Noncentral F-Distribution for an M-ary Phase Shift Keying Wedge-Shaped Region

Jung-Su Kim and Jong-Wha Chong

ABSTRACT—This letter presents an alternative analytical expression for computing the probability of an M-ary phase shift keying (MPSK) wedge-shaped region in an additive white Gaussian noise channel. The expression is represented by the cumulative distribution function of known noncentral F-distribution. Computer simulation results demonstrate the validity of our analytical expression for the exact computation of the symbol error probability of an MPSK system with phase error.

Keywords—Noncentral F-distribution, M-ary phase shift keying, cumulative distribution function, symbol error probability, phase error.

I. Introduction

Several papers have considered the cumulative distribution function (CDF) for computing the wedge-shaped region of an M-ary phase shift keying (MPSK) system in an additive white Gaussian noise (AWGN) channel. The Pawula F-function and two-dimensional Gaussian Q-function have been extensively used to compute the probability of the wedge-shaped region in various MPSK systems [1]-[4]. From the viewpoint of statistical distribution theory, however, it is possible to conceive an alternative solution for computing the probability of the wedge-shaped region of an MPSK system.

In this letter, we derive a CDF for computing the probability of an arbitrary wedge-shaped region over a Gaussian channel. The derived CDF is completely represented by the CDF of

Jung-Su Kim (phone: + 82 31 209 8603, email: jungx.kim@samsung.com) is with the Department of Electronics and Communication Engineering, Hanyang University, Seoul, Rep. of Korea, and also with the Channel Development Team, S.LSI Division, Samsung Electronics, Giheung, Rep. of Korea.

Jong-Wha Chong (email: jchong@hanyang.ac.kr) is with the Department of Electronics and Communication Engineering, Hanyang University, Seoul, Rep. of Korea.

noncentral F-distribution. To demonstrate our result, the symbol error probability (SEP) performance of an MPSK system with phase error is evaluated by using the *ncfcdf* function of a popular MATLAB program [5]. The newly derived CDF can be used to compute the wedge-shaped region of various MPSK systems in an AWGN channel.

II. Derivation

In this section, we derive an analytical CDF for computing the probability of the wedge-shaped region when a PSK signal is perturbed by Gaussian noise.

Figure 1 is a geometric representation of the wedge-shaped region characterized by a phase angle $\theta \in [0, 2\pi]$. We consider the wedge-shaped region in Fig. 1, $\angle AOX$, determined by the semi-infinite lines OA and OX in the presence of the MPSK signal having amplitude m_X perturbed by two Gaussian noises. In this case, the joint probability density function (PDF) of X and Y is given by

$$f(x,y) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{(x-m_X)^2 + y^2}{2\sigma^2} \right]. \qquad (1)$$

In (1), the two random variables X and Y are independent Gaussian random variables having mean m_X and zero mean, respectively, and two equal variances $\sigma_X^2 = \sigma_Y^2 = \sigma^2$.

Next, as shown in Fig. 1, the probability for the wedge-shaped region $\Pr\{\angle AOX\}$ can be expressed as

$$\begin{aligned} \Pr\{\angle AOX\} &= \Pr\{0 < Y < zX\} \\ &= \Pr\{Y > 0\} \Pr\{Y < zX | Y > 0\}. \end{aligned} \qquad (2)$$

Since Y is a Gaussian random variable with zero mean and variance σ^2, the probability $\Pr\{Y > 0\}$ becomes 1/2. Then,
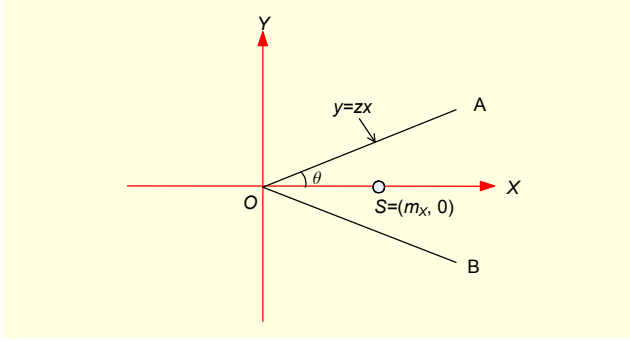
Fig. 1. Geometry of wedge-shaped region for the PSK signal, $S=(m_X, 0)$.

