - in terms of the entropies of the NUT and Bolt solutions found individually, i.e. $S_{\text{Nöether}} = S_{\text{Bolt,AdS/CFT}} - S_{\text{NUT,AdS/CFT}} + C$. It is important to compare the AdS/CFT counterterm results with these Nöether results, as the AdS/CFT correspondence is still a conjecture at this stage. Since the Nöether results yield a difference between the Bolt and NUT results, this comparison provides a confirmation of the AdS/CFT conjecture.

A natural next step from the work mentioned in the last paragraph is to test the so-called dS/CFT correspondence [18] using the Taub-NUT-dS metrics, which are only aLdS, but also still possess the same interesting topology. This provides an equally rigorous test of the proposed dS/CFT conjecture. The original intent of this was simply to test this correspondence, but there was an interesting consequence of calculating the entropy and conserved mass of Taub-NUT-dS spacetime.

For asymptotically de Sitter spacetimes, there exist two conjectures, one called the Bousso N-bound [19], and what was called in [20] - and which I will also call - the maximal mass conjecture. The N-bound states that *any asymptotically dS spacetime will have an entropy no greater than the entropy $\pi\ell^2$ of pure dS with cosmological constant $\Lambda = 3/\ell^2$ in $(3+1)$ dimensions.* Balasubramanian *et. al.* [21] derived the maximal mass conjecture assuming this N-bound. The maximal mass conjecture states that *any asymptotically dS spacetime with mass greater than dS has a cosmological singularity.*[6] In the original conjecture, the term *cosmological singularity* is not well defined. Here, I interpret it to mean that the scalar Riemann curvature invariants will diverge to form timelike regions of geodesic incompleteness whenever the conserved mass of a spacetime becomes larger than the zero value of pure dS.

Up until the calculation I performed [20] using Taub-NUT-dS spacetime, all asymptotically dS spacetimes respected both of these conjectures. However, as will be shown in chapter 4, the Taub-NUT-dS spacetime provides a counter-example to both of the conjectures, as for certain values of the NUT parameter, both the entropy and the conserved mass $\mathfrak{M}$ of Taub-NUT-dS spacetime are greater than the entropy and conserved mass of pure dS. Although the Taub-NUT-dS spacetime does have the quasi-regular singular-

[6] Of course, dS spacetime in four dimensions has $\mathfrak{M} = 0$, and so mass greater than dS means $\mathfrak{M} > 0$.

ities mentioned above, these singularities are the mildest form of singularity possible, and are not what I consider to be meant by the term "cosmological singularity". Thus, as stated, the maximal mass conjecture is violated. It was also suggested [22] that, since these spacetimes have CTC's, the Chronological Protection Conjecture (CPC) [23] would exclude the Taub-NUT-dS spacetime as a counterexample. This is because the CPC suggests that spacetimes that contain CTC's will develop singularities upon perturbation of the stress tensor, and thus our counterexample is at best a marginal one.

Following this, another counterexample to the maximal mass conjecture was obtained [24] using NUT-charged spacetimes without CTC's, with an overall global structure that is the same as de Sitter space. This NUT-charged spacetime can be found from the four dimensional Taub-NUT-dS metric I will use (4.3.1) through analytic continuation, and that the parameters of the metric exclude horizons and CTC's.

However, as stated in [25], the calculation of the conserved mass (for any dimension) doesn't depend on the existence of horizons or CTC's, since it is calculated at future infinity. Since the mass and the NUT charge in the spacetime are *a priori* independent, one can choose these quantities so as to preserve the global structure of pure de Sitter space, and violate the maximal mass conjecture. For example, one could choose Anderson's values [24], and recover his results. Mindful of this, I still consider the Taub-NUT-dS spacetime to be a violation of the conjecture, and present the results in chapter 4.

As well as being an aLAdS solution with which to test the AdS/CFT and an aLdS solution with which to test the dS/CFT, Taub-NUT spacetimes have played a key role in other M-Theory considerations. In eleven dimensional supergravity, the two types of branes that one can have are M2-branes (or membranes) and M5-branes, which are in fact dual to one another. Supergravity in $D = 11$ is important because it is believed to be the low-energy limit of fundamental M-theory. Thus there is a great deal of interest in extending our understanding of the different classical brane solutions that arise from M-theory or string theory. This includes $D = 11$ M-brane solutions that, after reduction down to ten dimensions, reduce simply to supersymmetric BPS saturated p-brane solutions. Supersymmetric solutions with two or three orthogonally intersecting M2 or M5 branes have been obtained [26].

Recently, however, an interesting supergravity solution for a localized D2/D6 intersecting brane system was found by Cherkis and Hashimoto [27]. The Taub-NUT metric was used in eleven dimensional supergravity to construct a solution that can be reduced to a D6-brane solution in type IIA string theory (specifically, the authors constructed the solution by lifting a D6 brane to a four dimensional Taub-NUT geometry embedded in M-theory, and then placed M2 branes in the Taub-NUT background geometry).

This construction is not restricted to the near core region of the D6 brane. By assuming a simple ansatz for the eleven dimensional metric, the equations of motion reduce to a separable partial differential equation that is solvable and admits proper boundary conditions.

Cherkis and Hashimoto only considered embedding one four dimensional (Euclidean) Taub-NUT metric into the eleven dimensional metric, when other combinations are possible. Indeed, even more combinations are possible if one does not require supersymmetry to be preserved. Taub-NUT space (1.4.8) is of use when embedded to eleven dimensions because it allows a reduction to ten dimensions along the τ coordinate that will automatically give rise to a D-brane with Ramond-Ramond (RR) field $C_{[1]}$ (proportional to the off-diagonal component $d\tau d\phi$ in the metric).

Since finding localized brane solutions aids in constructing SUGRA duals of gauge theories with fundamental matter, motivated by their work, I [28, 29] embedded combinations of Taub-NUT and Eguchi-Hanson metrics into and M2 brane and M5 brane metric (supersymmetry preserving) and also combinations of Taub-Bolt and higher dimensional Taub-NUT and Taub-Bolt metrics into M2 and M5 brane metrics (non-supersymmetry preserving). After suitably reducing to ten dimensions (and performing any necessary T-dualities), these solutions are localized brane intersections of D-branes or Neveu-Schwarz (NS)-branes with various other branes. I present some of these results in chapter 5.

Chapter 2

Review of Black Hole Thermodynamics and Quantum Gravity

My intent in this chapter is to present a review of black hole thermodynamics. In section 2.1, I will go over the general thermodynamics of black holes, with a review of the Nöether method in section 2.2. Then in sections 2.3 and 2.4, I will discuss thermodynamics of asymptotically AdS and dS spacetimes respectively, using the path integral approach. Finally, in section 2.5 I will review the proposed AdS/CFT and dS/CFT correspondences, and the counterterms that arise from these.

2.1 General Thermodynamics

The thermodynamic entropy of a system can be defined as a measure of the molecular disorder existing in the system. Mathematically, this works out to be

$$S = -k \langle \ln(P_r) \rangle = -k \sum_r P_r \ln(P_r) \tag{2.1.1}$$

where P_r is the thermodynamic probability [30][1], or *canonical distribution* [31][2]. Thus, the entropy can be regarded as a measure of the amount of chaos or disorder in a system, or in other words, is a description of the number of

[1]pg. 189. See also pg's 146-148, 170.

[2]pg's 53,54

accessible states available in a given system. Complete order ($S = 0$) will occur only when the system has no other choice but to be in a unique state ($P_r = 1$).

This means that, from (2.1.1), with an increase in entropy, a system goes from a state of lower probability (disorder) to a state of higher probability (disorder), or that the direction in which natural processes take place is governed by probability. Hence, the Second Law of Thermodynamics, which states that the total entropy of all matter in the universe can never decrease ($\delta S \geq 0$), is a *statistical law* - i.e. it is believed to be extremely likely to hold for systems with many degrees of freedom.

Take for example the process of heating a kettle on a stove. By the Second Law, the heat from the stove does not *have to* flow into the kettle, but rather it is highly probable that it will. This means that there is a - extremely small - non-zero probability that the heat will flow the other way.

The area law of Black Holes, on the other hand - which states that the total area of all Black Holes cannot decrease ($\delta A \geq 0$) - can be rigorously proven in the context of General Relativity. These two results, despite the differences in their mathematical rigor, are very similar. It would seem strange to compare such a mathematically proven result to a law that is inferred rather than shown, but ever since Bekenstein's suggestion [32] of the proportionality relationship between the area of the event horizon of a black hole and its physical entropy ($S = A/4$), the relationship between black holes and thermodynamics has been a fertile area of research.

It turns out that it is not only the Second Law of Thermodynamics that has an analogue in black hole thermodynamics. All four of the Laws of Thermodynamics are in some way applicable to black holes. I reproduce table 12.1 from Wald's book in Table 2.1 to demonstrate the involved relationship between black holes and thermodynamics that has actually been built up.

From the first law in table 2.1, we can see that the thermodynamic energy and the mass of a black hole are related; also that the angular momentum of a black hole is the "work" done by the black hole. This leaves the *surface gravity* κ, which plays the role of temperature in black hole thermodynamics. This surface gravity is a "measure of how fast the Killing vector is becoming spacelike"[3]. The main mathematical arguments involved in deriving the black hole thermodynamics are worth repeating, and I will do so now.

[3] [2], pg. 202.

2.1.1 Black Hole Thermodynamics

Zeroth Law

The surface gravity κ is defined on the horizon of an arbitrary, stationary black hole, and plays the role of temperature. For stationary black holes, there exists a Killing field χ^a, normal to the horizon (chapter 12.5, [33]),

$$\chi^a = \xi^a + \Omega_H \psi^a \tag{2.1.2}$$

where Ω_H is the angular velocity of the horizon, ξ^a is the stationary Killing field, and ψ^a is the axial Killing field. With this Killing vector, a simple formula for κ is easily found

$$\kappa^2 = -\frac{1}{2}\chi^{a;b}\chi_{a;b} \tag{2.1.3}$$

(for the full mathematical derivation of (2.1.3), see appendix A or [33, 34]). From this, the temperature of a black hole is given by $T = \frac{\hbar\kappa}{2\pi}$. The reason for the analogy between the zeroth law of thermodynamics and that of black holes given in table 2.1 is that, as shown in [33] and reviewed in appendix A, the surface gravity can be shown to be constant over the horizon of a black hole. It should be noted that classically, κ cannot physically represent a temperature, because of course a black hole is a perfect black body, and does not emit anything, and hence the temperature of a black hole would be

Table 2.1: Black Holes & Thermodynamics (Reproduced from [33], pg. 337.)

	Context	
Law	Thermodynamics	Black Holes
Zeroth	T constant throughout body in thermal equilibrium	κ constant over horizon of stationary black hole
First	$dE = TdS+$ work terms	$dM = \frac{\kappa}{8\pi}dA + \Omega_H dJ$
Second	$\delta S \geq 0$ in any process	$\delta A \geq 0$ in any process
Third	Impossible to achieve $T = 0$ by a physical process	Impossible to achieve $\kappa = 0$ by a physical process

absolute zero. However, Hawking showed [2] that a black hole can radiate quantum particles like a black body, with a spectrum at a temperature $T = \frac{\hbar\kappa}{2\pi}$.

First Law

The First Law of black hole thermodynamics can be stated mathematically such that the relation between the change in mass (M), area (A) and angular momentum (J) of a black hole is given by the equation

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J \tag{2.1.4}$$

This is directly analogous to the First Law of thermodynamics, which can be written mathematically as

$$\delta E = T\delta S - P\delta V \tag{2.1.5}$$

and indeed, the $\Omega_H \delta J$ term in (2.1.4) is the work done on a black hole, analogous to the work term $P\delta V$ in (2.1.5). Note that δA is analogous to the change in entropy δS in (2.1.5), and so κ is again seen to play the role of temperature in black hole thermodynamics.

A mathematical derivation of (2.1.4) following the results in [34] is given in appendix A

Second Law

The original version of the Second Law of black hole thermodynamics, as proven by Hawking [35], and stated by Bardeen, Carter and Hawking [36], was that the total entropy of a black hole cannot decrease with time, i.e.

$$\delta S \geq 0 \tag{2.1.6}$$

and was proven due to the relationship between the black hole entropy (S) and the area (A) of the horizon of the black hole, given by

$$S = \frac{1}{4}A \tag{2.1.7}$$

In other words, the statement that the entropy cannot decrease is equivalent to the statement that the area of the event horizon of the black hole cannot decrease, $\delta A \geq 0$. This version is of course stronger than the Second Law of thermodynamics, as in thermodynamics one can transfer entropy from one system to another, and the requirement is only that the overall or total entropy does not decrease. However, one cannot transfer area between two black holes, and classically, black holes cannot emit anything, and so do not lose mass, and hence the horizon area will not decrease.

However, Hawking [2] then discovered that black holes can radiate quantum mechanically, and hence black holes could lose mass and horizon area. This of course violates the direct relationship between the entropy and area of a black hole stated above. However around the same time, Bekenstein [37] suggested, and Hawking refined [2] a Generalized Second Law, such that the total entropy $\tilde{S}(= S_{BH} + S_{rad})$ - the entropy of the black hole horizon ($S_{BH} \propto A$) and the entropy of matter and gravitational radiation surrounding the black hole (S_{rad}) - could not decrease. (In the original version of the Generalized Second Law, Bekenstein did not suggest a black hole could emit as well as absorb particles - hence, this original version could be violated by, for example (as pointed out by Hawking [2]), immersing the black hole in black body radiation of lower temperatures. For the full Generalized Second Law then, one must include the absorbtion *and* emission of particles from the black hole.)

This Generalized Second Law was proven by Frolov and Page [38] for quasi-stationary changes of a generic, charged, rotating black hole that emits, absorbs and scatters any radiation in the Hawking semiclassical formalism.

Third Law

The Third Law of thermodynamics can be stated in several ways. One, due to Nernst, is that *The temperature of a system cannot be reduced to zero in a finite number of operations.* A stronger statement, by Planck, is *The entropy of any system tends, as $T \to 0$, to an absolute constant, which may be taken as zero.*

Although the Third Law of black hole thermodynamics is analogously stated by Bardeen, Carter and Hawking [36] in 1973, as *It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations*, they did not present a proof. It was Israel [39] in 1986 that presented a dynamical proof to the Third Law, reviewed in appendix A. Informally, Israel stated the Third Law as follows:

"*A non-extremal black hole cannot become extremal (i.e. lose its trapped* *surfaces) at a finite advanced time in any continuous process in which the stress-energy tensor of accreted matter stays bonded and satisfies the weak energy condition in a neighbourhood of the outer apparent horizon.*"

The proof depends, obviously, on the weak energy condition. It also depends on defining a "process" in a dynamical context as an interaction between a black hole and its environment, where the active phase has a finite time as seen by freely falling observers near the horizon of the black hole.

Period of a Black Hole

A simple example of a black hole metric is the Schwarzschild solution, given by the metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} + r^2 d\Omega^2 \tag{2.1.8}$$

Note that the apparent singularity at $r = 2m$ is a coordinate artifact that can be removed by a coordinate transformation. This metric has a Lorentzian signature $[-, +, +, +]$. One can obtain a Euclideanized form of the metric by Wick rotating the time coordinate $t \to \mathrm{i}\tau$, to give a positive-definite metric (for $r > 2m$)

$$ds^2 = \left(1 - \frac{2m}{r}\right) d\tau^2 + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} + r^2 d\Omega^2 \tag{2.1.9}$$

This metric will be regular at $r = 2m$ if τ is taken to be an angular variable, with period $8\pi m$. The manifold then defined by the ranges $r \geq 2m$, $0 \leq \tau \leq 8\pi m$ is the Euclidean section of the Schwarzschild solution (note the true singularity at $r = 0$ does not lie in the Euclidean section).

We demand regularity in the Euclidean section of any metric, and hence in general the Euclidean time τ will be a periodic variable with some period β. The period of a black hole can be found, for a general Euclideanized metric

$$ds^2 = G(r) d\tau^2 + \frac{dr^2}{F(r)} + d\Sigma^2 \quad ; \quad \tau = \mathrm{i}t \tag{2.1.10}$$

to be given by $\beta = 4\pi(|F'(a)G'(a)|)^{-1/2}$, where $r = a$ is the black hole horizon (and so $F(a) = G(a) = 0$). This is done by expanding $F(r)$ around $r = a$, and then using the fact that $G(a)$ has the same order of zero. This procedure will force τ to have a period $2\pi = \beta F'(a)G'(a)/2$. If $G(r) = F(r)$, then one arrives at

$$\beta = \frac{4\pi}{|F'(a)|} \tag{2.1.11}$$

The period can be shown to be equal to $\beta = 1/T$, where T is again the temperature of the black hole. The full derivation of (2.1.11) is given in appendix A, section A.5.

2.2 Nöether Charges

Note from the discussion in section 2.1.1 above that in the Euclideanized version of a black hole metric, the horizon is a place where the Killing field ∂_τ vanishes. This obstructs one from foliating the spacetime with constant-τ surfaces, and gives rise to entropy because it causes a difference in the Euclidean action I and βH, where β is the period of τ and H is the Hamiltonian.

In general, to calculate the entropy, one uses the Gibbs-Duhem relation

$$S = \beta H - I \tag{2.2.1}$$

(proven below using thermodynamic arguments in sections 2.3, 2.4 for asymptotically AdS/dS spacetimes). However, in general, the action and Hamiltonian are infinite, and one must introduce a background spacetime. The action and Hamiltonian of this background spacetime are also calculated, and subtracted off of the main spacetime in order to render a finite answer.

The Nöether method [40, 41] is an example of one of the methods used to calculate the thermodynamic properties of black hole spacetimes using background subtraction. In this method, the generalized gravitational entropy is related to the Nöether charge. A background of suitable topology must be introduced, to match with the main spacetime - for example, a Schwarzschild-AdS spacetime could be calculated relative to the pure-AdS spacetime. Provided that such a background can be introduced, conserved quantities are considered as being relative to the background.

Following the discussion in [40, 41], consider a $(d+1)$ dimensional action

$$I = -\int_M \mathbf{L} + \int_{\partial M} \mathbf{B} \tag{2.2.2}$$

where the action will depend in general on the metric and any matter fields present, collectively denoted Φ as in [41]. For diffeomorphism invariant theories[4] there is a $d-1$ form $\mathbf{Q}$ associated with the diffeomorphism invariance.

The action remains unchanged to first order by compact support variations of solutions to the equations of motion. Thus, there is a d-form $\mathbf{\Theta}(\Phi, \delta\Phi)$ such that

$$\delta \mathbf{L} = \mathbf{E}\delta\Phi + d\mathbf{\Theta} \tag{2.2.3}$$

where $\mathbf{E}$ represents all of the equations of motion. This implies

$$\begin{aligned} 0 &= \delta I = -\int_M [\mathbf{E}\delta\Phi + d\mathbf{\Theta}] + \int_{\partial M} \delta \mathbf{B} \\ 0 &= \int_{\partial M} [\delta \mathbf{B} - \mathbf{\Theta}] \end{aligned} \tag{2.2.4}$$

Now, if ξ^a is a smooth vector field on M, consider the field variation $\hat{\delta}\Phi = \mathcal{L}_\xi \Phi$. The diffeomorphism invariance of $\mathbf{L}$ implies that, under this variation,

$$\hat{\delta}\mathbf{L} = \mathcal{L}_\xi \mathbf{L} = d(\xi \cdot \mathbf{L}) \tag{2.2.5}$$

where the dot denotes contraction of the first index of the form[5]. Then there is an n-form Nöether current $\mathbf{J}$ associated with every diffeomorphism generated by ξ,

$$\mathbf{J}[\xi] = \mathbf{\Theta} - \xi \cdot \mathbf{L} \tag{2.2.6}$$

This then gives

$$\begin{aligned} d\mathbf{J} &= d\mathbf{\Theta} - d(\xi \cdot \mathbf{L}) \\ &= -\mathbf{E}\delta\Phi = 0 \end{aligned} \tag{2.2.7}$$

where (2.2.5) has been used, and the last equality follows when the equations of motion are satisfied. But, (2.2.7) implies that there exists a Nöether charge that is an $(n-1)$-form $\mathbf{Q}[\xi]$ such that

$$\mathbf{J}[\xi] = d\mathbf{Q}[\xi] \tag{2.2.8}$$

The Hamiltonian conjugate to a time-evolution vector field satisfying $t^a \nabla_a \tau = 1$ can be found to be

$$H = \int_\infty [\mathbf{Q}[t] - t \cdot \mathbf{B} + \mathbf{C}] \tag{2.2.9}$$

[4]From [40], for any diffeomorphism $\psi : M \to M$ we have $\mathbf{L}[\psi^*(\Phi)] = \psi^* \mathbf{L}[\Phi]$. Note on the LHS of this equation, ψ^* is *not* applied to ∇_a or any other non-dynamical fields that may appear in $\mathbf{L}$.

[5]This follows from the general identity, equation (5) of [40]: $\mathcal{L}_\xi \mathbf{\Lambda} = \xi \cdot d\mathbf{\Lambda} + d(\xi \cdot \mathbf{\Lambda})$

where $\mathbf{C}$ is any quantity with zero variation. This is implied through noting that the variation of the Hamiltonian is the integral of the symplectic current. In (2.2.9), $\mathbf{C}$ is chosen to be $-\bar{\mathbf{Q}}$ so that the Hamiltonian vanishes in the background (where a bar denotes a quantity evaluated in the background).

We want to foliate the manifold M with surfaces of constant τ - this may require the removal of a set of measure zero from M. The action is now

$$I = -\beta \left(\int_{\Sigma_\tau} t \cdot \mathbf{L} + \int_\infty t \cdot \mathbf{B} \right) \tag{2.2.10}$$

where the sign change in the $\mathbf{B}$ term is due to the orientation of ∂M. Then, inserting (2.2.9), (2.2.10) into (2.2.1),

$$S = \beta \left(\int_\infty \mathbf{Q} - \int_{\Sigma_\tau} \mathbf{J}[t] - \int_\infty \bar{\mathbf{Q}} \right) \tag{2.2.11}$$

and from (2.2.8), the integral of $\mathbf{J}$ depends on $\mathbf{Q}$ only on the boundary of Σ_τ. Recall now that all obstructions to the foliation were removed on the ∂M boundary before integration. Letting $\mathcal{O}$ represent the intersection of these obstructions with Σ_τ, we have

$$\int_{\Sigma_\tau} \mathbf{J}[t] = \int_\infty \mathbf{Q} + \int_{\mathcal{O}} \mathbf{Q} \tag{2.2.12}$$

giving

$$S = -\beta \left(\int_{\mathcal{O}} \mathbf{Q} + \int_\infty \bar{\mathbf{Q}} \right) \tag{2.2.13}$$

Thus, the entropy is seen to be related to the Nöether charge.

In practical, calculational terms then, the Nöether framework depends on the covariant first order Lagrangian, that can be written as the sum of three terms,

$$L = L_1 + L_2 + L_3 \tag{2.2.14}$$

where the terms are given by

$$\begin{aligned} L_1 &= \frac{1}{2\kappa}(R - \Lambda)\sqrt{g} ds && (2.2.15) \\ L_2 &= \frac{1}{2\kappa} d_\mu \left(\omega^\mu{}_{\alpha\beta} g^{\alpha\beta} \sqrt{g} \right) ds && (2.2.16) \\ L_3 &= -\frac{1}{2\kappa}(\overline{R} - \Lambda)\sqrt{\overline{g}} ds && (2.2.17) \end{aligned}$$

and each term is a covariant Lagrangian on its own. Here, $g_{\alpha\beta}$ is the dynamical metric, $\overline{g}_{\alpha\beta}$ is the background metric, and R, $\overline{R}$ are the Ricci scalars of the metric and background metric, respectively. The d_μ in (2.2.16) means differentiate with respect to x^μ, and $\omega^\mu{}_{\alpha\beta}$ is a tensorial quantity, defined by

$$\begin{aligned}
\omega^\mu{}_{\alpha\beta} &= u^\mu{}_{\alpha\beta} - \overline{u}^\mu{}_{\alpha\beta} \\
u^\mu{}_{\alpha\beta} &= \Gamma^\mu{}_{\alpha\beta} - \delta^\mu{}_{(\alpha}\Gamma^\lambda{}_{\beta)\lambda} \quad , \qquad \overline{u}^\mu{}_{\alpha\beta} = \overline{\Gamma}^\mu{}_{\alpha\beta} - \delta^\mu{}_{(\alpha}\overline{\Gamma}^\lambda{}_{\beta)\lambda}
\end{aligned} \tag{2.2.18}$$

The constant κ depends on unit conventions, and becomes relevant when coupling to matter fields.

This splitting of the Lagrangian will carry through to the conserved quantities calculated from it; in a geometrical, well-defined way, the total conserved quantities Q calculated from L will thus be given by $Q = Q_1 + Q_2 + Q_3$. Conserved quantities are defined by the following integral

$$Q_D(\xi) = \int_D U(\xi, g, \overline{g}) \tag{2.2.19}$$

of the superpotential (2.2.20). This superpotential, when integrated along the dynamical and background metrics, defines the conserved current (relative to the background) within a closed $(d-1)$ dimensional submanifold D of the spacetime.

The superpotential in (2.2.19) is given by

$$U(\xi) = U_1(\xi) + U_2(\xi) + U_3(\xi) \tag{2.2.20}$$

where the individual superpotentials are carried over from the individual Lagrangians,

$$L_1 \to U_1(\xi) = \frac{1}{2\kappa}\nabla^\alpha \xi^\beta \sqrt{g}\, ds_{\alpha\beta} \tag{2.2.21}$$

$$L_2 \to U_2(\xi) = \frac{1}{2\kappa}\xi^\alpha \omega^\beta{}_{\mu\nu} g^{\mu\nu} \sqrt{g}\, ds_{\alpha\beta} \tag{2.2.22}$$

$$L_3 \to U_3(\xi) = -\frac{1}{2\kappa}\overline{\nabla}^\alpha \xi^\beta \sqrt{\overline{g}}\, ds_{\alpha\beta} \tag{2.2.23}$$

Here, ∇^α and $\overline{\nabla}^\alpha$ are the covariant derivatives with respect to the dynamical metric and the background metric, respectively. Also, $ds_{\alpha\beta}$ is the surface element, defined by $ds_{\alpha\beta} = u_a n_b dS$, where u_a, n_b are the unit timelike/radial vectors, and $dS = d\theta d\phi$ in four dimensions, for example. The energy-momentum and the angular momentum of the spacetime (g) relative to the background spacetime ($\overline{g}$) can be obtained from (2.2.19), (2.2.20) by specifying ξ^α appropriately.

The First Law of black hole thermodynamics can be written in the following form,

$$\delta M = T\delta S \tag{2.2.24}$$

for the Taub-Bolt type solutions to be described later, and this formula can be integrated to obtain the entropy (relative to the background) of the spacetime. Note of course, since this entropy will be obtained through integration, it will depend on a constant that may depend on the model (e.g. on the cosmological constant), though of course not on the solution under investigation.

This entropy, from (2.2.24), can only be defined once the relevant thermodynamic potentials are provided, such as the temperature $T = \beta^{-1}$, and (in general) the momenta conjugate to the angular momentum along with other (gauge) charges. These must be provided through other physical means (e.g. from the radiation spectrum).

2.3 Asymptotically AdS Thermodynamics

2.3.1 Path Integral Approach

Here I wish to present an introduction to the use of the path integral in quantum gravity, for use later on in the calculation of the action in conjunction with the Gibbs-Duhem relation and the counterterms produced by the AdS/CFT correspondence. For other reviews, see for example [42].

The starting point for the path integral approach to quantum gravity is due to Feynman, for which

$$\langle g_2, \Phi_2, S_2 | g_1, \Phi_1, S_1 \rangle = \int D[g,\Phi] e^{iI[g,\phi]} \tag{2.3.1}$$

where the left hand side is the amplitude to go from some initial state of metric and matter fields g_1, Φ_1 on some (spacelike) surface S_1 to some final state of metric and matter fields g_2, Φ_2 on a surface S_2. This can be represented, as on the right hand side of (2.3.1), as a sum over all possible metric/field configurations g, Φ lying on all the surfaces S_i between S_1 and S_2 - see also figure 2.3.1 for a pictorial view, to be explained later. $D[g,\Phi]$ thus represents the measure of the space of all such field configurations, and

$I[g, \Phi]$ is the action taken over all fields having these values over the two surfaces. Asymptotically flat and asymptotically anti-de Sitter spacetimes have timelike tubes at some finite mean radii connecting the two surfaces, so that both the boundary and the contained region are compact. The amplitude for the entire spacetime can thus be obtained by letting the larger mean radii tend to infinity, and the smaller to zero.

The action (in $(d+1)$ dimensions) can be decomposed into two (later three - I_{ct} will be introduced in section 2.5) parts

$$I = I_B + I_{\partial B} \tag{2.3.2}$$

where the bulk and boundary actions are given by

$$\begin{aligned} I_B &= -\frac{1}{16\pi G}\int_{\mathcal{M}} d^{d+1}x\,\sqrt{-g}\,(R - 2\Lambda + \mathcal{L}_M(\Psi)) && (2.3.3) \\ I_{\partial B} &= -\frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^d x\,\sqrt{-\gamma}\Theta && (2.3.4) \end{aligned}$$

Here $\Lambda = -\frac{d(d-1)}{2\ell^2}$, and $\mathcal{L}_M$ is the matter Lagrangian if required (the matter Lagrangian won't be used here). Θ is the trace of the extrinsic curvature, where

$$\Theta_{\mu\nu} = \gamma_\mu{}^\sigma \gamma_\nu{}^\delta n_{\delta;\sigma}$$

and $\gamma_{\mu\nu}$ is the boundary metric. The unit radial vector is given by $n^a = [0, (\sqrt{g_{rr}})^{-1}, 0, 0]$. Note that G will be taken to be unity for all calculations.

The boundary action is present in order to correctly derive the Einstein equations. If one varies (2.3.2) with only the bulk action present, with only the condition $\delta g_{\alpha\beta} = 0$ on $\partial\mathcal{M}$, then an extra term in the equations of motion occurs (this term does not appear if one also requires $\delta\left(\partial g_{\alpha\beta}\right) = 0$). It turns out that this extra term is exactly equal to the negative of the variation of the boundary (2.3.4) term. Hence, (2.3.2) is the correct action to use to get the equations of motion. One can also just use I_B with the conditions that $\delta g_{\alpha\beta}$ and its first derivatives equal zero.

The presence of the bulk action can also be understood from the path integral viewpoint by considering the situation in figure 2.3.1 (see also [42]). Here, there is a transition from a surface S_1 with metric and matter fields $[g_1, \Phi_1]$ to an intermediate surface S_i with metric and matter fields $[g_i, \Phi_i]$, and then to a surface S_2 with $[g_2, \Phi_2]$. Thus, the amplitude of going from a

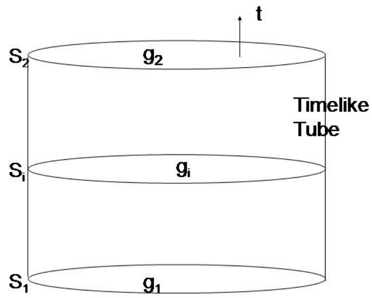

Figure 2.3.1: Amplitude to go from g_1, Φ_1 on S_1 to g_2, ϕ_2 on S_2.

surface $[g_1, \Phi_1, S_1]$ to a surface $[g_2, \Phi_2, S_2]$ can be written as a sum over all of the possible intermediate surfaces

$$\langle g_2, \Phi_2, S_2 |\, g_1, \Phi_1, S_1 \rangle = \sum_i \langle g_2, \Phi_2, S_2 |\, g_i, \Phi_i, S_i \rangle \langle g_i, \Phi_i, S_i |\, g_1, \Phi_1, S_1 \rangle \tag{2.3.5}$$

This will be true if and only if (iff) (i.e. will relate to (2.3.1) iff)

$$I\left[g_{12}, \Phi\right] = I\left[g_{1i}, \Phi\right] + I\left[g_{i2}, \Phi\right] \tag{2.3.6}$$

where g_{ij} is the metric that is between surfaces S_i, S_j, and the metric g_{12} is of course the full metric of the region between S_1, S_2. In general, g_{1i} and g_{i2} will have different normal derivatives - they will yield delta-function contributions to the Ricci tensor proportional to the difference between the extrinsic curvatures of the surfaces S_i in the metrics g_{1i}, g_{i2}. The boundary term in (2.3.2) is what compensates for this.

2.3.2 Thermodynamics

Next, I wish to relate the path-integral arguments above to the thermodynamic arguments needed to calculate the thermodynamic properties of the metrics in question. Consider a scalar quantum field ϕ - the amplitude for going from a state $|t_1, \phi_1\rangle$ to $|t_2, \phi_2\rangle$ can be expressed as an integral

$$\langle t_2, \phi_2 | t_1, \phi_1 \rangle = \int_1^2 d[\phi] e^{\mathrm{i}I[\phi]} \tag{2.3.7}$$

over all possible intermediate field configurations between the initial and final states. However, this amplitude can also be expressed as

$$\langle t_2, \phi_2 | t_1, \phi_1 \rangle = \langle \phi_2 | e^{-\mathrm{i}H(t_2 - t_1)} | \phi_1 \rangle \tag{2.3.8}$$

where H is the Hamiltonian. By imposing the periodicity condition $\phi_1 = \phi_2$ for $t_2 - t_1 = -i\beta$, we sum over ϕ_1 to obtain

$$\mathrm{Tr}(\exp(-\beta H)) = \int d[\phi] e^{-\hat{I}[\phi]} \tag{2.3.9}$$

The right-hand side is now a Euclidean path integral over all field configurations intermediate between the periodic boundary conditions because of the Wick rotation of the time coordinate, where $\hat{I}$ is the Euclidean action. Inclusion of gravitational effects can be carried out as described above, by considering the initial state to include a metric on a surface S_1 at time t_1 evolving to another metric on a surface S_2 at time t_2, yielding the relation (2.3.1).

Note that the left-hand side of (2.3.9) is simply the partition function Z for the canonical ensemble for a field at temperature β^{-1}. This connection with standard thermodynamic arguments [31] can be seen as follows. We start with the canonical distribution

$$P_r \equiv \frac{< n_r >}{\mathcal{N}} = \frac{e^{-\beta E_r}}{\sum_r e^{-\beta E_r}} \tag{2.3.10}$$

with β determined by considering the average total energy M

$$M = \frac{\sum_r E_r e^{-\beta E_r}}{\sum_r e^{-\beta E_r}} = -\frac{\partial}{\partial \beta} \ln \left\{ \sum_r e^{-\beta E_r} \right\} = -\frac{\partial}{\partial \beta} \ln Z \tag{2.3.11}$$

Also, the Helmholtz free energy $W = M - TS$ can be rearranged so that

$$M = W + TS = W - T \left(\frac{\partial W}{\partial T} \right)_{N,V} = -T^2 \left[\frac{\partial}{\partial T} \left(\frac{W}{T} \right) \right]_{N,V} \tag{2.3.12a}$$

$$= \frac{\partial}{\partial \beta} (\beta W) \tag{2.3.12b}$$

Comparing (2.3.11) and (2.3.12b), we get

$$-\beta W = \ln \left\{ \sum_r e^{-\beta E_r} \right\} = \ln Z \tag{2.3.13}$$

which can be interpreted as describing the partition function of a gravitational system at temperature β^{-1} contained in a (spherical) box of finite radius. Thus, using the expression for the Helmholtz free energy $M = W + TS$, and defining $T = 1/\beta$ (via thermodynamic arguments), we can re-arrange (2.3.13) into the more familiar form of the Gibbs-Duhem relation,

$$S = \beta M - I_{cl} \tag{2.3.14}$$

where recall from (2.3.9) that $Z = \int e^{-I_{cl}}$.

We therefore compute Z using an analytic continuation ("Wick rotation") of the action in (2.3.1) so that the axis normal to the surfaces S_1, S_2 is rotated clockwise by $\frac{\pi}{2}$ radians into the complex plane [3] (i.e. by rotating the time axis so that $t \to i\tau$) in order to obtain a Euclidean signature. The positivity of the Euclidean action ensures a convergent path integral in which one can carry out any calculations (of action, entropy, etc.). In order to achieve a physical result, one then Wick rotates back into the Lorentz frame at the end of the calculation.

2.4 Asymptotically dS Thermodynamics

2.4.1 Path-Integral

The idea of applying the thermodynamic analysis used on black holes to spacetimes with a positive cosmological constant was first suggested and used by Gibbons and Hawking [43]. Though pure and asymptotically dS spacetimes may not contain a black hole, one can calculate thermodynamic quantities using the cosmological event horizon in the same way as one does so using a black hole event horizon. This is because the cosmological event horizon acts in the same manner as the event horizon of a black hole.

As is well known for spacetimes containing a black hole, the region behind the event horizon is not visible to an observer outside the horizon. A similar effect is produced by the cosmological event horizon that each fundamental observer has in a spacetime with a positive cosmological constant. A positive cosmological constant causes the universe to expand so rapidly that there are regions where, for each observer, light can never reach him. In other words, for spacetimes with a positive cosmological constant, a given observer cannot receive information out of his cosmological horizon. Since cosmological spacetimes with a positive cosmological constant will approach dS spacetimes asymptotically at large times, thermodynamic analysis can be applied to asymptotically dS spacetimes as well as pure dS spacetime.

The future infinity of de Sitter spacetime is spacelike. Thus, an observer moving on a timelike world line will have an event horizon, beyond which he cannot detect anything - i.e. for the observers world line, the event horizon is the boundary of the past. As shown in [43], this cosmological horizon has many formal similarities with the event horizon of a black hole. It obeys laws similar to the Zeroth, First and Second laws, and also bounds the region where negative energy particles can exist with respect to the observer. Thus, particle creation with a thermal spectrum occurs in positive cosmological constant spacetimes, and thus a cosmological event horizon can be considered to have thermodynamic properties as with a black hole. It therefore also makes sense to have a path integral approach to an asymptotically de Sitter spacetime, which I will now explore.

Though the procedure and steps for the path integral approach in asymptotically de Sitter spacetimes is generally the same as that in the asymptotically AdS case above, there are difficulties that arise here that don't appear in AdS. The surfaces S_1, S_2 must be replaced with *histories*, or *time-lines*, H_1, H_2 that have spacelike unit normals. These histories are surfaces that form timelike boundaries of the spatial region, and describe particular histories of the d-dimensional subspaces of the full spacetime.

