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Dynamics in a Spiral FFAG with Tilted Cavities

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I develop a formulation for Hamiltonian dynamics in an accelerator with magnets whose edges follow a spiral. I demonstrate using this Hamiltonian that a spiral FFAG can be made perfectly "scaling." I describe how one computes the RF phase during a rapid acceleration cycle to keep the beam at the appropriate RF phase. I examine the effect of tilting an RF cavity with respect a radial line from the center of the machine, potentially with a different angle than the spiral of the magnets. I discuss partially the effects of the finite energy jumps on the dynamics. This is a status report of work that is still incomplete.

I. INTRODUCTION

A synchrotron is generally designed so that the closed orbit, independent of energy, passes through the center of the cavity, parallel to the cavity axis and therefore the accelerating fields. In an FFAG, particularly a scaling FFAG, it is generally not possible to achieve this. First of all, the closed orbit positions generally depend on energy. Secondly, the closed orbits are not circular, and therefore generally make an angle with respect to radial lines from the center of the machine. In a radial-sector FFAG, one can construct a triplet lattice, for example, which has a symmetry point where a cavity can be placed so that the electric fields lines are almost exactly in the same direction as the closed orbits.

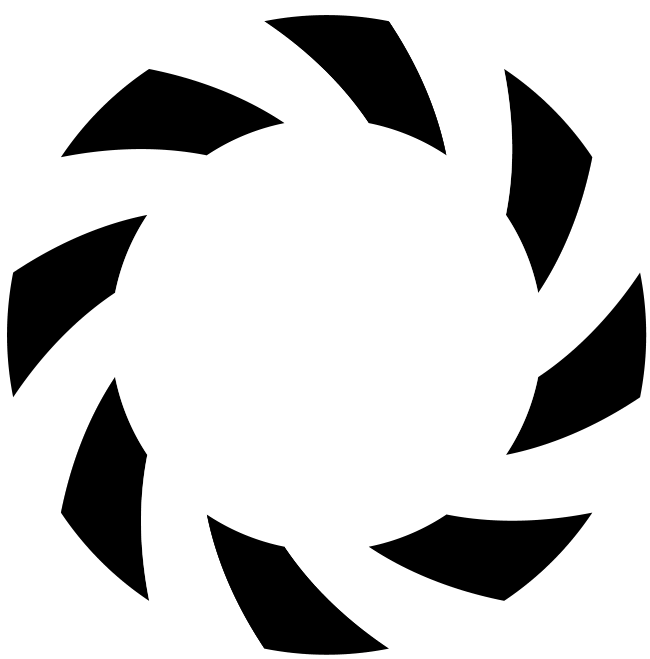

FIG. 1: A diagram of a spiral FFAG accelerator. Black regions are magnets.

One may construct a spiral-sector FFAG to attempt to make a more compact lattice than one could with a radial-sector FFAG. Such lattices generally have only a single magnet per cell, and make use of a spiral magnet edge to achieve additional vertical focusing. An example is shown in Fig. 1.

Ideally one would like to minimize the length of the drifts, and one way of doing this would be to tilt the cavity roughly parallel to the spiral. The fundamental concern with doing this is that now the electric field in the cavity is transverse to the energy-dependent closed orbits, and will thus affect the transverse motion of the beam. One would like to understand the magnitude of these effects.

In the first section, I will outline the basic theory of an accelerator in spiral coordinates, focusing particularly on the case of a logarithmic spiral. I will write down a Hamiltonian, a magnetic field expansion, and an RF cavity field expansion in these coordinates. In the following section, I will demonstrate that this Hamiltonian, when used with appropriately defined scaling fields, obeys the usual scaling laws. In the next section, I will discuss how to choose the RF cavity frequency (or more precisely, the phase) to keep particles synchronized to the RF. In the next section, I will discuss the effect of having a cavity tilted in the machine. Finally, I will examine the effect of closed orbit jumps due to the energy gain in the cavities on the beam motion.

