A CAPACITY EXPANSION STRATEGY ON PROJECT PLANNING

Un Gi Joo

CONTENTS

ABSTRACT

A capacity expansion planning problem with buy-or-lease decisions is considered. Demands for capacity are deterministic and are given period-dependently at each period. Capacity additions occur by buying or leasing a capacity, and leased capacity at any period is reconverted to original source after a fixed length of periods, say, lease period. All cost functions (buying, leasing and idle costs) are assumed to be concave. And shortages of capacity and disposals are not considered. The properties of an optimal solution are characterized. This is then used in a tree search algorithm for the optimal solution and other two algorithms for a near-optimal solution are added. And these algorithms are illustrated with numerical examples.

I. INTRODUCTION

In this paper, we consider a capacity expansion planning model with buy-or-lease decisions. This model treats a deterministic capacity expansion planning problem with a finite horizon of discrete time periods where a capacity addition plan is found to satisfy the demands for capacity at minimum cost. Demands are deterministic and known, but variate in time. Thus, the decisions are treated as dynamic variables.

Capacity addition can occur by buying or leasing a capacity at each time period. The leased capacity at any time period must be reconverted to the original supplier after a fixed leasing period, say, lease period. And all demands for capacity are satisfied by buying, leasing and idle capacity.

It is assumed that all involved cost functions (buying, leasing and holding cost) are concave. And capacity shortages and disposals are not allowed. The objective is to find a plan of timing and sizing for buying or leasing capacities which minimizes total (discounted) cost subject to satisfaction the given one type of demands for capacity.

Some studies have been done for capacity expansion planning models for two facility types with two demand types. For example, Kalotay [1], Erlenkotter [2], and Fong and Rao [3] have analyzed capacity expansion planning problems with two demand types under the assumption that a converted capacity was reconverted immediately at the end of each period. Gascon and Leachman [4] have developed a dynamic programming algorithm for production scheduling of time-varying deterministic demands on a single facility. Dutta and Lim[5] have adapted a Lagrangian relaxation for transmission capacity schedule in a communication network.

And Several models with two facilities for a single demand type have been developed. Lee [6] has considered a dynamic lot-size model with make-or-buy decisions having constraints on production and purchase capacities, and Sung [7] has developed a single-product parallel-facilities production-planning model, where demands in each period can be supplied by one of M facilities or some combination of them. Kamien and Li [8] have analyzed a dynamic lot-sizing problem with capacity conversion allowed to have a restriction on the usable amount of capacity at each period. Since such production planning models can be regarded as capacity expansion planning problems with two facility types for satisfying one demand type, they are similar to ours. But they differ from ours in the sense that a capacity added by leasing is reconverted after a fixed time period in ours, but the capacity added by buying operates through the planning horizon in those production planning models.

This model can be applied to problems with construction or subcontract decisions, where subcontract implies that a certain amount of demands for capacity is supplied from outside the system and a subcontract period corresponds to a lease period. This model may also be useful for a situation in which demands during a certain period (lease period) can be satisfied by renting a certain amount of capacity, and for a manpower scheduling of employing temporary workers for a fixed time length of period. And the capacity can be regarded as an optical fiber, transmission device (e.g., terminal multiplexer, add/drop multiplexer, digital cross-connector, etc.), or a set of optical fibers/transmission devices in a telecommunication system.

This paper is organized as follows : In Section II, we formulate the model. In Section III, we analyze properties of an optimal solution, upon which a tree search solution algorithm for an optimal solution is derived in Section IV. Section V describes a numerical example. In Section VI, we suggest two other algorithms for near-optimal solutions.

II. MODEL FORMULATION

Let us define some notations as follows :

t =index for a time period, (t=1,2,$\cdots$,T+1, where T is a planning horizon)

x_t =amount of buying capacity at the beginning of period t

y_t =amount of leasing capacity at the beginning of period t , let y_t =0 if $t \leq 0$

I_t =amount of idle capacity at the beginning of period t

τ= lease period which represents a period interval of leasing capacity at any time period t and hence being reconverted to the original source at the beginning of the period $t+\tau$, where τ is a given positive integer

r_t = increment of demand for additional capacity at the beginning of period t

$B_t(x_t)$ = cost function of buying the capacity x_t at the period t

$L_t(y_t)$ = cost function of leasing the capacity y_t at the period t

$H_t(I_{t+1})$ = holding cost idle capacity at period t.

Assume that all cost functions are non-decreasing concave functions and $B_t(0)=L_t(0)=H_t(0)=0$.

The objective is to find out the best strategy which minimizes the associated total cost while satisfying all the forecasted capacity demands during the project horizon T and to decide when and how much capacity must be expanded. The problem (denoted by P) can be formulated as follows:

$$(P)\text{: Min. } \sum_{t=1}^{T}[B_t(x_t)+L_t(y_t)+H_t(I_{t+1})]$$

$$\text{s.t. } I_{t+1} = I_t + x_t + y_t - r_t - y_{t-\tau} \qquad (1.1)$$

$$x_t \geq 0,\ y_t \geq 0, \qquad (1.2)$$

$$I_t \geq 0,\ I_1 = I_{T+1} = 0, \qquad (1.3)$$

$$\text{where } t = 1,2,\ldots,T.$$

The objective function in (P) minimizes the associated total discounted cost which consists of buying, leasing and idle capacity cost. Constraint (1.1) represents the idle capacity balance. Non-negative conditions for variables x_t, y_t and I_t imply that capacity disposals and shortages are not allowed, and the condition of I_1=I_{T+1} = 0 is added without loss of generality.

