# Improved Single-Tone Frequency Estimation by Averaging and Weighted Linear Prediction

Hing Cheung So and Hongqing Liu

**This paper addresses estimating the frequency of a cisoid in the presence of white Gaussian noise, which has numerous applications in communications, radar, sonar, and instrumentation and measurement. Due to the nonlinear nature of the frequency estimation problem, there is threshold effect, that is, large error estimates or outliers will occur at sufficiently low signal-to-noise ratio (SNR) conditions. Utilizing the ideas of averaging to increase SNR and weighted linear prediction, an optimal frequency estimator with smaller threshold SNR is developed. Computer simulations are included to compare its mean square error performance with that of the maximum likelihood (ML) estimator, improved weighted phase averager, generalized weighted linear predictor, and single weighted sample correlator as well as Cramér-Rao lower bound. In particular, with smaller computational requirement, the proposed estimator can achieve the same threshold and estimation performance of the ML method.**

**Keywords: Frequency estimation, threshold performance, linear prediction.**

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## I. Introduction

The problem of estimating the frequency of a complex tone in white noise has received considerable attention [1]-[10] because it has many applications in science and engineering. The complex single-tone model is:

$$x(n) = s(n) + q(n), \quad n = 1, 2, \cdots, N, \tag{1}$$

where $s(n) = Ae^{j(\omega n+\phi)}$ is the noise-free cisoid. The sinusoidal amplitude, normalized angular frequency, and phase are denoted by $A$>0, $\omega \in (-\pi, \pi)$, and $\phi \in (0, 2\pi]$, respectively, and they are considered as deterministic but unknown constants. The noise $q(n)$ is assumed to be a zero-mean complex white Gaussian process, that is, its real and imaginary components are real white processes with identical but unknown variances of $\sigma^2/2$ and uncorrelated with each other. Given the $N$ discrete-time noisy samples $\{x(n)\}$, the objective is to estimate $\omega$. Note that $\omega$ is assumed constant within the observation interval and all the investigated methods in this paper are of batch mode. If the frequency varies with time, then adaptive frequency estimation algorithms [11], [12] are needed for online parameter tracking.

In the presence of white Gaussian noise, the maximum likelihood (ML) estimate of the frequency is obtained from the periodogram maximum [5], but it involves extensive computations. As frequency estimation is nonlinear in nature, there is a threshold effect [4], [5] at sufficiently low signal-to-noise ratio (SNR) conditions. That is, below a threshold SNR, the estimation errors are several orders of magnitude greater than the performance benchmark of Cramér-Rao lower bound (CRLB) [4]. On the other hand, two fast and accurate frequency estimators, namely, weighted linear predictor (WLP) and weighted phase averager (WPA), based on cross-correlation and phase of $x(n)$, respectively, have been proposed in [6]. However, their threshold SNR is high. To lower the threshold SNR, So and Chan [7] extend the WLP by including higher-order cross-correlation terms and develop the so-called generalized WLP (GWLP). Nevertheless, the maximum estimation range of $\omega \in (-\pi, \pi)$ is not required in many applications. Frequency offset estimation in wireless communications [9] is a representative example where the nominal parameter only lies in the low-frequency range. Targeting for estimation with a reduced frequency range, Kim and others [8] have modified the WPA by employing the average of $x(n)$ while Tavares and others [9] consider weighting a higher-order sample correlation for threshold SNR reduction. Here, we refer to them as improved weighted phase averager (IWPA) and single weighted sample correlator (SWSC), respectively. In this paper, we combine the key ideas of GWLP and IWPA to devise an optimal frequency estimator with better threshold performance than [7]-[9], at the expense of reduction in frequency estimation range.