II. HAMILTONIAN IN SPIRAL COORDINATES

One first transforms into spiral coordinates

$$R = r \qquad Y = y \qquad \Theta = \theta - \int_{r_0}^{r} \frac{\tan\zeta(\bar{r})}{\bar{r}} d\bar{r}, \qquad (1)$$

where $\zeta(r)$ is the angle that the spiral faces make with respect to a radial line from the center of the machine, as a function of the radius r. For positive ζ, the magnet edges move in the direction of particle motion as the radius increases, assuming that the magnet edges are along lines of constant Θ. Note that Θ will eventually be the

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independent variable for the new system, but temporarily we will use time as the independent variable for the purpose of performing these transformations. This change of variables induces a change in the conjugate momenta to

$$p_R = p_r + \frac{p_\theta}{r}\tan\zeta \qquad p_Y = p_y \qquad p_\Theta = p_\theta \tag{2}$$

Note that p_R is different from p_r, despite the fact that $R = r$. Furthermore, note that the Hamiltonian with θ as the independent variable is $-p_\theta$, and the Hamiltonian with Θ as the independent variable is $-p_\Theta$.

It is important to understand the change of independent variable: it means that the question that one is asking of the equations of motion is changing. With θ as the independent variable, one is asking about the radius, vertical position, time, and their conjugate momenta *with respect to* θ at a given value of θ. With Θ as the independent variable, one is asking about the radius, vertical position, time, and their conjugate momenta *with respect to* Θ at a given value of Θ. Notice two things have changed in the question: where you are looking, and the nature of the conjugate momenta. Understanding this fact is essential to understanding why the spiral machine behaves differently than the radial sector machine.

The Hamiltonian in these coordinates, with Θ as the independent variable, is

$$\begin{aligned} -R\cos\zeta\Biggl\{&p_R\sin\zeta + qA_\Theta \\ &+\sqrt{\begin{aligned}&(E-q\Phi)^2/c^2-(p_R\cos\zeta-qA_R)^2\\&-(p_Y-qA_y)^2-(mc)^2\end{aligned}}\Biggr\}, \end{aligned} \tag{3}$$

where

$$A_\Theta = A_\theta\cos\zeta - A_r\sin\zeta \tag{4}$$
$$A_R = A_r\cos\zeta + A_\theta\sin\zeta. \tag{5}$$

These are the components of the vector potential perpendicular and parallel to the spirals, respectively.

A. Vector Potentials for Magnets

It is most convenient at this point to assume that ζ is constant, which is required for meeting the scaling condition in an FFAG. Writing the vector potentials in a power series about the midplane

$$A_R(R,Y,\Theta) = \sum_{n=0}^{\infty}\frac{1}{n!}A_{Rn}(R,\Theta)Y^n \tag{6}$$

$$A_y(R,Y,\Theta) = \sum_{n=0}^{\infty}\frac{1}{n!}A_{yn}(R,\Theta)Y^n \tag{7}$$

$$A_\Theta(R,Y,\Theta) = \sum_{n=0}^{\infty}\frac{1}{n!}A_{\Theta n}(R,\Theta)Y^n, \tag{8}$$

one can obtain a recursion relation for the coefficients from Maxwell's equations using the gauge $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$:

$$\begin{aligned} A_{R,n+2} = {}&-\frac{\partial}{\partial R}\left[\frac{1}{R}\frac{\partial}{\partial R}(RA_{Rn})\right] + \frac{2\tan\zeta}{R}\frac{\partial^2 A_{Rn}}{\partial\Theta\partial R} \\ &-\frac{\sec^2\zeta}{R^2}\frac{\partial^2 A_{Rn}}{\partial\Theta^2} + \frac{2}{R^2}\frac{\partial A_{\Theta n}}{\partial\Theta} \end{aligned} \tag{9}$$

$$\begin{aligned} A_{\Theta,n+2} = {}&-\frac{\partial}{\partial R}\left[\frac{1}{R}\frac{\partial}{\partial R}(RA_{\Theta n})\right] + \frac{2\tan\zeta}{R}\frac{\partial^2 A_{\Theta n}}{\partial\Theta\partial R} \\ &-\frac{\sec^2\zeta}{R^2}\frac{\partial^2 A_{\Theta n}}{\partial\Theta^2} - \frac{2}{R^2}\frac{\partial A_{Rn}}{\partial\Theta} \end{aligned} \tag{10}$$

$$\begin{aligned} A_{y,n+1} = {}&-\frac{\cos\zeta}{R}\frac{\partial}{\partial R}(RA_{Rn}) + \frac{\sin\zeta}{R}\frac{\partial}{\partial R}(RA_{\Theta n}) \\ &-\frac{\sec\zeta}{R}\frac{\partial A_{\Theta n}}{\partial\Theta} \end{aligned} \tag{11}$$