A network representation of (P) is depicted as in Fig.1.

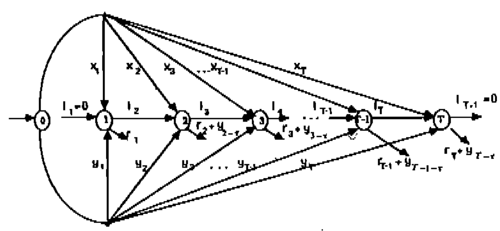


Fig.1. A network flow representation of (P)

III. PROPERTIES OF THE OPTIMAL SOLUTION

Since the constraints (1.1) through (1.3) of problem (P) form a non-empty convex set and it is assumed that all cost functions are concave

ETRI JOURNAL, VOL 15, NO 3/4, Jan. 1994

functions, an optimal solution for (P) occurs at an extreme point. Therefore, we will first find the characteristics of extreme points, and then characterize optimal solution properties.

The following Theorem 1 characterizes an extreme point of (P).

Theorem 1

A feasible point of (P) is an extreme point if and only if the point satisfies the following conditions:

$$x_t I_t = 0, \tag{2.1}$$

$$y_t I_t = 0, \tag{2.2}$$

$$x_t y_t = 0, \tag{2.3}$$

$$y_t \le \sum_{i=t}^{t+\tau-1} (r_i + y_{i-\tau}) \tag{2.4}$$

where $t = 1,2,\ldots,T$.

Proof

The detailed proof procedure is given in the Appendix. □

From Theorem 1, we can notice that it is not necessary to consider a point y_t such that $y_t > \sum_{i=t}^{t+\tau-1}(r_i + y_{i-\tau})$, for an extreme point, and we can find the followings:

If $x_t>0$, then $y_t=I_t=0$

If $I_t>0$, then $x_t=y_t=0$

If $y_t>0$, then $x_t=I_t=0$

But the values of $x_{t+\tau}$ and $y_{t+\tau}$ are affected by the size of y_t if y_t is positive.

Define that a time period t is a *capacity point* if the idle capacity at the time period t is zero, i.e., $I_t=0$.

We can derive Corollary 1 from Theorem 1. This corollary describes a structure of extreme points at a sub-problem which is a problem of only $t=u,u+1,\cdots, v,v+1$ for some integers of u and $v (1 \le u \le v \le T)$.

Corollary 1

Let u and $v+1$ are two consecutive capacity points. Then, for an extreme point, either x_u or y_u has only the following positive value with the corresponding $B_u(x_u) + L_u(y_u) + \sum_{t=u}^{v} H_t(I_{t+1})$ cost; either x_u or y_u has the value of $\sum_{t=u}^{v}(r_t + y_{t-\tau})$, and other variables of x_t and y_t $(t=u+1,\cdots,v)$ equal zeroes.

Corollary 2 describes which variable should be positive, by which computational burden will considerably be reduced in finding an optimal solution.

Corollary 2

Let $\sum_{t=u}^{v}(r_t + y_{t-\tau}) = a$ where u and $v+1$ are two consecutive capacity points.

$$\text{If } v-u \ge \tau,\ x_u = a \text{ and } y_u = 0. \tag{3.1}$$

$$\text{If } v-u < \tau \text{ and } T-u < \tau, \text{ then} \tag{3.2}$$

$$x_u = a, \text{ when } B_u(a) \le L_u(a)$$

$$y_u = a, \text{ when } B_u(a) > L_u(a)$$

If $v-u < \tau$ and $T-u \ge \tau$, then $x_u = a$,

when $B_u(a) \le L_u(a)$

$$+\,\mathrm{Min.}\left[\begin{array}{l} B_{u^*}(a+b) - B_{u^*}(b) + \sum_{t=u^*}^{u+\tau} H_t(a),\ x_{u^*} > 0, \\ L_{u^*}(a+b) - L_{u^*}(b) + \sum_{t=u^*}^{u+\tau} H_t(a),\ y_{u^*} > 0, \end{array}\right.$$

where (u^*, v^*+1) are two consecutive capacity points such that $v+1 \le u^* \le u+T \le v^*$ and

$$b = \sum_{t=u^*}^{v^*}(r_t + y_{t-\tau}) - a,$$

and if u^* is the last capacity addition point,

then $y_u = a$,

when $B_u(a) > L_u(a)$

$$+\,Min.\left[\begin{array}{l} B_{u^*}(a+b) - B_{u^*}(b) + \sum_{t=u^*}^{u+\tau} H_t(a),\ x_{u^*} > 0, \\ L_{u^*}(a+b) - L_{u^*}(b) + \sum_{t=u^*}^{u+\tau} H_t(a),\ y_{u^*} > 0. \end{array}\right. \tag{3.3}$$

The Equation (3.1) derived from Equation (2.4) in Theorem 1 represents the restriction of the size of y_t, and Equation (3.2) results from the fact that capacity expansion occurs through the cheaper facility when v-u<τ and T-u<τ. And Equation (3.3) is derived from the cost comparison between buying and leasing.