## II. Algorithm Development

There are two key ideas in our algorithm development. First, inspired by [8], we construct $y_K(n)$, which is a moving average of $x(n)$ to increase the SNR:

$$y_K(n) = \frac{\sum_{k=0}^{K-1} x(n+k)}{K}, \quad n = 1, 2, \cdots, N-K+1, \tag{2}$$

where $K \geq 2$ is a positive integer to represent the number of samples involved in the averaging. Second, the GWLP approach is utilized for frequency estimation with the use of $y_K(n)$. For simplicity but without loss of generality, we consider $K$=2, and $y_2(n)$ has the form of

$$y_2(n) = \frac{x(n+1)+x(n)}{2}, \quad n = 1, 2, \cdots, N-1. \tag{3}$$

The signal component of $y_2(n)$ is

$$\frac{s(n+1)+s(n)}{2} = A\cos\left(\frac{\omega}{2}\right) e^{j(\omega(n+\frac{1}{2})+\phi)}, \tag{4}$$

which has a power of $A^2 \cos^2(\omega/2)$. On the other hand, noting that $q(n+1)$ and $q(n)$ are uncorrelated, the noise power in $y_2(n)$, namely, $[q(n+1)+q(n)]/2$, is easily evaluated as $\sigma^2/2$. Thus, the SNR in $y_2(n)$ is

$$\frac{2A^2\cos^2\left(\frac{\omega}{2}\right)}{\sigma^2}. \tag{5}$$

Compared with the SNR in $x(n)$, namely, $A^2/\sigma^2$, we see that (5) is larger as long as $-\pi/2 < \omega < \pi/2$, which corresponds to an SNR gain of $2\cos^2(\omega/2)$. According to [8], there will be a threshold improvement of $10\log_{10}(2\cos^2(\omega/2))$ dB in the reduced frequency range of $(-\pi/2, \pi/2)$ because the input to the estimator, namely, $y_2(n)$, has higher SNR over $x(n)$, which is solely due to the sample averaging process. Nevertheless, there is no SNR improvement for $-\pi < \omega < -\pi/2$ and $\pi/2 < \omega < \pi$, and in particular, (5) tends to 0 when $\omega \to \pm\pi$.

Let $\rho = e^{j\omega}$. As the noise-free average is also a pure tone, (4) satisfies the linear prediction (LP) property:

$$s(n+1)+s(n) = \rho(s(n)+s(n-1)). \tag{6}$$

Based on (6), we construct the LP error as

$$y_2(n) - \tilde{\rho} y_2(n-1), \quad n = 2, 3, \cdots, N-1, \tag{7}$$

where $\tilde{\rho}$ is the variable for $\rho$. Let $\mathbf{y}_{2l} = [y_2(1)\ y_2(2) \cdots y_2(N-2)]^T$ and $\mathbf{y}_{2u} = [y_2(2)\ y_2(3) \cdots y_2(N-1)]^T$. To achieve accurate estimation of $\rho$, a weighted least squares (WLS) cost function constructed from (7) is needed [4]. Following [7] which utilizes the sinusoidal LP property and WLS, the optimal estimate of $\rho$, denoted by $\hat{\rho}$, is computed by

$$\begin{aligned}\hat{\rho} &= \arg\min_{\tilde{\rho}} (\mathbf{y}_{2l}\tilde{\rho} - \mathbf{y}_{2u})^H \mathbf{W}_2(\rho)(\mathbf{y}_{2l}\tilde{\rho} - \mathbf{y}_{2u}) \\ &= \frac{\mathbf{y}_{2l}^H \mathbf{W}_2(\rho)\mathbf{y}_{2u}}{\mathbf{y}_{2l}^H \mathbf{W}_2(\rho)\mathbf{y}_{2l}},\end{aligned} \tag{8}$$

where the optimum weighting matrix, $\mathbf{W}_2(\rho) \in \mathbf{C}^{(N-2)\times(N-2)}$, is constructed from the residual error of $\boldsymbol{e} = (\mathbf{y}_{2l}\rho - \mathbf{y}_{2u})$ and hence a function of $\rho$, which is commonly known as the Gauss-Markov estimate [4]. With the use of (6), we have $\boldsymbol{e} = (\mathbf{q}_{2l}\rho - \mathbf{q}_{2u})$ where $\mathbf{q}_{2l} = 1/2[q(2)+q(1)\cdots q(N-2)+q(N-1)]^T$ and $\mathbf{q}_{2u} = 1/2[q(3)+q(2)\cdots q(N-1)+q(N)]^T$. Utilizing the whiteness of $q(n)$, $\mathbf{W}_2(\rho)$ is determined as