the probability $\Pr\{Y < zX \mid Y > 0\}$ is rewritten with some algebraic manipulation as

$$\Pr\{Y < zX \mid Y > 0\} = 1 - \Pr\left\{\frac{X^2/\sigma^2}{Y^2/\sigma^2} < \frac{1}{z^2} \bigg| Y > 0\right\}, \quad (3)$$

where $z = \tan\theta = y/x$. Here, it is well-known that the distribution of $(X^2/\sigma^2 / Y^2/\sigma^2)$ has the noncentral F-distribution with two 1-degrees of freedom and the noncentral parameter m_X^2/σ^2 [6]. Applying the theory of noncentral F-distribution to (3) gives

$$\Pr\{Y < zX \mid Y > 0\} = 1 - F\left(\cot^2\theta \mid m_X^2/\sigma^2\right), \quad (4)$$

where the CDF of noncentral F-distribution $F(\alpha \mid \beta)$ is given as in [6] by

$$F(\alpha \mid \beta) = \sum_{j=0}^{\infty}\left[\frac{(0.5\beta)^j}{j!}\exp(-0.5\beta)\right] I\left(\frac{\alpha}{1+\alpha}\bigg| 0.5+j, 0.5\right), \quad (5)$$

in which $I(x \mid a, b)$ is the incomplete beta function with parameters a and b.

Finally, substituting (4) into (2) and using $\Pr\{Y > 0\} = 1/2$, we get an analytical expression for the probability of the wedge-shaped region $\Pr\{\angle AOX\}$ in the form of the CDF of noncentral F-distribution as

$$\Pr\{\angle AOX\} = \frac{1}{2} - \frac{1}{2}F\left(\cot^2\theta \mid m_X^2/\sigma^2\right). \quad (6)$$

To extend from the special case, $m_Y=0$, to the general case of an arbitrary PSK signal, $m_X \neq 0$ and $m_Y \neq 0$, we consider that the joint Gaussian PDF of X and Y centered at (m_X, m_Y) is given by

$$f(x, y) = \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{(x-m_X)^2 + (y-m_Y)^2}{2\sigma^2}\right\}. \quad (7)$$
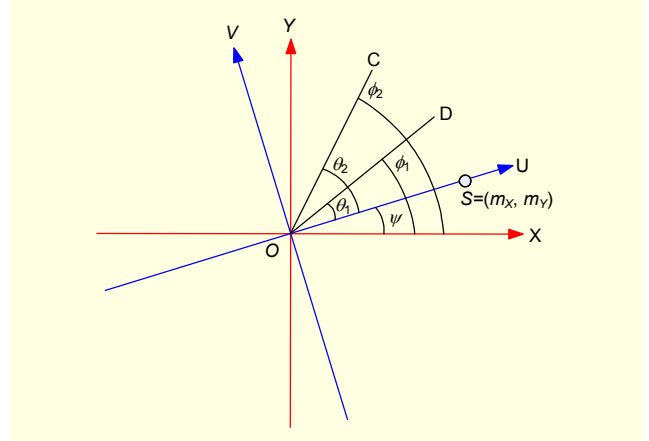
Note that when $m_Y=0$, (7) becomes (1).

Fig. 2. Geometry of an arbitrary wedge-shaped region for general PSK signal $S=(m_X, m_Y)$.

Figure 2 shows the geometry of an arbitrary wedge-shaped region, $\angle COD$ determined by $\phi_2 - \phi_1$. The probability $\Pr\{\angle COD\}$ is exactly equal to the probability $\Pr\{\phi_1 < \Theta < \phi_2\}$. Here, we rotate the Cartesian coordinates, as was done in [3], in such a way that

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}. \quad (8)$$

From the geometry of Fig. 2, the probability $\Pr\{\angle COD\}$ is characterized by the probability $\Pr\{\theta_1 < \Theta < \theta_2\}$, that is, $\theta_1 < \theta_2$, from the viewpoint of the UV Cartesian coordinate system. Using (6), we have

$$\Pr\{0 < \Theta < \theta_2\} = \frac{1}{2} - \frac{1}{2}F\left(\cot^2\theta_2 \mid m_U^2/\sigma_U^2\right), \quad (9)$$

where

$$\begin{cases} m_U = \sqrt{m_X^2 + m_Y^2} \\ \sigma_U^2 = \sigma^2 \end{cases}. \quad (10)$$

Thus, we easily obtain the probability $\Pr\{\theta_1 < \Theta < \theta_2\}$ as

$$\begin{aligned} &\Pr\{\theta_1 < \Theta < \theta_2\} \\ &= \frac{1}{2}F\left(\cot^2\theta_1 \mid m_U^2/\sigma^2\right) - \frac{1}{2}F\left(\cot^2\theta_2 \mid m_U^2/\sigma^2\right). \end{aligned} \quad (11)$$

III. Application

This section presents an alternative analytical expression and its numerical results for the SEP of a uniform MPSK system having a phase error ϕ in an AWGN channel. In this case, the phase angle between the MPSK transmitted signal vectors is π/M. As in [3], two different correction regions of the MPSK system are also determined by the two phase angles