Starting with the gauge choice $A_{R0}(R,\Theta) = 0$, one also has an equation for $A_{\Theta 0}$ from $B_y(R,0,\Theta)$:

$$B_y(R,0,\Theta) = -\frac{\cos\zeta}{R}\frac{\partial}{\partial R}(RA_{\Theta 0}). \tag{12}$$

If we have scaling fields, where

$$B_y(R,0,\Theta) = B_{y0}(\Theta)(R/r_0)^k, \tag{13}$$

then

$$A_{Rn}(R,\Theta) = \hat{A}_{Rn}(\Theta)(R/r_0)^{k+1-n} \tag{14}$$
$$A_{yn}(R,\Theta) = \hat{A}_{yn}(\Theta)(R/r_0)^{k+1-n} \tag{15}$$
$$A_{\Theta n}(R,\Theta) = \hat{A}_{\Theta n}(\Theta)(R/r_0)^{k+1-n}, \tag{16}$$

and the recursion relations become

$$\begin{aligned} \hat{A}_{R,n+2} = {}&-(k+2-n)(k-n)\hat{A}_{Rn} \\ &+2(k+1-n)\tan\zeta\frac{\partial\hat{A}_{Rn}}{\partial\Theta} \\ &-\sec^2\zeta\frac{\partial^2\hat{A}_{Rn}}{\partial\Theta^2} + 2\frac{\partial\hat{A}_{\Theta n}}{\partial\Theta} \end{aligned} \tag{17}$$

$$\begin{aligned} \hat{A}_{\Theta,n+2} = {}&-(k+2-n)(k-n)\hat{A}_{\Theta n} \\ &+2(k+1-n)\tan\zeta\frac{\partial\hat{A}_{\Theta n}}{\partial\Theta} \\ &-\sec^2\zeta\frac{\partial^2 A_{\Theta n}}{\partial\Theta^2} - 2\frac{\partial A_{Rn}}{\partial\Theta} \end{aligned} \tag{18}$$

$$\begin{aligned} \hat{A}_{y,n+1} = {}&-(k+2-n)\cos\zeta\hat{A}_{Rn} \\ &+(k+2-n)\sin\zeta\hat{A}_{\Theta n} - \sec\zeta\frac{\partial A_{\Theta n}}{\partial\Theta} \end{aligned} \tag{19}$$

$$B_{y0}(\Theta) = -(k+2)\cos\zeta\hat{A}_{\Theta 0}. \tag{20}$$

B. Vector Potentials for Cavities

For cavities, it is best to take the gauge with zero electric scalar potential, in which case the recursion relations for the power series in the midplane become

$$\begin{aligned} A_{R,n+2} = {} & -\frac{\partial}{\partial R}\left[\frac{1}{R}\frac{\partial}{\partial R}(RA_{Rn})\right] + \frac{2\tan\zeta}{R}\frac{\partial^2 A_{Rn}}{\partial\Theta\partial R} \\ & - \frac{\sec^2\zeta}{R^2}\frac{\partial^2 A_{Rn}}{\partial\Theta^2} + \frac{2}{R^2}\frac{\partial A_{\Theta n}}{\partial\Theta} + \frac{\omega^2}{c^2}A_{Rn} \end{aligned} \quad (21)$$

$$\begin{aligned} A_{\Theta,n+2} = {} & -\frac{\partial}{\partial R}\left[\frac{1}{R}\frac{\partial}{\partial R}(RA_{\Theta n})\right] + \frac{2\tan\zeta}{R}\frac{\partial^2 A_{\Theta n}}{\partial\Theta\partial R} \\ & - \frac{\sec^2\zeta}{R^2}\frac{\partial^2 A_{\Theta n}}{\partial\Theta^2} - \frac{2}{R^2}\frac{\partial A_{Rn}}{\partial\Theta} + \frac{\omega^2}{c^2}A_{\Theta n} \end{aligned} \quad (22)$$

$$\begin{aligned} A_{y,n+1} = {} & -\frac{\cos\zeta}{R}\frac{\partial}{\partial R}(RA_{Rn}) + \frac{\sin\zeta}{R}\frac{\partial}{\partial R}(RA_{\Theta n}) \\ & - \frac{\sec\zeta}{R}\frac{\partial A_{\Theta n}}{\partial\Theta}. \end{aligned} \quad (23)$$

