SUBMODULAR FUNCTIONS, GRAPHS AND INTEGER POLYHEDRA

by

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SUBMODULAR FUNCTIONS, GRAPHS AND INTEGER POLYHEDRA

Abstract

This thesis is a study of the faces of certain combinatorially-defined polyhedra. In particular, we examine the vertices and facets of these polyhedra.

Chapter 2 contains the essential mathematical background in polyhedral theory, linear programming and graph theory. We also discuss the existence of an integer-valued optimum solution to a linear program. This is essential for determining that the vertices of certain polyhedra are integer-valued, and for establishing related combinatorial min-max relations.

Chapter 4 deals with polymatroids. We discuss the relationship between (integral) polymatroids contained in $\mathbb{R}_+^E$ and (integer-valued) nonnegative, nondecreasing, submodular functions, called β_0-functions, of L_E, the family of all subsets of E. We prove that the vertices of the intersection of two integral polymatroids are integer-valued. We also characterize the facets of the intersection of two polymatroids in terms of the two β_0-functions defining these polymatroids. For any polymatroid $P \subseteq \mathbb{R}_+^E$ with corresponding β_0-function $f_P : L_E \to \mathbb{R}$ and for all nonempty $T \subseteq E$ the sets $\{x \in P : x(T) = f_P(T)\}$ and $\{x \in P : x(E) = f_P(E), x(T) = f_P(T)\}$ are faces of P. We determine the dimension of these faces in terms of f_P.

Chapter 3 is the application of the results of Chapter 4 to the polymatroid aspects of matroid theory. We characterize the vertices

and the facets of the intersection of two matroid polyhedra. We use the characterization of the facets of this intersection to derive a graph theoretic description of the facets of the polyhedron associated with branchings in a graph.

In Chapter 5 we discuss certain polyhedra which can be associated with strong k-covers and strong k-matchings of an acyclic graph. By proving the existence of integer-valued optimum solutions to particular primal-dual pairs of linear programs we are able to demonstrate certain combinatorial min-max relations.

Chapter 6 is a unification of the polyhedra described in Chapters 4 and 5. We give a combinatorial definition of a class of polyhedra which includes polymatroid intersection and the polyhedra associated with strong k-covers and strong k-matchings of an acyclic graph. We establish the existence of integer-valued optimum solutions to certain dual linear programs and thereby draw conclusions concerning the integrality of the vertices of particular polyhedra within this class. The applications include establishing the integrality of the vertices of the intersection of two integral polymatroids and the integrality of the vertices of strong k-cover and strong k-matching polyhedra.

Chapter 7 is a discussion of the facets of polyhedra defined in Chapter 6 and we obtain a description of the facets of a subclass of these polyhedra which includes a description of the facets of the intersection of two polymatroids and the facets of strong k-cover and strong k-matching polyhedra.

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CHAPTER 1

INTRODUCTION

1.1.1 In this thesis we study polyhedra which can be associated with certain combinatorial structures. In particular, we study the faces of these polyhedra.

1.1.2 The vertices of a polyhedron P are its faces of dimension zero. An important property of vertices of a polyhedron P is that if P contains a vertex and the linear objective function $c \cdot x$ has a maximum over $x \in P$ then that maximum is realized by a vertex of P. Therefore

1.1.3 If P contains a vertex then the maximum of $c \cdot x$ over $x \in P$ is equal to the maximum of $c \cdot x$ over the set of vertices of P. Also,

1.1.4 $x^0 \in P$ is a vertex of P if and only if there exists a linear objective function $c^0 \cdot x$ such that x^0 is the unique member of P maximizing $c^0 \cdot x$.

1.1.5 Frequently we will be determining that the vertices of P are integer-valued. Every polyhedron can be defined as the solution set of a linear system $Ax \leq b$, for some matrix A and some vector b. We denote the solution set of any linear system $Ax \leq b$ by $P\langle A,b\rangle$. Given a linear system $Ax \leq b$ and a linear objective function $c \cdot x$ consider the linear program

1.1.6 maximize $c \cdot x$ where x satisfies $Ax \leq b$.

If the vertices of $P\langle A,b\rangle$ are integer-valued then, by (1.1.3), the linear program (1.1.6) has an integer-valued optimum solution for all linear objective functions $c \cdot x$ such that (1.1.6) has an optimum solution. Conversely, suppose (1.1.6) has an integer-valued optimum solution for all linear objective functions $c \cdot x$ such that (1.1.6) has an optimum solution. Suppose x^0 is a vertex of P<A,b>. By (1.1.4), there exists a linear objective function $c^0 \cdot x$ such that x^0 is the unique member of P<A,b> maximizing $c^0 \cdot x$. Therefore, x^0 is integer-valued. We will often prove that the vertices of P<A,b> are integer-valued by showing that (1.1.6) has an integer-valued optimum solution for all linear objective functions $c \cdot x$ such that (1.1.6) has an optimum solution.

1.1.7 A useful sufficient condition for the vertices of P<A,b> to be integer-valued is that b is integer-valued and the dual linear program of (1.1.6) has an integer-valued optimum solution for every integer-valued c such that (1.1.6) has an optimum solution. We make frequent use of this condition to prove that the vertices of P<A,b> are integer-valued.

1.1.8 If the vertices of P<A,b> are integer-valued and the dual linear program of (1.1.6) has an integer-valued optimum solution for some linear objective function $c.x$ then, by the Strong L.P. Duality Theorem, the maximum of $c \cdot x$ subject to $Ax \le b$, x integer-valued is equal to the optimum value of the dual linear program subject to integer-valued dual feasible solutions. We establish certain combinatorial minimax theorems by proving that the vertices of a certain polyhedron are integer-valued and that a certain dual linear program has an integer-valued optimum solution.

1.1.9 Chapter 6 presents a class of combinatorially-defined polyhedra. Among the applications of these polyhedra are results about polymatroids and polyhedra associated with strong k-covers and strong k-matchings.

1.1.10 Chapter 3 motivates the discussion of polymatroids in Chapter 4 by outlining the polymatroid aspects of matroid theory. The major reference for these chapters is the work of Edmonds [E3]. Indeed, the inspiration for most of the research presented here has its roots in that paper. We prove many of the results given in [E3].

1.1.11 An <u>independence system</u> $M = (E, \mathfrak{F})$ is a set E together with nonempty family $\mathfrak{F}$ of <u>independent subsets</u> of E such that if $Y \subseteq Z \in \mathfrak{F}$ then $Y \in \mathfrak{F}$. For any set $S \subseteq E$ we can define the <u>rank</u> of S, $r(S)$, in M to be the maximum cardinality of an independent subset of S, and a <u>basis</u> of S is an independent subset of S having cardinality $r(S)$. A <u>matroid</u> is an independence system such that for all $S \subseteq E$ every maximal independent subset of S is a basis of S.

1.1.12 A <u>polyideal</u> is a compact subset P of $\mathbb{R}_+^E$ such that if $x' \in P$ and $0 \leq x^0 \leq x^1$ then $x^0 \in P$. For any vector $a \in \mathbb{R}_+^E$ we define the <u>rank</u> of a, $r(a)$, in P to be the maximum of $x(E) \equiv \Sigma(x_e : e \in E)$ over $x \in P$, $x \leq a$, and a <u>P-basis</u> of a to be a vector x of P, $x \leq a$, having $x(E) = r(a)$. A <u>polymatroid</u> is a polyideal such that for all $a \in \mathbb{R}_+^E$ every maximal vector $x \in P$ such that $x \leq a$ is a P-basis of a.

1.1.13 Let L_E be the family of subsets of E. Let $K_E \equiv L_E - \{\phi\}$. For any function $f: K_E \to \mathbb{R}$ let $P(K_E, f)$ be the solution set of the linear system.

1.1.14 $x_e \geq 0$ for all $e \in E$

$x(S) \leq f(S)$ for all $S \in K_E$,

where $x(S) \equiv \Sigma(x_e : e \in S)$.

1.1.15 The rank function $r: L_E \to \mathbb{R}$ of a matroid is nondecreasing, $r(\phi) = 0$ and $r(Y \cap Z) + r(Y \cup Z) \leq r(Y) + r(Z)$ for all $Y, Z \subseteq E$; i.e. r is <u>submodular</u> on L_E. Call any function $f: L_E \to \mathbb{R}$ which is nondecreasing, submodular and such that $f(\phi) = 0$ a β_0-function. In Chapter 4 we prove that if $f: L_E \to \mathbb{R}$ is a β_0-function then $P(K_E, f)$ is a polymatroid and if f is also integer-valued then $P(K_E, f)$ is also an <u>integral</u> polymatroid; i.e. for all integer-valued $a \in \mathbb{R}_+^E$ every maximal integer-valued vector $x \in P(K_E, f)$ such that $x \leq a$ is a P-basis of a. Conversely, if $P \subseteq \mathbb{R}_+^E$ is an (integral) polymatroid then there is an associated (integer-valued) β_0-function $f_P: L_E \to \mathbb{R}$ such that $P = P(K_E, f_P)$. This correspondence between β_0-functions and polymatroids is fundamental to the study of polymatroids (cf. [E3]).

1.1.16 In Chapter 3 we derive a construction of matroids from the construction of integral polymatroids from integer-valued β_0-functions and demonstrate that particular independence systems are matroids. For example, if for any indexed family $\{Q_j : j \in E'\}$ on E we let $\mathcal{F}$ be the family of subsets $J \subseteq E'$ such that $\{Q_j : j \in J\}$ has a transversal then $M = (E, \mathcal{F})$ is a matroid (cf. [E3]).

1.1.17 An important property of integer-valued β_0-functions f is that the vertices of $P(K_E,f)$ are integer-valued. In particular, where f is the rank function of a matroid $M = (E,\mathcal{F})$, the vertices of $P(K_E,f)$ are (0,1)-valued; i.e. the vertices of $P(K_E,f)$ are the vectors of the independent sets of M. Therefore, a linear system defining P(M), the convex hull of vectors of independent sets of M, is (1.1.14).

1.1.18 Let $f_1,f_2 : L_E \to \mathbb{R}$ be two β_0-functions. For any linear objective function $c\cdot x$ consider the linear program

1.1.19 maximize $c\cdot x$ where $x \in \mathbb{R}^E$ satisfies

$$x_e \geq 0 \text{ for all } e \in E$$

$$\left.\begin{array}{l} x(S) \leq f_1(S) \\ x(S) \leq f_2(S) \end{array}\right. \quad \text{for all } S \in K_E.$$

We prove that if c is integer-valued then the dual linear program of (1.1.19) has an integer-valued optimum solution. If f_1 and f_2 are integer-valued then, by (1.1.7), the vertices of $P(K_E,f_1) \cap P(K_E,f_2)$ are integer-valued (cf. [E3]). In Chapter 3 we discuss the case when f_1 and f_2 are the rank functions of matroids $M_1 = (E,\mathcal{F}_1)$ and $M_2 = (E,\mathcal{F}_2)$ respectively on E. Here the vertices of $P(K_E,f_1) \cap P(K_E,f_2)$ correspond to subsets of E which are independent in M_1 and in M_2.

1.1.20 Let $G = (V,E)$ be an acyclic connected graph. A <u>directed coboundary</u> of G is a nonempty subset of edges $D = \delta(S)$ for some $S \subseteq V$ such that $\delta(\bar{S}) = \phi$.

For any positive integer k let $P_k(G)$ be the solution set of the linear system

1.1.21 $\quad 0 \leq x_e \leq 1$ for all $e \in E$

$x(D) \geq k$ for all directed coboundaries D of G.

and let $P^k(G)$ be the solution set of the linear system

1.1.22 $\quad 0 \leq x_e \leq 1$ for all $e \in E$

$x(D) \leq k$ for all directed coboundaries D of G.

A main result is that the vertices of each of these polyhedra are integer-valued; i.e. they are (0,1)-valued. The vertices of $P_k(G)$ correspond to <u>strong k-covers</u>; i.e. subsets of edges which meet every directed coboundary of G at least k times. The vertices of $P^k(G)$ correspond to <u>strong k-matchings</u>, i.e. subsets of edges which contain at most k edges of each directed coboundary.

1.1.23 We prove that for any integer-valued c the dual linear programs of

minimize $c \cdot x$ where $x \in \mathbb{R}^E$ satisfies (1.1.21)

and

maximize $c \cdot x$ where $x \in \mathbb{R}^E$ satisfies (1.1.22)

each have integer-valued optimum solutions whenever an optimum solution exists. From this and the Strong L.P. Duality Theorem we are able to deduce certain combinatorial minmax relations, including theorems of Lucchesi and Younger [L2] on strong 1-covers and of Vidyasankar and Younger [V1] on strong 1-matchings.

1.1.24 Chapter 6 is a unification of polymatroids, the strong k-cover polyhedron $P_k(G)$ and the strong k-matching polyhedron $P^k(G)$. Given a graph $G = (V,E)$, let F be a family of subsets of V such that if $Y,Z \in F$, $Y \cap Z \neq \phi$ and $Y \cup Z \neq V$ then $Y \cap Z$ and $Y \cup Z \in F$. Let $f:F \to \mathbb{R}$ be a submodular function of F. Let $K,L \subseteq E$, $a \in \mathbb{R}^K$ and $d \in \mathbb{R}^L$. Finally, let P be set of solutions to the linear system

1.1.25 $$\begin{cases} x_e \leq a_e \text{ for all } e \in K \\ x_e \geq d_e \text{ for all } e \in L \\ x(\delta(S))-x(\delta(\bar{S})) \leq f(S) \text{ for all } S \in F. \end{cases}$$

In section 6.2 we prove that for all integer-valued c the dual linear program of

$$\text{maximize } c \cdot x \text{ where } x \text{ satisfies (1.1.25)}$$

has an integer-valued optimum solution whenever an optimum solution exists. If f is integer-valued, a_e is an integer for all $e \in K$, d_e is an integer for all $e \in L$ and P has a vertex then, by (1.1.7), the vertices of P are integer-valued. By making appropriate choices of the graph G, the family F, the function f, the subsets $K, L \subseteq E$ and the vectors $a \in \mathbb{R}^K$, $d \in \mathbb{R}^L$ we are able to deduce from this many of the key results of Chapters 3, 4 and 5. For example, we can show that the vertices of $P(K_E,f_1) \cap P(K_E,f_2)$ are integer-valued for any two integer-valued β_0-functions and that for any acyclic graph G the vertices of $P_k(G)$ and $P^k(G)$ are all integer-valued.

1.1.26 Another important aspect of polyhedral theory is the study of the facets of a polyhedron P. The facets of P are the faces of P which have dimension one less than the dimension of P. In the case that P is of "full dimension" the facets of P correspond to the unique (up to multiplication by a positive constant) minimal set of linear inequalities required to define P. Hence, characterizing the facets of a full dimensional polyhedron P is equivalent to characterizing the unique minimal linear system defining P.

1.1.27 $P(K_E,f)$ is defined by the linear system (1.1.14). For each $T \in K_E$,

$$P_T \equiv \{x \in P(K_E,f):x(T) = f(T)\}$$

is a face of $P(K_E,f)$. In section 4.4 we determine the dimension of P_T in terms of f, whenever f is a β_0-function for L_E. T is said to be <u>f-nonseparable</u> if there is no subset $S \in K_T-\{T\}$ such $f(T) = f(S)+f(T-S)$. There is a unique minimal partition $\{S:S \in F\}$ of T into nonempty subsets such that $f(T) = \Sigma(f(S):S \in F)$. Let $\mu_f \equiv |F|$. Also, there is a unique maximal set $S \subseteq E$ called the <u>closure</u> of T, cl(T), such that $T \subseteq S$ and $f(T) = f(S)$. We show that

1.1.28 $\dim(P_T) = |E|+|T|-|\mathrm{cl}(T)|-\mu_f(T)$.

In particular, when $P(K_E,f)$ is of full dimension, P_T is a facet of $P(K_E,f)$ if and only if $\dim(P_T) = |E|-1$; i.e. if and only $T = \mathrm{cl}(T)$ and $\mu_f(T) = 1$ (T is <u>f-closed</u> and f-nonseparable). Therefore, we have a description of the unique minimal subsystem of (1.1.14) defining $P(K_E,f)$; namely

$x_e \geq 0$ for all $e \in E$

$x(T) \leq f(T)$ for all $T \in K_E$ such that T is f-closed and f-nonseparable.

1.1.29 In section 3.4 we discuss $\dim(P_T)$ where f is the rank function of a matroid. We apply the formula (1.1.28) to the forest matroid of a graph $G = (V,E)$ (the independent sets are the edge sets of forests of G) to obtain a graph theoretic description of $\dim(P_T)$ for $T \in K_E$.

1.1.30 For each $T \in K_E$,

$$H_T \equiv \{x \in P(K\ ,f): x(E) = f(E), x(T) = f(T)\}$$

is a face of P_E and of $P(K_E,f)$. When f is a β_0-function we determine $\dim(H_T)$ in terms of f and therefore we can determine the facets of P_E. Section 3.4 also considers $\dim(H_T)$ where f is the rank function of a matroid M and we obtain a graph theoretic description of $\dim(H_T)$ for $T \in K_E$ when M is the forest matroid of a graph.

1.1.31 Consider the polyhedron $P \equiv P(K_E,f_1) \cap P(K_E,f_2)$ for two β_0-functions $f_1,f_2: L_E \to \mathbb{R}$. For each $S \subseteq E$ we can let $f(S) \equiv \max\{x(S): x \in P\}$. Clearly $P = P(K_E,f)$ and P is defined by the linear system (1.1.14). $T \in K_E$ is f-closed if for all $S \subseteq E$ such that $T \subset S$ we have $f(S) > f(T)$. In section 4.5 we prove that if P is of full dimension then for all $T \in K_T$, P_T is a facet of P if and only if T is f-closed and f-nonseparable. This generalizes the characterization of facets of $P(K_E,f)$ for a β_0-function f. In section 3.5 we consider the intersection of two matroids. As an application of the theory we describe the facets of $P(M)$, where $M = (E,\mathfrak{I})$ and $\mathfrak{I}$ is the family of matchings in a bipartite graph.

1.1.32 In section 3.6 we examine branchings in a graph. Given a graph $G = (V,E)$ for any $S \subseteq V$ let $\delta(S)$ denote the edges $e \in E$ with $t(e) \in S$, $h(e) \notin S$. A branching of G is a forest B of G such that for all $v \in V$, $|\delta(\bar{v})| \leq 1$. If we let $\mathfrak{I}$ be the family of edge sets of branchings then $M = (E,\mathfrak{I})$ is an independence system. Moreover, M is the intersection of two matroids. Edmonds [E1] introduced branchings and he demonstrated that a linear system defining $P(M)$ is

1.1.33 $x_e \geq 0$ for all $e \in E$

$x(\delta(\bar{v})) \leq 1$ for all $v \in V$

$x(\gamma(S)) \leq |S|-1$ for all $S \in K_V$,

where $\gamma(S)$ is the set of edges having both ends in S. Let $r:L_E \to \mathbb{R}$ be the rank function of M. We characterize the r-closed and r-nonseparable subsets of E, thereby determining the facets of $P(M)$ and the unique minimal subsystem of (1.1.33) which defines $P(M)$.

1.1.34 We also discuss the facets of the polyhedra $P_k(G)$ and $P^k(G)$ associated with an acyclic connected graph $G = (V,E)$. Chapter 7 represents a partial unification of the characterization of facets of $P(K_E,f_1) \cap P(K_E,f_2)$, where $f_1,f_2:L_E \to \mathbb{R}$ are two β_0-functions and the characterizations of the facets of $P_k(G)$ and $P^k(G)$. We are able to describe the facets of P, the solution set of (1.1.25), when we make certain restrictions on the combinatorial description of P. However, the class of these polyhedra for which we are able to describe the facets does include $P(K_E,f_1) \cap P(K_E,f_2)$, $P_k(G)$ and $P^k(G)$.

1.1.35 Chapter 2 is a presentation of the required basic background. A standard reference on polyhedra is Stoer and Witzgall [S1]. Most useful for our studies has been the exposition on polyhedra contained in Pulleyblank [P1]. Chapter 2 also contains a review of linear programming. For a thorough treatment see Dantzig [D1].

CHAPTER 2

FOUNDATIONS

2.1 Notation and Set Theory

2.1.1 We use the symbol "$\equiv$" to denote "is defined to be"; "$=$" to denote "is equal to"; "$\subseteq$" to denote "is a subset of "; and "$\subset$" to denote "is a proper subset of". The empty set is denoted by "ϕ".

2.1.2 The following definitions are all with respect to a fixed set X. If $Y \subseteq X$ then $\bar{Y}$ denotes the set X-Y. Subsets Y, Z of X are said to <u>meet</u> if $Y \cap Z \neq \phi$ and to <u>cross</u> if $Y \cap Z \neq \phi$, $Y \nsubseteq Z$, $Z \nsubseteq Y$ and $Y \cup Z \neq X$.

2.1.3 By a <u>family</u> on set X we mean a set of distinct subsets of X. L_X will denote the family of all subsets of X and K_X the family of all non-empty subsets of X. Where F is a family on X, F is a <u>nested</u> family means that for all $Y,Z \in F$ we have $Y \cap Z = \phi$, $Y \subseteq Z$ or $Z \subseteq Y$; F is a <u>crossing</u> family means that for any $Y,Z \in F$ which cross we have $Y \cap Z \in F$ and $Y \cup Z \in F$; F is a <u>cross-free</u> family means that no two elements of F cross.

2.1.4 We say that a family F on X is a <u>partition</u> of X if $X = \cup(S:S \in F)$ and $Y \cap Z = \phi$ for all distinct $Y,Z \in F$. If F is a partition of X, $S \neq \phi$ for all $S \in F$ and $|F| \geq 2$ then we call F a <u>nontrivial</u> partition of X.

2.1.5 If an expression involves a set consisting of a single element x then we will usually omit the parentheses enclosing x.

2.1.6 "$\square$" denotes the end or absence of a proof.

2.2 Linear Algebra

2.2.1 Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{Z}$ the set of integers. For $x \in \mathbb{R}$ we define the <u>floor</u> of x, $\lfloor x \rfloor$, to be $\max\{y \in \mathbb{Z}:y \leq x\}$ and the <u>ceiling</u> of x, $\lceil x \rceil$, to be $\min\{y \in \mathbb{Z}:x \leq y\}$.

2.2.2 For a finite set E let $\mathbb{R}^E$ denote the set of vectors $\{[x_e:e \in E]:x_e \in \mathbb{R}$ for all $e \in E\}$. For a vector $x = [x_e:e \in E] \in \mathbb{R}^E$ and for $e \in E$, x_e is called a <u>component</u> of x. The vector of $\mathbb{R}^E$ which is k in every component for some $k \in \mathbb{R}$ is also denoted by k, whenever there is no ambiguity. Let $\mathbb{Z}^E$ denote the set of vectors $\{[x_e:e \in E]:x_e \in \mathbb{Z}$ for all $e \in E\}$. If $x \in \mathbb{Z}^E$ then we say that x is <u>integer-valued</u>.

2.2.3 For $x,y \in \mathbb{R}^E$ we write $x \leq y$ if $x_e \leq y_e$ for all $e \in E$ and $x < y$ if $x_e < y_e$ for all $e \in E$. Clearly $\mathbb{R}^E$ is partially ordered by $\leq$ and we will use the term <u>maximal</u> vector with respect to this partial order. Let

$$\mathbb{R}_+^E \equiv \{x \in \mathbb{R}^E:0 \leq x\}$$

and

$$\mathbb{Z}_+^E \equiv \{x \in \mathbb{Z}^E:0 \leq x\}.$$

2.2.4 A set $X \subseteq \mathbb{R}^E$ is <u>linearly dependent</u> if there exists a set $\{\lambda_x:x \in X\}$ of real numbers such that $\lambda_x \neq 0$ for some $x \in X$ and $\Sigma(\lambda_x x:x \in X) = 0$. Otherwise, X is <u>linearly independent</u>. A <u>basis</u> of X is a maximal linearly independent subset of X. The following theorem is fundamental to the study of linear algebra.

2.2.5 <u>Theorem</u> (see Birkoff & MacLane [B2])

For all $X \subseteq \mathbb{R}^E$,

(i) All bases of X have the same cardinality called the <u>rank</u> of X, rank (X);

(ii) rank $(X) \leq |E|$. □

2.2.6 Let D, E be finite sets. If $A \in \mathbb{R}^{D\times E}$ is the matrix $[a_{de} \in \mathbb{R}: d \in D, e \in E]$ then for all $S \subseteq D$ let A_S denote the matrix $[a_{de}: d \in S, e \in E]$. Similarly, if $b = [b_d: d \in D] \in \mathbb{R}^D$ and $S \subseteq D$ then we denote $[b_d: d \in S] \in \mathbb{R}^S$ by b_S. Call $\{A_d: d \in D\}$ the <u>rows</u> of A and $\{[a_{de}: d \in D]: e \in E\}$ the <u>columns</u> of A. Let $I_D \in \mathbb{R}^{D\times D}$ denote the matrix $[a_{de}: d,e \in D]$ where for all $d,e \in D$,

$$a_{de} \equiv \begin{cases} 0 \text{ if } d \neq e \\ 1 \text{ if } d = e. \end{cases}$$

2.2.7 Where E and E' are disjoint sets and $A = [a_{de}: d \in D, e \in E] \in \mathbb{R}^{D\times E}$ and $B = [b_{de}: d \in D, e \in E'] \in \mathbb{R}^{D\times E'}$, $[A,B]$ denotes the matrix $[c_{de}: d \in D, e \in E \cup E'] \in \mathbb{R}^{D\times(E\cup E')}$ where

$$c_{de} \equiv \begin{cases} a_{de} \text{ if } e \in E \\ b_{de} \text{ if } e \in E'. \end{cases}$$

Similarly, if $a = [a_e: e \in E] \in \mathbb{R}^E$ and $b = [b_e: e \in E'] \in \mathbb{R}^{E'}$ then $[a,b]$ denotes the vector $[c_e: e \in E \cup E'] \in \mathbb{R}^{E\cup E'}$ where

$$c_e \equiv \begin{cases} a_e \text{ if } e \in E \\ b_e \text{ if } e \in E'. \end{cases}$$

2.2.8 If $x,y \in \mathbb{R}^E$ then we let $x \cdot y$ denote $\Sigma(x_e y_e: e \in E)$. If $x = [x_e: e \in E] \in \mathbb{R}^E$ and $A = [a_{de}: d \in D, e \in E] \in \mathbb{R}^{D\times E}$ then we define the product Ax to be the vector $y = [y_d: d \in D] \in \mathbb{R}^D$ where $y_d = A_d \cdot x$ for all $d \in D$.

2.2.9 For $A \in \mathbb{R}^{D\times E}$ we define the transpose of A to be the matrix $A^T \equiv [a^T_{ed}: e \in E, d \in D] \in \mathbb{R}^{E\times D}$ where $a^T_{ed} \equiv a_{de}$ for all $d \in D$, $e \in E$. Let the rank of matrix A, rank (A), be the rank of $\{A_d : d \in D\}$.

2.2.10 Theorem (see Birkhoff & MacLane [B2])

For all $A \in \mathbb{R}^{D\times E}$, $\mathrm{rank}(A) = \mathrm{rank}(A^T)$. □

2.3 Linear Inequalities and Polyhedra

2.3.1 An excellent development of the theory of linear inequalities and polyhedra which we will require is in the thesis of Pulleyblank [P1] and proofs of most of the following results can be found there.

2.3.2 Let D,E be finite sets, $A \in \mathbb{R}^{D\times E}$ and $b \in \mathbb{R}^D$. A linear system is the set of inequalities $Ax \le b$ and a polyhedron is the solution set of some linear system. For a given linear system $Ax \le b$ let $P\langle A,b\rangle \equiv \{x \in \mathbb{R}^E : Ax \le b\}$. For the remainder of the chapter we will assume that A,D,E and b are as above.

2.3.3 The following theorem is of fundamental importance to linear inequality theory and is one of several equivalent forms of Farkas' Lemma (cf. Stoer and Witzgall [S1], section 1.4).

2.3.4 Theorem Let $H \subseteq D$ and $K \subseteq E$. Exactly one of the following holds. Either

2.3.5 There exists $x \in \mathbb{R}^E$ such that $x_K \ge 0$, $A_H x \le b_H$ and $A_{\bar{H}} x = b_{\bar{H}}$

or

2.3.6 There exists $y \in \mathbb{R}^D$ such that $y_H \ge 0$, $A^T_K y \ge 0$, $A^T_{\bar{K}} y = 0$ and $b \cdot y < 0$.

□

2.3.7 For all $S \subseteq D$ let $q(S) \equiv \{x \in P<A,b>: A_S x = b_S\}$. Clearly, $q(S)$ is a polyhedron and is called a <u>face</u> of $P<A,b>$. $q(S)$ is a proper face of $P<A,b>$ if $q(S) \subset P<A,b>$.

2.3.8 <u>Theorem</u> (see [P1](2.1.5))

$H \subseteq P<A,b>$ is a face of $P<A,b>$ if and only if there exists $c \in \mathbb{R}^E$ and $\alpha \in \mathbb{R}$ such that $c \cdot x = \alpha$ for all $x \in H$ and $c \cdot x < \alpha$ for all $x \in P<A,b>-H$.

□

2.3.9 It follows from (2.3.8) that the faces of $P<A,b>$ depend only on the polyhedron and not on the defining linear system.

2.3.10 We will assume that the zero vector is not a row or column of A. For any linear system $Ax \le b$ there is a unique maximal set $D^0 \subseteq D$ such that $P<A,b> = q(D^0)$. The subsystem $A_{X^0}x \le b_{D^0}$ is called the <u>equality system</u> of $Ax \le b$. If $P<A,b> \ne \phi$ then we define the <u>dimension</u> of $P = P<A,b>$ to be $|E|-\mathrm{rank}(A_{D^0})$ and denote this by $\dim(P)$. If $P = \phi$ then we define $\dim(P) \equiv -1$.

2.3.11 If $\dim(P) = |E|$ then we say that P is of <u>full dimension</u>. If $\dim(P) = 0$ then P is a <u>vertex</u>. P is said to be <u>pointed</u> if P has a face which is a vertex. P is <u>bounded</u> if there exist $a,b \in \mathbb{R}^E$ such that $a \le x \le b$ for all $x \in P$. By the definition of dimension, $P<A,b>$ is a vertex if and only if $\mathrm{rank}\,(A_{D^0}) = |E|$. Therefore

2.3.12 $x \in P<A,b>$ is a vertex of $P<A,b>$ if and only if x is the unique solution to $A_S x = b_S$ for some $S \subseteq D$.

2.3.13 Let $A_{D^0}x \le b_{D^0}$ be the equality system of $Ax \le b$. We call $x \in P<A,b>$ an <u>interior point</u> of $P<A,b>$ if $A_{D-D^0}x < b_{D-D^0}$.

2.3.14 Proposition (see [P1] (2.1.9))

Every nonempty polyhedron contains an interior point. □

As a generalization of (2.3.12) we have

2.3.15 Proposition

Let H be a nonempty face of P<A,b>. Then H is a minimal nonempty face of P<A,b> if and only if $H = \{x \in \mathbb{R}^E : A_S x = b_S\}$, where S is the maximal subset of D such that H = q(S).

Proof Let $H = \{x \in \mathbb{R}^E : A_S x = b_S\}$. For all $T \subseteq S$, $\{x \in H : A_T x = b_T\} = H$. Therefore, H contains no proper nonempty faces.

Conversely, suppose that H is a minimal nonempty face of P<A,b> and for some $d \in D-S$ there exists $x^0 \in \{x \in \mathbb{R}^E : A_S x = b_S\}$ such that $A_d x^0 > b_d$. Let $j \in D-S$ be such that $A_j x^0 - b_j$ is minimal over all $d \in D-S$ such that $A_d x^0 > b_d$. By (2.3.14), H has an interior point x^1 and $A_j x^1 < b_j$. For some $\alpha \in (0,1)$, $x^2 \equiv \alpha x^0 + (1-\alpha)x^1$ is such that $A_S x^2 = b_S$, $A_j x^2 = b_j$ and $A_D x^2 \leq b_D$. But then $H' \equiv q(S \cup j)$ is a nonempty face of P<A,b> such that $H' \subset H$; a contradiction. □

2.3.16 Theorem Every minimal nonempty face of P<A,b> has the same dimension.

Proof Suppose that for two minimal nonempty faces H^1, H^2 of P<A,b> we have $\dim(H^1) < \dim(H^2)$. Let D^1, D^2 be the unique maximal subsets of D such that $H^1 = q(D^1)$, $H^2 = q(D^2)$. Then $\operatorname{rank}(A_{D^1}) > \operatorname{rank}(A_{D^2})$. Let $d \in D^1$ be such that A_d is not spanned by $\{A_j : j \in D^2\}$. Since $A_d x \leq b_d$ for all $x \in$ P<A,b>, there is no $x \in \mathbb{R}^E$ such that $A_{D^2} x = b_{D^2}$, $A_d x = b_d + 1$. Therefore, by (2.3.4), there exists $y \in \mathbb{R}^{D^2 \cup d}$ such that $A^T_{D^2 \cup d}\, y = 0$, $y \neq 0$.

Since $H^2 = \{x \in \mathbb{R}^E : A_{D^2}x = b_{D^2}\}$, we must have $y_d \neq 0$. But then A_{D^2} spans A_d; a contradiction. □

2.3.17 Corollary If polyhedron P is pointed then every nonempty face of P is pointed. □

2.3.18 Theorem (see [P1] (2.4.2))

If P is a nonempty bounded polyhedron then P is pointed. □

2.3.19 If P is a pointed polyhedron and $c \in \mathbb{R}^E$ is such that $c \cdot x$ has an upper bound over $x \in P$ then, by (2.3.8), this upper bound is achieved by the elements of a nonempty face of P. Therefore, by (2.3.17),

2.3.20 Theorem If P is a pointed polyhedron and $c \cdot x$ has an upper bound over $x \in P$ then there is a vertex of P which maximizes $c \cdot x$ over $x \in P$. □

2.3.21 Theorem (see [P1] (2.4.1))

$x^0 \in P$ is a vertex of P if and only if there is some $c \in \mathbb{R}^E$ such that x^0 is the unique member of P maximizing $c \cdot x$ over $x \in P$. □

2.3.22 A set $X \subseteq \mathbb{R}^E$ is affinely dependent if there exists a set $\{\lambda_x : x \in X\}$ such that $\lambda_x \neq 0$ for some $x \in X$, $\Sigma(\lambda_x : x \in X) = 0$ and $\Sigma(\lambda_x x : x \in X) = 0$. Otherwise, X is affinely independent. Clearly X is affinely independent if and only if $\{[x,1] : x \in X\}$ is linearly independent. Therefore, by (2.2.5), all maximal affinely independent subsets of X have the same cardinality, called the affine rank of X, and any $X \subseteq \mathbb{R}^E$ has affine rank at most $|E|+1$.

2.3.23 Theorem (see [P1] (2.2.14))

The dimension of P is one less than the affine rank of P. □

2.3.24 If $\dim(P) = k \geq 1$ and H is a face of P with $\dim(H) = k-1$ then H is a facet of P. As a corollary to (2.3.23) we have

2.3.25 Corollary If H is a proper face of P and $\dim(P) = k$ then H is a facet of P if and only if H contains k affinely independent vectors. □

2.3.26 We will be making use of the following construction of a set of affinely independent vectors. Let E, E' be disjoint sets, $K = \{x^m : 1 \leq m \leq k\} \subseteq \mathbb{R}^E$ and $L = \{z^m : 1 \leq m \leq \ell\} \subseteq \mathbb{R}^{E'}$. For $1 \leq m \leq k$ let $\bar{x}^m \equiv [x^m, z^1] \in \mathbb{R}^{E \cup E'}$ and for $2 \leq m \leq \ell$ let $\bar{z}^m \equiv [x^1, z^m] \in \mathbb{R}^{E \cup E'}$.

2.3.27 Theorem If K and L are each affinely independent sets of vectors then $R \equiv \{\bar{x}^m : 1 \leq m \leq k\} \cup \{\bar{z}^m : 2 \leq m \leq \ell\}$ is a set of $k+\ell-1$ affinely independent vectors.

Proof Suppose $\{\alpha_m : 1 \leq m \leq k\}$ and $\{\beta_m : 2 \leq m \leq \ell\}$ are real numbers such that

$$\Sigma(\alpha_m \bar{x}^m : 1 \leq m \leq k) + \Sigma(\beta_m \bar{z}^m : 2 \leq m \leq \ell) = 0$$

and

$$\Sigma(\alpha_m : 1 \leq m \leq k) + \Sigma(\beta_m : 2 \leq m \leq \ell) = 0.$$

Let $\bar{\alpha}_1 \equiv \Sigma(\beta_m : 2 \leq m \leq \ell)$. Then $\Sigma(\alpha_m : 1 \leq m \leq k) + \bar{\alpha}_1 = 0$ and

$$(\alpha_1 + \bar{\alpha}_1)x^1 + \Sigma(\alpha_m x^m : 2 \leq m \leq k) = 0.$$

Since K is affinely independent, $\alpha_1 + \bar{\alpha}_1 = 0$ and $\alpha_2 = \alpha_3 = \ldots = \alpha_k = 0$.

Let $\bar{\beta}_1 \equiv \Sigma(\alpha_m:1 \le m \le k)$. Then $\bar{\beta}_1 = \alpha_1$ and

$$\alpha_1 z^1 + \Sigma(\beta_m z^m : 2 \le m \le \ell) = 0.$$

Since $\alpha_1 + \bar{\alpha}_1 = 0$ and L is affinely independent, $\alpha_1 = \beta_2 = \beta_3 = \ldots = \beta_\ell = 0$. Therefore, R is affinely independent. Clearly $|R| = k+\ell-1$. □

2.3.28 For any $S \subseteq D$ the linear system $A_S x \le b_S$ is said to be essential for defining $P<A,b>$ if $P<A,b> \ne P<A_{\bar{S}},b_{\bar{S}}>$; otherwise, $A_S x \le b_S$ is nonessential for defining $P<A,b>$. In other words, $A_S x \le b_S$ is nonessential for defining $P<A,b>$ if and only if for all $x \in P<A_{\bar{S}},b_{\bar{S}}>$ we have $A_S x \le b_S$.

