DEPARTMENT OF ENERGY RESEARCH PROGRAM SUMMARY PROGRESS REPORT

Multidimensional Conservation Laws and Low Regularity Solutions

PI Name: Barbara Lee Keyfitz
PI Institution: University of Houston
PI Address: Houston, Texas 77204-3008

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This is the concluding report for the project, a continuation of research by Keyfitz and co-workers on multidimensional conservation laws, and applications of nonhyperbolic conservation laws in the two-fluid model for multiphase flow. The multidimensional research project was started with Sunčica Čanić, at the University of Houston and with Eun Heui Kim, now at California State University Long Beach. Two postdoctoral researchers, Katarina Jegdic and Allen Tesdall, also worked on this research. Jegdic's research was supported (for a total of one year) by this grant. Work on nonhyperbolic models for two-phase flows is being pursued jointly with Michael Sever, Hebrew University.

Background for the project is contained in earlier reports. Note that in 2006, the project received a one-year no-cost extension that will end in September, 2007. A new proposal, for continuation of the research and for new projects, will be submitted in the Fall of 2007, with funding requested to begin in the summer of 2008. The reason for the "funding gap" is Keyfitz's four-year stint as Director of the Fields Institute in Toronto, Canada. The research has continued, but has been supported by Canadian grant funds, as seems appropriate during this period.

1 Report on Research

1.1 Two-Dimensional Riemann Problems

It would be fair to say that the landscape of two-dimensional self-similar solutions has changed dramatically during the period of this grant. This is due in part to the research of the PI and the team of co-workers, and also in large measure to the fact that a large number of other researchers, working in several groups (Gui-Qiang Chen and Mikhail Feldman at Northwestern and Wisconsin; Yu-Xi Zheng and colleagues at Penn State; Marshall Slemrod and Seung-Yeal Ha at Wisconsin; Tai-Ping Liu and Volker Elling at Stanford; and others) have taken up the self-similar approach and have extended it in many directions. The specific features of our method that have lent themselves to extension are the centrality of a second-order quasi-linear partial differential equation that changes type (hyperbolic/elliptic) at a sonic line; the appearance of a reflected shock as a free boundary connecting a known constant flow to an undetermined elliptic (subsonic) flow, with the possible appearance of a sonic degeneracy in the elliptic equation; and the use of compactness methods, mainly the Schauder fixed-point theorem, to produce a convergent subsequence which is shown to lead to a solution. A list of our recent papers on this problem is given below.

Collectively, these results have created, for the first time, an understanding of the analytical properties of at least one regime in shock reflection problems, that of regular reflection.

There are significant differences between our approach and those of other groups: by writing the problem in potential form, Chen and Feldman have been able to simplify the boundary conditions at the free boundary, and have been able to produce global solutions. (Our method has required us to introduce a cut-off function, and thus the validity of our solution is confined to a finite neighbourhood of the reflection point.) However, we claim that, since the actual gas-dynamic flows are not of potential type (the curved shock necessarily introduces some vorticity), it will prove useful in the long run to work with some non-potential form of the equations.

At the moment, we are still trying to extend our approach to a case involving the equations of gas dynamics, where there is, again, a second-order equation that changes type. However, it is coupled to a set of evolution equations for the velocity variables. Obtaining the right framework for compactness and convergence has been much more difficult than we anticipated, but we think we are close to a solution and a draft is in preparation [13].

In a second direction, Allen Tesdall, working as a postdoc with our group, has been able to extend his earlier work with John Hunter. This work exhibited numerically a new phenomenon, named by the discoverers 'Guderley Mach reflection', consisting of a sequence of as many as six separate supersonic patches in the wake of a Mach-type shock reflection in a regime where neither regular nor standard Mach reflection is possible. Despite the reference to earlier work of Guderley that had predicted a single supersonic bubble, formed by a rarefaction reflecting off the sonic line as a shock, no one had expected to see the complex phenomenon of a series, possibly infinite, of such patches. Following the numerical solutions, the phenomenon has now been confirmed experimentally by Skews and Ashworth. The original numerical simulation found this solution in the model unsteady transonic small disturbance equations; since then, Tesdall, working with our group, has shown it in the so-called nonlinear wave system, an artificial system with characteristic structure similar to the isentropic compressible gas dynamics equations, and, most recently, in the adiabatic gas dynamics equations themselves.