To start the recursion sequence, we assume there are electric fields $E_{R0}(R,\Theta)\cos(\omega t+\phi)$ and $E_{\Theta 0}(R,\Theta)\cos(\omega t+\phi)$ in the midplane (parallel and perpendicular to the logarithmic spirals respectively), and thus

$$A_{\Theta 0} = -\frac{E_{\Theta 0}(R,\Theta)}{\omega}\sin(\omega t+\phi) \quad (24)$$

$$A_{R0} = -\frac{E_{R0}(R,\Theta)}{\omega}\sin(\omega t+\phi). \quad (25)$$

III. SCALING LAWS FOR SCALING FFAGS

Now we can see precisely what “scaling” means for a scaling FFAG. Assume that the vector potentials are described by Eqs. (6)–(8), with the coefficients in those equations given by Eqs. (14)–(16). The spiral angle ζ is assumed to be constant. The vector potentials are taken to be independent of time. Then for any of those vector potentials,

$$A(\lambda R, \lambda Y, \Theta) = \lambda^{k+1}A(R,Y,\Theta). \quad (26)$$

Change variables from E to Δ, where $E = E_0 + \Delta$, and assume that the scalar potential Φ is zero. The Hamiltonian is then

$$\begin{aligned} & -R\cos\zeta\Biggl\{p_R\sin\zeta + qA_\Theta \\ & + \sqrt{\begin{aligned} & p_0^2 + 2E_0\Delta/c^2 + \Delta^2/c^2 \\ & -(p_R\cos\zeta - qA_R)^2 - (p_Y - qA_y)^2 \end{aligned}}\Biggr\}, \end{aligned} \quad (27)$$

where $p_0^2 = E_0^2/c^2 - (mc)^2$.

Consider the following transformation

$$\begin{aligned} \hat{R} &= R(\hat{p}_0/p_0)^{1/(k+1)} \quad \hat{p}_R = p_R\hat{p}_0/p_0 \\ \hat{Y} &= Y(\hat{p}_0/p_0)^{1/(k+1)} \quad \hat{p}_Y = p_Y\hat{p}_0/p_0 \\ \hat{\Delta} &= \sqrt{\hat{E}_0^2 + 2cE_0(\hat{p}_0/p_0)^2\Delta + (\hat{p}_0/p_0)^2\Delta^2} - \hat{E}_0 \\ \hat{t} &= t\frac{p_0}{\hat{p}_0}\frac{\hat{E}_0+\hat{\Delta}}{E_0+\Delta}\left(\frac{\hat{p}_0}{p_0}\right)^{1/(k+1)}, \end{aligned} \quad (28)$$

where $\hat{p}_0^2 = \hat{E}_0^2/c^2 - (mc)^2$. The Hamiltonian which governs the new variables is

$$\begin{aligned} & -\hat{R}\cos\zeta\Biggl\{\hat{p}_R\sin\zeta + qA_\Theta \\ & + \sqrt{\begin{aligned} & \hat{p}_0^2 + 2\hat{E}_0\hat{\Delta}/c^2 + \hat{\Delta}^2/c^2 \\ & -(\hat{p}_R\cos\zeta - qA_R)^2 - (\hat{p}_Y - qA_y)^2 \end{aligned}}\Biggr\}, \end{aligned} \quad (29)$$

where now A_R, A_y, and A_Θ are all evaluated at $\hat{R}$ and $\hat{y}$ instead of R and y.

