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Title: SPARSE REPRESENTATIONS WITH DATA FIDELITY TERM VIA AN ITERATIVELY REWEIGHTED LEAST SQUARES ALGORITHM

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Sparse Representations with ℓ^1 Data Fidelity Term via an Iteratively Reweighted Least Squares Algorithm

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Abstract— **Basis Pursuit and Basis Pursuit Denoising, well established techniques for computing sparse representations, minimize an ℓ^2 data fidelity term subject to an ℓ^1 sparsity constraint or regularization term on the solution by mapping the problem to a linear or quadratic program. Basis Pursuit Denoising with an ℓ^1 data fidelity term has recently been proposed, also implemented via a mapping to a linear program. We introduce an alternative approach via an iteratively Reweighted Least Squares algorithm, providing greater flexibility in the choice of data fidelity term norm, and computational advantages in certain circumstances.**

I. Introduction

Recently, several authors have considered sparse approximation and image restoration problems with respect to error measures other than the usual ℓ^2 norm (Euclidean distance). Specifically, problems which include a ℓ^1 data fidelity term have attracted great attention.

In the case of sparse approximations, the Basis Pursuit (BP) and Basis Pursuit denoising (BPDN) paradigm, introduced in [1], consist in the following minimization problems:

$$\begin{array}{ll} \text{BP} & \min \left\|\mathbf{u}\right\|_1 \ \text{ subject to } \Phi\mathbf{u} = \mathbf{b} \\ \text{BPDN} & \min \ \frac{1}{2}\left\|\Phi\mathbf{u} - \mathbf{b}\right\|_2^2 + \lambda\left\|\mathbf{u}\right\|_1 \end{array}$$

where Φ is a $n \times p$ matrix form using the basis vectors from an (in general) overcomplete dictionary i.e. $p \gg n$, and it is assumed that rank$(\Phi) = n$. The BP and BPDN problems are solved using linear or quadratic programming via interior point methods (see [1]) or by iteratively solving a weighted ℓ^2 version of the original problem (see [2], [3]). The Affine Scaling Transformation (AST) algorithm ([4], [5]) was proposed to minimize a modified BP and BPDN cost functional; these functionals use the ℓ^p norm to measure the sparsity of the representation sought in the overcomplete dictionary Φ

$$\begin{array}{ll} \text{AST-BP} & \min \left\|\mathbf{u}\right\|_p^p \ \text{ subject to } \Phi\mathbf{u} = \mathbf{b} \\ \text{AST-BPDN} & \min \ \frac{1}{2}\left\|\Phi\mathbf{u} - \mathbf{b}\right\|_2^2 + \frac{\lambda}{p}\left\|\mathbf{u}\right\|_p^p \end{array}$$

and it can be summarized as:

$$\mathbf{u}^{(k)} = W^{(k)^2}\Phi^T\left(\Phi W^{(k)^2}\Phi^T + \lambda I\right)^{-1}\mathbf{b} \qquad (1)$$

where $W^{(k)} = \text{diag}\left(\left|\mathbf{u}^{(k-1)}\right|^{1-\frac{p}{2}}\right)$. Note that setting $p = 1$ and $\lambda = 0$ will solve the BP problem, whereas if $\lambda > 0$ will solve the BPDN problem. More recently, [6] studies the BPDN problem with an ℓ^1 data fidelity term

$$\text{BPDN } \ell^1 - \text{Fidelity} \qquad \min \ \left\|\Phi\mathbf{u} - \mathbf{b}\right\|_1 + \lambda\left\|\mathbf{u}\right\|_1$$

and shows the equivalence of the above minimization problem with Linear Programming; as an application, denoising in the presence of impulse noise is considered for (only) 1D signals.