The notion of a quantum field on a timelike boundary is a difficult one, indeed seemingly impossible to construct at first glance. The usual notion of a quantum field involves defining the field on a spacelike surface and having it propagate forward in time. So how does one define such a field on a timelike surface? Though this is just a conjecture, requiring further exploration and proof, it should be possible to define or describe a QM operator that evolves forward on some history H_i, but restricted to a specific point or region of space. Then, the notion of "correlation" that I will use below between two histories will occur when, for example, the light cones of two observers on two histories H_1, H_2 - originally spacelike, and hence completely causally, separated - meet. Once the light cones intersect, the two observers can compare their observations, made when they were causally separated, and see to what extent the measured observables match each other.

The starting amplitude (2.3.1) is thus altered to

$$\langle g_2, \Phi_2, H_2 |\, g_1, \Phi_1, H_1 \rangle = \int D\left[g, \Phi\right] e^{iI[g,\Phi]} \tag{2.4.1}$$

where the left hand side is again the amplitude to go from some initial state with a metric and matter fields g_1, Φ_1 to some final state with metric and matter fields g_2, Φ_2. The difference now is that the initial and final states are on some histories H_1, H_2. This can again be represented as sum over all possible metric and field configurations g, Φ, but again here, they lie on all histories H_i that lie between H_1, H_2. To make the boundary and interior region compact, the histories are joined by spacelike tubes at some initial and final time. In the limit that these initial and final times approach past/future infinity, the correlation between the complete histories $[g_1, \Phi_1, H_1]$ and $[g_2, \Phi_2, H_2]$ is obtained. Similar to the AdS case, the quantity $\langle g_2, \Phi_2, H_2 | \, g_1, \Phi_1, H_1 \rangle$ depend only on the hypersurfaces H_1 and H_2 - along with the metric and matter fields on these hypersurfaces - and not on any special hypersurface lying between H_1 and H_2.

The action can again be decomposed into two parts

$$I = I_B + I_{\partial B} \tag{2.4.2}$$

where the bulk and boundary actions are given as

$$\begin{aligned} I_B &= \frac{1}{16\pi G} \int_{\mathcal{M}} d^{d+1}x \, \sqrt{-g} \, (R - 2\Lambda + \mathcal{L}_M(\Psi)) && (2.4.3) \\ I_{\partial B^{\pm}} &= -\frac{1}{8\pi G} \int_{\partial \mathcal{M}^{\pm}} d^d x \, \sqrt{-\gamma} \Theta && (2.4.4) \end{aligned}$$

The dS case has a positive cosmological constant, $\Lambda = +\frac{d(d-1)}{2\ell^2}$. Θ is again the trace of the extrinsic curvature, and $\mathcal{L}_M$ the matter Lagrangian. Note that $\partial B^{\pm}$ is future/past infinity, so that $\int_{\partial B^{\pm}} = \int_{\partial B^-}^{\partial B^+}$ is the integral over a future boundary minus the integral over a past boundary, with appropriate boundary metrics and extrinsic curvatures.

The presence of the boundary action can be understood exactly as in the AdS case, through the need to cancel the extra term arising from the variation of the bulk action when one only requires $\delta g_{\alpha\beta} = 0$. It can also again be understood through the path-integral viewpoint by considering the correlation between the initial hypersurface $[g_1, \Phi_1, H_1]$ and some intermediate hypersurface $[g_i, \Phi_i, H_i]$, and also between this intermediate hypersurface and the final hypersurface $[g_2, \Phi_2, H_2]$. The correlation between the initial and final states should be found by the sum of products of correlations between all possible intermediate histories,

$$\langle g_2, \Phi_2, H_2|\, g_1, \Phi_1, H_1\rangle = \sum_i \langle g_2, \Phi_2, H_2|\, g_i, \Phi_i, H_i\rangle \langle g_i, \Phi_i, H_i|\, g_1, \Phi_1, H_1\rangle \tag{2.4.5}$$

This will again only hold iff (2.3.6) is true. Now, of course, g_{ij} is the metric between histories H_i, H_j, and the metric g_{12} is of course the full metric of the region between H_1, H_2. In general, g_{1i} and g_{i2} will have different normal spacelike derivatives, they will yield delta-function contributions to the Ricci tensor proportional to the difference between the extrinsic curvatures of the histories H_i in the metrics g_{1i}, g_{i2}. The boundary term in (2.4.2) is what compensates for this.

2.4.2 Thermodynamics

I will again link the arguments for the path-integral approach, now in asymptotically de Sitter spacetimes, with the usual thermodynamic arguments. Here however, these arguments require a greater degree of care because the action is in general negative definite near past and future infinity (outside of a cosmological horizon). The natural strategy would appear to be to analytically continue the coordinate orthogonal to the histories $[g_1, \Phi_1, H_1]$ and $[g_2, \Phi_2, H_2]$ to complex values by rotating the axis normal to the histories H_1, H_2 anticlockwise by $\frac{\pi}{2}$ radians into the complex plane. The action becomes pure imaginary and so $\exp(\mathrm{i}I\,[g,\Phi]) \longrightarrow \exp(+\hat{I}\,[g,\Phi])$, yielding a convergent path integral

$$Z' = \int e^{+\hat{I}} \tag{2.4.6}$$

since $\hat{I} < 0$. Furthermore, since we want a converging partition function, we must change (2.3.11) to

$$M = +\frac{\partial}{\partial\beta} \ln\left\{\sum_r e^{+\beta E_r}\right\} = +\frac{\partial}{\partial\beta}\ln Z' \tag{2.4.7}$$

Now comparing (2.4.7) with (2.3.12b) (since (2.3.12a,2.3.12b) won't change) one obtains

$$+\beta W = \ln\left\{e^{+\beta E_r}\right\} = \ln Z' \tag{2.4.8}$$

In the semi-classical approximation this will lead to $\ln Z' = +I_{cl}$. Substituting this and (2.3.12a) into (2.4.8),

$$\begin{aligned} \beta\,(M - TS) &= +I_{cl} \\ \beta M - S &= I_{cl} \\ S &= \beta M - I_{cl} \end{aligned} \tag{2.4.9}$$

As before, the presumed physical interpretation of the results is then obtained by rotation back to a Lorentzian signature at the end of the calculation. However there is an ambiguity here that is not present in the asymptotically flat and AdS cases. This occurs because outside the horizon, near past and future infinity, the signature of any asymptotically dS spacetime becomes $(+,-,+,+)$, and so the spacelike boundary tubes naturally have Euclidean signature. This leads to two possible approaches in evaluating physical quantities.

First, notice that if one does perform the Wick rotation in order to carry out the calculation of the action and entropy, the signature of the metric becomes $(-,-,+,+,\ldots)$ (in $(d+1)$ dimensions). Thus, the metric is not the Euclidean metric one would expect from the AdS case. However, the argument for proceeding in this manner would be that the purpose of performing the Wick rotation is not to attain a Euclidean signature, but rather its purpose is to achieve a convergent path integral and partition function. To ensure the absence of conical singularities, one also again identifies the period β with the temperature T. The presumed physical interpretation here, as before, would be to rotate back into the real plane at the end of the calculation. In [20] this was referred to as the C-approach, and I will continue to use this description.

In the second approach, one proceeds by noting that at future infinity, the Killing vector $\partial/\partial t = \partial_t$ is asymptotically spacelike. It was suggested in [18] that this means the purpose of the rotation into the complex plane is merely a mathematical device used in order to establish the Gibbs-Duhem relation (2.4.9), and is not actually required for actual calculation. In this approach, then, one simply calculates all quantities relative to the original signature of the metric at future infinity, and imposing periodicity in t consistent with regularity at the cosmological horizon (given by the surface gravity of the horizon of the $(+,-)$ section. This approach was referred to as the R-approach in [20].

A recent paper by Mann and Stelea [44] has explored the relationship between the C-approach and the R-approach as it applies to the Taub-NUT-dS spacetime in further depth. Since a full discussion of their results is easier once the Taub-NUT spacetime has been introduced, I will postpone this discussion until chapter 3, and the thermodynamic consequences of their results until chapter 4.

For now, suffice it to say that one can show that the C-approach results can be shown to be the analytic continuation of the R-approach results, and vice versa. Also, as will be shown, the NUT solution results found in the ds-C-approach, originally thought to be valid as it satisfies the first law of thermodynamics, is also shown not to be a solution as the root $\tau = N$ is not the largest root of the metric function. It also turns out that the two lower branches of the Bolt solutions found in [20] (from the C and R-approach) are not solutions either, as they are also not the largest roots of their respective metric functions. This too will be explored in chapters 3, 4.

2.5 Review of (A)dS/CFT Correspondence

2.5.1 General Review (AdS/CFT)

Since fully two-thirds of the results presented in this thesis involve the direct application of the counterterms that arise from the AdS/CFT or dS/CFT correspondences, it is pertinent to review the overall theory that gives rise to these counterterms. Hence, this section is intended to give a brief overview of the AdS/CFT and dS/CFT correspondences. For a full review, consult for example [45, 46, 47, 48], and references therein. Note that, since the dS/CFT is largely derived from the AdS/CFT, I will only discuss the latter here, and point out differences in section 2.5.3. The counterterms themselves, which will be used to derive the results in chapter 4, will be reviewed in sections 2.5.2, 2.5.3.

The action and energy-momentum of the spacetime, though of great importance in gravity, are very difficult to define and compute. One of the major stumbling blocks is that the action (2.3.2) diverges. A standard remedy for this has been the "background subtraction" method, for example the Noether approach. By bounding the spacetime by a surface and subtracting some reference spacetime with similar infinities in its action and possessing the same boundary geometry, one can in some cases successfully compute the action, which will be finite as the boundary is taken to infinity. The energy-momentum tensor can also be computed, by varying the action with respect to the boundary metric [49].

There are of course a number of drawbacks to this procedure. The first is that some spacetimes don't have suitable background metrics. Even for metrics with some suitable background, the asymptotic boundary geometries must be matched in order to get finite surface contributions [13, 50, 51, 52]. An alternative to this method was proposed by [53], and in the case of Anti-de Sitter spacetimes a completely iterative procedure, to be reviewed in the next section, was given by [48]. This new, "counterterm" procedure depends on the use of the proposed AdS/CFT correspondence.

The AdS/CFT correspondence, first proposed by Maldacena [45] and later enunciated more completely by Witten [46], is a holographic theory suggesting that the thermodynamics of quantum gravity in some spacetime of dimension $(d+1)$ can be successfully modelled using the thermodynamics of the corresponding field theory on the d dimensional boundary of the spacetime for large N, where the field theory will be described by a gauge theory with, for example, an $SU(N)$ gauge group. The behaviour of the fields at the boundary *uniquely* specifies the fields as they propagate in the bulk of the AdS spacetime. The correspondence is precisely formulated by the use of the partition functions of the bulk and boundary theories,

$$\begin{aligned} Z_{AdS}(\phi_{0,i}) &= \int_{[\gamma,\Psi_0]} D[g]D[\Psi]e^{-I(\phi_i)} = \left\langle \exp\left\{\int_{\partial M_d} d^d x\ \phi_{0,i}\mathcal{O}^i\right\}\right\rangle \\ &= Z_{CFT}(\phi_{0,i}) \end{aligned} \tag{2.5.1}$$

where $\phi_{0,i}$ represent, on the gravity side, the boundary values of the fields ϕ_i that propagate in the bulk. In the field theory, the $\phi_{0,i}$ are the external source currents that couple to the various CFT operators. In (2.5.1), $I(\phi_i)$ is the classical gravitational action, and $\mathcal{O}$ is a quasi-primary conformal operator defined on the boundary. The beauty of this conjecture is that it suggests the use of a counterterm action to offset the infinities in the Einstein-Hilbert and Gibbons-Hawking action (2.3.2). The counterterms, to be reviewed next, depend only on curvature invariants that are functionals of the intrinsic boundary geometry, thus leaving the equations of motion from varying (2.3.2) with respect to the bulk metric unchanged.

The conjecture has been verified for several important cases, providing an expectation that quantum gravity (at least in an AdS case) can be obtained by studying the holographic dual of the CFT.

2.5.2 AdS/CFT - Counterterms

The proposal from [53] suggests that in general, as mentioned above, for a manifold that has a boundary one can modify the action without altering the equations of motion by adding a coordinate invariant functional that is dependent only on the intrinsic boundary geometry - this of course could be done with or without the AdS/CFT conjecture. This counterterm cancels the divergences in the action (2.3.2) in a manner analogous to the counterterms used in field theory to cancel the infinities of the bare coupling constants.

The first few terms, usable for lower dimensional metrics, were found by [53] (see also [47]). A counterterm action usable in asymptotically flat cases was also suggested in [14]. However, there exists an iterative procedure, derived by Kraus, Larsen and Siebelink [48], that will provide the counterterms for any arbitrary dimension, at least for asymptotically AdS metrics (indeed, note the results in this section are from [48]). The procedure can provide the required counterterm action to fully cancel the divergences in (2.3.2). Thus, one alters this action by adding a counterterm action

$$I = I_B + I_{\partial B} + I_{ct} \tag{2.5.2}$$

Varying the original action (2.3.2) with respect to the boundary gives the energy-momentum tensor for the spacetime

$$\Pi_{\alpha\beta} = \frac{\delta S}{\delta \gamma^{\alpha\beta}} = \Theta_{\alpha\beta} - \gamma_{\alpha\beta}\Theta \tag{2.5.3}$$

which also diverges. Recall that the boundary action $I_{\partial B}$ was added by Gibbons and Hawking [3] to the action in order to achieve well defined equations of motion. The counterterm action will ruin this unless it depends only on the boundary metric γ. Further, since the counterterm action is taken as a series expansion in the radius of AdS space, dimensional analysis shows that the series can be truncated - only terms $< d/2$ contribute to the divergent part of the action. The divergent part of the stress tensor is

$$\tilde{\Pi}_{\alpha\beta} = \sum_{n=0}^{d/2} \tilde{\Pi}^{(n)}_{\alpha\beta} \quad ; \quad \tilde{\Pi}^{(n)}_{\alpha\beta} \propto \ell^{2n-1} \tag{2.5.4}$$

The procedure depends on the Gauss-Codazzi equations. From [33, 48], after using (2.5.3), the constraint equation reads

$$\frac{1}{d-1}\tilde{\Pi}^2 - \tilde{\Pi}_{\alpha\beta}\tilde{\Pi}^{\alpha\beta} = \frac{d(d-1)}{\ell^2} + R \tag{2.5.5}$$

where one always considers the bulk equations of motion. The counterterm energy-momentum tensor must be derived from the counterterm action, and so

$$\tilde{\Pi}_{\alpha\beta} = \frac{2}{\sqrt{-\gamma}} \frac{\delta}{\delta\gamma^{\alpha\beta}} \int d^d x \sqrt{-\gamma} \tilde{\mathcal{L}} \tag{2.5.6}$$

Under a Weyl rescaling $\delta_W \gamma_{\alpha\beta} = \sigma\gamma_{\alpha\beta}$, [48] show that (after some algebra)

$$\tilde{\mathcal{L}}^{(n)} = \frac{\tilde{\Pi}^{(n)}}{(d-2n)} \tag{2.5.7}$$

up to a total derivative.

Now, since the leading order term in (2.5.4) scales as ℓ^{-1}, the curvature term in (2.5.5) can be neglected, so that $\tilde{\Pi}^{(0)}_{\alpha\beta}$ is proportional to the metric,

$$\tilde{\Pi}^{(0)}_{\alpha\beta} = -\frac{(d-1)}{\ell}\gamma_{\alpha\beta} \tag{2.5.8}$$

Higher order terms are thus calculable through the iterative procedure from [48], restated here:

Step 1: Insert known terms into (2.5.5), giving a linear equation with the trace $\tilde{\Pi}^{(n)}$ being the only unknown.

Step 2: Integrate (2.5.6) to find $\mathcal{L}^{(n)}$ (i.e. use (2.5.7)).

Step 3: Take the functional derivative of $\mathcal{L}^{(n)}$ with respect to $\gamma_{\alpha\beta}$, to obtain $\tilde{\Pi}^{(n)}_{\alpha\beta}$ from (2.5.6).

An explicit computation of the first few terms in the counterterm action is done in [48], and they are

$$\begin{aligned} I_{ct} \;=\; & -\frac{1}{8\pi G}\int d^d x\,\sqrt{-\gamma}\Bigg\{-\frac{d-1}{\ell} - \frac{\ell\widetilde{\Theta}\,(d-3)}{2(d-2)}\mathcal{R} \\ & -\frac{\ell^3\widetilde{\Theta}\,(d-5)}{2(d-2)^2(d-4)}\left(\mathcal{R}_{ab}\mathcal{R}^{ab} - \frac{d}{4(d-1)}\mathcal{R}^2\right) \\ & +\frac{\ell^5\widetilde{\Theta}\,(d-7)}{(d-2)^3(d-4)(d-6)}\left(\frac{3d+2}{4(d-1)}\mathcal{R}\mathcal{R}^{ab}\mathcal{R}_{ab} - \frac{d(d+2)}{16(d-1)^2}\mathcal{R}^3\right. \\ & \left. -2\mathcal{R}^{ab}\mathcal{R}^{cd}\mathcal{R}_{acbd} + -\frac{d}{4(d-1)}\nabla_a\mathcal{R}\nabla^a\mathcal{R} + \nabla^c\mathcal{R}^{ab}\nabla_c\mathcal{R}_{ab}\right) + \ldots\Bigg\} \end{aligned} \tag{2.5.9}$$

Note in I_{ct} that $\widetilde{\Theta}(d)$ is a step function, equal to zero unless $d > 0$, where it is equal to one, and is not the extrinsic curvature in (2.3.4).

The conserved charges are found from the full stress tensor. The counterterms in the stress tensor can be found step by step from the above procedure along with the action, or by varying the action. Taking the variation of the action (2.5.2), and carefully keeping account of all boundary terms, the conserved charge is found to be

$$\mathfrak{Q}_\xi = \oint_\Sigma d^{d-1}S^\alpha \; \xi^\beta T^{eff}_{\alpha\beta} \tag{2.5.10}$$

(where $T^{eff}_{\alpha\beta} \equiv \tilde{\Pi}_{\alpha\beta}$). This is associated with a closed surface Σ (with unit normal n^a), provided the boundary geometry has an isometry generated by a Killing vector ξ^α. $T^{eff}_{\alpha\beta}$ is found by varying (2.3.2) at the boundary with respect to $\gamma^{\alpha\beta}$, and $\mathfrak{Q}_\xi$ is conserved between closed surfaces Σ distinguished by some foliation parameter τ. If the Killing vector is $\xi = \partial_t$, then $\mathfrak{Q}$ is the conserved mass/energy $\mathfrak{M}$, and if $\xi_\alpha = \partial_{\phi_i}$, then it is the conserved angular momentum $\mathfrak{J}$ in the ϕ_i direction, provided ϕ_i is periodic, and associated with Σ. Details of the formulation can be found in [48, 53, 54, 49, 55], with the first few terms of the expansion of $T^{eff}_{\alpha\beta}$ given by

$$T^{eff}_{\alpha\beta} = \Theta_{\alpha\beta} - \Theta\gamma_{\alpha\beta} + \frac{d-1}{\ell}\gamma_{\alpha\beta} + \frac{\ell}{d-2}\left(\mathcal{R}_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}\mathcal{R}\right) + \ldots \tag{2.5.11}$$

An expansion usable up to eight dimensions can be found in [56]. $\Theta_{\alpha\beta}$ is again the extrinsic curvature of the boundary, and $\gamma_{\alpha\beta}$ and $\mathcal{R}_{\alpha\beta}$ are the metric and curvature of the boundary, respectively.

The Gibbs-Duhem relation is given by (2.3.14), and can be used to find the now finite entropy associated with the metric in question, also now intrinsic to this metric.

It should be noted here that later I will show only the first term in (2.5.9) and the first three terms in (2.5.11) are needed for the calculations performed in this thesis.

2.5.3 dS/CFT - Counterterms

Asymptotically de Sitter spacetimes do not have a spatial infinity in the way that asymptotically AdS or flat spacetimes do. Also, although inside any cosmological horizon in a dS spacetime there is a timelike Killing vector, this vector becomes spacelike outside the horizon. For these reasons, the definition of a conserved charge is not well defined in asymptotically dS spacetimes, and hence it is unclear what the physical meaning of energy is outside the horizon.

There is a dS/CFT proposal that has been derived in analogy with the AdS/CFT correspondence described above. The method, analogous to the Brown-York prescription for asymptotically AdS spacetimes [53, 54, 49], yields suggestive information regarding the dual Euclidean CFT of asymptotically dS spacetimes.

Calculations for conserved charges for pure and asymptotically dS spacetimes have been carried out inside the cosmological horizon, where the Killing vector is timelike [3]. However, outside the horizon, the spacetime boundaries at early and late time infinity ($\mathcal{I}^{\pm}$) are Euclidean surfaces. One can adapt the coordinates [21] so that the "radial" normal $n^{a\pm}$ is proportional to the (now spacelike) boundary Killing vector, and hence use the notion of a conserved charge defined on the spacetime boundary at late (or early) time infinity, as in (2.5.14) below. These formulae are computed on surfaces of fixed time, and then time is sent to infinity so that it approaches the spacetime boundaries $\mathcal{I}^{\pm}$.

The notion of conservation of charges in de Sitter spacetimes outside the cosmological horizon, where the Killing vector to be used is now spacelike, needs clarification. In AdS spacetimes, of course, the Killing vector is timelike, and so the definition of a conserved charge $\mathfrak{Q}_{\xi}$ (for example the conserved mass/energy) is the usual one in that the mass is conserved as one moves in time. In de Sitter spacetime, however, with the Killing vector being spacelike, there is no longer a conservation with motion in time. There is now conservation with respect to position, i.e. multiple observers with a spacelike separation between them should now measure the same mass, for example, at the same time, and should also measure the same change in mass with changes in time.

By carrying out the calculation of the conserved charges using the proposed dS/CFT conjecture, a conserved charge that can thus be interpreted as the asymptotically dS mass can be calculated. Sample calculations performed by Balasubramanian *et. al.* [21] led them to what I have called the "maximal mass conjecture": *Any asymptotically dS spacetime with mass greater than that of dS has a cosmological singularity.* This conjecture has no exact proof, and I will show later that, at least as stated, there are counter-examples.

Along with the above difficulties, there is also the question of what thermodynamics means at future or past infinity. The approach of this thesis, to be shown below, is that the formalism allows one to define an action and a conserved mass, and hence the last element needed in order to get an entropy is the temperature. Calculating the temperature by analogy with the formalism from the AdS/CFT conjecture also allows one to calculate a quantity resembling the temperature in the dS case. As shown above in section 2.4, one can use this temperature in the C-approach, along with thermodynamic arguments, to calculate the Gibbs-Duhem relation (2.4.9) in the dS/CFT conjecture formalism. Hence, although it still needs to be proven that such quantities are valid in adS spacetimes, since the formalism allows the quantities to be calculated, I will take thermodynamics at future/past infinity in dS spacetimes to be valid throughout this thesis. As mentioned, doing so allows the calculation of quantities that resemble their thermodynamic counterparts from AdS spacetimes. Such calculations, even if they should prove to be something other than true thermodynamic information, will provide interesting insights into the dS/CFT conjecture.

The iterative procedure from [48] was shown by Ghezelbash and Mann [18] to apply for asymptotically dS metrics also. Thus, one alters the total action (2.4.2) to include a counterterm action

$$I = I_B + I_{\partial B} + I_{ct} \tag{2.5.12}$$

Except for differences in sign that arise due to going from asymptotically AdS to asymptotically dS spacetimes, the procedure for finding I_{ct} is as outlined above. The results of this procedure give the counterterm action as

$$\begin{aligned} I_{ct} \;=\; & -\frac{1}{8\pi G}\int d^d x\,\sqrt{-\gamma}\Bigg\{-\frac{d-1}{\ell}+\frac{\ell\Theta\,(d-3)}{2(d-2)}\mathcal{R} \\ & -\frac{\ell^3\Theta\,(d-5)}{2(d-2)^2(d-4)}\left(\mathcal{R}_{ab}\mathcal{R}^{ab}-\frac{d}{4(d-1)}\mathcal{R}^2\right) \\ & -\frac{\ell^5\Theta\,(d-7)}{(d-2)^3(d-4)(d-6)}\left(\frac{3d+2}{4(d-1)}\mathcal{R}\mathcal{R}^{ab}\mathcal{R}_{ab}-\frac{d(d+2)}{16(d-1)^2}\mathcal{R}^3\right. \\ & \left.-2\mathcal{R}^{ab}\mathcal{R}^{cd}\mathcal{R}_{acbd}-\frac{d}{4(d-1)}\nabla_a\mathcal{R}\nabla^a\mathcal{R}+\nabla^c\mathcal{R}^{ab}\nabla_c\mathcal{R}_{ab}\right)+\ldots\Bigg\} \end{aligned} \tag{2.5.13}$$

Taking the variation of the action (2.4.2), and carefully keeping account of all boundary terms, the conserved charge can be found to be

$$\mathfrak{Q}_{\xi}^{\pm} = \oint_{\Sigma^{\pm}} d^{d-1}\varphi^{\pm} \sqrt{\gamma^{\pm}} n^{a\pm} \xi^{b\pm} T_{ab}^{\pm} \tag{2.5.14}$$

This equation has the same interpretation as (2.5.10), where the $\pm$ again means take the calculation at the future boundary minus the calculation at the past boundary. The first few terms in the stress-tensor, found by varying the full action (2.4.2) with respect to the boundary metric, is given by

$$T_{ab} = \Theta_{ab} - \gamma_{ab}\Theta - \frac{d-1}{\ell}\gamma_{ab} + \frac{\ell}{d-2}\left(\mathcal{R}_{ab} - \frac{1}{2}\gamma_{ab}\mathcal{R}\right) + \ldots \tag{2.5.15}$$

with Θ_{ab} the extrinsic curvature on the boundary, and all other quantities again calculated on the boundary.

The Gibbs-Duhem relation (2.4.9) can be used to calculate the finite entropy intrinsic to the spacetime in question.

It is to be noted again that later, as in the AdS case, I will show only the first term in (2.5.13) and the first three terms in (2.5.15) will be needed to calculate the action and entropy of the Taub-NUT-dS metrics.

Chapter 3

Introduction to Taub-NUT Spacetimes

Taub-NUT spacetimes were discovered by Newman, Unti and Tamburino as a generalization of the Schwarzschild metric in 1963 [12]. Misner [57] then did an analysis of the Taub-NUT metric. More recently, Awad and Chamblin [17] demonstrated how the Taub-NUT-AdS metric can be generalized to $d+1$ (even) dimensions, which of course also easily demonstrates generalizations of pure Taub-NUT metrics to higher dimensions (by setting the cosmological constant to zero). Since this thesis is based on the Taub-NUT metric, I'll review the findings of those papers here. The Taub-NUT spacetime contains a gravitational analogue of the Dirac string from electromagnetic theory, called a Misner string. As such, it is also a good idea to briefly review the idea of Dirac Strings.

The outline of this chapter is therefore as follows. In section 3.1, I will review the idea of the Dirac string that comes about when one attempts to include a magnetic monopole. Then, in section 3.2, I will review the analogous Misner strings that arise in gravitational theory when one discusses a metric containing a NUT charge. Following this, a more overall review of the Taub-NUT spacetime itself, in $(3+1)$ dimensions, will be presented in 3.3. This will include a discussion of the NUT and the Bolt solutions that one can obtain, depending on the fixed point set of ∂_t. Next, the extension of the Taub-NUT spacetime to include a negative or positive cosmological constant, in $(3+1)$ dimensions, will be presented in sections 3.4, 3.5. Finally, section 3.6 will discuss the Taub-NUT spacetime in general $(d+1)$ dimensions, for the AdS and dS cases.

3.1 Dirac Strings

Dirac strings arise from Maxwell's equations when one attempts to consider the idea of a magnetic charge. Dirac first noticed in 1931 [58] that the Maxwell equation $\nabla \cdot \mathbf{B} = 0$ denies the existence of a magnetic charge. His solution to this (see also [60]) was to introduce the idea of a magnetic monopole[1], which is a point magnetic charge. Consider a magnetic monopole of strength g at the origin[2],

$$\mathbf{B} = \frac{g\mathbf{r}}{r^3} = -g\nabla\left(\frac{1}{r}\right) \tag{3.1.1}$$

Thus, recalling that $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\mathbf{r})$, this gives

$$\nabla \cdot \mathbf{B} = 4\pi g\delta^3(\mathbf{r}) \tag{3.1.2}$$

which corresponds to a magnetic monopole. The total flux through a sphere surrounding the origin will then be given by

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_V \nabla\cdot\mathbf{B}\; dV = 4\pi g \tag{3.1.3}$$

Consider now an electric particle with charge e moving through the field of the monopole. The wavefunction will change in the presence of an electromagnetic field to

$$\psi \to \psi \exp\left\{-\frac{\mathrm{i}e}{\hbar c}\mathbf{A}\cdot\mathbf{r}\right\} \tag{3.1.4}$$

or the phase α changes by $\alpha \to \alpha - \frac{e}{\hbar c}\mathbf{A}\cdot\mathbf{r}$. The total change in phase, for a closed path at fixed r, θ, and $0 \leq \phi \leq 2\pi$, is

$$\begin{aligned}\Delta\alpha &= \frac{e}{\hbar c}\oint \mathbf{A}\cdot d\mathbf{l} = \frac{e}{\hbar c}\int \nabla\times\mathbf{A}\cdot d\mathbf{S} = \frac{e}{\hbar c}\int \mathbf{B}\cdot d\mathbf{S} \\ &= \frac{e}{\hbar c}\Phi(r,\theta) \end{aligned} \tag{3.1.5}$$

If we take the flux through a cap at the top of a sphere, the situation will be as depicted in figure 3.1.1. Note that as $\theta \to 0$, the flux through the cap will approach zero. As the loop is passed over the sphere, the cap encloses more

[1] Sometimes also called the Dirac monopole

[2] Derivation from [59], pg. 402-405 or [61], pg. 14-17.

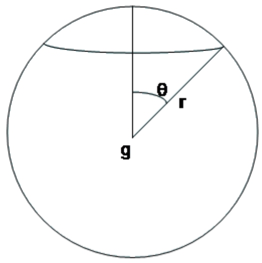

Figure 3.1.1: Magnetic Monopole Flux

of the sphere, and at $\theta \to \pi$, the flux should be (3.1.3). However, as $\theta \to \pi$, the cap has in fact shrunk back down to a point. Since the flux at π, $\Phi(r, \pi)$, is required to be finite, $\mathbf{A}$ must be singular at $\theta = \pi$. This argument holds for all spheres, of all possible radii, and hence it follows that $\mathbf{A}$ is singular along the entire z-axis. This is known as the *Dirac String*. Of course through a suitable choice of coordinates, the Dirac string can be chosen along any direction, and need not be straight, though it must be continuous.

In the current example, the wave function vanishing along the negative z-axis means its phase is indeterminate there. Thus, by (3.1.5), it is not necessary that as $\theta \to \pi$, $\Delta\alpha \to 0$. However, we must have $\Delta\alpha = 2\pi n$ for some integer n if ψ is to be single valued. So, from (3.1.5) and $\Phi(r, \pi) = 4\pi g$, we get,

$$\begin{aligned} 2\pi n &= \frac{e}{\hbar c} 4\pi g \\ eg &= \frac{n\hbar c}{2} \end{aligned} \tag{3.1.6}$$

This is the *Dirac quantization condition* for the magnetic monopole. It implies that, should a magnetic monopole exist, all electric charges are then quantized.

Now consider a vector potential A_μ that gives rise to the magnetic field

(3.1.1). From the argument above, it must obviously be singular, and can be constructed by considering the magnetic monopole to be the end-point of a string of magnetic dipoles going off to infinity. This will give a vector potential

$$A_x^N = -\frac{gy}{r(r+z)} \quad , \quad A_y^N = \frac{gx}{r(r+z)} \quad , \quad A_z^N = 0 \tag{3.1.7}$$

or equivalently in spherical coordinates[3]

$$A_r^N = A_\theta^N = 0 \quad , \quad A_\phi^N = \frac{g(1-\cos(\theta))}{r\sin(\theta)} \tag{3.1.8}$$

This is clearly singular along the $r = -z$ axis, reflecting a poor choice of coordinate system, (note $\nabla \times \mathbf{A} = g\mathbf{r}/r^3 + 4\pi g\delta(x)\delta(y)\theta(-z)\hat{r}$), and this singularity along the z axis is again the Dirac string. One can remove this singularity by dividing the sphere surrounding the monopole into two overlapping regions R^N and R^S, where R^N excludes the $-z$-axis (the "S" pole) and R^S excludes the z-axis (the "N" pole), as shown in figure 3.1.2. If we now let (3.1.8) hold for the region R^N, and define

$$A_r^S = A_\theta^S = 0 \quad , \quad A_\phi^S = -\frac{g(1+\cos(\theta))}{r\sin(\theta)} \tag{3.1.9}$$

to hold for a Dirac string at $\theta = 0$ ($+z$-axis), then $\mathbf{A}^N$ and $\mathbf{A}^S$ are both finite in their own regions. Where the two regions overlap, then the vector potentials are related by a gauge transformation ($\hbar = c = 1$), $\mathbf{A}^N - \mathbf{A}^S = \nabla\Lambda$, or

$$\mathbf{A}^N - \mathbf{A}^S = \frac{2g}{r\sin(\theta)}\hat{e}_\phi = \nabla(2g\phi) \tag{3.1.10}$$

where the gauge transformation connecting $\mathbf{A}^N$ and $\mathbf{A}^S$ is $\Lambda = 2g\phi$ (from the definition of the gradient). (The transformation is performed only at the boundary, $\theta = \frac{\pi}{2}$, and so the singularities in Λ at $\phi = 0, 2\pi$ don't matter.) This same trick of dividing the region of interest into two hemispheres will also be very useful in Taub-NUT spacetimes, which posses similar singularities.

[3](3.1.8) can be found from (3.1.7) by using $x = r\sin(\theta)\cos(\phi)$, $y = r\sin(\theta)\sin(\phi)$ and $z = r\cos(\theta)$, along with the unit vector transformations $\hat{x} = \sin(\theta)\cos(\phi)\hat{r} + \cos(\theta)\cos(\phi)\hat{\theta} - \sin(\theta)\hat{\phi}$, $\hat{y} = \sin(\theta)\sin(\phi)\hat{r} + \cos(\theta)\sin(\phi)\hat{\theta} + \cos(\theta)\hat{\phi}$, $\hat{z} = \cos(\theta)\hat{r} - \sin(\theta)\hat{\theta}$, and use of the double angle trig identities.

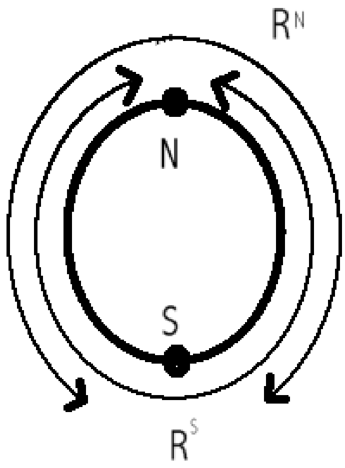

Figure 3.1.2: Diagram of region around magnetic monopole, showing division of sphere into a northern hemisphere "N" region that excludes $\theta = \pi$ and a southern hemisphere "S" region excluding $\theta = 0$.

3.2 Misner Strings

The general form of the Taub-NUT spacetime in four dimensions, with zero cosmological constant, and with a two-sphere S^2 as a base space, is given by equations (1.4.6), (1.4.7), and which I repeat here:

$$ds^2 = -F^2(r)\left[dt + 4n\sin^2\left(\frac{\theta}{2}\right)d\phi\right]^2 + \frac{dr^2}{F^2(r)} + (r^2+n^2)\left(d\theta^2 + \sin^2(\theta)d\phi^2\right) \tag{3.2.1}$$

where

$$F^2(r) = \frac{r^2 - n^2 - 2mr}{r^2+n^2} \tag{3.2.2}$$

The Riemann components are easily calculated (see appendix B.1, equations (B.1.39) for the tangent-space forms, for example), and from the scalar invariants, it can be shown that the metric has no curvature singularities for $n \neq 0$. Also from (B.1.39) the metric is seen to be asymptotically locally flat in the sense that the Riemann tensor vanishes as $r \to \infty$.

Though there are no curvature singularities, as mentioned the Taub-NUT spacetime contains a spurious singularity analogous to the Dirac string discussed above, commonly called a Misner string. These singularities are not associated with $F^2(r) = 0$, and are found from the basis vectors in the $e^{\hat{t}}$ component. We can see these singularities by manipulating (B.1.14) to give an expression for ∇t (this was first shown in [57]),

$$\begin{aligned} dt &= \frac{e^{\hat{t}}}{F(r)} - \frac{2n}{(r^2+n^2)^{1/2}} \tan\left(\frac{\theta}{2}\right) e^{\hat{\phi}} & (3.2.3) \\ \therefore -(\nabla t)^2 &= \frac{1}{F^2(r)} - \frac{(2n)^2}{r^2+n^2} \tan^2\left(\frac{\theta}{2}\right) & (3.2.4) \end{aligned}$$

and thus, though $(\nabla t)^2$ is regular as $\theta \to 0$, it diverges as $\theta \to \pi$.

These singularities are the same on all r =constant hypersurfaces, and can be removed by assigning to these hypersurfaces an S^3 topology. Thus, taking any r =const. hypersurface, we can study

$$e^{\hat{t}} = dt_N + 4n \sin^2\left(\frac{\theta}{2}\right) d\phi \qquad (3.2.5)$$

Note that t_N is regular as $\theta \to 0$ (the "north" pole), but that it diverges as $\theta \to \pi$, and hence I've changed $t = t_N$ in (B.1.14) to give (3.2.5) (on constant r hypersurfaces). We can introduce a new time coordinate $t_N = t_S - 4n\phi$ so that

$$e^{\hat{t}} = dt_S - 4n \cos^2\left(\frac{\theta}{2}\right) d\phi \qquad (3.2.6)$$

and through similar arguments t_S is singular as $\theta \to 0$, but regular as $\theta \to \pi$. One can now use t_N on a coordinate patch $0 \leq \theta < \pi$, and t_S on a coordinate patch $0 < \theta \leq \pi$, and combining the two patches, one gets a manifold where the metric is non-singular everywhere. However, recall that ϕ is only a regular function if it is periodic, $\phi \equiv \phi + 2\pi$, and this means that the time coordinate must also be periodic,

$$t_N \equiv t_N + 8\pi n \qquad (3.2.7)$$

and similarly for t_S. With this requirement, the manifold is also compact.