IV. A TREE SEARCH SOLUTION ALGORITHM

We can notice that the original problem can not be decomposed into smaller sub-problems which can be solved individually since the value of x_t and y_t affect the amount of capacity addition after the period t. Such a dependency results in the computation of the value of x_t and y_t to begin from period 1 and continue until T.

The computational complexity can be expressed as follows :

The number of total sub-problems is $T(T+1)$ and total number of eliminative sub-plans related to size of y_t is $\frac{(T-\tau)(T-\tau-1)}{2}$, where if $\tau \geq T$, then the problem becomes a two-facilities capacity expansion problem for one demand type. And there are 2^{T-1} possible capacity addition timing sequences. Thus, it is doubtful for existence of good algorithm for an optimal solution.

We developed a tree search algorithm using the concept of Baker *et al.* [9]'s algorithm. Baker *et al.*[9] have developed a tree search backward procedure for a single item problem with production capacity restrictions that can vary with time. We consider a tree structure in which nodes represent various sub-plans for any described time intervals, and arcs represent decompositions according to the last time points where demands are covered. And because of sequence dependency, we must adapt forward procedure.

Let us define the following notations to describe the algorithm:

$(u,v:x_u,y_u)$= numbers in a node which represent capacity addition sub-plan for covering demands over the interval from u to v, where u and v+1 are two consecutive capacity points; either x_u or y_u has the value $\sum_{i=u}^{v}(r_i+y_{i-\tau})$, from Corollary 1

$R_t=\sum_{i=u}^{v}(r_i+y_{i-\tau})$, where y_i=0, for i<0

d_{uv}= arc cost incurred over the time interval from u to v;

$$d_{uv}=B_u(x_u)+L_u(y_u)+\sum_{t=u}^{v}H_t(I_{t+1}),\ \text{from Corollary 1}$$

f_v= node cost incurred over the interval from 1 to v; $f_v=f_u+d_{uv}$.

We now formulate an algorithm for finding an optimal solution.

Step 1. Set v=0.

Step 2. (1) Set u=v+1.

(2) Compute $(u,v:x_u,y_u)$ according to Corollary 1, and Equations (3.1) and (3.2) in Corollary 2, for $v=u,u+1,\cdots,T$.

Step 3. If v=T, go to step 4. Otherwise, go to Step 2.

Step 4. Compose capacity addition plans which cover all demands. If there exists v such that v<T, go to Step 2. Otherwise, go to Step 5.

Step 5. (1) For each $(u,v:x_u,y_u)$, test which variable can be positive according to Equation (3.3) in Corollary 2, where either x_u or y_u can be positive.

(2) Eliminate the plans $(u,v:x_u,y_u)$ which can not be an optimal solution

according to the result of the above test and corresponding capacity addition plans.

Step 6. Set f_0=0, $g=\infty$ and v=0.

Step 7. Set $u=v+1$.

(1) Branch the node $(u,v:x_u,y_u),(v=u, u+1,\cdots,T)$.

(2) Calculate d_{uv} and $f_v,(v=u,u+1,\cdots,T)$.

(3) If $f_v \leq f_{v^*}$ and $R_v \leq R_{v^*}$, then node (u,v^*) need not be branched further.

(4) If $f_v > g$, then this node need not be considered further.

Step 8. If $v=T$ and $f_T \leq g$, then set $g=f_T$ and go to Step 9. Otherwise, go to Step 9.

Step 9. If the active list is empty, then stop. Current g is the optimal objective value and an optimal plan is the capacity addition sequence which results in g. Otherwise, select any active list and go to Step 7.

The computational load by this algorithm is heavily dependent on the selection of branching node, but it is difficult to show which node selection is the best [10]. By the way, we can apply the newest bound rule for the algorithm giving an advantage of less cumbersome book-keeping and greater opportunity to obtain the bound efficiently.

We illustrate the algorithm with a numerical example in the next section.

V. A NUMERICAL EXAMPLE

Consider the following 4-periods problem with τ=2 and (r_1, r_2, r_3, r_4)=(2,1,3,2). Associated cost functions are given as follows:

$B_t(x_t)=(20+10x_t)(0.9)^{t-1}$,

$L_t(y_t)=(15+8y_t)(0.9)^{t-1}$,

$H_t(I_{t+1})=5I_{t+1}(0.9)^{t-1}$,

where t=1,2,3,4.

The objective is to find a capacity addition plan for buying and leasing capacity at minimum cost. We will find such a plan according to the algorithm proposed in Section IV.

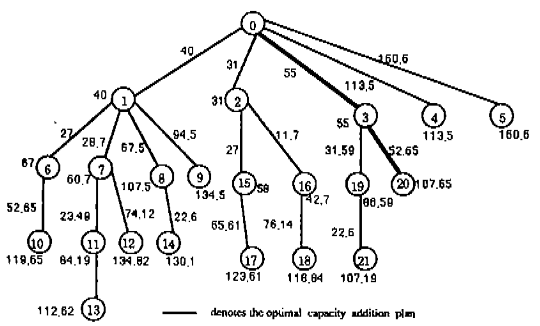
We will first find all sub-plans $(u,v:x_u,y_u)$ to be considered, according to Steps 1, 2 and 3.