The Hamiltonians for the two sets of variables are clearly identical, with the exception that p_0 is replaced by $\hat{p}_0$. The interpretation of this is that if you know the phase space dynamics near a total momentum p_0, you can find the phase space dynamics near any other total momentum $\hat{p}_0$ by applying the transformations (28). Several conclusions can be drawn from this:

- Transverse tunes are independent of reference momentum.
- Closed orbits for different momenta are geometrically similar, and their size is proportional to $p_0^{1/(k+1)}$.
- Normalized dynamic aperture in each plane is proportional to $p_0^{(k+2)/(k+1)}$. The shape of the dynamic aperture is independent of momentum, except that the transverse coordinate direction is proportional to to $p_0^{1/(k+1)}$, and the transverse momentum direction is proportional to p_0.
- The Courant-Snyder beta functions in normalized coordinates (with units of m/(eV·s)) are proportional to $p_0^{-k/(k+1)}$, and thus the usual beta functions (units of m) are proportional to $p_0^{1/(k+1)}$. The Courant-Snyder alpha function is independent of p_0.
- The momentum compaction is $1/(k+1)$, independent of energy.

IV. FREQUENCY PROGRAM FOR AN FFAG

Since the time for one turn along a fixed-energy closed orbit of an FFAG depends on the energy for that closed orbit, the frequency of the RF should be varied during the acceleration cycle. The simplest method of doing this is to adjust the frequency of the RF to correspond to the energy that one expects at a given time, based on the desired energy gain per turn. If for whatever reason, the current beam energy and the current RF frequency do not correspond, the phase of the beam will adjust itself to change the beam energy to achieve this correspondence. This will work well as long as the acceleration rate is sufficiently slow and the design phase is far enough from the RF crest.

First of all, I will define some variables and functions. The variable t is the time on the clock; it continually advances as we accelerate. $E(\Theta)$ is the reference energy, and is a function of the independent variable Θ described above, and should continually advance as we accelerate. The reason for its dependence on Θ is that generally one intends to accelerate a given amount on each turn. I will assume that the energy continuously increases, rather than in discrete steps. The time for one turn is $t_1(E)$, and is a function of the energy. The RF frequency should ideally be set to $1/t_1(E)$.

When the RF frequency is not constant, it is most convenient to speak of RF phase. The RF voltage $V(t,\Theta)$ and phase $\phi(t,\Theta)$ are defined so that the voltage as a function of time is

$$V(t,\Theta)\cos\bigl(\phi(t,\Theta)\bigr) = 2\pi\frac{dE}{d\Theta}. \tag{30}$$

V and $\cos\bigl(\phi(t,\Theta)\bigr)$ are periodic in Θ. Presumably there is some desired $E(\Theta)$, so this will be assumed to be known. The frequency is the time derivative of ϕ, divided by 2π.

Consider Θ to be a function of t. We know that

$$\frac{d\Theta}{dt} = \frac{2\pi}{t_1\bigl(E(\Theta)\bigr)}. \tag{31}$$

This equation can be integrated directly by writing it as a relationship between the differentials $d\Theta$ and dt:

$$\frac{t_1\bigl(E(\Theta)\bigr)}{2\pi}d\Theta = dt. \tag{32}$$

Integrating both sides,

$$t_r(\Theta) = \frac{1}{2\pi}\int_{\Theta_0}^{\Theta} t_1\bigl(E(\bar\Theta)\bigr)d\bar\Theta. \tag{33}$$

This function can be inverted to give $\Theta_r(t)$.

To meet the periodicity conditions on $\phi(t,\Theta)$, that function must take the form

$$\phi(t,\Theta) = \phi_s(t,\Theta) + h[\Theta_r(t) - \Theta], \tag{34}$$

where ϕ_s is a periodic function of Θ.

Let's take the case of a scaling FFAG with ϕ_s and V constant, with $V\cos\phi_s = \Delta E$. For the scaling FFAG,

$$t_1(E) = t_1(E_0)\frac{E}{E_0}\left(\frac{E^2-(mc^2)^2}{E_0^2-(mc^2)^2}\right)^{-\frac{k}{2k+2}} \tag{35}$$