3.3 Taub-NUT in $(3+1)$ dimensions

From now on, I will work with another form of the metric (3.2.1),

$$\begin{aligned} ds^2 &= -f(r)\left[dt + 2n\cos(\theta)\, d\phi\right]^2 + \frac{dr^2}{f(r)} + (r^2+n^2) d\Omega_2^2 & (3.3.1) \\ f(r) &\equiv F^2(r) = \frac{r^2 - n^2 - 2mr}{r^2+n^2} & (3.3.2) \end{aligned}$$

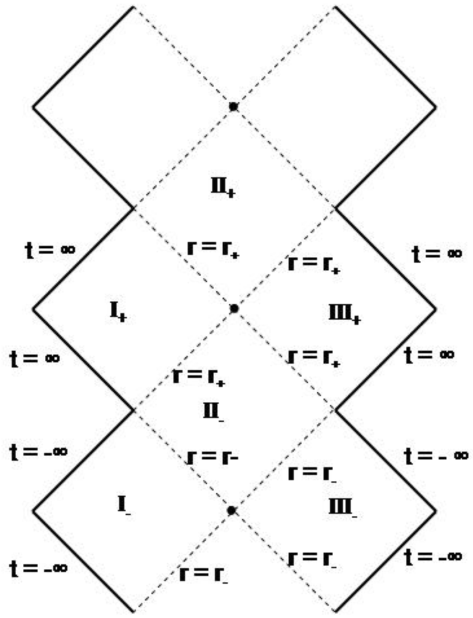

Figure 3.3.1: Penrose diagram of Taub-NUT spacetime.

easily found from (3.2.1). The Euclidean section is found by Wick rotating the time and the NUT charge ($t \to \mathrm{i}T,\ n \to \mathrm{i}N$), to give,

$$ds^2 = f(r)\left[dT + 2N\cos(\theta)\, d\phi\right]^2 + \frac{dr^2}{f(r)} + (r^2 - N^2)\left(d\theta^2 + \sin^2(\theta)d\phi^2\right) \tag{3.3.3}$$

and

$$f(r) = \frac{r^2 + N^2 - 2mr}{r^2 - N^2} \tag{3.3.4}$$

The spacetime (3.3.1) has singularities at $r = r_{\pm}$,

$$r_{\pm} = m \pm \sqrt{m^2 + n^2} \tag{3.3.5}$$

where $f(r) = 0$. The manifold defined by t and $r \in (r_-, r_+)$ ($\theta,\ \phi$ constant hypersurfaces) can be extended (see chapter 5 of [11]) to obtain a Penrose diagram as in figure 3.3.1. There are closed timelike curves (CTC's) in the regions $r < r_-$ and $r > r_+$. The region $r_- < r < r_+$ is compact, yet contains timelike and null geodesics that remain within this region and are incomplete, leading to quasi-regular singularities.

Quasi-regular singularities (see [62, 63]) are points of incomplete and inextensible geodesics that spiral infinitely around a topologically closed spatial dimension. These are the weakest form of singularity, in that the Riemann tensor is completely finite in all parallelly propagated orthonormal frames. No observer near a quasi-regular singularity, nor one who falls in to the singularity, feels unbounded tidal forces.

Taub-NUT spacetimes can be further separated into two "sub" spaces, depending on the fixed point set of ∂_t, and after Euclideanizing the metric. These solutions are the NUT and Bolt solutions, described below.

3.3.1 NUT solution ($(3+1)$ dimensions)

The metric (3.3.3) will describe a NUT solution if the fixed point set of ∂_t is zero-dimensional; i.e. the extra dimensions collapse to zero size. This condition occurs when $f(r)$ is fixed so that $f(N) = 0$. Solving for the mass in this case leads to

$$m_{n,flat} = N \tag{3.3.6}$$

giving, for the NUT,

$$f(r) = \frac{r-N}{r+N} \tag{3.3.7}$$

This solution is an example of a Euclidean, self-dual solution (i.e. can be shown to satisfy $\widetilde{R}_{\alpha\beta\mu\nu} = \varepsilon_{\alpha\beta}{}^{\gamma\delta} R_{\gamma\delta\mu\nu}$ where $\varepsilon = \sqrt{g}\epsilon$ is the Levi-Civita tensor). It is this self-dual nature of the NUT metric that makes it so useful in M-theory applications. Though this will be discussed more fully in chapter 5, in the full Lagrangian (in eleven dimensions, given by (5.1.1)), the fermion fields are included. However, maximal supersymmetry requires that the vacuum expectation values of the fermion fields vanish, i.e. $\langle\Psi\rangle = 0$, and the equation of motion for Ψ becomes an equation, given by (5.1.6), that checks the amount of supersymmetry preserved by any solution. Due to its self-duality, the Taub-NUT metric preserves 1/2 of its supersymmetry.

Though there are no curvature singularities associated with either the NUT or the Bolt solutions, both solutions will develop conical singularities unless the fibre closes smoothly at the NUT and bolt points. To ensure regularity in the (T, r) section, we impose the condition

$$\beta = \frac{4\pi}{F'(r = N)} = \frac{8\pi N}{q} \tag{3.3.8}$$

where q is a positive integer, and β is the period of the T component. The q is present because the period can't be greater than $8\pi N$, so that the Misner string singularities vanish - which is where the second equality comes from. However, the period can be less than $8\pi N$, as long as q is an integer. The period of the Lorentz frame can be recovered by Wick rotating back ($N \rightarrow \mathrm{i}n,\ q \rightarrow \mathrm{i}q$).

Of course, substituting (3.3.7) into (3.3.3) solves Einstein's vacuum equations.

3.3.2 Bolt Solution ($(3 + 1)$ dimensions)

The Bolt solution occurs when the fixed point set of the Killing field ∂_T is two-dimensional - as, obviously, the $(r^2 - N^2)$ term won't vanish at $r = r_b > N$. Here, the conditions for regularity at $r = r_b$ are given by

(i) $f(r = r_b) = 0$

(ii) $f'(r_b) = \frac{1}{2N}$

The second condition follows from the fact that we still want to avoid any conical singularities and from the Misner-string requirements (i.e. from the second equality in (3.3.8)). Imposing condition (i) forces the mass in the Bolt case to be

$$m_{b,flat} = \frac{r_b^2 + N^2}{2r_b} \tag{3.3.9}$$

Imposing condition (ii) then gives

$$r_b = 2N \tag{3.3.10}$$

The regularity requirement for the bolt will give the period

$$\beta_b = \frac{2\pi(r_b^2 - N^2)^2}{r_b^2 m - 2r_b N^2 + mN^2} \tag{3.3.11}$$

and this can be shown to give the same period as in the NUT case by substituting in (3.3.9) and (3.3.10). Substituting $r = r_b$ and (3.3.9) into (3.3.4) will give, for the bolt solution,

$$F(r = r_b) = \frac{2r_b^2 + 2N^2 - 5Nr_b}{2(r_b^2 - N^2)} \tag{3.3.12}$$

Using this in (3.3.3) will also satisfy Einstein's vacuum equations. It is important to note that, unlike the NUT solution, the Bolt solution is not self-dual, and so will not preserve any supersymmetry.

3.4 Taub-NUT-AdS Spacetimes in $(3+1)$ dimensions

The Taub-NUT metric can be adapted to satisfy Einstein's equations involving a cosmological constant. The four dimensional (AdS) case has been previously discussed in [47, 14] in relation to the AdS/CFT, but is easier to discuss here for demonstration purposes. I'll first deal with the Taub-NUT-AdS (TNAdS) case. In four $(3+1)$ dimensions, the discussion from section 3.2 still holds, and so we still have Misner strings present. The Euclidean section of the metric is again given by (3.3.3), where now

$$f(r) = \frac{r^2 + N^2 - 2mr + \ell^{-2}\left(r^4 - 6N^2r^2 - 3N^4\right)}{r^2 - N^2} \tag{3.4.1}$$

where the cosmological constant is given by $\Lambda = -\frac{3}{\ell^2}$. Note, of course, that to get the Euclidean metric from the Lorentzian, the Wick rotation is again $t \to \mathrm{i}T$, $n \to \mathrm{i}N$. A typical Penrose diagram of the TNAdS spacetime appears in figure 3.4.1. Note there are still quasi-regular singularities present (the solid dots) as well as CTC's. TNAdS spacetime, like the flat TN case, can also be divided into both a NUT solution and a Bolt solution, again depending on whether the fixed point set of ∂_T is zero- or two-dimensional.

3.4.1 TNAdS - NUT solution

The NUT solution, as in the flat case, occurs when the fixed point set of ∂_T is zero-dimensional. Solving (3.4.1) for the mass at the NUT ($r = N$), the

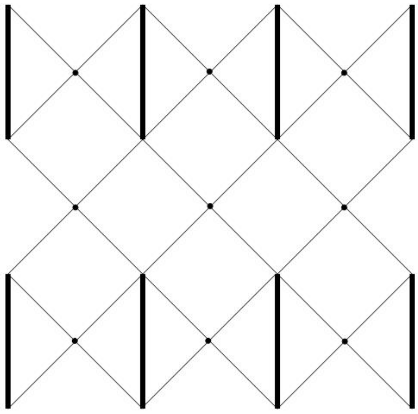

Figure 3.4.1: Penrose diagram of Taub-NUT-AdS spacetime.

NUT mass is found to be

$$m_{n,AdS} = \frac{N(\ell^2 - 4N^2)}{\ell^2} \tag{3.4.2}$$

This gives the metric function (3.4.1) for the NUT as;

$$F_{NUT}(r) = \frac{r^3 + Nr^2 + (\ell^2 - 5N^2)r + 3N^3 - N\ell^2}{\ell^2(r+N)} \tag{3.4.3}$$

Note that (3.4.3) $\to$ (3.3.7) as $\ell \to \infty$. The condition for regularity here is again given by (3.3.8), and so the period is again

$$\beta_{N,AdS} = \frac{8\pi N}{q} \tag{3.4.4}$$

as in the flat case. Substituting (3.4.3) back into (3.3.3) satisfies Einstein's equations with negative cosmological constant.

3.4.2 TNAdS - Bolt solution

The Bolt solution again occurs when the fixed point set of ∂_T is two dimensional, and the conditions for a regular bolt at $r = r_b > N$ are the conditions (i) and (ii) mentioned in the flat case on page 51. Imposing condition (i) gives the AdS bolt mass as

$$m_{bolt,AdS} = \frac{r_b^4 + (\ell^2 - 6N^2)r_b^2 + (\ell^2 - 3N^2)N^2}{2\ell^2 r_b} \tag{3.4.5}$$

Now, however, imposing condition (ii) leads to two possible solutions

$$r_{b\pm,AdS} = \frac{\ell^2 \pm \sqrt{\ell^4 - 48N^2\ell^2 + 144N^4}}{12N} \tag{3.4.6}$$

Since the solution is required to be real, equation (3.4.6) will impose an additional limit on the range of N, so that

$$0 < N < \frac{(3\sqrt{2} - \sqrt{6})\ell}{12} = N_{\max}$$

The regularity requirement for the Bolt-AdS is

$$\beta_{Bolt,AdS} = \frac{2\pi(r_b^2 - N^2)^2\ell^2}{r_b^5 - 2r_b^3N^2 + mr_b^2\ell^2 + N^2(9N^2 - 2\ell^2)r_b + m\ell^2N^2} \tag{3.4.7}$$

It is easily checked that substituting (3.4.5) and either of (3.4.6) into (3.4.7) gives the same period as in the NUT case (3.4.4).

3.5 Taub-NUT-dS Spacetimes in $(3 + 1)$ dimensions

Taub-NUT spacetimes can also be discussed with a positive cosmological constant, such that they are asymptotically de Sitter. The Taub-NUT-dS (TNdS) spacetimes will still contain CTC's, Misner strings, etc., and so regularity conditions must still be applied. Since all of the calculations performed in this document will be done outside the cosmological horizon (the region marked "X" in the Penrose diagram, figure 3.5.1), the metric will be written to reflect this. The metric in this case is thus given by

$$ds^2_{TNdS} = f(\tau)\left[dt + 2n\cos(\theta)d\phi\right]^2 - \frac{d\tau^2}{f(\tau)} + (\tau^2+n^2)(d\theta^2+\sin^2(\theta)d\phi^2) \tag{3.5.1}$$

(where, since we are working outside the cosmological horizon near future infinity, I've taken $r \to \tau$), $f(\tau)$ is given by

$$f(\tau) = \frac{\tau^4 + (6n^2 - \ell^2)\tau^2 + n^2(\ell^2 - 3n^2) + 2m\tau\ell^2}{(\tau^2 + n^2)\ell^2} \tag{3.5.2}$$

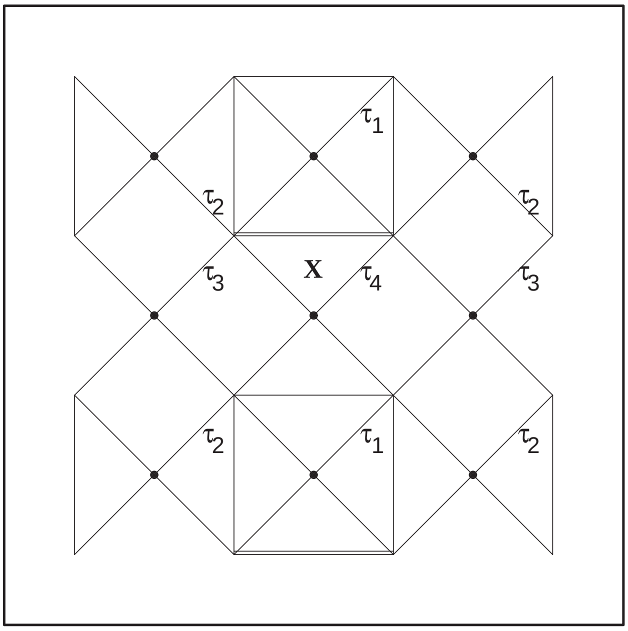

Figure 3.5.1: Penrose diagram of Taub-NUT-dS spacetime.

and the cosmological constant is $\Lambda = \frac{3}{\ell^2}$. The coordinate t parameterizes a circle (S^1) Hopf-fibred over this space, and must again meet the periodicity requirement

$$\beta_{TNdS,R} = \frac{4\pi}{|f'(\tau)|} = \frac{8\pi|n|}{q} \tag{3.5.3}$$

to avoid conical singularities (where again q is a positive integer), and to avoid the Misner string singularities. Constant-τ surfaces (spacelike hypersurfaces) are a Hopf-fibration of the circle over the base space (in four dimensions, S^2), well defined in a spacetime where $f(\tau) > 0$ outside of the past/future cosmological horizons.

The causal structure of TNdS spacetimes can be understood through a Penrose diagram (figure 3.5.1). The double line is future infinity $\tau = +\infty$, the single line is past infinity $\tau = -\infty$, and the roots of $f(\tau)$ are the horizons, denoted by the increasing sequence $\tau_1 < 0 < \tau_2 < \tau_3 < \tau_4 = \tau_c$. Quasi-regular singularities are again denoted by solid dots, and the region outside the cosmological horizon is the triangular region denoted by "X".

However, unlike in the asymptotically AdS and asymptotically flat cases, Wick rotation does not yield a Euclidean metric signature (in fact it gives the signature $(-,-,+,+)$), and so care must be taken when discussing this case. Also, as was discussed in [20], it is not clear, at least for the purposes of calculating the action and other quantities via the dS/CFT, whether one needs to Wick rotate the metric (indeed, it was shown in [44] that the quantities calculated by Wick rotating the metric are equivalent to those calculated when Wick rotation isn't used). However, since each case yields interesting discussions, and since most readers will be more familiar with the Wick rotated case from the AdS/CFT, I'll go through both (called the C-approach and R-approach, above). Also, as discussed previously, Wick rotation is necessary to prove the Gibbs-Duhem identity in the de Sitter case. Finally, I will go through both cases here since I will also use both cases to calculate the actions, entropies, conserved masses and specific heats in chapter 4.

3.5.1 TNdS C-approach

As mentioned above, the form of the metric in the C-approach is obtained from (3.5.1) by Wick rotating the time and the NUT parameter ($t \to \mathrm{i}T$, $n \to \mathrm{i}N$),

$$ds^2_{tndS,C} = -f(\rho)\left[dT + 2N\cos(\theta)d\phi\right]^2 - \frac{d\rho^2}{f(\rho)} + (\rho^2 - N^2)(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{3.5.4}$$

with $f(\rho)$ now given by

$$f(\rho) = \frac{\rho^4 - (6N^2 + \ell^2)\rho^2 - N^2(\ell^2 + 3N^2) + 2m\rho\ell^2}{(\rho^2 - N^2)\ell^2} \tag{3.5.5}$$

where now N is the non-vanishing NUT charge. T parameterizes a circle fibred over the non-vanishing sphere parameterized by (θ, ϕ), and must have a periodicity respecting

$$\beta_{tnds,C} = \frac{4\pi}{|f'(\rho)|} = \frac{8\pi|N|}{q} \tag{3.5.6}$$

(found from (3.5.3) by $n \to \mathrm{i}N$, $q \to \mathrm{i}q$). Note that this situation has a metric signature $(-,-,+,+)$, and so the geometry is no longer strictly speaking a Hopf fibration of S^1 over S^2, since T is now also timelike. The physical relevance of the C-approach is thus in question. However, since the metric is independent of T, calculations can still be carried out, provided one remembers the preceding considerations.

It appears at first glance that, as in the TNAdS situation, the C-approach can be sub-divided into two separate solutions, depending on the fixed point set of ∂_T. However, as discussed below, this turns out to not be the case.

NUT solution - $(3+1)$ dimensions

At first glance, (3.5.4) will be a NUT solution if the fixed point set of ∂_T is zero dimensional. Solving for the mass in the case when $f(\rho = N) = 0$ gives

$$m_{n,dS} = \frac{N(\ell^2 + 4N^2)}{\ell^2} \tag{3.5.7}$$

and substituting this back into (3.5.5) gives

$$f(\rho) = \frac{\rho^3 + N\rho^2 - (5N^2 + \ell^2)\rho + 3N^3 + N\ell^2}{(\rho + N)\ell^2} \tag{3.5.8}$$

Note that as $\ell \to \infty$, $f(\rho) \to -(3.3.7)$, as it should to recover (3.3.3) (recall the metric signatures). The condition for regularity is given by (3.5.6).

However, upon substitution of (3.5.7) back into (3.5.5) to give (3.5.8), one should be able to solve (3.5.8) for the largest root of $f(\rho)$ in the dS-NUT case, and get $\rho = N$. This is not the largest root one gets, however. Solving for the roots gives three (four - N is a double root) roots,

$$N\ ,\ \ \pm\sqrt{\ell^2 + 4N^2} - N \tag{3.5.9}$$

of which $\sqrt{\ell^2 + 4N^2} - N$ is the actual largest root. Thus, although the "NUT" solution $\rho = N$ can be shown to solve the First Law, since there exists a Bolt solution of larger radius, the NUT solution is not actually valid.

It should also be noted that, as mentioned in [44], the ds-NUT quantities (3.5.7), (3.5.8) can be computed, through analytic continuation of the cosmological parameter $\ell \to i\ell$, from the AdS-NUT quantities (3.4.2), (3.4.3) respectively (recalling that in (3.5.8), the negative sign that should be there is actually part of the metric (3.5.4), giving it the $(-,-,+,+)$ signature).

Bolt solution - $(3+1)$ dimensions

The Bolt solution occurs when the fixed point set of ∂_T is two dimensional. The conditions for a regular bolt are given by

(i) $f(\rho = \rho_b) = 0$

(ii) $f'(\rho = \rho_b) = \frac{q}{2|N|}$

(exactly the same as the conditions given on page 51, and for the same reasons). Imposing the first condition gives the bolt mass as

$$m_{bolt,dS(C)} = -\frac{(\rho_b^4 - (6N^2 + \ell^2)\rho_b^2 - (3N^2 + \ell^2)N^2)}{2\ell^2 \rho_b} \tag{3.5.10}$$

From condition (ii), ρ_b is given by

$$\rho_{b\pm} = \frac{q\ell^2 \pm \sqrt{q^2\ell^4 + 144N^4 + 48N^2\ell^2}}{12N} \tag{3.5.11}$$

Note that the discriminant in (3.5.11) is positive, and so there is no upper limit on N here, unlike in the TNAdS. The period for the Bolt is found from the first equality in (3.5.6),

$$\beta_{Bolt,AdS} = 2\pi \left| \frac{(\rho_b^2 - N^2)^2 \ell^2}{\rho_b^5 - 2\rho_b^3 N^2 + N^2(9N^2 + 2\ell^2)\rho_b - m\ell^2(\rho_b^2 + N^2)} \right| \tag{3.5.12}$$

Substituting (3.5.10) and either of (3.5.11) into this will give the same period (3.5.6) as in the NUT case.

Both solutions, the upper and lower branches $\rho_{b\pm}$ respectively, can be shown to solve the First Law once the entropy is calculated. However, although ρ_{b+} is the largest root of the upper branch, ρ_{b-} is not the largest root of the lower branch. There in fact exists two roots ([44]) ρ_1, ρ_2 of $f(\rho)$ such that $\rho_1 < \rho_{b-} < \rho_2$. Because of this, the lower branch solution, despite satisfying the first law, is not a valid solution. The fact that it satisfies the first law is a direct consequence of the fact that the lower branch C-approach is the analytic continuation of the lower branch solution in the Taub-NUT-AdS case.

3.5.2 TNdS R-approach

In this approach, the time coordinate and NUT charge are not rotated into complex space, and the metric is given by (3.5.1), (3.5.2). The geometry of a constant-τ surface is, here, a Hopf fibration of S^1 over S^2, and the metric describes the contraction/expansion (for $q = 1$) of this three-sphere in spacetime regions where $f(\tau) > 0$, outside of the past/future cosmological horizons. In the R-approach, we only get Bolt solutions, where $\tau = \tau_c$ will denote the horizon for any constant ϕ-slice. The conditions for a regular bolt are given on page 58. Again, the coordinate t parameterizes a circle fibred over the two-sphere with coordinates (θ, ϕ), and must have a period respecting

$$\beta_{tndS,R} = \frac{4\pi}{|f'(\tau)|} = \frac{8\pi|n|}{q} \tag{3.5.13}$$

yielding,

$$\beta_{tndS,R} = 2\pi \left| \frac{(\tau_c^2 + n^2)^2 \ell^2}{\tau_c^5 + 2n^2\tau_c^3 + n^2(9n^2 - 2\ell^2)\tau_c + m\ell^2(n^2 - \tau_c^2)} \right| \tag{3.5.14}$$

The condition $f(\tau) = 0$ gives the mass as

$$m_{bolt,dS(R)} = -\frac{(\tau_c^4 + (6n^2 - \ell^2)\tau_c^2 - 3n^4 + n^2\ell^2)}{2\ell^2\tau_c} \tag{3.5.15}$$

As in the C-approach, imposing condition (ii) appears to give two possible solutions $\tau = \tau_{c\pm}$

$$\tau_{c\pm} = \frac{q\ell^2 \pm \sqrt{q^2\ell^4 - 144n^4 + 48\ell^2n^2}}{12n} \tag{3.5.16}$$

Here, unlike in the C-approach bolt solution, the discriminant will sometimes be negative, forcing n to have the range,

$$0 < n < \frac{\ell\sqrt{6 + 3\sqrt{4 + q^2}}}{6} = n_{\max} \tag{3.5.17}$$

Again, substituting in (3.5.15) and either of (3.5.16) will give the period of t as $\frac{8\pi|n|}{q}$.

However, as in the C-approach, the lower branch $\tau_{c\pm}$ solution is not the largest root of the metric function $f(\tau)$. There again exist two roots τ_1, τ_2 of $f(\tau)$ such that $\tau_1 < \tau_{c-} < \tau_2$, and so the lower branch solution cannot be taken as a valid solution, even though this solution, once the thermodynamic quantities are calculated, can be shown to satisfy the first law. The fact that the lower branch solution is a solution of the first law is a consequence of the fact [44] that the lower branch solution is the analytic continuation of the lower branch solution of the TNAdS solution, if one were to calculate the thermodynamic properties of the TNAdS solution without first Wick rotating the t, n and q values (i.e. the TNAdS equivalent calculation of the R-approach).

3.5.3 C-approach from R-approach

It was also shown in [44] that the C-approach quantities can be derived from the R-approach quantities by direct analytic continuation of t, n and q in equations (3.5.14), (3.5.15) and (3.5.16). It is easy to see this; applying $t \to \mathrm{i}T$, $n \to \mathrm{i}N$ and $q \to \mathrm{i}q$ to these equations gives (3.5.6), (3.5.10) and (3.5.11) respectively, where now of course only the radii $\rho = \rho_{b+}$, $\tau = \tau_{c+}$ are to be used.

This trend will be later shown to hold for the thermodynamic quantities also.

3.6 General $(d+1)$ dimensional Taub-NUT spacetimes

The above discussion in $(3+1)$ dimensions can be extended to arbitrary even dimensions. Since the asymptotically flat case can be found from either of the TNAdS or TNdS general equations, I will only show the general equations for these cases, and point out how to get the flat case equations from these. The discussions involving Misner strings, quasi-regular singularities and CTC's apply in higher dimensions, and so won't be repeated here. It should also be pointed out that the discussion in this section are either taken from or are generalizations of the paper by Awad & Chamblin [17].

3.6.1 TNAdS in $(d+1)$ dimensions

The Euclidean section of the Taub-NUT-AdS metric in general (even) dimensions, for a $U(1)$ fibration over a series of two spheres as the base space $\otimes_{i=1}^{k} S^2$, is given by

$$\begin{aligned} ds^2_{TNAdS} &= f(r)\left[dT + 2N\sum_{i=1}^{k}\cos(\theta_i)d\phi_i\right]^2 + \frac{dr^2}{f(r)} \\ &\quad +(r^2-N^2)\sum_{i=1}^{k}\left(d\theta_i^2+\sin^2(\theta_i)d\phi_i^2\right) \end{aligned} \tag{3.6.1}$$

where $(d+1) = 2k+2$ is the total number of dimensions. The metric function has the general form

$$f(r) = \frac{r}{(r^2-N^2)^k}\int^r \left[\frac{(s^2-N^2)^k}{s^2} + \frac{(2k+1)}{\ell^2}\frac{(s^2-N^2)^{k+1}}{s^2}\right] ds - \frac{2mr}{(r^2-N^2)^k} \tag{3.6.2}$$

Note that to get the asymptotically flat metric for general $(d+1)$ dimensions, just take $\ell \to \infty$. In general, to ensure regularity in the (T, r) section, the condition (3.3.8) is generalized to

$$\beta_{AdS} = \frac{4\pi}{f'(r)} = \frac{2(d+1)\pi N}{q} \tag{3.6.3}$$

with q a positive integer, and β is the period of T. Note that the second equality is obtained when considering $f(r=N) = 0$ - i.e. obtained when we demand that the manifold be regular, so that the singularities at $\theta_i = 0, \pi$ are coordinate artifacts and there are no Misner string singularities.

The NUT and bolt solutions can also be generalized, depending on the fixed point set of ∂_T. For the Bolt solution, the fixed point set is $(d-1)$ dimensional, and for the NUT, it is zero dimensional.

General NUT solutions

A NUT solution arises in $(d+1)$ dimensions when the fixed point set of ∂_T is zero-dimensional (this is because it occurs at $r = N$, and so due to the $(r^2 - N^2)$, all of the other dimensions will vanish). The binomial theorem can be used on (3.6.2) to get a general expression for the NUT mass for $(d+1)$ dimensions,

$$\begin{aligned} m_{n,AdS} &= \frac{N^{2k-1}}{2\ell^2}\left[\ell^2 - (2k+2)N^2\right]\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^i}{(2k-2i-1)} \\ &= \frac{N^{2k-1}\left[\ell^2-(2k+2)N^2\right]}{\ell^2(2k-1)\sqrt{\pi}}\Gamma\left(\frac{3-2k}{2}\right)\Gamma(k+1) \end{aligned} \tag{3.6.4}$$

(where the identity, provable by induction

$$\sum_{i=0}^{k+1}\binom{k+1}{i}\frac{(-1)^i}{(2k-2i+1)} = -\left(\frac{2k+2}{2k+1}\right)\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^i}{(2k-2i-1)} \tag{3.6.5}$$

has been used). Note that, either by (3.6.5) or by noting the properties of the Gamma functions, it is easily seen that the general form of the NUT mass will remain $\propto [\ell^2 - (2k+2)N^2]$, but that the overall sign will change, alternating between (even) dimensions. The period of T is given by (3.6.3).

General Bolt solutions

The Bolt solution occurs when the fixed point set of ∂_T is $(d-1)$ dimensional. The conditions for regularity at the bolt radius $r = r_b > N$ are

(i) $f(r = r_b) = 0$

(ii) $f'(r = r_b) = \frac{2q}{(d+1)N}$

The second condition is required in general dimensions to avoid conical singularities, and the first will give rise to the bolt mass in the specific dimension. A general formula for the bolt mass can be found from (3.6.2),

$$\begin{aligned} m_{bolt,AdS} &= \frac{1}{2}\int^r \left[\frac{(s^2-N^2)^k}{s^2} + \frac{(2k+1)}{\ell^2}\frac{(s^2-N^2)^{k+1}}{s^2}\right] ds \\ &= \frac{1}{2}\left\{\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^i r_b^{2k-2i-1}N^{2i}}{(2k-2i-1)}\right. \\ &\quad \left. + \frac{(2k+1)}{\ell^2}\sum_{i=0}^{k+1}\binom{k+1}{i}\frac{(-1)^i r_b^{2k-2i+1}N^{2i}}{(2k-2i+1)}\right\} \end{aligned} \tag{3.6.6}$$

although in practical terms it is far easier to find $f(r)$ for a specific case, and then solve for m_b. It is also easier to solve for the two different horizon radii $r_{b\pm}$ in whichever specific case one is dealing with.

3.6.2 TNdS in $(d+1)$ dimensions

Taub-NUT spacetimes can also be written for general (even) dimension to include a positive cosmological constant. Here as in four dimensions however, we have the question of whether to rotate the time and NUT charge. I will cover both cases. Note that for simplicity again, the base space in either case will be taken to be a product of two-spheres $\otimes_{i=1}^{k} S^2$, where as in the TNAdS case, $(d+1) = 2k+2$ is the total number of dimensions.

R-approach in $(d+1)$ dimensions

The metric for the R-approach is the higher-dimensional generalization of (3.5.1), given by

$$\begin{aligned} ds^2_{TNdS,R} &= f(\tau)\left[dt + 2n\sum_{i=1}^{k}\cos(\theta_i)d\phi_i\right]^2 - \frac{d\tau^2}{f(\tau)} \\ &\quad +(\tau^2+n^2)\sum_{i=1}^{k}\left(d\theta_i^2 + \sin^2(\theta_i)d\phi_i^2\right) \end{aligned} \tag{3.6.7}$$

where the metric function can be found from the general formula

$$f(\tau) = \frac{2m\tau}{(\tau^2+n^2)^k} - \frac{\tau}{(\tau^2+n^2)^k}\int^{\tau}\left[\frac{(s^2+n^2)^k}{s^2} - \frac{(2k+1)}{\ell^2}\frac{(s^2+n^2)^{k+1}}{s^2}\right]ds \tag{3.6.8}$$

where again the largest root of $f(\tau)$ will be denoted by τ_c. The subspace for which $\tau = \tau_c$ is the fixed point set of ∂_t. Note that the fixed point set of ∂_t is always $(d-1)$ dimensional, and so the solutions for the general R-approach will always be bolt solutions.

The ∂_{ϕ_i} are Killing vectors, and so for any constant $(\phi_1, \ldots, \phi_k)$-slice near the horizon, additional conical singularities will be introduced in the (t, τ) section, unless we require that t has a period

$$\beta_{TNdS,R} = \frac{4\pi}{|f'(\tau_c)|} = \frac{2\pi(d+1)|n|}{q} \tag{3.6.9}$$

where note again that the second equality is to ensure that the Misner string singularities do not arise.

The two conditions for a regular bolt are

(i) $f(\tau = \tau_c) = 0$

(ii) $f'(\tau = \tau_c) = \frac{2q}{(d+1)|n|}$

Condition (ii) will have two solutions $\tau_c = \tau_{c\pm}$, both functions of n, and arises because (3.6.9) must match the condition for the vanishing of the Misner string. As mentioned above for the four dimensional case, and in [44], the τ_{c+} solution will be the largest root; the τ_{c-} solution, however, will not, as in any dimension, there will exist two roots of $f(\tau)$ such that $\tau_1 < \tau_{c+} < \tau_2$.

Hence, although this lower branch solution can be shown to satisfy the First Law, it is not a valid solution.

Condition (i) will give rise to a bolt (R-approach) mass - the general formula can be found,

$$\begin{aligned} m_{bolt,dS(R)} &= \frac{1}{2}\Bigg\{ \sum_{i=0}^{k} \binom{k}{i} \frac{n^{2i}\tau_c^{2k-2i-1}}{2k-2i-1} \\ &\quad - \frac{(2k+1)}{\ell^2} \sum_{i=0}^{k+1} \binom{k+1}{i} \frac{n^{2i}\tau_c^{2k-2i+1}}{2k-2i+1} \Bigg\} \end{aligned} \tag{3.6.10}$$

by using the binomial expansion on (3.6.8).

C-approach in $(d+1)$ dimensions

The metric here can be found from (3.6.7) by rotating the time coordinate and the NUT charge ($t \to \mathrm{i}T,\ n \to \mathrm{i}N$)

$$\begin{aligned} ds^2_{TNdS,C} &= -f(\rho)\left[dT + 2N\sum_{i=1}^{k}\cos(\theta_i)d\phi_i\right]^2 - \frac{d\rho^2}{f(\rho)} \\ &\quad + (\rho^2 - N^2)\sum_{i=1}^{k}\left(d\theta_i^2 + \sin^2(\theta_i)d\phi_i^2\right) \end{aligned} \tag{3.6.11}$$

where $f(\rho)$ can now be found from

$$f(\rho) = \frac{2m\rho}{(\rho^2-N^2)^k} - \frac{\rho}{(\rho^2-N^2)^k}\int^{\rho} ds\left[\frac{(s^2-N^2)^k}{s^2} - \frac{(2k+1)}{\ell^2}\frac{(s^2-N^2)^{k+1}}{s^2}\right] \tag{3.6.12}$$

The period of T is again obtained by setting

$$\beta_{TNdS,C} = \frac{4\pi}{|f'(\rho_+)|} = \frac{2(d+1)\pi|N|}{q} \tag{3.6.13}$$

to ensure regularity. In a general dimensional C-approach, as in four dimensions, one can solve for what appear to be NUT solutions, as well as Bolt solutions. However, the NUT solution will be invalid, as the root of the metric function $\rho = N$ used to solve for the NUT solution will not be the largest root of $f(\rho)$. However, a general "ds-NUT" solution, found through the analysis above, can be shown to be the analytic continuation of the AdS-NUT solution above.

General C-approach NUT solution

The fixed point set of ∂_T is again zero dimensional, for any $(d+1)$ dimensional metric, giving a NUT solution. Setting $\rho = N$ in (3.6.12) will give a general formula for the NUT mass

$$m_{N,TNdS} = \frac{N^{2k-1}\left[\ell^2 + (2k+2)N^2\right]}{\ell^2(2k-1)\sqrt{\pi}}\Gamma\left(\frac{3-2k}{2}\right)\Gamma\left(k+1\right) \qquad (3.6.14)$$

and the period will be given by (3.6.13). Note that this has no actual relevance - the root of $f(\rho)$ at $\rho = N$ is not the largest root, and the largest root, in any dimension, won't satisfy the First Law. However, (3.6.14) is in general, as stated above and in [44], the analytic continuation of the general AdS-NUT solution (3.6.4).

General C-approach Bolt solution

The fixed point set of ∂_T is $(d-1)$ dimensional here. The conditions for a regular bolt at $\rho = \rho_b > N$ in general dimensions are given by

(i) $f(\rho_b) = 0$

(ii) $f'(\rho_b) = \frac{2q}{(d+1)|N|}$

With (ii) arising for the same reasons as in the other cases. Condition (i) will again give a general bolt mass in $(d+1)$ dimensions,

$$\begin{aligned} m_{bolt,dS(C)} &= \frac{1}{2}\Bigg\{\sum_{i=0}^{k}\binom{k}{i}(-1)^i\frac{N^{2i}\rho_b^{2k-2i-1}}{2k-2i-1} \\ &\qquad -\frac{(2k+1)}{\ell^2}\sum_{i=0}^{k+1}\binom{k+1}{i}(-1)^i\frac{N^{2i}\rho_b^{2k-2i+1}}{2k-2i+1} \qquad (3.6.15)\end{aligned}$$

Also, as in four dimensions, one will get two different bolt horizon radii, $\rho_{b,\pm}$, as functions of N, in higher dimensions. However, also as in four dimensions, the lower branch solution $\rho_{b,\pm}$ won't be the largest root, and will hence be an invalid solution.

Chapter 4

Thermodynamics of Taub-NUT (A)dS Spacetimes

Here I consider the thermodynamic properties of $(d+1)$ dimensional Taub-NUT-(A)dS spacetimes. Calculation of the thermodynamics can be done both through the Nöether method, discussed briefly in section 2.2, and by using the (A)dS/CFT conjecture. Here, I will mainly discuss the results that arise from using the counterterm approach from the (A)dS/CFT's, though a comparison of the six dimensional TNAdS counterterm results with the Nöether results will be done in section 4.2.

This chapter is arranged as follows; I will first discuss the calculation of such quantities, for both Taub-NUT-AdS and Taub-NUT-dS, in general $(d+1)$ dimensions, in section 4.1. I will show that, in general, one only needs the first terms of the counterterm action in order to successfully calculate the action and thermodynamic quantities of Taub-NUT-(A)dS spacetimes. Then, in sections 4.2, 4.3, I will present specific examples, the TN-AdS $(5+1)$ dimensional solution and the TNdS $(3+1)$ dimensional solution, respectively. In section 4.3, it will also be shown that, for certain values of the NUT charge, in either the C-approach or the R-approach, the Taub-NUT-dS spacetime is a counter-example to both the maximal mass conjecture and the Bousso-N bound of asymptotically dS spacetimes.