2.3.29 Theorem (see [P1] (2.3.25))

If $d \in D$ is not in the equality set of $Ax \le b$ and $A_d x \le b_d$ is essential for defining $P<A,b>$ then $q(d)$ is a facet of $P<A,b>$. □

2.3.30 For each $d \in D$ let

$$p(d) \equiv \{j \in D : A_j = \alpha A_d, b_j = \alpha b_d \text{ for some } \alpha \in \mathbb{R},\ \alpha > 0\}.$$

In the case that P is of full dimension, (2.3.29) can be strengthened to

2.3.31 Theorem (see [P1] (2.3.30))

If $P = P<A,b>$ is of full dimension then for all $d \in D$, $A_{p(d)} x \le b_{p(d)}$ is essential for defining P if and only if $q(d)$ is a facet of P. □

2.3.32 Let $\{x^j : j \in J\} \subseteq \mathbb{R}^E$ be a finite set of vectors. We say that $x \in \mathbb{R}^E$ is a <u>convex combination</u> of $\{x^j : j \in J\}$ if there exist $\{\lambda_j : j \in J\}$ such that $\lambda_j \geq 0$ for all $j \in J$, $\Sigma(\lambda_j : j \in J) = 1$ and $x = \Sigma(\lambda_j x^j : j \in J)$. The <u>convex hull</u> of $\{x^j : j \in J\}$ is the set of vectors of $\mathbb{R}^E$ which are convex combinations of $\{x^j : j \in J\}$ and is denoted by $\mathrm{conv}(\{x^j : j \in J\})$.

2.3.33 <u>Theorem</u> (see [P1] (2.4.10))

If P is a bounded polyhedron then P is the convex hull of its set of vertices. □

2.3.34 <u>Theorem</u> If $H \subseteq \mathbb{R}^E$ is a finite set of vectors then conv(H) is a bounded polyhedron and the vertices of conv(H) are elements of H.

<u>Proof</u> See Stoer and Witzgall [S1], Theorem 2.11.4. □

2.4 Linear Programming

2.4.1 Let D,E be finite sets, $H \subseteq D$, $K \subseteq E$, $A \in \mathbb{R}^{D \times E}$, $b \in \mathbb{R}^D$, and $c \in \mathbb{R}^E$. A <u>(primal) linear program</u> is a problem of the form

2.4.2 maximize $c \cdot x$ where $x \in \mathbb{R}^E$ satisfies

2.4.3 $x_K \geq 0$

2.4.4 $A_H x \leq b_H$

2.4.5 $A_{\bar{H}} x = b_{\bar{H}}$.

2.4.6 The dual linear program of (2.4.2) is the linear program

2.4.7 minimize $b \cdot y$ where $y \in \mathbb{R}^D$ satisfies

2.4.8 $y_H \geq 0$

2.4.9 $A_K^T y \geq c_K$

2.4.10 $A_{\bar{K}}^T y = c_{\bar{K}}$.

2.4.11 A vector $x \in \mathbb{R}^E$ satisfying (2.4.3)-(2.4.5) is called a <u>feasible (primal) solution</u> and a vector $y \in \mathbb{R}^D$ satisfying (2.4.8)-(2.4.10) is called a <u>feasible dual solution</u>. A feasible primal solution which maximizes $c \cdot x$ over all feasible primal solutions is called an <u>optimum (primal) solution</u> to (2.4.2) and an <u>optimum dual solution</u> is defined analogously. If x^0 is an optimum solution to (2.4.2) then $c \cdot x^0$ is called the <u>optimum value</u> of (2.4.2). We have the following relationship between a feasible primal solution and a feasible dual solution.

2.4.12 <u>Weak L.P. Duality Theorem</u> For any feasible primal solution x and any feasible dual solution y we have $c \cdot x \leq b \cdot y$.

<u>Proof</u> Since x satisfies (2.4.3)-(2.4.5) and y satisfies (2.4.8)-(2.4.10) we have

2.4.13
$$\begin{aligned} 0 &\leq (A_K^T y - c_K) \cdot x_K + (A_{\bar{K}}^T y - c_{\bar{K}}) \cdot x_{\bar{K}} \\ &+ (b_H - A_H x) \cdot y_H + (b_{\bar{H}} - A_{\bar{H}} x) \cdot y_{\bar{H}} \\ &= b \cdot y - c \cdot x. \qquad \square \end{aligned}$$

2.4.14 <u>Corollary</u> If x^0 is a feasible primal solution and y^0 is a feasible dual solution such that $c \cdot x^0 = b \cdot y^0$ then x^0 is an optimum primal solution and y^0 is an optimum dual solution. $\square$

The following theorem is basic to linear programming.

2.4.15 Strong L.P. Duality Theorem If there exists a feasible primal solution x^1 and an upper bound α such that $c \cdot x \le \alpha$ for all feasible primal solutions x then there exists a feasible primal solution x^0 and a feasible dual solution y^0 such that $c \cdot x^0 = b \cdot y^0$.

Proof Suppose there is no $x \in \mathbb{R}^E$ and $y \in \mathbb{R}^D$ satisfying (2.4.3)-(2.4.5), (2.4.8)-(2.4.10) and $b \cdot y \le c \cdot x$. Then, by part (2.3.5) of (2.3.4), there exists $x^* \in \mathbb{R}^E$, $y^* \in \mathbb{R}^D$ and $z \in \mathbb{R}$ such that

$$x^*_K \ge 0$$

$$y^*_H \ge 0$$

$$z \ge 0$$

$$-A_H x^* + z b_H \ge 0$$

$$A^T_K y^* - z c_K \ge 0$$

$$-A_{\bar{H}} x^* - z b_{\bar{H}} = 0$$

$$A^T_{\bar{K}} y^* - z c_{\bar{K}} = 0$$

$$b \cdot y^* - c \cdot x^* < 0.$$

If $z = 0$ then $A^T_K y^* \ge 0$ and we have $0 \le (A^T y^*) \cdot x^1 = (Ax^1) \cdot y^* \le b \cdot y^*$ and so $c \cdot x^* > 0$. But if we let $x^0 \equiv x^1 + \beta x^*$ for any $\beta > 0$ then $x^0_K \ge 0$, $A_H x^0 = A_H x^1 + \beta A_H x^* \le A_H x^1 \le b_H$, $A_{\bar{H}} x^0 = A_{\bar{H}} x^1 = b_{\bar{H}}$ and $c \cdot x^0 = c \cdot x^1 + \beta c \cdot x^*$. Since $c \cdot x^* > 0$ we can choose β sufficiently large so that $c \cdot x^0 > \alpha$; a contradiction.

If $z > 0$ then let $x^0 \equiv \frac{x^*}{z}$, $y^0 \equiv \frac{y^*}{z}$. x^0 is a feasible primal solution and y^0 is a feasible dual solution such that $b \cdot y^0 < c \cdot x^0$; contradicting the Weak L.P. Duality Theorem.

Hence there exists a feasible primal solution x^0 and a feasible dual solution y^0 such that $c \cdot x^0 \geq b \cdot y^0$ and so, by the Weak L.P. Duality Theorem, we have $c \cdot x^0 = b \cdot y^0$. □

2.4.16 There are many results which are equivalent to the strong L.P. Duality Theorem and it is possible to derive our variant of Farkas' Lemma (2.3.4) from the Strong L.P. Duality Theorem (cf. Dantzig [D1] Theorem 6, p.137).

2.4.17 <u>Corollary</u> A feasible primal solution x^0 is an optimum primal solution if and only if there exists a feasible dual solution y^0 such that $c \cdot x^0 = b \cdot y^0$. □

2.4.18 Suppose P<A,b> is nonempty. Then for all $d \in D$ the linear system $A_{p(d)}x \leq b_{p(d)}$ is nonessential for defining P<A,b> if and only if the linear program

$$\text{maximize } A_d \cdot x \text{ where } x \in \mathbb{R}^E \text{ satisfies } A_{\overline{p(d)}}x \leq b_{\overline{p(d)}}.$$

has an optimum value and the optimum value is less than or equal to b_d. Therefore, as a corollary to the Strong L.P. Duality Theorem, we have

2.4.19 <u>Corollary</u> Let $P \equiv \{x \in \mathbb{R}^E : x \geq 0, Ax \leq b\}$. Then for all $d \in D$ the linear system $A_{p(d)}x \leq b_{p(d)}$ is nonessential for defining P if and only if the optimum value of the linear program

2.4.20 minimize $b \cdot y$ where $y \in \mathbb{R}^D$ satisfies $y \geq 0$, $y_{p(d)} = 0$, $A^T y \geq A_d$

is less than or equal to b_d. □

The following theorem is an important consequence of the Strong L.P. Duality Theorem.

2.4.21 <u>Complementary Slackness Theorem</u> A feasible solution x^0 to (2.4.2) and a feasible solution y^0 to (2.4.7) are optimum if and only if

2.4.22 For all $e \in K$, $x_e^0 > 0$ implies $A_e^T \cdot y^0 = c_e$,

2.4.23 For all $d \in H$, $y_d^0 > 0$ implies $A_d \cdot x^0 = b_d$.

<u>Proof</u> By the Strong L.P. Duality Theorem, x^0 and y^0 are optimum if and only if $c \cdot x^0 = b \cdot y^0$. $c \cdot x^0 = b \cdot y^0$ if and only if the inequality (2.4.13) is an equation, i.e.

2.4.24 $$0 = (A_K^T y^0 - c_K) x_K^0 + (b_H - A_H x^0)\, y_H^0.$$

But (2.4.24) holds if and only if each factor in each term of (2.4.24) is zero. Therefore x^0 and y^0 are optimum if and only if x^0 and y^0 satisfy (2.4.22) and (2.4.23). □

2.4.25 We remark that in proving the sufficiency of (2.4.22) and (2.4.23) we are using only the corollary (2.4.14) of the Weak L.P. Duality Theorem.

2.5 Totally Unimodular Matrices and Integer-Valued Optimum Solutions to a Linear Program

The following result is well-known.

2.5.1 Theorem If $A \in \mathbb{R}^{D\times E}$ and $b \in \mathbb{R}^D$ are rational-valued and the linear program (2.4.2) has an optimum solution then (2.4.2) has a rational-valued optimum solution.

Proof Let x^0 be an optimum solution to (2.4.2) and $\alpha \equiv c\cdot x^0$. Let $A'x \le b'$ represent the linear system (2.4.3)-(2.3.5). (Clearly, the equation $A_dx = b_d$ for $d \in H$ can be represented by $A_dx \le b_d$ and $-A_dx \le -b_d$). By (2.3.8), $L \equiv \{x \in P\langle A',b'\rangle : c\cdot x = \alpha\}$ is a nonempty face of $P\langle A',b'\rangle$. Let H be a minimal nonempty face of L. Then, by (2.3.15), there is a submatrix A_S of A' and a subvector b_S of b' such that $H = \{x \in \mathbb{R}^E : A_Sx = b_S\}$. Since A_S and b_S are rational-valued, there is a rational-valued vector $x^1 \in H$ and since $c\cdot x^1 = \alpha$, x^1 is an optimum solution to (2.4.2). □

The corresponding integer-valued statement of (2.5.1) is of course false. However, we have the following powerful relationship between the existence of integer-valued optimum solutions to primal and dual linear programs.

2.5.2 Theorem Let $b \in \mathbb{Z}^D$ and P, the polyhedron of feasible solutions to (2.4.2), be pointed. If the dual linear program (2.4.7) has an integer-valued optimum solution for every $c \in \mathbb{Z}^E$ such that (2.4.7) has an optimum solution then the vertices of P are integer-valued.

Proof Let x^0 be a vertex of P and suppose that for some $j \in E$, x_j^0 is not an integer. By (2.3.21), there exists $c^0 \in \mathbb{R}^E$ such that x^0 is the only optimum solution to (2.4.2) when $c = c^0$. In fact, we can choose

$c^0 \in \mathbb{Z}^E$. We can also multiply c^0 by a sufficiently large positive integer α so that x^0 is the only optimum solution to (2.4.2) when $c = c^1$, where c^1 is defined by

$$c^1_e \equiv \begin{cases} \alpha c^0_e & \text{if } e \neq j \\ \alpha c^0_e + 1 & \text{if } e = j \ . \end{cases}$$

By hypothesis, (2.4.7) has integer-valued optimum solutions y^0 and y^1 for $c = c^0$ and $c = c^1$ respectively. By the Strong L.P. Duality Theorem, $b \cdot y^0 = \alpha c^0 \cdot x^0$ and $b \cdot y^1 = \alpha c^0 \cdot x^0 + x^0_j$. But, since b, y^0 and y^1 are integer-valued, both $\alpha c^0 \cdot x^0$ and $\alpha c^0 \cdot x^0 + x^0_j$ must be integers; a contradiction. □

2.5.3 If P is pointed and (2.4.2) has an optimum solution then, by (2.3.20), (2.4.2) has an optimum solution which is a vertex of P. By (2.3.21), for every vertex $x^0 \in P$ there exists $c^0 \in \mathbb{R}^E$ such that x^0 is the only optimum solution to (2.4.2) when $c = c^0$. Therefore, (2.5.2) is equivalent to

2.5.4 Let $b \in \mathbb{Z}^D$ and P, the polyhedron of feasible solutions to (2.4.2), be pointed. If the dual linear program (2.4.7) has an integer-valued optimum solution for all $c \in \mathbb{Z}^E$ such that (2.4.7) has an optimum solution then for all $c \in \mathbb{R}^E$ such that the primal linear program (2.4.2) has an optimum solution, (2.4.2) has an integer-valued optimum solution.

We will be making frequent use of (2.5.2). Indeed, (2.5.2) is one of the main tools of the thesis. (2.5.2) was proved by Hoffman [H5] for (0,1)-matrices A and we have generalized his proof.

2.5.6 We will be discussing several properties of matrices $A \in \mathbb{R}^{D\times E}$ which involve the evaluation of determinants. However, these properties will be invariant under permutations of rows and columns of A. Hence when det(A), the determinant of A, is to be evaluated we may assume that $D = \{1,2,\ldots,|D|\}$ and $E = \{1,2,\ldots,|E|\}$.

2.5.7 The matrix $A \in \mathbb{R}^{D\times E}$ is said to be <u>totally unimodular</u> if for every square submatrix B of A we have $\det(B) \in \{0,1,-1\}$. The following properties of totally unimodular matrices are consequences of properties of determinants.

2.5.8 A is totally unimodular if and only if A^T is totally unimodular.

2.5.9 A is totally unimodular if and only if every submatrix of A is totally unimodular.

2.5.10 $A \in \mathbb{R}^{D\times E}$ is totally unimodular if and only if $[A,I_D]$ is totally unimodular.

2.5.11 Let B be obtained from A by multiplying a row of A by -1. Then A is totally unimodular if and only if B is totally unimodular.

2.5.12 Let $[B,I_D]$ be obtained from A by a sequence of elementary row operations. If A is totally unimodular then $[B,I_D]$ is totally unimodular.

2.5.13 To see (2.5.10), suppose A is totally unimodular and B is any submatrix of $[A,I_D]$. We can expand det(B) by the columns of B which are also subcolumns of I_D and we have $\det(B) = \pm\det(B')$ for some square submatrix B' of A. Therefore, $\det(B) \in \{0,1,-1\}$ and (2.5.10) follows.

2.5.14 To see (2.5.12), let C be a square submatrix of $[B,I_D]$. We can adjoin columns of I_D to C to obtain a submatrix C' of $[B,I_D]$ of side $|D|$ such that $\det(C) = \det(C')$. Let A' be the submatrix of A whose columns are indexed by the same set as C'. Then C' is obtained from A' by a sequence of elementary row operations and $\det(C') = \pm\det(A') \in \{0,1,-1\}$.

The relationship between totally unimodular matrices and the existence of integer-valued optimum solutions to a linear program was pointed out by Hoffman and Kruskal [H6].

2.5.15 Theorem If $A \in \mathbb{R}^{D\times E}$ is totally unimodular, $b \in \mathbb{Z}^D$ and $P\langle A,b\rangle$ is pointed then the vertices of $P\langle A,b\rangle$ are integer-valued.

Proof By (2.3.12) any vertex x of $P\langle A,b\rangle$ is the unique solution to the system $A_S x = b_S$ for some $S \subseteq D$. We may assume that A_S is a square submatrix of A and, when we use Cramer's Rule to solve for x, we see that, since b_S is integer-valued, x is integer-valued. □

2.5.16 Corollary If $A \in \mathbb{R}^{D\times E}$ is totally unimodular, $b \in \mathbb{Z}^D$ and (2.4.2) has an optimum solution then (2.4.2) has an integer-valued optimum solution.

Proof Let x^0 be an optimum solution to (2.4.2) and consider the linear program

2.5.17 maximize $c\cdot x$ where $x \in \mathbb{R}^E$ satisfies (2.4.3)-(2.3.5) and

2.5.18 $\lfloor x_e^0 \rfloor \le x_e \le \lceil x_e^0 \rceil$ for all $e \in E$.

If we represent the linear system of (2.4.3)-(2.4.5) and (2.5.18) as $A'x \leq b'$ and let $P = P\langle A';b'\rangle$ then clearly P is bounded. Therefore, by (2.3.18) and (2.3.20), there is a vertex x^1 of P which is an optimum solution to (2.5.17) and hence to (2.4.2). By (2.5.8)-(2.5.11) A' is totally unimodular and, since b' is integer-valued, x^1 is integer-valued by (2.5.15). □

2.6 Graphs

2.6.1 A *graph* $G = (V,E)$ is a finite set V of *nodes* and a finite set E of *edges* such that $V \cap E = \phi$ and every edge $e \in E$ has a *tail* $t(e) \in V$ and a *head* $h(e) \in V$. We denote the node set of G by V(G) and the edge set of G by E(G). The above object is more commonly called a directed graph (cf. Harary [H3]). If for some edge $e \in E$ we have $t(e) = h(e)$ then e is called a *loop* and if G has no loops then G is said to be *loopless*.

2.6.2 A graph H is said to be a *subgraph* of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and for every $e \in E(H)$ the tail and head of e in H are the same as the tail and head respectively of e in G. H is a *spanning* subgraph of G if $V(H) = V(G)$. Where S is a set of subgraphs of G, $H \in S$ is a *maximal subgraph of G in S* if there is no $K \in S$ such that $E(H) \subset E(K)$.

2.6.3 For any $S \subseteq V$ let $\delta(S) \equiv \{e \in E: t(e) \in S, h(e) \notin S\}$; $\gamma(S) \equiv \{e \in E: t(e) \in S, h(e) \in S\}$ and G[S] denote the *subgraph of G induced by S*, where $G[S] \equiv (S,\gamma(S))$. For any $T \subseteq E$ let G(T) denote the *subgraph of G spanned by T*, where $V(G(T)) \equiv \{t(e): e \in T\} \cup \{h(e): e \in T\}$ and $E(G(T)) \equiv T$.

2.6.4 A <u>path π in $G = (V,E)$ from v_0 to v_n</u> is a sequence $(v_0,e_1,v_1,\ldots,v_{n-1},e_n,v_n)$ such that (i) $v_i \in V$ and $v_i \neq v_j$ for all $i,j \in \{0,1,\ldots,n\}$, $i \neq j$ and (ii) $e_i \in \gamma(\{v_{i-1},v_i\})$ for all $i \in \{1,2,\ldots,n\}$. Such a path is a <u>directed</u> path if (iii) $h(e_i) = v_i$ for all $i \in \{1,2,\ldots,n\}$. G is <u>connected</u> if for all $v,w \in V$ there is a path in G from v to w. A <u>component</u> of G is a maximal connected subgraph of G. Let $p_0(G)$ denote the number of components of G.

2.6.5 G is <u>strongly connected</u> if for all $v,w \in V$ there is a directed path in G from v to w.

2.6.6 Node $v \in V$ is a <u>cutnode</u> of G if $G[V-v]$ has more components than G. A <u>block</u> H of G is a maximal connected subgraph of G such that H contains no cutnodes, no loops and $|V(H)| \geq 2$ or such that H is a loop. Let $\beta(G)$ denote the number of blocks of G.

2.6.7 A <u>polygon</u> Q of G is a sequence $(v_0,e_1,v_1,\ldots,v_{n-1},e_n,v_n = v_0)$ such that (i) $v_i \in V$ and $v_i \neq v_j$ for all $i,j \in \{0,1,\ldots,n\}$, $i \neq j$ except $v_0 = v_n$ and (ii) $e_i \in \gamma(\{v_{i-1},v_i\})$ for all $i \in \{1,2,\ldots,n\}$. Such a polygon is a <u>directed</u> polygon if (iii) $h(e_i) = v_i$ for all $i \in \{1,2,\ldots,n\}$. Clearly, there is no loss in meaning if we refer to a subgraph of G as being a polygon. A <u>forest</u> is a graph which contains no polygons and a <u>tree</u> is a connected forest. G is <u>acyclic</u> if G contains no directed polygons. The following results are well-known.

2.6.8 $G = (V,E)$ is a forest if and only if $|\gamma(S)| \leq |S|-1$ for all $S \in K_V$ and G is a tree if and only if G is a forest and $|V| = |E|+1$.

2.6.9 If $G = (V,E)$ is a nontrivial forest then there are at least two nodes $v \in V$ such that $|\delta(v)| + |\delta(\bar{v})| = 1$.

2.6.10 G is connected if and only if G contains a spanning tree.

2.6.11 There are large classes of totally unimodular matrices which can be obtained from graphs. Given a graph $G = (V,E)$ and a subset $S \subseteq V$ we define the <u>coboundary vector</u> of S, $cv(S) = [cv(S)_e : e \in E]$, by

$$cv(S)_e \equiv \begin{cases} -1 & \text{if } e \in \delta(\bar{S}) \\ 1 & \text{if } e \in \delta(S) \\ 0 & \text{otherwise} \end{cases} .$$

The matrix $A \in \mathbb{R}^{V \times E}$ with rows $[cv(v) : v \in V]$ is called the <u>(incidence) matrix</u> of G. The following theorem is a special case of a theorem in Heller and Thomkins [H4].

2.6.12 <u>Theorem</u> For any graph $G = (V,E)$ the matrix of G is totally unimodular.

<u>Proof</u> Let A be the matrix of G and let B be an $m \times m$ submatrix of A. We prove by induction on m that $\det(B) \in \{0,1,-1\}$, this obviously being the case when $m = 1$. Now assume that every $(m-1) \times (m-1)$ submatrix B' of A has determinant 0, 1 or -1. If a column of B is 0 then $\det(B) = 0$. If every column of B has two nonzero components then, since each column of B sums to 0, $\det(B) = 0$. Finally, if B has a column with exactly one nonzero component then when we expand $\det(B)$ by that column we obtain $\det(B) = \pm\det(B')$ where B' is an $(m-1) \times (m-1)$ submatrix of A. Hence, by our induction hypothesis, $\det(B) \in \{0,1,-1\}$.

□

2.6.13 Corollary If $A \in \mathbb{R}^{D \times E}$ and $B \in \mathbb{R}^{D \times E'}$ have the property that all components of each row of A and B are zero except possibly one component being equal to one then [A,B] is totally unimodular.

Proof It is easy to see that $[-A,B]^T$ is a submatrix of the matrix of a graph. Therefore, by (2.5.7)-(2.5.11) and (2.6.12), [A,B] is totally unimodular. □

2.6.14 Let T = (V,E) be a tree. For a given edge $e \in E$ the node $v \in V$ is said to be above e if the unique path in T from v to t(e) contains e. Otherwise, v is below e. Let

$$\alpha(e) \equiv \{v \in V : v \text{ is above } e\}$$

$$\omega(e) \equiv \{v \in V : v \text{ is below } e\}.$$

2.6.15 Theorem Let T = (V,E') be a spanning tree of G = (V,E). Then the matrix $A \in \mathbb{R}^{E' \times E}$ with rows $[cv_G(\omega(e)) : e \in E']$ is totally unimodular.

Proof Since $cv_G(\omega(e)) = -cv_G(\alpha(e))$ for all $c \in E'$, multiplying a row of A by -1 is equivalent to reversing the edge of E' corresponding to that row and, by (2.5.11), the resulting matrix is totally unimodular if and only if A is totally unimodular. Since T is a tree, we can multiply appropriate rows of A by -1 and assume that for every node $v \in V$, $|\delta(v) \cap E'| \leq 1$. Therefore,

2.6.16 For every edge $e \in E'$,

$$cv_G(\omega(e)) = cv_G(t(e)) + \Sigma(cv_G(\omega(j)) : j \in E', h(j) = t(e)).$$

$\Sigma(|\delta(v) \cap E'|):v \in V) = |E'| = |V|-1$, so there is exactly one node $w \in V$ such that $\delta(w) \cap E' = \phi$. Let A' have rows $[cv_G(v):v \in V-w]$. A' is a submatrix of the matrix of G. Hence, by (2.5.9) and (2.6.12), A' is totally unimodular. By (2.6.16), A can be obtained from A' by a sequence of additions of one row to another. For any two edges $e,j \in E'$, $e \notin \delta(\alpha(j))$ and $e \in \delta(\omega(j))$ if and only if $e = j$. Therefore, the columns of A indexed by E' form an identity submatrix of A having side $|E'|$. Therefore, by (2.5.12), A is totally unimodular. □

CHAPTER 3

MATROIDS

In this chapter we summarize many results concerning matroids. Most of these results are direct consequences of a more general theory of polymatroids and we leave their proofs for Chapter 4.

3.1 Independence Systems and Matroids

3.1.1 An independence system $M = (E,\mathfrak{J})$ is a finite set E and a non-empty family $\mathfrak{J}$ on E of sets called independent sets of M such that if $Y \subseteq Z \in \mathfrak{J}$ then $Y \in \mathfrak{J}$. For all $S \subseteq E$ the rank of S, $r(S)$, is the maximum cardinality of an independent subset of S and an independent subset of S of maximum cardinality is called a basis of S.

3.1.2 A matroid (on E) is an independence system $M = (E,\mathfrak{J})$ such that for all $S \subseteq E$ every maximal independent subset of S is a basis of S. That is, every independent subset of S can be extended to a basis of S.

3.1.3 Clearly, the rank function $r:L_E \to \mathbb{R}$ of an independence system $M = (E,\mathfrak{J})$ satisfies the following:

3.1.4 If $Y \subseteq Z \subseteq E$ then $r(Y) \leq r(Z)$; i.e. r is nondecreasing on L_E.

3.1.5 If $Y,Z \subseteq E$ then $r(Y \cup Z) \leq r(Y) + r(Z)$; i.e. r is subadditive on L_E.

We have the following important characterization of matroids:

3.1.6 Proposition An independence system $M = (E,\mathfrak{J})$ with rank function $r:L_E \to \mathbb{R}$ is a matroid if and only if for all $Y,Z \subseteq E$ we have $r(Y \cap Z) + r(Y \cup Z) \leq r(Y) + r(Z)$; i.e. r is submodular on L_E.

Proof. Suppose M is a matroid and let $Y, Z \subseteq E$. Let J_0 be a basis of $Y \cap Z$ and extend J_0 to a basis J_1 of $Y \cup Z$. Then

$$\begin{aligned} r(Y \cap Z) + r(Y \cup Z) &= |J_0| + |J_1| \\ &= |J_1 \cap (Y \cap Z)| + |J_1 \cap (Y \cup Z)| \\ &= |J_1 \cap Y| + |J_1 \cap Z| \\ &\leq r(Y) + r(Z). \end{aligned}$$

Conversely, suppose that $M = (E, \mathfrak{I})$ is an independence system with rank function r and suppose that for some $S \subseteq E$, J is a maximal independent subsets of S such that $|J| < r(S)$. Note that for all maximal independent subsets A of S and any $e \in S-A$ we have $r(A \cup e) = |A|$. So let S_0 be a maximal subset of S such that $J \subseteq S_0$ and $r(S_0) = |J|$. Then $S_0-J \neq \phi$ and, since $r(S) > |J|$, $S-S_0 \neq \phi$. Let $e \in S-S_0$ and let S_1 be a maximal subset of S such that $J \cup e \subseteq S_1$ and $r(S_1) = |J|$. Then

$$r(S_0 \cup S_1) > |J| = r(S_0) + r(S_1) - r(S_0 \cap S_1);$$

hence r is not submodular. □

3.1.7 There are many independence systems which are matroids. One of the most general constructions of matroids is as follows. For any lattice L with minimum element $\dot{m}$ let $L^0 \equiv L-\{m\}$. (Clearly $K_E = L^0_E$.) A function $f: L \to \mathbb{R}$ is said to be $\underline{\sigma\text{-function}}$ if

3.1.8 $f(a) \geq 0$ for all $a \in L^0$.

3.1.9 For all $a,b \in L^0$ such that $a \wedge b \neq m$ we have

$$f(a \wedge b) + f(a \vee b) \leq f(a) + f(b);$$

i.e. f is <u>submodular</u> on L^0.

Note that the conditions for a σ-function are independent of $f(m)$ and $f(m)$ can assume any value.

3.1.10 Let L be any family on E containing E and ϕ such that if $Y,Z \in L$ then $Y \cap Z \in L$. Then L is a lattice, called a <u>closure system on E</u>, when we define $Y \wedge Z \equiv Y \cap Z$ and $Y \vee Z \equiv \cap(S \in L:Y,Z \subseteq S)$.

3.1.11 <u>Theorem</u> Let L be a closure system on E and let $f:L \to \mathbb{R}$ be an integer-valued σ-function of L. Then $\mathfrak{J} \equiv \{J \subseteq E:|J \cap S| \leq f(S) \text{ for all } S \in L^0\}$ is the family of independent sets of a matroid $M = (E,\mathfrak{J})$. For all $T \subseteq E$ the rank of T in M is given by

$$r(T) = \min\{\Sigma\ (f(S):S \in F)+|T-\cup(S:S \in F)|:F \subseteq L^0\}.$$

<u>Proof</u> See (4.1.10). (3.1.11) is a special case of (4.1.4) which will be proved in Chapter 4. □

3.1.12 For any lattice L with minimum element m, a function $f:L \to \mathbb{R}$ is a <u>β-function</u> if f is a σ-function which satisfies

3.1.13 For all $a,b \in L^0$ such that $a \leq b$ we have $f(a) \leq f(b)$; i.e. f is <u>nondecreasing</u> on L^0, and

3.1.14 For all $a,b \in L^0$ we have $f(a \vee b) \leq f(a) + f(b)$; i.e. f is <u>subadditive</u> on L^0.

3.1.15 We remark that if $f:L \to \mathbb{R}$ is a nondecreasing σ-function such that $f(m) = 0$ and $f(a \wedge b) + f(a \vee b) \leq f(a) + f(b)$ for all $a,b \in L$ then f is subadditive on L and therefore a β-function. A β-function f of a lattice L with minimum element m is called a <u>β_0-function</u> if $f(m) = 0$.

3.1.16 <u>Corollary</u> Let L be a closure system on E and let $f:L \to \mathbb{R}$ be an integer-valued β_0-function of L. Then $\mathfrak{J} \equiv \{J \subseteq E: |J \cap S| \leq f(S) \text{ for all } S \in L\}$ is the family of independent sets of a matroid $M = (E,\mathfrak{J})$. For all $T \subseteq E$ the rank of T in M is given by

$$r(T) = \min\{f(U) + |T-U| : U \in L\}.$$

<u>Proof</u> By (3.1.11), $M = (E,\mathfrak{J})$ is a matroid. Suppose that for $T \subseteq E$, $F \subseteq L$ is such that $r(T) = \Sigma(f(S):S \in F) + |T - \cup(S:S \in F)|$. Let $U \equiv \vee(S:S \in F)$. By the submodularily of f and since $f(\phi) = 0$,

$$r(T) \leq f(U) + |T-U|. \text{ The corollary now follows.} \qquad \square$$

3.1.17 If f is a β_0-function of L_E then for any $J \subseteq E$, $|J \cap S| \leq f(S)$ for all $S \subseteq E$ if and only if $|A| \leq f(A)$ for all $A \subseteq J$. Furthermore, if for some $T \subseteq E$, $U \subseteq E$ is such that $r(T) = f(U)+|T-U|$ then, because f is nondecreasing, $r(T) \leq f(T \cap U) + |T-U|$. Therefore, we may assume that $U \subseteq T$ and we have

3.1.18 <u>Corollary</u> Let f be an integer-valued β_0-function of L_E. Then $\mathfrak{J} \equiv \{J \subseteq E: |A| \leq f(A) \text{ for all } A \subseteq J\}$ is the family of independent sets of a matroid $M = (E,\mathfrak{J})$ and for all $T \subseteq E$ the rank of T in M is given by

$$r(T) = \min\{f(U) + |T-U| : U \subseteq T\}. \qquad \square$$

3.1.19 If $f:L_E \to \mathbb{R}$ is an integer-valued β_0-function of L_E such that $f(\{e\}) = 0$ or 1 for all $e \in E$ then for all $T \subseteq E$ and for all $U \subseteq T$ we have

$$f(U) + |T-U| \geq f(U) + \Sigma(f(\{e\}):e \in T-U) \geq f(T).$$

Therefore, the rank function $r:L_E \to \mathbb{R}$ of the matroid constructed by (3.1.18) is identical to f. If $J \in \mathfrak{I}$ then $|J| = r(J) = f(J)$. If $J \subseteq E$ and $|J| = f(J)$ then for any $A \subseteq J$ we have

$$|A| = |J| - |J-A| \leq f(J)-f(J-A) \leq f(A).$$

Hence $J \in \mathfrak{I}$ if and only if $J \subseteq E$ and $|J| = f(J)$. Thus, by (3.1.6),

(3.1.20) <u>Corollary</u> A function $f:L_E \to \mathbb{R}_+$ is the rank function of a matroid if and only if f is an integer-valued β_0-function such that

$$f(\{e\}) = 0 \text{ or } 1 \text{ for all } e \in E. \qquad \square$$

Thus we see that every matroid appears as an instance of (3.1.11). What is of interest is that (3.1.11) can be used to demonstrate that certain independence systems which are not obviously matroids are indeed matroids.

(3.1.21) As a first application of (3.1.11) let $G = (V,E)$ be a graph and let L be the family of sets $S \subseteq E$ such that $S = \gamma(T)$ for some $T \in K_V$. Clearly E and ϕ are elements L. For all $T,U \subseteq V$ we have $\gamma(T \cap U) = \gamma(T) \cap \gamma(U)$. Therefore, L is a closure system on E and for all $S \in L^0$ there is a unique minimal subset $T_S \in K_V$, namely $V(G(J))$, such that $S = \gamma(T_S)$. For all $S \in L^0$ let $f(S) \equiv |T_S|-1$. Let $f(\phi) \equiv -1$. For all $R,S \in L$ we have

$$f(R \wedge S) + f(R \vee S) \leq |T_R \cap T_S|-1 + |T_R \cup T_S|-1$$
$$= |T_R|-1 + |T_S|-1$$
$$= f(R) + f(S).$$

Hence f is a σ-function of L. By (3.1.11), $\mathfrak{J} \equiv \{J \subseteq E: |J \cap \gamma(S)| \leq |S|-1 \text{ for all } S \in K_V\}$ is the family of independent sets of a matroid. Therefore, by (2.6.8), the edge-sets of forests of G are the independent sets of a matroid M(G), called the <u>forest matroid</u> of G. If $r:L_E \to \mathbb{R}$ is the rank function of M(G) then it is easy to verify that for all $T \subseteq E$ there is a forest B of G(T) such that $|E(B)| = |V(G(T))|-p_0(G(T))$. (Recall that $p_0(G(T))$ is the number of components of G(T)). Hence $r(T) \geq |V(G(T))|-p_0(G(T))$. If the components of G(T) are $G_1, G_2, \ldots, G_k$ then, by (3.1.11),

$$r(T) \leq \Sigma(f(\gamma_G(V(G_i))):1 \leq i \leq k)$$
$$= \Sigma(|V(G_i)|-1:1 \leq i \leq k)$$
$$= |V(G(T))|-p_0(G(T)).$$

Therefore, for all $T \subseteq E$,

3.1.22 $\quad r(T) = |V(G(T))|-p_0(G(T)).$

3.1.23 <u>Theorem</u> Let $f:L_E \to \mathbb{R}$ be a β_0-function and for all $j \in E'$ let Q_j be a subset of E. For all $S \subseteq E'$ let $U(S) \equiv \cup(Q_j:j \in S)$. Then $f'(S) \equiv f(U(S))$ for all $S \subseteq E'$ is a β_0-function of $L_{E'}$.

Proof Clearly, $f'(S) \geq 0$ for all $S \subseteq E'$. If $S \subseteq T \subseteq E$; then $U(S) \subseteq U(T)$ and $f'(S) \leq f'(T)$. For all $S,T \subseteq E'$ we have $U(S \cap T) \subseteq U(S) \cap U(T)$ and $U(S \cup T) = U(S) \cup U(T)$ and so

$$\begin{aligned} f'(S \cap T)+f'(S \cup T) &= f(U(S \cap T))+f(U(S \cup T)) \\ &\leq f(U(S) \cap U(T))+f(U(S) \cup U(T)) \\ &\leq f(U(S))+f(U(T)) \\ &= f'(S)+f'(T). \end{aligned}$$

Therefore f' is a β_0-function of $L_{E'}$, by (3.1.15). □

3.1.24 A transversal of an indexed family $\{Q_j : j \in J\}$ of not necessarily distinct subsets of E is a set $\{e_j : j \in J\}$ of distinct elements of E such that $e_j \in Q_j$ for all $j \in J$.

3.1.25 Theorem For any finite indexed family $\{Q_j : j \in E'\}$ on E the sets $J \subseteq E'$ such that $\{Q_j : j \in J\}$ has a transversal is the set of independent sets of a matroid $M = (E', \mathfrak{I})$ on E'. The rank of $T \subseteq E'$ in matroid M is given by

3.1.26 $r(T) = \min\{|U(S)|+|T-S| : S \subseteq T\}$.