$$\begin{aligned}\mathbf{W}_2(\rho) &= [E\{\boldsymbol{e}\boldsymbol{e}^H\}]^{-1} \\ &= \frac{1}{\sigma^2}[\text{Toeplitz}([a_0, a_1, a_2, 0, \cdots, 0])]^{-1} \\ &= \frac{1}{\sigma^2}\begin{bmatrix} a_0 & a_1^* & a_2^* & 0 & 0 & 0 & \cdots & 0 \\ a_1 & a_0 & a_1^* & a_2^* & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & a_1^* & a_2^* & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & a_2 & a_1 & a_0 & a_1^* \\ 0 & \cdots & \cdots & \cdots & 0 & a_2 & a_1 & a_0 \end{bmatrix}^{-1},\end{aligned} \tag{9}$$

where $a_0 = 2 - \rho - \rho^* + 2|\rho|^2$, $a_1 = 1 - 2\rho + |\rho|^2$, $a_2 = -\rho$, $E$ denotes expectation, and Toeplitz is the Toeplitz operator. Note that the value of $\sigma^2$ is not required as it will be canceled out in (8).

As $\mathbf{W}_2(\rho)$ is characterized by the unknown parameter $\rho$, we follow [7] to estimate $\omega$ in an iterative manner, and the estimation procedure is summarized as follows.

i) Find an initial frequency estimate $\hat{\omega}$ using the WLP [6]. Set $\hat{\rho} = e^{j\hat{\omega}}$.
ii) Use $\rho = \hat{\rho}$ to construct (9).
iii) Compute an updated estimate of $\hat{\rho}$ using (8).
iv) Repeat steps (ii) and (iii) for a few iterations.
v) Use the finalized $\hat{\rho}$ to estimate the frequency as $\hat{\omega} = \angle(\hat{\rho})$.

Note that when the sample size and/or SNR are large enough, the initial frequency estimate based on the WLP will be sufficiently close to the true frequency value. According to the convergence analysis in [7], the GWLP iterative algorithm achieves global convergence when the initial frequency estimate is close to ω at $N \to \infty$. This means that in principle the proposed iterative method will provide large frequency errors only for a sufficiently small sample size and/or SNR conditions, as in the feedforward estimators such as IWPA and SWSC.

Mean and variance analysis of the proposed method is conducted as follows. Based on (8), we construct

$$\begin{aligned} f(\tilde{\rho}) &= \mathbf{y}_{2l}^H \mathbf{W}_2(\tilde{\rho}) \mathbf{y}_{2u} - \mathbf{y}_{2l}^H \mathbf{W}_2(\tilde{\rho}) \mathbf{y}_{2l} \tilde{\rho} \\ &= \mathbf{y}_{2l}^H \mathbf{W}_2(\tilde{\rho})(\mathbf{y}_{2u} - \tilde{\rho}\mathbf{y}_{2l}). \end{aligned} \quad (10)$$

Upon parameter convergence, $\hat{\rho}$ should satisfy $f(\hat{\rho}) = 0$. For sufficiently large SNR and/or $N$, $\hat{\rho}$ will be located at a reasonable proximity of $\rho$. Using Taylor's series to expand $f(\hat{\rho})$ around $\rho$ up to the first-order term, we get

$$0 = f(\hat{\rho}) \approx f(\rho) + f'(\rho)(\hat{\rho} - \rho), \quad (11)$$

where $f'(\rho)$ is the first derivative of $f(\tilde{\rho})$ evaluated at the true value. Let $\mathbf{s}_{2l} = 1/2[s(2)+s(1) \cdots s(N-2)+s(N-1)]^T$ and $\mathbf{s}_{2u} = 1/2[s(3)+s(2) \cdots s(N-1)+s(N)]^T$, and noting that $\mathbf{s}_{2u} = \rho \mathbf{s}_{2l}$, $f(\rho)$ is approximated as

$$\begin{aligned} f(\rho) &= (\mathbf{s}_{2l} + \mathbf{q}_{2l})^H \mathbf{W}_2(\rho)((\mathbf{s}_{2u} + \mathbf{q}_{2u}) - \rho(\mathbf{s}_{2l} + \mathbf{q}_{2l})) \\ &\approx \mathbf{s}_{2l}^H \mathbf{W}_2(\rho)(\mathbf{q}_{2u} - \rho \mathbf{q}_{2l}). \end{aligned} \quad (12)$$