We then find

$$t_r(\Theta) = t_1(E_0)\frac{k+1}{k+2}\frac{E_0^2-(mc^2)^2}{E_0\Delta E} \left[\left(\frac{E^2(\Theta)-(mc^2)^2}{E_0^2-(mc^2)^2}\right)^{\frac{k+2}{2k+2}} - 1\right], \tag{36}$$

with

$$E(\Theta) = E_0 + \frac{\Theta\Delta E}{2\pi}. \tag{37}$$

Inverting, we find

$$\Theta_r(t) = 2\pi\frac{E_r(t)-E_0}{\Delta E} \tag{38}$$

$$E_r(t) = \sqrt{p_r^2(t)c^2 + (mc^2)^2} \tag{39}$$

$$p_r(t) = p_0\left[1 + \frac{t}{t_1(E_0)}\frac{k+2}{k+1}\frac{E_0\Delta E}{p_0^2c^2}\right]^{\frac{k+1}{k+2}} \tag{40}$$

$$p_0 = \sqrt{(E_0/c)^2 - (mc^2)^2}. \tag{41}$$

V. ANALYSIS OF CAVITY PLACEMENT

Start with a cavity which makes an angle of ζ_C with respect to radial lines. To be able to produce some analytic results, I assume ζ_C to be constant. The cavity thus has a logarithmic spiral shape, unless $\zeta_C = 0$. The center of the cavity is given by

$$\theta = \theta_C + \tan\zeta_C \ln(r/r_C). \tag{42}$$

Assume that the midplane electric fields in the cavity are perpendicular to lines that make an angle ζ_C with respect to radial lines. We will define the cavity fields in coordinates which are along the logarithmic spirals making angle ζ_C with respect to radial lines and which are along curves which are perpendicular to those spirals. We define these coordinates to be r_1 and θ_1 as follows:

$$\theta_1 = \theta - \theta_C - \tan\zeta_C\ln(r/r_C) \tag{43}$$

$$r_1 = r^{\cos^2\zeta_C} r_C^{\sin^2\zeta_C} \exp[(\theta-\theta_C)\cos\zeta_C\,\sin\zeta_C] \tag{44}$$

Now, assume that the magnitude of the electric field in the midplane is of the form

$$c(r_1)d(\theta_1) \tag{45}$$

Integrating in θ_1 to find the on-crest energy gain in the cavity (ignoring the time dependence of the electric field), one finds the voltage to be

$$r_1 c(r_1)\int d(\theta_1)\exp(-\theta_1\sin\zeta_C\,\cos\zeta_C)\,d\theta_1. \tag{46}$$

If one wishes the energy gain to be independent of the line along which you integrated, then $c(r_1) \propto r_1^{-1}$.

Thus, in terms of r and θ, one can write the electric field component in the θ_1 direction to be

$$(r/r_C)^{-\cos^2\zeta_C}\exp[-(\theta-\theta_C)\sin\zeta_C\,\cos\zeta_C]\\ d\bigl(\theta-\theta_C-\tan\zeta_C\ln(r/r_C)\bigr). \quad (47)$$

Redefining d to eliminate the exponential, one can rewrite this as

$$\frac{r_C}{r}E_0\bigl(\theta-\theta_C-\tan\zeta_C\ln(r/r_C)\bigr) \quad (48)$$

On performing the integral (46), we find the voltage to be

$$r_C\cos\zeta_C\int E_0(\theta)d\theta \quad (49)$$

In terms of the spiral coordinates for the Hamiltonian and the electric field components used earlier (and taking $r_C=r_0$ and ζ constant),

$$E_\Theta(R,\Theta)=\frac{r_0}{R}\cos(\zeta-\zeta_C)\\ E_0\bigl(\Theta-\theta_C+(\tan\zeta-\tan\zeta_C)\ln(R/r_0)\bigr) \quad (50)$$

$$E_R(R,\Theta)=\frac{r_0}{R}\sin(\zeta-\zeta_C)\\ E_0\bigl(\Theta-\theta_C+(\tan\zeta-\tan\zeta_C)\ln(R/r_0)\bigr) \quad (51)$$

Note that if $\zeta=\zeta_C$, then $E_R=0$ (thus $A_R=0$ as well), and the only R dependence remaining in E_Θ (and therefore A_Θ) is an inverse dependence in R. Thus, such a cavity following the spiral coordinates will not modify the fixed-energy dynamics.

VI. CLOSED ORBIT SHIFT IN ACCELERATION

Since there is nonzero dispersion at the cavities, when a particle is accelerated, the orbit about which it is undergoing betatron oscillations changes. This in principle can lead to emittance growth.