$\pi/M + \phi$ and $\pi/M - \phi$, respectively. Thus, the SEP of the MPSK system can be expressed as

$$P_E = 1 - \Pr\{0 < \Theta < \pi/M + \phi\} - \Pr\{0 < \Theta < \pi/M - \phi\}. \quad (12)$$

Substituting $m_U = \sqrt{E_S}$ and $\sigma_U^2 = \sigma^2 = N_0/2$ into (9) leads to the noncentral parameter of noncentral F-distribution $2E_S/N_0$, which represents twice the signal-to-noise ratio per symbol. Using (9) with $\theta_2 = \pi/M \pm \phi$, we find the SEP of the MPSK system to be

$$P_E = \frac{1}{2}F\left[\cot^2(\pi/M + \phi)\big|2E_S/N_0\right]$$
$$+ \frac{1}{2}F\left[\cot^2(\pi/M - \phi)\big|2E_S/N_0\right]. \quad (13)$$

For $\phi = 0$, (13) reduces to

$$P_E = F\left[\cot^2(\pi/M)\big|2E_S/N_0\right]. \quad (14)$$

Note that (14) is an analytical expression for the SEP of the perfect MPSK system having no phase error.

To demonstrate the validity of our results, we compute the SEP of 8-PSK with various values of phase error ϕ using (13) and the *ncfcdf* function provided by MATLAB [5]. Figure 3 shows numerically good agreement between (13) and the previous results presented in (11) of [3] for various values of ϕ. We observed that the effect of phase error ϕ increases as the signal-to-noise ratio per symbol, E_S/N_0, increases. These computer simulation results demonstrate that the derived CDF of noncentral F-distribution is available in computing the probability for the wedge-shaped region of the MPSK signal vector in an AWGN channel.
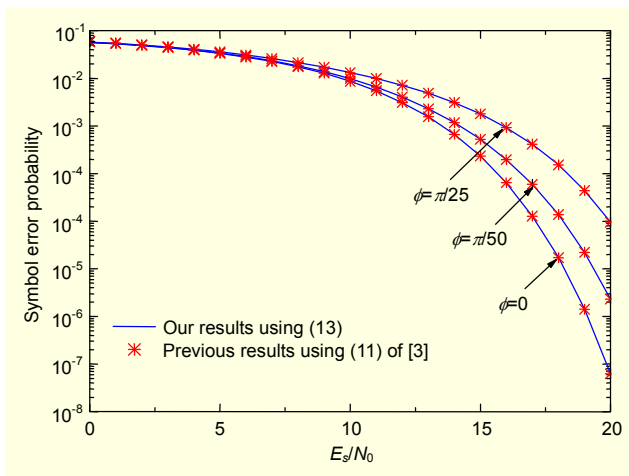
Also, the noncentral F-distribution of $(X^2/\sigma^2/Y^2/\sigma^2)$ can be applied to compute $\Pr\{X^2 < aY^2\}$ in a situation in which two voltages, *X* and *Y*, exist in dual diversity over log-normal fading channels.

IV. Conclusion

In this letter, we presented the CDF of noncentral F-distribution for computing an arbitrary wedge-shaped region of an MPSK system having a phase error over a Gaussian channel. We investigated the effect of phase error for MPSK SEP using our analytical results. Computer simulations show very good agreement between our results and previous results. We hope that this work provides an additional tool for computing the probability of an arbitrary wedge-shaped region in the presence of two uncorrelated Gaussian random variables with two different means and two equal variances.

Acknowledgment

References

[1] R.F. Pawula, S.O. Rice, and J.H. Roberts, "Distribution of the Phase Angle between Two Vectors Perturbed by Gaussian Noise," *IEEE Trans. Commun.*, vol. COM-30, no. 8, Aug. 1982, pp. 1828-1841.

[2] R.F. Pawula, "Distribution of the Phase Angle between Two Vectors Perturbed by Gaussian Noise II," *IEEE Trans. Veh. Technol.*, vol. 50, no. 2, Mar. 2001, pp. 576-583.

[3] S. Park and S.H. Cho, "Probability of an Arbitrary Wedge-Shaped Region of the MPSK System in the Presence of Quadrature Error," *IEEE Commun. Lett.*, vol. 6, no. 2, Feb. 2005, pp. 196-197.

[4] S. Park and S.H. Cho, "Computing the Average Symbol Error Probability of the MPSK System Having Quadrature Error," *ETRI Journal*, vol. 28, no. 6, Dec. 2006, pp.793-795.

[5] Matwork Inc., *MATLAB User's Guides*, version 7.0, 2006.

[6] R. Price, "Some Non-central F Distributions Expressed in Closed Forms," *Biometrika*, vol. 51, no. 1-2, 1964, pp. 107-122.

Fig. 3. Comparison of 8-PSK SEP in an AWGN channel with results from [3].