While $f'(\rho)$ is

$$\begin{aligned} f'(\rho) &= -\mathbf{y}_{2l}^H \mathbf{W}_2(\rho) \mathbf{y}_{2l} - \mathbf{y}_{2l}^H \mathbf{W}_2'(\rho)(\mathbf{y}_{2u} - \rho \mathbf{y}_{2l}) \\ &\approx -\mathbf{s}_{2l}^H \mathbf{W}_2(\rho) \mathbf{s}_{2l}. \end{aligned} \quad (13)$$

Combining (11) through (13) yields

$$\hat{\rho} \approx \rho - \frac{f(\rho)}{f'(\rho)} \approx \rho + \frac{\mathbf{s}_{2l}^H \mathbf{W}_2(\rho)(\mathbf{q}_{2u} - \rho \mathbf{q}_{2l})}{\mathbf{s}_{2l}^H \mathbf{W}_2(\rho) \mathbf{s}_{2l}}. \quad (14)$$

As $\mathbf{s}_{2l}^H \mathbf{W}_2(\rho) \mathbf{s}_{2l}$ is deterministic while $\mathbf{q}_{2l}$ and $\mathbf{q}_{2u}$ are of zero mean, we have $E\{\hat{\rho}\} \approx \rho$, and hence $\hat{\omega}$ is an approximately unbiased frequency estimate. Employing (14), the variance or mean square error (MSE) of $\hat{\rho}$ is derived as

$$\begin{aligned} \mathrm{var}(\hat{\rho}) &= E\{(\hat{\rho} - \rho)(\hat{\rho} - \rho)^*\} \\ &\approx \frac{\mathbf{s}_{2l}^H \mathbf{W}_2(\rho) E\{(\mathbf{q}_{2u} - \rho \mathbf{q}_{2l})(\mathbf{q}_{2u} - \rho \mathbf{q}_{2l})^H\} \mathbf{W}_2(\rho) \mathbf{s}_{2l}}{(\mathbf{s}_{2l}^H \mathbf{W}_2(\rho) \mathbf{s}_{2l})^2} \\ &= \frac{1}{\mathbf{s}_{2l}^H \mathbf{W}_2(\rho) \mathbf{s}_{2l}}. \end{aligned} \quad (15)$$

Based on the variance relationship between $\hat{\rho}$ and $\hat{\omega}$ [10], the variance of $\hat{\omega}$ is

$$\mathrm{var}(\hat{\omega}) \approx \frac{\mathrm{var}(\hat{\rho})}{2} \approx \frac{1}{2\mathbf{s}_{2l}^H \mathbf{W}_2(\rho) \mathbf{s}_{2l}}. \quad (16)$$

Although there is no closed-form expression for $\mathrm{var}(\hat{\omega})$, empirical studies show that (16) is very close to CRLB for single frequency estimation [6], which is given as $6\sigma^2/(A^2 N(N^2-1))$.

Finally, extension to an arbitrary value of $K$ is summarized as follows. The general form of (4) is

$$\frac{\sum_{k=0}^{K-1} s(n+k)}{K} = A e^{j[\omega(n+\frac{K-1}{2})+\phi]} \left( e^{j\omega\frac{K-1}{2}} + \cdots + e^{-j\omega\frac{K-1}{2}} \right), \quad (17)$$

where the SNR gain of $y_K(n)$ over $x(n)$ is $\left( e^{j\omega\frac{K-1}{2}} + \cdots + e^{-j\omega\frac{K-1}{2}} \right)^2 / K$ at the expense of a reduced frequency estimation range of approximately $(-\pi/K, \pi/K)$. The general conceptual WLS solution for (8) is

$$\hat{\rho} = \frac{\mathbf{y}_{Kl}^H \mathbf{W}_K(\rho) \mathbf{y}_{Ku}}{\mathbf{y}_{Kl}^H \mathbf{W}_K(\rho) \mathbf{y}_{Kl}}, \quad (18)$$