4.1 General Calculations in $(d+1)$ dimensions

One can calculate the action, entropy, specific heat and conserved mass for a specific dimension, as done in [20, 16]. However, also shown in [20, 16] was that the calculations can be done for arbitrary even dimension, leaving all quantities general, so that one has a set of equations that apply for any dimension (and I have checked that these formula agree with calculations done in specific dimensions up to 20 dimensions, for both TNAdS and TNdS - though it should be noted that this check depends on the proof, to be shown later, that the counterterm contribution to the finite action comes only from the first term in the counterterm expansion). Since it is easier to perform this calculation in general, and then apply the results as needed to a specific dimension, I will discuss these general results first, and then perform the calculation in four dimensions to demonstrate their accuracy. Also, at the end, I will discuss the general trend that occurs in $4k$ and $4k+2$ dimensions (k is a positive integer) in both TNAdS and TNdS.

4.1.1 Taub-NUT-AdS

The general form of the Taub-NUT/Bolt class of metrics for a $U(1)$ fibration over k two-spheres $\otimes_{i=1}^{k} S^2$ is given by (3.6.1), along with (3.6.2), where $(d+1) = 2k+2$, and m is an integration constant. From (3.6.2), the Ricci scalar and metric determinant can be found for an arbitrary k,

$$g = (r^2 - N^2)^{2k} \prod_{i=1}^{k} \sin^2(\theta_i) \tag{4.1.1}$$

$$R = -\frac{d(d+1)}{\ell^2} \tag{4.1.2}$$

where $\Lambda = -\frac{d(d-1)}{2\ell^2}$. Also needed are the determinant of the d-dimensional boundary metric (the boundary is taken to be $r \to \infty$) and the Ricci scalar on the boundary, found to be

$$\gamma = f(r)(r^2 - N^2)^{2k} \prod_{i=1}^{k} \sin^2(\theta_i) \tag{4.1.3}$$

$$\mathcal{R}(\gamma) = 2k\left[\frac{1}{(r^2 - N^2)} - \frac{f(r)N^2}{(r^2 - N^2)^2}\right] \tag{4.1.4}$$

Finally, for the contribution from the boundary action (2.3.4), the trace of the extrinsic curvature can be calculated for general dimension

$$\Theta = \frac{f'(r)}{2\sqrt{f(r)}} + \frac{2kr\sqrt{f(r)}}{(r^2 - N^2)} \tag{4.1.5}$$

Before calculating the specific terms, it should be noted that the counterterm action (2.5.9) is designed to cancel divergences in the bulk and boundary actions. This means that, since after cancellation the action is finite, only the finite contributions from (2.3.3) and (2.3.4), as well as from (2.5.9), need to be calculated.

Substituting (4.1.1) into (2.3.3) gives, for the bulk action,

$$I_{B,AdS} = \frac{(2k+1)(4\pi)^k\beta}{8\pi\ell^2}\int dr\ (r^2 - N^2)^{2k} \tag{4.1.6}$$

and note that the $\prod \sin^2(\theta_i)$ from $\sqrt{g}$ (and later $\sqrt{\gamma}$) contributes to the volume element $(4\pi)^k$. The binomial theorem can now be used on this to integrate term by term, over the range $r = (r_+, r')$, where since $r' \to \infty$, any terms involving r' will be cancelled by the counterterm action. This will give the finite bulk action contribution as

$$I_{B,AdS} = -\frac{(2k+1)(4\pi)^k\beta}{8\pi\ell^2}\sum_{i=0}^{k}\binom{k}{i}(-1)^i N^{2i}\frac{r_+^{(2k-2i+1)}}{2k-2i+1} \tag{4.1.7}$$

where r_+ is the largest positive root of $f(r)$, determined by the fixed point set of ∂_T, and β is the period of T.

Next, expanding $\sqrt{\gamma}\Theta$ for large r, the general finite contribution from the boundary action can be calculated also,

$$I_{\partial B,AdS} = \frac{(2k+1)(4\pi)^k\beta}{8\pi}m \tag{4.1.8}$$

Finally, from (2.5.9), the counterterm action can be calculated. However, in general only the first term in (2.5.9) needs to be calculated, as all of the other terms in the expansion will diverge as $r \to \infty$. This can be seen by noting that $f(r)$ can be written as an expansion (for large r)

$$f(r) \sim \frac{r^2}{\ell^2} - \frac{2m}{r^{2k-1}} + \sum_{i=1}\frac{A_i}{r^{2k+2i}} - 2m\sum_{i=1}\frac{B_i}{r^{2k+2i+1}} \tag{4.1.9}$$

where $A_i,\ B_i$ are constants depending on $\ell,\ N$. This means that the expansion of $\sqrt{\gamma}$ (the first term in (2.5.9)) is

$$\sqrt{\gamma} \sim \frac{r^{2k+1}}{2\ell} - m\ell - \sum_j \frac{C_j r^{2k-2j+1}}{2} + \text{(terms that vanish at large } r) \quad (4.1.10)$$

The first and third terms will cancel divergences in the bulk and boundary actions, and thus only the second term will contribute to the finite action. All powers of r in this expansion of $\sqrt{\gamma}$ are odd (or zero) - and from (4.1.4) it can be seen that any expansion of the Ricci scalar on the boundary will be even, and so multiplying $\sqrt{\gamma}R(\gamma)$ will give only terms that contain non-zero powers of r. Thus, the second term in (2.5.9) will have no finite terms - all the terms will either vanish at large r or will diverge, and hence be used to cancel terms from the bulk and boundary. Since multiples of the Riemann and Ricci tensors will also contain even powers of r, later terms in (2.5.9) will also not give any finite contributions, and so the only term that needs to be calculated is the first term in the expansion (2.5.9). This will give a finite contribution of

$$I_{ct,AdS} = -\frac{2k(4\pi)^k\beta}{8\pi}m \quad (4.1.11)$$

to the total action.

Adding together (4.1.7), (4.1.8) and (4.1.11) gives, for the total Taub-NUT-AdS finite action,

$$I_{AdS} = \frac{(4\pi)^k\beta}{8\pi\ell^2}\left[m\ell^2 - (2k+1)\sum_{i=0}^{k}\binom{k}{i}(-1)^i N^{2i}\frac{r_+^{2k-2i+1}}{2k-2i+1}\right] \quad (4.1.12)$$

The conserved mass can also be found in general, from formulae (2.5.10) and (2.5.11),

$$\mathfrak{M} = \frac{1}{8\pi}\int d^{d-1}x\ \sqrt{\gamma}\left\{\Theta_{ab} - \gamma_{ab}\Theta + \frac{d-1}{\ell}\gamma_{ab} + \dots\right\}u^a\xi^b N_{lpse} \quad (4.1.13)$$

Note that, of course, in this case there are no further conserved quantities to be calculated. In (4.1.13), ξ^a is the timelike Killing vector, $u^a = \left[\frac{1}{\sqrt{g_{tt}}}, 0, \dots\right]$ is the timelike unit normal, and N_{lpse} is the square root of the lapse function, here equal to $1/\sqrt{f(r)}$. It turns out, for the same reasons as used for calculating the action, that only those terms in (4.1.13) will contribute to the finite conserved mass, and the rest will only be used to cancel divergences. Thus, using (4.1.5) and $\Theta_{tt},\ \gamma_{tt}$, the general formula for the conserved mass in any (even) dimension is given by

$$\mathfrak{M}_{TNAdS}=\frac{2k(4\pi)^k}{8\pi}m \tag{4.1.14}$$

Using (4.1.12), (4.1.14), the Gibbs-Duhem relation (2.3.14) gives a general formula for the entropy in (even) dimensions:

$$S_{TNAdS}=\frac{(4\pi)^k\beta}{8\pi\ell^2}\left[(2k-1)m\ell^2+(2k+1)\sum_{i=0}^{k}\binom{k}{i}(-1)^iN^{2i}\frac{r_+^{2k-2i+1}}{2k-2i+1}\right] \tag{4.1.15}$$

Note that, since r_+ will depend on N (in the NUT case, it will equal N, and in the Bolt case, $r_+=r_{b\pm}(N)$), and since the formula to find the specific heat[1], $C=-\beta\partial_\beta S$, will be a differentiation of the entropy with respect to N (since β is a function of N), a general formula for the specific heat cannot be found before specifying to the NUT solution (since $r_{b\pm}$ cannot be found in general, a general solution for the specific heat cannot be found for the Bolt).

General formulae can also be found for the cases of specifying our solution to either the NUT or Bolt.

General NUT solution

The general form of the NUT mass for arbitrary even dimension has already been found in (3.6.4). Using this, and substituting $r_+=N$ into (4.1.12), the general form

$$I_{\mathrm{NUT},AdS}=\frac{(4\pi)^k\beta N^{2k-1}}{16\pi^{3/2}\ell^2}\left[2kN^2-\ell^2\right]\Gamma\left(\frac{1-2k}{2}\right)\Gamma(k+1) \tag{4.1.16}$$

for the NUT action for general dimension $(d+1)=2k+2$ can be found. The general NUT entropy can be found similarly from (4.1.15),

$$S_{\mathrm{NUT},AdS}=\frac{(4\pi)^k\beta N^{2k-1}}{16\pi^{3/2}\ell^2}\left[2k(2k+1)N^2-(2k-1)\ell^2\right]\Gamma\left(\frac{1-2k}{2}\right)\Gamma(k+1) \tag{4.1.17}$$

[1]Which can be found from the usual thermodynamic specific heat formula $C=T\partial_T S$ ([31], pg. 15, for either constant volume or pressure) by noting that $T\partial_T=\beta\partial_\beta$, since β is equal to the inverse of the temperature T.

A general expression for the specific heat can also be found, by using the relation $C = -\beta\partial_\beta S$ and the general form of the period (3.6.3);

$$C_{\mathrm{NUT},AdS} = -\frac{(4\pi)^k \beta N^{2k-1}}{16\pi^{3/2}\ell^2}\Gamma\left(\frac{1-2k}{2}\right)\Gamma(k+1) \left[2k(2k+1)(2k+2)N^2 - 2k(2k-1)\ell^2\right] \tag{4.1.18}$$

Analysis

Tables 4.1, 4.2 shows the NUT quantities for four to ten dimensions, with the general dimensions given at the end (note in the table, $n = (d+1)$).

From (4.1.16), (4.1.17) and (4.1.18), it is easy to notice that the action, entropy and specific heat for any NUT solution will vanish in the high temperature limit (note that the high temp. limit is $N \to 0$, because $T = 1/\beta$). Also, due to the nature of Gamma functions, $\Gamma\left(\frac{1-2k}{2}\right)$ will produce a negative sign for all odd k. This means that for dimensions $4k$ $(4, 8, 12, \ldots)$, the entropy and specific heat will both be greater than zero, for N given by

$$\ell\sqrt{\frac{(2k-1)}{(2k+2)(2k+1)}} < N < \ell\sqrt{\frac{(2k-1)}{2k(2k+1)}} \tag{4.1.19}$$

This means that in these dimensions, the NUT solutions will be thermodynamically stable for this range of N, though note that the range becomes increasingly narrow as k increases. Since no minus sign will be produced for even k, (i.e. in $4k+2$ dimensions, $6, 10, 14, \ldots$), the entropy will be positive for $N > \ell\sqrt{\frac{2k-1}{2k(2k+1)}}$, and the specific heat for $N < \ell\sqrt{\frac{(2k-1)}{(2k+1)(2k+2)}}$. Thus, for dimensions with even k, the NUT solutions are all by definition thermodynamically unstable.

The flat space limits ($\ell \to \infty$) of the NUT quantities can also be found,

$$I_{NUT,AdS} \to -\frac{(4\pi)^k \beta}{16\pi^{3/2}}\Gamma\left(\frac{1-2k}{2}\right)\Gamma(k+1)\, N^{2k-1} \tag{4.1.20}$$

$$S_{NUT,AdS} \to -\frac{(4\pi)^k \beta}{16\pi^{3/2}}\Gamma\left(\frac{1-2k}{2}\right)\Gamma(k+1)\, N^{2k-1}(2k-1) \tag{4.1.21}$$

$$C_{NUT,AdS} \to \frac{(4\pi)^k \beta}{16\pi^{3/2}}\Gamma\left(\frac{1-2k}{2}\right)\Gamma(k+1)\, N^{2k-1}2k(2k-1) \tag{4.1.22}$$

With the same Gamma function property taken into account, these equations show that in any dimension, the asymptotically locally flat pure NUT solutions will always be thermally unstable, for any dimension.

General Bolt Solution

The conditions for arbitrary $(d+1)$ dimensions that give a regular Bolt at $r = r_b > N$ are given on page 62. Recall that condition (i) implies

$$\begin{aligned} m_{b,AdS} &= \frac{1}{2}\Bigg[\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^i r_b^{2k-2i-1}N^{2i}}{(2k-2i-1)} \\ &\quad +\frac{(2k+1)}{\ell^2}\sum_{i=0}^{k+1}\binom{k+1}{i}\frac{(-1)^i N^{2i} r_b^{2k-2i+1}}{(2k-2i+1)}\Bigg] \end{aligned} \tag{4.1.23}$$

from which a general expression for the Bolt action can be obtained

$$\begin{aligned} I_{Bolt,AdS} &= \frac{(4\pi)^{(n-2)/2}\beta}{16\pi\ell^2}\Bigg\{-\frac{(2k+1)(-1)^{k+1}N^{2k+2}}{r_b} \\ &\quad +\sum_{i=0}^{k}\binom{k}{i}(-1)^i N^{2i} r_b^{2k-2i}\Bigg[\frac{\ell^2}{r_b(2k-2i-1)} \\ &\quad -\frac{r_b(2k+1)(k-2i+1)}{(2k-2i+1)(k-i+1)}\Bigg]\Bigg\} \end{aligned} \tag{4.1.24}$$

by substituting $m = m_b$ in (4.1.12). Next, using the Gibbs-Duhem relation (2.3.14), and substituting $m = m_b$ into (4.1.14) (or directly from (4.1.15)), we find

$$\begin{aligned} S_{Bolt,AdS} &= \frac{(4\pi)^{(n-2)/2}\beta}{16\pi\ell^2}\Bigg\{-\frac{(2k-1)(2k+1)(-1)^{k+1}N^{2k+2}}{r_b} \\ &\quad +\sum_{i=0}^{k}\binom{k}{i}(-1)^i N^{2i} r_b^{2k-2i}\Bigg[\frac{(2k-1)\ell^2}{r_b(2k-2i-1)} \\ &\quad +\frac{(2k+1)(2k^2+3k-2i+1)r_b}{(2k-2i+1)(k-i+1)}\Bigg]\Bigg\} \end{aligned} \tag{4.1.25}$$

for the general expression for the Bolt entropy in $n = d+1$ dimensions. The explicit expression for the specific heat is extremely cumbersome, and I will not include it here. Note, for both of the equations above ((4.1.24), (4.1.25)), that $r_b = r_{b\pm}$ must be substituted in to get the upper/lower branch solutions and analysis.

Table 4.1: Summary of NUT Period, Mass and Action for TNAdS

Dim.	Period.	Mass	Action
4	$8\pi N$	$\frac{N(\ell^2-4N^2)}{\ell^2}$	$\frac{4\pi N^2(\ell^2-2N^2)}{\ell^2}$
6	$12\pi N$	$\frac{4N^3(6N^2-\ell^2)}{3\ell^2}$	$\frac{32\pi^2 N^4(4N^2-\ell^2)}{\ell^2}$
8	$16\pi N$	$\frac{8N^5(\ell^2-8N^2)}{5\ell^2}$	$\frac{1024\pi^3 N^6(\ell^2-6N^2)}{5\ell^2}$
10	$20\pi N$	$\frac{64N^7(10N^2-\ell^2)}{35\ell^2}$	$\frac{8192\pi^4 N^8(8N^2-\ell^2)}{7\ell^2}$
n	$2n\pi N$	$\frac{N^{n-3}\Gamma\left(\frac{5-n}{2}\right)\Gamma\left(\frac{n}{2}\right)[\ell^2-nN^2]}{\sqrt{\pi}\ell^2(n-3)}$	$\frac{(4\pi)^{(n-2)/2}\beta\Gamma\left(\frac{3-n}{2}\right)\Gamma\left(\frac{n}{2}\right)N^{n-3}[(n-2)N^2-\ell^2]}{16\pi^{3/2}\ell^2}$

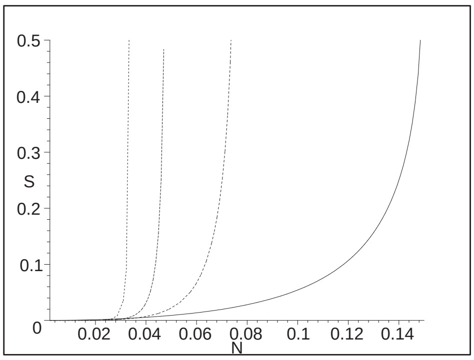

Figure 4.1.1: Plot of the Relative entropies for (from right to left) 4 to 10 dimensions for TNAdS - Bolt case.

Table 4.2: Summary of NUT Entropy and Specific Heat for TNAdS

Dim.	Entropy	Specific Heat
4	$\frac{4\pi N^2(\ell^2-6N^2)}{\ell^2}$	$\frac{8\pi N^2(12N^2-\ell^2)}{\ell^2}$
6	$\frac{32\pi^2 N^4(20N^2-3\ell^2)}{\ell^2}$	$\frac{384\pi^2 N^4(\ell^2-10N^2)}{\ell^2}$
8	$\frac{1024\pi^3 N^6(5\ell^2-42N^2)}{5\ell^2}$	$\frac{6144\pi^3 N^6(56N^2-5\ell^2)}{5\ell^2}$
10	$\frac{8192\pi^4 N^8(72N^2-7\ell^2)}{7\ell^2}$	$\frac{65536\pi^4 N^8(7\ell^2-90N^2)}{7\ell^2}$
n	$\frac{B[(n-1)(n-2)N^2-(n-3)\ell^2]}{16\pi^{3/2}\ell^2}$	$\frac{B[(n-2)(n-3)\ell^2-n(n-1)(n-2)N^2]}{16\pi^{3/2}\ell^2}$
	$A = N^{n-3}\Gamma\left(\frac{5-n}{2}\right)\Gamma\left(\frac{n}{2}\right), \quad B = (4\pi)^{(n-2)/2}\beta\Gamma\left(\frac{3-n}{2}\right)\Gamma\left(\frac{n}{2}\right)N^{n-3}$	

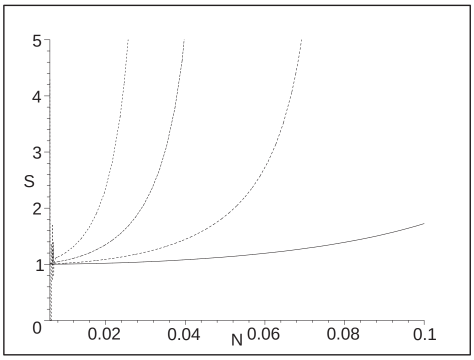

Figure 4.1.2: Plot of the Re-scaled Relative entropies for (from right to left) 4 to 10 dimensions for TNAdS - Bolt case.

An analysis of the general bolt case is somewhat awkward, though it is possible to deduce some general trends. For example, it can be seen that the relative entropies ($S_{Rel} = S_{Bolt}(r = r_{b-}) - S_{NUT}$) increase (with increasing N) faster as we increase the number of dimensions. This can be seen in figure 4.1.1, where we plot the relative entropies from four to ten dimensions. From small values of N, the entropy increases with decreasing dimensionality. However this rapidly changes once N becomes sufficiently large, in which case the entropy rapidly increases with increasing dimensionality like N^{d-2}. The plot in figure 4.1.2 shows the relative entropies with the pre-factor of N^{d-2} scaled out (so that all relative entropies are unity at $N = 0$) - and it can be seen that entropy still increases with increasing dimensionality. The general Bolt quantities for four to ten dimensions are summarized in table 4.3 (where note again for this table, $n = (d + 1)$ dimensions).

4.1.2 Taub-NUT-dS: R-approach

The general metric for the Taub-NUT metric with positive cosmological constant is given by (3.6.7) with (3.6.8). As above, the general forms for the metric determinant and Ricci scalar are easily found

$$g_{dS,R} = -(\tau^2 + n^2)^{2k} \prod_{i=1}^{k} \sin^2(\theta_i) \tag{4.1.26}$$

$$R_{dS,R} = \frac{d(d+1)}{\ell^2} \tag{4.1.27}$$

With these, the finite contribution from the bulk action (2.4.3) can be calculated in general, using the same steps as in the TNAdS case, where the binomial expansion is again used;

$$I_{B,ds(R)} = -\frac{(2k+1)\beta(4\pi)^k}{8\pi\ell^2} \sum_{i=0}^{k} \binom{k}{i} n^{2i} \frac{\tau_c^{2k-2i+1}}{2k-2i+1} \tag{4.1.28}$$

where τ_c is the largest positive root of $f(\tau)$, found from the fixed point set of ∂_t. The finite boundary and counterterm contributions can also be calculated, by finding the general forms for the determinant of the boundary

Table 4.3: Summary of Bolt quantities for TNAdS

Dim.	Mass	Action	Entropy
4	$\frac{r_b^4+(\ell^2-6N^2)r_b^2+N^2(\ell^2-3N^2)}{2\ell^2 r_b}$	$\frac{-\pi(r_b^4-\ell^2 r_b^2+N^2(3N^2-\ell^2))}{3r_b^2-3N^2+\ell^2}$	$\frac{\pi(3r_b^4+(\ell^2-12N^2)r_b^2+N^2(\ell^2-3N^2))}{3r_b^2-3N^2+\ell^2}$
6	$\frac{1}{6r_b\ell^2}\Big[3r_b^6+(\ell^2-15N^2)r_b^4 -3N^2(2\ell^2-15N^2)r_b^2 -3N^4(\ell^2-5N^2)\Big]$	$\frac{-4\pi^2}{3(5r_b^2-5N^2+\ell^2)}\Big[3r_b^6 -(5N^2+\ell^2)r_b^4 -3N^2(5N^2-2\ell^2)r_b^2 +3N^4(\ell^2-5N^2)\Big]$	$\frac{4\pi^2}{3(5r_b^2-5N^2+\ell^2)}\Big[15r_b^6 -(65N^2-3\ell^2)r_b^4 +3N^2(55N^2-6\ell^2)r_b^2 +9N^4(5N^2-\ell^2)\Big]$
8	$\frac{1}{10\ell^2 r_b}\Big[5r_b^8 +(\ell^2-28N^2)r_b^6 +5N^2(14N^2-\ell^2)r_b^4 +5N^4(3\ell^2-28N^2)r_b^2 +5N^6(\ell^2-7N^2)\Big]$	$\frac{-16\pi^3}{5(7r_b^2-7N^2+\ell^2)}\Big[5r_b^8 -(\ell^2+14N^2)r_b^6 +5N^2r_b^4\ell^2 -5N^4(3\ell^2-14N^2)r_b^2 -5N^6(\ell^2-7N^2)\Big]$	$\frac{16\pi^3}{5(7r_b^2-7N^2+\ell^2)}\Big[35r_b^8 +(5\ell^2-182N^2)r_b^6 -5N^2(5\ell^2-84N^2)r_b^4 -5N^4(154N^2-15\ell^2)r_b^2 +25N^6(\ell^2-7N^2)\Big]$
10	$\frac{1}{70\ell^2 r_b}\Big[35r_b^{10} +5(\ell^2-45N^2)r_b^8 +14N^2(45N^2-2\ell^2)r_b^6 +70N^4(\ell^2-15N^2)r_b^4 +35N^6(45N^2-4\ell^2)r_b^2 +35N^8(9N^2-\ell^2)\Big]$	$\frac{-64\pi^4}{35(9r_b^2-9N^2+\ell^2)}\Big[35r_b^{10} -5(\ell^2+27N^2)r_b^8 +14N^2(2\ell^2+9N^2)r_b^6 -70N^4(\ell^2-3N^2)r_b^4 -35N^6(27N^2-4\ell^2)r_b^2 -35N^8(9N^2-\ell^2)\Big]$	$\frac{64\pi^4}{35(9r_b^2-9N^2+\ell^2)}\Big[315r_b^{10} +5(7\ell^2-387N^2)r_b^8 +14N^2(369N^2-14\ell^2)r_b^6 +70N^4(7\ell^2-117N^2)r_b^4 +35N^6(333N^2-28\ell^2)r_b^2 +245N^8(9N^2-\ell^2)\Big]$
n	$\frac{1}{2}\Big[\sum_1 A_1 \frac{(-1)^i N^{2i} r_b^{2k-2i-1}}{(2k-2i-1)} + \frac{(2k+1)}{\ell^2}\sum_2 A_2 \frac{(-1)^i N^{2i} r_b^{2k-2i+1}}{(2k-2i+1)}\Big]$	$V\Big\{\frac{(2k+1)(-1)^k N^{2k+2}}{r_b} + \sum_3 A_3\Big[\frac{\ell^2}{r_b(2k-2i-1)} - \frac{r_b(2k+1)(k-2i+1)}{(2k-2i+1)(k-i+1)}\Big]\Big\}$	$V\Big\{\frac{(2k-1)(2k+1)(-1)^k N^{2k+2}}{r_b} + \sum_3 A_3\Big[\frac{(2k-1)\ell^2}{r_b(2k-2i-1)} + \frac{(2k+1)(2k^2+3k-2i+1)r_b}{(2k-2i+1)(k-i+1)}\Big]\Big\}$
	$\sum_1 A_1 = \sum_{i=0}^{k}\binom{k}{i}$, $\quad \sum_2 A_2 = \sum_{i=0}^{k+1}\binom{k+1}{i}$, $\quad \sum_3 A_3 = \sum_{i=0}^{k}\binom{k}{i}(-1)^i N^{2i} r_b^{2k-2i}$, $V = \frac{(4\pi)^{(n-2)/2}\beta}{16\pi\ell^2}$		

metric and the Ricci scalar of the boundary,

$$\gamma_{dS,R} = f(\tau)(\tau^2+n^2)^{2k}\prod_{i=1}^{k}\sin^2(\theta_i) \tag{4.1.29}$$

$$\mathcal{R}_{dS,R}(\gamma) = 2k\left[\frac{1}{(\tau^2+n^2)} - \frac{n^2 f(\tau)}{(\tau^2+n^2)^2}\right] \tag{4.1.30}$$

as well as finding the trace of the extrinsic curvature

$$\Theta_{dS,R} = -\left[\frac{f'(\tau)}{2\sqrt{f(\tau)}} + \frac{2k\tau\sqrt{f(\tau)}}{(\tau^2+n^2)}\right] \tag{4.1.31}$$

(4.1.29) and (4.1.31) will give the finite contribution from the boundary action as

$$I_{dS,R(\partial B)} = -\frac{2k\beta(4\pi)^k}{8\pi}m \tag{4.1.32}$$

The finite contribution from the counterterm action can be found from (2.5.13). Here, as in the TNAdS case, only the first term from the counterterm expansion is needed, which can be understood through exactly the same reasoning as used above. Thus, the finite contribution from I_{ct} can be written

$$I_{dS,R(ct)} = \frac{2k\beta(4\pi)^k}{8\pi}m \tag{4.1.33}$$

Adding together these three contributions, the finite action for general (even) dimensions in the R-approach is given by the formula

$$I_{ds,R\ \mathrm{finite}} = -\frac{\beta(4\pi)^k}{8\pi}\left[m + \frac{d}{\ell^2}\sum_{i=0}^{k}\binom{k}{i}n^{2i}\frac{\tau_c^{2k-2i+1}}{2k-2i+1}\right] \tag{4.1.34}$$

The only non-vanishing conserved charge will be the mass, associated with ξ_t. Using (2.5.14), we get upon including the counterterms,

$$\mathfrak{M}_{dS} = \frac{1}{8\pi}\int d^{d-1}x\,\sqrt{\gamma}\left\{\Theta_{ab} - \Theta\gamma_{ab} - \frac{(d-1)}{\ell}\gamma_{ab} + \ldots\right\}\xi^a u^b N_{lpse} \tag{4.1.35}$$

where here, as in the TNAdS case, only these terms are necessary to compute the finite mass, for the same reasons. The finite conserved mass in general dimensions $(d+1) = 2k+2$ is then given by

$$\mathfrak{M}_{dS,R} = -\frac{2k(4\pi)^k}{8\pi}m \tag{4.1.36}$$

where m is solved for in terms of τ and the NUT parameter through the first condition for a regular bolt, given on page 63.

Using the Gibbs-Duhem relation and (4.1.36), (4.1.34), the entropy formula for general dimensions is given by

$$S_{dS,R} = \frac{(4\pi)^k\beta}{8\pi}\left[\frac{(2k+1)}{\ell^2}\sum_{i=0}^{k}\binom{k}{i}n^{2i}\frac{\tau_c^{2k-2i+1}}{2k-2i+1} - (2k-1)m\right] \quad (4.1.37)$$

Note that again, since $\tau_c = \tau_c(n)$, a general formula for the specific heat is very difficult to find, and is best left to calculation in the specific dimension of interest.

It should also be noted that none of the above formulae required the use of the consistency condition (ii) on page 63, which can also be written

$$|f'(\tau_c)| = \frac{2q}{(d+1)|n|} \quad (4.1.38)$$

This equation will have, in general, four solutions for τ_c, (two positive, two negative), yielding two possible relationships between the parameters m and n from the positive pair. This will give two distinct spacetimes with two distinct sets of characteristics with regards to the entropy and conserved mass. While it is easy to solve this equation in a specific dimension, it is difficult to do so in closed form in general dimensions.

4.1.3 Taub-NUT-dS: C-approach

The metric here is given by (3.6.11), (3.6.12). This metric is the same as the R-approach metric with a few signs changed, and so exactly the same reasoning can be used to calculate the general finite contributions for the C-approach. (Note also that one could find the finite formulae through appropriate continuation and sign counting of the TNAdS quantities here). I am working at future infinity. The metric determinant and Ricci scalar are

$$g_{dS,C} = (\rho^2 - N^2)\prod_{i=1}^{k}\sin^2(\theta_i) \quad (4.1.39)$$

$$R_{dS,C} = \frac{d(d+1)}{\ell^2} \quad (4.1.40)$$

Through the use of the binomial expansion, the finite bulk action is found to be given by

$$I_{B,C\ \text{finit}} = -\frac{2k(4\pi)^k\beta}{8\pi\ell^2}\sum_{i=0}^{k}\binom{k}{i}(-1)^i N^{2i}\frac{\rho_+^{2k-2i+1}}{2k-2i+1} \tag{4.1.41}$$

where ρ_+ is the largest positive root of $f(\rho)$. The boundary metric is given by $\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$, so that

$$\begin{aligned} \gamma_{dS,C} &= -f(\rho)(\rho^2 - N^2)^{2k}\prod_{i=1}^{k}\sin^2(\theta_i) && (4.1.42)\\ \mathcal{R}_{dS,C} &= 2k\left[\frac{1}{(\rho^2-N^2)} + \frac{N^2 f(\rho)}{(\rho^2-N^2)^2}\right] && (4.1.43)\end{aligned}$$

are the boundary metric determinant and Ricci scalar, respectively. The trace of the extrinsic curvature can also be found,

$$\Theta_{dS,C} = -\left[\frac{f'(\rho)}{2\sqrt{f(\rho)}} + \frac{(2k+1)\rho\sqrt{f(\rho)}}{(\rho^2-N^2)}\right] \tag{4.1.44}$$

Using the same steps as above, the boundary and counterterm contributions can be found to be the same as in the R-approach, (4.1.32) and (4.1.33), and so the general action is

$$I_{dS,C\ \text{finite}} = -\frac{(4\pi)^k\beta}{8\pi}\left[m + \frac{d}{\ell^2}\sum_{i=0}^{k}\binom{k}{i}(-1)^i N^{2i}\frac{\rho_+^{2k-2i+1}}{2k-2i+1}\right] \tag{4.1.45}$$

The conserved mass is also found through the same steps,

$$\mathfrak{M}_{dS,C} = -\frac{2k(4\pi)^k}{8\pi}m \tag{4.1.46}$$

The general formula for the entropy in the C-approach is again found through use of the Gibbs-Duhem relation,

$$S_{dS,C} = \frac{(4\pi)^k\beta}{8\pi}\left[\frac{d}{\ell^2}\sum_{i=0}^{k}\binom{k}{i}(-1)^i N^{2i}\frac{\rho_+^{2k-2i+1}}{2k-2i+1} - (2k-1)m\right] \tag{4.1.47}$$

In the C-approach, however, we now have NUT and Bolt solutions analogous to the TNAdS case, depending on the co-dimensionality of the fixed point set of ∂_T.

General C-approach; NUT

For the NUT solution, the fixed point set of ∂_T is zero dimensional at $\rho_+ = N$. By substituting in (3.6.14) and $\rho_+ = N$, general expressions for the NUT action and entropy in $(d+1)$ dimensions can be found from (4.1.45) and (4.1.47). The action is

$$I_{dS(C),NUT} = \frac{(4\pi)^k \beta N^{2k-1}}{16\pi^{3/2}\ell^2}\left[\ell^2 + 2kN^2\right]\Gamma\left(\frac{1-2k}{2}\right)\Gamma(k+1) \qquad (4.1.48)$$

and the entropy

$$S_{dS,NUT} = \frac{(4\pi)^k \beta N^{2k-1}}{16\pi^{3/2}\ell^2}\left[(2k-1)\ell^2 + 2k(2k+1)N^2\right]\Gamma\left(\frac{1-2k}{2}\right)\Gamma(k+1) \qquad (4.1.49)$$

With the specific equation (4.1.49), using $C = -\beta\partial_\beta S$ the general equation for the NUT specific heat can be found;

$$\begin{aligned} C_{dS,NUT} &= -\frac{(4\pi)^k \beta N^{2k-1}}{16\pi^{3/2}\ell^2}\Gamma\left(\frac{1-2k}{2}\right)\Gamma(k+1)\cdot \\ &\quad \left[2k(2k-1)\ell^2 + 2k(2k+1)(2k+2)N^2\right] \end{aligned} \qquad (4.1.50)$$

Note, as should be expected from the discussions above, all of these can be found by taking $\ell \to \mathrm{i}\ell$ in the AdS quantities (4.1.16), (4.1.17) and (4.1.18). Indeed, it can be shown in specific dimensions that, although the above formulae satisfy the First Law, the root at $\rho = N$ is not the largest root of $f(\rho)$, and hence, in any dimension, the ds-NUT solution is not a valid solution. The fact that the above entropy satisfies the First Law is merely a consequence of the fact that this entropy is derivable by analytic continuation ($\ell \to \mathrm{i}\ell$) of the AdS-NUT entropy.

General C-approach: Bolt

Recall that the first condition for a regular bolt solution will force the general bolt mass parameter to be (3.6.15). Thus, general formulae for the bolt action and entropy in the C-approach can be found by substituting (3.6.15) into (4.1.45) and (4.1.47). The general action is thus

$$\begin{aligned} I_{bolt,dS,C} &= -\frac{(4\pi)^k\beta}{16\pi\ell^2}\Bigg\{\frac{(2k+1)(-1)^{k+1}N^{2k+2}}{\rho_b} \\ &+\sum_{i=0}^{k}\binom{k}{i}(-1)^i N^{2i}\rho_b^{2k-2i}\left[\frac{\ell^2}{\rho_b(2k-2i-1)}\right. \\ &\left.+\frac{(2k+1)(k-2i+1)\rho_b}{(k-i+1)(2k-2i+1)}\right]\Bigg\} \end{aligned} \tag{4.1.51}$$

and the general entropy is

$$\begin{aligned} S_{bolt,dS,C} &= \frac{(4\pi)^k\beta}{16\pi\ell^2}\Bigg\{-\frac{(2k+1)(2k-1)(-1)^{k+1}N^{2k+2}}{\rho_b} \\ &+\sum_{i=0}^{k}\binom{k}{i}(-1)^i N^{2i}\rho_b^{2k-2i}\left[-\frac{\ell^2(2k-1)}{\rho_b(2k-2i-1)}\right. \\ &\left.+\frac{(2k^2+3k-2i+1)(2k+1)\rho_b}{(k-i+1)(2k-2i+1)}\right]\Bigg\} \end{aligned} \tag{4.1.52}$$

Again here, a general expression for the specific heat is difficult to find, and also analysis is best left to specific dimensions. Note also that in employing these formulae, one must further specify to the upper bolt solution (ρ_+) as in the AdS case. Note that, though there is always a lower bolt solution, it is not a valid solution as it can be shown that this lower bolt root is not the largest root of the metric function.

4.2 Six dimensional Example - TNAdS

Calculations in a specific dimension can be done by taking the above general forms, and using the appropriate k to get the dimension of interest. However, obviously the calculation can also be done in full in the specific dimension. As an example, here I will calculate the six dimensional actions, entropies, etc., as a specific demonstration of the AdS/CFT (and next for the four dimensional example in the dS/CFT) counterterm method.

The Euclidean section of the six dimensional metric, using $S^2 \times S^2$ as a base space, is given by

$$\begin{aligned} ds^2 &= f(r)\left[d\tau + 2N\cos(\theta_1)d\phi_1 + 2N\cos(\theta_2)d\phi_2\right]^2 + \frac{dr^2}{f(r)} \\ &\quad +(r^2 - N^2)(d\theta_1^2 + \sin^2(\theta_1)d\phi_1^2 + d\theta_2^2 + \sin^2(\theta_2)d\phi_2^2) \end{aligned} \tag{4.2.1}$$

with

$$f(r) = \frac{3r^6 + (\ell^2 - 15N^2)r^4 - 3N^2(2\ell^2 - 15N^2)r^2 - 6mr\ell^2 - 3N^4(\ell^2 - 5N^2)}{3\ell^2(r^2 - N^2)^2} \tag{4.2.2}$$

Another form, mentioned in [17], is to use $\mathcal{B} = \mathbb{CP}^2$ as a base space,

$$ds^2 = f(r)\left[d\tau + A\right]^2 + \frac{dr^2}{f(r)} + (r^2 - N^2)d\Sigma_2^2 \tag{4.2.3}$$

where $d\Sigma_2^2$ is the metric for the $\mathbb{CP}^2$ space,

$$\begin{aligned} d\Sigma_2^2 &= \frac{du^2}{(1+u^2/6)^2} + \frac{u^2}{4(1+u^2/6)^2}\left[d\psi + \cos(\theta)d\phi\right]^2 \\ &\quad + \frac{u^2}{4(1+u^2/6)}\left(d\theta^2 + \sin^2(\theta)d\phi^2\right) \end{aligned} \tag{4.2.4}$$

A is the one-form, given by

$$A = \frac{u^2 N}{2(1+u^2/6)}(d\psi + \cos(\theta)d\phi) \tag{4.2.5}$$

This metric (4.2.3) also uses (4.2.2), and yields the same action and total mass, with appropriate change in the volume element.