Proof For all $S \subseteq E$ let $f(S) \equiv |S|$. Then, where $f'(S) \equiv |U(S)|$ for all $S \subseteq E'$, $f' : L_{E'} \to \mathbb{R}$ is an integer-valued β_0-function by (3.1.23). If we let $\mathfrak{I} \equiv \{J \subseteq E' : |A| \leq |U(A)|$ for all $A \subseteq J\}$ then, by (3.1.18), $M \equiv (E', \mathfrak{I})$ is a matroid with rank function r where for all $T \subseteq E'$,

$$\begin{aligned} r(T) &= \min\{|U(S)|+|T-S| : S \subseteq E'\} \\ &= \min\{|U(S)|+|T-S| : S \subseteq T\}. \end{aligned}$$

By Hall's Theorem [H1], $\{Q_j : j \in J\}$ has a transversal if and only if $|S| \leq |\mathfrak{U}(S)|$ for all $S \subseteq J$. Therefore,

$$\mathfrak{I} = \{J \subseteq E' : \{Q_j : j \in J\} \text{ has a transversal}\}. \qquad \square$$

3.1.27 A matroid of the form described by (3.1.25) is called a <u>transversal matroid</u>.

As a final example of an application of (3.1.11) we have the following:

3.1.28 <u>Theorem</u> Let $\{M_i = (E, \mathfrak{I}_i) : i \in I\}$ be a family of matroids on E with rank functions $\{r_i : i \in I\}$ and let

$$\mathfrak{I} \equiv \{J \subseteq E : |A| \leq \Sigma(r_i(A) : i \in I) \text{ for all } A \subseteq J\}.$$

Then $M = (E, \mathfrak{I})$ is a matroid with rank function r, where for all $T \subseteq E$

3.1.29 $r(T) = \min\{\Sigma(r_i(S) : i \in I) + |T-S| : S \subseteq T\}$.

<u>Proof</u> This theorem follows immediately from (3.1.18) where for all $S \subseteq E$ we let $f(S) \equiv \Sigma(r_i(S) : i \in I)$. Clearly $f : L_E \to \mathbb{R}$ is an integer-valued β_0-function and the theorem holds. $\square$

The matroid $M = (E, \mathfrak{I})$ of (3.1.28) is called the <u>sum</u> of $\{M_i : i \in I\}$. Edmonds [E2] proved that for all $J \subseteq E$ we have $|A| \leq \Sigma(r_i(A) : i \in I)$ for all $A \subseteq J$ if and only if J can be partitioned into possibly empty sets $\{J_i : i \in I\}$ such that $J_i \in \mathfrak{I}_i$ for all $i \in I$. Therefore

3.1.30 <u>Theorem</u> Let $\{M_i = (E, \mathfrak{I}_i)\}$ be a family of matroids on E with rank functions $\{r_i : i \in I\}$ and let

$$\mathfrak{I} \equiv \{J \subseteq E : J \text{ can be partitioned into sets } \{J_i \in \mathfrak{I}_i : i \in I\}\}$$

Then $M = (E, \mathfrak{I})$ is a matroid on E with rank function r given by (3.1.29). $\square$

3.2 Matroid Polyhedra

3.2.1 There is a natural polyhedron which one can associate with any independente system $M = (E, \mathfrak{I})$. For all $S \subseteq E$ the vector of S, x^S, is defined by

$$x_e^S \equiv \begin{cases} 1 & \text{if } e \in S \\ 0 & \text{if } e \notin S \end{cases}$$

for all $e \in E$. Let $P(M) \equiv \text{conv}(\{x^J : J \in \mathfrak{I}\})$. By (2.3.34), $P(M)$ is a polyhedron.

3.2.2 For any family $F \subseteq K_E$ and any function $f: F \to \mathbb{R}_+$ let

$$P(F,f) \equiv \{x \in \mathbb{R}_+^E : x(S) \leq f(S) \text{ for all } S \in F\}.$$

3.2.3 If $r: L_E \to \mathbb{R}$ is the rank function of an independence system $M = (E, \mathfrak{I})$ and $J \in \mathfrak{I}$ then for all $S \in K_E$, $x^J(S) = |J \cap S| \leq r(S)$. Therefore,

3.2.4 $P(M) \subseteq P(K_E, r)$.

3.2.5 Proposition If $r: L_E \to \mathbb{R}$ is the rank function of an independence system $M = (E, \mathfrak{I})$ then for all $J \in \mathfrak{I}$, x^J is a vertex of $P(K_E, r)$.

Proof For any $J \in \mathfrak{I}$, x^J is the unique solution to the system of equations

$$x_e = r(\{e\}) \text{ for all } e \in J$$
$$x_e = 0 \qquad \text{for all } e \notin J.$$

Therefore, by (2.3.12), x^J is a vertex of $P(K_E, r)$. $\square$

3.2.6 Since for every $J \in \mathfrak{J}$, x^J is a vertex of $P(K_E,r)$ and $x^J \in P(M)$, $P(M) \subseteq P(K_E,r)$ implies the following:

3.2.7 <u>Proposition</u> If $r:L_E \to \mathbb{R}$ is the rank function of an independence system $M = (E,\mathfrak{J})$ then for all $J \in \mathfrak{J}$, x^J is a vertex of $P(M)$. □

3.2.8 In general $P(K_E,r)$ will have many more vertices than $P(M)$. However, in the case of matroids we have

3.2.9 <u>Theorem</u> If $r:L_E \to \mathbb{R}$ is the rank function of a matroid $M = (E,\mathfrak{J})$ then $P(M) = P(K_E,r)$.

<u>Proof</u> See (4.2.13). (3.2.9) is a special case of (4.2.12). □

3.2.10 For any independence system $M = (E,\mathfrak{J})$ and any $c \in \mathbb{R}^E$ we can consider the combinatorial problem:

3.2.11 maximize $c(J)$ over $J \in \mathfrak{J}$.

In section 4.2 we will show that the following algorithm solves (3.2.11), whenever M is a matroid (see 4.2.10).

3.2.12 <u>Matroid Greedy Algorithm</u> Starting with $J = \phi$, at each step augment J by finding an element $j \notin J$ with largest positive weight c_j among those elements $e \notin J$ such that $J \cup e \in \mathfrak{J}$ and adding j to J. Stop when there are no more positive weight elements $e \notin J$ such that $J \cup e \in \mathfrak{J}$.

3.2.13 Not only does the Matroid Greedy Algorithm solve (3.2.11) for a matroid $M = (E,\mathfrak{J})$, the optimum solution $J \in \mathfrak{J}$ to (3.2.11) it produces is such that x^J is an optimum solution to the linear program:

3.2.14 maximize $c \cdot x$ where $x \in \mathbb{R}^E$ satisfies

$$x_e \geq 0 \text{ for all } e \in E$$

$$x(S) \leq r(S) \text{ for all } S \in K_E,$$

where $r: L_E \to \mathbb{R}$ is the rank function of M. Therefore, by (2.3.21), the vertices of $P(K_E, r)$ must be the vectors of sets $J \in \mathfrak{J}$ and $P(K_E, r) \subseteq P(M)$. By (3.2.4), $P(M) = P(K_E, r)$. This is precisely the technique we will use in section 4.2 to prove (3.2.9).

3.2.15 The Greedy Algorithm is well-defined for any independence system $M = (E, \mathfrak{J})$ but it will not in general solve (3.2.11). Note that matroids are precisely those independence systems for which the Greedy Algorithm works for all (0,1)-vectors $c \in \mathbb{R}^E$.

3.3 Matroid Intersection

3.3.1 Let $M_1 = (E, \mathfrak{J}_1)$ and $M_2 = (E, \mathfrak{J}_2)$ be two independence systems on E and $M = (E, \mathfrak{J})$ where $\mathfrak{J} \equiv \{J \subseteq E : J \in \mathfrak{J}_1 \cap \mathfrak{J}_2\}$. M is called the intersection of M_1 and M_2 and is denoted by $M_1 \cap M_2$. We will be studying the case where M_1 and M_2 are matroids on E.

(3.2.9) is remarkable in that there are very few classes of independence systems $M = (E, \mathfrak{J})$ with rank function r for which it is known that $P(M) = P(K_E, r)$. It is even more remarkable that (3.2.9) can be extended to include the intersection of two matroids.

3.3.2 Theorem If $r_1, r_2 : L_E \to \mathbb{R}$ are the rank functions of matroids $M_1 = (E, \mathfrak{J}_1)$ and $M_2 = (E, \mathfrak{J}_2)$ respectively then

$$P(M_1 \cap M_2) = P(M_1) \cap P(M_2) = P(K_E, r)$$

where $r: L_E \to \mathbb{R}$ is the rank function of $M_1 \cap M_2$. Equivalently,

3.3.3 The vertices of $P(K_E, r_1) \cap P(K_E, r_2)$ are integer-valued, i.e. the vectors x^J for $J \in \mathfrak{F}_1 \cap \mathfrak{F}_2$.

<u>Proof</u> (3.3.2) is a special case of (4.3.8), which we prove later.

□

3.3.4 As we outlined in the previous section, one method of proving (3.3.2) would be to display, for each $c \in \mathbb{R}^E$, an optimum solution to the linear program

3.3.5 maximize $c \cdot x$ where $x \in \mathbb{R}^E$ satisfies

$$\begin{aligned} x_e &\geq 0 \\ x(S) &\leq r_1(S) \quad \text{for all } S \in K_E \\ x(S) &\leq r_2(S) \quad \text{for all } S \in K_E \end{aligned}$$

which is the vector x^J of some set $J \in \mathfrak{F}_1 \cap \mathfrak{F}_2$. This is the method used by Edmonds [E4] to prove (3.3.2).

3.3.6 An alternate approach is the following. The dual linear program of (3.3.5) is

3.3.7 minimize $r_1 \cdot y^1 + r_2 \cdot y^2$ where $y^1, y^2 \in \mathbb{R}^{K_E}$ satisfy

$$\begin{aligned} &y^i_S \geq 0 \text{ for all } S \in K_E,\ i = 1,2 \\ &y^1(K_E, e) + y^2(K_E, e) \geq c_e \text{ for all } e \in E, \end{aligned}$$

where for any family F on E, function $f: F \to \mathbb{R}$, $y \in \mathbb{R}^F$ and $e \in E$,

$$f \cdot y \equiv \Sigma(f(S)y_S : S \in F)$$

and

$$y(F,e) \equiv \Sigma(y_S : e \in S \in F).$$

If we could show that (3.3.7) has an integer-valued optimum solution for all $c \in \mathbb{Z}^E$ then (3.3.2) would follow from (2.5.2).

It is not necessarily true that the vertices of the set of feasible solutions to (3.3.7) are integer-valued. However,

3.3.8 Theorem If $c \in \mathbb{Z}^E$ then (3.3.7) always has an integer-valued optimum solution.

Proof (3.3.8) will follow from (4.3.4). □

3.3.9 By (3.3.3) and (2.3.20), for all $c \in \mathbb{R}^E$ the combinatorial problem

3.3.10 maximize $c(J)$ over $J \in \mathfrak{J}_1 \cap \mathfrak{J}_2$

always has an optimum solution J such that x^J is an optimum solution to (3.3.5) and (3.3.5) always has an optimum solution x^J for some $J \in \mathfrak{J}_1 \cap \mathfrak{J}_2$. Therefore, by (3.3.8) and the Strong L.P. Duality Thoerem,

3.3.11 Theorem If $r_1, r_2 : L_E \to \mathbb{R}$ are the rank functions of matroids $M_1 = (E, \mathfrak{J}_1)$ and $M_2 = (E, \mathfrak{J}_2)$ respectively and $c \in \mathbb{Z}^E$ then

$$\max\{c(J) : J \in \mathfrak{J}_1 \cap \mathfrak{J}_2\}$$

$$= \min\{r_1 \cdot y^1 + r_2 \cdot y^2 : [y^1, y^2] \text{ is a nonnegative integer-valued vector such that } y^1(K_E,e) + y^2(K_E,e) \geq c_e \text{ for all } e \in E\}. \quad □$$

3.3.12 If $c = 1$ then (3.3.8) implies we can always find a (0,1)-valued optimum solution $[y^1, y^2]$ to (3.3.7). Since r_1 is subadditive we can let $\bar{y}^1$ be defined by

$$\bar{y}^1_S \equiv \begin{cases} 1 & \text{if } S = \cup(T \in K_E : y^1_T = 1) \\ 0 & \text{otherwise} \end{cases}$$

for all $S \in K_E$ and $[\bar{y}^1, y^2]$ is also an optimum solution to (3.3.7). Hence we may assume that there is at most one set $S \in K_E$ such that $y^1_S = 1$ and at most one set $T \in K_E$ such that $y^2_T = 1$. Since r_1 and r_2 are non-decreasing, we may assume that $S \cap T = \phi$. Since $r_1(\phi) = r_2(\phi) = 0$, we have

3.3.13 Theorem If $r_1, r_2 : L_E \to \mathbb{R}$ are the rank functions of matroids $M_1 = (E, \mathfrak{I}_1)$ and $M_2 = (E, \mathfrak{I}_2)$ respectively and $r : L_E \to \mathbb{R}$ is the rank function of $M_1 \cap M_2$ then

$$r(E) = \max\{|J| : J \in \mathfrak{I}_1 \cap \mathfrak{I}_2\} = \min\{r_1(S) + r_2(\bar{S}) : S \subseteq E\}. \qquad \square$$

3.3.14 Given a graph $G = (V,E)$ and $S \subseteq E$ let $t(S) \equiv \{t(e) : e \in S\}$ and $h(S) \equiv \{h(e) : e \in S\}$. G is said to be bipartite if there exists a partition $V = V_1 \cup V_2$ such that $t(E) \subseteq V_1$ and $h(E) \subseteq V_2$.

V_1 and V_2 are called the parts of G and are unique if every component of G has at least one edge.

3.3.15 Given a bipartite graph $G = (V,E)$ with parts V_1 and V_2, let $\mathfrak{I}_1 \equiv \{J \subseteq E : |J \cap \delta(v)| \leq 1 \text{ for all } v \in V_1\}$. Clearly $M_1 = (E, \mathfrak{I}_1)$ is an independence system with rank function r_1 where for all $S \subseteq E$, $r_1(S) = |t(S)|$. Therefore $M_1 = (E, \mathfrak{I}_1)$ is a matroid. Similarly, if we let $\mathcal{J}_2 \equiv \{J \subseteq E: |J \cap \delta(\bar{v})| \leq 1$ for all $v \in V_2\}$ then $M_2 = (E, \mathcal{J}_2)$ is a matroid with rank function r_2 where for all $S \subseteq E$, $r_2(S) = |h(S)|$. The independent sets of $M_1 \cap M_2$ are called the <u>matchings</u> of G and $M_1 \cap M_2$ is called the <u>matching independence system</u> of G.

As a particular instance of (3.3.13) we have Konig's Theorem.

3.3.16 <u>Theorem</u> For any bipartite graph $G = (V,E)$ the maximum cardinality of a matching in G is equal to the minimum cardinality of a set $T \subseteq V$ such that every edge of E meets a node of T.

<u>Proof</u> Let the parts of G be V_1 and V_2. Let $M_1 = (E, \mathcal{J}_1)$ and $M_2 = (E, \mathcal{J}_2)$ be the matroids defined in (3.3.15). Then, by (3.3.13),

$$\begin{aligned} &\max\{|J| : J \in \mathcal{J}_1 \cap \mathcal{J}_2\} \\ = \;& \min\{r_1(S) + r_2(\bar{S}) : S \subseteq E\} \\ = \;& \min\{|t(S)| + |h(\bar{S})| : S \subseteq E\} \\ = \;& \min\{|T_1| + |T_2| : T_1 \subseteq V_1, T_2 \subseteq V_2 \text{ and for all } e \in E, \\ & \qquad\qquad t(e) \in T_1 \text{ or } h(e) \in T_2\}. \qquad \square \end{aligned}$$

3.4 <u>Faces of Matroid Polyhedra</u>

3.4.1 The linear system

3.4.2
$$x_e \geq 0 \text{ for all } e \in E$$
$$x(S) \leq r(S) \text{ for all } S \in K_E$$

which defines $P(K_E, r)$ for an independence system $M = (E, \mathcal{J})$ with rank function r is very large and it is reasonable to ask for a minimal linear system defining $P(K_E, r)$. If we assume that $P(K_E, r)$ is of full dimension then, by (2.3.31), characterizing a minimal linear system defining $P(K_E,r)$ is equivalent to characterizing the facets of $P(K_E,r)$.

3.4.3 By (3.2.4), $P(M) \subseteq P(K_E,r)$. Therefore, $\dim(P(M)) \leq \dim(P(K_E,r))$. In fact, $\dim(P(M)) = \dim(P(K_E,r))$.

3.4.4 Suppose for $S,T \subseteq E$ we have $r(S) = r(T) = 0$. Then, since r is nonnegative and subadditive, we have

$$0 \leq r(S \cup T) \leq r(S) + r(T) = 0.$$

Therefore there is a unique maximal set $U \subseteq E$ such that $r(U) = 0$. Call U the <u>kernel</u> of M.

3.4.5 <u>Proposition</u> Let $r:L_E \to \mathbb{R}$ be the rank function of an independence system $M = (E,\mathfrak{F})$. Let $U \subseteq E$ be the kernel of M. Then $\dim(P(M)) = \dim(P(K_E,r)) = |E|-|U|$.

<u>Proof</u> Let $j \in U$. The inequality $x_j \leq 0$ is satisfied by all $x \in P(K_E,r)$ and so $x_j \leq 0$ is in the equality system of the linear system (3.4.2) defining $P(K_E,r)$. The vectors $\{x^{\{j\}}:j \in U\}$ are linearly independent. Therefore, by definition (2.3.10) of dimension, $\dim(P(K_E,r)) \leq |E|-|U|$.

For each $j \notin U$, $x^{\{j\}}$ is a vertex of $P(M)$, by (3.2.7). Therefore,

$$\{x^{\{j\}}:j \notin U\} \cup \{0\}$$

is a set of $|E - U|+1$ affinely independent vectors of $P(M)$. Hence, by (2.3.23), $\dim(P(M)) \geq |E|-|U|$. Hence,

$$|E|-|U| \leq \dim(P(M)) \leq \dim(P(K_E,r)) \leq |E|-|U|$$

and the proposition follows. □

3.4.6 The kernel of M is the empty set if and only if $r(\{e\}) = 1$ for all $e \in E$. Therefore, by (3.4.5), we have

3.4.7 Proposition Let $M = (E, \mathfrak{F})$ be an independence system with rank function r. Then $P(M)$ and $P(K_E,r)$ are of full dimension if and only if $r(\{e\}) = 1$ for all $e \in E$. □

3.4.8 For each $j \in E$ the inequality $x_j \geq 0$ is satisfied by all $x \in P(M)$ and by all $x \in P(K_E,r)$. Therefore,

$$Q_j \equiv \{x \in P(M):x_j = 0\}$$

is a face of $P(M)$ and

$$R_j \equiv \{x \in P(K_E,r):x_j = 0\}$$

is a face of $P(K_E,r)$.

3.4.9 It is easily seen that $Q_j \subseteq R_j$. If $r(\{j\}) = 0$; i.e. j is an element of the kernel of M, then clearly $Q_j = P(M)$ and $R_j = P(K_E,r)$.

3.4.10 Proposition Let $r:L_E \to \mathbb{R}$ be the rank function of an independence system $M = (E, \mathfrak{F})$ and U be the kernel of M. Then for all $j \notin U$, Q_j is a facet of $P(M)$ and R_j is a facet of $P(K_E,r)$.

Proof By definition (2.3.24) of facet and by (3.4.5), Q_j is a facet of $P(M)$ if and only if $\dim(Q_j) = |E|-|U|-1$ and R_j is a facet of $P(K_E,r)$ if and only if $\dim(R_j) = |E|-|U|-1$. The proof that $\dim(Q_j) = \dim(R_j) = |E|-|U|-1$ is essentially the same as the proof that $\dim(P(M)) = \dim(P(K_E,r)) = |E|-|U|$.

For all $g \in U \cup j$, $x_g \leq 0$ is in the equality system of the linear system

$$x_e \geq 0 \text{ for all } e \in E$$
$$x_j \leq 0$$
$$x(S) \leq f(S) \text{ for all } S \in K_E$$

which defines R_j. Therefore, $\dim(R_j) \leq |E|-|U|-1$.

For each $g \notin U \cup j, x^{\{g\}}$ is an element of Q_j. Therefore

$$\{x^{\{g\}}: g \notin U \cup j\} \cup \{0\}$$

is a set of $|E|-|U|$ affinely independent vectors of Q_j. By (2.3.23), $\dim(Q_j) \geq |E|-|U|-1$. Since $Q_j \subseteq R_j$, $\dim(Q_j) \leq \dim(R_j)$. Hence,

$$|E|-|U|-1 \leq \dim(Q_j) \leq \dim(R_j) \leq |E|-|U|-1. \qquad \square$$

3.4.11 For each $j \notin U$ the facet Q_j of $P(M)$ might be called a "trivial facet" of $P(M)$ and there has been a great deal of research directed towards determining the remaining "nontrivial" facets of $P(M)$ (see, for example, [B1], [H2] and [W1]). For the remainder of this section we will be restricting ourselves to the study of $P(M)$, where M is a matroid.

3.4.12 Let $r: L_E \to \mathbb{R}$ be the rank function of an independence system $M = (E, \mathcal{F})$. An r-separation of $T \in K_E$ is a partition F of T into non-empty sets such that $r(T) = \Sigma(r(S): S \in F)$. If T has a nontrivial r-separation then we say that T is r-separable; otherwise, T is r-nonseparable. An r-separation F of T is minimal if each $S \in F$ is r-nonseparable.

3.4.13 Proposition If $r:L_E \to \mathbb{R}$ is the rank function of a matroid then every $T \in K_E$ has a unique minimal r-separation.

Proof (3.4.13) is a special case of (4.4.14), which we prove later. □

3.4.14 Let $T \subseteq E$ is r-closed if for all $e \notin T$ we have $r(T \cup e) > r(T)$. We assert (see (4.4.16)) that if M is a matroid then there is a unique maximal set $S \subseteq E$, called the closure of T, cl(T), such that $T \subseteq S$, and $r(T) = r(S)$. Where F is the unique minimal r-separation of T let $\mu_r(T) \equiv |F|$ and

$$\Delta(T) \equiv |E|+|T|-\mu_r(T)-|cl(T)|.$$

3.4.15 Theorem Let $r:L_E \to \mathbb{R}$ be the rank function of a matroid $M = (E,\mathcal{F})$. Then for all $T \in K_E$, $\dim(P_T) = \Delta(T)$, where $P_T \equiv \{x \in P(M):x(T) = r(T)\}$.

Proof (3.4.15) is a special case of (4.4.17), since $P(M) = P(K_E,r)$. □

3.4.16 Corollary Let $r:L_E \to \mathbb{R}$ be the rank function of a matroid $M = (E,\mathcal{F})$ such that P(M) is of full dimension. Then for all $T \in K_E$, P_T is a facet of P(M) if and only if T is r-closed and r-nonseparable.

Proof Since P(M) is of full dimension, P_T is a facet of P(M) if and only if $\dim(P_T) = |E|-1$. By (3.4.15), $\dim(P_T) = |E|-1$ if and only if $\mu_r(T) = 1$ and $cl(T) = T$; i.e. if and only if T is r-closed and r-nonseparable. □

3.4.17 Let $G = (V,E)$ be a loopless graph and M be the forest matroid of G. Then there is a simple graph theoretic description of $\Delta(T)$ for all $T \in K_E$; namely,

$$\Delta(T) = |E|+|T|-\beta(G(T))-|\cup(\gamma_G(V(G_i)):1 \le i \le k)|,$$

where the components of $G(T)$ are $G_1, G_2, \ldots, G_k$. We first require the following results.

3.4.18 Lemma Let M be the forest matroid of a graph $G = (V,E)$. Let $r:L_E \to \mathbb{R}$ be the rank function of M. Then $\mu_r(E) = \beta(G)$ and if the blocks of G are $G_1, G_2, \ldots, G_{\beta(G)}$ then the minimal r-separation of E is $\{E(G_1), E(G_2), \ldots, E(G_{\beta(G)})\}$.

Proof If the edge e is a loop of G then $\{e\}$ must be a set in the minimal r-separation of E since $r(\{e\}) = 0$. Therefore, we may assume that G is loopless. If G_0 is a component of G then $\{E(G_0), E-E(G_0)\}$ is clearly an r-separation of E. Therefore, we may assume that G is connected.

The proof is by induction on $\beta(G)$. Suppose $\beta(G) = 1$, but for some $T \in K_E-\{E\}$ we have $r(E) = r(T)+r(\bar{T})$. Let $G_1, G_2, \ldots, G_k$ be the components of $G(T)$ and $H_1, H_2, \ldots, H_m$ be those of $G(\bar{T})$. Since $\beta(G) = 1$, each of $V(G_1), V(G_2), \ldots, V(G_k)$ must contain at least two nodes of $W \equiv V(G(T)) \cap V(G(\bar{T}))$. Because $V(G_i) \cap V(G_j) = \phi$ for $i \neq j$, $|W| \ge 2k$. Similarly, $|W| \ge 2m$ and so $|W| \ge k+m$. However, by (3.1.22),

$$\begin{aligned} W &= |V(G(T)) \cap V(G(\bar{T}))| \\ &= |V(G(T))| + |V(G(\bar{T}))| - |V| \\ &= r(T)+k+r(\bar{T})+m-r(E)-1 \\ &= k+m-1; \end{aligned}$$

a contradiction. Therefore $\mu_r(E) = \beta(G)$ and E is r-nonseparable.

If $\beta(G) \geq 2$ then assume as our induction hypothesis that the theorem holds for all graphs with less than $\beta(G)$ blocks. It is easy to show that

3.4.19 If $\beta(G) \geq 2$ then for some block, say G_1, of G the graph $G' \equiv G(E-E(G_1))$ is connected and $|V(G_1) \cap V(G')| = 1$. Furthermore, the blocks of G' are $G_2, G_3, \ldots, G_{\beta(G)}$.

By our induction hypothesis, $\{E(G_2), E(G_3), \ldots, E(G_{\beta(G)})\}$ is the minimal r-separation of $E(G')$. By (3.1.22),

$$r(E) = |V|-1 = |V(G_1)|+|V(G')|-2 = r(E(G_1))+r(E(G')).$$

Therefore, $\{E(G_1), E(G')\}$ is an r-separation of $E(G)$. Since $\beta(G_1) = 1$, $E(G_1)$ is r-nonseparable. Hence $\{E(G_1), E(G_2), \ldots, E(G_{\beta(G)})\}$ is the minimal r-separation of E and the theorem holds by induction. □

3.4.20 <u>Lemma</u> Let $r: L_E \to \mathbb{R}$ be the rank function of the forest matroid $M(G)$ of a loopless graph $G = (V,E)$. For all $T \in K_E$, if the components of $G(T)$ are $G_1, G_2, \ldots, G_k$ then

$$cl(T) = \cup(\gamma_G(V(G_i)) : 1 \leq i \leq k).$$

<u>Proof</u> Let $e \notin T$. If $h(e) \notin V(G_i)$ for $i = 1,2,\ldots,k$ then, by (3.1.22), $r(T \cup e) = r(T)+1$. Similarly, if $t(e) \notin V(G_i)$ for $i = 1,2,\ldots,k$ then $r(T \cup e) = r(T)+1$ and if $t(e) \in V(G_i)$, $h(e) \in V(G_j)$ for some $i \neq j$ then $r(T \cup e) = r(T)+1$. If $t(e),h(e) \in V(G_i)$ for some $i = 1,2,\ldots,k$ then, by (3.1.22), $r(T \cup e) = r(T)$. Hence, $e \in cl(T)$ if and only if $e \in \gamma_G(G_i)$ for some $i = 1,2,\ldots,k$ and the lemma follows. □

By (3.4.18) and (3.4.20) we have

3.4.21 <u>Theorem</u> If $r:L_E \to \mathbb{R}$ is the rank function of the forest matroid of a loopless graph $G = (V,E)$, $T \in K_E$ and the components of $G(T)$ are $G_1,G_2,\ldots,G_k$ then

$$\Delta(T) = |E|+|T|-|\cup(\gamma_G(V(G_i)):1 \le i \le k)|-\beta(G(T)).$$

T is r-closed and r-nonseparable if and only if $T = \gamma(S)$ for some $S \subseteq V$ such that $\beta(G[S]) = 1$. □

3.4.22 Let $r:L_E \to \mathbb{R}$ be the rank function of a matroid $M = (E,\mathcal{J})$ and consider the face $P_E = \{x \in P(M):x(E) = r(E)\}$ of $P(M)$. If $x^0 \in \mathbb{R}^E$ is a vertex of P_E then x^0 is a vertex of $P(M)$ such that $x^0(E) = r(E)$. x^0 must be the vector of a basis of E. For each $T \in K_E$ let

$$H_T \equiv \{x \in P_E:x(T) = r(T)\}.$$

Then H_T is a face of P_E and we can determine $\dim(H_T)$ in terms of r.

3.4.23 For any matroid $M = (E,\mathcal{J})$ and $T \subseteq E$ let B be a basis of $\bar{T}$. Let $\mathcal{J}' \equiv \{J \subseteq T:J \cup B \in \mathcal{J}\}$. Clearly $M \times T \equiv (E,\mathcal{J}')$ is an independence system on T. $M \times T$ is called the <u>contraction of M to T</u>.

3.4.24 Proposition Let $M = (E, \mathcal{I})$ be a matroid with rank function r. For all $T \subset E$, $M \times T$ is a matroid and $\mathcal{I}'$ is independent of the choice of basis B of $\bar{T}$. Furthermore, for all $S \subseteq T$ the rank of S in $M \times T$, $(r \times T)(S)$ is equal to $r(S \cup \bar{T}) - r(\bar{T})$.

Proof Suppose that for some $J \subseteq T$ and two basis B and B' of $\bar{T}$ we have $J \cup B \in \mathcal{I}'$ but $J \cup B' \notin \mathcal{I}'$. Then $J \cup B$ is a basis of $J \cup \bar{T}$ but $J \cup B'$ properly contains a maximal independent subset of $J \cup \bar{T}$, which is impossible. Therefore $\mathcal{I}'$ is independent of the choice of B.

Let $S \subseteq T$ and let J be a maximal $(M \times T)$-independent subset of S. Then for any basis B of $\bar{T}$, $J \cup B \in \mathcal{I}$. Furthermore, $J \cup B$ must be a basis of $S \cup \bar{T}$. Therefore $|J| = |J \cup B| - |B| = r(S \cup \bar{T}) - r(\bar{T})$ and all maximal $(M \times T)$-independent subsets of S have the same cardinality. Hence the proposition holds. □

Contractions of matroids can be used to describe the dimension of H_T for all $T \in K_E - \{E\}$.

3.4.25 Theorem Let $r : L_E \to \mathbb{R}$ be the rank function of a matroid $M = (E, \mathcal{I})$. Then for all $T \in K_E - \{E\}$,

$$\dim(\mathbb{H}_T) = |E| - \mu_r(T) - \mu_{r \times \bar{T}}(\bar{T}).$$

Proof (3.4.25) is a special case (4.4.24). □

3.4.26 If M is the forest matroid of a graph $G = (V, E)$ then for all $T \subseteq E$ there does exist a graph such that $M \times T$ is the forest matroid of that graph. For $T \subseteq E$ let $G \times T$, G contracted to T, be obtained from G by successively deleting the edges of $\bar{T}$ and identifying their heads and tails. It is easy to verify that the order in which the elements of $\bar{T}$ are deleted does not alter $G \times T$.

3.4.27 Proposition Let $G = (V,E)$ be a graph and $T \subseteq E$. Then $M(G) \times T = M(G \times T)$.

Proof It is sufficient to prove the result when $T = E-e$ for $e \in E$. Let $S \subseteq T$. It is easy to verify that, where $r:L_E \to \mathbb{R}$ is the rank function of $M(G)$ and $r':L_T \to \mathbb{R}$ is the rank function of $M(G \times T)$, we have, by (3.1.22),

$$\begin{aligned} r'(S) &= |V(G \times T(S))|-p_0(G \times T(S)) \\ &= |V(G(S \cup e))|-p_0(G(S \cup e))-[|V(G(e))|-p_0(G(e))] \\ &= r(S \cup e)-r(e). \end{aligned}$$

The proposition now follows from (3.4.24). □

By (3.4.18), (3.4.25) and (3.4.27) we have

3.4.28 Theorem Let $r:L_E \to \mathbb{R}$ be the rank function of the forest matroid of a graph $G = (V,E)$. Then for all $T \in K_E-\{E\}$,

$$\dim(H_T) = |E|-\beta(G(T))-\beta(G \times \bar{T}). \qquad \square$$

3.5 Faces of Matroid Intersection Polyhedra

3.5.1 Let $M_1 = (E, \mathfrak{F}_1)$ and $M_2 = (E, \mathfrak{F}_2)$ be two matroids on E with rank functions r_1 and r_2 respectively of L_E. In section 3.3 we stated that where $r:L_E \to \mathbb{R}$ is the rank function of $M = M_1 \cap M_2$ we have $P(M) = P(K_E,r)$ (see (3.3.2)). A linear system defining $P(M)$ is

3.5.2 $\quad x_e \geq 0$ for all $e \in E$

$\qquad x(S) \leq r(S)$ for all $S \in K_E$.

As in the previous section, we can characterize the sets $T \in K_E$ such that $P_T = \{x \in P(M): x(T) = r(T)\}$ is a facet of $P(M)$.

3.5.3 $\quad$ <u>Theorem</u> Let $r: L_E \to \mathbb{R}$ be the rank function of $M = M_1 \cap M_2$, where $M_1 = (E, \mathfrak{I}_1)$ and $M_2 = (E, \mathfrak{I}_2)$ are matroids on E. Let $P(M)$ be of full dimension. Then for all $T \in K_E$, P_T is a facet of $P(M)$ if and only if T is r-closed and r-nonseparable.

<u>Proof</u> $\quad$ Since $P(M) = P(K_E, r)$, (3.5.3) is a special case of (4.5.5). $\square$

3.5.4 $\quad$ As an example of (3.5.3) consider the matching independence system $M = (E, \mathfrak{I})$ of a bipartite graph $G = (V,E)$ with parts V_1 and V_2. If $r: L_E \to \mathbb{R}$ is the rank function of M then, by (3.3.16), for some $T_1 \subseteq V_1$ and $T_2 \subseteq V_2$ such that $E = \delta(T_1) \cup \delta(\bar{T}_2)$ we have

$$\begin{aligned} |T_1|+|T_2| &= r(E) \\ &\leq \Sigma(r(\delta(v)):v \in T_1)+\Sigma(r(\delta(\bar{v})):v \in T_2) \\ &\leq |T_1|+|T_2|. \end{aligned}$$

Therefore, if $T \in K_E$ is r-nonseparable then $T = \delta(v)$ for some $v \in V_1$ or $T = \delta(\bar{v})$ for some $v \in V_2$. Let

$$W_1 \equiv \{v \in V_1 : |\delta(v)| \geq 2 \text{ or } |\delta(V-h(\delta(v)))| = 1\}$$

and

$$W_2 \equiv \{v \in V_2 : |\delta(\bar{v})| \geq 2 \text{ or } |\delta(t(\delta(\bar{v}))| = 1\}.$$

3.5.5 Theorem Let $r:L_E \to \mathbb{R}$ be the rank function of the matching independence system of a bipartite graph $G = (V,E)$ with parts V_1 and V_2. Then $T \in K_E$ is r-closed and r-nonseparable if and only if $T = \delta(v)$ for some $v \in W_1$ or $T = \delta(\bar{v})$ for some $v \in W_2$.

Proof Suppose $T = \delta(v)$ for $v \in W_1$. Since $r(T) = 1$, T is r-nonseparable. For any edge $e \notin T$, $t(e) \neq v$ and for some edge $j \in T$, $h(e) \neq h(j)$. Therefore, $\{e,j\}$ is a matching of G and $r(T \cup e) = r(T)+1$. Therefore, T is r-closed. Similarly, if $T = \delta(\bar{v})$ for some $v \in W_2$ then T is r-closed and r-nonseparable.

Conversely, suppose T is r-closed and r-nonseparable. Then $T = \delta(v)$ for some $v \in V_1$ or $T = \delta(\bar{v})$ for some $v \in V_2$. If $T = \delta(v)$ for some $v \in V_1 - W_1$ then $T \subset \delta(\overline{h(T)})$ and t is not r-closed; a contradiction. Similarly, if $T = \delta(\bar{v})$ for some $v \in V_2$ then $v \in W_2$. □

By (3.5.3),

3.5.6 Corollary Let $r:L_E \to \mathbb{R}$ be the rank function of the matching independence system of a bipartite graph $G = (V,E)$ with parts V_1 and V_2. Then for all $T \in K_E$, P_T is a facet of P(M), if and only if $T = \delta(v)$ for some $v \in W_1$ or $T = \delta(\bar{v})$ for some $v \in W_2$. □

We remark that (3.5.6) is a very simple instance of a more general theory of matching polyhedra for arbitrary graphs and we refer the reader to Pulleyblank [P1].

3.6 Branchings

3.6.1 For any graph $G = (V,E)$ a branching of G is a forest B of G such that for all $v \in V$, $|\delta(\bar{v}) \cap E(B)| \leq 1$. If we let $\mathcal{J} \equiv \{J \subseteq E : J$ is the edge set of a branching of $G\}$ then we can describe the independence system $M = (E,\mathcal{J})$ as the intersection of two matroids $M_1 = (E,\mathcal{J}_1)$ and $M_2 = (E,\mathcal{J}_2)$ on E. In particular, let $M_1 = M(G)$, the forest matroid of G, and $M_2 = (E,\mathcal{J}_2)$, where $\mathcal{J}_2 \equiv \{J \subseteq E : |\delta(\bar{v}) \cap J| \leq 1$ for all $v \in V\}$. As was the case for matching independence systems, M_2 is a matroid. Clearly, $M = M_1 \cap M_2$ and we can apply the results of sections 3.3 and 3.5. M is called the branching independence system of G.

3.6.2 In this section, we develop a graph theoretic description of those sets $T \in K_E$ which are r-closed and r-nonseparable where $r : L_E \to \mathbb{R}$ is the rank function of the branching independence system $M = (E,\mathcal{J})$ of a loopless graph $G = (V,E)$. If G is loopless then, by (3.4.7), P(M) is of full dimension. Therefore, by (3.5.3), characterizing the sets which are r-closed and r-nonseparable is equivalent to characterizing the nontrivial facets of P(M).