These dramatic numerical results cry out both for numerical confirmation by a different method (currently being pursued by a Canadian postdoc of the PI, Mary Chern), and for some analytical backing. This last challenge appears to be related to a problem that we had examined some time ago but had not completed [16]: study of the free boundary created at a sonic line where the solution is continuous, but where the position is influenced by the flow in both the supersonic and the subsonic regions. This is a type of two-phase Stefan problem, as distinct from the one-phase problem that defines shock reflection. In both cases, the fact that the equations are quasilinear rather than semilinear means that standard methods do not apply. We are now taking another look at this problem.

1.2 Nonhyperbolic Conservation Laws and Low Regularity Solutions

The proposal in 2002 set out a path of investigation of shocks we had discovered in a simple model for incompressible two-phase flow. The results have appeared in two papers, [1, 2]. A number of other questions, of some mathematical interest, were listed in the proposal. However, interest in non-hyperbolic models from the engineering community has waned, as scientists have proposed more complex models for the interaction of the dispersed and continuous phases. At this point, we feel that our work simply gives an explanation for why numerical simulations based on two-fluid models do not give divergent results (as linear theory would predict). Our results cannot help people to devise better models, and thus pursuing a more detailed explanation of the role of balance and viscous terms would have limited impact.

Instead, research on singular shocks (that is, low regularity solutions containing measures) has moved in a different direction. Work with Michael Sever has advanced the mathematical theory of problems with singular shocks, and we are completing an existence theorem for the Cauchy problem for a hyperbolic model equation with singular shocks [15]. This project has turned out to be much more complex than anticipated. In devising a front-tracking algorithm, we have found that a number of new kinds of waves can potentially appear, including local blow-up of the total variation in two different ways. Both are associated with 'critical shocks': the largest shock waves that are not singular. Small and moderate-sized waves interacting with critical and nearly-critical waves can lead to rapid amplification of the total variation. This is not surprising, but requires delicate estimates on the rate of blow-up. By dealing suitably with these possible singularities in the front-tracking approximation, we find that a weak solution can be constructed for all time.

Although our results are only for a single model system, the problem is interesting because it is possible that any large data problem in which the wave speeds are not strictly separated will exhibit such phenomena. Thus, the obstructions we find may well be difficulties typical of large data problems in general systems.

2 Honours, Outreach and Synergistic Activities

2.1 Career Update

As mentioned, I will spend a total of four years as Director of the Fields Institute, and then plan to return to a faculty position at the University of Houston in the summer of 2008. During the first two years of my visit to Toronto, two postdoctoral visitors, Jegdic and Tesdall, divided their time between Houston and Toronto. They both currently have faculty positions in Texas, and our joint research is continuing, as is my collaboration with Čanić in Houston. Sever has visited Toronto several times to carry out the research on singular shocks.

2.2 Awards

I received the 2005 Krieger-Nelson Prize of the Canadian Mathematical Society. I was also honoured to receive the 2006 Farfel Award of the University of Houston. This is the highest award given to faculty at the University of Houston, and is based on research, teaching and service. Besides the Krieger-Nelson lecture, I have also received a number of prestigious invitations to lecture on the research funded by this grant. These include

> Invited lecture, PASI Americas VI meeting on Non Linear Analysis and Differential Equations, Santiago, Chile, Jan 17-21, 2005; *Self-Similar Solutions of Two-Dimensional*

Conservation Laws.

Invited talk, Frontiers of Applied Analysis - A Conference in the Occasion of the 15th Anniversary of the CNA, Carnegie-Mellon University, September 8-10, 2005 *Can self-similar problems tell us anything about multi-dimensional conservation laws?*

Invited talk, Conference on Differential Equations: From Theory to Computational Science and Engineering, in honor of Rolf Jeltsch's 60th birthday, ETH-Zurich, October 20-22, 2005; *Multidimensional Conservation Laws: Can Analysis and Numerical Methods be Friends?*

Invited talk, First East Asia SIAM Symposium and Second International conference on Scientific Computing and Partial Differential Equations, Hong Kong, December 12-16, 2005; *Why Are Multidimensional Conservation Laws So Difficult?*.

Invited speaker, Conference on Advances in PDE in honor of the eightieth birthdays of Peter Lax and Louis Nirenberg, Toledo, Spain, June 7-10, 2006; *Some interesting questions in multidimensional conservation laws.*

Joint Plenary speaker, Analysis of PDE Conference and Topical speaker, SIAM Annual Meeting, Boston, July 9-12, 2006; *Multidimensional Conservation Laws.*

Invited speaker, International Congress on Industrial and Applied Mathematics, Zurich, 16-20 July, 2007; *Hyperbolic Conservation Laws: Past and Future.*

Invited speaker, conference on 'Perspectives in Numerical Analysis, TKK', in honour of Olavi Nevanlinna and Juhani Pitkäranta, Helsinki University of Technology, May 27-29, 2008.