where $\mathbf{y}_{Kl} = [y_K(1)\ y_K(2) \cdots y_K(N-K)]^T$, $\mathbf{y}_{Ku} = [y_K(2)\ y_K(3) \cdots y_K(N-K+1)]^T$, and $\mathbf{W}_K(\rho)$ has the form of

$$\mathbf{W}_K(\rho) = [\mathrm{Toeplitz}([a_0, a_1, \cdots, a_K, 0, \cdots, 0])]^{-1}, \quad (19)$$

where

$$\begin{aligned} a_0 &= K|\rho|^2 + K - (K-1)\rho^* - (K-1)\rho, \\ a_i &= (K-i)|\rho|^2 + (K-i) - (K-i-1)\rho^* - (K-i+1)\rho, \\ i &= 1, 2, \cdots, K-1, \ \text{and} \ a_K = -\rho. \end{aligned}$$

Following (10) through (16), the approximate unbiasedness of (18) can be shown, and the corresponding frequency variance is

$$\mathrm{var}(\hat{\omega}) \approx \frac{1}{2\mathbf{s}_{Kl}^H \mathbf{W}_K(\rho) \mathbf{s}_{Kl}}, \quad (20)$$

where $\mathbf{s}_{Kl}$ is the signal component of $\mathbf{y}_{Kl}$.

## III. Results

Computer simulations have been conducted to evaluate the proposed approach for frequency estimation of a single cisoid in white Gaussian noise. Five iterations are employed in the estimation procedure as no significant improvement is observed for more iterations. We compare its MSE performance with that of the ML estimator [5], GWLP [7], IWPA [8], SWSC [9], as well as CRLB. As the ML cost function is multimodal, we employ discrete Fourier transform peak as the initial estimate and then apply Newton's method to search for the maximum point. Note that the computational requirement for the ML method is the highest, which corresponds to $O(N^3)$, while those of the proposed scheme and GWLP, and IWPA and SWSC, are $O(N^2)$ and $O(N)$, respectively. The tone amplitude and phase are $A$=1 and $\phi=1$ rad, respectively, while the complex noise is constructed as $q(n)=\mu(n)+jv(n)$ where $\mu(n)$ and $v(n)$ are independent real white Gaussian processes with identical variances of $\sigma^2/2$. Different SNRs are obtained by proper scaling $q(n)$ with SNR = $A^2/\sigma^2$. The data length is $N$=30. All results are based on an average of 1,000 independent runs.

Figure 1 shows the MSEs of frequency for the five estimators as well as CRLB versus SNR, both in dB scales, at $\omega=0.04\pi$ rad. Note that the dB values are computed as $10\log_{10}$(MSE) and $10\log_{10}$(SNR). Although all methods show their optimality at sufficiently high SNRs, the proposed scheme and ML estimator have the best threshold performance, followed by GWLP, IWPA, and SWSC. Their average computation times for a single trial are measured as 0.0531 s, 0.0497 s, 0.0972 s, 0.0241 s, and 0.0365s, respectively, which agree with the complexity analysis. In particular, the threshold improvement of the proposed method over GWLP is around 2 dB which aligns with the analysis in section II, namely, $10\log_{10}(2\cos^2(\omega/2))$ dB.

Figures 2 and 3 show the MSE performance versus $\omega$ at SNR=10 dB and SNR=0 dB, respectively. In the former figure, we observe that the ML estimator, GWLP, and proposed method give optimum performance except when $\omega$ is very close to $-\pi$ and $\pi$. On the other hand, the IWPA performance approaches CRLB only for $\omega\in(-\pi/2,\pi/2)$ and SWSC is suboptimum for all frequencies. Their smaller admissible frequency estimation range also explains why they have smaller MSEs at SNR $\in[-10,-5]$ dB in Fig. 1. Using the IWPA as an illustration, it always produces a frequency estimate with a value between $-\pi/2$ and $\pi/2$, even when it is of large estimation error before the threshold SNR. In the latter figure, the performance of the proposed method and periodogram is close to CRLB at approximately $\omega\in(-\pi/2,\pi/2)$ and