3.6.3 If a branching B of G is a tree then, since $|V(B)| = |E(B)|+1$ and $|E(B) \cap \delta(\bar{v})| \leq 1$ for all $v \in V(B)$, there must be exactly one node $r \in V(B)$ such that $E(B) \cap \delta(\bar{r}) = \phi$ and r is called the root of B. B is said to be rooted at r. It is easy to check that

3.6.4 If B is a branching rooted at r then for all $v \in V(B)$ there is a unique directed path in B from r to v.

3.6.5 Lemma For a graph $G = (V,E)$ the following are equivalent:

3.6.6 G is strongly connected (i.e. for all $v,w \in V$ there is a directed path in G from v to w).

3.6.7 For all $S \in K_V - \{V\}$, $\delta(S) \neq \phi$.

3.6.8 For every $r \in V$ there exists a spanning branching of G rooted at r.

Proof (3.6.6) implies (3.6.7). Suppose G is strongly connected. If for some $S \in K_V - \{V\}$ we have $\delta(S) = \phi$ then there can be no directed path in G from a node of S to a node of $\bar{S}$; a contradiction.

(3.6.7) implies (3.6.8). Suppose $\delta(S) \neq \phi$ for all $S \in K_V - \{V\}$ but for some $r \in V$ there is no spanning branching of G rooted at r. Let B be a maximal branching of G rooted at r. Since B is maximal, $\delta(V(B)) = \phi$. But $V(B) \in K_V - \{V\}$; a contradiction.

(3.6.8) implies (3.6.6). Suppose (3.6.8) holds and let $v,w \in V$. If B is a spanning branching of G rooted at v then, by (3.6.4), there is a directed path in G from v to w. Therefore, G is strongly connected. □

3.6.9 Lemma A connected graph $G = (V,E)$ is strongly connected if and only if every block of G is strongly connected.

Proof The proof is by induction on $\beta(G)$, the result obviously being true when $\beta(G) = 1$. Suppose $\beta(G) \geq 2$ and the blocks of G are $G_1, G_2, \ldots, G_{\beta(G)}$. As our induction hypothesis we assume that the lemma is true for all connected graphs with $\beta(G)-1$ blocks. Since G is connected there is, by (3.4.19), a block, say G_1, of G such that $G' \equiv G(E-E(G_1))$ is connected, the blocks of G' are $G_2, G_3, \ldots, G_{\beta(G)}$ and $|V(G_1) \cap V(G')| = 1$. Let $V(G_1) \cap V(G') = \{u\}$.

Suppose G is strongly connected and let v,w be distinct nodes of $V(G')$. If π is a directed path in G from v to w then, since the nodes of π are distinct, we must have $V(\pi) \subseteq V(G')$. Therefore, G' is strongly connected. By the induction hypothesis each of $G_2, G_3, \ldots, G_{\beta(G)}$ is strongly connected. Similarly, G_1 is strongly connected.

Suppose each of $G_1, G_2, \ldots, G_{\beta(G)}$ is strongly connected. By the induction hypothesis G' is strongly connected. For $v \in V(G_1)$ and $w \in V(G')$ let π_1 be a directed path in G_1 from v to u and π_2 be a directed path in G' from u to w. Then (π_1, π_2) is a directed path in G from v to w. Similarly, there is a directed path in G from w to v, so G is strongly connected. □

3.6.10 <u>Lemma</u> Let $r: L_E \to \mathbb{R}$ be the rank function of the branching independence system of a graph $G = (V,E)$. If E is r-nonseparable then either $E = \delta(\bar{v})$ for some $v \in V$ or G is strongly connected and $\beta(G) = 1$.

<u>Proof</u> Suppose E is r-nonseparable, for all $v \in V$, $E \neq \delta(\bar{v})$ and G is not strongly connected. By (3.6.5), there is a set $S \in K_V - \{V\}$ such that $\delta(S) = \phi$. G must be connected, so $\delta(\bar{S}) \neq \phi$. Let $e \in \delta(\bar{S})$ and $T \equiv \delta(\overline{h(e)})$. Since $E \neq \delta(\overline{h(e)})$, $\bar{T} \neq \phi$. Let B_1 be a branching of $G(\bar{T})$. Let $B \equiv (V(B_1) \cup \{t(e), h(e)\}, E(B_1) \cup e)$. We claim that B is a branching of G. Clearly, $|E(B) \cap \delta(\bar{v})| \leq 1$ for all $v \in V$. If B contains a polygon Q then because $|E(Q) \cap \delta(\bar{v})| \leq 1$ for all $v \in V(Q)$, Q must be a directed polygon.

We must also have $e \in E(Q)$. But then there is a directed path in G from $h(e)$ to $t(e)$, which is impossible since $h(e) \in S$ and $t(e) \in \bar{S}$. In particular, where $E(B_1)$ is a basis of $\bar{T}$ we have $r(E) \geq |E(B)| = r(T)+r(\bar{T})$. Thus E is r-separable; a contradiction.

Hence, if E is r-nonseparable then either $E = \delta(\bar{v})$ for some $v \in V$ or G is strongly connected. Suppose G is strongly connected. Let the blocks of G be $G_1, G_2, \ldots, G_{\beta(G)}$ and B be a spanning branching of G. For each $i = 1,2,\ldots,\beta(G)$, $B_i \equiv (V(B) \cap V(G_i), E(B) \cap E(G_i))$ is a branching. Moreover, $|E(B_i)| = |V(G_i)| - 1$, so B_i is a spanning branching of G_i for $i = 1,2,\ldots,\beta(G)$. Since $E(G_i) \cap E(G_j) = \phi$ for $i \neq j$ we have

$$
\begin{aligned}
r(E) &= |E(B)| \\
&= \Sigma(|E(B_i)| : 1 \leq i \leq \beta(G)) \\
&= \Sigma(r(E(G_i)) : 1 \leq i \leq \beta(G))
\end{aligned}
$$

If $\beta(B) \geq 2$ then E is r-separable. The lemma follows. □

3.6.11 Given a graph $G = (V,E)$ let

$$U \equiv \{v \in V : \text{ if } |t(\delta(\bar{v}))| = 1 \text{ then for all } e \in E \text{ we have } t(e) \neq v \text{ or } h(e) \neq t(\delta(\bar{v}))\},$$

$$W \equiv \{S \subseteq K_V : |S| \geq 2,\ G[S] \text{ is strongly connected and } \beta(G[S]) = 1\}.$$

3.6.12 <u>Theorem</u> Let $r : L_E \to \mathbb{R}$ be the rank function of the branching independence system of a loopless graph $G = (V,E)$ and let $T \in K_E$. Then T is r-closed and r-nonseparable if and only if $T = \delta(\bar{v})$ for some $v \in U$ or $T = \gamma(S)$ for some $S \in W$.

Proof Suppose $T \in K_E$ is r-closed and r-nonseparable. Clearly $G(T)$ must be connected. If $T \subseteq \delta(\bar{v})$ for some $v \in V$ then, since $r(T) = r(\delta(\bar{v})) = 1$ and T is r-closed, $T = \delta(\bar{v})$. If $|t(T)| = 1$ and for some $e \in E$ we have $t(e) = v$ and $h(e) = t(T)$. Then $r(T \cup e) = 1$ and T is not r-closed; a contradiction.

If $T \neq \delta(\bar{v})$ for any $v \in U$ then, by (3.6.10), $G(T)$ is strongly connected and $\beta(G(T)) = 1$. Hence, by (3.6.5), $r(T) = |V(G(T))|-1$. If $e \in \gamma(V(G(T)))-T$ then $r(T \cup e) \leq |V(G(T))|-1 = r(T)$. Thus $r(T \cup e) = r(T)$; a contradiction. Therefore $T = \gamma(V(G(T)))$ and $V(G(T)) \in W$.

Conversely, suppose $T = \delta(\bar{v})$ for some $v \in U$. Then $r(T) = 1$ and clearly T is r-nonseparable. Let $j \in \bar{T}$. If $|t(T)| \geq 2$ then for some $e \in T$, $t(e) \neq h(j)$ and $\{e,j\}$ is the edge-set of a branching. If $|t(T)| = 1$ then since $v \in U$, for any $e \in T$ either $t(e) \neq h(j)$ or $h(e) \neq t(j)$ and $\{e,j\}$ is the edge-set of a branching. Therefore, $r(T \cup j) \geq 2$ for all $j \in \bar{T}$ and T is r-closed.

Suppose $T = \gamma(S)$ for some $S \in W$. Let $j \in \bar{T}$. If $h(j) \notin S$ then for any branching B of $G(T)$, $E(B) \cup j$ is the edge set of a branching of G so $r(T \cup j) = r(T)+1$. If $h(j) \in S$ then $t(j) \notin S$. Since $G[S]$ is strongly connected there is a spanning branching B of $G[S]$ rooted at $h(j)$ and $E(B) \cup j$ is again the edge set of a branching of G and $r(T \cup j) = r(T)+1$. Therefore, T is r-closed.

Suppose that for some $R \in K_T-\{T\}$ we have $r(T) = r(R)+r(T-R)$. Let the components of $G(R)$ be $G_1,G_2,\ldots,G_k$ and those of $G(T-R)$ be $H_1,H_2,\ldots,H_m$. Since $\beta(G(T)) = 1$, each of $V(G_1),V(G_2),\ldots,G(G_k)$ must contain at least two nodes of $X \equiv V(G(R)) \cap V(G(T-R))$. Because $V(G_i) \cap V(G_j) = \phi$ for $i \neq j$, $|X| \geq 2k$. Similarly, $|X| \geq 2m$ and so

3.6.13 $\quad |V(G(R)) \cap V(G(T-R))| \geq k+m.$

We claim that every component of $G(R)$ is strongly connected. To see this, suppose that G_1 is not strongly connected. Then there exists a nonempty subset V_1 of $V(G_1)$ such that $V(G_1)-V_1 \neq \phi$ and $\delta_{G_1}(V_1) = \phi$. Let $e \in \delta_{G_1}(V(G_1)-V_1)$ and B be a spanning branching of $G[S]$ rooted at $h(e)$. We claim that $J \equiv (E(B) \cap E(G_1)) \cup e$ is the edge set of a branching of G_1. Clearly, $|J \cap \delta_{G_1}(\bar{v})| \leq 1$ for all $v \in V(G_1)$. If J contains the edge set of a directed polygon Q then $e \in E(Q)$. But then there is a directed path in G_1 from $h(e)$ to $t(e)$ which is impossible since $h(e) \in V_1$ and $t(e) \in V(G_1)-V_1$. Hence, J is the edge set of a branching of G_1 and so

$$\begin{aligned} r(T) &= |E(B)| \\ &= |E(B) \cap R| + |E(B) \cap (T-R)| \\ &= \Sigma(|E(B) \cap E(G_i)| : 1 \leq i \leq k) + |E(B) \cap (T-R)| \\ &< \Sigma(r(E(G_i) : 1 \leq i \leq k) + r(T-R) \\ &= r(R) + r(T-R); \end{aligned}$$

a contradiction. Similarly, every component of $G(T-R)$ is strongly connected. This means that $r(R) = |V(G(R))|-k$ and $r(T-R) = |V(G(T-R))|-m$. Hence,

$$\begin{aligned} |V(G(R)) \cap V(G(T-R))| &= |V(G(R))| + |V(G(T-R))| - |V(G(T))| \\ &= r(R)+k+r(T-R)+m-r(T)-1 \\ &= k+m-1. \end{aligned}$$

But this contradicts (3.6.13) and so T is r-nonseparable. □

By (3.5.5) we have

3.6.14 <u>Corollary</u> Let $r: L_E \to \mathbb{R}$ be the rank function of the branching independence system M of a loopless graph $G = (V,E)$. Let $T \in K_E$. Then P_T is a facet of $P(M)$ if and only if $T = \delta(\bar{v})$ for some $v \in U$ or $T = \gamma(S)$ for some $S \in W$. □

CHAPTER 4

POLYMATROIDS

In this chapter we will generalize many of the results of Chapter 3 and prove those of Chapter 3 which are referred to this chapter.

4.1 Polymatroids and Submodular Functions

4.1.1 We can translate the definitions of independence system and matroid into the language of vectors. A polyideal P is a compact subset of $\mathbb{R}_+^E$ such that if $x^1 \in P$ and $0 \le x^0 \le x^1$ then $x^0 \in P$. For all vectors $a \in \mathbb{R}_+^E$ the rank of a, $r(a)$, is the maximum of $x(E)$ over $x \in P$, $x \le a$. A vector $x \in P$, $x \le a$, which maximizes $x(E)$ is called a P-basis of a.

4.1.2 A polymatroid is a polyideal $P \subseteq \mathbb{R}_+^E$ such that for all $a \in \mathbb{R}_+^E$ every maximal vector $x \in P$ such that $x \le a$ is a P-basis of a. In other words, for all $a \in \mathbb{R}_+^E$ and for any $x^0 \in P$ such that $x^0 \le a$ there exists a P-basis x^1 of a such that $x^0 \le x^1$ and we say that x^0 can be extended to a P-basis x^1 of a. A polymatroid $P \subseteq \mathbb{R}_+^E$ is said to be integral if for all $a \in \mathbb{Z}_+^E$ every maximal integer-valued vector $x \in P$ such that $x \le a$ is a P-basis of a.

4.1.3 We have the following general construction of polymatroids. Recall that for any lattice L a function $f:L \to \mathbb{R}$ is a σ-function if $f(a) \ge 0$ for all $a \in L^0$ and f is submodular on L^0. Also recall that $P(L^0,f) = \{x \in \mathbb{R}_+^E : x(S) \le f(S) \text{ for all } S \in L^0\}$.

4.1.4 Theorem Let L be a closure system on E and $f:L \to \mathbb{R}$ be a σ-function. Then $P(L^0,f)$ is a polymatroid. For all $a \in \mathbb{R}_+^E$, 4.1.5 $\quad r(a) = \min\{f\cdot y + a\cdot z:[y,z]$ is a $(0,1)$-vector satisfying

$$y(L^0,e) + z_e \geq 1 \text{ for all } e \in E\},$$

where $y(L^0,e) \equiv \Sigma(y_S:e \in S \in L^0)$. If $f(S)$ is an integer for all $S \in L^0$ then $P(L^0,f)$ is an integral polymatroid.

To prove (4.1.4) we first require a lemma.

4.1.6 $\quad$ Lemma Let L be a closure system on E and $f:L \to \mathbb{R}$ be a σ-function. If $Y,Z \in L$ are such that $Y \wedge Z \neq \phi$ and $x \in P(L^0,f)$ is such that $x(Y) = f(Y)$ and $x(Z) = f(Z)$ then $x(Y \wedge Z) = f(Y \wedge Z)$ and $x(Y \vee Z) = f(Y \vee Z)$.

Proof $\quad$ We have
$$\begin{aligned} x(Y \wedge Z) + x(Y \vee Z) &\leq f(Y \wedge Z) + f(Y \vee Z) \\ &\leq f(Y) + f(Z) \\ &= x(Y) + x(Z) \\ &= x(Y \cap Z) + x(Y \cup Z) \\ &\leq x(Y \wedge Z) + x(Y \vee Z). \end{aligned}$$

Therefore, the above inequalities must be equations and the lemma follows.

□

Proof of (4.1.4) $\quad$ Since $P(L^0,f)$ is a polyhedron and $E \in L^0$, $P(L^0,f)$ is closed and bounded. Clearly $P(L^0,f)$ is a polyideal. Let $a \in \mathbb{R}^E_+$ and let $x^0 \in \mathbb{R}^E$ be a maximal vector of $P(L^0,f)$ such that $x^0 \leq a$. Let $E^0 \equiv \{e \in E:x^0_e < a_e\}$. For each $e \in E^0$ there exists a set $S_e \in L^0$ such that $x^0(S_e) = f(S_e)$ (otherwise, x^0 would not be a maximal vector of $P(L^0,f)$ less than or equal to a). If we assume that for each $e \in E^0, J_e$ is a maximal such set then, by (4.1.6), the family of distinct sets of $\{J_e:e \in E^0\}$, call it F, is a family of pairwise disjoint sets and $E^0 \subseteq \cup(S:S \in F) \equiv J$. Because x^0 is maximal, $x^0_e = a_e$ for all $e \notin J$.

$r(a)$ is the optimum value of the linear program:

4.1.7 maximize $1 \cdot x$ where $x \in \mathbb{R}^E$ satisfies

$$0 \le x_e \le a_e \text{ for all } e \in E$$

$$x(S) \le f(S) \text{ for all } S \in L^0.$$

The dual linear program of (4.1.7) is

4.1.8 minimize $f \cdot y + a \cdot z$ where $y \in \mathbb{R}^{L^0}$ and $z \in \mathbb{R}^E$ satisfy

$$y_S \ge 0 \text{ for all } S \in L^0$$

$$z_e \ge 0 \text{ for all } e \in E$$

$$y(L^0, e) + z_e \ge 1 \text{ for all } e \in E.$$

Let $[y^0, z^0]$ be defined by

$$y_S^0 \equiv \begin{cases} 1 \text{ for all } S \in F \\ 0 \text{ for all } S \in L^0 - F \end{cases}$$

$$z_e^0 \equiv \begin{cases} 1 \text{ for all } e \notin J \\ 0 \text{ for all } e \in J. \end{cases}$$

$[y^0, z^0]$ is a feasible solution to (4.1.8) and, since F is a partition of J,

$$\begin{aligned} x^0(E) &= \Sigma(x^0(S) : S \in F) + \Sigma(x_e^0 : e \notin J) \\ &= \Sigma(f(S) : S \in F) + a(\bar{J}) \\ &= f \cdot y^0 + a \cdot z^0. \end{aligned}$$

Therefore, by Corollary (2.4.14) to the Weak L.P. Duality Theorem, x^0 is an optimum solution to (4.1.7), i.e. a P-basis of a, and $[y^0, z^0]$ is an optimum solution to (4.1.8). Thus (4.1.5) holds.

Suppose $f(S)$ is an integer for all $S \in L^0$. Let $a \in \mathbb{Z}_+^E$ and x^0 be a maximal integer-valued vector of $P(L^0,f)$ such that $x^0 \leq a$. Let $E^0 \equiv \{e \in E: x_e^0 < a_e\}$. Since x^0 and a are integer-valued, for each $e \in E^0$ there exists a set $S_e \in L^0$ such that $x^0(S_e) = f(S)$. Now precisely the same argument as above holds and $x^0(E) = r(a)$. □

4.1.9 Let $P \subseteq \mathbb{R}_+^E$ be an integral polymatroid. Let $a \in \mathbb{R}_+^E$ be a $(0,1)$-vector, i.e. the vector of some set $S \subseteq E$. Then, by definition, every maximal $(0,1)$-vector x of P such that $x \leq a$ has the same sum $x(E) = r(a)$. Hence, if we let $\mathfrak{F} \equiv \{J \subseteq E: x^J \in P\}$ then $M = (E,\mathfrak{F})$ is a matroid and for all $T \subseteq E$ the rank of T in M is equal to the rank of x^T in P.

4.1.10 Let L be a closure system on E and $f:L \to \mathbb{R}$ be a σ-function such that $f(S)$ is an integer for all $S \in L^0$. By (4.1.4), $P(L^0,f)$ is an integral polymatroid. By (4.1.9), $\mathfrak{F} \equiv \{J \subseteq E: x^J \in P(L^0,f)\}$ is the family of independent sets of a matroid $M = (E,\mathfrak{F})$. But, by definition of $P(L^0,f)$, $x^J \in P(L^0,f)$ if and only if for all $S \in L^0$ we have $x^J(S) = |J \cap S| \leq f(S)$. Therefore, $\mathfrak{F} = \{J \subseteq E: |J \cap S| \leq f(S) \text{ for all } S \in L^0\}$. By (4.1.4) and (4.1.9), the rank of $T \subseteq E$ in M is equal to

4.1.11 $$\min\{f\cdot y + x^J\cdot z : [y,z] \text{ is a } (0,1)\text{-vector satisfying } y(L^0,e) + z_e \geq 1 \text{ for all } e \in E\}.$$

We can always choose an optimum solution $[y^0,z^0]$ to (4.1.11) so that $z_e^0 = 0$ for all $e \in E$ such that $y^0(L^0,e) \geq 1$. Then $x^J\cdot z^0$ is equal to $|T - \cup(S:S \in F)|$, where $F \equiv \{S \in L^0 : y_S^0 = 1\}$. Therefore, the rank of $T \subseteq E$ in M is equal to $\min\{\Sigma(f(S):S \in F) + |T - \cup(S:S \in F)| : F \subseteq L^0\}$. This is precisely (3.1.11).

4.1.12 As a converse to (4.1.4) we will show how any polymatroid $P \subseteq \mathbb{R}_+^E$ induces a β_0-function $f_P : L_E \to \mathbb{R}$; a β_0-function being a function $f : L_E \to \mathbb{R}$ such that $f(\phi) = 0$, f is nondecreasing on L_E and f is submodular on L_E.

4.1.13 For any two vectors $a, b \in \mathbb{R}^E$ let

$$a \wedge b \equiv [\min\{a_e, b_e\} : e \in E] \in \mathbb{R}^E$$

and

$$a \vee b \equiv [\max\{a_e, b_e\} : e \in E] \in \mathbb{R}^E.$$

Under this meet and join, $\mathbb{R}_+^E$ is a lattice with minimum element 0.

4.1.14 Let $r : \mathbb{R}_+^E \to \mathbb{R}$ be the rank function of a polyideal $P \subseteq \mathbb{R}_+^E$. If $a, b \in \mathbb{R}_+^E$ are such that $a \leq b$ then clearly for any P-basis x of a we have $x \leq b$. Therefore $r(a) = x(E) \leq r(b)$ and r is nondecreasing on $\mathbb{R}_+^E$. Let x be a P-basis of $a \vee b$. Then

$$r(a \vee b) = x(E) \leq (x \wedge a)(E) + (x \wedge b)(E) \leq r(a) + r(b),$$

so $r(a \vee b) \leq r(a) + r(b)$ and r is subadditive on $\mathbb{R}_+^E$.

4.1.15 <u>Lemma</u> If $P \subseteq \mathbb{R}_+^E$ is a polymatroid then for all $a, b \in \mathbb{R}_+^E$, $r(a \wedge b) + r(a \vee b) \leq r(a) + r(b)$, i.e. r is submodular on $\mathbb{R}_+^E$.

<u>Proof</u> Let x^0 be a P-basis of $a \wedge b$ and extend x^0 to a P-basis x^1 of $a \vee b$. Then

$$\begin{aligned} r(a \wedge b) + r(a \vee b) &= x^0(E) + x^1(E) \\ &= (x^1 \wedge (a \wedge b))(E) + (x^1 \wedge (a \vee b))(E) \\ &= (x^1 \wedge a)(E) + (x^1 \wedge b)(E) \\ &\leq r(a) + r(b). \qquad \square \end{aligned}$$

4.1.16 For a polyideal $P \subseteq \mathbb{R}_+^E$ and $S \subseteq E$ let

$$f_P(S) \equiv \max\{x(S): x \in P\}.$$

$x \in P$ is an <u>f_P-basis</u> of S if $x(S) = f_P(S)$ and $x_e = 0$ for all $e \notin S$.
For $a \in R_+^E$ and $S \subseteq E$ let $a|S \in R_+^E$ be the vector with components

$$(a|S)_e \equiv \begin{cases} a_e & \text{if } e \in S \\ 0 & \text{if } e \notin S. \end{cases}$$

4.1.17 Let $P \subseteq \mathbb{R}_+^E$ be a polymatroid and $S \subseteq E$. Let $a \in \mathbb{R}_+^E$ be such that $x < a$ for all $x \in P$. Then $f_P(S) = r(a|S)$ and $x^0 \in P$ is a P-basis of $a|S$ if and only if x^0 is an f_P-basis of S. Therefore, if $x^0 \in P$ is such that $x_e^0 = 0$ for all $e \notin S$ then x^0 can be extended to an f_P-basis x^1 of S.

4.1.18 <u>Theorem</u> Let $P \subseteq \mathbb{R}_+^E$ be a polymatroid. Then $f_P: L_E \to \mathbb{R}$ is a β_0-function and $P = P(K_E, f_P)$.

<u>Proof</u> Clearly f_P is nonnegative and nondecreasing on L_E. By (4.1.15), f_P is submodular on L_E and, since $f_P(\phi) = 0$, f_P must be a β_0-function of L_E.

For all $x \in P$ and $S \subseteq E$, $x(S) \leq f_P(S)$. Therefore, $P \subseteq P(K_E, f_P)$. Suppose there exists some $a \in \mathbb{R}_+^E$ such that $a \in P(K_E, f_P) - P$. Let x^0 be a P-basis of a which maximizes $|E^0|$ where $E^0 \equiv \{e \in E: x_e^0 < a_e\}$. Let $b \equiv \frac{1}{2}(x^0 + a)$. Then x^0 must be a P-basis of b and every P-basis of b is a P-basis of a.

Then

$$x^0(E^0) < b(E^0) < a(E^0) \leq f_P(E^0).$$

Therefore, $x^0|E^0$ is not an f_P-basis of E^0 and we can extend $x^0|E^0$ to an f_P-basis x^1 of E^0. But then $(x^1 \wedge b)(E^0) > (x^0 \wedge b)(E^0)$ and $x^0|E^0$ is not a P-basis of $b|E^0$. Extend $x^0|E^0$ to a P-basis x^2 of $b|E^0$ and then extend x^2 to a P-basis x^3 of b. Then

$$x^3(E^0) = x^2(E^0) > x^0(E^0).$$

Since $x^3(E) = x^0(E)$ there must exist $e \in E-E^0$ such that $x^3_e < x^0_e$. However, because $x^3_e < a_e$, we have a contradiction to the maximality of $|E^0|$. Hence, $P(K_E,f_P) \subseteq P$ and the theorem follows. □

It is not necessarily true that every polyideal is a polyhedron. However, since $P(K_E,f)$ is a polyhedron, we have as a corollary to (4.1.18)

4.1.19 Corollary Every polymatroid is a polyhedron. □

4.1.20 By (4.1.4) and (4.1.18) we see that for every σ-function $f:L \to \mathbb{R}$ of a closure system L on E there exists a β_0-function $f':L_E \to \mathbb{R}$ such that $P(L^0,f) = P(K_E,f')$. Hence, we will usually be restricting ourselves to β_0-functions of L_E.

4.1.21 It is not necessarily true that for any σ-function $f:L_E \to \mathbb{R}$ we have

$$f_{P(K_E,f)}(S) = f(S) \quad \text{for all } S \in K_E,$$

where, by definition (4.1.16),

$$f_{P(K_E,f)}(S) = \max\{x(S): x \in P(K_E,f)\}.$$

For example, let $E = \{d,e\}$, $f(\phi) = 0$, $f(d) = f(e) = 1$ and $f(E) = 3$. Then f is a σ-function of L_E, but $f_{P(K_E,f)}(E) = 2$. However,

4.1.22 <u>Theorem</u> If $f: L_E \to \mathbb{R}$ is a β_0-function then for all $S \subseteq E$ we have

$$f_{P(K_E,f)}(S) = f(S).$$

<u>Proof</u> Let $S \subseteq E$ and $a \in \mathbb{R}_+^E$ be such that $x < a$ for all $x \in P$. Let $b \equiv a|S$. As we observed in (4.1.17), $f_{P(K_E,f)}(S)$ is equal to the rank of b in $P(K_E,f)$. By (4.1.4),

$$f_{P(K_E,f)}(S) =$$

4.1.23 $$\min\{f\cdot y + b\cdot z : [y,z] \text{ is a } (0,1)\text{-vector satisfying } y(K_E,e) + z_e \geq 1 \text{ for all } e \in E\}.$$

Let $[y^0,z^0]$ be an optimum solution to (4.1.23). Let $F \equiv \{S \in K_E : y^0_S = 1\}$. Because f is subadditive,

$$f(\cup(S:S \in F)) \leq \Sigma(f(S):S \in F) = f\cdot y^0.$$

Therefore, we may assume that there is at most one set $T \subseteq E$ such that $y^0_T = 1$. Because f is nondecreasing and $b_e = 0$ for all $e \notin S$ we may assume that $T \subseteq S$ and $z^0_e = 1$ for all $e \notin S$.

Suppose x^0 is an $P(K_E,f)$-basis of b. If $z_e^0 = 1$ for some $e \in S$ then, by the Complementary Slackness Theorem, $x_e^0 = b_e$; a contradiction. Therefore $T = S$ and $f_{P(K_E,f)}(S) = f(S)$. □

Thus there is a one-to-one correspondence between polymatroids contained in $\mathbb{R}_+^E$ and β_0-functions $f:L_E \to \mathbb{R}$.

4.1.24 <u>Corollary</u> If $f_1,f_2:L_E \to \mathbb{R}$ are two β_0-functions and $P(K_E,f_1) = P(K_E,f_2)$ then $f_1(S) = f_2(S)$ for all $S \subseteq E$. □

4.2 <u>The Polymatroid Greedy Algorithm</u>

4.2.1 For any β_0-function $f:L_E \to \mathbb{R}$ and any $c \in \mathbb{R}^E$ consider the linear program

4.2.2 maximize $c \cdot x$ where $x \in \mathbb{R}^E$ satisfies

$$x_e \geq 0 \text{ for all } e \in E$$
$$x(S) \leq f(S) \text{ for all } S \in K_E.$$

Note that when f is the rank function of a matroid, (4.2.2) is the linear program (3.2.14) which we claimed was solved by the Matroid Greedy Algorithm. We claim that the following algorithm is a generalization of the Matroid Greedy Algorithm and that this algorithm solves (4.2.2).

4.2.3 <u>Polymatroid Greedy Algorithm</u> Let the elements of E be ordered $\{e_1,e_2,\ldots\}$ so that $c_{e_1} \geq c_{e_2} \geq \ldots \geq c_{e_k} > 0 \geq c_{e_{k+1}} \geq \ldots$ For each $i \in \{1,2,\ldots,k\}$ let $A_i \equiv \cup(e_j:1 \leq j \leq i)$. Let $x^0 \in \mathbb{R}_+^E$ be defined by

$$x^0_{e_1} \equiv f(A_1)$$

$$x^0_{e_i} \equiv f(A_i)-f(A_{i-1}) \text{ for all } i \in \{2,3,\ldots,k\}$$

$$x^0_{e_i} \equiv 0 \text{ for all } i \geq k+1.$$

4.2.4 We prove that for $i \in \{1,2,\ldots,k\}$ we have $x^0|A_i \in P \equiv P(K_E,f)$ by induction. Clearly $x^0|A_1 \in P$. Now assume $x^0|A_i \in P$ and consider $x^0|A_{i+1}$. Let $S \subseteq E$. If $e_{i+1} \notin S$ then, since $x^0|A_i \in P$,

$$(x^0|A_{i+1})(S) = (x^0|A_i)(S) \leq f(S).$$

If $e_{i+1} \in S$ then

$$\begin{aligned}(x^0|A_{i+1})(S) &= (x^0|A_i)(S) + f(A_{i+1}) - f(A_i)\\ &\leq f(A_i \cap S) + f(A_i \cup S) - f(A_i)\\ &\leq f(S).\end{aligned}$$

Therefore, $x^0|A_{i+1} \in P$ and, by induction, $x^0 = x^0|A_k \in P$ and x^0 is a feasible solution to (4.2.2).

4.2.5 In order to prove that x^0 is an optimum solution to (4.2.2) it is sufficient, by corollary (2.4.14) to the Weak L.P. Duality Theorem, to produce a feasible solution y^0 to the dual linear program of (4.2.2) such that $c \cdot x^0 = f \cdot y^0$, where the dual linear program of (4.2.2) is

4.2.6 minimize $f \cdot y$ where $y \in \mathbb{R}^{K_E}$ satisfies

$$y_S \geq 0 \text{ for all } S \in K_E$$

$$y(K_E,e) \geq c_e \text{ for all } e \in E.$$

4.2.7 <u>Dual Polymatroid Greedy Algorithm</u> Define $y^0 \in \mathbb{R}^{K_E}$ by

$$y^0_{A_i} \equiv c_{e_i} - c_{e_{i+1}} \quad \text{for all } i \in \{1,2,\ldots,k-1\}$$

$$y^0_{A_k} \equiv c_{e_k}$$

$$y^0_S \equiv 0 \text{ for all other } S \in K_E.$$

4.2.8 For all $i \in \{1,2,\ldots,k\}$,

$$y^0(K_E, e_i) = \Sigma(y^0_{A_j} : i \leq j \leq k) = c_{e_i}$$

and for all $i \in \{k+1, k+2, \ldots, |E|\}$, $y^0(K_E, e_i) = 0$. Therefore, y^0 is a feasible solution to (4.2.6). Furthermore,

$$\begin{aligned} f \cdot y^0 &= \Sigma(f(A_i)(c_{e_i} - c_{e_{i+1}}) : 1 \leq i \leq k-1) + f(A_k)c_{e_k} \\ &= \Sigma(f(A_i)c_{e_i} : 1 \leq i \leq k) - \Sigma(f(A_{i-1})c_{e_i} : 2 \leq i \leq k) \\ &= f(A_1) + \Sigma((f(A_i) - f(A_{i-1}))c_{e_i} : 2 \leq i \leq k) \\ &= c \cdot x^0. \end{aligned}$$

Therefore, x^0 and y^0 are indeed optimum primal and dual solutions respectively.

4.2.9 Note that the Dual Polymatroid Greedy Algorithm produces an optimum solution y^0 to (4.2.6) with the property that the family F of sets $S \in K_E$ such that $y^0_S > 0$ is a nested family.

4.2.10 When f is the rank function of a matroid $M = (E, \mathfrak{J})$ then the Polymatroid Greedy Algorithm produces an optimum solution x^0 to (4.2.2) which is (0,1)-valued, i.e. the vector of some set $J \in \mathfrak{J}$. Therefore, the Matroid Greedy Algorithm is a special case of the Polymatroid Greedy Algorithm and does produce an optimum solution to (3.2.11), as we asserted in (3.2.10).

By the construction of x^0 and y^0 given by the Polymatroid and Dual Polymatroid Greedy Algorithms we have proved

4.2.11 <u>Theorem</u> Let $f: L_E \to \mathbb{R}$ be a β_0-function and $c \in \mathbb{R}^E$. If $f(S)$ is an integer for all $S \subseteq E$ then (4.2.2) always has an integer-valued optimum solution. If $c \in \mathbb{Z}^E$ then (4.2.6) always has an integer-valued optimum solution. □

By (2.3.21), for every vertex $x^0 \in P(K_E, f)$ there exists a vector $c \in \mathbb{R}^E$ such that x^0 is the unique optimum solution to (4.2.2). Therefore, by (4.2.11),

4.2.12 <u>Theorem</u> If $f: L_E \to \mathbb{R}$ is a β_0-function such that $f(S)$ is an integer for all $S \subseteq E$ then the vertices of $P(K_E, f)$ are integer-valued. □

4.2.13 In the special case that f is the rank function of a matroid $M = (E, \mathfrak{J})$, (4.2.12) implies that the vertices of $P(K_E, f)$ are (0,1)-vectors, i.e. the vectors of sets $J \in \mathfrak{J}$. This is precisely the statement of (3.2.9).

4.3 Polymatroid Intersection

4.3.1 Given β_0-functions $f_1, f_2 : L_E \to \mathbb{R}$ and $c \in \mathbb{R}^E$ consider the linear program

4.3.2 maximize $c \cdot x$ where $x \in \mathbb{R}^E$ satisfies

$$x_e \geq 0 \text{ for all } e \in E$$

$$\left.\begin{aligned} x(S) &\leq f_1(S) \\ x(S) &\leq f_2(S) \end{aligned}\right\} \text{for all } S \in K_E.$$

The dual linear program of (4.3.2) is

4.3.3 minimize $f_1 \cdot y^1 + f_2 \cdot y^2$ where $y^1, y^2 \in \mathbb{R}^{K_E}$ satisfy

$$y^i_S \geq 0 \text{ for all } S \in K_E,\ i = 1,2$$

$$y^1(K_E, e) + y^2(K_E, e) \geq c_e \text{ for all } e \in E.$$

4.3.4 <u>Theorem</u> For any two β_0-functions $f_1, f_2 : L_E \to \mathbb{R}$ and for all $c \in \mathbb{Z}^E$ the linear program (4.3.3) always has an integer-valued optimum solution.

<u>Proof</u> Let $[y^1, y^2]$ be an optimum solution to (4.3.3) and for $i = 1,2$ and $e \in E$ let $c^i_e \equiv y^i(K_E, e)$. Then, as we observed in (4.2.9), the Dual Polymatroid Greedy Algorithm produces an optimum solution $\bar{y}^i$ to the linear program

$$\text{minimize } f_i \cdot y^i \text{ where } y^i \in \mathbb{R}_+^{K_E} \text{ satisfies}$$

$$y^i(K_E, e) \geq c^i_e \text{ for all } e \in E$$

with the property that the family F^i of sets $S \in K_E$ such that $\bar{y}^i_S > 0$ is a nested family.

Consider the linear program

4.3.5 minimize $f_1 \cdot y^1 + f_2 \cdot y^2$ where $y^1, y^2 \in \mathbb{R}^{K_E}$ satisfy

4.3.6
$$\begin{cases} y^i_S \geq 0 \text{ for all } S \in K_E,\ i = 1,2 \\ y^i_S \leq 0 \text{ for all } S \in K_E - F^i,\ i = 1,2 \\ y^1(F^1,e) + y^2(F^2,e) \geq c_e \text{ for all } e \in E. \end{cases}$$

Any feasible solution to (4.3.5) is a feasible solution to (4.3.3) and, since $[\bar{y}^1, \bar{y}^2]$ is an optimum solution to (4.3.3) and a feasible solution to (4.3.5), every optimum solution to (4.3.5) must be an optimum solution to (4.3.3).

4.3.7 The matrix A with rows $[x^S : S \in F^i,\ i = 1,2]$ is totally unimodular.

Any square submatrix B of A is of the form $[x^S : S \in H^i,\ i = 1,2]$ for some subset E' of E and nested families H^1, H^2 of E'. We can iteratively subtract the row of a minimal set $S \in H^i$ from the rows of other sets of H^i containing S to obtain a matrix C of the form $[x^S : S \in K^i,\ i = 1,2]$ for two families K^1, K^2 of pairwise disjoint subsets of E' such that $\det(B) = \pm\det(C)$. By (2.6.13), $\det(C) \in \{0,1,-1\}$ and so A is totally unimodular.

If we represent the linear system (4.3.6) as $A'x \leq b$ for an appropriate choice of A' and b then, by (2.5.8)-(2.5.11) and (4.3.7), A' is totally unimodular. Therefore, since c is integer-valued, (4.3.5) and hence (4.3.3) has an integer-valued optimum solution by (2.5.16).