Calculating the action (2.3.2), and including the counterterm action to cancel the divergences, the finite, six dimensional action is

$$I = \frac{2\pi\beta(-3r_+^5 + 10N^2r_+^3 - 15N^4r_+ + 3m\ell^2)}{3\ell^2} \tag{4.2.6}$$

where r_+ is as usual the largest positive root of $f(r)$, determined from the fixed point set of ∂_τ. The period of τ is

$$\beta = 12\pi N \tag{4.2.7}$$

here. Note that as pointed out in section 3.6.1 below equation (3.6.3), (4.2.7) is determined by demanding regularity of the manifold so that the singularities at $\theta_i = 0, \pi$ are coordinate artifacts (for $i = 1, 2$). Since the metric is not rotating, we only have the conserved mass to calculate. Using the full expansion of the boundary stress-energy $T_{\mu\nu}$ for six dimensions (see [56]), the mass can be calculated

$$\mathfrak{M}_\xi = \frac{(4\pi)^2}{8\pi} \int_0^\pi d\theta_1 \int_0^\pi d\theta_2 \sqrt{\gamma} \left[\Theta_{\mu\nu} - \gamma_{\mu\nu}\Theta - \frac{(d-1)}{\ell}\gamma_{\mu\nu} + \ldots \right] u^\mu \xi^\nu$$

where $\xi^\mu = [1, 0, 0, 0, 0, 0]$ is the timelike Killing vector, and u^μ is the timelike unit vector. Upon explicit calculation, the finite conserved mass is found to be

$$\mathfrak{M} = 8\pi m \tag{4.2.8}$$

4.2.1 Taub-NUT-AdS solution

The zero dimensional fixed point set of ∂_τ occurs when $r = N$, giving the NUT solution. This implies $f(N) = 0$, which will give, for the mass parameter [17]

$$m_n = \frac{4N^3(6N^2 - \ell^2)}{3\ell^2} \tag{4.2.9}$$

Note that the overall sign of this mass is minus the overall sign from four dimensions (see table 4.1).

Inserting $r = N$ and the period (4.2.7) into the action, we get

$$I_{\text{NUT}} = \frac{32\pi^2 N^4(4N^2 - \ell^2)}{\ell^2} \tag{4.2.10}$$

By inserting (4.2.9) into the conserved mass, we can use (4.2.8) and (4.2.10) in the Gibbs-Duhem relation (2.3.14) to obtain the entropy for the NUT solution

$$S_{\text{NUT}} = \frac{32\pi^2 N^4(20N^2 - 3\ell^2)}{\ell^2} \tag{4.2.11}$$

Finally, the specific heat $C = -\beta\partial_\beta S$ can now be found (using (4.2.7))

$$C_{\text{NUT}} = \frac{384\pi^2 N^4(\ell^2 - 10N^2)}{\ell^2} \tag{4.2.12}$$

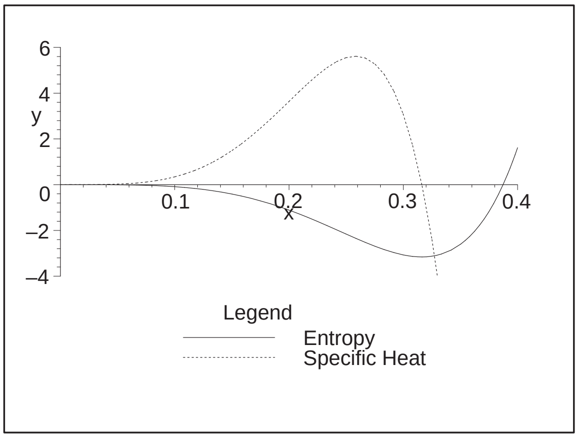

Figure 4.2.1: Plot of the NUT Entropy and Specific Heat vs. N in 6 dimensions

The above equations have been checked to verify they satisfy the first law of black hole thermodynamics, $dS = \beta dH$. Here, the Hamiltonian H is equal to the conserved mass $\mathfrak{M}$, with m given by (4.2.9).

The entropy and specific heat can now be plotted as functions of N (figure 4.2.1). From the figure, it can be seen that the entropy is negative for $N < \ell\sqrt{3/20}$, and the specific heat is negative for $N > \ell/\sqrt{10}$. More importantly, from the figure it can be seen that nowhere are the entropy and the specific heat both positive for the same range of N. Since it is necessary for both to be positive for the solution to be thermodynamically stable, this means that the NUT solution in six dimensions is not thermally stable. Note that this is different from the four dimensional solution, where there does exist a range where both are positive. This trend continues: for dimensions $8, 12, 16, \ldots$, the solution is thermally stable in a given range (as in four dimensions), and for dimensions $10, 14, 18, \ldots$, there is no range of stability, as in six dimensions.

4.2.2 Taub-Bolt-AdS solution

The Bolt solution requires the fixed point set of ∂_τ to be four dimensional, implying $r_+ = r_b > N$ and giving

$$m_b = \frac{3r_b^6 + (\ell^2 - 15N^2)r_b^4 - 3N^2(2\ell^2 - 15N^2)r_b^2 - 3N^4(\ell^2 - 5N^2)}{6r_b\ell^2} \quad (4.2.13)$$

The conditions in six dimensions for a regular Bolt solutions, given in [17], are (i) $f(r_b) = 0$ and (ii) $f'(r_b) = 1/(3N)$. From (ii), r_b is given as a function of N as

$$r_{b\pm} = \frac{\ell^2 \pm \sqrt{\ell^4 - 180N^2\ell^2 + 900N^4}}{30N} \tag{4.2.14}$$

(where note the $q = 1$ here). Since we require the solution to be real, the discriminant of the square root forces the range of N to be

$$N \leq \left(\frac{\sqrt{15}}{15} - \frac{\sqrt{30}}{30}\right)\ell = N_{max} \tag{4.2.15}$$

and of course greater than zero.

The action is obtained from (4.2.6)

$$I_{Bolt} = \frac{-4\pi^2(3r_b^6 - (5N^2 + \ell^2)r_b^4 - 3N^2(5N^2 - 2\ell^2)r_b^2 + 3N^4(\ell^2 - 5N^2))}{3(5r_b^2 + \ell^2 - 5N^2)} \tag{4.2.16}$$

where regularity requires that the period be

$$\begin{aligned} \beta_{Bolt} &= \frac{4\pi}{F'(r_b)} \\ &= \left(6\pi\ell^2(r_b^2 - N^2)^3\right)\left[3r_b^7 - 9r_b^5N^2 + N^2(4\ell^2 - 15N^2)r_b^3\right. \\ &\quad \left.+9m\ell^2r_b^2 + 3N^4(4\ell^2 - 25N^2)r_b + 3m\ell^2N^2\right]^{-1} \end{aligned} \tag{4.2.17}$$

The temperature of the NUT and Bolt solutions is of course the same, as can be seen by substituting m_b and either of $r_{b\pm}$ into β_{Bolt}.

The Bolt entropy is

$$S_{Bolt} = \frac{4\pi^2(15r_b^6 - (65N^2 - 3\ell^2)r_b^4 + 3N^2(55N^2 - 6\ell^2)r_b^2 + 9N^4(5N^2 - \ell^2))}{3(5r_b^2 + \ell^2 - 5N^2)} \tag{4.2.18}$$

and this can also be checked to satisfy the first law (though each branch $r_{b\pm}$ must be checked separately). The specific heat can be computed for each branch specifically, giving

$$\begin{aligned} C_{Bolt}(r_{b\pm}) &= \frac{-8\pi^2}{50625}\left[\frac{\ell^6(90N^2 - \ell^2)}{N^4} \pm \left((\ell^2 + 30N^2)(-\ell^2 + 30N^2) \times \right.\right. \\ &\quad \left.\left.\frac{(\ell^8 - 180\ell^6N^2 + 5400\ell^4N^4 - 162000\ell^2N^6 + 810000N^8)}{N^4\ell^2\sqrt{\ell^4 + 900N^4 - 180N^2\ell^2}}\right)\right] \end{aligned} \tag{4.2.19}$$

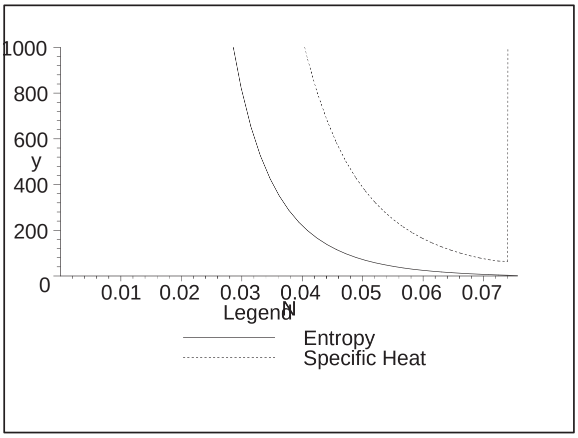

Figure 4.2.2: Plot of Entropy and Specific Heat vs. N for the upper branch Bolt solutions in 6 dimensions.

Note that r_{b+} diverges as $\ell \to \infty$, so only the lower branch r_{b-} need be considered in the flat space limit. We found $r_{b-} \to r_0 = 3N$ and

$$\begin{array}{rcl} I_{Bolt}(r_b = r_{b-}) & \to & 32\pi^2 N^4 \\ S_{Bolt}(r_b = r_{b-}) & \to & 96\pi^2 N^4 \\ C_{Bolt}(r_b = r_{b-}) & \to & -384\pi^2 N^4 \end{array}$$

and the Bolt mass parameter (4.2.13) is

$$m_b \to \frac{(r_b^4 - 6N^2 r_b^2 - 3N^4)}{6r_b} = \frac{4}{3}N^3$$

confirming the results of ref. [17].

The upper and lower branch entropies and specific heats are plotted vs. N, in figures 4.2.2, 4.2.3 and 4.2.4. Note that the upper branch plot, figure 4.2.2, shows that the specific heat and entropy are everywhere positive, the lower branch entropy is always positive, and the lower branch specific heat is always negative. This means the upper branch is thermodynamically stable, while the lower branch is unstable.

4.2.3 Nöether Results for TNAdS 6

The results here were first presented in [16]. I reproduce them here as a comparison to the counterterm calculation presented above.

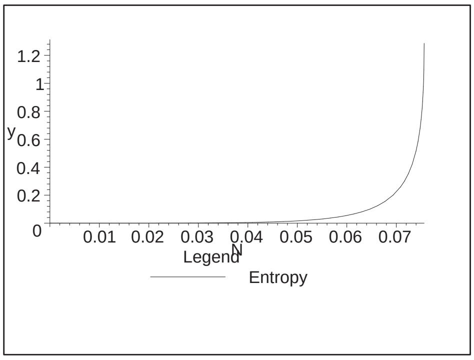

Figure 4.2.3: Plot of lower branch Bolt Entropy vs. N in 6 dimensions

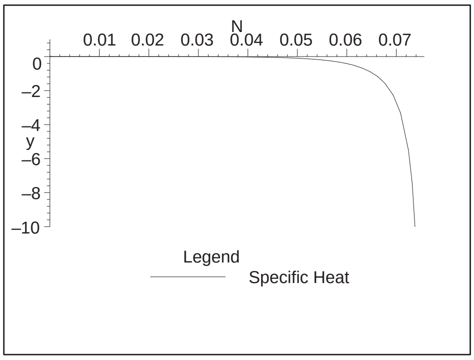

Figure 4.2.4: Plot of lower branch Bolt Specific Heat vs. N in 6 dimensions

Consider now taking the Taub-Bolt-AdS solution (4.2.13), with metric (4.2.1) and metric function (4.2.2), as the dynamical metric to be studied. The reference spacetime is then chosen to be the NUT solution (4.2.9), with (4.2.1) and (4.2.2) again used. Note that despite the use of the term "mass" for m_b and m_n, these have nothing to do *a priori* with the mass of the solution, though the Nöether theorem does justify the description "mass".

The following spacetime vector field,

$$\xi = \partial_\tau + a\partial_{\theta_1} + b\partial_{\theta_2} \tag{4.2.20}$$

will (by definition) produce the Nöether conserved quantity $Q = m + aJ_1 + bJ_2$ of the dynamical Taub-Bolt-AdS metric relative to the background Taub-NUT-AdS metric. Here, of course, $J_1 = J_2 = 0$. The superpotential (2.2.20) is evaluated on $(g, \overline{g})$, and then integrated on the spatial region $(\tau = \tau_0, r = r_0)$. The solution is then Taylor series expanded around $r_0 = \infty$. The three separate conserved charges are thus given by:

$$\begin{aligned} Q_1 &= \frac{4\pi^2}{\kappa\ell^2}\left(2r_0^5 - 4N^2r_0^3 + \frac{2r_0}{3}N^2(4\ell^2 - 21N^2) + \frac{1}{r_b}(3r_b^6 + (\ell^2 - 15N^2)r_b^4\right. \\ &\quad \left. +3N^2(15N^2 - 2\ell^2)r_b^2 + 3N^4(5N^2 - \ell^2))\right) + \mathcal{O}\left(\frac{1}{r_0}\right) \end{aligned} \tag{4.2.21}$$

$$\begin{aligned} Q_2 &= \frac{4\pi^2}{3\kappa\ell^2 r_b}\left(3r_b^6 + (\ell^2 - 15N^2)r_b^4 + 3N^2(15N^2 - 2\ell^2)r_b^2\right. \\ &\quad \left. +8N^3(\ell^2 - 6N^2)r_b + 3N^4(5N^2 - \ell^2)\right) + \mathcal{O}\left(\frac{1}{r}\right) \end{aligned} \tag{4.2.22}$$

$$\begin{aligned} Q_3 &= \frac{4\pi^2}{\kappa\ell^2}\left(-2r_0^5 + 4N^2r_0^3 + \frac{2r_0}{3}N^2(21N^2 - 4\ell^2) + 8N^3(\ell^2 - 6N^2)\right) \\ &\quad +\mathcal{O}\left(\frac{1}{r}\right) \end{aligned} \tag{4.2.23}$$

Note that $Q_1, Q_3 \to \infty$ as $r_0 \to \infty$, but that adding these together, the total conserved quantity does not. The finite conserved charge is in fact

$$\begin{aligned} Q &= \frac{16\pi^2}{\kappa\ell^2}\left(r_b^5 + \frac{1}{3}(\ell^2 - 15N^2)r_b^3 + N^2(15N^2 - 2\ell^2)r_b\right. \\ &\quad \left. +\frac{8}{3}N^3(\ell^2 - 6N^2) + \frac{N^4}{r_b}(5N^2 - \ell^2)\right) + \mathcal{O}\left(\frac{1}{r_0}\right) \end{aligned} \tag{4.2.24}$$

This is in fact equal to

$$Q = \frac{32\pi^2}{\kappa}(m_b - m_n) \tag{4.2.25}$$

justifying the interpretation of Q as the relative mass.

From the first law of thermodynamics,

$$\delta S = \beta \delta M = 12\pi N \delta Q \tag{4.2.26}$$

which can be interpreted as, and indeed is shown to be, the relative entropy between the Bolt and NUT solutions. Note that in taking δQ, one must remember that $r_b = r_b(N)$ for both $r_{b\pm}$. Substituting in either $r_{b\pm}$ into (4.2.24) and integrating to give the entropy will give the same answer,

$$\begin{aligned} S &= \frac{16\pi^3}{3375\kappa N^3 \ell^2}\left(\left(\ell^8 - 90N^2\ell^6 - 300\ell^4 N^4 - 27000N^6\ell^2 + 540000N^8\right) r_b \right. \\ & \left. \quad + N(30N^2 - \ell^2)(3\ell^6 + 80N^2\ell^4 + 1500N^4\ell^2 - 18000N^6)\right) & (4.2.27) \\ &= \frac{4\pi}{\kappa}\left(S_{Bolt} - S_{NUT} - \frac{112\pi^2}{675}\ell^4\right) & (4.2.28) \end{aligned}$$

This result shows that the classical entropy, computed through either the counterterm or Nöether method, is the same, up to a constant of integration.

4.3 Four dimensional Example - TNdS

4.3.1 R-approach

Recall that the metric for the R-approach in $(3+1)$ dimensions, with an S^2 base space, is given by

$$ds_R^2 = f(\tau)\left[dt + 2n\cos(\theta)d\phi\right]^2 - \frac{d\tau^2}{f(\tau)} + (\tau^2 + n^2)(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{4.3.1}$$

with

$$f(\tau) = \frac{\tau^4 + (6n^2 - \ell^2)\tau^2 + n^2(\ell^2 - 3n^2) + 2m\tau\ell^2}{(\tau^2 - N^2)\ell^2} \tag{4.3.2}$$

The period must respect the condition (3.5.3), and yields the equation (3.5.14). The finite action can be calculated from either the general formula (4.1.34) or directly from (2.5.12), to give

$$I_{R,4d} = -\frac{\beta}{2\ell^2}(m_R\ell^2 + \tau_c^2 + 3n^2\tau_c) \tag{4.3.3}$$

and recall that m_R is given by (3.5.15). The finite conserved mass can also be found

$$\mathfrak{M}_{R,4d} = -m_R \tag{4.3.4}$$

near future infinity. The entropy, either through use of the Gibbs-Duhem relation or from the general formula above, can also be found

$$S_{R,4d} = -\frac{\beta(m_R\ell^2 - 3n^2\tau_c - \tau_c^3)}{2\ell^2} \tag{4.3.5}$$

Note that these results are satisfied by the solution to the second condition for a regular bolt, from page 63, the solution given by $(3.5.16)^+$, rewritten here for convenience:

$$\tau_c^+ = \frac{q\ell^2 + \sqrt{q^2\ell^4 - 144n^4 + 48n^2\ell^2}}{12n} \tag{4.3.6}$$

The high temperature ($n \to 0$) and flat space ($\ell \to \infty$) limits of τ_c^+ are infinite.

From these results, one can straightforwardly obtain the mass and temperature by substituting τ_c^+ into (3.5.15) and (3.5.14) respectively. Following the notation in [20], I will call these solutions R_4^+ with action I_4^+ and entropy S_4^+.

From figure 4.3.1, it can be seen that $\mathfrak{M}_R^+$ is always positive. This means, since the mass of pure dS space in four dimensions $\mathfrak{M}_{ds} = 0$, that this solution R_4^+ violates the maximal mass conjecture of [21], at least as written. Note that the solution violates the conjecture for all q, since $\mathfrak{M}_R^+ > 0$.

Note also that, as mentioned in the introduction, the use of (2.5.14) to calculate (4.3.4) did not depend on the existence of horizons, nor on the CTC's present in the TNdS spacetime.

From (4.3.3), using (3.5.14) and (3.5.16), the R^+ action is

$$\begin{aligned} I_{R,4d}^+(\tau_0 = \tau_0^+) &= -\frac{\pi\ell^2}{216}\frac{(72n^2 + q^2\ell^2)}{n^2} \\ &\quad + \frac{\pi}{216}\frac{(-q^2\ell^4 + 144n^4 - 48n^2\ell^2)\sqrt{q^2\ell^4 - 144n^4 + 48n^2\ell^2}}{n^2q\ell^2} \end{aligned} \tag{4.3.7}$$

and from (4.3.5), the entropy is

$$S_{R,4d}^+ = \frac{\pi\ell^2(24n^2 + q^2\ell^2)}{72n^2} + \frac{\pi(144n^4 + q^2\ell^4)\sqrt{q^2\ell^4 - 144n^4 + 48n^2\ell^2}}{72n^2q\ell^2} \tag{4.3.8}$$

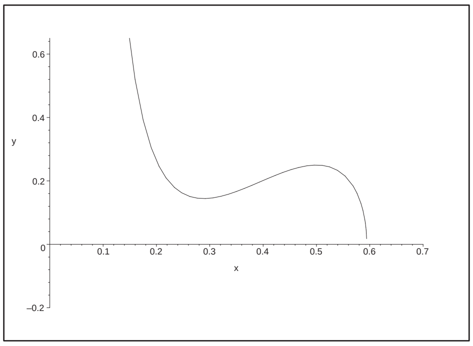

Figure 4.3.1: Plot of the bolt ($\tau_b = \tau_{b+}$) mass for R-approach.

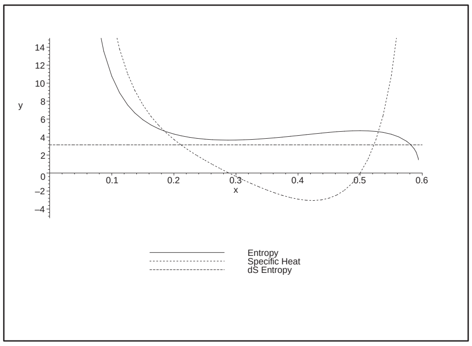

Figure 4.3.2: Plot of the bolt entropy and specific heat (for $q = 1$). Horizontal line is pure dS entropy.

It can be easily checked that this satisfies the first law. From the entropy and the relation $C^+_{R,4d} = -\beta^+_R \partial_{\beta^+_R} S^+_{R,4d}$, we find for the specific heat

$$C^+_R(\tau^+_0) = \frac{\pi\ell^4 q^2}{36n^2} + \frac{\pi(-144q^2\ell^4 n^4 + 41472n^8 - 10368n^6\ell^2 + 24n^2\ell^6 q^2 + q^4\ell^8)}{36q\ell^2 n^2\sqrt{q^2\ell^4 - 144n^4 + 48n^2\ell^2}} \tag{4.3.9}$$

The plot for the entropy/specific heat is in figure 4.3.2.

It can be seen in figure 4.3.2 that the entropy is always positive (and almost always greater than $\pi\ell^2$, except for NUT charge in a range near the maximal value $n_{max} = .5941\ell$), but the specific heat is positive only outside the range $.2886751346\ell < n < .5\ell$; thus, this solution is only stable for n outside this range.

The pure dS entropy is shown in figure 4.3.2 as the horizontal line. As can be seen, the entropy of the TNdS solution is greater than the dS entropy for all n except near n_{max}. This means the N-bound is violated for all n except near n_{max} by the TNdS solution.

4.3.2 C-approach

Here, as mentioned, the metric is obtained from the R-approach metric by a Wick rotation of the time and NUT parameters, to give

$$ds^2 = -f(\rho)\,[dT + 2N\cos(\theta)d\phi]^2 - \frac{d\rho^2}{f(\rho)} + (\rho^2 - N^2)(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{4.3.10}$$

with the metric function

$$f(\rho) = \frac{\rho^4 - (\ell^2 + 6N^2)\rho^2 + 2m\rho\ell^2 - N^2(\ell^2 + 3N^2)}{(\rho^2 - N^2)\ell^2} \tag{4.3.11}$$

and the cosmological constant is now $\Lambda = \frac{3}{\ell^2}$. Recall that the period must satisfy the condition (3.5.6).

The action can be found to be

$$I_{C,4d} = -\frac{\beta_c(\rho_+^3 - 3N^2\rho_+ + m\ell^2)}{2\ell^2} \tag{4.3.12}$$

before specifying to the bolt solution, where as usual ρ_+ is the largest positive root of $f(\rho)$. The finite conserved mass is given by

$$\mathfrak{M}_{C,4d} = -m \tag{4.3.13}$$

near future infinity, and applying the Gibbs-Duhem relation, the entropy is given by

$$S_{C,4d} = \frac{\beta_c(\rho_+^3 - 3N^2\rho_+ - m\ell^2)}{2\ell^2} \tag{4.3.14}$$

Bolt solution

The mass, period and the two radii solutions were (3.5.10), (3.5.12) and (3.5.11) respectively. Recall that the lower branch solution is invalid, and so I will present only the upper branch solution here. Substituting in $\rho_b = \rho_{b+}$ into $\mathfrak{M}_{C,b}$, we can see that the Taub-Bolt-C solution is also a counter-example to the maximal mass conjecture of [21] for certain values of N. The conserved mass $\mathfrak{M}(\rho_b = \rho_{b+})$ is positive for $N < 0.2066200733$, and thus the solution

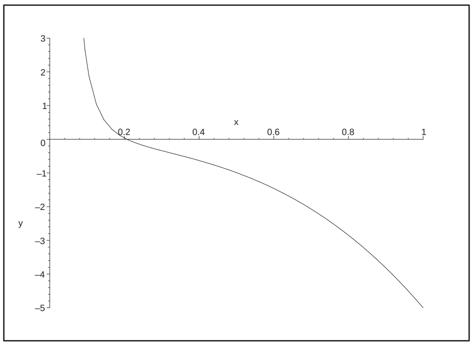

Figure 4.3.3: Plot of ($\rho_b = \rho_{b+}$) bolt mass $\mathfrak{M}_{b+}$ (for $q = 1$).

violates the conjecture for N less than this value. (This trend holds for higher values of q, with the cross-over point for the solution increasing with increasing q).

Note here, as in the R-approach, the calculation of (4.3.13) is again done at future infinity, and does not depend on the existence of the horizon of the spacetime, nor the CTC's present in the spacetime.

The bolt action is, using (4.3.12)) and (3.5.12)

$$\begin{aligned} I_{C,bolt4d}(\rho_b = \rho_{b+}) &= -\frac{\pi(\rho_b^4 + \ell^2\rho_b^2 + N^2(\ell^2 + 3N^2))}{\rho_b}\left|\frac{\rho_b}{3\rho_b^2 - 3N^2 - \ell^2}\right| \\ &= -\frac{\pi}{216}\left[\frac{(q^2\ell^2 + 72N^2)\ell^2}{N^2}\right. \\ &\quad \left. + \frac{(q^2\ell^4 + 144N^4 + 48N^2\ell^2)^{(3/2)}}{N^2 q\ell^2}\right] \end{aligned} \tag{4.3.15}$$

and from (4.3.14), the bolt entropy is

$$\begin{aligned} S^{+}_{C,bolt4d} &= \frac{\pi(3\rho_b^4 - (\ell^2 + 12N^2)\rho_b^2 - N^2(\ell^2 + 3N^2))}{\rho_b}\beta \\ &= \frac{\pi}{72}\left[\frac{(q^2\ell^2 + 24N^2)\ell^2}{N^2}\right. \\ &\quad \left. + \frac{(q\ell^2 - 12N^2)(q\ell^2 + 12N^2)\sqrt{q^2\ell^4 + 144N^4 + 48N^2\ell^2}}{\ell^2 q N^2}\right] \end{aligned} \tag{4.3.16}$$

It can again be checked that this satisfies the first law. From this entropy,

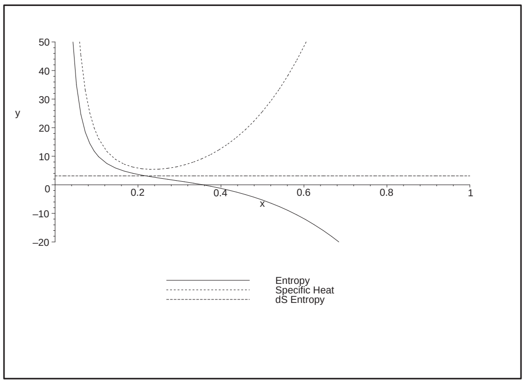

Figure 4.3.4: Plot of the upper branch bolt entropy and specific heat (for $q = 1$). Horizontal line is pure dS entropy.

the specific heat can be found for the bolt;

$$
\begin{aligned}
C^{+}_{C,bolt4d} &= \frac{\pi}{36N^2}\Big[q^2\ell^4 \\
&+ \frac{(144q^2\ell^4N^4 + 41472N^8 + 10368N^6\ell^2 + 24N^2\ell^6q^2 + q^4\ell^8)}{q\ell^2\sqrt{q^2\ell^4 + 144N^4 + 48N^2\ell^2}}\Big]
\end{aligned} \tag{4.3.17}
$$

A plot of the entropy and specific heat (for $q = 1$) appears in figure 4.3.4. From the figure, we can see that the entropy is positive for $N < .3562261982\ell$, and the specific heat is always positive; thus this solution is thermodynamically stable for $N < .3562261982\ell$. Note that this trend continues for $q > 1$. Also note that the upper branch entropy violates the N-bound for $N < .2180098653\ell$. This can be seen in figure 4.3.4, where the horizontal line is the pure dS entropy.

Chapter 5

M-Branes and Taub-NUT Metrics

The Taub-NUT metric also has another use when one considers eleven dimensional supergravity. This interesting idea was considered recently by Cherkis and Hashimoto [27]. They constructed an M2-brane solution by lifting a D6 brane to a four dimensional Taub-NUT geometry embedded in M-theory and then placed M2 branes in the Taub-NUT background geometry. However, they only considered a single four dimensional Taub-NUT metric. One can also embed a Taub-NUT or similar metric (in [28, 29], we used Taub-NUT and Eguchi-Hanson metrics) into the eleven dimensional equations of motion, either those that describe an M2-brane (a "membrane"), or those describing M5-branes. These metrics can then be solved for their harmonic functions (sometimes only numerically), supplying a full solution to the eleven-dimensional equations of motion. Further, due to the form of such metrics, once embedded into eleven dimensions they can be used to reduce the solution down to a ten dimensional type IIA metric, using the well known Kaluza-Klein (KK) reduction. Embedding the four dimensional Taub-NUT or Eguchi-Hanson metrics preserves 1/8 of the supersymmetry. The four dimensional Taub-Bolt metric can also be embedded into both the M2 and M5 brane solutions; also, higher dimensional Taub-NUT/Bolt metrics can be embedded into the M2 brane solution. These embeddings do not preserve any supersymmetry, but are however interesting in that they exhibit properties qualitatively similar to the supersymmetry preserving cases. For instance, the harmonic function of the M-brane behaves the same way near the brane core, as well as at infinity.

A map of all possible D-brane combinations giving rise to supersymmetric solutions has been assembled [64], and it should be noted that some of the non-supersymmetric solutions discussed in [28, 29] and to be discussed here have a brane structure listed in this map. Thus, each metric must be explicitly checked to see if it preserves any supersymmetry, and cannot be simply compared to the map. For example, by embedding the four dimensional Taub-Bolt into eleven dimensions, and reducing to ten dimensions, the result achieved is a $2 \perp 6(2)$ solution (i.e. a D2 brane intersecting a D6 brane along 2 tangential directions), but this specific embedding is not supersymmetric.

The layout of this chapter will be as follows: in section 5.1, I will briefly discuss the equations of motion for eleven dimensional supergravity, and introduce the equations for M-branes. Then, in section 5.2, I'll go over the Kaluza-Klein reduction down to ten dimensions, as well as discuss the type IIA (and IIB) D-branes given by the ten dimensional string theory. Finally, in sections 5.3 and 5.4, I'll give examples of the embeddings that are supersymmetric and non-supersymmetric, respectively.

5.1 M-theory and Eleven dimensional Supergravity

As mentioned in the introduction, M-theory would appear to be the current, best candidate for the "theory of everything". Eleven dimensional supergravity, the topic of this chapter, is in general understood to be the low energy limit of M-theory. The M-branes contained in M-theory can be found in eleven dimensions, both the two dimensional (M2) and five dimensional (M5) branes. These M-branes are dual to one another, with the M2 brane being the "electric" type brane, and the M5 brane being the "magnetic" type dual to the M2 brane.

However, if M-theory, and specifically eleven dimensional supergravity are to be useful, it must be reducible to the already successful ten dimensional string theories, as well as to four dimensional spacetimes. The procedure of reduction down to a lesser dimension is given by the Kaluza-Klein ansatz, where the extra dimensions are wrapped up to extremely small size around a sphere, torus, etc..

To begin one must start with the equations of motion in eleven dimensions.

5.1.1 Eleven dimensional Equations of Motion

An excellent introduction to the eleven dimensional equations of motion and the KK-reduction method is given by Duff *et. al.* in [7]. Here, I will be using slightly different notation (mainly lower case letters $m, n, \ldots$ for world indices and $a, b, \ldots$ for tangent space indices, instead of upper case letters). The Lagrangian for $N = 1$ supergravity in eleven dimensions is given by

$$\begin{aligned} L &= \frac{1}{4}eE_a{}^m E_b{}^n R_{mn}{}^{ab}(\omega) - \frac{1}{2}\mathrm{i}e\bar{\Psi}_m\hat{\Gamma}^{mnp}D_n\left[\frac{1}{2}\left(\omega+\bar{\omega}\right)\right]\Psi_p \\ &\quad -\frac{1}{48}eF_{mnpq}F^{mnpq} + \frac{2}{(12)^4}e\varepsilon^{m_1\ldots m_{11}}F_{m_1\ldots m_4}F_{m_5\ldots m_8}A_{m_9m_{10}m_{11}} \\ &\quad \frac{3}{4(12)^2}e\left[\bar{\Psi}_m\hat{\Gamma}^{mnwxyz}\Psi_n + 12\bar{\Psi}^w\hat{\Gamma}^{xy}\Psi^z\right]\left(F_{wxyz}+\tilde{F}_{wxyz}\right) \end{aligned} \tag{5.1.1}$$

where the metric signature is $(-, +, \ldots)$, and with $m, n, \ldots/a, b, \ldots$ being $d = 11$ world/tangent space indices, respectively. Here, $e = \det E_m{}^a$, $\bar{\Psi} = \Psi^\dagger\hat{\Gamma}_0$, $D_m(\omega) = \partial_m - \frac{1}{4}\omega_m{}^{ab}\hat{\Gamma}_{ab}$, and the ω's are the spin-connection coefficients. Note also $\varepsilon = \sqrt{g}\epsilon$, where ϵ is the usual Levi-Civita symbol (not to be confused with the Killing spinor, below). The $\hat{\Gamma}$ matrices satisfy the usual Clifford algebra

$$\left\{\hat{\Gamma}_a, \hat{\Gamma}_b\right\} = -2\eta_{ab} \tag{5.1.2}$$

and the notation $\hat{\Gamma}_{a_1\ldots a_p} = \hat{\Gamma}_{[a_1}\ldots\hat{\Gamma}_{a_p]}$ is used.

To get the equations of motion, one varies the Lagrangian (5.1.1) with respect to $E_m{}^a$, $\bar{\Psi}_m$ and A_{mnp}. However, these simplify when one requires maximal supersymmetry. Maximal supersymmetry will occur when the vacuum expectation value (VEV) of all fermion fields vanish, so

$$\langle\Psi_m\rangle = 0 \tag{5.1.3}$$

Also, note that the low energy classical limit can be taken when $\langle\Psi_m\rangle = 0$.

Thus, the equations of motion become

$$R_{mn} - \frac{1}{2}g_{mn}R = \frac{1}{3}\left[F_{mpqr}F_n{}^{pqr} - \frac{1}{8}g_{mn}F_{pqrs}F^{pqrs}\right] \tag{5.1.4}$$

$$\nabla_m F^{mnpq} = -\frac{1}{576}\varepsilon^{m_1\ldots m_8npq}F_{m_1\ldots m_4}F_{m_5\ldots m_8} \tag{5.1.5}$$

Since the VEV's (5.1.3) vanish, F_{mnpq} here is the unmodified four-form field strength.

The equation of motion for $\bar{\Psi}_m$ then becomes an equation checking the amount of supersymmetry that is preserved by any solution. The number of non-trivial solutions to this Killing spinor equation [65]

$$0 = \partial_m \epsilon + \frac{1}{4}\omega_{mab}\hat{\Gamma}^{ab}\epsilon + \frac{1}{144}\Gamma_m{}^{npqr}F_{npqr}\epsilon - \frac{1}{18}\hat{\Gamma}^{pqr}F_{mpqr}\epsilon \tag{5.1.6}$$

determine the number of supersymmetries that are preserved, where ϵ is the anti-commuting parameter of the supersymmetry transformation. My conventions for the Cartan algebra are

$$\begin{aligned} de^a &= g^a{}_{bc}e^b \wedge e^c \\ \omega^a{}_{bc} &= \frac{1}{2}\left(g^a{}_{bc} + g^a{}_{ca} - g^c{}_{ab}\right) \\ \omega_{dbm} &= \omega^a{}_{bc}\eta_{ad}e^c{}_m \end{aligned} \tag{5.1.7}$$

where the usual definitions and properties

$$e^a = e^a{}_m dx^m \qquad g_{mn} = \eta_{ab}e^a e^b \tag{5.1.8}$$

(*etc.*) are taken to hold.

5.2 Kaluza-Klein Reduction to type IIA D-branes

The metric ansatz for an M2 brane solution can be written in the general form

$$\begin{aligned} ds^2_{M2} &= H^{-2/3}\left(-dt^2 + dx_1^2 + dx_2^2\right) + H^{1/3}\left(ds^2_{\text{mtrc1}} + ds^2_{\text{mtrc2}}\right) && (5.2.1) \\ F_{mnpq} &= 4\partial_{[m}A_{npq]} \\ &= \frac{1}{2}\left[A_{mnp,q} - A_{npq,m} + A_{pqm,n} - A_{qmn,p}\right] && (5.2.2) \end{aligned}$$

In general, the metric function $H = H(x_3, \ldots, x_{10})$ will depend on the eight coordinates transverse to the brane, and $A_{tx_1x_2} = 1/H$. The eight-dimensional space not part of the brane world-volume is labelled by the two metrics $ds^2_{\text{mtrc}_i}$, where these are some combination of (Euclideanized) flat space or curved metrics, totalling eight dimensions. These metrics will contain one or more coordinates upon which I will compactify in order to reduce down to ten dimensions. Fully localize solutions are difficult unless one chooses these transverse spaces to be spherical in nature, allowing one to define H in terms of the radii of the transverse spaces.

For an M5 brane, the ansatz is given by

$$\begin{aligned} ds^2_{M5} &= H^{-1/3}\left(-dt^2 + dx_1^2 + \ldots + dx_5^2\right) + H^{2/3}\left(dy^2 + ds_4^2\right) && (5.2.3) \\ F_{m_1 \ldots m_4} &= \frac{\alpha}{2}\varepsilon_{m_1 \ldots m_5}\partial^{m_5} H && (5.2.4) \end{aligned}$$

where again ds_4^2 is a flat space or curved (Euclideanized) metric, H again depends only on the coordinates transverse to the M5 brane, and $\alpha = \pm 1$ corresponding to an M5/anti M5 brane, respectively. The metrics I will use for the $ds^2_{\mathrm{mtrc}_i}$ or ds_4^2 will of course be either flat space, k-dimensional Taub-NUT/bolt, or Eguchi-Hanson.