Define the closed orbit as a function of angular position and energy to be $\boldsymbol{z}_0(\theta;E)$. Define the linear map about that closed orbit from θ_0 to θ_1 to be $M(\theta_1,\theta_0,E)$. Define the transformation that normalizes M at the point θ to be $A(\theta,E)$; A transforms normalized coordinates into unnormalized coordinates. The map in the normalized coordinates is then $R(\theta_1,\theta_0,E)$.

To determine if there is emittance growth, we want to examine the evolution of the beam in normalized coordinates. Say that on each turn, at a point θ, the energy jumps from E_n to E_{n+1}. Define $\boldsymbol{z}_n$ to be the phase space coordinates at that point, just before the energy jump, $\boldsymbol{z}_{0n}=\boldsymbol{z}_0(\theta,E_n)$, $M_n=M(\theta+2\pi,\theta,E_n)$, $A_n=A(\theta,E_n)$, and $R_n=R(\theta+2\pi,\theta,E_n)$. Then

$$\boldsymbol{z}_{n+1}=M_{n+1}(\boldsymbol{z}_n-\boldsymbol{z}_{0,n+1})+\boldsymbol{z}_{0,n+1}. \quad (52)$$

In normalized coordinates,

$$\begin{aligned}\boldsymbol{w}_{n+1}&=R_{n+1}A_{n+1}^{-1}(A_n\boldsymbol{w}_n+\boldsymbol{z}_{0n}-\boldsymbol{z}_{0,n+1}) && (53)\\ &=R_{n+1}A_{n+1}^{-1}A_n\boldsymbol{w}_n\\ &\quad+R_{n+1}A_{n+1}^{-1}(\boldsymbol{z}_{0n}-\boldsymbol{z}_{0,n+1}) && (54)\end{aligned}$$

where $\boldsymbol{z}_n=A_n\boldsymbol{w}_n+\boldsymbol{z}_{0n}$.

To estimate the evolution of the magnitude of $\boldsymbol{w}_n$, one can make one of two assumptions: that between turns, the beam filaments completely into a circular disk, or that the motion remains perfectly linear between turns. Reality is probably closer to the latter assumption, but it is instructive to assume the former first. Call λ_n the magnitude of the largest eigenvalue of $R_{n+1}A_{n+1}^{-1}A_n$. Let $\boldsymbol{b}_n=R_{n+1}A_{n+1}^{-1}(\boldsymbol{z}_{0n}-\boldsymbol{z}_{0,n+1})$. Then

$$|\boldsymbol{w}_{n+1}|\leqslant\lambda_n|\boldsymbol{w}_n|+|\boldsymbol{b}_n|, \quad (55)$$

and thus

$$|\boldsymbol{w}_n|\leqslant|\boldsymbol{w}_0|\prod_{j=0}^{n-1}\lambda_j+\sum_{m=0}^{n-1}|b_m|\prod_{j=m+1}^{n-1}\lambda_j \quad (56)$$

Assuming that R_{n+1} is sufficiently far from linear resonances, $\lambda_n=1$.

If instead the motion remains linear between turns, examine Eq. (54): it is of the form

$$\boldsymbol{w}_{n+1}=B_n\boldsymbol{w}_n+\boldsymbol{b}_n. \quad (57)$$

If B_n and $\boldsymbol{b}_n$ were constant, the solution to the equation would be linear oscillations about

$$(I-B_n)^{-1}\boldsymbol{b}_n. \quad (58)$$

If B_n is nearly a rotation (it generally is), then to the accuracy of that approximation, the maximum increase in magnitude of $\boldsymbol{w}$ is the magnitude of Eq. (58). B_n and b_n are not constant, but they should be varying very slowly, and thus this is a good approximation.

In fact, Eq. (58) gives the coordinates for what one may call an "accelerated orbit." As long as $\boldsymbol{b}_n$ and B_n vary slowly, the beam should follow this accelerated orbit, and the changes in the normalized amplitude should be small.