□

From (2.5.2) it follows that

4.3.8 <u>Corollary</u> If $f_1, f_2 : L_E \to \mathbb{R}$ are two integer-valued β_0-functions then the vertices of $P(K_E, f_1) \cap P(K_E, f_2)$ are integer-valued. □

4.3.9 If $f_1, f_2 : L_E \to \mathbb{R}$ are the rank functions of two matroids $M_1 = (E, \mathfrak{J}_1)$ and $M_2 = (E, \mathfrak{J}_2)$ then, by (4.3.8), the vertices of $P(K_E, f_1) \cap P(K_E, f_2)$ are integer-valued, i.e. the vectors of sets $J \in \mathfrak{J}_1 \cap \mathfrak{J}_2$. This is the statement of (3.3.3). (3.3.8) is a special case of (4.3.4).

4.3.10 If for two β_0-functions $f_1, f_2 : L_E \to \mathbb{R}$ we let

$$f(S) \equiv \max\{x(S) : x \in P(K_E, f_1) \cap P(K_E, f_2)\}$$

for all $S \subseteq E$ then clearly $P(K_E, f_1) \cap P(K_E, f_2) = P(K_E, f)$. Call f the <u>rank function</u> of $P(K_E, f_1) \cap P(K_E, f_2)$.

4.3.11 <u>Theorem</u> If $f_1, f_2 : L_E \to \mathbb{R}$ are two β-functions and $f : L_E \to \mathbb{R}$ is the rank function of $P(K_E, f_1) \cap P(K_E, f_2)$ then for all $T \subseteq E$,

$$f(T) = \min\{f_1(S) + f_2(T-S) : S \subseteq T\}.$$

<u>Proof</u> By (4.3.4),

$$f(T) = \min\{f_1 \cdot y^1 + f_2 \cdot y^2 : [y^1, y^2] \text{ is a } (0,1)\text{-vector satisfying } y^1(K_E, e) + y^2(K_E, e) \geq 1 \text{ for all } e \in T\}.$$

Since f_1 and f_2 are subadditive we may assume that $y^1_Y = 1$ for at most one $Y \in K_E$ and $y^2_Z = 1$ for at most one $Z \in K_E$. Also, since f_1 and f_2 are nondecreasing, we may assume that $Y \cap Z = \phi$ and $T = Y \cup Z$. Therefore, $Y = T-Z$ and the theorem follows. □

4.3.12 <u>Theorem</u> Let $f_1, f_2 : L_E \to \mathbb{R}$ be two β_0-functions and $f : L_E \to \mathbb{R}$ be the rank function of $P(K_E, f_1) \cap P(K_E, f_2)$. For all $c \in \mathbb{R}^E$ the linear program (4.3.3) has an integer-valued optimum solution if and only if the linear program

4.3.13 minimize $f \cdot y$ where $y \in \mathbb{R}^E$ satisfies

$$y_S \geq 0 \text{ for all } S \in K_E$$

$$y(K_E, e) \geq c_e \text{ for all } e \in E$$

has an integer-valued optimum solution.

<u>Proof</u> The dual linear program of (4.3. 3) is (4.3.2). Therefore, by the Strong L.P. Duality Theorem, the optimum value of (4.3.3) is $\max\{c \cdot x : x \in P(K_E, f_1) \cap P(K_E, f_2)\}$. The dual linear program of (4.3.13) is

4.3.14 maximize $c \cdot x$ where $x \in \mathbb{R}^E$ satisfies

$$x_e \geq 0 \text{ for all } e \in E$$

$$x(S) \leq f(S) \text{ for all } S \in K_E.$$

Again, by the Strong L.P. Duality Theorem, the optimum value of (4.3.13) is $\max\{c \cdot x : x \in P(K_E, f)\}$. Since $P(K_E, f_1) \cap P(K_E, f_2) = P(K_E, f)$, the optimum value of (4.3.3) is equal to the optimum value of (4.3.13).

Let $[\bar{y}^1, \bar{y}^2]$ be an integer-valued optimum solution to (4.3.3). Let $\bar{y} \in \mathbb{R}^{K_E}$ be defined by

$$\bar{y}_T \equiv \bar{y}^1_T + \bar{y}^2_T , \quad \text{for all } T \in K_E.$$

Clearly $\bar{y}$ is an integer-valued feasible solution to (4.3.13). By (4.3.11), $f(T) = \min\{f_1(S) + f_2(T-S) : S \subseteq T\}$ for all $T \subseteq E$. Therefore, $f(T) \leq \min\{f_1(T), f_2(T)\}$ for all $T \subseteq E$ and

$$f\cdot\bar{y} = \Sigma(f(T)[\bar{y}^1_T+\bar{y}^2_T]:T \in K_E)$$
$$\le \Sigma(f_1(T)\bar{y}^1_T + f_2(T)y^2_T:T \in K_E)$$
$$= f_1\cdot\bar{y}^1 + f_2\cdot\bar{y}^2.$$

Therefore, $\bar{y}$ must be an optimum solution to (4.3.13).

Conversely, let $\bar{y}$ be an integer-valued optimum solution to (4.3.13). By (4.3.11), for each $T \in K_E$ we can fix $T_S \subseteq T$ so that $f(T) = f_1(T_S) + f_2(T-T_S)$. Let $[\bar{y}^1,\bar{y}^2]$ be defined by

$$\bar{y}^1_S \equiv \Sigma(\bar{y}_T:S = T_S,\ T \in K_E)$$
$$\bar{y}^2_S \equiv \Sigma(\bar{y}_T:S = T-T_S, T \in K_E)$$

Then $[\bar{y}^1,\bar{y}^2]$ is an integer-valued feasible solution to (4.3.3). Since $f_1\cdot\bar{y}^1 + f_2\cdot\bar{y}^2 = f\cdot\bar{y}$, $[\bar{y}^1,\bar{y}^2]$ is an optimum solution to (4.3.3). □

4.3.15 Note that according to the proof of (4.3.12) if (4.3.3) has an (integer-valued) optimum solution $[\bar{y}^1,\bar{y}^2]$ such that for some $T \in K_E$, $\bar{y}^1_T = \bar{y}^2_T = 0$ then (4.3.13) has an (integer-valued) optimum solution $\bar{y}$ such that $\bar{y}_T = 0$.

4.3.16 Corollary Let $f:L_E \to \mathbb{R}$ be the rank function of $P(K_E,f_1) \cap P(K_E,f_2)$ for two β_0-functions $f_1,f_2:L_E \to \mathbb{R}$. Then for all $c \in \mathbb{Z}^E$ the linear program (4.3.13) has an integer-valued optimum solution.

Proof By (4.3.4), (4.3.3) has an integer-valued optimum solution. Therefore, by (4.3.12), (4.3.13) has an integer-valued optimum solution.

□

4.4 Faces of Polymatroids

4.4.1 For any function $f:L_E \to \mathbb{R}$ consider the polyhedron $P(K_E,f)$. For any $T \in K_E$ let

$$P_T \equiv \{x \in P(K_E,f):x(T) = f(T)\}.$$

and

$$H_T \equiv \{x \in P_E:x(T) = f(T)\}.$$

By definition (2.3.7), P_T and H_T are faces of $P(K_E,f)$.

4.4.2 Suppose f is a β_0-function of L_E. By (4.1.22), for all $T \in K_E$ every f-basis x^0 of T is such that $x^0(T) = f(T)$. Therefore, P_T is nonempty. As we noted in (4.1.17), any f-basis of T can be extended to an f-basis of E. Therefore, H_T is also nonempty. Clearly $H_E = P_E$. In this section we will determine the dimension of the faces P_T and H_T in terms of f. In particular, we determine the facets of $P(K_E,f)$.

4.4.3 Proposition For any function $f:L_E \to \mathbb{R}_+$, $P(K_E,f)$ is of full dimension if and only if $f(S) > 0$ for all $S \in K_E$.

Proof Suppose $f(S) = 0$ for some $S \in K_E$. Then for all $x \in P(K_E,f)$, $x(S) = 0$ and, by definition, $\dim(P(K_E,f)) < |E|$.

Conversely, if $f(S) > 0$ for all $S \in K_E$ then there exists $\alpha > 0$ such that for all $e \in E$ the vector $\alpha x^{\{e\}}$ is an element of $P(K_E,f)$.

Therefore

$$\{\alpha x^{\{e\}}:e \in E\} \cup \{0\}$$

is a set of $|E|+1$ affinely independent vectors of $P(K_E,f)$. Thus, by (2.3.23), $\dim(P(K_E,f)) = |E|$. □

4.4.4 Given a function $f:L_E \to \mathbb{R}_+$ a set $T \in K_E$ is <u>f-separable</u> if there exists a set $S \in K_T-\{T\}$ such that $f(S)+f(T-S) \le f(T)$; otherwise, T is <u>f-nonseparable</u>. T is <u>f-closed</u> if for all $S \subseteq E$ such that $T \subset S$ we have $f(T) < f(S)$.

4.4.5 <u>Proposition</u> Let $P(K_E,f)$ be of full dimension and let $T \in K_E$. If P_T is a facet of $P(K_E,f)$ then T is f-closed and f-nonseparable.

<u>Proof</u> By (2.3.31), P_T is a facet of $P(K_E,f)$ if and only if $x(T) \le f(T)$ is essential for defining $P(K_E,f)$. Suppose T is not f-closed, i.e. there exists $S \subseteq E$ such that $T \subset S$ and $f(T) \ge f(S)$. Then for all $x \in P(K_E,f)$ we have $x(T) \le x(S) \le f(S) \le f(T)$ and $x(T) \le f(T)$ is not essential for defining $P(K_E,f)$.

Suppose T is f-separable, i.e. there exists $S \in K_T - \{T\}$ such that $f(S)+f(T-S) \le f(T)$. Then for all $x \in P(K_E,f)$ we have $x(T) = x(S)+x(T-S) \le f(S)+f(T-S) \le f(T)$ and $x(T) \le f(T)$ is nonessential for defining $P(K_E,f)$. □

4.4.6 Notice that if $f:L_E \to \mathbb{R}_+$ is subadditive and $T \in K_E$ then for all $S \subseteq T$ such that $f(S) + f(T-S) \le f(T)$ we have $f(S) + f(T-S) = f(T)$. If f is nondecreasing and $T \subset S \subseteq E$ are such that $f(S) \le f(T)$ then for all $e \in S-T$, $f(T \cup e) = f(T)$.

In the case that $f:L_E \to \mathbb{R}$ is a β_0-function we can strengthen (4.4.5) to

4.4.7 Theorem Let $f:L_E \to \mathbb{R}$ be a β_0-function such that $P(K_E,f)$ is of full dimension. For all $T \in K_E$, P_T is a facet of $P(K_E,f)$ if and only if T is f-closed and f-nonseparable.

Proof If P_T is a facet of $P(K_E,f)$ then, by (4.4.5), T is f-closed and f-nonseparable. Suppose T is f-closed and f-nonseparable but P_T not a facet of $P(K_E,f)$, i.e. $\dim(P_T) \leq |E|-2$. A linear system defining P_T is

4.4.8 $$\begin{cases} x_e \geq 0 \text{ for all } e \in E \\ x(S) \leq f(S) \text{ for all } S \in K_E \\ x(T) \geq f(T). \end{cases}$$

If $\dim(P_T) \leq |E|-2$ then, by definition (2.3.10) of dimension, there exists $e \in E$ such that the inequality $x_e \geq 0$ is in the equality system of (4.4.8) or there exists $S \in K_E-\{T\}$ such that $x(S) \leq f(S)$ is in the equality system of (4.4.8).

Let x^0 be an f-basis of T. Then, as we observed in (4.4.2), $x^0 \in P_T$. For $e \notin T$ extend x^0 to an f-basis x^1 of $T \cup e$. $x^1 \in P_T$ and, since T is f-closed, $x^1_e > 0$. For $e \in T$ let x^2 be an f-basis of $\{e\}$ and extend x^2 to a f-basis x^3 of T. $x^3 \in P_T$ and, since $P(K_E,f)$ is of full dimension, $x^3_e > 0$. Therefore, for all $e \in E$ there exists $x \in P_T$ such that $x_e > 0$ and so $x_e \geq 0$ cannot be in the equality system of (4.4.8).

Let $S \in K_E-\{T\}$ and suppose $x(S) = f(S)$ for all $x \in P_T$. In particular, $x^0(S) = f(S)$. By (4.1.15), $x^0(S \cap T) = f(S \cap T)$ and

$x^0(S \cup T) = f(S \cup T)$. If $T \subset S \cup T$ then $f(T) = x^0(T) = x^0(S \cup T) = f(S \cup T)$ and T is not f-closed; a contradiction. Therefore, $S \subset T$. Let x^4 be an f-basis of T-S and extend x^4 to an f-basis x^5 of T. Then $x^5 \in P_T$ and, since $x^5(S) = f(S)$,

$$f(T) = x^5(T) = x^5(S) + x^4(T-S) = f(S) + f(T-S).$$

But then T is f-separable; a contradiction. Therefore, for all $S \in K_E-\{T\}$ the inequality $x(S) \leq f(S)$ cannot be in the equality system of (4.4.8) which contradicts our supposition. $\square$

4.4.9 When $P(K_E,f)$ is of full dimension and P_T is a facet of $P(K_E,f)$ we know, by definition (2.3.24) of facet, that $\dim(P_T) = |E|-1$. Therefore, (4.4.7) describes $\dim(P_T)$ whenever $P(K_E,f)$ is of full dimension and T is f-closed and f-nonseparable. We will use (4.4.7) to determine the dimension of the face P_T for any $T \in K_E$.

4.4.10 Given a function $f:L_E \to \mathbb{R}$ an <u>f-separation</u> of $T \in K_E$ is a partition F of T into nonempty sets such that $f(T) = \Sigma(f(S):S \in F)$.

4.4.11 <u>Proposition</u> If $f:L_E \to \mathbb{R}$ is a subadditive function and F is an f-separation of E then for all $F' \subseteq F$, F' is an f-separation of $\cup(S:S \in F')$; i.e. $f(\cup(S:S \in F')) = \Sigma(f(S):S \in F')$.

<u>Proof</u> It is sufficient to prove that for all $T \in F$,

$$f(\cup(S:S \in F-\{T\}) = \Sigma(f(S):S \in F-\{T\}).$$

But

$$
\begin{aligned}
f(\cup(S:S \in F-\{T\})) + f(T) &\geq f(E) \\
&= \Sigma(f(S):S \in F) \\
&= \Sigma(f(S):S \in F-\{T\})+f(T) \\
&\geq f(\cup(S:S \in F-\{T\})+f(T).
\end{aligned}
$$

Hence equality holds everywhere and the lemma follows. □

4.4.12 Proposition Let $f:L_E \to \mathbb{R}$ be a β_0-function. If $f(E) = f(T)+f(\bar{T})$ for some $T \subseteq E$ then for all $S \subseteq E$, $f(S) = f(S \cap T)+f(S \cap \bar{T})$.

Proof We have the following:

$$
\begin{aligned}
f(E) &= f(T) + f(\bar{T}) \\
&\geq f(S \cap T) + f(S \cup T) - f(S) + f(S \cap \bar{T}) + f(S \cup \bar{T}) - f(S) \\
&= [f(S \cap T) + f(S \cap \bar{T}) - f(S)] + f(S \cup T) + f(S \cup \bar{T}) - f(S) \\
&\geq f(S \cup T) + f(S \cup \bar{T}) - f(S) \\
&\geq f(E).
\end{aligned}
$$

Therefore, $f(S) = f(S \cap T) + f(S \cap \bar{T})$. □

4.4.13 For any subadditive function $f:L_E \to \mathbb{R}$ and $T \in K_E$, an f-separation F of T is minimal if each $S \in F$ is f-nonseparable. Suppose F is an f-separation of $T \in K_E$ and for some $R \in F$ there exists $U \in K_R-\{R\}$ such that $f(R) = f(U)+f(R-U)$. Then $(F-\{R\}) \cup \{U,R-U\}$ is an f-separation of T since $f(T) = \Sigma(f(S):S \in F-\{R\})+f(U)+f(R-U)$. Therefore every $T \in K_E$ has a minimal f-separation.

4.4.14 <u>Theorem</u> If $f:L_E \to \mathbb{R}$ is a β_0-function then every $T \in K_E$ has a unique minimal f-separation.

<u>Proof</u> It is sufficient to prove that E has a unique minimal f-separation. Suppose F and H are two distinct f-separations of E. Since F and H are distinct partitions of E there exist $S \in F$ and $T \in H$ such that $S \cap T \neq \phi$ and $S \neq T$. We may assume $S \nsubseteq T$. Then $S \cap \bar{T} \neq \phi$. By (4.4.11), $f(E) = f(T)+f(\bar{T})$. By (4.4.12), $f(S) = f(S \cap T) + f(S \cap \bar{T})$. But then S is f-separable; contradicting the minimality of F. □

4.4.15 <u>Proposition</u> Let $f:L_E \to \mathbb{R}$ be a β_0-function and let $T \subseteq E$ be such that $f(E) = f(T)+f(\bar{T})$. Then for all $x \in \mathbb{R}_+^E$, x is an f-basis of E if and only if $x|T$ is an f-basis of T and $x|\bar{T}$ is an f-basis of $\bar{T}$.

<u>Proof</u> Let x be an f-basis of E. Then

$$f(E) = x(E) = x(T)+f(\bar{T}) \leq f(T)+f(\bar{T}) = f(E).$$

Therefore, $x(T) = f(T)$ and $x(\bar{T}) = f(\bar{T})$. Hence, $x|T$ is an f-basis of T and $x|\bar{T}$ is an f-basis of $\bar{T}$.

Conversely, suppose $x|T$ is an f-basis of T and $x|\bar{T}$ is an f-basis of $\bar{T}$. By (4.4.12), for any $S \subseteq E$ we have

$$\begin{aligned} x(S) &= x(S \cap T)+x(S \cap \bar{T}) \\ &= (x|T)(S \cap T)+(x|\bar{T})(S \cap \bar{T}) \\ &\leq f(S \cap T)+f(S \cap \bar{T}) \\ &= f(S). \end{aligned}$$

Therefore, $x \in P(K_E, f)$. Since

$$x(E) = (x|T)(T) + (x|\bar{T})(\bar{T}) = f(T) + f(\bar{T}) = f(E),$$

x is an f-basis of E. □

4.4.16 Proposition Let $f:L_E \to \mathbb{R}$ be a β_0-function and let $T \subseteq E$. Then there is a unique maximal subset $S \subseteq E$ such that $T \subseteq S$ and $f(T) = f(S)$.

Proof Suppose $Y, Z \subseteq E$ are such that $T \subseteq Y$, $T \subseteq Z$ and $f(T) = f(Y) = f(Z)$. Then

$$\begin{aligned} f(T) &\leq f(Y \cup Z) \\ &\leq f(Y) + f(Z) - f(Y \cap Z) \\ &\leq f(Y) + f(Z) - f(T) \\ &= f(T). \end{aligned}$$

Therefore, $f(T) = f(Y \cup Z)$ and $S \equiv \cup(Y \subseteq E : T \subseteq Y, f(T) = f(Y))$ is the unique maximal set such that $T \subseteq S$ and $f(T) = f(S)$. □

4.4.17 For any β_0-function $f:L_E \to \mathbb{R}$ and any $T \subseteq E$ call the unique maximal subset $S \subseteq E$ such that $T \subseteq S$ and $f(T) = f(S)$ the closure of T, $cl(T)$. By (4.4.14), T has a unique minimal f-separation F. Let $\mu_f(T)$ denote $|F|$, the number of subsets in the minimal f-separation of T. Let

$$\Delta(T) \equiv |E| + |T| - |cl(T)| - \mu_f(T)$$

4.4.18 Theorem Let $f:L_E \to \mathbb{R}$ be a β_0-function. Then for all $T \in K_E$, $\dim(P_T) = \Delta(T)$.

<u>Proof</u> Let F be a minimal f-separation of T. We first show $\dim(P_T) \le \Delta(T)$. Let $x \in P_T$. By (4.4.15), $x(S) = f(S)$ for all $S \in F$. For each $e \in cl(T)-T$, $x(T \cup e) = f(T) = f(T \cup e)$. Now a defining linear system for P_T is

$$4.4.19 \quad \begin{cases} x_e \ge 0 \text{ for all } e \in E \\ x(S) \le f(S) \text{ for all } S \in K_E \\ x(T) \ge f(T). \end{cases}$$

Hence, for each $S \in F$ the inequality $x(S) \le f(S)$ is in the equality system of (4.4.19) and for each $e \in cl(T)-T$ the inequality $x(T \cup e) \le f(T \cup e)$ is in the equality system of (4.4.19). The vectors $\{x^S : S \in F\} \cup \{x^{T\cup e} : e \in cl(T)-T\}$ are linearly independent. Therefore, by definition (2.3.10) of dimension,

$$\dim(P) \le |E|-|F|-|cl(T)-T| = \Delta(T).$$

We next show $\dim(P_T) \ge \Delta(T)$. Let $F' \equiv \{S \in F : f(S) > 0\}$. Then $S \in F-F'$ if and only if $S = \{e\}$ for some $e \in T$ such that $f(\{e\}) = 0$. For all $S \in F'$, $P(K_S,f|S) \subseteq \mathbb{R}^S$ is of full dimension, by (4.4.3). Each $S \in F'$ must be f-nonseparable. Therefore, by (4.4.7), for all $S \in F'$, $\{x \in P(K_S,f|S) : x(S) = f(S)\}$ is a facet of $P(K_S,f|S)$. By (2.3.25) there exist $|S|$ affinely independent vectors

$$\{x^{S,m} \in \mathbb{R}^E : 1 \le m \le |S|\}$$

such that

$$4.4.20 \quad x^{S,m} \text{ is an f-basis of } S \text{ for } 1 \le m \le |S|.$$

Let $x^1 \in \mathbb{R}^E$ be defined by

$$x_e^1 \equiv \begin{cases} x_e^{S,1} & \text{if } e \in S \in F' \\ 0 & \text{otherwise} \end{cases}$$

By (4.4.15), x^1 is an f-basis of T.

For each $S \in F'$ and for $2 \le m \le |S|$ let $\bar{x}^{S,m}$ be defined by

$$\bar{x}_e^{S,m} \equiv \begin{cases} x_e^{U,1} & \text{if } e \in U \in F'-\{S\} \\ x_e^{S,m} & \text{if } e \in S \\ 0 & \text{otherwise} \end{cases} .$$

Again, by (4.4.15), $\bar{x}^{S,m}$ is an f-basis of T.

Clearly x^1 is an f-basis of cl(T). For each $j \notin cl(T)$ extend x^1 to an f-basis x^j of $T \cup j$. Since $f(T \cup j) > f(T)$, we must have $x_j^j > 0$. $x^j(T) = x^1(T) = f(T)$, so $x^j \in P_T$ for all $j \notin cl(T)$. Let

$$K \equiv \{x^1\} \cup \{\bar{x}^{S,m} : S \in F', 2 \le m \le |S|\} \cup \{x^j : j \notin cl(T)\}.$$

By (2.3.27), K is a set of affinely independent vectors contained in P_T. Therefore, by (2.3.23),

$$\begin{aligned} \dim(P_T) &\ge |K|-1 \\ &= \Sigma(|S|-1 : S \in F') + |E-cl(T)| \\ &= \Sigma(|S|-1 : S \in F) + |E| - |cl(T)| \\ &= |E| + |T| - \mu_f(T) - |cl(T)| \\ &= \Delta(T). \end{aligned}$$

□

4.4.21 Let $f:L_E \to \mathbb{R}$ be any function and $T \subseteq E$ be a fixed set. Let the function $f \times T:L_T \to \mathbb{R}$, <u>the contraction of f to T</u>, be defined by

$$(f \times T)(U) \equiv f(\bar{T} \cup U)-f(\bar{T}),$$

for all $U \subseteq T$.

4.4.22 <u>Theorem</u> Let $f:L_E \to \mathbb{R}$ be a β_0-function and let $T \subseteq E$. Then $f \times T:L_T \to \mathbb{R}$ is a β_0-function and for all $x^0 \in \mathbb{R}^T_+$, $x^0 \in P(K_T,f \times T)$ if and only if for all $x^1 \in P((K_{\bar{T}},f|\bar{T})$ the vector $[x^0,x^1]$ is an element of $P(K_E,f)$.

<u>Proof</u> Clearly $(f \times T)(\phi) = 0$. If f is nondecreasing on L_E then $f \times T$ must be nondecreasing on L_T. For all $Y,Z \subseteq T$ we have

$$\begin{aligned}(f \times T)(Y \cap Z)+(f \times T)(Y \cup Z) &= f(\bar{T} \cup (Y \cap Z))-f(\bar{T})+f(\bar{T} \cup (Y \cup Z))-f(\bar{T}) \\ &= f((\bar{T} \cup Y)\cap(\bar{T} \cup Z))+f((\bar{T} \cup Y)\cup(\bar{T} \cup Z))-2f(\bar{T}) \\ &\leq f(\bar{T} \cup Y)+f(\bar{T} \cup Z)-2f(\bar{T}) \\ &= (f \times T)(Y)+ (f \times T)(Z).\end{aligned}$$

Therefore, by (3.1.15), $f \times T$ is a β_0-function of L_T.

Suppose $x^0 \in P(K\ ,f \times T)$ and $x^1 \in P(K_{\bar{T}},f|\bar{T})$. Then for all $S \subseteq E$,

$$\begin{aligned}[x^0,x^1](S) &= x^0(S \cap T)+x^1(S \cap \bar{T}) \\ &\leq (f \times T)(S \cap T)+f(S \cap \bar{T}) \\ &= f((S \cap T) \cup \bar{T})-f(\bar{T})+f(S \cap \bar{T}) \\ &\leq f(S).\end{aligned}$$

Hence $[x^0,x^1] \in P(K_E,f)$.

Conversely, suppose $x^0 \in \mathbb{R}^T$ and for all $x^1 \in P(K_{\bar{T}}, f|\bar{T})$ we have $[x^0, x^1] \in P(K_E, f)$. Then, in particular, when x^1 is a basis of $a|\bar{T}$ we have

$$\begin{aligned} x^0(U) &= [x^0, x^1](\bar{T} \cup U) - x^1(\bar{T}) \\ &\le f(\bar{T} \cup U) - f(\bar{T}) \\ &= (f \times T)(U) \end{aligned}$$

for all $U \in K_T$. Therefore, $x^0 \in P(K_T, f \times T)$ and the theorem is proved.

□

4.4.23 We remark that since $f \times T$ is a β_0-function of L_T for any β_0-function $f: L_E \to \mathbb{R}$ and for any $T \subseteq E$, for all $U \subseteq T$ there exists $x^0 \in P(K_T, f \times T)$ such that $x^0(U) = (f \times T)(U)$, by (4.1.22).

We now use contractions of a β_0-function $f: L_E \to \mathbb{R}$ to determine the dimension of $H_T = \{x \in P(K_E, f) : x(E) = f(E),\ x(T) = f(T)\}$ for all $T \in K_E - \{T\}$. The methods used are similar to those used to determine $\dim(P_T)$.

4.4.24 <u>Theorem</u> Let $f: L_E \to \mathbb{R}$ be a β_0-function and let $T \in K_E - \{E\}$. Then $\dim(H_T) = |E| - \mu_f(T) - \mu_{f \times \bar{T}}(\bar{T})$.

<u>Proof</u> Let F be the minimal f-separation of T and F' be the minimal $(f \times \bar{T})$-separation of $\bar{T}$. By (4.4.15), for all $S \in F$ and for all $x \in H_T$ we have $x(S) = f(S)$.

By (4.4.11), for all $S \in F'$,

$$\begin{aligned} f(E) - f(T) &= (f \times \bar{T})(\bar{T}) \\ &= (f \times \bar{T})(S) + (f \times \bar{T})(\bar{T} - S) \\ &= f(T \cup S) - f(T) + f(\bar{S}) - f(T). \end{aligned}$$

Therefore, $f(T)+f(E) = f(T \cup S)+f(\bar{S})$ and for all $x \in H_T$,

$$\begin{aligned} x(T \cup S)+x(\bar{S}) &\le f(T \cup S)+f(\bar{S}) \\ &= f(T)+f(E) \\ &= x(T)+x(E) \\ &= x(T \cup S)+x(\bar{S}). \end{aligned}$$

Thus, $x(T \cup S) = f(T \cup S)$ for all $S \in F'$ and for all $x \in H_T$.

A linear system which defines H_T is

$$4.4.25 \quad \begin{cases} x_e \ge 0 \text{ for all } e \in E \\ x(S) \le f(S) \text{ for all } S \in K_E \\ x(E) \ge f(E) \\ x(T) \ge f(T). \end{cases}$$

For each $S \in F$ the inequality $x(S) \le f(S)$ is in the equality system of (4.4.25) and for each $S \in F'$ the inequality $x(T \cup S) \le f(T \cup S)$ is in the equality system of (4.4.25). The set of vectors $\{x^S : S \in F\} \cup \{x^{T \cup S} : S \in F'\}$ is linearly independent. Therefore, by definition (2.3.10) of dimension,

$$\dim(H_T) \le |E|-|F|-|F'| = |E|-\mu_f(T)-\mu_{f\times\bar{T}}(\bar{T}).$$

We now show $\dim(H_T) \ge |E|-\mu_f(T)-\mu_{f\times\bar{T}}(\bar{T})$. Consider the face $P_1 \equiv \{x \in P(K_T,f|T) : x(T) = f(T)\}$ of $P(K_T,f|T) \subseteq \mathbb{R}_+^T$. By (4.4.18), $\dim(P_1) = |T|-\mu_f(T)$. By (2.3.23), there exist $|T|-\mu_f(T)+1$ affinely independent vectors $\{x^m : 1 \le m \le |T|-\mu_f(T)+1\}$ of P_1. Similarly, there exist $|\bar{T}|-\mu_{f\times\bar{T}}(\bar{T})+1$ affinely independent vectors $\{z^m : 1 \le m \le |\bar{T}|-\mu_{f\times\bar{T}}(\bar{T})+1\}$ of $P_2 \equiv \{x \in P(K_{\bar{T}},f \times \bar{T}) : x(\bar{T}) = (f \times \bar{T})(\bar{T})\}$.

For $1 \le m \le |T|-\mu_f(T)+1$ let $\bar{x}^m \equiv [x^m, z^1]$. By (4.4.22), $\bar{x}^m \in P(K_E, f)$. Furthermore,

$$\bar{x}^m(T) = x^m(T) = f(T)$$

and

$$\bar{x}^m(E) = x^m(T) + z^1(\bar{T}) = f(T) + (f \times \bar{T})(\bar{T}) = f(E).$$

Therefore, $\bar{x}^m \in H_T$.

For $2 \le m \le |\bar{T}|-\mu_{f\times\bar{T}}(\bar{T})+1$ let $\bar{z}^m \equiv [x^1, z^m]$. As above, $\bar{z}^m \in H_T$. By (2.3.27),

$$K \equiv \{\bar{x}^m : 1 \le m \le |T|-\mu_f(T)+1\} \cup \{\bar{z}^m : 2 \le m \le |\bar{T}|-\mu_{f\times\bar{T}}(\bar{T})+1\}$$

is a set of $|E|-\mu_f(T)-\mu_{f\times\bar{T}}(\bar{T})+1$ affinely independent vectors of H_T. Therefore, by (2.3.23),

$$\dim(H_T) \ge |E|-\mu_f(T)-\mu_{f\times\bar{T}}(\bar{T}). \qquad \square$$

4.4.26 Corollary Let $f: L_E \to \mathbb{R}$ be a β_0-function such that $P(K_E, f)$ is of full dimension and let $T \in K_E - \{E\}$. If P_E is a facet of $P(K_E, f)$ then H_T is a facet of P_E if and only if T is f-nonseparable and $\bar{T}$ is $(f \times \bar{T})$-nonseparable.

Proof Since $\dim(P_E) = |E|-1$, H_T is a facet of P_E if and only if $\dim(H_T) = |E|-2$. By (4.4.24), $\dim(H_T) = |E|-2$ if and only if $\mu_f(T) = 1$ and $\mu_{f\times\bar{T}}(\bar{T}) = 1$. $\square$

4.4.27 Suppose $P(K_E,f)$ is of full dimension and for $T \in K_E$, P_T is a facet of $P(K_E,f)$. Then, since $\dim(P_T) = |E|-1$, the only inequalities of the linear system (4.4.19) defining P_T which can be in the equality system of (4.4.19) are $x(T) \leq f(T)$ and $x(T) \geq f(T)$.

4.4.28 Suppose $\dim(H_T) = |E|-2$ for $T \in K_E-\{E\}$. The inequalities $x(T) \leq f(T)$, $x(T) \geq f(T)$, $x(E) \leq f(E)$ and $x(E) \geq f(E)$ are in the equality system of (4.4.25). For all $S \in K_E-\{T,\bar{T},E\}$ the vectors $\{x^S,x^T,x^E\}$ are linearly independent. Therefore, by the definition (2.3.10) of dimension, the only other possible inequalities in the equality system of (4.4.25) are $x(\bar{T}) \leq f(\bar{T})$ or, in the case that $|T| = |E|-1$, $x_e \geq 0$, where $T \cup e = E$. If $x(\bar{T}) \leq f(\bar{T})$ is in the equality system of (4.4.25) then for all $x \in H_T$ we have

$$f(E) = x(E) = x(T) + x(\bar{T}) = f(T) + f(\bar{T})$$

and $\{T,\bar{T}\}$ is an f-separation of E. If $x_e \geq 0$ is in the equality system of (4.4.25), where $T \cup e = E$, then for all $x \in H_T$,

$$f(E) = x(E) = x(T) + x_e = f(T)$$

and T is not f-closed.

4.4.29 For each $j \in E$ let

$$N_j \equiv \{x \in P_E : x_j = 0\}.$$

Then N_j is a face of P_E. If $f(\{j\}) = 0$ then $N_j = P_E$. If $E-j$ is f-closed then for all $x \in P(K_E,f)$ such that $x_j = 0$ we have

$$x(E) = x(E-j) \leq f(E-j) < f(E)$$

and $N_j = \phi$. Finally, if E-j is not f-closed then for all $x \in N_j$ we have

$$x(E-j) = x(E) = f(E-j)$$

and $N_j = H_{E-j}$. Since $\mu_{f\times\{j\}}(\{j\}) = 1$, we have

$$\dim(N_j) = |E|-\mu_f(E-j)-1,$$

by (4.4.24).

4.5 Faces of Polymatroid Intersection

4.5.1 Let $f_1,f_2:L_E \to \mathbb{R}$ be two β_0-functions. Recall that in (4.3.10) we defined the rank function of $P(K_E,f_1) \cap P(K_E,f_2), f:L_E \to \mathbb{R}$, by

$$f(T) = \max\{x(T):x \in P(K_E,f_1) \cap P(K_E,f_2)\}.$$

A linear system defining $P(K_E,f_1) \cap P(K_E,f_2) = P(K_E,f)$ is

$$x_e \geq 0 \text{ for all } e \in E$$
$$x(S) \leq f(S) \text{ for all } S \in K_E.$$

4.5.2 In this section we determine the sets $T \in K_E$ such that

$$P_T \equiv \{x \in P(K_E,f):x(T) = f(T)\}$$

is a facet of $P(K_E,f)$. By (4.4.5), if P_T is a facet of $P(K_E,f)$ then T must be f-closed and f-nonseparable. We intend to show the converse; i.e. if P_T is not a facet of $P(K_E,f)$ then T is not f-closed or T is f-separable.

4.5.3 Lemma Let $f: L_E \to \mathbb{R}_+^E$ be nondecreasing subadditive function. Let $P(K_E,f)$ be of full dimension. If for $T \in K_E$, P_T is not a facet of $P(K_E,f)$ and the linear program

4.5.4 minimize $f \cdot y$ where $y \in \mathbb{R}^{K_E}$ satisfies

$$y_S \geq 0 \text{ for all } S \in K_E$$
$$y_T \leq 0$$
$$y(K_E,e) \geq 1 \text{ for all } e \in T$$

has an integer-valued optimum solution then T is not f-closed or T is not f-separable.

Proof Since $P(K_E,f)$ is of full dimension, P_T is a facet of $P(K_E,f)$ if and only if the inequality $x(T) \leq f(T)$ is essential for defining $P(K_E,f)$, by (2.3.31). By (2.4.19), $x(T) \leq f(T)$ is nonessential for defining $P(K_E,f)$ if and only if the optimum value of (4.5.4) is $f(T)$. Therefore, if P_T is not a facet of $P(K_E,f)$ then the optimum value of (4.5.4) is $f(T)$.

Let $y^0 \in \mathbb{Z}^{K_E}$ be an optimum solution to (4.5.4). Then y^0 must be (0,1)-valued. Let $F \equiv \{S \in K_E : y_S^0 = 1\}$. Then

$$f(T) = f \cdot y^0 = \Sigma(f(S) : S \in F).$$

Let $U \equiv \cup(S : S \in F)$. If $T \subset U$ then, by the subadditivity of f, $f(T) = f(U)$ and T is not f-closed. If $T = U$ then, since f is nondecreasing, we may assume that F is a nontrivial partition of T. Since $f(T) = \Sigma(f(S) : S \in F)$, T is f-separable. $\square$

4.5.5 Theorem Let $f_1, f_2 : L_E \to \mathbb{R}$ be two β_0-functions and let $f : L_E \to \mathbb{R}$ be the rank function of $P(K_E, f_1) \cap P(K_E, f_2)$. Let $P(K_E, f)$ be of full dimension. Then for all $T \in K_E$, P_T is a facet of $P(K_E, f)$ if and only if T is f-closed and f-nonseparable.

Proof As we observed in (4.5.3), all we need show is that if P_T is not a facet of $P(K_E, f)$ then T is not f-closed or T is f-separable.

Suppose that P_T is not a facet of $P(K_E, f)$. Then, since $P(K_E, f)$ is of full dimension, the inequality $x(T) \le f(T)$ is nonessential for defining $P(K_E, f)$, by (2.3.29). Therefore, the inequalities $x(T) \le f_1(T)$ and $x(T) \le f_2(T)$ together are nonessential for defining $P(K_E, f_1) \cap P(K_E, f_2)$. By (2.4.19), the optimum value of the linear program

4.5.6 minimize $f_1 \cdot y^1 + f_2 \cdot y^2$ where $y^1, y^2 \in \mathbb{R}^{K_E}$ satisfy

$$y^i_S \ge 0 \text{ for all } S \in K_E,\ i = 1,2$$

$$y^1_T = y^2_T = 0$$

$$y^1(K_E, e) + y^2(K_E, e) \ge 1 \text{ for all } e \in T$$

is equal to f(T). We now show that (4.5.6) has an integer-valued optimum solution. Then, as we noted in (4.3.15), (4.5.4) has an integer-valued optimum solution. From (4.5.3) we can then deduce that T is not f-closed or T is f-separable.