For example, the sub-plan $(1,3:x_1,y_1)$ is derived according to Steps 1 and 2 as follows: either x_1 or y_1 has the value $\sum_{t=1}^{3}(r_t+y_{t-2})=6$. Therefore, $(1,3:x_1,y_1)$ ={(1,3:6,0),(1,3:0,6)}. But (1,3:0,6) is eliminated by Equation (3.1), since $v-u=2=\tau$

And the sub-plan $(3,3:3+y_1,0)$ is eliminated by Equation (3.2), since $B_t(a)>L_t(a)$ in the problem, for any t and a, $(a>0)$. By the similar way, we can obtain Table 1. And we can obtain all capacity addition plans to be considered which are found from Steps 4 and 5, where there are eliminative plans resulted from Step 5. For example, plans (y_1=2: x_2=1: y_3=5: y_4=2) and (y_1=2: y_2=1: y_3=5: y_4=3) are eliminated because $B_1(2)$ =40 < $L_1(2)+[L_3(5)-L_3(3)]$=31+19.44 = 50.44, and the plan (x_1=2: x_2=1: y_3=3: y_4=2) is eliminated because $B_2(1)=27 > L_2(1)+[L_4(3)-L_4(2)]$=26.53 and $v^*=4=T$. We will now find an optimal plan by cost comparisons among uneliminated plans which resulted from Step 1 through Step 5. Such cost comparisons are made by a tree structure search. For example, arc cost of the sub-plan (1, 4:8,0) is $B_1(8)+H_1(6)+H_2(5)+H_3(2)$ = 160.6 and g = 160.6 = f_4, since $v=4=T$. Node (3,4:0,5) gives the arc cost of $L_3(5)+H_3(2)$=52.65 and node cost of $f_4=f_3+d_{34}$ = 55+52.65 = 107.65 < 112.62 = g. Thus, g becomes the current cost f_4 = 107.65. And node (1,3:6,0) is disregarded, since f_3 = 113. 5 > g = 107.65. By the similar way, we can form a tree structure as shown in Fig.2 and find the optimal solution $(x_1,y_1:x_2,y_2:x_3,y_3:x_4,y_4)$ = (3,0:0, 0:0,5:0,0), with the objective value of 107.65.

Table 1. Capacity addition sub-plans

$(u,v:x_u,y_u)$	possible capacity addition sub-plans
$(1,v:x_1,y_1)$	(1,1:2,0),(1,1:0,2),(1,2:3,0), (1,2:0,3),(1,3:6,0),(1,4:8,0)
$(2,v:x_2,y_2)$	(2,2:1,0),(2,2:0,1),(2,3:4+y_1,0), (2,3:0,4+y_1),(2,4:6+y_1,0)
$(3,v:x_3,y_3)$	(3,3:0,3+y_1),(3,4:0,5+y_1+y_2)
$(4,v:x_4,y_4)$	(4,4:0,2+y_2)



node number	$(u,v:x_u,y_u)$	node number	$(u,v:x_u,y_u)$	node number	$(u,v:x_u,y_u)$
1	(1,1:2,0)	8	(2,3:4,0)	15	(2,2:1,0)
2	(1,1:0,2)	9	(2,4:6,0)	16	(2,2:0,1)
3	(1,2:3,0)	10	(3,4:0,5)	17	(3,4:0,7)
4	(1,3:6,0)	11	(3,3,:0,3)	18	(3,4:0,8)
5	(1,4:8,0)	12	(3,4:0,6)	19	(3,3:0,3)
6	(2,2 :1,0)	13	(4,4:0,3)	20	(3,4:0,5)
7	(2,2:0,1)	14	(4,4:0,2)	21	(4,4:0,2)

Fig. 2. Tree of sub-plans

VI. ALGORITHMS FOR NEAR-OPTIMAL SOLUTIONS

Since this model is not solved in polynomial time, it requires the development of a heuristically efficient algorithm to find a good solution. We developed two heuristic algorithms for near-optimal solutions, where one is a tree search algorithm which we will call the *simplified tree search algorithm* and the other one is a *piecewise linear approximation approach.*

1. SIMPLIFIED TREE SEARCH ALGORITHM

In this heuristic algorithm, we will find a near-optimal solution by considering only a subset of extreme points. The basic idea of this heuristic algorithm is as follows. Construct a sub-optimal solution by using optimal plans of sub-problems. For such a solution, the algorithm in Section IV can be used with some modifications as follows:

Step 1. Set $f_0 = 0$, $g = \infty$ and $v = 0$.

Step 2. If the active list is empty, then stop. Current g is a good objective value and the resulting plan is a capacity addition sequence associated with g. Otherwise, select any active list and go to Step 3.

Step 3. Set u=v+1.

(1) Compute $(u,v:x_u,y_u)$ according to Corollary 1 and Equation (3.1) in Corollary 2, v=u,u+1,$\cdots$,T.

(2) Test $(u,v:x_u,y_u)$ according to Equation (3.2.1), not necessary Equation (3.3) in Corollary 2, where (3.2.1) is derived as follows:

If $v - u < T$,

$$\text{then} \begin{cases} x_u = a, \text{ when } B_u(a) \leq L_u(a). \\ y_u = a, \text{ when } B_u(a) > L_u(a). \end{cases} \tag{3.2.1}$$

(3) Calculate d_{uv} and f_v, v=u,u+1,$\cdots$,T.