Either metric can be decomposed into one of the following forms

$$\hat{g}_{mn} = \begin{bmatrix} e^{-2\Phi/3}\left(g_{\alpha\beta} + e^{2\Phi} C_\alpha C_\beta\right) & \nu e^{4\Phi/3} C_\alpha \\ \nu e^{4\Phi/3} C_\beta & \nu^2 e^{4\Phi/3} \end{bmatrix} \quad (5.2.5a)$$

$$= \begin{bmatrix} e^{\Phi/6}\left(g_{\alpha\beta} + e^{-3\Phi/2} C_\alpha C_\beta\right) & \nu e^{-4\Phi/3} C_\alpha \\ \nu e^{-4\Phi/3} C_\beta & \nu^2 e^{-4\Phi/3} \end{bmatrix} \quad (5.2.5b)$$

where (5.2.5a) is the reduction to the string frame, and (5.2.5b) is the reduction to the Einstein frame. Here, ν is the winding number, representing the number of times the brane is wrapped around the compactified dimension [66]. The Kaluza-Klein reduction of 11-D supergravity down to 10-D is then

$$\begin{aligned} ds^2_{(1,10),s} &= e^{-2\Phi/3} ds^2_{(1,9)} + e^{4\Phi/3}\left(\nu dx_{10} + C_\alpha dx^\alpha\right)^2 && (5.2.6) \\ F_{[4]} &= \mathcal{F}_{[4]} + \nu \mathcal{H}_{[3]} \wedge dx_{10} && (5.2.7) \end{aligned}$$

in the string frame, and

$$\begin{aligned} ds^2_{(1,10),E} &= e^{\Phi/6} ds^2_{(1,9)} + e^{-4\Phi/3}\left(\nu dx_{10} + C_\alpha dx^\alpha\right)^2 && (5.2.8) \\ F_{[4]} &= \mathcal{F}_{[4]} + \nu \mathcal{H}_{[3]} \wedge dx_{10} && (5.2.9) \end{aligned}$$

where $\mathcal{F}_{(4)}$ and $\mathcal{H}_{(3)}$ are the Ramond-Ramond (RR) four-form and Neveu-Schwarz Neveu-Schwarz (NSNS) three-form field strengths corresponding to $\mathcal{A}_{\alpha\beta\gamma}$ and $\mathcal{B}_{\alpha\beta}$ respectively, and x_{10} is the coordinate of the compactified manifold. I will take this compactified manifold to be a circle with radius R_∞, parameterized as $x_{10} = R_\infty \psi$, $0 < \psi < 2\pi$. Thus, the RR (C_α, $\mathcal{A}_{\alpha\beta\gamma}$) and NSNS ($\Phi$, $\mathcal{B}_{\alpha\beta}$ and $g_{\alpha\beta}$) fields can easily be found.

By using either of these forms of the metric in the bosonic action in eleven dimensions

$$\hat{S}_{11} = \int d^{11}x \, \sqrt{-\hat{g}} \left\{ \hat{R} - \frac{1}{48} F_{[4]}^2 + \frac{2}{(12)^4} F_{[4]} \wedge F_{[4]} \wedge A_{[3]} \right\} \tag{5.2.10}$$

one can find the ten dimensional action, for example

$$\begin{aligned} S_{10} &= \int d^{10}x \, \sqrt{-g} \left\{ \mathcal{R} - \frac{1}{2} \left(\partial_\mu \Phi\right) \left(\partial^\mu \Phi\right) - \frac{1}{4} e^{-3\Phi/2} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \right. \\ & \left. - \frac{1}{48} e^{-\Phi/2} \mathcal{F}_{\mu\nu\rho\sigma} \mathcal{F}^{\mu\nu\rho\sigma} - \frac{1}{4} e^{\Phi} \mathcal{H}_{\mu\nu\rho} \mathcal{H}^{\mu\nu\rho} \right\} \end{aligned} \tag{5.2.11}$$

represents the action in the Einstein frame in ten dimensions (note $\mathcal{F}_{[2]} = dC_{[1]}$). Varying this action with respect to the RR and NSNS fields gives the following equations of motion

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta g^{\alpha\beta}} = 0 &= \left[\mathcal{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \mathcal{R} \right] - \frac{1}{2} \left[\left(\partial_\alpha \Phi\right) \left(\partial_\beta \Phi\right) - \frac{1}{2} g_{\alpha\beta} \left(\partial_\gamma \Phi\right) \left(\partial^\gamma \Phi\right) \right] \\ & - \frac{1}{2} e^{-3\Phi/2} \left[\mathcal{F}_{\alpha\gamma} \mathcal{F}_\beta{}^\gamma - \frac{1}{4} g_{\alpha\beta} \mathcal{F}_{\gamma\delta} \mathcal{F}^{\gamma\delta} \right] \\ & - e^{\Phi} \left[\mathcal{H}_{\alpha\gamma\delta} \mathcal{H}_\beta{}^{\gamma\delta} - \frac{1}{6} g_{\alpha\beta} \mathcal{H}_{\gamma\delta\rho} \mathcal{H}^{\gamma\delta\rho} \right] \\ & - \frac{1}{3} e^{-\Phi/2} \left[\mathcal{F}_{\alpha\gamma\delta\sigma} \mathcal{F}_\beta{}^{\gamma\delta\sigma} - \frac{1}{8} g_{\alpha\beta} \mathcal{F}_{\gamma\delta\sigma\rho} \mathcal{F}^{\gamma\delta\sigma\rho} \right] \end{aligned} \tag{5.2.12}$$

$$\frac{\delta \mathcal{L}}{\delta \mathcal{A}_{[3]}} = 0 = \nabla_\delta \left(e^{-\Phi/2} \mathcal{F}^{\alpha\beta\gamma\delta} \right) \tag{5.2.13}$$

$$\frac{\delta \mathcal{L}}{\delta \mathcal{B}_{[2]}} = 0 = \nabla_\gamma \left(e^{\Phi} \mathcal{H}^{\alpha\beta\gamma} \right) \tag{5.2.14}$$

$$\frac{\delta \mathcal{L}}{\delta C_{[1]}} = 0 = \nabla_\beta \left(e^{-3\Phi/2} \mathcal{F}^{\alpha\beta} \right) \tag{5.2.15}$$

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta \Phi} = 0 &= \Box \Phi + \frac{1}{24} e^{-\Phi/2} \mathcal{F}_{\alpha\beta\gamma\delta} \mathcal{F}^{\alpha\beta\gamma\delta} \\ & - \frac{1}{3} e^{\Phi} \mathcal{H}_{\alpha\beta\gamma} \mathcal{H}^{\alpha\beta\gamma} + \frac{3}{8} e^{-3\Phi/2} \mathcal{F}_{\alpha\beta} \mathcal{F}^{\alpha\beta} \end{aligned} \tag{5.2.16}$$

Note of course that, as long as the ten dimensional metric and fields satisfy the Einstein frame equations of motion, they also satisfy the string frame EOM. Thus, since the string frame demonstrates more clearly the relations between the D-branes in ten dimensions, I will use the string frame reduction in the following sections, but will check the ten dimensional fields versus the above EOM in the Einstein frame.

5.3 Supersymmetric Solutions

In [28, 29] several new supersymmetric M-brane solutions were found. Here, due to space considerations, I will only demonstrate two such solutions - the solution found by embedding two Taub-NUT metrics into the eight extra dimensions space of an M2 brane, and the solution found by embedding a Taub-NUT metric into the five extra dimensions space of an M5 brane. Other possible solutions include embedding an Eguchi-Hanson metric, embedding TN4 $\otimes$ Eguchi-Hanson, and embedding an Eguchi-Hanson $\otimes$ Eguchi-Hanson.

5.3.1 M2 Brane with $TN_4 \otimes TN_4$ embedded

Several localized and supersymmetric M2 brane solutions are possible through embedding Hyper-Kähler metrics into the eleven dimensional metric equations. The case of embedding one four dimensional Taub-NUT metric was done by Cherkis and Hashimoto [27]. Here, I consider embedding two four dimensional Taub-NUT metrics. The metric is then

$$\begin{aligned} ds_{11}^2 &= H(r_1, r_2)^{-2/3}\left(-dt^2 + dx_1^2 + dx_2^2\right) + H(r_1, r_2)^{1/3}\left(ds_{tn_1}^2 + ds_{tn_2}^2\right) \\ F_{tx_1x_2r_i} &= -\frac{1}{2H^2}\frac{\partial H}{\partial r_i} \end{aligned} \tag{5.3.1}$$

where the TN metrics can be written in one of two forms:

$$\begin{aligned} ds_{tn_i}^2 &= \frac{1}{f_i(r_i)}\left[d\Psi_i + 2n_i\cos(\theta_i)d\phi_i\right]^2 + f_i(r_i)dr^2 \\ &\quad +(r_i^2 - n_i^2)\left(d\theta_i^2 + \sin^2(\theta_i)d\phi_i^2\right) \\ f_i(r_i) &= \frac{r_i + n_i}{r_i - n_i} \end{aligned} \tag{5.3.2}$$

or

$$
\begin{aligned}
ds^2_{tn_i} &= \frac{(4n_i)^2}{\tilde{f}_i(r_i)}\left[d\psi_i + \frac{1}{2}\cos(\theta_i)d\phi_i\right]^2 \\
&\quad + \tilde{f}_i(r_i)\left(dr_i^2 + r_i^2\left(d\theta_i^2 + \sin^2(\theta_i)d\phi_i^2\right)\right) \qquad (5.3.3) \\
\tilde{f}_i(r_i) &= 1 + \frac{2n_i}{r_i}
\end{aligned}
$$

One can go from (5.3.2) to (5.3.3) by taking $\Psi_i \to 4n_i\psi_i$, $r_i \to r_i - n_i$. The form (5.3.2) is more useful for checking the supersymmetry of the solutions, and (5.3.3) is more useful for the reduction of the solution down to ten dimensions, as the period of ψ_i is 2π. (5.3.1) can be checked to solve the eleven dimensional equations of motion (5.1.4). Thus, with (5.3.3) the ranges of the coordinates are $\{r_i \in [0,\infty)\}$, $\{\psi_i,\ \phi_i \in [0,2\pi]\}$ and $\{\theta_i \in [0,\pi]\}$.

The metric (5.3.1) satisfies the eleven dimensional equations of motion if the following differential equation for the Harmonic function is satisfies

$$
\begin{aligned}
0 &= (r_1 + 2n_1)\frac{\partial H(r_1,r_2)}{\partial r_2} + \frac{r_2(r_1+2n_1)}{2}\frac{\partial^2 H(r_1,r_2)}{\partial r_2^2} \\
&\quad + (r_2 + 2n_2)\frac{\partial H(r_1,r_2)}{\partial r_1} + \frac{r_1(r_2+2n_2)}{2}\frac{\partial^2 H(r_1,r_2)}{\partial r_1^2} \qquad (5.3.4)
\end{aligned}
$$

This equation is separable, and a suitable choice for the harmonic function is $H(r_1,r_2) = 1 + Q_{M2}R_1(r_1)R_2(r_2)$. With this choice, as $r_1, r_2 \to \infty$, the harmonic function can be chosen to approach unity. Substitution of this function into (5.3.4) gives two differential equations

$$
0 = 2\frac{dR_i(r_i)}{dr_i} + r_i\frac{d^2R_i(r_i)}{dr_i^2} \pm c^2(r_i + 2n_i)R_i(r_i) \qquad (5.3.5)
$$

where each equation is set equal to either $\pm c^2$ so that they both sum to zero. One of the equations will produce a complex solution of the form (in full)

$$
\begin{aligned}
R_1(r_1) &= \frac{C_1}{r_1}\mathcal{W}_M\left(-\mathrm{i}n_1c, \frac{1}{2}, 2\mathrm{i}cr_1\right) + \frac{C_2}{r_1}\mathcal{W}_W\left(-\mathrm{i}n_1c, \frac{1}{2}, 2\mathrm{i}cr_1\right) \qquad (5.3.6) \\
&= D_1e^{-\mathrm{i}cr_1}\mathcal{M}\left(1 + \mathrm{i}cn_1, 2, 2\mathrm{i}cr_1\right) \\
&\quad + D_2e^{-\mathrm{i}cr_1}\mathcal{U}\left(1 + \mathrm{i}cn_1, 2, 2\mathrm{i}cr_1\right) \qquad (5.3.7)
\end{aligned}
$$

where $\mathcal{W}_{M,W}$ are the Whittaker M, W functions, and $\mathcal{M}$,$\mathcal{U}$ are the Kummer M,U functions (see [67] for details). We need the part of the solution that is finite at $r = 0$, and undergoes damped oscillations and vanishes as $r \to \infty$. Thus, the constants $C_2 = D_2 = 0$. Similarly, the other differential equation will produce a real (full) solution of

$$\begin{aligned} R_2(r_2) &= \frac{C_3}{r_2}\mathcal{W}_M\left(-cn_2, \frac{1}{2}, 2cr_2\right) + \frac{C_4}{r_2}\mathcal{W}_W\left(-cn_2, \frac{1}{2}, 2cr_2\right) \quad (5.3.8) \\ &= D_3 e^{-cr_2}\mathcal{M}(1 + cn_2, 2, 2cr_2) \\ &\quad + D_4 e^{-cr_2}\mathcal{U}(1 + cn_2, 2, 2cr_2) \quad (5.3.9) \end{aligned}$$

Here again we want the solution that is finite at $r = 0$ and vanishes at $r \to 0$, so $C_3 = D_3 = 0$.

The most general solution is the product of the exponentially decaying real solution and the damped oscillating complex solution, which must be summed over all possible values of c,

$$\begin{aligned} & H(r_1, r_2) \quad (5.3.10) \\ = \; & 1 + Q_{M2}\int_0^\infty dc\; f(c)e^{-\mathrm{i}cr_1}\mathcal{M}\left(1 + \mathrm{i}cn_1, 2, 2\mathrm{i}cr_1\right)e^{-cr_2}\mathcal{U}(1 + cn_2, 2, 2cr_2) \end{aligned}$$

Here, the function $f(c)$ is present because this solution must match the solution in the near horizon limit. The metric of the transverse space in the near horizon limit ($r_i \ll n_i$) reduces to $R^4 \otimes R^4$, with

$$ds^2 = dz_1^2 + z_1^2 d\Omega_3^2 + dz_2^2 + z_2^2 d\Omega_3'^2 \quad (5.3.11)$$

(where $z_i^2 = 8n_i r_i$) and hence, near the horizons of the Taub-NUT spacetimes, the metric function (5.3.10) should coincide with the solution for an M2-brane with a flat transverse space - $(1 + Q_{M2}/R^6)$, where $R = \sqrt{r_1^2 + r_2^2}$. This means,

$$\begin{aligned} & \lim_{r_i \ll n_i}\int_0^\infty dc\; f(c)e^{-\mathrm{i}cr_1}\mathcal{M}\left(1 + \mathrm{i}cn_1, 2, 2\mathrm{i}cr_1\right)e^{-cr_2}\mathcal{U}(1 + cn_2, 2, 2cr_2) \\ = \; & \frac{1}{(z_1^2 + z_2^2)^3} \end{aligned}$$

Here one can use the limiting values on hypergeometric and confluent hypergeometric functions [67], this can be re-written

$$\int_0^\infty dc\; f(c)\frac{\mathrm{i}I_1\left(2\mathrm{i}c\sqrt{2n_1 r_1}\right)}{c}\frac{2K_1\left(2c\sqrt{2n_2 r_2}\right)}{c\Gamma(cn_2)} = \frac{\sqrt{2n_1 r_1}\sqrt{2n_2 r_2}}{512\left(n_1 r_1 + n_2 r_2\right)^3} \quad (5.3.12)$$

Integrating the left-hand side with $f(c) = -(\mathrm{i}c^5\Gamma(cn_2)/64)$ gives the right-hand side, so the complete harmonic metric function is

$$\begin{aligned} &H(r_1, r_2) = 1 \qquad (5.3.13)\\ + \quad & Q_{M2}\int_0^\infty dc\frac{c^5\Gamma(cn_2)}{64}e^{-\mathrm{i}cr_1}\mathcal{M}\left(1+\mathrm{i}cn_1, 2, 2\mathrm{i}cr_1\right)e^{-cr_2}\mathcal{U}(1+cn_2, 2, 2cr_2) \end{aligned}$$

Note again that taking $c \to \mathrm{i}c$ will simply switch the solutions $R_1 \leftrightarrow R_2$.

Supersymmetry

The supersymmetry of the solution must also be checked. Here, equation (5.1.6) is used to generate eleven equations, that must be checked separately. The veilbeins are given by

$$\begin{aligned} e^{\hat{t}} &= H(r_1, r_2)^{-1/3}dt \quad , \quad e^{\hat{x}_i} = H(r_1, r_2)^{-1/3}dx_i \\ e^{\hat{\psi}_i} &= \frac{H(r_1, r_2)^{1/6}}{F(r_i)}\left[d\psi_i + 2n_i\cos(\theta_i)d\phi_i\right] \\ e^{\hat{r}_i} &= H(r_1, r_2)^{1/6}F(r_i)dr_i \\ e^{\hat{\theta}_i} &= H(r_1, r_2)^{1/6}\sqrt{r_i^2 - n_i^2}d\theta_i \\ e^{\hat{\phi}_i} &= H(r_1, r_2)^{1/6}\sqrt{r_i^2 - n_i^2}\sin(\theta_i)d\phi_i \end{aligned}$$

where I am currently using (5.3.2) for the Taub-NUT metric, and have defined $F^2(r_i) \equiv F_i^2 = f(r_i)$. These veilbeins can be used in the Cartan algebra (5.1.7) to calculate the spin-connection coefficients ω_{mab}, and along with the four-form field strengths and the use of the Clifford algebra, we get eleven equations. The first four, from $m = t, x_1, x_2$;

$$0 = \frac{\Gamma^{\hat{t}\hat{r}_1}}{6H^{3/2}F_1}\frac{\partial H}{\partial r_1}\left[1+\Gamma^{\hat{t}\hat{x}_1\hat{x}_2}\right]\epsilon + \frac{\Gamma^{\hat{t}\hat{r}_2}}{6H^{3/2}F_2}\frac{\partial H}{\partial r_2}\left[1+\Gamma^{\hat{t}\hat{x}_1\hat{x}_2}\right]\epsilon \qquad (5.3.14)$$

$$0 = \frac{\Gamma^{\hat{r}_1\hat{x}_1}}{6H^{3/2}F_1}\frac{\partial H}{\partial r_1}\left[1+\Gamma^{\hat{t}\hat{x}_1\hat{x}_2}\right]\epsilon + \frac{\Gamma^{\hat{r}_2\hat{x}_1}}{6H^{3/2}F_2}\frac{\partial H}{\partial r_2}\left[1+\Gamma^{\hat{t}\hat{x}_1\hat{x}_2}\right]\epsilon \qquad (5.3.15)$$

$$0 = \frac{\Gamma^{\hat{r}_1\hat{x}_2}}{6H^{3/2}F_1}\frac{\partial H}{\partial r_1}\left[1+\Gamma^{\hat{t}\hat{x}_1\hat{x}_2}\right]\epsilon + \frac{\Gamma^{\hat{r}_2\hat{x}_2}}{6H^{3/2}F_2}\frac{\partial H}{\partial r_2}\left[1+\Gamma^{\hat{t}\hat{x}_1\hat{x}_2}\right]\epsilon \qquad (5.3.16)$$

These three equations are zero through the use of the projection operator

$$\left[1+\Gamma^{\hat{t}\hat{x}_1\hat{x}_2}\right]\epsilon = 0 \qquad (5.3.17)$$

This projection operator will remove half of the supersymmetry.

The two equations from $m = r_i$, $i = 1, 2$ require the Killing spinor to be $\epsilon H^{-1/6}\epsilon'$, and give (for r_1; for r_2, let $r_1 \leftrightarrow r_2$)

$$0 = -\frac{1}{6H}\frac{\partial H}{\partial r_1}\left[1+\Gamma^{\hat{t}\hat{x}_1\hat{x}_2}\right]\epsilon + \frac{F_1\Gamma^{\hat{r}_1\hat{r}_2}}{12HF_2}\frac{\partial H}{\partial r_2}\left[1+\Gamma^{\hat{t}\hat{x}_1\hat{x}_2}\right]\epsilon \quad (5.3.18)$$

This is again zero using (5.3.17). The final six equations are from $m = \psi_i, \theta_i, \phi_i$ (where $i, j = 1, 2$, $i \neq j$, and no sum over repeated indices)

$$\begin{aligned} 0 &= \left\{\partial_{\psi_i}\epsilon - \frac{1}{2F_i^3}\frac{dF_i}{dr_i}\Gamma^{\hat{\psi}_i\hat{r}_i}\epsilon + \frac{n_i}{2F_i^2(r_i^2-n_i^2)}\Gamma^{\hat{\theta}_i\hat{\phi}_i}\epsilon\right\} \qquad (5.3.19)\\ &\quad + \frac{\Gamma^{\hat{\psi}_i\hat{r}_i}}{12HF_i^2}\frac{\partial H}{\partial r_i}\left[1+\Gamma^{\hat{t}\hat{x}_1\hat{x}_2}\right]\epsilon + \frac{\Gamma^{\hat{\psi}_i\hat{r}_j}}{12HF_iF_j}\frac{\partial H}{\partial r_j}\left[1+\Gamma^{\hat{t}\hat{x}_1\hat{x}_2}\right]\epsilon \\ 0 &= \left\{\partial_{\theta_i}\epsilon - \frac{r_i\Gamma^{\hat{r}_i\hat{\theta}_i}}{2F_i\sqrt{r_i^2-n_i^2}}\epsilon + \frac{n_i\Gamma^{\hat{\psi}_i\hat{\phi}_i}}{2F_i\sqrt{r_i^2-n_i^2}}\epsilon\right\} \qquad (5.3.20)\\ &\quad - \frac{\sqrt{r_i^2-n_i^2}\Gamma^{\hat{r}_i\hat{\theta}_i}}{12HF_i}\frac{\partial H}{\partial r_i}\left[1+\Gamma^{\hat{t}\hat{x}_1\hat{x}_2}\right]\epsilon - \frac{\sqrt{r_i^2-n_i^2}\Gamma^{\hat{r}_j\hat{\theta}_i}}{12HF_2}\frac{\partial H}{\partial r_j}\left[1+\Gamma^{\hat{t}\hat{x}_1\hat{x}_2}\right]\epsilon \end{aligned}$$

$$\begin{aligned} 0 = &\left\{\partial_{\phi_i}\epsilon - \frac{n_i\sin(\theta_i)}{2F_i\sqrt{r_i^2-n_i^2}}\Gamma^{\hat{\phi}_i\hat{\theta}_i}\epsilon - \frac{r_i\sin(\theta_i)}{2F_i\sqrt{r_i^2-n_i^2}}\Gamma^{\hat{r}_i\hat{\phi}_i}\epsilon - \frac{\cos(\theta_i)}{2}\Gamma^{\hat{\theta}_i\hat{\phi}_i}\epsilon\right.\\ &\left. + \frac{n_i^2\cos(\theta_i)}{F_i^2(r_i^2-n_i^2)}\Gamma^{\hat{\theta}_i\hat{\phi}_i}\epsilon - \frac{n_i\cos(\theta_i)}{F_i^3}\frac{dF_i}{dr_i}\Gamma^{\hat{\psi}_i\hat{r}_i}\epsilon\right\} \qquad (5.3.21)\\ &+ \left\{\frac{n_i\cos(\theta_i)}{6HF_i^2}\frac{\partial H}{\partial r_i}\Gamma^{\hat{\psi}_i\hat{r}_i} - \frac{\sqrt{r_i^2-n_i^2}\sin(\theta_i)}{12HF_i}\frac{\partial H}{\partial r_i}\Gamma^{\hat{r}_i\hat{\phi}_i}\right\}\left[1+\Gamma^{\hat{t}\hat{x}_1\hat{x}_2}\right]\epsilon\\ &+ \left\{\frac{n_i\cos(\theta_i)}{6HF_iF_j}\frac{\partial H}{\partial r_j}\Gamma^{\hat{\psi}_i\hat{r}_j} - \frac{\sqrt{r_i^2-n_i^2}\sin(\theta_i)}{12HF_j}\frac{\partial H}{\partial r_j}\Gamma^{\hat{r}_j\hat{\phi}_i}\right\}\left[1+\Gamma^{\hat{t}\hat{x}_1\hat{x}_2}\right]\epsilon \end{aligned}$$

The projection operator (5.3.17) can again be used, and substituting in

F_1, F_2, these reduce to

$$\begin{aligned}
0 &= \partial_{\psi_i}\epsilon + \frac{n_i}{2(r_i+n_i)^2}\Gamma^{\hat{\psi}_i\hat{r}_i}\epsilon + \frac{n_i}{2(r_i+n_i)^2}\Gamma^{\hat{\theta}_i\hat{\phi}_i}\epsilon && (5.3.22)\\
0 &= \partial_{\theta_i}\epsilon + \frac{r_i}{2(r_i+n_i)}\Gamma^{\hat{r}_i\hat{\theta}_i}\epsilon - \frac{n_i}{2(r_i+n_i)}\Gamma^{\hat{\psi}_i\hat{\phi}_i}\epsilon && (5.3.23)\\
0 &= \partial_{\phi_i}\epsilon - \frac{n_i\sin(\theta_i)}{2(r_i+n_i)}\Gamma^{\hat{\psi}_i\hat{\theta}_i}\epsilon - \frac{r_i\sin(\theta_i)}{2(r_i+n_i)}\Gamma^{\hat{r}_i\hat{\phi}_i}\epsilon - \frac{\cos(\theta_i)}{2}\Gamma^{\hat{\theta}_i\hat{\phi}_i}\epsilon \\
&\quad + \frac{n_i^2\cos(\theta_i)}{(r_i+n_i)^2}\Gamma^{\hat{\theta}_i\hat{\phi}_i}\epsilon + \frac{n_i^2\cos(\theta_i)}{(r_i+n_i)^2}\Gamma^{\hat{\psi}_i\hat{r}_i}\epsilon && (5.3.24)
\end{aligned}$$

These are solvable through a Lorentz rotation and using the projection operators

$$\begin{aligned}
\Gamma^{\hat{\psi}_1\hat{r}_1\hat{\theta}_1\hat{\phi}_1}\epsilon &= \epsilon && (5.3.25)\\
\Gamma^{\hat{\psi}_2\hat{r}_2\hat{\theta}_2\hat{\phi}_2}\epsilon &= \epsilon && (5.3.26)
\end{aligned}$$

This would suggest that the supersymmetry is reduced by a further quarter; however, in eleven dimensions, (5.3.17) and (5.3.25) imply the third projection operator, and so the use of these two projection operators only reduces the supersymmetry by a half. Since the use of these projection operators makes (5.3.22) zero, $\epsilon \neq \epsilon(\psi)$, and all that remains is the Lorentz transformation(s) to solve (5.3.23) and (5.3.24) (for $i = 1, 2$, again no sum). This rotation is given by

$$\epsilon = \exp\left\{-\frac{\theta_i}{2}\Gamma^{\hat{\psi}_i\hat{\phi}_i}\right\}\exp\left\{\frac{\phi_i}{2}\Gamma^{\hat{\theta}_i\hat{\phi}_i}\right\}\widetilde{\epsilon} \tag{5.3.27}$$

This transformation can be calculated from (5.3.23) and (5.3.24) by using the Baker-Campbell-Hausdorff identity

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \ldots \tag{5.3.28}$$

along with the following easily calculated identity

$$\left[\Gamma^{ab}, \Gamma^{cd}\right] = 2\eta^{ad}\Gamma^{bc} + 2\eta^{bc}\Gamma^{ad} - 2\eta^{ac}\Gamma^{bd} - 2\eta^{bd}\Gamma^{ac} \tag{5.3.29}$$

Thus, with the use of the projection operators, we have reduced the total preserved supersymmetry of the solution to 1/4.

Kaluza-Klein Reduction

Now, dimensionally reducing the solution along one of the ψ_i directions using the ansatz (5.2.7) will reduce (5.3.1) to a type-IIA string theory in ten dimensions. Since there are two Taub-NUT metrics, each with a ψ_i with a period of 2π, we can compactify to ten dimensions using either of the ψ_i's. Since either choice will produce the same results (with the indexes' $(1 \leftrightarrow 2)$ switched), I'll compactify on ψ_2. The radius of the circle as $r \to \infty$ with line element $R_\infty^2 \left[d\psi_2 + \frac{1}{2}\cos\theta_2 d\phi_2\right]^2$ is thus

$$R_\infty = 4n_2 \tag{5.3.30}$$

The reduction gives the RR fields

$$\begin{aligned} C_{[1]} &= 2n_2 \cos(\theta_2) d\phi_2 \\ \mathcal{A}_{[3]} &= \frac{1}{H(r_1, r_2)} dt \wedge dx_1 \wedge dx_2 \end{aligned} \tag{5.3.31}$$

and the NSNS fields

$$\begin{aligned} \Phi &= \frac{3}{4}\ln\left(\frac{H^{1/3}}{\tilde{f}_2}\right) \\ \mathcal{B}_{\mu\nu} &= 0 \end{aligned} \tag{5.3.32}$$

with metric

$$\begin{aligned} ds_{10}^2 &= H^{-1/2}\tilde{f}_2^{-1/2}\left(-dt^2 + dx_1^2 + dx_2^2\right) + H^{1/2}\tilde{f}_2^{-1/2} ds_{TN_1}^2 \\ &\quad + H^{1/2}\tilde{f}_2^{1/2}\left(dr_2^2 + r_2^2\left(d\theta_2^2 + \sin^2(\theta_2) d\phi_2^2\right)\right) \end{aligned} \tag{5.3.33}$$

This is a D2⊥D6(2) brane system[1] with $H(r_1, r_2) = (5.3.13)$. This solution solves the ten dimensional EOM[2].

Decoupling Limit

The dynamics of the D2 brane decouple from the bulk at low energies, and the region close to the D6 branes corresponds to a range of energy scales governed by the IR fixed point [69] (See also [45, 70]). In the IR region,

[1] D2⊥D6(2) means the D2 brane and the D6 brane intersect along 2 coordinates.
[2] The field C_n is what determines that this is a D6 brane - see [68], eq. (14).

one can discuss the dependence on the number of branes. Here, in order to obtain a reliable classical geometric description, the number of D2 branes, N_2, must be large.

The different phases of the theory can then be seen by varying the number of D6 branes relative to N_2. For $N_6 < N_2$, the theory is weakly coupled, and eleven dimensional, with an AdS$_4\times$ the orbifold $S_7/\mathbb{Z}_{N_6}$ geometry. Increasing N_6, one goes to a ten dimensional phase, where the geometry is now AdS$_4$ *fibred* over a six dimensional compact base manifold X_6. When $N_6 \gg N_2$, we get a very highly curved ten dimensional geometry, where it is expected that this means that there is a transition to a weakly coupled phase of the gauge theory.

Note that the field theory on the D2 branes (extended in the $x_0 \ldots x_2$ directions), in the absence of the D6 branes, is three dimensional $N = 8$ super Yang-Mills (SYM) theory, with $U(N_2)$ gauge group. When the D6 branes are added (extended along the $x_0 \ldots x_6$ directions), one breaks half the susy, meaning that there is now $N = 4$ susy on the D2 branes. This is shown above by the required use of the first projection operator (5.3.17).

In the solution presented above, I am interested in the D2 branes localized on the D6 branes. Note that as can be seen from the above reduction down to ten dimensions, a set of N_6 coinciding D6 branes corresponds to a Kaluza-Klein monopole in the M-Theory, given by the $C_{[1]}$ field.

Thus, near the D2 brane ($H \gg 1$), the field theory limit is given by

$$g^2_{YM2} = g_s \ell_s^{-1} = \text{fixed} \tag{5.3.34}$$

In the field theory limit, the gauge couplings in the bulk go to zero and so the dynamics in the bulk decouple. The radial coordinates of the metric r_i are scaled so that

$$U_i = \frac{r_i}{\ell_s^2} \tag{5.3.35}$$

are fixed ($i = 1, 2$). This will change the harmonic function for the D6 brane to the following

$$\begin{aligned} f_2(r_2) &= \left(1 + \frac{2n_2}{r_2}\right) = \left(1 + \frac{g_s \ell_s}{2r_2}\right) = \left(1 + \frac{g_s}{2\ell_s U_2}\right) \\ &= \left(1 + \frac{g^2_{YM2} N_6}{2U_2}\right) = f(U_2) \end{aligned} \tag{5.3.36}$$

where I have used the asymptotic radius of the eleventh dimension $R_\infty = 4n_2 = g_s \ell_s$. I have also generalized to the case of N_6 D6 branes.

The D2 harmonic function (5.3.13) can be shown to scale as $H(Y,U_2) = \ell_s^{-4} h(Y,U_2)$, with $h(U_1,U_2)$ given by

$$
\begin{aligned}
h_{(TN)^2}(U_1,U_2) &= \frac{\pi^2 N_2 g_{YM2}^2}{2} \int_0^\infty dP\ P^5 \Gamma\left(\frac{g_{YM2}^2 P}{4}\right) e^{-\mathrm{i}PU_1} \cdot \\
&\quad \cdot \mathcal{M}(1+\mathrm{i}Pm_1, 2, 2\mathrm{i}PU_1) e^{-PU_2} \mathcal{U}\left(1+\frac{g_{YM2}^2 P}{4}, 2, 2PU_2\right)
\end{aligned}
\tag{5.3.37}
$$

where I have rescaled $n_1 = m_1 \ell_s^2$ and $c = P\ell_s^{-2}$, and the M2 brane charge has been rewritten

$$
Q_{M2} = 32\pi^2 N_2 \ell_p^6 = 32\pi^2 N_2 g_{YM2}^2 \ell_s^8 \tag{5.3.38}
$$

Then, the supersymmetric metric in ten dimensions is given by inserting (5.3.36), (5.3.37) into (5.3.33), (where the scaling $n_1 = m_1 \ell_s^2$ has to be used in the $ds_{TN_1}^2$ metric)

$$
\begin{aligned}
\frac{ds_{10}^2}{\ell_s^2} &= h(U_1,U_2)^{-1/2} f(U_2)^{-1/2} \left(-dt^2 + dx_1^2 + dx_2^2\right) \\
&\quad + h(U_1,U_2)^{1/2} f(U_2)^{-1/2} ds_{TN_1}^2 \\
&\quad + h(U_1,U_2)^{1/2} f(U_2)^{1/2} \left(dU_2^2 + U_2^2 (d\theta_2^2 + \sin^2(\theta_2) d\phi_2^2)\right)
\end{aligned}
\tag{5.3.39}
$$

where note that there is only an overall factor of ℓ_s^2 in this metric. This is expected for a supergravity solution dual to a CFT.

5.3.2 M5 Brane with TN$_4$ embedded

One can also embed the Taub-NUT solutions (5.3.2) or (5.3.3) into the metric for an M5-brane in eleven dimensions. The metric will then be

$$
\begin{aligned}
ds_{M5}^2 &= H(y,r)^{-1/3} \left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2\right) \\
&\quad + H(y,r)^{2/3} \left(dy^2 + ds_{TN}^2\right)
\end{aligned}
\tag{5.3.40}
$$

If (5.3.2) is used, then the four-form field strength has the non-zero components

$$
F_{\psi\theta\phi y} = -\frac{(r-n)^2 \sin(\theta)}{2} \frac{\partial H}{\partial r} \quad , \quad F_{\psi\theta\phi r} = \frac{(r^2-n^2)\sin(\theta)}{2} \frac{\partial H}{\partial y} \tag{5.3.41}
$$

If instead one uses (5.3.3), the form of the field strength components is given by

$$
F_{\psi\theta\phi y} = 2nr^2 \sin(\theta) \frac{\partial H}{\partial r} \quad , \quad F_{\psi\theta\phi r} = -2nr(r+2n)\sin(\theta) \frac{\partial H}{\partial y} \tag{5.3.42}
$$

Here, the metric (5.3.40) (with (5.3.3)) solves the eleven dimensional equations of motion if the harmonic function $H(y,r)$ solves the differential equation

$$0 = \frac{r}{2(r+2n)}\frac{\partial^2 H}{\partial r^2} + \frac{1}{r+2n}\frac{\partial H}{\partial r} + \frac{1}{2}\frac{\partial^2 H}{\partial y^2} \tag{5.3.43}$$

This equation is separable - substituting in $H(y,r) = 1 + Q_{M5}Y(y)R(r)$ (where Q_{M5} is the M5-brane charge), the differential equations to be solved are

$$0 = \frac{d^2Y(y)}{dy^2} + c^2Y(y) \tag{5.3.44}$$

$$0 = r\frac{d^2R(r)}{dr^2} + 2\frac{dR(r)}{dr} - c^2(r+2n)R(r) \tag{5.3.45}$$

The solutions to these two equations are (where note (5.3.45) is the same as in the M2 case),

$$Y(y) = C_1\cos(cy) + C_2\sin(cy) \tag{5.3.46}$$

$$R(r) = D_1e^{-cr}\mathcal{M}(1+cn,2,2cr) + D_2e^{-cr}\mathcal{U}(1+cn,2,2cr) \tag{5.3.47}$$

As in the M2 case, we again require that the solution decay for large r, and hence $D_1 = 0$. Thus, the final solution for the harmonic function is again a superposition

$$H(y,r) = 1 + Q_{M5}\int_0^\infty dc\,(f_1(c)\cos(cy) + f_2(c)\sin(cy))\,e^{-cr}\mathcal{U}(1+cn,2,2cr) \tag{5.3.48}$$

Also as in the M2 case, the Taub-NUT metric becomes flat

$$ds^2_{TN} = dz^2 + z^2d\Omega_3^2 \tag{5.3.49}$$

Thus, $H(y,r)$ must match the flat transverse-space solution

$$\begin{aligned} &\lim_{z^2\ll 8n^2} Q_{M5}\int_0^\infty dc\,(f_1(c)\cos(cy) + f_2(c)\sin(cy))\,e^{-cr}\mathcal{U}(1+cn,2,2cr) \\ =\;& Q_{M5}\int_0^\infty dc\,(f_1(c)\cos(cy) + f_2(c)\sin(cy))\,e^{-cr}\frac{2K_1(2c\sqrt{2nr})}{c\Gamma(cn)\sqrt{2nr}} \\ =\;& \frac{Q_{M5}}{R^3} \end{aligned} \tag{5.3.50}$$

where $R = \sqrt{y^2 + z^2} = \sqrt{y^2 + 8nr}$. This requires that $C_2 = 0$ in (5.3.46), and gives $f_1(c) = \frac{c^2\Gamma(cn)}{2\pi}$, and so the final solution is

$$H(y, r) = 1 + Q_{M5} \int_0^\infty \frac{c^2\Gamma(cn)}{2\pi} \cos(cy) e^{-cr} \mathcal{U}(1 + cn, 2, 2cr) \tag{5.3.51}$$

Another solution is also possible here - one can let $c \to \mathrm{i}p$ to change the form of the differential equations (5.3.44) and (5.3.45). This will give a second solution

$$\widetilde{H}(y, r) = 1 + \frac{Q_{M5}}{2} \int_0^\infty dp p^2 e^{\mathrm{i}pr} \mathcal{M}(1 + \mathrm{i}pn, 2, 2\mathrm{i}pr) e^{-py} \tag{5.3.52}$$

arrived at through exactly the same steps as (5.3.51).