2.3 Outreach and Synergistic Activities

Continuing outreach and networking activities include

- A second term as Treasurer of ICIAM. This will begin in Fall 2007, and will continue for four years; however, I have already indicated my interest in becoming president-elect of ICIAM in 2009.
- I served as founding Chair of the SIAM Activity Group in the Analysis of Partial Differential Equations, which held its first meeting in December, 2004, in Houston. The activity group will have its third meeting in December, 2007.
- As Chair of Section A of the American Association for the Advancement of Science, I coordinated an ambitious program for the 2006 meeting, and supervised the transition to a new secretary. The section has also begun a "media" program that is generating non-technical articles about mathematical events at the meeting.
- I also served as President of the Association for Women in Mathematics for two years (2005 and 2006) and am now past-president.

- As Director of Fields, I am responsible for a number of outreach activities, including coordinating IMSI, the informal organization of International Mathematical Sciences Institutes.

3 Publications Citing the Grant

1. B. L. Keyfitz, R. Sanders and M. Sever, 'Lack of Hyperbolicity in the Two-Fluid Model for Two-Phase Incompressible Flow', Discrete and Continuous Dynamical Systems - B, **3** (2003), 541-563.

2. B. L. Keyfitz, M. Sever and F. Zhang, 'Viscous Singular Shock Structure for a Nonhyperbolic Two-Fluid Model', *Nonlinearity*, **17** (2004), 1731-1747.

3. B. L. Keyfitz, 'Self-Similar Solutions of Two-Dimensional Conservation Laws', *Journal of Hyperbolic Differential Equations*, **1** (2004), 445-492.

4. S. Čanić, B. L. Keyfitz and E. H. Kim, 'Free Boundary Problems for Nonlinear Wave Equations: Mach Stems for Interacting Shocks', *SIAM Journal on Mathematical Analysis*, **37** (2005), 1947-1977.

5. K. Jegdić, B. L. Keyfitz and S. Čanić, 'Transonic regular reflection for the nonlinear wave system', *Journal of Hyperbolic Differential Equations*,**3** (2006) 443-474.

6. A. Tesdall, R. Sanders and B. L. Keyfitz, 'The Triple Point Paradox for the Nonlinear Wave System', 14 pages, *SIAM Journal on Applied Mathematics*, **67** (2006), 321-336.

7. B. L. Keyfitz, 'The Legacy of Olga Oleĭnik in Hyperbolic Conservation Laws', extended abstract, 2006.

8. K. Jegdic, 'Remarks on a free boundary value problem for the nonlinear wave system: weak regular reflection', *Proceedings of The 5th International Conference on Differential Equations and Dynamical Systems, University of Texas - Pan American*, 2006, to appear.

9. K. Jegdic, 'Weak regular reflection for the nonlinear wave system', *Journal of Hyperbolic Differential Equations*, to appear.

10. K. Jegdic, 'Numerical solutions to a two-dimensional Riemann problem for gas dynamics equations', *Proceedings of the 11th WSEAS International Conference on Applied Mathematics*, 2007 to appear.

11. K. Jegdić, B. L. Keyfitz and S. Čanić, 'Transonic regular reflection for the Unsteady Transonic Small Disturbance Equation - details of the subsonic solution', *Free and Moving Boundaries: Analysis, Simulation and Control*, (Roland Glowinski and Jean Paul Zolesio, editors), CRC Press, Boca Raton, 2006, to appear.

12. B. L. Keyfitz and N. Popivanov, 'Nonlocal Regularization for the 3-D Morawetz-Protter Problem', draft, 2007.

13. K. Jegdic, B. L. Keyfitz and S. Čanić, ‘A Free Boundary Problem for the Isentropic Gas Dynamics Equations - Transonic Regular Reflection’, in preparation.

14. A. Tesdall, R. Sanders and B. L. Keyfitz, ‘Self-similar Solutions for The Triple Point Paradox in Gas Dynamics’, in preparation.

15. M. Sever and B. L. Keyfitz, ‘An Existence Theorem for the Cauchy Problem for a Model System Exhibiting Singular Shocks’, in preparation.

16. S. Čanić, B. L. Keyfitz, and E. H. Kim, ‘A Global Riemann Solution with Noninteracting Rarefaction Waves for the Two-Dimensional Nonlinear Wave System’, in preparation.