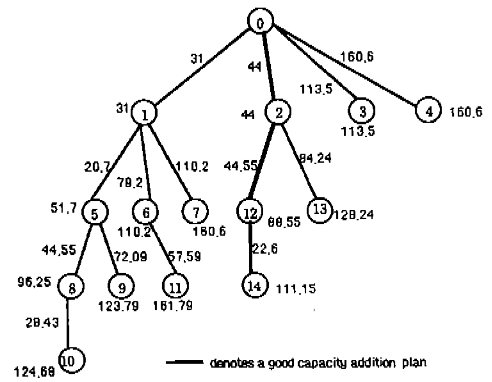
(4) If $f_{v^*} \leq f_{v'}$ and $R_{v^*} \leq R_{v'}$, then node (u,v^*) need not be branched further.

(5) If $f_{v^*} > g$, then this node need not be considered further.

Step 4. If $v=T$ and $f_T \leq g$, then set $g=f_T$ and go to Step 2. Otherwise, go to Step 2.

Since this heuristic algorithm always has a unique candidate at a sub-plan for forming a good solution, it will reduce the number of capacity addition sequences to be considered. Thus we can find a good solution easily. But in this heuristic algorithm, sequence dependency exists where the value of d_{uv} changes according to the previous addition sequence.

To illustrate the heuristic algorithm, we will reconsider the example given in Section V. We form the tree structure as given in Fig.3 obtained from the proposed algorithm.



node number	$(u,v{:}x_u,y_v)$	node number	$(u,v{:}x_u,y_v)$	node number	$(u,v{:}x_u,y_v)$
1	(1,1:0,2)	6	(2,3:0,1)	11	(4,4:0,3)
2	(1,2:0,3)	7	(2,4:8,0)	12	(3,3:0,5)
3	(1,3:6,0)	8	((3,3:0,5)	13	(3,4:0,8)
4	(1,4:8,0)	9	(3,4:0,8)	14	(4,4:0,2)
5	(2,2:0,1)	10	(4,4:0,3)		

Fig. 3. Tree structure for capacity addition plans

For example, the sub-plan (1,1:2,0) is disregarded at Step 3(2), because $B_1(2)$=40 > $L_1(2)$ =31, and costs associated with sub-plan (1,1:0,2) are computed at Step 3(3) such that d_{11} = $L_1(2)$=31 and f_1 = 0+d_{11} = 31. And the sub-plan (1,3:6,0) is fathomed, since the current value of g=111.15 is smaller than the value f_3 = 113.5, and hence it need not be branched further. By the similar way, we can find a good solution such that $(x_1,y_1{:}x_2,y_2{:}x_3,y_3{:}x_4,y_4)$ = (0,3:0,0:0,5:0,2) and the corresponding cost is 111.15 which has 3.25% relative error to the optimal solution.

2. PIECE-WISE LINEAR APPROXIMATION APPROACH

In this section, we discuss a piece-wise linear approximation(relaxation) to the solution for handling the concave objective function by adapting a mixed integer programming technique.

For this purpose, we define the following notations:

R = number of intervals

r,t = indices for the segment interval and time period, respectively, where r=1,2,$\cdots$,R and t=1,2,$\cdots$,T

M_{rt} = length of interval r at period t

b_{rt}, l_{rt}, h_{rt} = slopes of interval r at period t at cost function $B_t(.)$, $L_t(.)$ and $H_t(.)$, respectively

x_{rt}, y_{rt}, I_{rt} = decision variables corresponding to the amount of capacity in interval r at period t such that

$$x_t = \sum_{r=1}^{R} x_{rt},\ y_t = \sum_{r=1}^{R} y_{rt} \text{ and } I_t = \sum_{r=1}^{R} I_{rt}.$$

To illustrate this formulation, consider the concave function, $B_t(x_t)$.

For our problem, x_{rt} must satisfy the following conditions:

If $x_{rt} > 0$, then $x_{it} = M_{it}$, (i=1,2,$\cdots$$r$-1), and $x_{rt} \leq M_{rt}$.

These conditions can be mathematically expressed by introducing 0-1 integer variables U_{rt} as follows:

$x_{rt} \leq M_{rt}U_{rt}$,

$M_{r-1,t}U_{rt} \leq x_{r-1,t}$,

where $r=1,2,\cdots,R$ and $U_{rt} = 0$ or 1.

Therefore, for each t, $B_t(x_t)$ is approximated as follows:

$B_t(x_t) \cong \sum_{r=1}^{R} b_{rt}x_{rt}$, where $x_{rt} \leq M_{rt}U_{rt}$, $M_{r-1,t}U_{rt} \leq x_{r-1,t}$ and $U_{rt} = 0$ or 1, for $r = 1,2,\ldots, R$.

Likewise, the approximation for other functions, $L(y_t)$ and $H_t(I_{t+1})$, can be accomplished.