By (2.5.1), (4.5.6) has a rational-valued optimum solution $[y^1, y^2]$. Apply the following transformation:

4.5.7 Starting with $j = 2$, suppose $Y, Z \in K_E$, $Y \cap Z \neq \phi$, $Y \nsubseteq Z$, $Z \nsubseteq Y$ and

$$0 < y_Y^j \leq y_Z^j$$

For each $S \in K_E$ define y_S^{j+1} by

$$y_S^{j+1} \equiv \begin{cases} y_S^j + y_Y^j & \text{if } S \in \{Y \cap Z, Y \cup Z\} \\ y_S^j - y_Y^j & \text{if } S \in \{Y, Z\} \\ y_S^j & \text{otherwise} \end{cases}$$

It is easy to check that $[y^1, y^{j+1}]$ is a feasible solution to the linear program

4.5.8 minimize $f_1 \cdot y^1 + f_2 \cdot y^2$ where $y^1, y^2 \in \mathbb{R}^{K_E}$ satisfy

$y_S^i \geq 0$ for all $S \in K_E$, $i = 1,2$

$y^1(K_E, e) + y^2(K_E, e) \geq 1$ for all $e \in T$.

By the submodularity of f_2,

$$\begin{aligned} f_2 \cdot y^{j+1} &= f_2 \cdot y^j + y_Y^j [f_2(Y \cap Z) + f_2(Y \cup Z) - f_2(Y) - f_2(Z)] \\ &\leq f_2 \cdot y^j. \end{aligned}$$

Therefore $[y^1, y^{j+1}]$ is an optimum solution to (4.5.8). (We may have $y_T^{j+1} > 0$, in which case $[y^1, y^{j+1}]$ is not an optimum solution to (4.5.6).)

Let α be a common denominator of $\{y_S^2 : S \in K_E\}$. Let $w^2 \equiv \alpha y^2$ and for each vector y^{j+1} constructed according to (4.5.7) let

$w^{j+1} \equiv \alpha y^{j+1}$. Then $w^{j+1} \in \mathbb{Z}_+^E$, since $y^{j+1} \geq 0$. Since $1 \cdot y^{j+1} = 1 \cdot y^j$, we have $1 \cdot w^{j+1} = 1 \cdot w^j$. There can be only a finite number of distinct vectors $w \in \mathbb{Z}_+^E$ having the same sum $1 \cdot w$. Therefore, there can be only a finite number of distinct vectors in the sequences $\{y^2, y^3, \ldots, y^j, y^{j+1}, \ldots\}$. Since

$$\begin{aligned}(y_S^{j+1}|S|^2 : S \in K_E) &= \Sigma(y_S^j|S|^2 : S \in K_E) + y_Y^j[|Y \cap Z|^2 + |Y \cup Z|^2 - |Y|^2 - |Z|^2] \\ &> \Sigma(y_S^j|S|^2 : S \in K_E)\end{aligned}$$

the sequence has only finitely many terms.

Thus there exists an optimum solution $[y^1, \bar{y}^2]$ to (4.5.8) with the property that for all $Y, Z \in K_E$ such that $\bar{y}_Y^2 > 0$ and $\bar{y}_Z^2 > 0$ we have $Y \cap Z = \phi$ or $Y \subseteq Z$ or $Z \subseteq Y$, i.e. the family F^2 of sets $S \in K_E$ such that $\bar{y}_S^2 > 0$ is a nested family. Similarly, there is an optimum solution $[\bar{y}^1, \bar{y}^2]$ to (4.5.8) with the property that the family F^1 of sets $S \in K_E$ such that $\bar{y}_S^1 > 0$ is a nested family.

By the construction of $[\bar{y}^1, \bar{y}^2]$, for $i = 1$ or $i = 2$ there is a set $S \in K_E - \{T\}$ such that $\bar{y}_S^i > 0$. Since $f_1 \cdot \bar{y}^1 + f_2 \cdot \bar{y}^2 = f(T)$ we must have $\beta \equiv \bar{y}_T^1 + \bar{y}_T^2 < 1$. Let $[\hat{y}^1, \hat{y}^2]$ be defined by

$$\hat{y}_S^i \equiv \begin{cases} \dfrac{\bar{y}_S^i}{1-\beta} & \text{if } S \neq T \\ 0 & \text{if } S = T \end{cases}$$

for all $S \in K_E$, $i = 1,2$. For all $e \in T$ we have

$$\hat{y}^1(K_E,e)+\hat{y}^2(K_E,e) = \frac{1}{1-\beta}[\bar{y}^1(K_E,e)+\bar{y}^2(K_E,e)-\beta]$$
$$\geq \frac{1}{1-\beta}(1-\beta)$$
$$= 1.$$

Therefore $[\hat{y}^1,\hat{y}^2]$ is a feasible solution to (4.5.6). Moreover,

$$f_1\cdot\hat{y}^1 + f_2\cdot\hat{y}^2 = \frac{1}{1-\beta}[f_1\cdot\bar{y}^1+f_2\cdot\bar{y}^2 - \beta f(T)]$$
$$= \frac{1}{1-\beta}[f(T)-\beta f(T)]$$
$$= f(T).$$

Hence $[\hat{y}^1,\hat{y}^2]$ is an optimum solution to (4.5.6).

Let $F_i \equiv F^i-\{T\}$ for $i = 1,2$ (i.e. $F_i = \{S \in K_E:\hat{y}^i_S > 0\}$).

Consider the linear program

4.5.9 minimize $f_1\cdot y^1+f_2\cdot y^2$ where $[y^1,y^2]$ satisfies

4.5.10
$$\begin{cases} y^i_S \geq 0 & \text{for all } S \in K_E,\ i = 1,2 \\ y^i_S \leq 0 & \text{for all } S \in K_E-\{F_i\},\ i = 1,2 \\ y^1(F_1,e)+y^2(F_2,e) \geq 1 & \text{for all } e \in T. \end{cases}$$

Any feasible solution to (4.5.9) is a feasible solution to (4.5.6). Since $[\hat{y}^1,\hat{y}^2]$ is an optimum solution to (4.5.6) and a feasible solution to (4.5.9), every optimum solution to (4.5.9) must be an optimum solution to (4.5.6).

If we represent the linear system of (4.5.10) as $Ax \le b$ for an appropriate choice of A and b then, since F_1 and F_2 are nested families, A is totally unimodular, by (4.3.7). Therefore, by (2.5.16), there exists an integer-valued (and hence (0,1)-valued) optimum solution $[\tilde{y}^1, \tilde{y}^2]$ to (4.5.9) and hence to (4.5.6). □

4.5.11 Proposition Let $f: L_E \to \mathbb{R}$ be the rank function of $P(K_E, f_1) \cap P(K_E, f_2)$ for two β_0-functions $f_1, f_2: L_E \to \mathbb{R}$. Then $T \in K_E$ is f-separable if and only if there exist $S \in K_T - \{T\}$, $i = 1$ or 2 and $j = 1$ or 2 such that $f(T) = f_i(S) + f_j(T-S)$.

Proof If $S \in K_T$, $i = 1$ or 2 and $j = 1$ or 2 are such that $f(T) = f_i(S) + f_j(T-S)$, then we have

$$f(T) = f_i(S) + f_j(T-S) \ge f(S) + f(T-S) \ge f(T)$$

and T is f-separable.

Conversely, suppose T is f-separable and $S \in K_T$ is such that $f(T) = f(S) + f(T-S)$. Let $S_1 \subseteq S$ be such that $f(S) = f_1(S_1) + f_2(S-S_1)$ and let $S_2 \subseteq T-S$ be such that $f(T-S) = f_1(S_2) + f_2((T-S)-S_2)$. Then

$$\begin{aligned} f(T) &= f(S) + f(T-S) \\ &= f_1(S_1) + f_2(S-S_1) + f_1(S_2) + f_2((T-S)-S_2) \\ &\ge f_1(S_1 \cup S_2) + f_2(T-(S_1 \cup S_2)) \\ &\ge f(T). \end{aligned}$$

Hence $f(T) = f_1(S_1 \cup S_2) + f_2(T-(S_1 \cup S_2))$. If $S_1 \cup S_2 \in K_T - \{T\}$ then we are done. Therefore, we may assume that $S_1 \cup S_2 = T$. This implies that $S_1 = S$ and $S_2 = T-S$. Therefore $f(T) = f_1(S) + f_1(T-S)$. □

4.5.12 <u>Theorem</u> Let $f_1, f_2 : L_E \to \mathbb{R}$ be two β_0-functions such that for all $e \in E$, $f_1(e) > 0$ and $f_2(e) > 0$. Let $f : L_E \to \mathbb{R}$ be the rank function of $P(K_E, f_1) \cap P(K_E, f_2)$. Then for all $T \in K_E$, T is f-closed and f-nonseparable if and only if there is no $S \in K_T - \{T\}$, $i = 1$ or 2 and $j = 1$ or 2 such that $f(T) = f_i(S) + f_j(T-S)$ and T is f_k-closed for each $k = 1$ or 2 such that $f(T) = f_k(T)$.

<u>Proof</u> Suppose T is f-closed and f-nonseparable. By (4.5.11), there is no $S \in K_T - \{T\}$, $i = 1$ or 2 and $j = 1$ or 2 such that $f(T)$ $f(T) = f_i(S) + f_j(T-S)$. Let $f(T) = f_k(T)$ for $k = 1$ or 2. Then for all $j \notin T$,

$$f_k(T \cup j) \geq f(T \cup j) > f(T) = f_k(T),$$

so T is f_k-closed.

Conversely, suppose T is not f-closed or T is f-separable. If T is f-separable then, by (4.5.11), there exists $S \in K_T - \{T\}$, $i = 1$ or 2 and $j = 1$ or 2 such that $f(T) = f_i(S) + f_j(T-S)$. If T is not f-closed then for some $e \notin T$ we have $f(T \cup e) = f(T)$. Let $U \subseteq T \cup e$ be such that $f(T \cup e) = f_1(U) + f_2((T \cup e)-U)$. We may assume that $e \in U$ and then

$$f(T) = f(T \cup e) = f_1(U)+f_2((T \cup e)-U) \geq f_1(U-e)+f_2(T-(U-e)) \geq f(T).$$

Let $S \equiv U-e$. If $S = \phi$ then $U = \{e\}$ and

$$f(T) = f_1(e) + f_2(T) > f(T);$$

which is impossible. If $S = T$ then $U = T \cup e$ and

$$f(T) = f_1(T \cup e) = f_1(T).$$

Therefore, T is not f_1-closed. Finally, if $S \in K_T-\{T\}$ then $f(T) = f_1(S) + f_2(T-S)$. □

4.5.13 Corollary Let $f_1,f_2:L_E \to \mathbb{R}$ be two β_0-functions and let $f:L_E \to \mathbb{R}$ be the rank function of $P(K_E,f_1) \cap P(K_E,f_2)$. Let $P(K_E,f)$ be of full dimension. Then for all $T \in K_E$, P_T is facet of $P(K_E,f)$ if and only if there is no $S \in K_T-\{T\}$ such that $f(T) = f_1(S) + f_2(T-S)$ and P_T^i is a facet of $P(K_E,f_i)$ for each $i = 1$ or 2 such that $f(T) = f_i(T)$, where

$$P_T^i \equiv \{x \in P(K_E,f_i):x(T) = f_i(T)\}.$$

Proof By (4.4.3), $P(K_E,f)$ is of full dimension if and only if $f_1(e) > 0$ and $f_2(e) > 0$ for all $e \in E$. By (4.5.5), P_T is a facet of $P(K_E,f)$ if and only if T is f-closed and f-nonseparable. By (4.4.5), for $i = 1$ or 2, P_T^i is a facet of $P(K_E,f_i)$ if and only if T is f_i-closed and f_i-nonseparable. The corollary now follows from (4.5.12). □

4.6 Polymatroid Partition

4.6.1 In this section we generalize the sum of matroids described in section 3.1 to a sum of polymatroids. For any family $\{P_i : i \in I\}$ of subsets of $\mathbb{R}^E$ let

$$\Sigma(P_i : i \in I) \equiv \{\Sigma(x^i : i \in I) : x^i \in P_i \text{ for all } i \in I\}.$$

4.6.2 Theorem Let I be a finite set and for each $i \in I$ let f_i be a β_0-function. Let $P_i \equiv P(K_E, f_i)$. Then for $u \in \mathbb{R}_+^E$ we have $u \in \Sigma(P_i : i \in I)$ if and only if $u(S) \le \Sigma(f_i(S) : i \in S)$ for all $S \subseteq E$. Furthermore, if P_i is an integral polymatroid for all $i \in I$ and $u \in \Sigma(P_i : i \in I)$ is integer-valued then there exists $\{x^i : i \in I\}$ such that $x^i \in P_i \cap \mathbb{Z}^E$ for all $i \in I$ and $u = \Sigma(x^i : i \in I)$.

Proof Let $P \equiv \Sigma(P_i : i \in I)$. Let

$$Q_1 \equiv \{x_e^i \in \mathbb{R}_+^{I \times E} : X^i(S) \le f_i(S) \text{ for all } S \in K_E, \text{ for all } i \in I\},$$

$$Q_2 \equiv \{x_e^i \in \mathbb{R}_+^{I \times E} : \Sigma(x_e^i : i \in I) \le u_e \text{ for all } e \in E\}.$$

It is easy to check that Q_1 and Q_2 are polymatroids. For any $x \in Q_1 \cap Q_2$ we have $1 \cdot x \le u(E)$. Furthermore, $u \in P$ if and only if the optimum value of

4.6.3 $\max \{1 \cdot x : x \in Q_1 \cap Q_2\}$

is $u(E)$, since any vector $x \in Q_1 \cap Q_2$ such that $1 \cdot x = u(E)$ corresponds to a set $\{x^i \in P_i : i \in I\}$ of vectors such that $u = \Sigma(x^i : i \in I)$. A defining linear system for $Q_1 \cap Q_2$ is

$$x_e^i \geq 0 \text{ for all } e \in E \text{ and for all } i \in I$$

$$x^i(S) \leq f_i(S) \text{ for all } S \in K_E \text{ and for all } i \in I$$

$$\Sigma(x_e^i : i \in I) \leq u_e \text{ for all } e \in E.$$

By (4.3.11) and the Strong L.P. Duality Theorem, the optimum value of (4.6.3) is u(E) if and only if the optimum value of

4.6.4 $$\min \{(f_i(S_i):S_i \subseteq E, i \in I) + u(T):S_i \cup T = E \text{ for all } e \in E\}$$

is u(E).

We can choose an optimum solution $\{S_i^0 \subseteq E : i \in I\}$ and T^0 to (4.6.4) so that $T^0 = \cap(S_i^0 : i \in I)$. Since f_i is nondecreasing, we may assume that $S_i^0 = S_j^0$ for all $i, j \in I$. Therefore, the optimum value of (4.6.4) is u(E) if and only if for all $S \in K_E$ we have

$$u(S) \leq \Sigma(f_i(S):i \in I).$$

If $\{P_i : i \in I\}$ are integral polymatroids and $u \in P$ is integer-valued then, by (4.3.8) and (2.3.20), we can choose an optimum solution to (4.6.3) to be integer-valued, and this corresponds to integer-valued $\{x^i \in P_i : i \in I\}$ such that $u = \Sigma(x^i : i \in I)$. □

4.6.5 We see from (4.6.2) that a linear system defining $\Sigma(P_i : i \in I)$ is

$$x_e \geq 0 \text{ for all } e \in E$$
$$x(S) \leq \Sigma(f_i(S) : i \in I) \text{ for all } S \in K_E.$$

For all $S \subseteq E$ let $f(S) \equiv \Sigma(f_i(S) : i \in I)$. Clearly f is a β_0-function of L_E. By (4.1.4) we have

4.6.6 Theorem Let $f_i : L_E \to \mathbb{R}$ be a β_0-function for all $i \in I$ and let $P_i \equiv P(K_E, f_i)$. Then $P \equiv \Sigma(P_i : i \in I)$ is a polymatroid and $P = P(K_E, f)$, where $f(S) \equiv \Sigma(f_i(S) : i \in I)$ for all $S \subseteq E$. If P_i is an integral polymatroid for all $i \in I$ then P is an integral polymatroid. If, in addition, $u \in P$ is integer-valued then there exists $\{x^i : i \in I\}$ such that $x^i \in P_i \cap \mathbb{Z}^E$ for all $i \in I$ and $u = \Sigma(x^i : i \in I)$. □

4.6.7 Suppose that $\{M_i = (E, \mathcal{F}_i) : i \in I\}$ is a collection of matroids on E with rank functions $\{r_i : i \in I\}$. Let $P \equiv \Sigma(P(M_i) : i \in I)$. By (4.6.6), P is an integral polymatroid. By (4.1.9),

$$\mathcal{F} \equiv \{J \subseteq E : x^J \in P\}$$

is the family of independent sets of a matroid $M = (E, \mathcal{F})$. Again by (4.6.6), for any $J \subseteq E$, $x^J \in P$ if and only if there exist integer-valued, i.e. (0,1)-valued, $\{x^i \in P(M_i) : i \in I\}$ such that $x^J = \Sigma(x^i : i \in I)$. Thus $J \in \mathcal{F}$ if and only if there exists a partition $\{J_i \in \mathcal{F}_i : i \in I\}$ of J into possibly empty sets. This proves (3.1.30).

4.7 Polymatroid Translation

4.7.1 The following construction of polymatroids is analagous to the contraction of a matroid $M = (E,\mathcal{J})$ to $\bar{J}$ for some $J \in \mathcal{J}$ (recall that we discussed matroid contraction in (3.4.23)). For any $S \subseteq \mathbb{R}^E$ let $S_+ \equiv \{x \in S : x \geq 0\}$. For a polymatroid $P \subseteq \mathbb{R}^E$ and a vector $k \in P$, the <u>translation of P by k</u> is the polyhedron $(P - k)_+$.

4.7.2 <u>Proposition</u> Let $f : L_E \to \mathbb{R}$ be the rank function of a polymatroid P and let $k \in P$. Then for all $k \in P$, $P' \equiv (P-k)_+$ is a polymatroid. If P is an integral polymatroid and $k \in P$ is integer-valued then P' is an integral polymatroid. For all $S \subseteq E$, $f_{P'}(S) = \min\{f(T) - k(T) : S \subseteq T\}$.

<u>Proof</u> Clearly a defining linear system for $P - k$ is

$$x_e \geq -k_e \text{ for all } e \in E$$
$$x(S) \leq f(S) - k(S) \text{ for all } S \in K_E.$$

Let $g(S) \equiv f(S) - k(S)$ for all $S \in E$. Clearly g is a submodular function of L_E. Since $k \in P$, $g(S) \geq 0$ for all $S \subseteq E$. Thus g is a σ-function of L_E. A defining linear system for $(P - k)_+$ is

$$x_e \geq 0 \text{ for all } e \in E$$
$$x(S) \leq g(S) \text{ for all } S \in K_E.$$

By (4.1.4), P' is a polymatroid and for all $S \subseteq E$ we have $f_{P'}(S) = \min\{f(T) - k(T) : S \subseteq T\}$. Furthermore, if f is integer-valued and $k \in P$ is integer-valued then g is an integer-valued σ-function and, by (4.1.4), P' is an integral polymatroid.

Polymatroid partition and polymatroid translation can be combined to give the following construction of polymatroids.

4.7.3 Theorem Let $P_1, P_2 \subseteq \mathbb{R}^E$ be two polymatroids. Let $W \equiv \{x \in P_2 : x(E) = r(E)\}$. Then for all $k \in W$, $P' \equiv (P_1 + W - k)_+$ is a polymatroid and $P' = (P_1 + P_2 - k)_+$. For all $S \subseteq E$,

4.7.4 $$f_{P'}(S) = \min\{f_1(T) + f_2(T) - k(T) : S \subseteq T\}.$$

Furthermore, if P_1 and P_2 are integral polymatroids and $k \in \mathbb{Z}^E$ then P' is an integral polymatroid.

Proof By (4.6.2) and (4.7.2), $(P_1 + P_2 - k)_+$ is a polymatroid. Therefore it is sufficient to prove that $P' = (P_1 + P_2 - k)_+$. Clearly $P' \subseteq (P_1 + P_2 - k)_+$. Let $x \in (P_1 + P_2 - k)_+$. Then there exist $x^1 \in P_1$, $x^2 \in P_2$ such that $x = x^1 + x^2 - k$. Since $x \geq 0$, we have $x^1 + x^2 \geq k$. Since k is a P_2-basis of $x^1 + x^2$, any P_2-basis of $x^1 + x^2$ is an element of W. Extend x^2 to a P_2-basis x^3 of $x^1 + x^2$. Then $x^3 \in W$ and $x^4 \equiv x^1 + x^2 - x^3 \in P_1$. Therefore $x \in (P_1 + W - k)_+$. □

4.8 Network Polymatroids

4.8.1 Throughout this section we let $G = (V,E)$ be an loopless graph, let A be the matrix of G and $a \in \mathbb{R}^E_+$. Let

$$Y \equiv \{y \in \mathbb{R}^V : Ax = y \text{ for some } x \in \mathbb{R}^E \text{ such that } 0 \leq x \leq a\}.$$

The following is a well-known result from network flow theory (see, for example, Ford and Fulkerson [F1]).

4.8.2 Theorem For any $y^0 \in \mathbb{R}^V$, $y \in Y$ if and only if $y^0(V) = 0$ and $y^0(S) \leq a(\delta(S))$ for all $S \subseteq V$. Furthermore, if $y^0 \in Y$ is integer-valued and $a \in \mathbb{Z}^E_+$ then there exists integer-valued $x \in \mathbb{R}^E$ such that $Ax = y^0$ and $0 \leq x \leq a$. □

4.8.3 It follows from (4.8.2) that a linear system defining Y is

$$y(S) \leq a(\delta(S)) \text{ for all } S \in K_V$$
$$y(V) = 0.$$

It is easily seen that the function $a(\delta): L_V \to \mathbb{R}$ is a submodular function of L_V. Let $k \in \mathbb{R}^V$. It is also easy to check that if we define $f(S) \equiv a(\delta(S)) + k(S)$ for $S \subseteq V$ then f is a submodular function of L_V. We can choose k to be a sufficiently large vector so that f is a β_0-function of L_V (For example, for each $v \in V$ let $k_v \equiv a(\delta(\bar{v}\ )))$. Then, by (4.14), $P(K_V,f)$ is a polymatroid and $Y = W - k$, where $W \equiv \{x \in P(K_V,f) : x(E) = f(E)\}$.

4.8.4 If we let the polymatroid $P(K_V, f)$ of (4.8.3) be the polymatroid P_2 of (4.7.3) then, by (4.7.3) and (4.8.2), we have the following application of the previous polymatroid constructions.

4.8.5 <u>Theorem</u> (Woodall [W2]) Let $G = (V;E)$ be a graph and let A be the matrix of G. Let $a \in \mathbb{R}_+^E$. Let $P_1 \subseteq \mathbb{R}_+^V$ be a polymatroid with rank function $f_1 : L_V \to \mathbb{R}$. Let

$$P' \equiv \{y \in \mathbb{R}_+^V : \text{there exists } t \in P_1,\ x \in \mathbb{R}^E \text{ such that } 0 \leq x \leq a \text{ and } t + Ax = y\}$$

Then P' is a polymatroid and for all $S \subseteq V$,

4.8.6 $$f_{P'}(S) = \min \{f_1(T) + a(\delta(T)) : S \subseteq T\}.$$

Furthermore, if $a \in \mathbb{Z}_+^E$ and P_1 is an integral polymatroid then P' is an integral polymatroid. For all integer-valued $y \in P'$ there exists an integer-valued $t \in P_1$, $x \in \mathbb{Z}_+^E$ such that $0 \leq x \leq a$ and $t + Ax = y$. □

CHAPTER 5

STRONG k-COVERS and STRONG k-MATCHINGS

5.1 Strong k-Covers

5.1.1 Let $G = (V,E)$ be a graph. $D \in K_E$ is a *directed coboundary* if $D = \delta(S)$ for some $S \subset V$ such that $\delta(\bar{S}) = \phi$. For any positive integer k, $J \subseteq E$ is a *strong k-cover* if $|J \cap D| \geq k$ for every directed coboundary D. Lucchesi and Younger [L2] have shown the following result concerning strong 1-covers.

5.1.2 *Theorem* For any graph G the minimum cardinality of a strong 1-cover of G is equal to the maximum number of pairwise disjoint directed coboundaries. □

5.1.3 Let $D(G) \equiv \{S \subseteq V : \delta(S)$ is a directed coboundary$\}$. By the Strong L.P. Duality Theorem we have

5.1.4 $\min\{1\cdot x : x \in \mathbb{Z}_+^E, x(\delta(S)) \geq 1 \text{ for all } S \in D(G)\}$

$\geq$

5.1.5 $\min\{1\cdot x : x \in \mathbb{R}_+^E, x(\delta(S)) \geq 1 \text{ for all } S \in D(G)\}$

$=$

5.1.6 $\max\{1\cdot y : y \in \mathbb{R}_+^{D(G)}, \Sigma(y_S : e \in \delta(S), S \in D(G)) \leq 1 \text{ for all } e \in E\}$

$\geq$

5.1.7 $\max\{1\cdot y : y \in \mathbb{Z}_+^{D(G)}, \Sigma(y_S : e \in \delta(S), S \in D(G)) \leq 1 \text{ for all } e \in E\}$.

Any optimum solution to (5.1.4) must be (0,1)-valued and therefore the vector of a strong 1-cover. Any feasible solution to (5.1.7) must be the vector of a family $F \subseteq D(G)$ such that $\delta(S) \cap \delta(T) = \phi$ for all $S,T \in F$. Therefore the optimum value of (5.1.7) is the maximum number of pairwise edge-disjoint directed coboundaries. By (5.1.2), any optimum solution to (5.1.4) is an optimum solution to (5.1.5) and any optimum solution to (5.1.7) is an optimum solution to (5.1.6). Equivalently,

5.1.8 Both linear programs (5.1.5) and (5.1.6) have integer-valued optimum solutions.

5.1.9 More generally, let $c \in \mathbb{R}^E$ and consider the linear program

5.1.10 minimize $c \cdot x$ where $x \in \mathbb{R}^E$ satisfies

$x_e \geq 0$ for all $e \in E$

$x(\delta(S)) \geq 1$ for all $S \in D(G)$.

If $c_j < 0$ for some $j \in E$ then we can always find a feasible solution x to (5.1.10) such that x_j is arbitrarily large, and so (5.1.10) is unbounded. Therefore, we may assume $c \in \mathbb{R}^E_+$. The dual linear program of (5.1.10) is

5.1.11 maximize $1 \cdot y$ where $y \in \mathbb{R}^{D(G)}$ satisfies

$y_S \geq 0$ for all $S \in D(G)$

$\Sigma(y_S : e \in \delta(S), S \in D(G)) \leq c_e$ for all $e \in E$.

Clearly 0 is a feasible solution to (5.1.11) and $1 \cdot y$ has an upper bound for all $c \in \mathbb{R}^E_+$. Therefore, by the Strong L.P. Duality Theorem, (5.1.11) has an optimum solution. In the next chapter we will prove the following result.

5.1.12 <u>Theorem</u> For any graph $G = (V,E)$ and for all $c \in \mathbb{Z}^E_+$, (5.1.11) has an integer-valued optimum solution.

<u>Proof</u> See (6.3.20) . $\square$

Let $P_{D(G)} \equiv \{x \in \mathbb{R}_+^E : x(\delta(S)) \geq 1 \text{ for all } S \in D(G)\}$. Let J be a minimal strong 1-cover of G. Then for each $j \in J$ there is a set $S_j \in D(G)$ such that $J \cap D(G) = \{j\}$. Therefore, x^J is the unique solution to the system of linear equations

$$x_e = 0 \text{ for all } e \notin J$$

$$x(\delta(S_j)) = 1 \text{ for all } j \in J$$

Thus x^J is a vertex of $P_{D(G)}$ and we have

5.1.13 If $J \subseteq E$ is a minimal strong 1-cover of G then x^J is a vertex of $P_{D(G)}$.

5.1.14 By (2.5.2) and (5.1.12), the vertices of $P_{D(G)}$ are integer-valued. Let x^0 be a vertex of $P_{D(G)}$. By (2.3.21) there exists $c^0 \in \mathbb{R}_+^E$ such that x^0 is the only optimum solution to (5.1.10). Therefore, x^0 must be (0,1)-valued and the vector of a minimal strong 1-cover of G. Together with (5.1.13) this implies

5.1.15 <u>Theorem</u> For any graph $G = (V,E)$, $x \in \mathbb{R}^E$ is a vertex of $P_{D(G)}$ if and only if x is the vector of a minimal strong 1-cover of G. $\square$

5.1.16 By (2.3.20), for every $c \in \mathbb{R}_+^E$, (5.1.10) has an optimum solution which is a vertex of $P_{D(G)}$. Therefore, by (5.1.12), (5.1.15) and the Strong L.P. Duality Theorem we have

5.1.17 <u>Theorem</u> Let $G = (V,E)$ be a graph and $c \in \mathbb{R}_+^E$. Then

$$\min\{c(J) : J \text{ is a strong 1-cover of } G\}$$

$$=$$

$$\max\{1 \cdot y : y \in \mathbb{R}_+^{D(G)} : \Sigma(y_S : e \in \delta(S), S \in D(G)) \leq c_e \text{ for all } e \in E\}.$$

Furthermore, if $c \in \mathbb{Z}_+^E$ then we can choose $y \in \mathbb{Z}_+^{D(G)}$. □

Clearly when $c = 1$ we have (5.1.2).

5.1.18 In order to generalize to k-covers we consider the following linear program.

5.1.19 minimize $c \cdot x$ where $x \in \mathbb{R}^E$ satisfies

$$x_e \geq 0 \text{ for all } e \in E$$
$$x_e \leq 1 \text{ for all } e \in E$$
$$x(\delta(S)) \geq k \text{ for all } S \in D(G).$$

Any integer-valued feasible solution to (5.1.19) must be the vector of a strong k-cover of G. Notice that (5.1.19) has a feasible solution if and only if $k \leq |\delta(S)|$ for all $S \in D(G)$. Also, the polyhedron of feasible solutions to (5.1.19) is bounded. Therefore, by the Strong L.P. Duality Theorem, (5.1.19) always has an optimum solution whenever $k \leq \min\{|\delta(S)| : S \in D(G)$

The dual linear program of (5.1.19) is

5.1.20 maximize $k \cdot y - 1 \cdot z$ where $y \in \mathbb{R}_+^{D(G)}$ and $z \in \mathbb{R}_+^E$ satisfy

$$y_S \geq 0 \text{ for all } S \in D(G)$$
$$z_e \geq 0 \text{ for all } e \in E$$
$$\Sigma(y_S : e \in \delta(S),\ S \in D(G)) - z_e \leq c_e \text{ for all } e \in E.$$

Together with (5.1.12) we will prove

5.1.21 Theorem Let $G = (V,E)$ be a graph and k be a positive integer such that $k \leq |\delta(S)|$ for all $S \in D(G)$. Then for all $c \in \mathbb{Z}^E$ the linear program (5.1.20) has an integer-valued optimum solution.

<u>Proof</u> See (6.3.21). □

5.1.22 Let

$$P_k(G) \equiv \{x \in \mathbb{R}_+^E : x_e \le 1 \text{ for all } e \in E, x(\delta(S)) \ge k \text{ for all } S \in D(G)\}.$$

If J is a strong k-cover of G then x^J is the unique solution to the system of linear equations

$$x_e = 0 \text{ for all } e \notin J$$
$$x_e = 1 \text{ for all } e \in J$$

and must be a vertex of $P_k(G)$. By (2.5.2) and (5.1.22), the vertices of $P_k(G)$ are integer-valued, i.e. the vectors of strong k-covers of G. Hence,

5.1.23 <u>Theorem</u> Let $G = (V,E)$ be a graph. Then $x \in \mathbb{R}^E$ is a vertex of $P_k(G)$ if and only if x is the vector of a strong k-cover of G. □

By (5.1.21), (5.1.23) and the Strong L.P. Duality Theorem we have

5.1.24 <u>Theorem</u> Let $G = (V,E)$ be a graph and k be a positive integer such that $k \le |\delta(S)|$ for all $S \in D(G)$. Then for all $c \in \mathbb{R}^E$,

$$\min\{c(J) : J \text{ is a strong k-cover of } G\}$$

$$=$$

$$\max\{k\cdot y - 1\cdot z : y \in \mathbb{R}_+^{D(G)}, z \in \mathbb{R}_+^E \text{ satisfy}$$
$$\Sigma(y_S : e \in \delta(S), S \in D(G)) - z_e \le c_e \text{ for all } e \in E\}.$$

Furthermore, if $c \in \mathbb{Z}^E$ then we can choose $y \in \mathbb{Z}_+^{D(G)}$, $z \in \mathbb{Z}_+^E$. □

When $c = 1$, we have, as a corollary to (5.1.24),

5.1.25 Corollary If the graph $G = (V,E)$ has a strong k-cover for a positive integer k then the minimum cardinality of a strong k-cover of G is equal to

$$\max\{k\cdot y - 1\cdot z : y \in \mathbb{Z}_+^{D(G)}, z \in \mathbb{Z}_+^{E} \text{ satisfy } \Sigma(y_S : e \in \delta(S), S \in D(G)) - z_e \leq 1 \text{ for all } e \in E\}.$$

□

5.2 Strong k-Matchings

5.2.1 Let $G = (V,E)$ be a graph. For any positive integer k, a strong k-matching of G is a subset $J \subseteq E$ such that for every directed coboundary D of G, $|J \cap D| \leq k$. We can treat strong k-matchings in exactly the same manner as strong k-covers were in the previous section.

5.2.2 The integer-valued feasible solutions to the following linear program are the vectors of strong k-matchings of G.

5.2.3 maximize $c\cdot x$ where $x \in \mathbb{R}^E$ satisfies

$$x_e \geq 0 \text{ for all } e \in E$$

$$x_e \leq 1 \text{ for all } e \in E$$

$$x(\delta(S)) \leq k \text{ for all } S \in D(G).$$

If $c_j < 0$ for any $j \in E$ then for any optimum solution x^0 to (5.2.3) we have $x_j^0 = 0$. Hence we may assume that $c \in \mathbb{R}_+^E$. If j is in no directed coboundary then clearly $x_j^0 = 1$ for any optimum solution x^0 to (5.2.3). Therefore we may assume that every edge is contained in a directed coboundary.

5.2.4 Proposition Any edge of a graph $G = (V,E)$ is an element of either a directed polygon or a directed coboundary (but not both).

Proof Let $e \in E$. It is easy to see that e cannot belong to both a directed polygon and a directed coboundary. Let
$W \equiv \{v \in V$:there exists a directed path in G from $h(e)$ to $v\}$.
If $t(e) \in W$ then e is an element of a directed polygon and if $t(e) \notin W$ then, since $\delta(W) = \phi$, $\delta(\bar{W})$ is a directed coboundary containing e. □

Let j be an edge of $G = (V,E)$ and $G' \equiv G \times (E-j)$. It is easy to verify the following:

5.2.5 If D is a directed coboundary of G' then D is a directed coboundary of G.

5.2.6 If $e \in E-j$ is in a directed polygon of G then e is in a directed polygon of G'.

5.2.7 If j is in a directed polygon of G and D is a directed coboundary of G then D is directed coboundary of G'.

5.2.8 Let $T \equiv \{e \in E$:e is an element of a directed polygon$\}$. It follows from (5.2.5)-(5.2.7) that $G \times \bar{T}$ is acyclic and for $c \in \mathbb{R}^E_+$ any optimum solution to the linear program

maximize $c \cdot x$ where $x \in \mathbb{R}^E$ satisfies

$x_e \geq 0$ for all $e \in E$

$x_e \leq 1$ for all $e \in E$

$x_e \geq 1$ for all $e \in T$

$x(\delta(S)) \leq k$ for all $S \in D(G \times \bar{T})$.

is an optimum solution to (5.2.3). Therefore, we may assume that G is acyclic.

The dual linear program of (5.2.3) is

5.2.9 minimize $k\cdot y + 1\cdot z$ where $y \in \mathbb{R}^{D(G)}$ and $z \in \mathbb{R}^{E}$ satisfy

$$y_S \geq 0 \text{ for all } S \in D(G)$$

$$z_e \geq 0 \text{ for all } e \in E$$

$$\Sigma(y_S : e \in \delta(S), S \in D(G)) + z_e \geq c_e \text{ for all } e \in E.$$

In chapter 6 we will prove

5.2.10 <u>Theorem</u> Let $G = (V,E)$ be an acyclic graph and k a positive integer. Then for all $c \in \mathbb{Z}^E$, (5.2.9) has an integer-valued optimum solution.