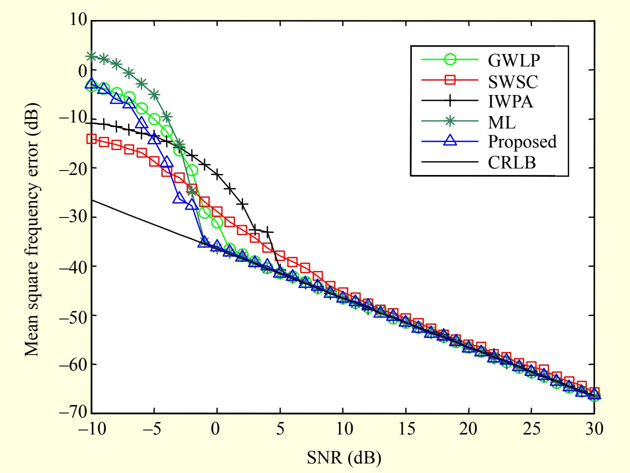


Fig. 1. Mean square frequency errors versus SNR at $\omega=0.04\pi$ rad.

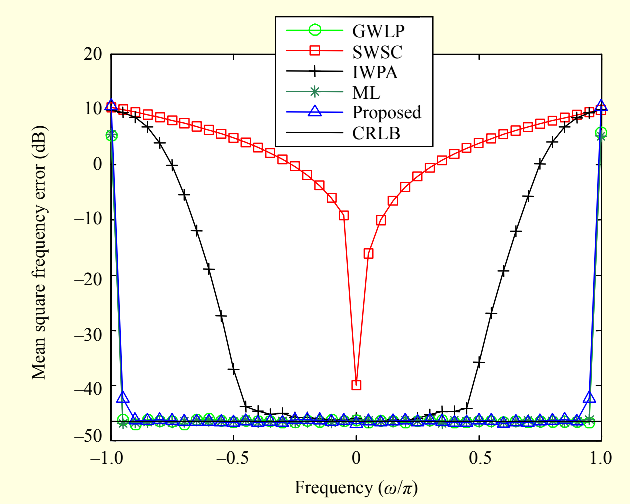


Fig. 2. Mean square frequency errors versus $\omega$ at SNR=10 dB.

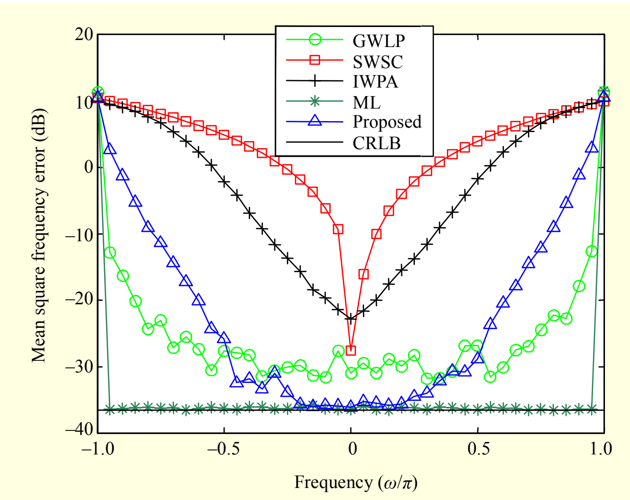


Fig. 3. Mean square frequency errors versus $\omega$ at SNR=0 dB.

$\omega \in (-\pi, \pi)$, respectively, while the remaining estimators are suboptimal in the whole frequency range. Note that as (16) almost overlaps with the CRLB, the theoretical value of $\text{var}(\hat{\omega})$ has not been provided in the plots. In summary, the proposed method gives optimum estimation performance for all frequencies at sufficiently high SNRs. At lower SNR conditions, it is optimum only when $\omega$ is approximately between $-\pi/2$ and $\pi/2$. That is, it is advantageous to employ the devised scheme particularly when the frequency parameter lies in the low-frequency range and/or the SNR is small enough.

## IV. Conclusion

A new single frequency estimator based on averaging and weighted linear predictor is devised and analyzed. The superiority of its threshold performance within the reduced estimation range is demonstrated by comparing it with ML estimator, GWLP, IWPA, and SWSC.

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