As the result, the piece-wise linear approximation of the original problem is formulated as follows:

$$\text{Min.} \sum_{t=1}^{T}\sum_{r=1}^{R}[b_{rt}x_{rt} + l_{rt}y_{rt} + h_{rt}I_{r,t+1}]$$

$$\text{s.t.} \sum_{r=1}^{R} I_{r,t+1} = \sum_{r=1}^{R} I_{rt} + \sum_{r=1}^{R} x_{rt} + \sum_{r=1}^{R} y_{rt} - r_t - \sum_{r=1}^{R} y_{r,t-\tau}$$

$$x_{rt} \leq M_{rt}\ U_{rt}, \quad M_{r-1,t}\ U_{rt} \leq x_{r-1,t},$$

$$y_{rt} \leq M_{rt}\ U_{rt}, \quad M_{r-1,t}\ U_{rt} \leq y_{r-1,t},$$

$$I_{rt} \leq M_{rt}\ U_{rt}, \quad M_{r-1,t}\ U_{rt} \leq I_{r-1,t},$$

$$x_{rt}\ y_{rt},\ I_{rt} \geq 0,$$

$$I_{ri} = I_{r,T+1} = 0,$$

$$U_{rt}, V_{rt}, W_{rt} = 0 \text{ or } 1,$$

where $r = 1,2,\ldots, R$ and $t = 1,2,\ldots T$.

The effectiveness of the piecewise linear approximation technique depends heavily on the degree of non-linearity to be approximated. Precision in the approximation increases as the size of the linear segments decreases. In principle, the model can be formulated with an arbitrary number of linear segments, where both model size and precision of the resulting model increase with the number of such segments. Since slopes of incorporated cost functions are nonincreasing with amounts of capacity, an efficient approach may require the size of segments to be small initially, and then to increase with amount of capacity.

We must consider, then, how segment sizes vary as the slopes of cost functions decrease. We can suggest one way such as doubling the size of segments sequentially, except for the first segment, with an identical rate of variation in slope average [11].

For example, the example problem considered in Section V is solved by the above approach, which results in the same optimal solution in Section V.

The formulation for the example is made with some modifications to handle set-up cost as shown below and others are formulated by the above way. We select R=1 and $M_1 = \sum_{t=1}^{4} r_t = 8$, since this problem itself has piecewise linear cost functions for each time t, and formulate $B_t(x_t)$ and $L_t(y_t)$ as follows:

$$B_t(x_t) = (20 + 10x_t)(0.9)^{t-1},$$

$$= 20(0.9)^{t-1}p_t + 10(0.9)^{t-1}\sum_{r=1}^{R} x_{rt},$$

$$= 20(0.9)^{t-1}p_t + 10(0.9)^{t-1}x_{1t} \text{ and}$$

$$L_t(y_t) = (15 + 8y_t)(0.9)^{t-1},$$

$$= 15(0.9)^{t-1}q_t + 8(0.9)^{t-1}y_{1t},$$

where $p_t,\ q_t = 0 \text{ or } 1,$

$$x_{1t} \leq 8p_t,$$

$$y_{1t} \leq 8q_t \text{ and}$$

$$t = 1,2,3,4.$$

We can now obtain a Mixed Integer Programming(MIP) formulation for the given problem as follows:

$$\text{Min.} \sum_{t=1}^{4}[20(0.9)^{t-1}p_t + b_{1t}x_{1t} + 15(0.9)^{t-1}q_t + l_{1t}y_{1t} + h_{1t}I_{1,t+1}]$$

$$\text{s.t } I_{1,t+1} = I_{1t} + x_{1t} + y_{1t} - r_t - y_{1,t-2},$$

$$x_{1t} \leq 8U_{1t},\ x_{1t} \leq 8p_t,$$

$$y_{1t} \leq 8V_{1t},\ y_{1t} \leq 8q_t,$$

$I_{1t} \leq 8W_{1t}$,
$x_{1t}, y_{1t}, I_{1t} \geq 0$,
$I_{1,1} = I_{1,5} = 0$,
$U_{1t}, V_{1t}, W_{1t}, p_t$ *and* q_t *equal to* 0 *or* 1,
where t = 1, 2, 3, 4.

The coefficients of objective functions, b_{1t}, l_{1t} and h_{1t}, are computed as follows:

$b_{1t} = [B_t(8)-20(0.9)^{t-1}]/8 = 10(0.9)^{t-1}$,
$l_{1t} = [L_t(8)-15(0.9)^{t-1}]/8 = 8(0.9)^{t-1}$,
$h_{1t} = H_t(8)/8 = 5(0.9)^{t-1}$,
where t=1, 2, 3, 4.

Then, we can find a solution for such MIP problems by using a software available in the market.

VII. CONCLUSION

This paper described a deterministic capacity expansion planning model with buy-or-lease decisions for one demand type. These decisions are treated as dynamic variables. The problem has sequence dependency in decisions. But if $\tau \geq$T, then this model becomes a two facilities capacity expansion planning model without any sequence dependency as studied by Lee [6] and Sung [7].

We analyzed the characteristics of an extreme point such that capacity addition can occur only at a capacity point and there is a restriction to the size of leasing capacity for an extreme point. We showed that this problem can not be solved in polynomial time, and developed a tree search forward procedure for an optimal solution. And two heuristic algorithms for a near-optimal solution such as a simplified tree search algorithm and a piecewise linear approximation approach were added.

Based on the major contribution of this paper, the first investigation (mathematical formulation and characterization of the optimal solution) of the buy-or-lease capacity expansion situations, some further researches can be considered. A problem with integral capacity additions is a further study subject. And an extension by allowing capacity disposals and shortages may be considered. But such models may still have the sequence dependency of the leased capacity. Therefore, some efficient heuristic algorithms need to be developed.