<u>Proof</u> See (6.3.13). □

5.2.11 Let

$$P^k(G) \equiv \{x \in \mathbb{R}_+^E : x_e \leq 1 \text{ for all } e \in E,$$
$$x(\delta(S)) \leq k \text{ for all } S \in D(G)\}.$$

Clearly $P^k(G)$ is nonempty, bounded and pointed. By (2.5.2) and (5.2.10) the vertices of $P^k(G)$ are integer-valued, i.e. the vectors of strong k-matchings of G. If $J \subseteq E$ is a strong k-matching of G then x^J is the unique solution to the linear system

$$x_e = 0 \text{ for all } e \notin J$$

$$x_e = 1 \text{ for all } e \in J$$

and so x^J is a vertex of $P^k(G)$. Therefore

5.2.12 Theorem Let $G = (V,E)$ be an acyclic graph and k a positive integer. Then $x \in \mathbb{R}^E$ is a vertex of $P^k(G)$ if and only if x is the vertex of a strong k-matching of G. □

5.2.13 For any $c \in \mathbb{R}^E$, (5.2.3) has an optimum solution which is a vertex of $P^k(G)$, by (2.3.20). Hence, by (5.2.10), (5.2.12) and the Strong L.P. Duality Theorem, we have

5.2.14 Theorem Let $G = (V,E)$ be an acyclic graph and k a positive integer. Then for all $c \in \mathbb{R}^E$,

$$\max\{c(J) : J \text{ is a strong } k\text{-matching of } G\}$$

$$=$$

$$\min\{k \cdot y + 1 \cdot z : y \in \mathbb{R}_+^{D(G)},\ z \in \mathbb{R}_+^E \text{ satisfy}$$
$$\Sigma(y_S : e \in \delta(S), S \in D(G)) + z_e \geq c_e \text{ for all } e \in E\}.$$

Furthermore, if $c \in \mathbb{Z}^E$ then we can choose $y \in \mathbb{Z}_+^{D(G)}, z \in \mathbb{Z}_+^E$. □

5.2.15 For the case of $k = 1$ the constraints $x_e \leq 1$ for all $e \in E$ are implied by $x(\delta(S)) \leq 1$ for all $S \in D(G)$ and $x_e \geq 0$ for all $e \in E$. If $[y^0, z^0]$ is an optimum solution to (5.2.9) and $z_j^0 > 0$ for some $j \in E$ then we can let T be any element of $D(G)$ such that $j \in \delta(T)$ and define $[y^1, z^1]$ by

$$y_S^1 \equiv \begin{cases} y_S^0 + z_j^0 & \text{if } S = T \\ y_S^0 & \text{otherwise} \end{cases}$$

and

$$z_e^1 \equiv \begin{cases} 0 & \text{if } e = j \\ z_e^0 & \text{otherwise.} \end{cases}$$

It is easy to see that $[y^1,z^1]$ must also be an optimum solution to (5.2.9). Therefore, by (5.2.14), we have

5.2.16 Theorem Let $G = (V,E)$ be an acyclic graph and $c \in \mathbb{R}^E$. Then

$$\max\{c(J): J \text{ is a strong 1-matching of } G\}.$$

5.2.17 $$\min\{1\cdot y : y \in \mathbb{R}_+^{D(G)} \text{ satisfies } \Sigma(y_S : e \in \delta(S), S \in D(G)) \geq c_e \text{ for all } e \in E\}.$$

Furthermore, if $c \in \mathbb{Z}^E$ then we can choose $y \in \mathbb{Z}_+^{D(G)}$. □

For $c = 1$, any optimum solution y to (5.2.17) is (0,1)-valued and therefore the vector of a family $F \subseteq D(G)$ such that $E = \cup(\delta(S) : S \in F)$. Thus we obtain as a corollary to (5.2.16) a result due to Vidyasankar and Younger [V1].

5.2.18 Corollary For an acyclic graph $G = (V,E)$ the maximum cardinality of a strong 1-matching of G is equal to the minimum cardinality of a family of directed coboundaries whose union is E. □

5.3 Facets of Strong k-Cover and Strong k-Matching Polyhedra

In this section we describe the facets of the polyhedra defined in sections 5.1 and 5.2. We shall see that the descriptions of the facets of $P^k(G)$, the polyhedron associated with the strong k-matchings or a graph G, and of $P_{D(G)}$, a polyhedron associated with the strong 1-covers of G, are actually a description of the facets of polyhedra we can associate with any family F on a set E.

5.3.1 To begin with we consider $P^k(G)$, defined by the linear system

$$x_e \geq 0 \quad \text{for all } e \in E$$
$$x_e \leq 1 \quad \text{for all } e \in E$$
$$x(\delta(S)) \leq k \quad \text{for all } S \in D(G).$$

More generally, let F be any family of nonempty subsets of E and let k be a positive integer. Let $P^k(F)$ be the solution set of the linear system

$$\text{5.3.2} \quad \begin{cases} x_e \geq 0 & \text{for all } e \in E \\ x_e \leq 1 & \text{for all } e \in E \\ x(S) \leq k & \text{for all } S \in F. \end{cases}$$

Clearly $P^k(G)$ is of this form, where $F = \{\delta(S) : S \in D(G)\}$.

5.3.3 Proposition For any family F on E and for any positive integer k, $P^k(F)$ is of full dimension.

<u>Proof</u> For all $j \in E$ we have $x^{\{j\}} \in P^k(F)$. Therefore

$$\{x^{\{j\}}:j \in E\} \cup \{0\}$$

is a set of $|E|+1$ affinely independent vectors of $P^k(F)$. By (2.3.23), $P^k(F)$ is of full dimension. □

For each $j \in E$ let $P_j \equiv \{x \in P^k(F):x_j = 0\}$. Then P_j is a face of $P^k(F)$. Moreover,

5.3.4 <u>Proposition</u> For all $j \in E$, P_j is a facet of $P^k(F)$.

<u>Proof</u> Since $\dim(P^k(F)) = |E|$, it is sufficient to find $|E|$ affinely independent vectors of P_j, by (2.3.25).

$$\{x^{\{g\}}:g \neq j\} \cup \{0\}$$

is such a set of vectors. □

For each $j \in E$ let $P^j \equiv \{x \in P^k(F):x_j = 1\}$.

5.3.5 <u>Proposition</u> If $k \geq 2$ then for all $j \in E$, P^j is a facet of $P^k(F)$.

<u>Proof</u> By (2.3.31), it is sufficient to prove that the inequality $x_j \leq 1$ is essential for defining $P^k(F)$. The vector $2x^{\{j\}}$ satisfies all the inequalities of (5.3.2) except $x_j \leq 1$. Therefore, $x_j \leq 1$ is essential for defining $P^k(G)$. □

5.3.6 Consider $P^1(F)$. We may assume that $E = \cup(S:S \in F)$, since we can always consider single element sets as members of F without altering $P^k(F)$ for any k. Then a defining linear system for $P^1(F)$ is

5.3.7
$$
\begin{aligned}
&x_e \geq 0 \quad \text{for all } e \in E \\
&x(S) \leq 1 \text{ for all } S \in F,
\end{aligned}
$$

since every $j \in E$ is an element of some $S_j \in F$ and the inequalities $x_e \geq 0$ for all $e \in S_j-\{j\}$ and $x(S_j) \leq 1$ imply $x_j \leq 1$. Therefore, the "nontrivial" facets of $P^1(F)$ are of the form

$$Q_T^1 \equiv \{x \in P^1(F):x(T) = 1\}$$

for sets $T \in F$.

5.3.8 Proposition For all $T \in F$, Q_T^1 is a facet of $P^1(F)$ if and only if T is a maximal member of F.

Proof If T is not maximal then there is some $S \in F$ such that $T \subset S$. The inequalities $x_e \geq 0$ for all $e \in S-T$ and $x(S) \leq 1$ imply $x(T) \leq 1$ and $x(T) \leq 1$ is nonessential for defining $P^1(F)$. By (2.3.31), Q_T^1 is not a facet of $P^1(F)$.

Conversely, suppose T is a maximal member of F. If $|T| = 1$ then let $\alpha \equiv 2$ and if $|T| \geq 2$ then let $\alpha \equiv \frac{1}{|T|-1}$. Let $x' \equiv \alpha x^T$. For any $S \in F-\{T\}$ we have

$$x'(S) = \alpha|S \cap T| \leq \alpha[|T|-1] = 1$$

Hence x' satisfies all the inequalities of (5.3.7) except $x(T) \le 1$. Thus $x(T) \le 1$ is essential for defining $P^1(F)$ and Q_T^1 is a facet of $P^1(F)$, by (2.3.31). □

Similarly, we can decide which of the sets

$$Q_T^k \equiv \{x \in P^k(F) : x(T) = k\}$$

are facets of $P^k(F)$ for $k \ge 2$. Clearly if $|T| < k$ then $Q_T^k = \phi$.

5.3.9 Proposition If $k \ge 2$ then for all $T \in F$, Q_T^k is a facet of $P^k(F)$ if and only if T is a maximal member of F and $|T| > k$.

Proof Suppose T is not a maximal member of F and $S \in F$ is such that $T \subset S$. Then the inequalities $x_e \ge 0$ for all $e \in S-T$ and $x(S) \le k$ imply $x(T) \le k$ and $x(T) \le k$ is nonessential for defining $P^k(F)$. By (2.3.31), Q_T^k is not a facet of $P^k(F)$. If $|T| = k$ then the inequalities $x_e \le 1$ for all $e \in T$ imply $x(T) \le k$ and again Q_T^k is not a facet of $P^k(F)$.

Conversely, suppose T is a maximal member of F and $|T| > k$.

$$x' \equiv \frac{k}{|T|-1} x^T.$$

Since $|T|-1 \ge k$, $x'_e \le 1$ for all $e \in E$. Since T is a maximal member of F, for all $S \in F-\{T\}$ we have

$$\begin{aligned} x'(S) &= |S \cap T|\left[\frac{k}{|T|-1}\right] \\ &\le [|T|-1]\left[\frac{k}{|T|-1}\right] \\ &= k. \end{aligned}$$

Since

$$x'(T) = |T|\lceil\tfrac{k}{|T|-1}\rceil$$
$$> k,$$

x' satisfies all the inequalities of (5.3.2) except $x(T) \le k$. Therefore, $x(T) \le k$ is essential for defining $P^k(F)$. Since $P^k(F)$ is of full dimension, Q^k_T is a facet of $P^k(F)$, by (2.3.31). □

By (5.3.3), $P^k(F)$ is of full dimension. (5.3.4), (5.3.5), (5.3.8) and (5.3.9) characterize the facets of $P^k(F)$. By (2.3.31) we have

5.3.10 Proposition Let F be a family of nonempty subsets of E such that $E = \cup(S : S \in F)$. The unique minimal linear system defining $P^1(F)$ is

$$x_e \ge 0 \text{ for all } e \in E$$
$$x(S) \le f(S) \text{ for all maximal members } S \text{ of } F.$$

The unique minimal linear system defining $P^k(F)$ for any integer $k \ge 2$ is

$$x_e \ge 0 \text{ for all } e \in E$$
$$x_e \le 1 \text{ for all } e \in E$$
$$x(S) \le k \text{ for all } S \in F \text{ such that } S \text{ is a maximal member of } F \text{ and } |S| > k. \qquad \square$$

5.3.11 We have characterized the facets of $P^k(G)$ by treating the more general polyhedron $P^k(F)$ for any family F on E. Similarly, we can generalize a characterization of the facets of the polyhedron

$$P_{D(G)} \equiv \{x \in \mathbb{R}^E_+ : x(\delta(S)) \ge 1 \text{ for all } S \in D(G)\}$$

for any graph G. More generally, for any family F of nonempty subsets of E let P_F denote the solution set of the linear system

5.3.12 $$\begin{cases} x_e \geq 0 \text{ for all } e \in E \\ x(S) \geq 1 \text{ for all } S \in F. \end{cases}$$

$P_{D(G)}$ is the solution set of (5.3.12) when $F = \{\delta(S) : S \in D(G)\}$.

5.3.13 Proposition For any family F of nonempty subsets of E, P_F is of full dimension.

Proof It is easy to see that for all $e \in E$ there exists $x \in P_F$ such that $x_e > 0$. Similarly, for all $S \in F$ there exists $x \in P_F$ such that $x(S) > 1$. Therefore, there are no inequalities in the equality system of (5.3.12) and, by definition, $\dim(P_F) = |E|$. □

5.3.14 Proposition For all $j \in E$, $H_j \equiv \{x \in P_F : x_j = 0\}$ is a facet of P_F if and only if $\{j\} \notin F$.

Proof If $\{j\} \in F$ then $H_j = \phi$ since $x_j \geq 1$ for all $x \in P_F$. Conversely, suppose $\{j\} \notin F$. Let x' be defined by

$$x'_e \equiv \begin{cases} -1 \text{ if } e = j \\ 2 \text{ otherwise .} \end{cases}$$

x' satisfies all the inequalities of (5.3.12) except $x_e \geq 0$. Therefore, $x_e \geq 0$ is essential for defining P_F and, by (2.3.31), H_j is a facet of P_F. □

5.3.15 <u>Proposition</u> For all $T \in F$, $Q_T \equiv \{x \in P_F : x(T) = 1\}$ is a facet of P_F if and only if T is a minimal member of F.

<u>Proof</u> Suppose T is not a minimal member of F and for $S \in F$ we have $S \subset T$ then the inequalities $x_e \geq 0$ for all $e \in T-S$ and $x(S) \geq 1$ imply $x(T) \geq 1$. Therefore, $x(T) \geq 1$ is nonessential for defining P_F and, by (2.3.31), Q_T is not a facet of P_F.

Conversely, suppose T is a minimal member of F. For all $S \in F-\{T\}$ we have

$$x^{\bar{T}}(S) = |\bar{T} \cap S| \geq 1.$$

Therefore, $x^{\bar{T}}$ satisfies all the inequalities of (5.3.12) except $x(T) \geq 1$. Hence $x(T) \geq 1$ is essential for defining P_F. By (2.3.31), Q_T is a facet of P_F. $\square$

By (2.3.31) and (5.3.13)-(5.3.15) we have

5.3.16 <u>Proposition</u> For any family F of nonempty subsets of E, the unique minimal linear system defining P_F is

$$x_e \geq 0 \text{ for all } e \in E \text{ such that } \{e\} \notin F$$
$$x(S) \geq 1 \text{ for all } S \text{ such that } S \text{ is a minimal member of } F. \qquad \square$$

5.3.17 The ease with which we can describe the facets of $P^k(G)$ and $P_{D(G)}$ serves as a contrast to the rather involved methods we used to determine the facets of the intersection of two polymatroids. It is also in constrast with the methods used to characterize the facets of $P_k(G)$, the solution set of the linear system

$x_e \geq 0$ for all $e \in E$

$x_e \leq 1$ for all $e \in E$

$x(\delta(S)) \geq k$ for all $S \in D(G)$.

For any family F on E let $P_k(F)$ be the solution set of the linear system

5.3.18
$$\begin{cases} x_e \geq 0 \text{ for all } e \in E \\ x_e \leq 1 \text{ for all } e \in E \\ x(S) \geq k \text{ for all } S \in F \end{cases}$$

5.3.19 Proposition For any positive integer k and for any family F on E, $P_k(F)$ is of full dimension if and only if $k < |S|$ for all $S \in F$.

Proof If $k > |S|$ for some $S \in F$ then clearly $P_k(F) = \phi$. If $k = |T|$ for $T \in F$ then for all $e \in T$ then inequality $x_e \leq 1$ is in the equality system of (5.3.18), and so $\dim(P_k(F)) < |E|$.

Conversely, suppose $k < |S|$ for all $S \in F$. Then

$$\{x^{E-\{j\}} : j \in E\} \cup \{1\}$$

is a set of $|E|+1$ affinely independent vectors of $P_k(F)$. By (2.3.23), $P_k(G)$ is of full dimension. □

For each $j \in E$ let $R_j^k \equiv \{x \in P_k(F) : x_j = 0\}$.

5.3.20 <u>Proposition</u> Let F be a family on E and k be a positive integer such that $P_k(F)$ is of full dimension. Then for all $j \in E$, R_j^k is a facet of $P_k(F)$ if and only if there is no $T \in F$ such that $j \in T$ and $|T| = k+1$.

<u>Proof</u> A linear system defining R_j^k is

$$\text{5.3.21} \quad \begin{cases} x_e \geq 0 \text{ for all } e \in E \\ x_j \leq 0 \\ x_e \leq 1 \quad \text{for all } e \in E \\ x(S) \geq k \text{ for all } S \in F \end{cases}$$

Suppose $T \in F$ is such that $j \in T$ and $|T| = k+1$. If $x_j = 0$ then $x_e \geq 1$ for all $e \in T-j$ and the inequality $x_e \leq 1$ of (5.3.21) is in the equality system of (5.3.21). Since $x_j \leq 0$ is also in the equality system of (5.3.21), $\dim(R_j^k) \leq |E|-2$. Since $P_k(F)$ is of full dimension, R_j^k is not a facet of $P_k(F)$.

Conversely, suppose there is no $T \in F$ such that $j \in T$ and $|T| = k+1$. Then

$$\{x^{E-\{g,j\}}\} \cup \{x^{E-\{j\}}\}$$

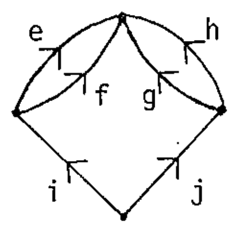
is a set of $|E|$ affinely independent vectors of R_j^k. Hence, by (2.3.23), $\dim(R_j^k) = |E|-1$. Since $P_k(F)$ is of full dimension, R_j^k is a facet of $P_k(F)$. $\square$

5.3.22 For any $T \in F$ let $L_T^k \equiv \{x \in P_k(F) : x(T) = k\}$. We have the following characterization of when L_T^1 is a facet of $P_1(F)$.

5.3.23 <u>Proposition</u> Let $P_1(F)$ be of full dimension. Then for all $T \in F$, L_T^1 is a facet of $P_1(F)$ if and only if T is a minimal member of F.

<u>Proof</u> The proof of (5.3.23) is identical to that of (5.3.15). □

5.3.24 A characterization of when L_T^k is a facet of $P_k(F)$ appears to be difficult for the case $k \geq 2$. It is certainly not true that for $k \geq 2$, L_T^k is a facet of $P_k(F)$ if and only if T is minimal member of F, even when F is the family of directed coboundaries of a graph G. As an example of this, consider the following graph:



G

Let $k = 2$. The family of directed coboundaries of G contains $\{e,f,j\}$, $\{g,h,i\}$ and $\{e,f,g,h\}$. Therefore $T \equiv \{e,f,g,h\}$ is a minimal directed coboundary of G. However, we have

$$\begin{array}{rcl}
x_e + x_f \qquad\qquad\qquad + x_j & \geq & 2 \\
x_g + x_h + x_i \qquad & \geq & 2 \\
- x_i \qquad & \geq & -1 \\
- x_j & \geq & -1 \\
\hline
x_e + x_f + x_g + x_h \qquad\qquad & \geq & 2.
\end{array}$$

Hence, $x(T) \geq 2$ is nonessential for defining $P_2(G)$. Since $P_2(G)$ is of full dimension, $L^2_{\delta(T)}$ is not a facet of $P_2(G)$, by (2.3.31).

5.3.25 However, we do have a characterization of the sets $T \in D(G)$ such that $L^k_{\delta(T)}(G)$ is a facet of $P_k(G)$ for $k \geq 2$.

5.3.2 <u>Theorem</u> Let $G = (V,E)$ be a connected graph and k be a positive integer such that $P_k(G)$ is of full dimension. Then for any $T \in D(G)$, $L^k_{\delta(T)}$ is a facet of $P_k(G)$ if and only if there is no family $F \subseteq D(G)-\{T\}$ such that

$$\Sigma(|\delta(S)|-k : S \in F) + |\delta(T) - \cup(\delta(S) : S \in F)| \leq |\delta(T)|-k.$$

<u>Proof</u> See (7.2.7). $\square$

CHAPTER 6

SUBMODULAR FUNCTIONS ON GRAPHS

In this chapter we amalgamate concepts of Chapters 3 and 4 into a common theory. We will show that many "integer-valued results" of the previous chapters are special consequences of this theory; by an "integer-valued result" we mean a result of the form "this linear program has an integer-valued optimum solution" (cf.(4.3.4), (5.1.12) and (5.2.10)).

6.1 A Class of Totally Unimodular Matrices

6.1.1 You may recall that in Chapter 4 we required the total unimodularity of a particular matrix A (specifically we are referring to the proofs of (4.3.4) and (4.5.5)). A was obtained from the matrix of a bipartite graph by multiplying certain rows by -1. In this section we extend the class of matrices of graphs to a larger class of totally unimodular matrices. This class is also obtained from graphs and a construction is attributed to N. Robertson by Lovász [L1].

6.1.2 Given a tree T and a set V, a V-labelling of T is a function $\ell: V \to V(T)$. For $S \subseteq V$ let $\ell(S) \equiv \{\ell(v): v \in S\}$ and for $e \in E(T)$ let $\lambda(e) \equiv \{v \in V: \ell(v) \in \omega(e)\}$.

6.1.3 Lemma If ℓ is a V-labelling of a tree T and $S \subseteq V$ then either there exist subtrees T_1, T_2 of T and a node $v \in V(T)$ such that

6.1.4 $V(T_1) \cap V(T_2) = \{v\}, V(T_1) \cup V(T_2) = V(T), \ell(S) \subseteq V(T_1)$ and $\ell(\bar{S}) \subseteq V(T_2)$.

or there exists an edge $e \in E(T)$ such that

6.1.5 each of $\alpha(e) \cap \ell(S)$, $\alpha(e) \cap \ell(\bar{S})$, $\omega(e) \cap \ell(S)$, $\omega(e) \cap \ell(\bar{S})$ is nonempty.

Proof Let T_1', T_2' be the unique minimal subtrees of T such that $\ell(S) \subseteq V(T_1')$ and $\ell(\bar{S}) \subseteq V(T_2')$. If $|V(T_1') \cap V(T_2')| \leq 1$ then we can extend T_1' and T_2' to subtrees T_1 and T_2 respectively for which (6.1.4) holds.

If $|V(T_1') \cap V(T_2')| \geq 2$ then there is an edge $e \in E(T_1') \cap E(T_2')$. But $e \in E(T_1')$ if and only if $\alpha(e) \cap \ell(S) \neq \phi$ and $\omega(e) \cap \ell(S) \neq \phi$ and $e \in E(T_2')$ if and only if $\alpha(e) \cap \ell(\bar{S}) \neq \phi$ and $\omega(e) \cap \ell(\bar{S}) \neq \phi$. Hence (6.1.5) holds for e. $\square$

6.1.6 Lemma A family F of set V is a cross-free family if and only if

6.1.7 There exists a tree T and a V-labelling ℓ of T such that $F = \{\lambda(e) : e \in E(T)\}$.

Proof Suppose F is defined by (6.1.7) and let $Y = \lambda(e)$ and $Z = \lambda(j)$ for distinct edges $e, j \in E(T)$. Let π be the path in T from $h(e)$ to $h(j)$. There are four possibilities. If π contains neither $t(e)$ nor $t(j)$ then $Y \cap Z = \phi$. If π contains $t(e)$ but not $t(j)$ then $Z \subseteq Y$. If π contains $t(j)$ but not $t(e)$ then $Y \subseteq Z$. If π contains both $t(e)$ and $t(j)$ then $Y \cup Z = V$. Hence F is a cross-free family of V.

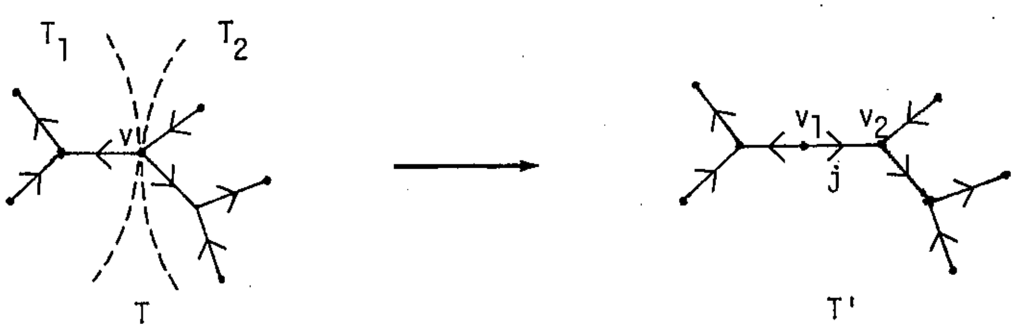
We prove that if F is a cross-free family then (6.1.7) holds by induction on $|F|$. If F consists of just one set S then let T consist of a single edge e and for each $v \in V$ let

$$\ell(v) \equiv \begin{cases} t(e) \text{ if } v \in S \\ h(e) \text{ if } v \notin S \ . \end{cases}$$

Clearly T and ℓ satisfy (6.1.7). Now assume that $|F| \geq 2$ and (6.1.7) holds for all cross-free families H with $|H| < |F|$. Let $S \in F$ and consider the family $H \equiv F-\{S\}$. By the induction hypothesis there exists a tree T and a V-labelling ℓ of T such that $H = \{\lambda(e):e \in E(T)\}$. If for some edge $e \in E(T)$ each of $\alpha(e) \cap \ell(S)$, $\alpha(e) \cap \ell(\bar{S})$, $\omega(e) \cap \ell(S)$ and $\omega(e) \cap \ell(\bar{S})$ is nonempty then $\lambda(e)$ and S cross; a contradiction. Hence, by (6.1.3), there exist subtrees T_1 and T_2 of T and a node $v \in V(T)$ such that (6.1.4) holds. Let T' be the tree with $V(T') \equiv (V(T)-\{v\}) \cup \{v_1,v_2\}$, $E(T') \equiv E(T) \cup j$ where $v_1,v_2 \notin V$, $j \notin E(T)$ and for each $e \in E(T')$,

$$t'(e) \equiv \begin{cases} t(e) & \text{if } t(e) \neq v \text{ and } e \neq j \\ v_1 & \text{if } t(e) = v \text{ and } e \in E(T_1), \text{ or } e = j \\ v_2 & \text{if } t(e) = v \text{ and } e \in E(T_2) \end{cases}$$

$$h'(e) \equiv \begin{cases} h(e) & \text{if } h(e) \neq v \text{ and } e \neq j \\ v_1 & \text{if } h(e) = v \text{ and } e \in E(T_1) \\ v_2 & \text{if } h(e) = v \text{ and } e \in E(T_2), \text{ or } e = j . \end{cases}$$



For $w \in V$ let

$$\ell'(w) \equiv \begin{cases} \ell(w) & \text{if } \ell(w) \neq v \\ v_1 & \text{if } \ell(w) = v \text{ and } w \in S \\ v_2 & \text{if } \ell(w) = v \text{ and } w \notin S. \end{cases}$$

It is easy to check that F, T' and ℓ' satisfy (6.1.7). Therefore the lemma holds. □

6.1.8 Theorem If $G = (V,E)$ is a graph and F is a cross-free family of V then the matrix $A \in \mathbb{R}^{F \times E}$ with rows $[cv(S): S \in F]$ is totally unimodular.

Proof By (6.1.6) there is a tree T and a V-labelling ℓ of T such that $F = \{\lambda(e): e \in T\}$. Consider the graph H where $V(H) \equiv V(T)$, $E(H) \equiv E(G) \cup E(T)$ and for all $e \in E(G)$, $t_H(e) \equiv \ell(t_G(e))$ and $h_H(e) \equiv \ell(h_G(e))$. T is a spanning tree of H and so, by (2.6.15), the matrix B with rows $[cv_H(\omega(e)): e \in E(T)]$ is totally unimodular. However, for each $e \in E(T)$, $\delta_H(\omega(e)) = \delta_G(\lambda(e)) \cup e$ and $\delta_H(\alpha(e)) = \delta_G(\overline{\lambda(e)})$. Hence, $B = [A, I_F]$ and, by (2.5.10), A is totally unimodular. □

6.2 Submodular Functions on Graphs

6.2.1 Given a connected graph $G = (V,E)$ recall that $D(G)$ is the family of sets $S \subseteq V$ such that $\delta(S)$ is a directed coboundary of G. An important property of $D(G)$ is that it is a crossing family of V; i.e. if $S,T \in D(G)$, $S \cap T \neq \phi$ and $S \cup T \neq V$ then $S \cap T, S \cup T \in D(G)$. A proof of this property of $\mathbb{D}(G)$ is in the next section (see (6.3.12)).

6.2.2 The important point about polymatroids was the submodularity of the functions $f:L_E \to \mathbb{R}$ which determined polymatroids. Now we will blend the concepts of submodular functions and crossing families.

6.2.3 Let $G = (V,E)$ be a graph, F a crossing family on V and $f:F \to \mathbb{R}$ a submodular function. Let a,d be vectors with possibly infinite components. For a given vector $c \in \mathbb{R}^E$ consider the linear program

6.2.4 maximize $c \cdot x$ where $x \in \mathbb{R}^E$ satisfies

$$\text{6.2.5} \quad \begin{cases} x_e \le a_e \text{ for all } e \in E \\ x_e \ge d_e \text{ for all } e \in E \\ cv(S) \cdot x \le f(S) \text{ for all } S \in F. \end{cases}$$

The dual linear program of 6.2.4 is

6.2.6 minimize $f \cdot y + a \cdot z - d \cdot w$ where $y \in \mathbb{R}^F$ and $w,z \in \mathbb{R}^E$ satisfy

$$\text{6.2.7} \quad \begin{cases} y_S \ge 0 \text{ for all } S \in F \\ w_e, z_e \ge 0 \text{ for all } e \in E \\ F(y,e)+z_e-w_e = c_e \text{ for all } e \in E, \end{cases}$$

where for any family F on V, $y \in \mathbb{R}^F$ and $e \in E$

$$F(y,e) \equiv \Sigma(y_S : e \in \delta(S), S \in F) - \Sigma(y_S : e \in \delta(\bar{S}), S \in F).$$

(We interpret the dual variable of an infinite constraint as being zero.)

6.2.8 We next show that if $c \in \mathbb{Z}^E$ and (6.2.6) has an optimum solution then (6.2.6) has an integer-valued optimum solution. We can represent the linear system (6.2.7) by $Ax \le b$ for an appropriate choice of A and b.

In general, A will not be totally unimodular. However, we can manipulate an optimum solution to (6.2.6) as we did in the proof of (4.5.5) to obtain a linear program such that any optimum solution to this linear program will be an optimum solution to (6.2.6). Moreover, the constraint matrix of the new linear program will be totally unimodular, so it and (6.2.6) will have integer-valued optimum solutions.

6.2.9 Theorem If $c \in \mathbb{Z}^E$ and (6.2.6) has an optimum solution then (6.2.6) has an integer-valued optimum solution.

Proof[1]. If $c \in \mathbb{Z}^E$ and (6.2.6) has an optimum solution then, by (2.5.1), (6.2.6) has a rational-valued optimum solution $[y^0, z^0, w^0]$.

6.2.10 Starting with $j = 0$, suppose $Y, Z \in F$, Y and Z cross and $0 < y^j_Y \le y^j_Z$. Then, since F is a crossing family, $Y \cap Z \in F$ and $Y \cup Z \in F$. For $S \in F$ define y^{j+1}_S by

$$y^{j+1}_S \equiv \begin{cases} y^j_S + y^j_Y & \text{if } S \in \{Y \cap Z, Y \cup Z\} \\ y^j_S - y^j_Y & \text{if } S \in \{Y, Z\} \\ y^j_S & \text{otherwise.} \end{cases}$$

It is easy to check that $F(y^{j+1}, e) = F(y^j, e)$ for all $e \in E$. Therefore $[y^{j+1}, z^0, w^0]$ is a feasible solution to (6.2.6). Furthermore,

$$\begin{aligned} f \cdot y^{j+1} &= f \cdot y^j + y^j_Y[f(Y \cap Z) + f(Y \cup Z) - f(Y) - f(Z)] \\ &\le f \cdot y^j, \end{aligned}$$

---

[1] This is a generalization of a method used by Lovasz [L1] to prove (5.1.2) and of the method used in Chapter 4 to prove (4.3.4).

by the submodularity of f. Hence $[y^{j+1},z^0,w^0]$ must also be an optimum solution to (6.2.6).

As in the proof of(4.5.5), let α be a common denominator of $\{y_S^0 : S \in F\}$. Let $u^0 \equiv \alpha y^0$ and for each vector y^{j+1} constructed according to (6.2.10) let $u^{j+1} \equiv \alpha y^{j+1}$. Since $y^{j+1} \geq 0$, $u^{j+1} \in \mathbb{Z}_+^F$. Since $1 \cdot y^{j+1} = 1 \cdot y^j$ we have $1 \cdot u^{j+1} = 1 \cdot u^j$. There can only be a finite number of distinct vectors $u \in \mathbb{Z}_+^E$ having the same sum $1 \cdot u$. Therefore, there can be only a finite number of distinct vectors in the sequence $\{y^0,y^1,\ldots,y^j,y^{j+1},\ldots\}$. Since

$$\Sigma(y_S^{j+1}|S|^2 : S \in F) = \Sigma(y_S^j|S|^2 : S \in F) + y_Y^j[|Y \cap Z|^2 + |Y \cup Z|^2 - |Y|^2 - |Z|^2] > \Sigma(y_S^j|S|^2 : S \in F),$$

the sequence has only finitely many terms.

Therefore, there is an optimum solution $[y^\ell,z^0,w^0]$ to (6.2.6) with the property that the family $F^\ell \equiv \{S \in F : y_S > 0\}$ is a cross-free family. $[y^\ell,z^0,w^0]$ is a feasible solution to the linear program

6.2.11 minimize $f \cdot y + a \cdot z - d \cdot w$ where $y \in R^F$ and $w,z \in R^E$ satisfy

6.2.12
$$\begin{cases} y_S \geq 0 \text{ for all } S \in F \\ w_e, z_e \geq 0 \text{ for all } e \in E \\ y_S \leq 0 \text{ for all } S \in F - F^\ell \\ F^\ell(y,e) + z_e - w_e = c_e \text{ for all } e \in E. \end{cases}$$

Any feasible solution to (6.2.11) is a feasible solution to (6.2.6). Since $[y^{\ell},z^0,w^0]$ is an optimum solution to (6.2.6), $[y^{\ell},z^0,w^0]$ must be an optimum solution to (6.2.11). Therefore, any optimum solution to (6.2.11) is an optimum solution to (6.2.6).

Represent the linear system (6.2.12) by $Ax \le b$ for an appropriate choices of A and b. Consider the submatrix A' of A corresponding to the constraints $F^{\ell}(y,e) + z_e - w_e = c_e$ for all $e \in E$; omitting the columns corresponding to w_e, z_e. It is easily seen that the columns of A' are of the form $[cv(S):S \in F^{\ell}]$. Since F^{ℓ} is a cross-free family, A' is totally unimodular, by (2.5.8) and (6.1.8). Therefore, by (2.5.8)-(2.5.11), A is totally unimodular. By (2.5.16), (6.2.11) has an integer-valued optimum solution $[\bar{y},\bar{z},\bar{w}]$. Therefore, $[\bar{y},\bar{z},\bar{w}]$ is an integer-valued optimum solution to (6.2.6). □

In general, the polyhedron of feasible solutions to (6.2.4) may not be pointed. However, in the case that it is we have, by (2.5.2),

6.2.13 Corollary If $a,d \in \mathbb{Z}^E$, f is integer-valued and the polyhedron P of feasible solutions to (6.2.4) is pointed then the vertices of P are integer-valued.

6.2.14 Theorem If $a,d \in \mathbb{Z}^E$, f(S) is an integer for all $S \in F$ then (6.2.4) has an integer-valued optimum solution for every $c \in \mathbb{R}^E$ such that (6.2.4) has an optimum solution.

<u>Proof</u> Let x^0 be an optimum solution to (6.2.4) and for each $e \in E$ let $a_e^0 \equiv \lceil x_e^0 \rceil$, $d_e^0 \equiv \lfloor x_e^0 \rfloor$. Clearly $d_e^0 \geq d_e$ and $a_e^0 \leq a_e$ for all $e \in E$. For any $c^0 \in \mathbb{R}^E$ consider the linear program

6.2.15 maximize $c^0 \cdot x$ where $x \in \mathbb{R}^E$ satisfies

$$x_e \leq a_e^0 \text{ for all } e \in E$$
$$x_e \geq d_e^0 \text{ for all } e \in E$$
$$cv(S) \cdot x \leq f(S) \text{ for all } S \in F.$$

Since x^0 is an element of the polyhedron P of feasible solutions to (6.2.15), P is nonempty and bounded. Therefore, by (2.3.18), P is pointed. By (6.2.13), (6.2.15) has an integer-valued optimum solution for every $c^0 \in \mathbb{R}^E$. In particular, for $c^0 = c$, (6.2.15) has an integer-valued optimum solution x^1. Clearly, any feasible solution to (6.2.15) is a feasible solution to (6.2.4). Since x^0 is a feasible solution to (6.2.15), x^1 must also be an optimum solution to (6.2.4). □

Combining (6.2.9), (6.2.14) and the Strong L.P. Duality Theorem we have

6.2.16 <u>Corollary</u> If $a, d \in \mathbb{Z}^E$, $c \in \mathbb{Z}^E$, $f(S)$ is an integer for all $S \in F$ and (6.2.4) has an optimum solution then

$$\max\{c \cdot x : x \in \mathbb{Z}_+^E,\ x \text{ satisfies (6.2.5)}\}$$
$$=$$
$$\min\{f \cdot y + a \cdot z - d \cdot w : y \in \mathbb{Z}_+^F, w, z \in \mathbb{Z}_+^E, [y,z,w] \text{ satisfies (6.2.7)}\}.$$

□

6.3 Applications

In this section we show how certain key results in the theory of polymatroid intersection, strong k-covers and strong k-matchings are consequences of results in the previous section. We will indicate how some of the linear programs we associated with polymatroid intersection, strong k-covers and strong k-matchings are special instances of (6.2.4) and thereby be able to apply (6.2.9).

Polymatroid Intersection

6.3.1 Recall that in section 4.3 we considered the linear program

6.3.2 maximize $c \cdot x$ where $x \in \mathbb{R}^E$ satisfies

$x_e \geq 0$ for all $e \in E$

6.3.3 $x(S) \leq f_1(S)$ for all S K_E

6.3.4 $x(S) \leq f_2(S)$ for all $S \in K_E$,

where $f_1, f_2 : L_E \to \mathbb{R}$ are β_0-functions and $c \in \mathbb{R}^E$ (see (4.3.1)).

6.3.5 We now show how (6.3.2) is a special instance of (6.2.4). Let E be the edge set of a loopless graph where each component of G is a single edge together with its head and tail. For each $S \in K_E$ let $S_1 \equiv \{t(e) : e \in S\}$, $S_2 \equiv \{t(e) : e \in E\} \cup \{h(e) : e \notin S\}$, $f(S_1) \equiv f_1(S)$ and $f(S_2) \equiv f_2(S)$. Let $F \equiv \{S_1 : S \in K_E\} \cup \{S_2 : S \in K_E\}$.

6.3.6 If for $S, T \in K_E$ we have $S_1 \cap T_1 \neq \phi$ then $S_1 \cap T_1 = (S \cap T)_1$ and $S_1 \cup T_1 = (S \cup T)_1$ are elements of F. If $S_2 \cup T_2 \neq V$ then

$S \cap T \neq \phi$ and $S_2 \cap T_2 = (S \cup T)_2$ and $S_2 \cup T_2 = (S \cap T)_2$ are elements of F. Since $S_1 \subseteq T_2$, F is a crossing family.