Supersymmetry

The supersymmetry must again be checked. Using the (5.3.2) form for the Taub-NUT metric, and hence (5.3.41), one again gets eleven equations that must be solved for the Killing spinor. The equations for $m = t, m = x_i$ $(i = 1, \ldots 5)$ are

$$0 = \frac{\Gamma^{\hat{t}\hat{y}}}{12H^{3/2}} \frac{\partial H}{\partial y} \left[1 - \Gamma^{\hat{y}\hat{\psi}\hat{r}\hat{\theta}\hat{\phi}}\right] \epsilon + \frac{\Gamma^{\hat{t}\hat{r}}}{12H^{3/2}F} \frac{\partial H}{\partial r} \left[1 - \Gamma^{\hat{y}\hat{\psi}\hat{r}\hat{\theta}\hat{\phi}}\right] \epsilon \tag{5.3.53}$$

$$0 = -\frac{\Gamma^{\hat{x}_i\hat{y}}}{12H^{3/2}} \frac{\partial H}{\partial y} \left[1 - \Gamma^{\hat{y}\hat{\psi}\hat{r}\hat{\theta}\hat{\phi}}\right] \epsilon - \frac{\Gamma^{\hat{x}_i\hat{r}}}{12H^{3/2}F} \frac{\partial H}{\partial r} \left[1 - \Gamma^{\hat{y}\hat{\psi}\hat{r}\hat{\theta}\hat{\phi}}\right] \epsilon \tag{5.3.54}$$

where again I have taken $F(r)^2 = f(r)$. These two equations are solvable using the projection operator

$$\left[1 - \Gamma^{\hat{y}\hat{\psi}\hat{r}\hat{\theta}\hat{\phi}}\right] \epsilon = 0 \tag{5.3.55}$$

or, because we are in eleven dimensions, this can be re-written as

$$\left[1 - \Gamma^{\hat{t}\hat{x}_1\hat{x}_2\hat{x}_3\hat{x}_4\hat{x}_5}\right] \epsilon = 0 \tag{5.3.56}$$

Thus, there is half of the supersymmetry remaining.

After requiring the Killing spinor have the form $\epsilon = H^{-1/12}\epsilon'$, the equations for $m = y, m = r$ are

$$0 = -\frac{1}{12H}\frac{\partial H}{\partial y}\left[1-\Gamma^{\hat{y}\hat{\psi}\hat{r}\hat{\theta}\hat{\phi}}\right]\epsilon + \frac{\Gamma^{\hat{y}\hat{r}}}{6HF}\frac{\partial H}{\partial r}\left[1-\Gamma^{\hat{y}\hat{\psi}\hat{r}\hat{\theta}\hat{\phi}}\right]\epsilon \quad (5.3.57)$$

$$0 = -\frac{1}{12H}\frac{\partial H}{\partial r}\left[1-\Gamma^{\hat{y}\hat{\psi}\hat{r}\hat{\theta}\hat{\phi}}\right]\epsilon - \frac{F\Gamma^{\hat{y}\hat{r}}}{6H}\frac{\partial H}{\partial y}\left[1-\Gamma^{\hat{y}\hat{\psi}\hat{r}\hat{\theta}\hat{\phi}}\right]\epsilon \quad (5.3.58)$$

These are zero due to (5.3.55). Finally, the equations for $m = \psi, \theta, \phi$ take the same form as in the M2 case above; (5.3.22), (5.3.23) and (5.3.24), and through the use of the same projection operator

$$\Gamma^{\hat{\psi}\hat{r}\hat{\theta}\hat{\phi}}\epsilon = \epsilon \quad (5.3.59)$$

and the lorentz rotation

$$\epsilon = \exp\left\{-\frac{\theta}{2}\Gamma^{\hat{\psi}\hat{\phi}}\right\}\exp\left\{\frac{\phi}{2}\Gamma^{\hat{\theta}\hat{\phi}}\right\}\widetilde{\epsilon} \quad (5.3.60)$$

these are again solvable. Thus, since two projection operators were needed to solve the Killing spinor equations, 1/4 of the supersymmetry is preserved by this M5 solution.

Kaluza Klein Reduction

This metric can also be reduced to ten dimensions. The radius of the circle of compactification is again found from the line element $R_\infty^2\left[d\psi + \frac{1}{2}\cos(\theta)\right]^2$

$$R_\infty = 4n = g_s\ell_s \quad (5.3.61)$$

using (5.3.3). The reduction to ten dimensions will give the NSNS dilaton as

$$\Phi = \frac{3}{4}\ln\left\{\frac{H^{3/2}}{\nu^2 f}\right\} \quad (5.3.62)$$

The NSNS field strength from the NS5-brane that appears because of the reduction is given by

$$\mathcal{H}_{[3]} = \frac{F_{\theta\phi y\psi}}{4n}d\theta\wedge d\phi\wedge dy + \frac{F_{\theta\phi r\psi}}{4n}d\theta\wedge d\phi\wedge dr \quad (5.3.63)$$

which gives the NSNS two-form field

$$\mathcal{B}_{[2]} = r^2 \cos(\theta) \frac{\partial H}{\partial r} dy \wedge d\phi + r(r+2n)\cos(\theta) \frac{\partial H}{\partial y} d\phi \wedge dr \tag{5.3.64}$$

The RR fields are

$$\begin{aligned} C_{[1]} &= 2n \cos(\theta) d\phi & (5.3.65) \\ \mathcal{A}_{[3]} &= 0 & (5.3.66) \end{aligned}$$

where $C_{[1]}$ is the field associated with the D6 brane. The metric in ten dimensions is

$$\begin{aligned} ds_{10}^2 &= \frac{1}{\nu}\left[f^{-1/2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2\right) + Hf^{-1/2}dy^2 \right. \\ & \left. +Hf^{1/2}\left(dr^2 + r^2 d\Omega_2^2\right)\right] \qquad (5.3.67) \end{aligned}$$

It can be seen from the metric, as well as (5.3.64), (5.3.65), that this is a NS5⊥D6(5) brane solution. This ten dimensional solution solves the equations of motion for ten dimensional supergravity.

Decoupling Limit

The dynamics of the type IIA NS5 branes found above in equation (5.3.67) also decouple from the bulk at low energies. Here, near the NS5 brane ($H \gg 1$), one is interested in the behaviour of the NS5-branes in the limit where the string coupling vanishes,

$$g_s \to 0 \quad , \quad \ell_s = \text{ fixed} \tag{5.3.68}$$

In these limits, in order to keep the radial coordinates fixed, they are rescaled to

$$Y = \frac{y}{g_s \ell_s^2} \quad , \quad U = \frac{r}{g_s \ell_s^2} \tag{5.3.69}$$

This will cause the harmonic functions to become (where $R_\infty = 4n = g_s \ell_s$ is also used)

$$f(r) = 1 + \frac{2n}{r} = 1 + \frac{N_6}{2U\ell_s} \equiv f(U) \tag{5.3.70}$$

$$\begin{aligned} H_{TN_4}(Y,U) &\approx Q_{M5} \int dc \frac{c^2\Gamma(cn)}{2\pi} \cos(cy) e^{-cr} \mathcal{U}(1+cn,2,2cr) \\ &= \frac{\pi N_5}{g_s^2 \ell_s^3} \int dP\ P^2 \Gamma\left(\frac{P}{4\ell_s}\right) \cos(PY) e^{-PU} \mathcal{U}\left(1+\frac{P}{4\ell_s},2,2PU\right) \\ &= \frac{h(Y,U)}{g_s^2} \end{aligned} \tag{5.3.71}$$

where I have generalized to the case of N_5 NS5-branes and N_6 D6-branes, $\ell_p = g_s^{1/3}\ell_s$ has been used to rewrite

$$Q_{M5} = \pi N_5 \ell_p^3 = \pi N_5 g_s \ell_s^3 \tag{5.3.72}$$

and c has been rescaled so that $c = P g_s^{-1} \ell_s^{-2}$. The decoupled metric is then found by substituting in (5.3.70), (5.3.71) and the above limits into the metric (5.3.67), to give

$$\begin{aligned} ds_{10}^2 &= f^{-1/2}(U)\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2\right) \\ &\quad + \ell_s^2\left[h(Y,U) f^{-1/2}(U) dY^2 + h(Y,U) f^{1/2}(U)\left(dU^2 + U^2 d\Omega_2^2\right)\right] \end{aligned} \tag{5.3.73}$$

Though outside the scope of this thesis, it should be noted that in the decoupling limit, the decoupled free theory on the NS5-branes should then be a little string theory. A little string theory is a six dimensional, non-gravitational theory, where modes on the 5-brane interact amongst themselves, decoupled from the bulk [71].

5.4 Non-Supersymmetric Solutions

There are quite a few solutions possible that don't preserve any supersymmetry. Embedding a six or eight dimensional Taub-NUT metric, or a four, six or eight dimensional Taub-Bolt (TB) metric (or a TN4 and a TB4, or two TB4 metrics, or a TB4 with an Eguchi-Hanson metric) into an M2 brane metric will give a solution to the eleven dimensional equations of motion, but will not preserve any supersymmetry. Also, embedding a TB4 metric into the M5 brane metric provides another non-supersymmetric example. The solution I present below is the embedding of the eight dimensional Taub-NUT metric into the transverse space of an M2 brane.

5.4.1 M2 Brane with TN_8

The metric for an M2 brane with an eight dimensional Taub-NUT metric background is given by

$$\begin{aligned} ds^2 &= H^{-2/3}(r)\left(-dt^2+dx_1^2+dx_2^2\right)+H^{1/3}(r)ds_{tn8}^2 \\ F_{tx_1x_2y} &= -\frac{1}{2H^2}\frac{\partial H}{\partial y} \quad , \qquad F_{tx_1x_2r} = -\frac{1}{2H^2}\frac{\partial H}{\partial r} \end{aligned} \tag{5.4.1}$$

where

$$\begin{aligned} ds_{tn8}^2 &= \frac{(8n)^2}{f(r)}\left[d\psi+\frac{1}{4}\cos(\theta_1)d\phi_1+\frac{1}{4}\cos(\theta_2)d\phi_2+\frac{1}{4}\cos(\theta_3)d\phi_3\right]^2 \\ &\quad +f(r)dr^2+r(r+2n)\left(d\Omega_{2(1)}^2+d\Omega_{2(2)}^2+d\Omega_{2(3)}^2\right) \\ f(r) &= \frac{5(r+2n)^3}{r(r^2+6nr+10n^2)} \\ d\Omega_{2(i)}^2 &= d\theta_i^2+\sin^2(\theta_i)d\phi_i^2 \end{aligned} \tag{5.4.2}$$

Note that with this form of the Taub-NUT metric, the minimum value of r is zero. This metric is a solution to the eleven dimensional supergravity field equations if the metric function $H(r)$ satisfies the differential equation

$$0 = r(r^2+6rn+10n^2)\frac{d^2H}{dr^2}+2(3r^2+15rn+20n^2)\frac{dH}{dr} \tag{5.4.3}$$

This has the analytic solution

$$\begin{aligned} H(r) &= \frac{1}{30n^2r^3}-\frac{3}{100n^3r^2}+\frac{13}{500n^4r} \\ &\quad +\frac{1}{2500n^5}\left(12\ln\left(\frac{r^2}{r^2+6nr+10n^2}\right)-7\tan^{-1}\left(\frac{r}{n}+3\right)+\frac{7\pi}{2}\right) \end{aligned} \tag{5.4.4}$$

The small r/large r behaviour of this solution is

$$\begin{aligned} \lim_{r\to 0} H(r) &\sim \frac{1}{r^3} \\ \lim_{r\to\infty} H(r) &\sim \frac{1}{r^5} \end{aligned}$$

Kaluza-Klein Reduction

Here, the reduction down to ten dimensions gives the NSNS fields

$$\begin{aligned} \Phi &= \frac{3}{4}\ln\left(\frac{H^{1/3}}{f}\right) && (5.4.5)\\ \mathcal{B}_{\alpha\beta} &= 0 \end{aligned}$$

and RR fields

$$C_{[1]} = 2n\cos(\theta_1)d\phi_1 + 2n\cos(\theta_2)d\phi_2 + 2n\cos(\theta_3)d\phi_3 \tag{5.4.6}$$

$$\mathcal{A}_{[3]} = \frac{1}{H}dt\wedge dx_1\wedge dx_2 \tag{5.4.7}$$

with the metric in ten dimensions given by

$$\begin{aligned} ds_{10}^2 &= H^{-1/2}f^{-1/2}\left(-dt^2+dx_1^2+dx_2^2\right) && (5.4.8)\\ &\quad +H^{1/2}f^{-1/2}\left(fdr^2+r(r+2n)\left(d\Omega_{2(1)}^2+d\Omega_{2(2)}^2+d\Omega_{2(3)}^2\right)\right) \end{aligned}$$

This ten dimensional solution is a D2 brane localized at a point of the seven dimensional background space.

Note that there is of course no extra D-brane for the D2 brane to decouple from in this case.

Chapter 6

Discussion and Closing

6.1 Review

The Taub-NUT metric has been shown to have several useful applications to M-theory constructions. As I discussed in chapter 3, the Taub-NUT spacetime (in any $d+1$ (even) dimension) possesses many features that render it a non-trivial metric, such as its closed timelike curves and the quasi-regular singularities. The metric also has the “NUT” charge that acts as a magnetic type of mass, giving rise to Misner string singularities.

The Euclidean form of the metric gives rise to two separate solutions; the NUT solution, when the fixed point set of ∂_T is zero dimensional, and the Bolt solution that exists when the fixed point set of ∂_T is $(d-1)$ dimensional.

The Taub-NUT metric has been shown to be a non-trivial test of both the AdS/CFT and the dS/CFT conjectures, along with some interesting consequences when one has a positive cosmological constant. Also, Taub-NUT metrics provide new, localized brane solutions when embedded into eleven dimensional supergravity. I recap these results now.

6.1.1 Taub-NUT-AdS

When a negative cosmological constant is added to the metric, the resulting Taub-NUT-AdS metrics provide a non-trivial, successful test of the AdS/CFT correspondence, that posits the relationship between a $(d+1)$ dimensional AdS metric in the bulk and its field theory dual in d dimensions on the boundary. There is consistency between the thermodynamic results I calculated and the Nöether -charge approach, despite the *a priori* distinction between the two approaches (I show the specific six dimensional example in section 4.2). Several of the results should be re-emphasized.

First, I was able to obtain a general formula for the action and entropy, that applies to any even dimension. This allows an analysis of the TNAdS spacetimes in general. For the NUT case in $(d+1)$ dimensions the thermodynamics vary depending if one is in $4k$ or $4k+2$ dimensions (where k is a positive integer). In $4k$ dimensions the entropy and specific heat will both be positive inside a specific range,

$$\ell\sqrt{\frac{(n-3)}{n(n-1)}} < N < \ell\sqrt{\frac{(n-3)}{(n-1)(n-2)}} \tag{6.1.1}$$

(where $(n = d+1)$ here). Outside of this range, one or the other of the entropy or specific heat becomes negative, and hence the solution is no longer considered to be thermodynamically stable. In $4k+2$ dimensions, no such range exists; the entropy or specific heat or both are always negative for a given value of N/ℓ. The Bolt case, on the other hand, always has a region of thermodynamic stability, for both the upper and lower branch solutions.

Also, recall that the Nöether method involves the subtraction of results from a background metric, in order to make all quantities well-defined. While a Nöether result for general $d+1$ dimensions was not obtained in [16], some general trends were noticed that bear mentioning. Based on the results in $4, 6, 8$ and 10 dimensions, the Nöether charge can be conjectured to be

$$Q = \frac{(d-1)(4\pi)^{(d-1)/2}}{8\pi}(m_b - m_n) = \mathfrak{M}_{\text{Bolt}} - \mathfrak{M}_{\text{NUT}} \tag{6.1.2}$$

for general $d+1$ dimensions. From this, the entropy is calculated through the first law, up to an overall constant. Again generalizing from the results in 4 to 10 dimensions, the entropy can be conjectured to be

$$S_Q = \frac{4\pi}{\kappa}\left(S_{\text{Bolt}} - S_{\text{NUT}}\right) + c_\ell \tag{6.1.3}$$

where c_ℓ is a constant not depending on N, but it can depend on ℓ.

The results of the calculation of the entropy from the counterterm approach give negative entropy, and in the case of the NUT solution this happens over quite a broad range, which is potentially troubling. However, the subtraction of the NUT entropy from the Bolt entropy (6.1.3) will always give a positive result.

6.1.2 Taub-NUT-dS

Perhaps the most interesting of the results presented in this thesis are those calculated from the de Sitter version of the Taub-NUT metric. Recall that there are two conjectures that apply to asymptotically de Sitter spacetimes. The first is the Bousso N-bound [19], which states that *any asymptotically dS spacetime will have an entropy no greater than the entropy* $\pi\ell^2$ *of pure dS with cosmological constant* $\Lambda = 3/\ell^2$ *in* $(3+1)$ *dimensions.* The second is the maximal mass conjecture, based in part upon the N-bound, by Balasubramanian *et. al.* [21], and states that *any asymptotically dS spacetime with mass greater than dS has a cosmological singularity.*

The Taub-NUT-dS metric provides a counter-example to both of these conjectures for certain ranges of the NUT charge n (or $N = \mathrm{i}n$, depending on which approach one uses), though some of the solutions originally calculated in [20] have since been shown by Mann and Stelea [44] to be invalid. Specifically, it was shown that there is in fact no NUT solution analogous to the NUT solution found in the AdS case. The NUT solution at $r = N$ is in fact not the largest root of the metric in the NUT case, and hence isn't a valid solution. The fact that the entropy and conserved mass at $r = N$ satisfy the first law can be supposed to be a consequence of the fact that the dS-NUT solution is the analytic continuation $\ell \to \mathrm{i}\ell$ of the AdS-NUT solution.

Also, the two lower branch bolt solutions found through either the C-approach or the R-approach are not the largest roots of the lower branches. There exist two roots of the metric function such that the lower branch radius lies between these two. Similar to the NUT case, the fact that the lower branch C- and R-approach solutions solve the first law is a consequence of the fact that they can be found through analytic continuations of the lower branch TNAdS case.

Finally, it was also shown by Mann and Stelea [44] that the C-approach and R-approach quantities are in fact exactly the same, and are simply analytic continuations of one another.

Despite these new findings, the surviving thermodynamic properties found from the Taub-NUT-dS solution provide successful counter-examples to the two asymptotically dS conjectures, as shown in section 4.3. Also note that, as in the AdS case, the thermodynamic behaviour of the solutions in $4k$ dimensions are qualitatively similar to each other, and the behaviour of the solutions in $4k+2$ are also similar to each other.

The reason for the violation of the N-bound by the TNdS spacetime is due to the presence of the Misner string. In any asymptotically dS spacetime, the usual entropy/area law $S = A/4$ is respected. However, the Misner string gives an extra contribution to the entropy of the TNdS spacetime [14], and it is this extra contribution that violates the N-bound. In the R-approach, for example, and in four dimensions, the N-bound is violated for almost all values of n, except near the maximal value $n_{max} = 0.5941\ell$. Note that the specific heat is only positive for $0.2886751346\ell < n < 0.5\ell$, and hence the TNdS metric (R-approach), though it violates the N-bound over a broad range of n, is only thermodynamically stable for n outside of these values.

Again taking the R-approach as an example, the maximal mass conjecture is violated for all values of the NUT charge as it can be seen (figure 4.3.1) that the conserved mass $\mathfrak{M}_R^+$ is always positive.

6.1.3 M-Branes

The embedding of Taub-NUT metrics into eleven dimensional supergravity, with either an existing M2 or M5 brane present, provides new brane solutions, some of which preserve some supersymmetry, and others completely breaking supersymmetry. Though I have not presented all of the solutions found in [28, 29], the results are in appendix C, and include combinations of embeddings of the Taub-NUT metric in four dimensions, the four dimensional Eguchi-Hanson metric, and for the M2 branes, higher dimensional Taub-NUT metrics, though some of the metric function solutions are numerical.

With the M2 branes, the common feature found in all of the cases is that the brane function is found to be a combination of an exponentially decaying "radial" function and a damped oscillating one; for example, see equation (5.3.10) in the case of embedding two TN_4 metrics. The radial parts of these metric functions will diverge near the brane core, and will vanish as r_i approaches infinity.

By reducing these solutions down to ten dimensions, one gets a variety of fully localized Type IIA string theory D2⊥D6 systems. In all cases involving combinations of the Taub-NUT metric or the Eguchi-Hanson - both of which have self-dual Riemann curvature - these solutions preserve 1/4 of the supersymmetry. In all of the other cases, involving the four dimensional Taub-Bolt metric, or higher dimensional Taub-NUT or Taub-Bolt metrics, the M-brane and resulting D-brane systems are not supersymmetric. It is interesting to note, however, that even in these non-supersymmetric cases, the metric functions found display the same behaviour as their supersymmetric counterparts.

The metric functions for the M5 brane solutions also display the exponential decay and damped oscillation combinations. Here, however, the reduction down to ten dimensions gives rise to localized NS5⊥D6 brane intersections. Again the radial function vanishes far from the M5 brane and diverges near the brane core. The M5 and NS5⊥D6 configurations found from embedding the Taub-NUT or Eguchi-Hanson metrics preserve 1/4 of the supersymmetry; again, though the Taub-Bolt embedding preserves none of the supersymmetry, the general structure of the brane is the same as in the supersymmetric cases.

6.2 Closing

In the preceding thesis, I have attempted to provide justification for the use of Taub-NUT spacetimes in testing M-theory applications such as the (A)dS/CFT conjectures, as well as expanding our knowledge of supergravity brane solutions in an attempt to better understand the as yet unknown M-theory itself. There are, as ever, directions for future work suggested by the material left out of this work.

It would be interesting to perform an off-shell calculation of the Taub-NUT-AdS results using both the counterterm approach from the AdS/CFT as well as the Nöether charge approach, and compare the results so computed. Also, the use of only the first term of the counterterm action to calculate the general formula for the action in $d+1$ dimensions, though I believe fully justified, could do with a more rigorous mathematical proof. It would be interesting to see if one could show that the finite contribution to any asymptotically AdS spacetime's action comes only from these first few terms.

As for the dS/CFT, besides the points just noted for the Taub-NUT-AdS calculations that would also apply, it should be emphasized that the calculations of the thermodynamic properties here were due to an extension of the path-integral formulation to regions outside of the cosmological horizon. This extension was used to justify the Gibbs-Duhem relation $S = \beta\mathfrak{M} - I_{cl}$ in an asymptotically de Sitter spacetime. This involved evaluating the path-integral between two histories or time-lines, as opposed to the evaluation between two surfaces as in AdS spacetimes. The justification of the Gibbs-Duhem relation is accomplished here through thermodynamic arguments, as in section 2.4, but a more rigorous proof of this dS path-integral remains an open problem for future work; the arguments for this approach should be taken as suggestive evidence, but not fact, until such a proof is presented.

Also, the maximal mass conjecture is just a conjecture - no proof of it has yet been presented. Before further tests or counterexamples are presented, it would be instructive to provide a definite proof of the conjecture, as well as a hard definition of the singularities meant by "cosmological" singularities.

New brane solutions should also be quite easily calculable based on an extension of the work presented here. For example, more non-supersymmetric solutions in the form of embedding higher dimensional Eguchi-Hanson metrics should certainly be possible. Another consideration that should be taken into account, though it has been pointed out to me that it would be quite difficult to do in eleven dimensional supergravity, should be to include the fermions in a brane solution instead of setting the VEV's of the fermion fields to zero. Such a solution, were it possible to find, would be a far more complete solution to the eleven dimensional equations of motion.

Finally, an exploration of the little string theory found in the decoupling limit of the NS5-branes of the reduced M5-brane solutions would also be interesting, in both the type IIA theory and the dual type IIB theory.

Appendix A

Derivation of Laws of Black Hole Thermodynamics

Note that the results derived here are drawn from [33], [34] and [39], though most of the mathematical steps will be shown.

A.1 Surface Gravity

As mentioned in the main text, section 2.1.1, κ is calculated on the horizon of an arbitrary, stationary black hole. Since the horizon of a black hole is a null hypersurface, and the Killing vector χ^α is normal to the horizon, $\chi^\alpha \chi_\alpha = 0$. This implies $(\chi^\gamma \chi_\gamma)_{;\alpha}$ is also normal to the horizon, and thus there exists a function κ such that

$$(\chi^\gamma \chi_\gamma)^{;\alpha} = -2\kappa \chi^\alpha \tag{A.1.1}$$

Recall that the Lie derivative of a tensor $A^{\alpha\beta}$ with respect to a vector u^ν is given by

$$\mathcal{L}_u A^{\alpha\beta} = A^{\alpha\beta}{}_{;\gamma} u^\gamma - A^{\gamma\beta} u^\alpha{}_{;\gamma} - A^{\alpha\gamma} u^\beta{}_\gamma \tag{A.1.2}$$

So, take the Lie derivative of (A.1.1),

$$\begin{aligned}
-2\chi^\alpha \mathcal{L}_\chi \kappa - 2\kappa \left\{ \chi^\alpha{}_{;\beta} \chi^\beta - \chi^\alpha{}_{;\beta} \chi^\beta \right\} &= \mathcal{L}_\chi \left[\chi^{\gamma;\alpha} \chi_\gamma + \chi^\gamma \chi_\gamma{}^{;\alpha} \right] \\
-2\chi^\alpha \mathcal{L}_\chi \kappa &= 2\mathcal{L}_\chi \left[\chi^{\gamma;\alpha} \chi_\gamma \right] \\
-\chi^\alpha \mathcal{L}_\chi \kappa &= 0 \\
\mathcal{L}_\chi \kappa &= 0
\end{aligned} \tag{A.1.3}$$

where the last line follows because $\chi^\alpha \neq 0$.

Now, note that (A.1.1) can be rewritten as

$$\kappa\chi_\alpha = \chi_{\alpha;\gamma}\chi^\gamma = -\chi_{\gamma;\alpha}\chi^\alpha \tag{A.1.4}$$

and recall the geodesic equation

$$u^\alpha{}_{;\beta}u^\beta = \kappa u^\alpha \tag{A.1.5}$$

For $\kappa \neq 0$, (A.1.5) is the non-affine parameterization. Thus, κ is a measure of the failure of the Killing parameter v to agree with the affine parameter λ along null geodesic generators of the horizon, where the Killing parameter v is defined by

$$\chi^\alpha v_{;\alpha} = 1 \tag{A.1.6}$$

If we now define

$$k^\alpha = e^{-\kappa v}\chi^\alpha \tag{A.1.7}$$

then

$$\begin{aligned} k^\beta k^\alpha{}_{;\beta} &= e^{-2\kappa v}\left\{\chi^\beta\chi^\alpha{}_{;\beta} - \chi^\alpha\chi^\beta\kappa_{;\beta}v - \chi^\alpha\chi^\beta\kappa v_{;\beta}\right\} \\ &= e^{-2\kappa v}\left\{\kappa\chi^\alpha - \chi^\alpha(\mathcal{L}_\chi\kappa)v - \chi^\alpha\kappa\right\} \\ k^\beta k^\alpha{}_{;\beta} &= 0 \end{aligned} \tag{A.1.8}$$

where the second equality follows from using the Lie derivative of a scalar (κ), (A.1.4) and (A.1.6). This means k^α is the affinely parameterized tangent to the null geodesic generators of the horizon. So, on the horizon, the relation between λ and v is, for $\kappa \neq 0$,

$$\lambda \propto e^{\kappa v} \tag{A.1.9}$$

Now, recall that, by the Frobenius theorem ([33], pg. 435), a necessary and sufficient condition that ξ^α be hypersurface-orthogonal is that ξ^α satisfy

$$0 = \xi_{[\alpha}\nabla_\beta\xi_{\gamma]} \tag{A.1.10}$$

Since, on the horizon of the black hole, χ^α is hypersurface-orthogonal, we then have (on the horizon)

$$\begin{aligned} 0 &= \chi_{[\alpha}\nabla_\beta\chi_{\gamma]} \\ 0 &= 2\chi_\alpha\nabla_\beta\chi_\gamma + 2\chi_\gamma\nabla_\alpha\chi_\beta - 2\chi_\beta\nabla_\alpha\chi_c \\ \therefore \chi_\gamma\nabla_\alpha\chi_\beta &= -2\chi_{[\alpha}\nabla_{\beta]}\chi_\gamma \end{aligned} \tag{A.1.11}$$

If (A.1.11) is now contracted with $\nabla^\alpha \chi^\beta$, we get

$$\begin{aligned}
(\nabla^\alpha \chi^\beta)\chi_\gamma(\nabla_\alpha \chi_\beta) &= -2\chi_{[\alpha}\nabla_{\beta]}\chi_\gamma(\nabla^\alpha \chi^\beta) = -2(\nabla^\alpha \chi^\beta)\chi_\alpha(\nabla_\beta \chi_\gamma) \\
\chi_\gamma(\nabla^\alpha \chi^\beta)(\nabla_\alpha \chi_\beta) &= -2\kappa\chi^\beta(\nabla_\beta \chi_\gamma) \quad \text{(by } (A.1.4)\text{)} \\
\chi_\gamma(\nabla^\alpha \chi^\beta)(\nabla_\alpha \chi_\beta) &= -2\kappa^2\chi_\gamma \quad \text{(by } (A.1.4)\text{)}
\end{aligned}$$

and thus

$$\kappa^2 = -\frac{1}{2}(\nabla^\alpha \chi^\beta)(\nabla_\alpha \chi_\beta) \tag{A.1.12}$$

which is (2.1.3).

A.2 Zeroth Law

The Zeroth Law of black hole thermodynamics is that this surface gravity κ is constant over the surface of the horizon. This can be shown as follows (see also [33, 34]).

$\varepsilon^{\alpha\beta\gamma\delta}\chi_\delta$ is tangent to the horizon, since $\chi_\gamma(\varepsilon^{\alpha\beta\gamma\delta}\chi_\delta) = 0$. Thus, $\varepsilon^{\alpha\beta\gamma\delta}\chi_\delta\nabla_\gamma \equiv \chi_{[\delta}\nabla_{\gamma]}$ can be applied to any equation holding on the horizon.

Also, for any Killing field, the relation

$$\nabla_\alpha\nabla_\beta\xi_\gamma = -R_{\beta\gamma\alpha}{}^\delta\xi_\delta \tag{A.2.1}$$

is true. Thus, apply $\chi_{[\delta}\nabla_{\gamma]}$ to (A.1.4);

$$\begin{aligned}
\underbrace{\chi_\alpha\chi_{[\delta}\nabla_{\gamma]}\kappa}_{b'} + \underbrace{\kappa\chi_{[\delta}\nabla_{\gamma]}\chi_\alpha}_{a'} &= (\chi_{[\delta}\nabla_{\gamma]}\chi^\beta)\nabla_\beta\chi_\alpha + \chi^\beta\chi_{[\delta}\nabla_{\gamma]}\nabla_\beta\chi_\alpha \\
&= \underbrace{(\chi_{[\delta}\nabla_{\gamma]}\chi^\beta)\nabla_\beta\chi_\alpha}_{a} + \underbrace{\chi^\beta R_{\alpha\beta[\gamma}{}^\sigma\chi_{\delta]}\chi_\sigma}_{b} \qquad \text{(A.2.2)}
\end{aligned}$$

But, a can be written

$$\begin{aligned}
a = (\chi_{[\delta}\nabla_{\gamma]}\chi^\beta)\nabla_\beta\chi_\alpha &= -\frac{1}{2}(\chi^\beta\nabla_\delta\chi_\gamma)(\nabla_\beta\chi_\alpha) \quad \text{(by } (A.1.11)\text{)} \\
&= -\frac{1}{2}\kappa\chi_\alpha\nabla_\delta\chi_c \quad \text{(by } (A.1.4)\text{)} \\
&= \kappa\chi_{[\delta}\nabla_{\gamma]}\chi_\alpha \quad \text{(by } (A.1.11)\text{)} \\
&= a' \qquad \text{(A.2.3)}
\end{aligned}$$

and so (A.2.2) becomes

$$\chi_\alpha \chi_{[\delta} \nabla_{\gamma]} \kappa = \chi^\beta R_{\alpha\beta[\gamma}{}^\sigma \chi_{\delta]} \chi_\sigma \tag{A.2.4}$$

Applying $\chi_{[\delta}\nabla_{\sigma]}$ to (A.1.11) gives

$$\begin{aligned}\underbrace{(\chi_{[\delta}\nabla_{\sigma]}\chi_\gamma)\nabla_\alpha\chi_\beta}_{A} + \underbrace{\chi_\gamma(\chi_{[\delta}\nabla_{\sigma]}\nabla_\alpha\chi_\beta)}_{B} \quad = \quad & \underbrace{-2(\chi_{[\delta}\nabla_{\sigma]}\chi_{[\alpha})\nabla_{\beta]}\chi_\gamma}_{A'} \\ & - \underbrace{2(\chi_{[\delta}\nabla_{\sigma]}\nabla_{[\beta}\chi_{|c|})\chi_{\alpha]}}_{B'}\end{aligned}$$

Here, A will cancel A' after repeated use of (A.1.11), and using (A.2.1),

$$-\chi_\gamma R_{\alpha\beta[\sigma}{}^\rho \chi_{\delta]} \chi_\rho = 2\chi_{[\alpha} R_{\beta]\gamma[\sigma}{}^\rho \chi_{\delta]} \chi_\rho \tag{A.2.5}$$

Next, multiplying this by $g^{\gamma\sigma}$, and contracting over γ, σ, the LHS will vanish, and after some more algebra,

$$-\chi_{[\alpha} R_{\beta]}^{\rho} \chi_\delta \chi_\rho = \chi_{[\alpha} R_{\beta]\gamma\delta}{}^\rho \chi^\gamma \chi_\rho \tag{A.2.6}$$

Note that the RHS of (A.2.4) is equal to the RHS of (A.2.6), and so we can equate the LHS's, to give

$$\chi_{[\delta} \nabla_{\gamma]} \kappa = -\chi_{[\delta} R_{\gamma]}{}^\rho \chi_\rho \tag{A.2.7}$$

Now, from the Raychaudhuri equation (see eq's (9.2.32), (9.2.33) in [33]) on the horizon,

$$0 = R_{\alpha\beta} k^\alpha k^\beta \tag{A.2.8}$$

and along with the Einstein equations, this implies $T^\alpha{}_\beta \chi^\beta \chi_\alpha = 0$. Thus, $-T^\alpha{}_\beta \chi^\beta$ must point in the direction of χ^α, or

$$0 = \chi_{[\gamma} T_{\alpha]\beta} \chi^\beta$$

Thus, again using Einstein's equations, the RHS of (A.2.7) is zero, and so

$$\chi_{[\delta} \nabla_{\gamma]} \kappa = 0 \tag{A.2.9}$$

showing that κ is indeed a constant over the horizon of the black hole.

A.3 First Law

Before going into the derivation, some equations are needed. First, Raychaudhuri's equation[1] is given by

$$\frac{d\theta}{dv} = \kappa\theta - 8\pi T_{\alpha\beta}\chi^{\alpha}\chi^{\beta} \tag{A.3.1}$$

where quadratic terms have been neglected (and recall the definitions of κ and χ^{α} from sections A.1, A.2, and equation (2.1.2)). θ is the fractional rate of change in the cross-sectional area,

$$\theta = \frac{1}{dS}\frac{d(dS)}{dv} \tag{A.3.2}$$

Also, the differential surface element is given by

$$d\Sigma_{\alpha} = -\chi_{\alpha} dS\ dv \tag{A.3.3}$$

So, consider a quasi-static transformation of a black hole from a state with mass M, angular momentum J and surface area A to a state $M + \delta M$, $J + \delta J$ and $A + \delta A$. In other words, suppose an initially stationary black hole is perturbed by a small quantity of matter given by the infinitesimal stress-tensor. Thus, the change in the mass and angular momentum will be given by[2]

$$\begin{aligned} \delta M &= -\int_H T^{\alpha}{}_{\beta}\xi^{\beta} d\Sigma_{\alpha} && \text{(A.3.4)} \\ \delta J &= \int_H T^{\alpha}{}_{\beta}\psi^{\beta} d\Sigma_{\alpha} && \text{(A.3.5)} \end{aligned}$$

where integration is over the event horizon H, we want first order in $T_{\alpha\beta}$, and recall $\chi^{\alpha} = \xi^{\alpha} + \Omega_H\psi^{\alpha}$. Thus,

$$\begin{aligned} (A.3.4) - \Omega_H(A.3.5) &= \delta M - \Omega_H \delta J \\ \delta M - \Omega_H \delta J &= -\int dv \oint_{\mathcal{H}} dS T_{\alpha\beta}\chi^{\alpha}\chi^{\beta} && \text{(A.3.6)} \end{aligned}$$

[1] See for example [33], pg. 218 or [34], chpt. 5.
[2] See [34].

Substituting in (A.3.1), this becomes

$$\begin{aligned}\delta M - \Omega_H \delta J &= -\frac{1}{8\pi}\int dv \oint_{\mathcal{H}} \left(\frac{d\theta}{dv} - \kappa\theta\right) dS \\ &= -\frac{1}{8\pi}\oint_{\mathcal{H}} \theta dS\Bigg|_{v=-\infty}^{v=\infty} + \frac{\kappa}{8\pi}\int dv \oint_{\mathcal{H}} \theta dS \qquad \text{(A.3.7)}\end{aligned}$$

Here, however, the first term on the right is zero, since the black hole is stationary before and after the perturbation ($\therefore \theta(v = \pm\infty) = 0$). So,

$$\begin{aligned}\delta M - \Omega_H \delta J &= +\frac{\kappa}{8\pi}\int dv \oint_{\mathcal{H}} \left(\frac{1}{dS}\frac{d(dS)}{dv}\right) dS = \frac{\kappa}{8\pi}\oint_{\mathcal{H}} dS\Bigg|_{v=-\infty}^{v=\infty} \\ &= \frac{\kappa}{8\pi}\delta A \\ \therefore \delta M &= \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J \qquad \text{(A.3.8)}\end{aligned}$$

which is the First Law of black hole thermodynamics (2.1.4).