A. Scaling FFAG

Consider the case of a scaling FFAG. Then

$$A_n=\begin{bmatrix}(p_n/p_0)^{-\frac{k}{2k+2}} & 0\\ 0 & (p_n/p_0)^{\frac{k}{2k+2}}\end{bmatrix}A_0 \quad (59)$$

$$\boldsymbol{z}_{0n}=\begin{bmatrix}(p_n/p_0)^{\frac{1}{k+1}} & 0\\ 0 & p_n/p_0\end{bmatrix}\boldsymbol{z}_{00} \quad (60)$$

$$R_n=R_0. \quad (61)$$

Therefore

$$\boldsymbol{b}_n = \left(\frac{p_{n+1}}{p_0}\right)^{\frac{k+2}{2k+2}} RA_0^{-1} \begin{bmatrix} (p_n/p_{n+1})^{\frac{1}{k+1}} - 1 & 0 \\ 0 & (p_n/p_{n+1}) - 1 \end{bmatrix} \boldsymbol{z}_{00} \quad (62)$$

$$B_n = RA_0^{-1} \begin{bmatrix} (p_n/p_{n+1})^{-\frac{k}{2k+2}} & 0 \\ 0 & (p_n/p_{n+1})^{\frac{k}{2k+2}} \end{bmatrix} A_0 \quad (63)$$

For small changes in the momentum per turn Δp, these can be approximated as

$$\boldsymbol{b}_n \approx -\Delta p\, p_0^{-\frac{k+2}{2k+2}} p_{n+1}^{-\frac{k}{2k+2}} RA_0^{-1} \begin{bmatrix} \frac{1}{k+1} & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{z}_{00} \quad (64)$$

$$B_n \approx R. \quad (65)$$

Near a symmetry point where the closed orbit has no p_R component,

$$A_0^{-1} \begin{bmatrix} \frac{1}{k+1} & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{z}_{00} \approx \frac{r_0}{k+1}\sqrt{\frac{p_0}{\beta_0}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (66)$$

where β_0 is the Courant-Snyder beta function at the initial momentum and the symmetry point, and r_0 is the radius of the closed orbit at the initial momentum and the symmetry point.

Under the assumption that the beam filaments completely between turns,

$$|\boldsymbol{w}_n| \leqslant |\boldsymbol{w}_0| + \frac{2k+2}{k+2}\left[\left(\frac{p_n}{p_0}\right)^{\frac{k+2}{2k+2}} - 1\right] \left| A_0^{-1} \begin{bmatrix} \frac{1}{k+1} & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{z}_{00} \right|, \quad (67)$$

which at a symmetry point is just

$$|\boldsymbol{w}_n| \leqslant |\boldsymbol{w}_0| + \frac{2}{k+2}\left[\left(\frac{p_n}{p_0}\right)^{\frac{k+2}{2k+2}} - 1\right] \frac{r_0}{\sigma_0}\sqrt{\epsilon_n mc}, \quad (68)$$

where ϵ_n is the transverse normalized emittance, and σ_0 is the RMS beam size. The average of $|\boldsymbol{w}_0|^2$ over the beam distribution is $2\epsilon_n mc$. Thus, there is a large relative increase in the beam size (by a factor comparable to the ratio of the injection radius to the beam size!) under this pessimistic assumption.

If instead we assume that the motion is linear between turns, the magnitude of the closed orbit shift between turns at the symmetry point is just

$$\left(\frac{1}{k+1}\frac{\Delta p}{p_0}\frac{r_0}{\sigma_0}\right)\csc\mu_x\sqrt{\epsilon_n mc}. \quad (69)$$

The term in parentheses is just the orbit separation between turns divided by the RMS beam size. μ_x is the horizontal phase advance per turn. This is the result that one expects trivially. As long as this quantity is small, one expects the beam to follow the accelerated orbit described above.

VII. DISCUSSION AND CONCLUSIONS

I have developed a Hamiltonian formulation for dynamics in a spiral machine. In particular, it appears that having any RF cavities follow the spiral of the magnets will minimize longitudinal-transverse effects.

However, this latter conclusion is still somewhat speculative. There is longitudinal-transverse coupling that arises from the having dispersion in the RF cavities. It is conceivable that giving the cavity a different angle would be able to reduce this coupling. However, it initially appears that the two effects do not come into the Hamiltonian in the same way. However, to verify this, the longitudinal-transverse coupling due to finite dispersion in the cavities should be computed.

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