6.3.7 For $S,T \in K_E$ we have

$$\begin{aligned} f(S_1 \cap T_1)+f(S_1 \cup T_1) &= f_1(S \cap T)+f_1(S \cup T) \\ &\leq f_1(S)+f_1(T) \\ &= f(S_1)+f(T_1) \end{aligned}$$

and

$$\begin{aligned} f(S_2 \cap T_2)+f(S_2 \cup T_2) &= f((S \cup T)_2)+f((S \cap T)_2) \\ &= f_2(S \cup T)+f_2(S \cap T) \\ &\leq f_2(S)+f_2(T) \\ &= f(S_2)+f(T_2). \end{aligned}$$

Since $S_1 \subseteq T_2$, $f(S_1 \cap T_2)+f(S_1 \cup T_2) = f(S_1)+f(T_2)$. Therefore, f is a submodular function of F.

6.3.8 Because $\delta(S_1) = S$ and $\delta(V-S_1) = \phi$ for all $S \in K_E$, the inequality $x(S) \leq f_1(S)$ is equivalent to $cv(S_1)\cdot x \leq f(S_1)$. Similarly, $\delta(S_2) = S$ and $\delta(V-S_2) = \phi$, and so the inequality $x(S) \leq f_2(S)$ is equivalent to $cv(S_2)\cdot x \leq f(S_2)$. Therefore (6.3.2) is indeed a special case of (6.2.4), with $d \equiv 0$, $a \equiv +\infty$.

6.3.9 By (6.2.9), if $c \in \mathbb{Z}^E$ then the dual linear program of (6.3.2)

6.3.10 minimize $f_1 \cdot y^1 + f_2 \cdot y^2$ where $y^1, y^2 \in \mathbb{R}^{K_E}$ satisfy

$$y^i_S \geq 0 \text{ for all } S \in K_E,\ i = 1,2$$

$$y^1(K_E,e) + y^2(K_E,e) \geq c_e \text{ for all } e \in E$$

always has an integer-valued optimum solution. This is the statement of (4.3.4), an important result of Chapter 4.

6.3.11 If $f_1(S)$ and $f_2(S)$ are integers for all $S \in K_E$ then, by (6.2.13), the vertices of the set of feasible solutions to (6.3.2), i.e. the vertices of $P(K_E,f_1) \cap P(K_E,f_2)$ are integer-valued. This is the statement of (4.3.8).

Strong k-Matchings

6.3.12 Recall that for any graph $G = (V,E)$, $D(G)$ was the family of subsets $S \subseteq V$ such that $\delta(S) \neq \phi$, $\delta(\bar{S}) = \phi$. In the previous section we asserted:

6.3.13 Proposition For any connected graph $G = (V,E)$, $D(G)$ is a crossing family.

Proof Suppose $S,T \subseteq V$ are such that S and T cross, $\delta(\bar{S}) = \delta(\bar{T}) = \phi$, $\delta(S) \neq \phi$ and $\delta(T) \neq \phi$. If $e \in \delta(\overline{S \cap T})$ then $e \in \delta(\bar{S})$ or $e \in \delta(\bar{T})$, which is impossible. Therefore, $\delta(S \cap T) = \phi$. Similarly, $\delta(S \cup T) = \phi$. Since G is connected, $S \cap T \neq \phi$ and $S \cup T \neq V$, both $\delta(S \cap T)$ and $\delta(S \cup T)$ are nonempty. Therefore, $S \cap T$, $S \cup T \in F$. □

6.3.14 If we let $f(S) \equiv k$ for all $S \in D(G)$ then clearly f is submodular function of $D(G)$. Since $\delta(\bar{S}) = \phi$ for all $S \in D(G)$, the inequality $x(\delta(S)) \le k$ is equivalent to $cv(S)\cdot x \le f(S)$. If we now let $a \equiv 1$, $d \equiv 0$ then (6.2.4) becomes

6.3.15 maximize $c \cdot x$ where $x \in \mathbb{R}^E$ satisfies

$$0 \le x_e \le 1 \text{ for all } e \in E$$

$$x(\delta(S)) \le k \text{ for all } S \in D(G),$$

which is (5.2.3). The key result concerning strong k-matchings was that if G is acyclic and k is a positive integer then

6.3.16 For all $c \in \mathbb{Z}^E$ the dual linear program of (6.3.15) has an integer-valued optimum solution. (See (5.2.10)).

We may assume w.l.o.g. that G is connected and then (6.3.16) follows from (6.2.9).

Strong k-covers

6.3.17 For any family F on set V let $\bar{F} \equiv \{\bar{S} : S \in F\}$.

6.3.18 Proposition If F is crossing family on V then $\bar{F}$ is a crossing family on V.

Proof Suppose $S,T \in \bar{F}$ and S,T cross. Since $S \cap T \ne \phi$, $\bar{S} \cup \bar{T} \ne V$. Since $S \cup T \ne V$, $\bar{S} \cap \bar{T} \ne \phi$. Hence, because F is a crossing family, we have $\bar{S} \cap \bar{T} = \overline{S \cup T} \in F$ and $\bar{S} \cup \bar{T} = \overline{S \cap T} \in F$. Therefore, $S \cap T$, $S \cup T \in \bar{F}$ and $\bar{F}$ is a crossing family on V. □

6.3.19 In studying strong k-covers of a graph $G = (V,E)$ we may assume w.l.o.g. that G is connected. One of the linear programs we associated with strong 1-covers of G was:

6.3.20 minimize $c \cdot x$ where $x \in \mathbb{R}^E$ satisfies

$$x_e \geq 0$$

$$x(\delta(S)) \geq 1 \text{ for all } S \in D(G),$$

where $c \in \mathbb{R}^E_+$. (See (5.1.10)).

6.3.21 Let $F \equiv \{S \subseteq V : \delta(\bar{S}) \neq \phi, \delta(S) = \phi\}$. Then $F = \overline{D(G)}$ and, by (6.3.13) and (6.3.18), F is a crossing family on V. For all $S \in F$ let $f(S) \equiv -1$. Clearly $f: F \to \mathbb{R}$ is a submodular function. For all $S \in F$, the inequality $x(\delta(\bar{S})) \geq 1$ is equivalent to $cv(S) \cdot x \leq f(S)$. Therefore, (6.3.20) is an instance of (6.2.4). By (6.2.9), the dual linear program of (6.3.20):

maximize $1 \cdot y$ where $y \in \mathbb{R}^{D(G)}$ satisfies

$$y_S \geq 0 \text{ for all } S \in D(G)$$

$$\Sigma(y_S : e \in \delta(S), S \in D(G)) \leq c_e$$

has an integer-valued optimum solution for all $c \in \mathbb{Z}^E_+$. That is the statement of (5.1.12).

6.3.22 In exactly the same manner one can show that for any positive integer k the linear program

6.3.23 minimize $c \cdot x$ where $x \in \mathbb{R}^E$ satisfies

$$0 \le x_e \le 1 \text{ for all } e \in E$$
$$x(\delta(S)) \ge k \text{ for all } S \in D(G)$$

is a special instance of (6.2.4). Therefore, by (6.2.9), the dual linear program of (6.3.23) has an integer-valued optimum solution for every $c \in \mathbb{Z}^E$, provided $k \le |\delta(S)|$ for all $S \in D(G)$. This is (5.1.21).

Supermodular Functions on Graphs

6.3.24 For any family F on set V a function $f:F \to \mathbb{R}$ is said to be *supermodular* if

$$f(Y \cap Z) + f(Y \cup Z) \ge f(Y) + f(Z)$$

for all $Y,Z \in F$ such that $Y \cap Z$, $Y \cup Z \in F$. Clearly, f is supermodular of F if and only if $-f$ is submodular function of F. We can treat supermodular functions on graphs just as submodular functions on graphs in the previous section.

6.3.25 Let $G = (V,E)$ be a graph, F a crossing family on V and $f:F \to \mathbb{R}$ be a supermodular function. Let $a,d \in \mathbb{R}^E$ with possibly infinite components. For any $c \in \mathbb{R}^E$ consider the linear program

6.3.26 minimize $c \cdot x$ where $x \in \mathbb{R}^E$ satisfies

$$x_e \le a_e \text{ for all } e \in E$$
$$x_e \ge d_e \text{ for all } e \in E$$
$$cv(S) \cdot x \ge f(S) \text{ for all } S \in F.$$

6.3.27 We can translate (6.3.26) into an equivalent submodular version in the form of (6.2.4). Let $\bar{F} \equiv \{\bar{S}:S \in F\}$. By (6.3.18), $\bar{F}$ is also a crossing family of V. For any $S \subseteq V$, $cv(S) = -cv(\bar{S})$. Therefore, for all $S \in F$ the inequality $cv(S)\cdot x \geq f(S)$ is equivalent to $-cv(\bar{S})\cdot x \geq f(S)$. If we let $f'(S) \equiv -f(\bar{S})$ for all $S \in \bar{F}$ then for all $Y,Z \in \bar{F}$ such that $Y \cap Z$, $Y \cup Z \in \bar{F}$ we have

$$\begin{aligned} f'(Y \cap Z)+f'(Y \cup Z) &= -f(\overline{Y \cap Z})-f(\overline{Y \cup Z}) \\ &= -f(\bar{Y} \cup \bar{Z})-f(\bar{Y} \cap \bar{Z}) \\ &\leq -f(\bar{Y}) - f(\bar{Z}) \\ &= f'(V) + f'(Z). \end{aligned}$$

Hence, f' is a submodular function of $\bar{F}$. For all $S \in F$ the inequality $cv(S)\cdot x \geq f(S)$ is equivalent to $cv(\bar{S})\cdot x \leq f'(\bar{S})$ and (6.3.26) is equivalent to

6.3.28 maximize $-c\cdot x$ where $x \in \mathbb{R}^E$ satisfies

$$\begin{aligned} &x_e \leq a_e \text{ for all } e \in E \\ &x_e \geq d_e \text{ for all } e \in E \\ &cv(S)\cdot x \leq f'(S) \text{ for all } S \in \bar{F}. \end{aligned}$$

Clearly (6.3.28) is a linear program in the form of (6.2.4). The dual linear program of (6.3.26) is

6.3.29 maximize $f\cdot y - a\cdot z + d\cdot w$ where $y \in \mathbb{R}^F$ and $w,z \in \mathbb{R}^E$ satisfy

$$\begin{aligned} &y_S \geq 0 \text{ for all } S \in F \\ &w_e,z_e \geq 0 \text{ for all } e \in E \\ &F(y,e)-z_e+w_e = c_e \text{ for all } e \in E. \end{aligned}$$

By (6.2.9),

6.3.30 <u>Theorem</u> If $c \in \mathbb{Z}^E$ and (6.3.29) has an optimum solution then (6.3.29) has an integer-valued optimum solution. $\square$

By (6.2.14),

6.3.31 <u>Theorem</u> If $a, d \in \mathbb{Z}^E$ (with possibly infinite components) and $f(S)$ is an integer for all $S \in F$ then (6.3.26) has an integer-valued optimum solution for all $c \in \mathbb{R}^E$ such that (6.3.26) has an optimum solution. $\square$

<u>Network Flows</u>

6.3.32 Let $G = (V,E)$ be a loopless graph, $a \in \mathbb{R}^E_+$, $q \in \mathbb{R}^V$. A <u>feasible flow</u> in G is a vector $x \in \mathbb{R}^E$ which satisfies

$$6.3.33 \quad \begin{cases} x_e \geq 0 \text{ for all } e \in E \\ x_e \leq a_e \text{ for all } e \in E \\ cv(v) \cdot x = q_v \text{ for all } v \in V. \end{cases}$$

Let P be the solution set of (6.3.33). We can describe P using a submodular function of a cross-free family on V.

6.3.34 Let $F_1 \equiv \{\{v\}: v \in V\}$ and $F_2 \equiv \{V-\{v\}: v \in V\}$. For each $\{v\} \in F_1$ let $f(\{v\}) \equiv q_v$ and for each $V - \{v\} \in F_2$ let $f(V - \{v\}) \equiv -q_v$. Let $F \equiv F_1 \cup F_2$. It is easily seen that F is a cross-free family and that f is a submodular function of F. For any $v \in V$ the inequalities $cv(\{v\}) \cdot x \leq f(\{v\}) = q_v$ and $cv(V - \{v\}) \cdot x \leq f(V - \{v\}) = -q_v$ imply $cv(v) \cdot x = q_v$. Thus (6.3.33) is an instance of (6.2.4).

CHAPTER 7

FACETS FOR SUBMODULAR FUNCTIONS ON GRAPHS

7.1 Some Facets for Submodular Functions on Graphs

7.1.1 Let $G = (V,E)$ be a graph, F a crossing family on V and $f:F \to \mathbb{R}$ a submodular function. Let $a,d \in \mathbb{R}^E$ with possibly infinite components. For this section P will denote the set of solutions of the linear system

7.1.2 $x_e \leq a_e$ for all $e \in E$

7.1.3 $x_e \geq d_e$ for all $e \in E$

7.1.4 $cv(S) \cdot x \leq f(S)$ for all $S \in F$;

i.e. P is the set of feasible solutions to the linear program (6.2.4). By placing some restrictions on the combinatorial description of P we can determine the facets of P.

7.1.5 If for some $j \in E$, a_j is finite then

$$P^j \equiv \{x \in P : x_j = a_j\}$$

is a face of P. It may be that for some $S \in F$ we have $\delta(S) = \{j\}$, $\delta(\bar{S}) = \{\phi\}$ and $f(S) = a_j$, in which case the inequality $cv(S) \cdot x \leq f(S)$ is equivalent to $x_j \leq a_j$. If we assume that there is no such $S \in F$ then we can determine whether or not P^j is a facet of P.

7.1.6 Theorem Let P, the set of solutions to (7.1.2)-(7.1.4), be of full dimension. Let $j \in E$ be such that a_j is finite. If there is no $S \in F$ such that $\delta(S) = \{j\}$, $\delta(\bar{S}) = \phi$ and $f(S) = a_j$ then P^j is a facet of P if and only if there is no integer-valued $[y,z,w]$ satisfying

$$7.1.7 \quad \begin{cases} y_S \geq 0 \quad \text{for all } S \in F \\ w_e, z_e \geq 0 \quad \text{for all } e \in E \\ z_j \leq 0 \\ F(y,j)-w_j = 1 \\ F(y,e)+z_e-w_e = 0 \quad \text{for all } e \neq j \end{cases}$$

and

$$7.1.8 \qquad f\cdot y + a\cdot z - d\cdot w \leq a_j.$$

<u>Proof</u> Suppose P^j is a facet of P. Since there is no $S \in F$ such that $\delta(S) = \{j\}$, $\delta(\bar{S}) = \phi$ and $f(S) = a_j$, the inequality $x_j \leq a_j$ is not equivalent to $cv(S)\cdot x \leq f(S)$ for any $S \in F$. Therefore, because P is of full dimension, the inequality $x_j \leq a_j$ of (7.1.2) is essential for defining P, by (2.3.31). By (2.4.19), there can be no $[y,z,w]$ which satisfies (7.1.7) and (7.1.8).

Conversely, suppose P is not a facet of P. By (2.3.29), $x_j \leq a_j$ is nonessential for defining P. By (2.4.18) and the Strong L.P. Duality Theorem, the optimum value of the linear program

7.1.9 minimize $f\cdot y + a\cdot z - d\cdot w$ where $[y,z,w]$ satisfies (7.1.7)

is less than or equal to a_j. We now show that (7.1.9) has an integer-valued optimum solution. By (2.5.1), (7.1.9) has a rational-valued optimum solution $[y^0,z^0,w^0]$.

7.1.10 As in the proof of (6.2.9), we can find an optimum solution $[y^\ell, z^0, w^0]$ with the property that the family $F^\ell \equiv \{S \in F : y_S^\ell > 0\}$ is a cross-free family. Hence, by an argument similar to that outlined in the proof of (6.2.9), (7.1.9) has an integer-valued optimum solution $[\bar{y}, \bar{z}, \bar{w}]$. Since $f \cdot \bar{y} + a \cdot \bar{z} - d \cdot \bar{w} \le a_j$, $[\bar{y}, \bar{z}, \bar{w}]$ satisfies (7.1.7) and (7.1.8). □

7.1.11 Similarly, if d_j is finite for some $j \in E$ then

$$P_j \equiv \{x \in P : x_j = d_j\}$$

is a face of P. We could imitate the proof of (7.1.6) to prove the following, but we will apply (7.1.6) directly.

7.1.12 Theorem Let P, the set of solutions to (7.1.2)-(7.1.4) be of full dimension. Let $j \in E$ be such that d_j is finite. If there is no $S \in F$ such that $\delta(S) = \phi$, $\delta(\bar{S}) = \{j\}$ and $f(S) \equiv -d_j$ then P_j is a facet of P if and only if there is no integer-valued $[y,z,w]$ satisfying

$$\text{7.1.13} \quad \begin{cases} y_S \ge 0 \quad \text{for all } S \in F \\ w_e, z_e \ge 0 \text{ for all } e \in E \\ w_j \le 0 \\ F(y,j) + z_j = -1 \\ F(y,e) + z_e - w_e = 0 \text{ for all } e \ne j. \end{cases}$$

and

7.1.14 $\quad f \cdot y + a \cdot z - d \cdot w \le -d_j.$

Proof Let $\hat{G}$ be obtained from G by reversing the edge j. Let $\hat{a}, \hat{d} \in \mathbb{R}^E$ be defined by

$$\hat{a}_e \equiv \begin{cases} -d_e & \text{if } e = j \\ a_e & \text{otherwise} \end{cases}$$

$$\hat{d}_e \equiv \begin{cases} -a_e & \text{if } e = j \\ d_e & \text{otherwise.} \end{cases}$$

Let $\hat{P}$ be the solution set of the linear system

$$x_e \geq \hat{d}_e \quad \text{for all } e \in E$$
$$x_e \leq \hat{a}_e \quad \text{for all } e \in E$$
$$cv_{\hat{G}}(S) \cdot x \in f(S) \text{ for all } S \in F.$$

Let $g: \mathbb{R}^E \to \mathbb{R}^E$ be the linear transformation defined by

$$(g(x))_e \equiv \begin{cases} -x_e & \text{if } e = j \\ x_e & \text{otherwise,} \end{cases}$$

for all $x \in \mathbb{R}^E$ and for all $e \in E$. It is straightforward to verify that $g(P) = \hat{P}$. Since g is nonsingular, a set $K \subseteq \mathbb{R}^E$ is affinely independent if and only if $g(K)$ is affinely independent. Therefore, P contains k affinely independent vectors if and only if $\hat{P}$ contains k affinely independent vectors. Hence, by (2.3.23), $\dim(P) = \dim(\hat{P})$.

Consider $\hat{P}^j = \{x \in \hat{P}: x_j = \hat{a}_j\}$. Since $\hat{a}_j = -d_j$, $cv_G(S) \cdot x = cv_{\hat{G}}(S) \cdot g(x)$ for all $x \in P^j$ and for all $S \in F$, $g(P_j) = \hat{P}^j$ and, as above, $\dim(P_j) = \dim(\hat{P}^j)$. Therefore, P_j is a facet of P if and only if $\hat{P}^j$ is a facet of $\hat{P}$.

Since P is of full dimension, $\hat{P}$ is of full dimension. Since there is no $S \in F$ such that $\delta_G(S) = \phi$, $\delta_G(\bar{S}) = \{j\}$ and $f(S) = -d_j$, there is no $S \in F$ such that $\delta_{\hat{G}}(S) = \{j\}$, $\delta_{\hat{G}}(S) = \phi$ and $f(S) = \hat{a}_j$. By (7.1.6) $\hat{P}^j$ is a facet of $\hat{P}$ if and only if there is no integer-valued $[y,\hat{z},\hat{w}]$ satisfying

$$7.1.15 \quad \begin{cases} y_S \geq 0 \text{ for all } S \in F \\ \hat{w}_e, \hat{z}_e \geq 0 \text{ for all } e \in E \\ \hat{z}_j \leq 0 \\ F(y,j)_{\hat{G}} - \hat{w}_j = 1 \\ F(y,e)_{\hat{G}} + \hat{z}_e - \hat{w}_e = 0 \text{ for all } e \neq j \\ f \cdot y + \hat{a} \cdot \hat{z} - \hat{d} \cdot \hat{w} \leq \hat{a}_j, \end{cases}$$

where $F(y,e)_{\hat{G}} \equiv \Sigma(y_S : e \in \delta_{\hat{G}}(S), S \in F) - \Sigma(y_S : e \in \delta_{\hat{G}}(\bar{S}), S \in F)$.

For $[y,\hat{z},\hat{w}]$ satisfying (7.1.15), let $w', z' \in \mathbb{R}^E$ be defined by

$$w'_e \equiv \begin{cases} \hat{z}_e \text{ if } e = j \\ \hat{w}_e \text{ otherwise} \end{cases}$$

$$z'_e \equiv \begin{cases} \hat{w}_e \text{ if } e = j \\ \hat{z}_e \text{ otherwise.} \end{cases}$$

Since for all $y \in \mathbb{R}^F$ we have $F(y,j)_{\hat{G}} = -F(y,j)_G$ and $F(y,e)_{\hat{G}} = F(y,e)_G$ for all $e \neq j$, there exists an integer-valued $[y,\hat{z},\hat{w}]$ satisfying (7.1.5) if and only if there exists a corresponding integer-valued $[y,z',w']$ satisfying

$y_S \geq 0$ for all $S \in F$

$w'_e, z'_e \geq 0$ for all $e \in E$

$w'_j \geq 0$

$F(y,j) + z'_j = -1$

$F(y,e) + z'_e - w'_e = 0$ for all $e \neq j$

$f \cdot y + a \cdot z' - d \cdot w' \leq -d_j.$

The theorem now follows. □

7.1.16 For all $T \in F$ let

$$Q_T \equiv \{x \in P: cv(T) \cdot x = f(T)\}.$$

We are able to determine when Q_T is a facet of P only when we place further restrictions on P. For any $T \in F$ let

$$F_T \equiv \{S \in F: cv(S) = cv(T),\ f(S) = f(T)\}.$$

7.1.17 <u>Theorem</u> Let $G = (V,E)$ be a graph, F a crossing family on V, $f: F \to \mathbb{R}$ a submodular function and $a \in \mathbb{R}^E$ with possibly infinite components such that

7.1.18 $\delta(\bar{S}) = \phi$ for all $S \in F$,

7.1.19 there are no $Y,Z \in F$ which cross such that $cv(Y) = cv(Z)$,

7.1.20 there is no $S \in F$, $j \in E$ such that $\delta(S) = \{j\}$ and $f(S) = a_j$.

We further assume that $d_e = 0$ for all $e \in E$. If P, the set of feasible solutions to (7.1.2)-(7.1.4), is of full dimension then for all $T \in F$, Q_T is a facet of P if and only if there is no (0,1)-valued [y,z] satisfying

7.1.21
$$\begin{cases} y_S \geq 0 & \text{for all } S \in F \\ y_S \leq 0 & \text{for all } S \in F_T \\ z_e \geq 0 & \text{for all } e \in E \\ F(y,e) + z_e \geq 1 & \text{for all } e \in \delta(T) \end{cases}$$

and

7.1.22 $\quad f \cdot y + a \cdot z \leq f(T).$

7.1.23 Note If there exist $S \in F$, $j \in E$ such that $\delta(S) = \{j\}$ and $f(S) = a_j$ then the inequality $x_j \leq a_j$ is equivalent to $cv(S) \cdot x \leq f(S)$. Hence, we can still determine whether or not Q_T is a facet of P by redefining $a_j \equiv +\infty$.

Proof of (7.1.17) Suppose Q_T is a facet of P. Since P is of full dimension, the set of inequalities $\{cv(S) \cdot x \leq f(S) : S \in F_T\}$ of (7.1.4) is essential for defining P, by (2.3.31). By (2.4.19) there can be no $[y,z]$ satisfying (7.1.21) and (7.1.22).

Conversely, suppose Q_T is not a facet of P. By (2.3.29), $\{cv(S) \cdot x \leq f(S) : S \in F_T\}$ is nonessential for defining P. By (2.4.19), the optimum value of the linear program

7.1.24 minimize $f \cdot y + a \cdot z$ where $[y,z]$ satisfies (7.1.21)

is less than or equal to $f(T)$. We show that (7.1.24) has an integer-valued optimum solution. By (2.5.1), (7.1.24) has a rational-valued optimum solution $[y^0, z^0]$. Apply the following transformation to $[y^0, z^0]$. (This is generalization of the proof of the corresponding result for the intersection of two polymatroids (4.5.5)).

7.1.25 Starting with $j = 0$, suppose $Y,Z \in F$, Y and Z cross and $0 < y^j_Y \leq y^j_Z$. Then, since F is a crossing family, $Y \cap Z \in F$ and $Y \cup Z \in F$. For $S \in F$ define y^{j+1}_S by

$$y^{j+1}_S \equiv \begin{cases} y^j_S + y^j_Y & \text{if } S \in \{Y \cap Z, Y \cup Z\} \\ y^j_S - y^j_Y & \text{if } S \in \{Y,Z\} \\ y^j_S & \text{otherwise.} \end{cases}$$

$[y^{j+1}, z^0]$ is a feasible solution to the linear program

7.1.26 minimize $f \cdot y + a \cdot z$ where $y \in \mathbb{R}^F$, $z \in \mathbb{R}^E$ satisfy

$$y_S \geq 0 \text{ for all } S \in F$$

$$z_e \geq 0 \text{ for all } e \in E$$

$$F(y,e) + z_e \geq 1 \text{ for all } e \in \delta(T).$$

By the submodularity of f, $f \cdot y^{j+1} \leq f \cdot y^j$. The dual linear program of (7.1.26) is

7.1.27 maximize $cv(T) \cdot x$ where $x \in \mathbb{R}^E$ satisfies (7.1.2)-(7.1.4).

Since $\{cv(S) \cdot x \leq f(S) : S \in F_T\}$ is nonessential for defining P, the optimum value of (7.1.27) is equal to the optimum value of

7.1.28 maximize $cv(T) \cdot x$ where $x \in \mathbb{R}^E$ satisfies

$$x_e \geq 0 \text{ for all } e \in E$$

$$x_e \leq a_e \text{ for all } e \in E$$

$$cv(S) \cdot x \leq f(S) \text{ for all } S \in F - F_T.$$

The dual linear program of (7.1.28) is equivalent to (7.1.24). Therefore, by the Strong L.P. Duality Theorem, the optimum value of (7.1.24) is equal to the optimum value of (7.1.26). Hence $[y^{j+1}, z^0]$ must be an optimum solution to (7.1.26).

As in the proofs of (4.5.5) and (6.2.9), the sequence $\{y^0, y^1, \ldots, y^j, y^{j+1}, \ldots\}$ constructed according to (7.1.25) has only finitely many terms.

Therefore, (7.1.26) has an optimum solution $[y^\ell, z^0]$ with the property that the family $F^\ell \equiv \{S \in F : y^\ell_S > 0\}$ is a cross-free family. There may be some set $S \in F_T$ such that $y^\ell_S > 0$. However, if for some set $R \in F - F_T$ we have $y^j_R > 0$ then, since for any two $Y, Z \in F$ which cross we have $cv(Y \cap Z) \neq cv(Y \cup Z)$, either $Y \cap Z \notin F_T$ or $Y \cup Z \notin F_T$. Hence, for some set $U \in F - F_T$ we have $y^{j+1}_U > 0$. By induction on j, for some $R \in F - F_T$ we have $y^\ell_R > 0$

Let $\beta \equiv \Sigma(y^\ell_S : S \in F_T)$. Since P is of full dimension, we must have $f(S) > 0$ for all $S \in F$ (compare (4.4.3)). Therefore,

$$\Sigma(f(S)y^\ell_S : S \in F - F_T) > 0.$$

If $\beta \geq 1$ then

$$f \cdot y^\ell + a \cdot z = \beta f(T) + \Sigma(f(S)y^\ell_S : S \in F - F_T) + a \cdot z^0 > f(T);$$

which is impossible. Therefore, $0 \leq \beta < 1$. Let $\hat{y} \in \mathbb{R}^F$ be defined by

$$\hat{y}_S \equiv \begin{cases} \dfrac{y_S^\ell}{1-\beta} & \text{if } S \in F-F_T \\ 0 & \text{if } S \in F_T. \end{cases} \qquad \square$$

Let $\hat{z} \equiv \frac{1}{1-\beta} z^0$. For all $e \in \delta(T)$ we have

$$\begin{aligned} F(\hat{y},e) + \hat{z}_e &= \frac{1}{1-\beta}[\Sigma(y_S^\ell : e \in \delta(S), S \in F-F_T) + z_e^0] \\ &\geq \frac{1}{1-\beta}(1-\beta) \\ &= 1. \end{aligned}$$

Therefore, $[\hat{y},\hat{z}]$ is a feasible solution to (7.1.24). Moreover,

$$\begin{aligned} f\cdot\hat{y} + a\cdot\hat{z} &= \frac{1}{1-\beta}[\Sigma(f(S)y_S^\ell : S \in F-F_T) + a\cdot z^0] \\ &\leq \frac{1}{1-\beta}[f(T)-\beta f(T)] \\ &= f(T). \end{aligned}$$

Hence, $[\hat{y},\hat{z}]$ satisfies (7.1.21) and (7.1.22).

The family $\hat{F} \equiv \{S \in F : \hat{y}_S > 0\}$ is a cross-free family. By an argument analogous to that in the proof of (6.2.9), the linear program

7.1.29 minimize $f\cdot y + a\cdot z$ where $[y,z]$ satisfies

$$\begin{aligned} &y_S \geq 0 \text{ for all } S \in F \\ &y_S \leq 0 \text{ for all } S \in F-\hat{F} \\ &z_e \geq 0 \text{ for all } e \in E \\ &y(\hat{F},e) + z_e \geq 1 \text{ for all } e \in \delta(T). \end{aligned}$$

has an integer-valued optimum solution $[\tilde{y},\tilde{z}]$. Since $[\hat{y},\hat{z}]$ is a feasible solution to (7.1.29) we have $f\cdot\tilde{y} + a\cdot\tilde{z} \le f\cdot\hat{y} + a\cdot\hat{z} \le f(T)$ and so $[\tilde{y},\tilde{z}]$ satisfies (7.1.21)-(7.1.22). $[\tilde{y},\tilde{z}]$ must be (0,1)-valued. □

(7.1.6), (7.1.12) and (7.1.17) characterize the facets of only a subclass of the polyhedra described by submodular functions on graphs. In view of the nature of the facet characterizations we do have we make the conjecture

7.1.30 Conjecture Let P, the set of solutions to (7.1.2)-(7.1.4) be of full dimension. For all $T \in F$, $Q_T = \{x \in P : cv(T)\cdot x = f(T)\}$ is a facet of P if and only if there is no integer-valued $[y,z,w]$ satisfying

$$y_S \ge 0 \text{ for all } S \in F$$
$$w_e, z_e \ge 0 \text{ for all } e \in E$$
$$y_S \le 0 \text{ for all } S \in F_T$$
$$F(y,e) + z_e - w_e = cv(T)_e \text{ for all } e \in E$$

and

$$f\cdot y + c\cdot z - d\cdot w \le f(T).$$

7.2 Applications

In this section we show how our characterizations of the facets of the intersection of two polymatroids and of $P_k(G)$ are direct consequences of (7.1.17).

Polymatroid Intersection

7.2.1 Let $f_1, f_2 : L_E \to \mathbb{R}$ be two β_0-functions such that $P \equiv P(K_E, f_1) \cap P(K_E, f_2)$ is of full dimension. In section 6.3 we saw how P could be described as the solution set of the linear system

$x_e \geq 0$ for all $e \in E$

$cv(S) \cdot x \leq f'(S)$ for all $e \in E$.

for an appropriate choice of graph $G = (V,E)$, crossing family F on V and submodular function f' of F. Recall that the graph G and family F on V are such that $\delta(\bar{S}) = \phi$ for all $S \in F$ and for any $Y,Z \in F$ which cross, $cv(Y \cap Z) \neq cv(Y \cup Z)$ (see (6.3.5)-(6.3.8)). Therefore, (7.1.18)-(7.1.20) are satisfied and we can apply (7.1.17) to determine for which sets $S \in F$ the set $Q_S = \{x \in P : cv(S) \cdot x = f'(S)\}$, i.e. for which sets $T \in K_E$ and $i = 1$ or 2 the face $\{x \in P : x(T) = f_i(T)\}$, is a facet of P.

7.2.2 Let $f : L_E \to \mathbb{R}$ be the rank function of P (for all $S \subseteq E$, $f(S) = \max\{x(S) : x \in P\}$. Then $P = P(K_E, f)$. If $P_T = \{x \in P : x(T) = f(T)\}$ is a facet of P then, by (4.4.5), T is f-nonseparable and f-closed.

7.2.3 Conversely, suppose P_T is not a facet of P. By (4.3.11), $f(T) = f_1(S) + f_2(T-S)$ for some $S \subseteq T$. If $S \in K_T - \{T\}$ then, by (4.5.11), T is f-separable. Hence, we may assume $f(T) = f_1(T)$.

Therefore, $\{x \in P : x(T) = f_1(T)\}$ is not a facet of P and, by (7.1.17), there exists an integer-valued vector $y = [y^1, y^2]$ satisfying

7.2.4
$$\begin{cases} y^i_S \geq 0 \text{ for all } S \in K_T,\ i = 1 \text{ or } 2 \\ y^1_T = y^2_T = 0 \\ y^1(K_E, e) + y^2(K_E, e) \geq 1 \text{ for all } e \in T \end{cases}$$

7.2.5 $\quad f_1 \cdot y^1 \cdot f_2 \cdot y^2 \leq f(T),$

and we may choose $[y^1, y^2]$ to minimize $f_1 \cdot y^1 + f_2 \cdot y^2$ subject to (7.2.4). By (4.5.3) and (4.5.4), T is not f-closed or T is f-separable. We have proved (4.5.5), a characterization of the nontrivial facets of $P(K_E, f)$.

Strong k-Cover Polyhedra

7.2.6 Recall that in section 5.3 we asserted the following theorem concerning $P_k(G)$ (see (5.3.26)).

7.2.7 Theorem Let G be a connected graph such that $P_k(G)$ is of full dimension. Then for any $T \in D(G)$, $L^k_{\delta(T)} = \{x \in P_k(G): x(\delta(T)) = k\}$ is a facet of $P_k(G)$ if and only if there does not exist a family $F \subseteq D(G)-\{T\}$ such that

7.2.8 $\Sigma(|\delta(S)|-k : S \in F) \le |\delta(T) \cap U(F)|-k,$

where $U(F) \equiv \cup(\delta(S) : S \in F)$.

Proof Let $f(S) \equiv |\delta(S)|-k$ for all $S \in D(G)$. For $Y, Z \in D(G)$ such that $Y \cap Z \ne \phi$ and $Y \cup Z \ne V$ we have

$$\begin{aligned} f(Y \cap Z) + f(Y \cup Z) &= |\delta(Y \cap Z)|-k + |\delta(Y \cup Z)|-k \\ &= |\delta(Y)|-k + |\delta(Z)|-k \\ &= f(Y) + f(Z). \end{aligned}$$

Therefore f is submodular on D(G). Let P be the solution set of the linear system

7.2.9 $$\begin{cases} x_e \ge 0 \text{ for all } e \in E \\ x_e \le 1 \text{ for all } e \in E \\ cv(S) \cdot x \le f(S) \text{ for all } S \in D(G). \end{cases}$$

Let $g(x) \equiv 1-x$ for all $x \in \mathbb{R}^E$. Suppose $x \in P_k(G)$. Clearly $g(x) \geq 0$ and $g(x) \leq 1$. For any $S \in D(G)$ we have

$$cv(S)\cdot g(x) = cv(S)\cdot(1-x) = |\delta(S)|-x(\delta(S)) \leq |\delta(S)|-k,$$

since $x(\delta(S)) \geq k$. Therefore $g(P_k(G)) \subseteq P$. Similarly $P \subseteq g(P_k(G))$ and so $g(P_k(G)) = P_k(G)$.

Since g is a nonsingular affine transformation, $P_k(G)$ is of full dimension if and only if P is of full dimension. Consider the face $Q_T = \{x \in P: cv(T)\cdot x = f(T)\}$. As above, $g(L_T^k) = Q_T$. Hence, L_T^k is a facet of $P_k(G)$ if and only if Q_T is a facet of P (compare the proof of (7.1.12)).

Because $\delta(\bar{S}) = \phi$ for all $S \in D(G)$, $cv(Y) \neq cv(Z)$ for all $Y,Z \in D(G)$ which cross, and $|\delta(S)| \geq 2$ for all $S \in D(G)$, we can apply (7.1.17). Hence Q_T is a facet of P if and only if there does not exist $(0,1)$-valued $y \in \mathbb{R}^{D(G)}$ and $z \in \mathbb{R}^E$ satisfying

$$\text{7.2.10}\quad \left\{\begin{array}{l} y_S \geq 0 \text{ for all } S \in D(G) \\ y_T \leq 0 \\ z_e \geq 0 \text{ for all } e \in E \\ D(G)(y,e) + z_e \geq 1 \text{ for all } e \in \delta(T) \\ f\cdot y + 1\cdot z \leq f(T). \end{array}\right.$$

For any $(0,1)$-valued $[y^0,z^0]$ satisfying (7.2.10) let $F \equiv \{S \in D(G): y_S^0 = 1\}$. Clearly, we may assume that $z_e^0 = 1$ if and only if $e \in \delta(T)-U(F)$. Then

$$f \cdot y^0 + 1 \cdot z^0 = \Sigma(|\delta(S)-k : S \in F)| + |\delta(T) - U(F)| - k$$
$$\leq |\delta(T)| - k.$$

Thus F satisfies (7.2.8).

Conversely, suppose $F \subseteq D(G) - \{T\}$ satisfies (7.2.8). Then let $y^0 \in \mathbb{R}^{D(G)}$ be defined by

$$y^0_S \equiv \begin{cases} 1 & \text{if } S \in F \\ 0 & \text{otherwise.} \end{cases}$$

and $z^0 \in \mathbb{R}^E$ be defined by

$$z^0_e \equiv \begin{cases} 1 & \text{if } e \in \delta(T) - U(F) \\ 0 & \text{otherwise.} \end{cases}$$

It is easily verified that $[y^0, z^0]$ satisfies (7.2.10). □

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