APPENDIX : PROOF FOR THEOREM 1

Proof

First, we will prove "only if" part. Since problem (P) is a single commodity network problem, as shown in Fig. 1, each node has at most one positive input to be an extreme flow [12]. Thus, the Conditions (2.1), (2.2) and (2.3) are sufficient conditions for an extreme point. And if Condition (2.4) is not satisfied, then neither Condition (2.1) nor (2.2) is satisfied for a feasible solution. Thus, Condition (2.4) must be satisfied.

Now we will prove "if" part by contraposition. The proof is accomplished by the following procedure. Let $Z = (X,Y) = (x_1,y_1,x_2,y_2,\cdots,x_i,y_i,\cdots x_T,y_T)$. Assume that $Z = (X,Y)$ is a feasible but not an extreme point vector. Then there exist feasible points Z^1 and Z^2 such that

$$Z = \frac{Z^1 + Z^2}{2}, (Z^1 \neq Z^2).$$

We will show that Z does not satisfy the Conditions (2.1), (2.2), (2.3) and (2.4). According to the above procedure, it is sufficient to consider the following five cases:

Case (1) $Z^1=Z+\varepsilon\mu_{x_i}-\varepsilon\mu_{x_j}$ and $Z^2=Z-\varepsilon\mu_{x_i}+\varepsilon\mu_{x_j}$

Case (2) $Z^1=Z+\varepsilon\mu_{x_i}-\varepsilon\mu_{y_j}$ and $Z^2=Z-\varepsilon\mu_{x_i}+\varepsilon\mu_{y_j}$

Case (3) $Z^1=Z+\varepsilon\mu_{x_i}-\varepsilon\mu_{y_i}$ and $Z^2=Z-\varepsilon\mu_{x_i}+\varepsilon\mu_{y_i}$

Case (4) $Z^1=Z+\varepsilon\mu_{y_i}-\varepsilon\mu_{x_j}$ and $Z^2=Z-\varepsilon\mu_{y_i}+\varepsilon\mu_{x_j}$

Case (5) $Z^1=Z+\varepsilon\mu_{y_i}-\varepsilon\mu_{y_j}$ and $Z^2=Z-\varepsilon\mu_{y_i}+\varepsilon\mu_{y_j}$

where μ_{x_i} and μ_{y_i} represent a $1\times 2T$ unit vector in which i-th elements of X and Y are one and others equal zero, respectively, i.e.,

$$\mu_{xi}=(\mu_1^1,\mu_2^1,\ldots,\mu_{2(i-1)}^1,\mu_{2i}^1,\ldots\mu_{2T}^1),$$
$$\mu_{yi}=(\mu_1^2,\mu_2^2,\ldots,\mu_{2(i-1)}^2,\mu_{2i}^2,\ldots\mu_{2T}^2),$$

where $\mu_{2(i-1)}^1=\mu_{2i}^2=1$, others equal zeros, and $1\leq i<j\leq T$. First, we will work on Case (1). Assume that Z is a feasible but not an extreme point, then there exist feasible points as follows :

$$Z^1=Z+\varepsilon\mu_{xi}-\varepsilon\mu_{xj} \text{ and } Z^2=Z-\varepsilon\mu_{xi}+\varepsilon\mu_{xj}$$
$$\text{i.e., } Z^1=(x_1,y_1,\ldots,x_i+\varepsilon,y_i,\ldots,x_j-\varepsilon,y_j,\ldots,x_T,y_T)$$
$$Z^2=(x_1,y_1,\ldots,x_i-\varepsilon,y_i,\ldots,x_j+\varepsilon,y_j,\ldots,x_T,y_T)$$

To find feasible conditions of Z^1 and Z^2, denote idle capacity results given from the plan Z^i (i=1, 2) by I^i. Then, we can derive the followings:

$$I_t^1=I_t^2=I_t,\ \text{for } 1\leq t\leq i,$$
$$I_t^1=I_t+\varepsilon \text{ and } I_t^2 = I_t+\varepsilon,\ \text{for } i+1\leq t\leq j,$$
$$I_t^1=I_t^2=I_t,\ \text{for } j+1\leq t\leq T+1.$$

Thus, for feasibility of Z^1 and Z^2, we can select an ε ($\varepsilon>0$) such that

$$\varepsilon=\frac{1}{2}Min.[x_i,x_j,\ \min_{i+1\leq t\leq j}[I_t]].$$

There exists such an ε when $x_i\geq\varepsilon>0$, $x_j\geq\varepsilon>0$ and $I_t\geq\varepsilon>0$, for $i+1\leq t\leq j$. However, $x_jI_j\neq 0$. Therefore, the feasible point Z must satisfy Condition (2.1) so as to be an extreme point.

We now work on Case (2). Assume that Z is a feasible but not an extreme point. Consider the following feasible points.

$$Z^1=Z+\varepsilon\mu_{xi}-\varepsilon\mu_{yj} \text{ and } Z^2=Z-\varepsilon\mu_{xi}+\varepsilon\mu_{yj}$$
$$\text{i.e., } Z^1=(x_1,y_1,\ldots,x_i+\varepsilon,y_i,\ldots,y_j-\varepsilon,y_j,\ldots,x_T,y_T)$$
$$Z^2=(x_1,y_1,\ldots,x_i-\varepsilon,y_i,\ldots,y_j+\varepsilon,y_j,\ldots,x_T,y_T)$$

The idle capacities of Z^1 and Z^2 are derived as follows:

$$I_t^1=I_t^2=I_t,\ \text{for } 1\leq t\leq i,$$
$$I_t^1=I_t+\varepsilon \text{ and } I_t^2 = I_t-\varepsilon,\ \text{for } i+1\leq t\leq j,$$
$$I_t^1=I_t^2=I_t,\ \text{for } j+1\leq t\leq T+1.$$
$$\text{where } T-j<\tau.$$

Note that if $T-j\geq\tau$, then $I_{T+1}^1=\varepsilon$ and $I_{T+1}^2=-\varepsilon$. Thus, it is an infeasible plan.