In the case of image restoration, [7] considers the minimization of a cost functional with an ℓ^1 norm for both the fidelity and regularization terms. More specifically, for Total Variation (TV), the inclusion of the ℓ^1 data fidelity term [8], [9] has a number of advantages, including superior denoising performance with salt and pepper (speckle) noise [10]; there are a fair amount of algorithms [11], [9], [10], [12], [13], [14] to tackle this problem.

In this paper we propose a simple yet flexible and computationally efficient algorithm to solve the generalized BP problem

$$\text{Generalized BPDN} \qquad \min \ \frac{1}{q}\left\|\Phi\mathbf{u} - \mathbf{b}\right\|_q^q + \frac{\lambda}{p}\left\|\mathbf{u}\right\|_p^p$$

this algorithm, called Iteratively Reweighted Norm for BPDN (or IRN-BPDN), which is an extension to the AST algorithm [4], [5], is closely related to the Iteratively Reweighted Norm for Total Variation (or IRN-TV, see [11]).

II. IRN-BPDN Algorithm

A. *Fidelity term*

$$W_F = \text{diag}\left(\frac{2}{q}f_F(\Phi\mathbf{u} - \mathbf{b})\right),$$

where f_F is defined (for some small ϵ_F) as

$$f_F(x) = \begin{cases} |x|^{1-\frac{q}{2}} & \text{if } |x| > \epsilon_F \\ \epsilon_F^{1-\frac{q}{2}} & \text{if } |x| \le \epsilon_F, \end{cases}$$

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Algorithm 1 IRN-BPDN algorithm

Initialize

$$\mathbf{u}^{(0)} = \Phi^T \left(\Phi\Phi^T + \lambda I\right)^{-1} \mathbf{b}$$

Iterate

$$W_F^{(k)} = \mathrm{diag}\left(\frac{2}{q} f_F(\Phi \mathbf{u}^{(k-1)} - \mathbf{b})\right)$$

$$W_R^{(k)} = \mathrm{diag}\left(\frac{2}{p} f_R\left(\mathbf{u}^{(k-1)}\right)\right)$$

$$\boldsymbol{\chi}^{(k)} = \left(\Phi W_R^{(k)} \Phi^T + \lambda W_F^{(k)}\right)^{-1} \mathbf{b}$$

$$\mathbf{u}^{(k)} = \left(W_R^{(k)}\right)^2 \Phi^T \boldsymbol{\chi}^{(k)}$$

B. Sparsity term

See [4], [5] for details.

$$W_R = \mathrm{diag}\left(\frac{2}{p} f_R(\mathbf{u})\right),$$

where f_R is defined (for some small ϵ_R) as

$$f_R(x) = \begin{cases} |x|^{1-\frac{p}{2}} & \text{if } |x| > \epsilon_R \\ 0 & \text{if } |x| \leq \epsilon_R \end{cases}$$

C. Derivation

$$\frac{1}{q} \|\Phi \mathbf{u} - \mathbf{b}\|_q^q + \frac{\lambda}{p} \|\mathbf{u}\|_p^p$$

$$\frac{1}{2} \left\|W_F{}^{-1} \Phi \mathbf{u} - W_F{}^{-1} \mathbf{b}\right\|_2^2 + \frac{\lambda}{2} \left\|W_R{}^{-1} \mathbf{u}\right\|_2^2$$

Set $\mathbf{u} = W_R \boldsymbol{\nu}$

$$\frac{1}{2} \left\|W_F{}^{-1} \Phi W_R \boldsymbol{\nu} - W_F{}^{-1} \mathbf{b}\right\|_2^2 + \frac{\lambda}{2} \|\boldsymbol{\nu}\|_2^2$$

gradient

$$W_R \Phi^T W_F{}^{-2} \Phi W_R \boldsymbol{\nu} - W_R \Phi^T W_F{}^{-2} \mathbf{b} + \lambda \boldsymbol{\nu} = 0$$