A.4 Third Law

A more formal definition of the Third Law (again due to Israel [39]) is

"*In a strongly future asymptotically predictable black hole spacetime, there is a continuous process in which $\mathcal{S}(\tau)$ contains trapped surfaces for all $\tau < \tau_1$, but none for $\tau > \tau_1$. Then the weak energy condition is necessarily violated in a neighbourhood of the apparent horizon on $\mathcal{S}(\tau_1)$.*"

The proof depends on the following *Lemma*:
If S_0 is a trapped two surface, and is extended to a three cylinder Σ that is foliated by two-sections $S(\tau)$, with the following properties;

(i) The extension of S_0 is *semi-rigid* (i.e. Lie transport along the normal to $S(\tau)$ preserves elements of the two-area).

(ii) Σ is regular

(iii) The weak energy condition, $T_{\alpha\beta}u^\alpha u^\beta \geq 0$ holds on Σ.

Then, Σ is everywhere spacelike, and thus all subsequent two-sections of $S(\tau)$ are trapped.

So, if S_0 is one of the outermost trapped surfaces $\mathcal{S}(\tau_1 - \epsilon)$, and S_0 is extended semi-rigidly to the future (i) with (iii) assumed to hold, then not all subsequent two-sections $\mathcal{S}(\tau)$ are trapped, and we get a contradiction.

A.5 Calculation of period β of a Black Hole

Here I wish to demonstrate the calculation of the result (2.1.11) for the period of the Euclidean section of a black hole. For a general Euclideanized metric,

$$ds^2 = G(r)d\tau^2 + \frac{dr^2}{F(r)} + d\Sigma^2 \quad ; \quad \tau = it \tag{A.5.1}$$

where $d\Sigma$ is the spherical part of the metric. If $F(r)$ and $G(r)$ have the same zeroes at $r = a$, then this can be re-written

$$ds^2 = (r-a)g(r)d\tau^2 + \frac{dr^2}{(r-a)f(r)} + d\Sigma^2 \tag{A.5.2}$$

with $F(r)$ expanded as

$$\begin{aligned} F(r)|_{r=a} &\approx F(a) + (r-a)F'(a) + \dots \\ &= 0 + (r-a)\left[f(r) + (r-a)f'(r)\right]_{r=a} + \dots \\ &\approx (r-a)f(a) \end{aligned} \tag{A.5.3}$$

Next, define a new radial coordinate

$$\begin{aligned} \rho &= \int \frac{dr}{\sqrt{(r-a)f(a)}} = \frac{2\sqrt{(r-a)}}{\sqrt{f(a)}} \\ \therefore (r-a) &= \frac{\rho^2 f(a)}{4} \end{aligned}$$

So, $G(r)$ can be similarly expanded

$$G(r)|_{r=a} \approx (r-a)g(a) \tag{A.5.4}$$

$$= \frac{\rho^2 f(a)g(a)}{4} \tag{A.5.5}$$

Substituting these into the metric gives,

$$ds^2 \approx \frac{\rho^2 f(a)g(a)}{4}d\tau^2 + d\rho^2 + d\Sigma^2 \tag{A.5.6}$$

and letting $\theta = \frac{\sqrt{f(a)g(a)}}{2}\tau$,

$$ds^2 = \underbrace{\rho^2 d\theta^2 + d\rho^2}_{S^1} + d\Sigma^2 \tag{A.5.7}$$

The first two terms here are S^1, and θ has a period of 2π. Thus, τ must have a period

$$\beta = \frac{4\pi}{\sqrt{|f(a)g(a)|}} = \frac{4\pi}{\sqrt{|F'(a)G'(a)|}} \tag{A.5.8}$$

If we have that $G(r) = F(r)$, we recover the standard form of the period,

$$\beta = \frac{4\pi}{|F'(a)|} \tag{A.5.9}$$

or equation (2.1.11).

Appendix B

Manifolds and Forms

B.1 Cartan Algebra

Here I wish to introduce the Cartan algebra, which uses the language of differential forms to calculate GR quantities such as the Riemann curvature tensor, Ricci Tensor, etc., and allows one to do so more easily than using tensor notation, at least for metrics with a lot of symmetries. This is useful in chapter 5 for calculating the preserved supersymmetry of the M-brane solutions, as well as in chapter 3 for double-checking calculations (see below).

The operators $\{\partial/\partial x\}$ at $x = p$ form a basis for the tangent space $T_p(M)$ of a manifold M at a point $p \in M$ (see [72]). The cotangent space $T_p^*(M)$ of a manifold at $p \in M$ is the dual vector space to the tangent space. The basis vectors of the cotangent space are $\{dx\}$. The inner product is then given by

$$\left\langle \partial/\partial x^i, dx^j \right\rangle = \delta^i_j \tag{B.1.1}$$

We define one forms e^a (where here $a, b, \ldots$ are tangent space indices, and $\mu, \nu, \ldots$ are coordinate indices - note for later, if coordinates are specified, hatted indices $\hat{t}, \hat{x}, \ldots$ will be tangent space indices, and non-hatted will be coordinate indices)

$$e^a = e^a{}_\mu dx^\mu \tag{B.1.2}$$

which takes the coordinate basis dx^μ of $T_x^*(M)$ into an orthonormal basis of $T_x^*(M)$ (note e^a is not necessarily an exact 1-form). The metric

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu = \eta_{ab} e^a e^b \tag{B.1.3}$$

is decomposed into these *vierbeins* or *tetrads* $e^a{}_\mu(x)$

$$g_{\mu\nu} \quad = \quad \eta_{ab} e^a{}_\mu e^b{}_\nu \tag{B.1.4}$$

where η_{ab} is (usually) a flat Minkowski or Euclidean metric. The inverse of $e^a{}_\mu$ is defined by

$$E_a{}^\mu = \eta_{ab} g^{\mu\nu} e^b{}_\nu \tag{B.1.5}$$

and $e^a{}_\mu$, $E_a{}^\mu$ are used to inter-convert tangent and coordinate indices where necessary. $E_a{}^\mu$ is thus a transformation from the basis $\partial/\partial x^\mu$ of $T_x(M)$ to the orthonormal basis of $T_x(M)$

$$E_a = E_a{}^\mu \frac{\partial}{\partial x^\mu} \tag{B.1.6}$$

We can now introduce the *affine spin connection one-form* $\omega^a{}_b$, and write Cartan's structure equations [72]

$$\begin{aligned} de^a + \omega^a{}_b \wedge e^b \quad &= \quad de^a + \omega^a{}_{bc} e^b \wedge e^c \equiv T^a && \text{(B.1.7)} \\ \mathcal{R}^a{}_b \quad &= \quad d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2}\mathcal{R}^a{}_{bcd} e^c \wedge e^d && \text{(B.1.8)} \end{aligned}$$

where T^a is called the *torsion* 2-form, and $\mathcal{R}^a{}_b$ is the *curvature* 2-form. The Riemann tensor is then

$$R^\mu{}_{\nu\alpha\beta} = \mathcal{R}^a{}_{bcd} E_a{}^\mu e^b{}_\nu e^c{}_\alpha e^d{}_\beta \tag{B.1.9}$$

In GR, the Levi-Civita connection is determined by two conditions, which affect ω_{ab} in an analogous way; the conditions are (i) metricity: $\omega_{ab} = -\omega_{ba}$ (from $g_{\mu\nu;\alpha} = 0$), and (ii) no torsion: $T^a = 0$.

One can calculate the spin connections by observation, though there is a procedure for calculating the spin connections. The procedure is as follows: express (B.1.7) as

$$de^a = g^a{}_{bc} e^b \wedge e^c \tag{B.1.10}$$

where $g^a{}_{bc} = -g^a{}_{cb}$, and then one can show that the spin coefficients can be calculated via

$$\omega^a{}_{bc} = \frac{1}{2}\left[g^a{}_{bc} + g^b{}_{ca} - g^c{}_{ab}\right] \tag{B.1.11}$$

(note that for $a = c$,

$$\omega^a{}_{ba} = \frac{1}{2}\left[g^a{}_{ba} + g^b{}_{aa} - g^c{}_{ab}\right] = g^a{}_{ba} \tag{B.1.12}$$

Also note that $\omega^a{}_{bc}$ is anti-symmetric on the a, b indices, not the b, c indices). This will then give the spin connections (up to a sign) as

$$\omega^a{}_b = \omega^a{}_{bc} e^c \tag{B.1.13}$$

I say up to a sign as I have found that one must substitute (B.1.13) back into (B.1.7) to make sure that the sign of the calculated $\omega^a{}_b$ is correct.

As a concrete example (both to demonstrate the above and for my own education), I will now use the above method to calculate the curvature of four-dimensional Taub-NUT space, also done (slightly differently) in [57]. Again taking the metric in the form (3.2.1). Take the covariant basis vectors to be

$$\begin{aligned} e^{\hat{t}} &= F(r)\left[dt + 4n\sin^2\left(\frac{\theta}{2}\right)d\phi\right] && \text{(B.1.14)} \\ e^{\hat{r}} &= F^{-1}(r)dr && \text{(B.1.15)} \\ e^{\hat{\theta}} &= (r^2+n^2)^{1/2}d\theta && \text{(B.1.16)} \\ e^{\hat{\phi}} &= (r^2+n^2)^{1/2}\sin(\theta)d\phi && \text{(B.1.17)} \end{aligned}$$

This will give

$$\begin{aligned} de^{\hat{t}} &= F'dr \wedge \left[dt + 4nF\sin^2\left(\frac{\theta}{2}\right)d\phi\right] + 4n\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)d\theta\wedge d\phi \\ &= F'e^{\hat{r}}\wedge e^{\hat{t}} + \frac{2nF}{(r^2+n^2)}e^{\hat{\theta}}\wedge e^{\hat{\phi}} && \text{(B.1.18)} \end{aligned}$$

From (B.1.11) and (B.1.12) this gives

$$\begin{aligned} \omega^{\hat{t}}{}_{\hat{r}\hat{t}} &= F' \quad \rightarrow \quad \omega^{\hat{t}}{}_{\hat{r}} = F'e^{\hat{t}} && \text{(B.1.19)} \\ g^{\hat{t}}{}_{\hat{\theta}\hat{\phi}} &= \frac{2nF}{(r^2+n^2)} && \text{(B.1.20)} \end{aligned}$$

Similarly, the other three basis vectors can be differentiated to give

$$\begin{aligned} de^{\hat{r}} &= F'dr\wedge dr = 0 \\ de^{\hat{\theta}} &= \frac{rF}{(r^2+n^2)}e^{\hat{r}}\wedge e^{\hat{\theta}} \\ de^{\hat{\phi}} &= \frac{rF}{(r^2+n^2)}e^{\hat{r}}\wedge e^{\hat{\phi}} + \frac{\cos(\theta)}{(r^2+n^2)^{1/2}\sin(\theta)}e^{\hat{\theta}}\wedge e^{\hat{\phi}} \end{aligned}$$

(B.1.11), (B.1.12), (B.1.20) and the above relations then combine to give

$$\omega^{\hat{t}}{}_{\hat{r}\hat{t}} = F' \tag{B.1.21}$$

$$\omega^{\hat{t}}{}_{\hat{\theta}\hat{\phi}} = \frac{nF}{(r^2+n^2)} \tag{B.1.22}$$

$$\omega^{\hat{\theta}}{}_{\hat{\phi}\hat{t}} = -\frac{nF}{(r^2+n^2)} \tag{B.1.23}$$

$$\omega^{\hat{\phi}}{}_{\hat{t}\hat{\theta}} = \frac{nF}{(r^2+n^2)} \tag{B.1.24}$$

$$\omega^{\hat{\theta}}{}_{\hat{r}\hat{\theta}} = \frac{rF}{(r^2+n^2)} \tag{B.1.25}$$

$$\omega^{\hat{\phi}}{}_{\hat{r}\hat{\phi}} = \frac{rF}{(r^2+n^2)} \tag{B.1.26}$$

$$\omega^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}} = \frac{\cos(\theta)}{(r^2+n^2)^{1/2}\sin(\theta)} \tag{B.1.27}$$

Note, though, upon inserting (B.1.24) back into (B.1.7) and comparing with (B.1.19), one finds that the sign of (B.1.24) should be negative. Similarly, the sign of (B.1.23) should be positive. (Also note here, $\eta_{ab} = [-1,1,1,1]$, and so $\omega^{\hat{\phi}}{}_{\hat{t}\hat{\theta}} = +\omega^{\hat{t}}{}_{\hat{\phi}\hat{\theta}}$). With these corrections, the spin connections are:

$$\begin{aligned}
\omega^{\hat{t}}{}_{\hat{r}} &= F'e^{\hat{t}} \\
\omega^{\hat{t}}{}_{\hat{\theta}} &= \frac{nF}{(r^2+n^2)}e^{\hat{\phi}} \\
\omega^{\hat{t}}{}_{\hat{\phi}} &= -\frac{nF}{(r^2+n^2)}e^{\hat{\theta}} \\
\omega^{\hat{r}}{}_{\hat{\theta}} &= -\frac{rF}{(r^2+n^2)}e^{\hat{\theta}} \\
\omega^{\hat{r}}{}_{\hat{\phi}} &= -\frac{rF}{(r^2+n^2)}e^{\hat{\phi}} \\
\omega^{\hat{\theta}}{}_{\hat{\phi}} &= \frac{nF}{(r^2+n^2)}e^{\hat{t}} - \frac{\cos(\theta)}{(r^2+n^2)^{1/2}\sin(\theta)}e^{\hat{\phi}}
\end{aligned} \tag{B.1.28}$$

The components of the curvature two-form can then be calculated from

(B.1.8). For example,

$$\begin{aligned}
\mathcal{R}^{\hat{t}}{}_{\hat{r}} &= d\omega^{\hat{t}}{}_{\hat{r}} + \omega^{\hat{t}}{}_{c} \wedge \omega^{c}{}_{\hat{r}} \\
&= -2\left[\frac{(F^2)''}{4}\right] e^{\hat{t}} \wedge e^{\hat{r}} + 2\left[\frac{n(F^2)'}{2(r^2+n^2)} - \frac{rnF^2}{(r^2+n^2)^2}\right] e^{\hat{\theta}} \wedge e^{\hat{\phi}} \\
&= -2Ae^{\hat{t}} \wedge e^{\hat{r}} + 2De^{\hat{\theta}} \wedge e^{\hat{\phi}} \qquad \text{(B.1.29)}
\end{aligned}$$

Similarly, the other components are

$$\begin{aligned}
\mathcal{R}^{\hat{t}}{}_{\hat{\theta}} &= -Ce^{\hat{t}} \wedge e^{\hat{\theta}} + De^{\hat{r}} \wedge e^{\hat{\phi}} \qquad \text{(B.1.30)} \\
\mathcal{R}^{\hat{t}}{}_{\hat{\phi}} &= -De^{\hat{r}} \wedge e^{\hat{\theta}} - Ce^{\hat{t}} \wedge e^{\hat{\phi}} \qquad \text{(B.1.31)} \\
\mathcal{R}^{\hat{r}}{}_{\hat{\theta}} &= -Ce^{\hat{r}} \wedge e^{\hat{\theta}} + De^{\hat{t}} \wedge e^{\hat{\phi}} \qquad \text{(B.1.32)} \\
\mathcal{R}^{\hat{r}}{}_{\hat{\phi}} &= -Ce^{\hat{r}} \wedge e^{\hat{\phi}} - De^{\hat{t}} \wedge e^{\hat{\theta}} \qquad \text{(B.1.33)} \\
\mathcal{R}^{\hat{\theta}}{}_{\hat{\phi}} &= -2De^{\hat{t}} \wedge e^{\hat{r}} + 2Be^{\hat{\theta}} \wedge e^{\hat{\phi}} \qquad \text{(B.1.34)}
\end{aligned}$$

where (as in [57])

$$\begin{aligned}
A &= \frac{(F^2)''}{4} \qquad \text{(B.1.35)} \\
B &= \frac{1}{2}\left[\frac{1}{(r^2+n^2)} + \frac{4n^2F^2}{(r^2+n^2)^2} - \frac{F^2}{(r^2+n^2)}\right] \qquad \text{(B.1.36)} \\
C &= \frac{r(F^2)'}{2(r^2+n^2)} + \frac{n^2F^2}{(r^2+n^2)^2} \qquad \text{(B.1.37)} \\
D &= \frac{n(F^2)'}{2(r^2+n^2)} - \frac{rnF^2}{(r^2+n^2)^2} \qquad \text{(B.1.38)}
\end{aligned}$$

Now, the curvature 4-forms can be read off from (B.1.29) - (B.1.34) using (B.1.8):

$$\begin{aligned}
\mathcal{R}^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} &= -4A \;,\quad \mathcal{R}^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} = -2C \;,\quad \mathcal{R}^{\hat{t}}{}_{\hat{\phi}\hat{t}\hat{\phi}} = -2C \\
\mathcal{R}^{\hat{r}}{}_{\hat{\theta}\hat{r}\hat{\theta}} &= -2C \;,\quad \mathcal{R}^{\hat{r}}{}_{\hat{\phi}\hat{r}\hat{\phi}} = -2C \;,\quad \mathcal{R}^{\hat{\theta}}{}_{\hat{\phi}\hat{\theta}\hat{\phi}} = +4B \qquad \text{(B.1.39)} \\
\mathcal{R}^{\hat{t}}{}_{\hat{r}\hat{\theta}\hat{\phi}} &= 4D \;,\quad \mathcal{R}^{\hat{\theta}}{}_{\hat{\phi}\hat{t}\hat{r}} = -4D \\
\mathcal{R}^{\hat{t}}{}_{\hat{\theta}\hat{r}\hat{\phi}} &= 2D \;,\quad \mathcal{R}^{\hat{r}}{}_{\hat{\phi}\hat{t}\hat{\theta}} = -2D \\
\mathcal{R}^{\hat{t}}{}_{\hat{\phi}\hat{r}\hat{\theta}} &= -2D \;,\quad \mathcal{R}^{\hat{r}}{}_{\hat{\theta}\hat{t}\hat{\phi}} = 2D
\end{aligned}$$

(Note in (B.1.39) that the last three lines were read off directly from (B.1.29) - (B.1.34), but do show the correct (anti-)symmetries). It is sufficient to show zero curvature in the Taub-NUT metric to calculate the curvature 0-form, given by:

$$\begin{aligned} \mathcal{R} &= \mathcal{R}^{ab}{}_{ab} \\ &= 16\,[A + B - 2C] \end{aligned} \qquad \text{(B.1.40)}$$

and upon substituting in $F(r)$, this is indeed shown to be zero.

Appendix C

M-brane solution summary

Presented here, without calculation, are all of the M-brane solutions found in [28, 29]. I have divided them into M2 branes and M5 branes, and also by supersymmetric/non-supersymmetric, and have excluded the M2 $\mathrm{TN}_4 \otimes \mathrm{TN}_4$, TN_8 and the M5 TN_4 solutions already presented in chapter 5. Note that all of the solutions can be reduced down to ten dimensions (not done here).

All of these solutions can be found using the steps shown for the M2 and M5 branes presented in chapter 5.

C.1 M2 Branes

The general form of the metric for an M2 brane in eleven dimensional supergravity is

$$\begin{aligned} ds^2_{11} &= H(r_1, r_2)^{-2/3} \left(-dt^2 + dx_1^2 + dx_2^2\right) \\ &\quad + H(r_1, r_2)^{1/3} \left(ds_1^2(r_1) + ds_2^2(r_2)\right) \\ A_{tx_1x_2} &= \frac{1}{H(r_1, r_2)} dt \wedge dx_1 \wedge dx_2 \end{aligned} \tag{C.1.1}$$

where $ds_i^2(r_i)$ are four or higher dimensional metrics that depend on some radii r_i, and in my case can be some combination of flat space, TN_4, TB_4, TN_6, TB_6, TN_8, TB_8, and Eguchi-Hanson.

C.1.1 Supersymmetric Metrics

TN_4

The TN_4 metric was originally embedded into an M2 brane configuration by Cherkis and Hashimoto [27]. The embedded metrics are

$$
\begin{aligned}
ds_1^2 &= \tilde{f}_4(r)(dr^2 + r^2 d\Omega_2^2) + \frac{(4n)^2}{\tilde{f}_4(r)}\left(d\psi + \frac{1}{2}\cos(\theta)d\phi\right)^2 && \text{(C.1.2)}\\
ds_2^2 &= dy^2 + y^2 d\Omega_3^2 && \text{(C.1.3)}\\
\tilde{f}_4(r) &= 1 + \frac{2n}{r} && \text{(C.1.4)}
\end{aligned}
$$

The metric function can be found from solving the differential equation arising from the eleven dimensional supergravity equations of motion (5.1.4), (5.1.5). This differential equation is separable, giving two differential equations;

$$
\begin{aligned}
0 &= 2\frac{\partial R(r)}{\partial r} + r\frac{\partial^2 R(r)}{\partial r^2} - c^2(r+2n)R(r) && \text{(C.1.5)}\\
0 &= \frac{\partial^2 Y(y)}{\partial y^2} + \frac{3}{y}\frac{\partial Y(y)}{\partial y} + c^2 Y(y) && \text{(C.1.6)}
\end{aligned}
$$

Solving both of these gives the metric function for the M2 brane

$$
H_{TN_4}(y,r) = 1 + Q_{M2}\int_0^\infty dc \frac{(cy)^2 J_1(cy)}{4\pi^2 y^3} R_c(r) \qquad \text{(C.1.7)}
$$

where

$$
\begin{aligned}
R_c(r) &= \frac{\pi^2 c}{16}\Gamma(cn)\frac{\mathcal{W}_W(-cn, 1/2, 2cr)}{r}\\
&= \frac{\pi^2 c^2}{8}\Gamma(cn)e^{-cr}\mathcal{U}(1+cn, 2, 2cr) && \text{(C.1.8)}
\end{aligned}
$$

and $J_1(py)$ is the Bessel function that solves (C.1.6). Here, $\mathcal{W}_W$ is the Whittaker-Watson function, related to the Kummer U-function or confluent hypergeometric function $\mathcal{U}$.

Another solution, not discussed in [27], can be found by taking $c = \mathrm{i}c$, giving

$$
\tilde{H}_{TN_4}(y,r) = 1 + \frac{Q_{M2}}{16}\int_0^\infty dc\; c^4 e^{-\mathrm{i}cr}\mathcal{G}(1+\mathrm{i}cn, 2, 2\mathrm{i}cr)\frac{K_1(cy)}{y} \qquad \text{(C.1.9)}
$$

where $\mathcal{G}$ is a hypergeometric function, finite at $r = 0$ and undergoing damped oscillations, vanishing at $r = \infty$. The function $K_1(cy)$ is the modified Bessel function, diverging at $y = 0$ and vanishing at $y = \infty$.

4d Eguchi-Hanson

The four dimensional Eguchi-Hanson metric is another self-dual, asymptotically flat metric that, when embedded into eleven dimensions, will give a localized brane solution. The embedded metrics are thus

$$ds_1^2 = \frac{r^2}{4g(r)}\left(d\psi + \cos(\theta)d\phi\right)^2 + g(r)dr^2 + \frac{r^2}{4}d\Omega_2^2 \quad \text{(C.1.10)}$$

$$ds_2^2 = dy^2 + y^2 d\Omega_3^2 \quad \text{(C.1.11)}$$

$$g_4(r) = \left(1 - \frac{a^4}{r^4}\right)^{-1} \quad \text{(C.1.12)}$$

The metric function, again found from solving the differential equation arising from the eleven dimensional supergravity equations of motion was found to be

$$H_{EH_4}(y,r) = 1 + Q_{M2}\int_0^\infty dc\left(\frac{c^4}{8}\right)R_c(r)\frac{J_1(cy)}{y} \quad \text{(C.1.13)}$$

where in this case, $R_c(r)$ is a solution to the differential equation

$$0 = \frac{(r^4 - a^4)}{r^4}\frac{d^2R_c(r)}{dr^2} + \frac{(3r^4 + a^4)}{r^5}\frac{dR_c(r)}{dr} - c^2R_c(r) \quad \text{(C.1.14)}$$

and must be solved for numerically, and J_1 is again the solution to (C.1.6). For large r, the solution of this equation that vanishes at infinity is $K_1(cr)/r$. $J_1(py)$ is of course the Bessel function.

Another solution can be found by taking $c = \mathrm{i}c$ in (C.1.6), (C.1.14), giving

$$\tilde{H}_{EH_4}(y,r) = 1 + Q_{M2}\int_0^\infty dc\;\left(\frac{c^4}{8}\right)\tilde{R}_c(r)\frac{K_1(cy)}{y} \quad \text{(C.1.15)}$$

with $\tilde{R}_c(r)$ is a damped oscillating function.

Eguchi-Hanson $\otimes$ Eguchi-Hanson

Here, one embeds two Eguchi-Hanson metrics, so that

$$ds_i^2 = \frac{r_i^2}{4g(r_i)}\left(d\psi_i + \cos(\theta_i)d\phi_i\right)^2 + g(r_i)dr_i^2 + \frac{r_i^2}{4}d\Omega_{2(i)}^2 \quad \text{(C.1.16)}$$

$$g_4(r_i) = \left(1 - \frac{a_i^4}{r_i^4}\right)^{-1} \quad \text{(C.1.17)}$$

The metric function of the M-brane is given by

$$H_{(EH)^2}(r_1, r_2) = 1 + Q_{M2}\int_0^\infty dc\left(\frac{c^5}{8}\right)\mathcal{R}_{1c}(r_1)\mathcal{R}_{2c}(r_2) \quad \text{(C.1.18)}$$

The functions $\mathcal{R}_{ic}$ are solutions to the equation (C.1.14) with either $\pm c^2$ multiplying the last R_c in (C.1.14). Here, switching $c \to \mathrm{i}c$ simply switches $\mathcal{R}_{1c}$, $\mathcal{R}_{2c}$.

$\mathbf{TN}_4 \otimes \mathbf{EH}_4$

The final supersymmetric M2 brane configuration involves embedding a four dimensional Taub-NUT metric (C.1.2) (denote $\psi, r, \theta, \phi = \psi_1, r_1, \theta_1, \phi_1$) and the Eguchi-Hanson metric (C.1.16) (denote $\psi, r, \theta, \phi = \psi_2, r_2, \theta_2, \phi_2$). The first metric function of the M-brane is

$$H_A(r_1, r_2) = 1 + Q_{M2}\int_0^\infty dc\frac{c^5}{16}\Gamma(cn_1)e^{-cr_1}\mathcal{U}(1 + n_1c, 2, 2cr_1)\mathcal{R}_2(r_2) \quad \text{(C.1.19)}$$

found by solving (C.1.5) and (C.1.14) (again (C.1.14) is solved numerically). The second solution, taking $c \to \mathrm{i}c$, is given by

$$H_B(r_1, r_2) = 1 + Q_{M2}\int_0^\infty dc\frac{c^5}{8}e^{-\mathrm{i}cr_1}\mathcal{G}(1 + \mathrm{i}n_1c, 2, 2\mathrm{i}cr_1)\mathcal{R}_1(r_2) \quad \text{(C.1.20)}$$

C.1.2 Non-supersymmetric Metrics

Embedding TB_4 metric

The embedded metrics in this case are

$$
\begin{aligned}
ds_1^2 &= \frac{16n^2}{f(r)}\left[d\psi + \frac{1}{2}\cos(\theta)d\phi\right]^2 + f(r)dr^2 + r(r+2n)d\Omega_2^2 && \text{(C.1.21)} \\
ds_2^2 &= dy^2 + y^2 d\Omega_3^2 && \text{(C.1.22)} \\
f(r) &= \frac{2r(r+2n)}{(r-n)(2r+n)} && \text{(C.1.23)}
\end{aligned}
$$

The differential equation separates into (C.1.6) and

$$0 = (2r^2 - n^2 - rn)R'' + (4r - n)R' - 2rc^2(r+2n)R \qquad \text{(C.1.24)}$$

and (C.1.24) must be solved numerically. The metric function here is

$$H_{TB_4}(y,r) = 1 + Q_{M2}\int_0^\infty dc\ p(c)R_c(r)\frac{J_1(cy)}{y} \qquad \text{(C.1.25)}$$

Here, $p(c)$ cannot be solved for exactly. However, by dimensional analysis, $p(c) = p_0c^4$, with p_0 a constant that can be absorbed into Q_{M2}. The behaviour of $R_c(r)$ is qualitatively the same as the $R_c(r)$ in the Taub-NUT case (both diverge at the brane location, and vanish at infinity), the behaviour here is the same as the TN_4 case shown above.

The second solution, taking $c \to \mathrm{i}c$, is

$$\tilde{H}_{TB_4} = 1 + Q_{M2}\int_0^\infty dc\ \tilde{p}(c)\tilde{R}_c(r)\frac{K_1(cy)}{y} \qquad \text{(C.1.26)}$$

where $\tilde{R}_c(r)$ is a damped, oscillating function. Again through dimensional analysis, $\tilde{p}(c) = \tilde{p}_0c^4$.

Embedding TN_6 and TB_6

The Taub-NUT and Taub-Bolt metrics in six dimensions can be re-written so that they differ by a plus/minus sign. Thus, the embedded metrics here are

$$
\begin{aligned}
ds_1^2 = ds^2_{TN6\pm} &= g_{6\pm}(r)dr^2 + \frac{36n^2}{g_{6\pm}(r)}\left[d\Psi + \frac{1}{3}\cos(\theta_1)d\phi_1 + \frac{1}{3}\cos(\theta_2)d\phi_2\right]^2 \\
& \quad r(r\pm 2n)\left(d\theta_1^2 + \sin^2(\theta_1)d\phi_1^2 + d\theta_2^2 + \sin^2(\theta_2)d\phi_2^2\right) && \text{(C.1.27)} \\
ds_2^2 &= dy^2 + y^2 d\alpha^2 && \text{(C.1.28)} \\
g_{6\pm}(r) &= \frac{3(r\pm 2n)^2}{r(r\pm 4n)} && \text{(C.1.29)}
\end{aligned}
$$

and the coordinate r belongs to $[0,\infty)$ for the NUT solution and to $[4n,\infty)$ for the Bolt.

The $Y(y)$ function is $\sim J_0(qy)$, and the $R(r)$ differential equation is

$$
0 = (t^2 - 4n^2)R''_\pm(t) + 4(t\pm n)R'_\pm(t) - 3q^2t^2R_\pm(t) \qquad \text{(C.1.30)}
$$

after a coordinate change $t = r\pm 2n$. This must again be solved numerically. The most general solutions are then

$$
H_{TN6\pm}(y,r) = 1 + Q_{M2}\int_0^\infty dc\ s_\pm(c)R_{c\pm}(r)J_0(cy) \qquad \text{(C.1.31)}
$$

where $s_\pm(c) = s_{0\pm}c^5$. The second set of solutions, again found by taking $c\to \mathrm{i}c$, is

$$
\tilde{H}_{TN6\pm}(y,r) = 1 + Q_{M2}\int_0^\infty dc\ \tilde{s}_\pm(c)\tilde{R}_{c\pm}(r)K_0(cy) \qquad \text{(C.1.32)}
$$

Embedding $\mathbf{TN_8}$ and $\mathbf{TB_8}$

The embedded metrics here are

$$
\begin{aligned}
ds_1^2 = ds^2_{TN_8} &= \frac{64n^2}{f(r)}\left[d\Psi + \frac{1}{4}\cos(\theta_1)d\phi_1 + \frac{1}{4}\cos(\theta_2)d\phi_2 + \frac{1}{4}\cos(\theta_3)d\phi_3\right]^2 \\
& \quad f(r)dr^2 + r(r+2n)\left(d\Omega^2_{2(1)} + d\Omega^2_{2(2)} + d\Omega^2_{2(3)}\right) && \text{(C.1.33)} \\
d\Omega_{2(i)} &= d\theta_i^2 + \sin^2(\theta_i)d\phi_i^2 && \text{(C.1.34)} \\
ds_2^2 &= 0 && \text{(C.1.35)} \\
f_N(r) &= \frac{5(r+2n)^3}{r(r^2+6nr+10n^2)} && \text{(C.1.36)} \\
f_B(r) &= \frac{5r^3(r+2n)^3}{r^6+6nr^5+10n^2r^4-\frac{2997}{4}n^5(r+n)} && \text{(C.1.37)}
\end{aligned}
$$

where r belongs to $[0,\infty)$ for the NUT solution and to $[3n,\infty)$ for the Bolt.

The NUT case is done in section 5.4. The Bolt solution must satisfy the following differential equation

$$\begin{aligned} 0 &= \left[4r^6 + 24nr^5 + 40n^2r^4 - 2997n^5(r+n)\right] H'' \\ &+ (24r^5 + 120nr^4 + 160n^2r^3 - 2997n^5)H' \end{aligned} \tag{C.1.38}$$

which has the solution (found using the Maple software)

$$H = \int \frac{d\tilde{r}}{(16875n^5 + 13500\tilde{r}n^4 + 4800\tilde{r}^2n^3 + 940\tilde{r}^3n^2 + 96\tilde{r}^4n + 4\tilde{r}^5)\tilde{r}} \tag{C.1.39}$$

where $r \to \tilde{r} - 3n$ so that the Bolt is at $\tilde{r} = 0$.

C.2 M5 Branes

The general form of an M5 brane metric in eleven dimensions is given by

$$\begin{aligned} ds_{11}^2 &= H(y,r)^{-1/3}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2\right) \\ &+ H(y,r)^{2/3}\left(dy^2 + ds_4^2(r)\right) \end{aligned} \tag{C.2.1}$$

$$F_{m_1\ldots m_4} = \frac{\alpha}{2}\epsilon_{m_1\ldots m_5}\partial^{m_5}H \tag{C.2.2}$$

where $ds_4^2(r)$ is a four-dimensional (Euclideanized) metric in spherical coordinates, and $\alpha = \pm 1$ corresponds to the M5 brane being an M5/anti-M5 brane, respectively. The case of $ds_4^2(r)$ being the Taub-NUT metric is done in section 5.3.2. Below, I show (without proof) the solutions for embedding an Eguchi-Hanson metric and a Taub-Bolt metric.

C.2.1 Supersymmetric Metric - EH$_4$

The embedded metric here is the Eguchi-Hanson metric, given by

$$ds_4^2 = \frac{r^2}{4g(r)}\left(d\psi + \cos(\theta)d\phi\right)^2 + g(r)dr^2 + \frac{r^2}{4}d\Omega_2^2 \tag{C.2.3}$$

$$g_4(r) = \left(1 - \frac{a^4}{r^4}\right)^{-1} \tag{C.2.4}$$

The differential equations that must be solved, coming from the supergravity equations of motion which are separable, are

$$\begin{aligned} 0 &= \frac{d^2Y(y)}{dy^2} + c^2 Y(y) && \text{(C.2.5)} \\ 0 &= r(r^4-a^4)\frac{d^2R(r)}{dr^2} + (3r^4+a^4)\frac{dR(r)}{dr} - c^2 r^5 R(r) && \text{(C.2.6)} \end{aligned}$$

The general solution to the $Y(y)$ equation is given by

$$Y(y) = C_1\cos(cy) + C_2\sin(cy) \tag{C.2.7}$$

The solution for $R(r)$ does not have an explicit solution; the power series solution near $r=a$ is given by

$$\begin{aligned} R(r) = {} & \left(\tilde{C}_1 \ln\left(\frac{r}{a}-1\right)\right)\left[1 + \frac{c^2a^2}{4}\left(\frac{r}{a}-1\right) + \frac{c^2a^2}{8}\left(\frac{r}{a}-1\right)^2 \right. \\ & \left. + \frac{c^4a^4}{64}\left(\frac{r}{a}-1\right)^2 + \cdots\right] + \tilde{C}_2\left[-\frac{1+c^2a^2}{2}\left(\frac{r}{a}-1\right) \right. \\ & \left. -\frac{1+c^2a^2}{8}\left(\frac{r}{a}-1\right)^2 - \frac{3c^4a^4}{64}\left(\frac{r}{a}-1\right)^2 + \cdots\right] + \\ & + \mathcal{O}((r-a)^3) \end{aligned} \tag{C.2.8}$$

and the solution of interest logarithmically diverges at $r=a$.

The most general solution for the metric function is then

$$H_{EH}(y,r) = 1 + \frac{Q_{M5}}{\pi}\int_0^\infty dc\left(2c^2\cos(cy)R_c(y)\right) \tag{C.2.9}$$

By taking $c \to \mathrm{i}\tilde{c}$, a second solution is

$$\tilde{H}_{EH}(y,r) = 1 + Q_{M5}\int_0^\infty d\tilde{c}\ \tilde{f}(\tilde{c})e^{-\tilde{c}y}\mathcal{R}_{\tilde{c}}(r) \tag{C.2.10}$$

where $\mathcal{R}_{\tilde{c}}(r)$ is the solution of (C.2.6) with $c \to \mathrm{i}\tilde{c}$.

C.2.2 Non-supersymmetric Metric - TB_4

The embedded metric here is

$$\begin{aligned} ds_4^2 &= \frac{(4n)^2}{f(r)}\left[d\psi + \frac{1}{2}\cos(\theta)d\phi\right]^2 + f(r)dr^2 + r(r+2n)d\Omega_2^2 && \text{(C.2.11)} \\ f(r) &= \frac{2r(r+2n)}{(r-n)(2r+n)} && \text{(C.2.12)} \end{aligned}$$

The separated differential equations are

$$
\begin{aligned}
0 &= \frac{d^2Y(y)}{dy^2} + c^2Y(y) &\text{(C.2.13)}\\
0 &= (r-n)(2r+n)\frac{d^2R(r)}{dr^2} + (4r-n)\frac{dR(r)}{dr}\\
&\quad -2c^2r(r+2n)R(r) &\text{(C.2.14)}
\end{aligned}
$$

The $Y(y)$ equation has the same solution (C.2.7). The $R(r)$ equation doesn't have an explicit solution - a power series solution near the origin yields two solutions, one with a logarithmic divergence at $r = n$. The metric function has the general solution

$$H_{TB_4}(y,r) = 1 + Q_{M5}\int_0^\infty dc\; p(c)\cos(cy)R_c(r) \tag{C.2.15}$$

where $p(c) = p_0c^2$ by dimensional analysis. The second solution is

$$\tilde{H}_{TB_4}(y,r) = 1 + Q_{M5}\int_0^\infty d\tilde{c}\; \tilde{p}(\tilde{c})e^{-\tilde{c}y}R_{\tilde{c}}(r) \tag{C.2.16}$$

where again $\tilde{p}(\tilde{c}) = \tilde{p}_0\tilde{c}^2$.

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