We can select an ε for feasibility of Z^1 and Z^2 such that

$$\varepsilon=\frac{1}{2}Min.[x_i,x_j,\ \min_{i+1\leq t\leq j}[I_t]]>0.$$

Such an ε exists when $x_i\geq\varepsilon>0$, $y_j\geq\varepsilon>0$ and $I_t\geq\varepsilon>0$, ($i+1\leq t\leq j$ and $T<j+\tau$).

However, $y_jI_j\neq 0$, which contradicts to Condition (2.2). Thus, the feasible point Z must satisfy Condition (2.2) so as to be an extreme point.

By the similar way, we can show that Case (3) contradicts to Condition (2.3) and Case (4) contradicts to Condition (2.1). Likewise, Case (5) contradicts to Condition (2.2). Thus, Conditions (2.1), (2.2) and (2.3) must be satisfied for an extreme point solution.

It remains to show that Condition (2.4) is a necessary condition. This is accomplished by showing that at least one of the Conditions (2.1), (2.2) and (2.3) are not satisfied for a feasible solution if the condition (2.4) is not satisfied. That is to say, if $y_t=\sum_{i=t}^{t+\tau-1+j}(r_i+y_{i-\tau})y_{r,t-\tau}$ for $j\geq 1$ and y_i=0 ($t+1\leq i\leq t+\tau+j$), then a capacity addition must occur at period $t+\tau$ to satisfy the demand. But it contradicts to Condition (2.1) or (2.2). And if there exist at least one i such that y_i>0, ($t+1\leq i\leq t+\tau+j$), then this point contradicts to

Condition (2.2), though it need not have a capacity addition at period $t+\tau$. As a result, the feasible point Z must satisfy Conditions (2.1), (2.2), (2.3) and (2.4) so as to be an extreme point. Thus, the proof is completed.

ACKNOWLEDGEMENTS

The author is grateful to the anonymous referees for their comments. And the author also would like to thank the Head of Transmission Systems Section for his considerations.

REFERENCES

[1] Kalotay, A.J., "Capacity Expansion and Specialization," *Management Science*, Vol.20, No.1, pp.56-64, 1973.

[2] Erlenkotter, D., "A Dynamic Approach to Capacity Expansion with Specialization," *Management Science*, Vol.21, No.3, pp.360-362, 1974.

[3] Fong, C.O. and Rao, M.R., "Capacity Expansion with Two Producing Regions and Concave Costs," *Management Science*, Vol.22, No.3, pp.291-303, 1979.

[4] Gascon, A. and Leachman, R.C., "A Dynamic Programming Solution to the Dynamic, Multi-item Single-machine Scheduling Problem," *Operations Research*, Vol.36, No.1, pp.50-56, 1988.

[5] Dutta, A. and Lim, J.I., "A Multiperiod Capacity Planning Model for Backbone Computer Communication Networks," *Operations Research*, Vol.40, No.4, pp. 689-705, 1992.

[6] Lee, S.B., "A Dynamic Lot-Size Model with Make-or-Buy Decisions" in *Topics in Dynamic Lot-Sizing*, Ph. D. dissertation, Graduate School of Arts and Sciences, University of Columbia, 1985, pp.9-26.

[7] Sung, C.S., "A Single-Product Parallel-Facilities Production-Planning Model," *International Journal of Systems Science*, Vol.17, No.7, pp.983-989, 1986.

[8] Kamien, M.I. and Li, L., "Subcontracting, Coordination, Flexibility, and Production Smoothing in Aggregate Planning," *Management Science*, Vol.36, No.11, pp. 1352-1363, 1990.

[9] Baker, K.R., Dixon, P., Magazine M.J. and Silver, E.A., "An Algorithm for The Dynamic Lot-Size Problem with Time-Varying Production Capacity Constraints," *Management Science*, Vol.24, No.16, pp.1710-1720, 1978.

[10] Ibaraki, T., "The Power of Dominance Relations in Branch-and-Bound Algorithms," *Journal of the Association for Computing Machinery*, Vol.24, No.2, pp. 264-279, 1977.

[11] Hiller, R.S. and Shapiro, J.F., "Optimal Capacity Expansion Planning when There Are Learning Effects," *Management Science*, Vol.32, No.9, pp.1153-1163, 1986.

[12] Zangwill, W.I., "Minimum Concave Cost Flows in Certain Networks," *Management Science*, No.7, pp.429-450, 1968.

Un Gi Joo received a B.S. degree in industrial engineering from Sung Kyun Kwan University, in 1986, the M.S. and Ph.D. degrees in industrial engineering from Korea Advanced Institute of Science and Technology(KAIST), in 1988 and 1992, respectively. Dr. Joo is currently a senior member of technical staff in the Transmission Systems Section at ETRI, and his research field includes project scheduling, quality assurance, and network survivability.