set $\boldsymbol{\nu} = W_R \Phi^T \boldsymbol{\chi}$

$$W_R \Phi^T W_F{}^{-2} \Phi W_R \boldsymbol{\nu} - W_R \Phi^T W_F{}^{-2} \mathbf{b} + \lambda W_R \Phi^T \boldsymbol{\chi} = 0$$

$$W_R \Phi^T W_F{}^{-2} \left(\Phi W_R{}^2 \Phi^T \boldsymbol{\chi} - \mathbf{b} + \lambda W_F{}^2 \boldsymbol{\chi}\right) = 0$$

solve

$$\boldsymbol{\chi} = \left(\Phi W_R{}^2 \Phi^T + \lambda W_F{}^2\right)^{-1} \mathbf{b}$$

Replace for $\boldsymbol{\nu}$ and $\mathbf{u}$

III. Results

Granai and Vandergheynst [6] have previously discussed the advantages of BPDN with an ℓ^1 data fidelity term. Here were provide additional evidence based on examples computed using the proposed algorithm. A simple cubic phase cosine image is displayed in Figure 1, and the same image after addition of 5% speckle noise is displayed in Figure 2. This simple example is intended to illustrate the advantages of BPDN when an appropriate dictionary is available, in this case the DCT. Figures 3(a), 3(b), and 3(c) display denoising results for standard ℓ^2 BPDN, ℓ^1-TV, and ℓ^1 BPDN via the proposed algorithm. Note that even though the ℓ^1-TV result has a slightly higher SNR than BPDN with ℓ^1 data fidelity term, the former has a superior visual quality.

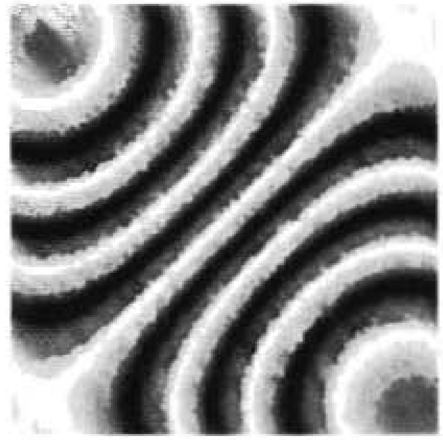

Fig. 1. 128×128 Cubic phase image: $I(x, y) = \cos(x^3 - y^3 - 9x - 9y)$

Fig. 2. Cubic image with 5% speckle noise. SNR: 9.91dB.

IV. Conclusions

The proposed algorithm provides a flexible and computationally efficient means of solving the generalized BPDN problem, including the ℓ^1 BPDN problem.

Note

This is an early draft version of this paper, submitted for an LA-UR number at this stage due to an approaching submission deadline.

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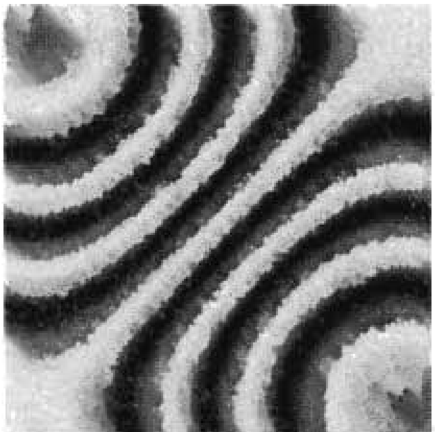

(a) Denoised image via (standard) BPDN using an overcomplete DCT dictionary. SNR: 16.02dB.

(b) Denoised image via ℓ^1-TV. SNR: 25.77

(c) Denoised image via BPDN with ℓ^1 data fidelity term using an overcomplete DCT dictionary. SNR: 25.62

Fig. 3. Comparison of 3 denoising strategies: 3(a) (standard) BPDN, 3(b) ℓ^1-TV and 3(c) BPDN with ℓ^1 data fidelity term. Note that even though the ℓ^1-TV result has a slightly higher SNR than BPDN with ℓ^1 data fidelity term, the former has a superior visual quality.

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