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Fully Electromagnetic Nonlinear Gyrokinetic Equations for Tokamak Edge Turbulence

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Abstract

An energy conserving set of the fully electromagnetic nonlinear gyrokinetic Vlasov equation and Maxwell's equations, which is applicable to both L-mode turbulence with large amplitude and H-mode turbulence in the presence of high $\mathbf{E} \times \mathbf{B}$ shear has been derived. The phase-space action variational Lie perturbation method ensures the preservation of the conservation laws of the underlying Vlasov-Maxwell system. Our generalized ordering takes $\rho_i \ll \rho_{\theta i} \sim L_E \sim L_p \ll R$ (here ρ_i is the thermal ion Larmor radius and $\rho_{\theta i} = \frac{B}{B_\theta} \rho_i$), as typically observed in the tokamak H-mode edge, with L_E and L_p being the radial electric field and pressure gradient lengths. We take $k_\perp \rho_i \sim 1$ for generality, and keep the relative fluctuation amplitudes $e\delta\phi/T_i \sim \delta B/B$ up to the second order. Extending the electrostatic theory in the presence of high $\mathbf{E} \times \mathbf{B}$ shear [Hahm, Phys. Plasmas **3**, 4658 (1996)], contributions of electromagnetic fluctuations to the particle charge density and current are explicitly evaluated via pull-back transformation from the gyrocenter distribution function in the gyrokinetic Maxwell's equation.

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I. INTRODUCTION

Understanding tokamak microturbulence is required for developing a predictive capability of tokamak transport. The nonlinear gyrokinetic formulations¹⁻⁷ have provided theoretical foundations for recent advances in nonlinear gyrokinetic simulations of tokamak microturbulence. Both formulations and simulations have traditionally focused on tokamak core turbulence in which the fluctuation amplitude is relatively small, i.e., less than or comparable to $\delta n/n_0 \sim 10^{-2}$ and the gradients in macroscopic parameters such as pressure are relatively mild.⁸

As experience in enhancing tokamak confinement⁹⁻¹³ accumulates, for instance by H-mode operation, tokamak edge plasmas became hotter and less collisional. Therefore, traditional edge turbulence simulations¹⁴⁻¹⁶ based on Braginskii-type fluid equations are not strictly applicable in some of the leading tokamaks in the present and future. For more accurate simulations of such collisionless edge plasmas, kinetic effects associated with long mean free path and finite orbit sizes should be included properly. However, the following obvious challenges must be faced when one tries to apply the existing nonlinear gyrokinetic formalism derived mainly for core turbulence:

i) Fluctuation amplitudes in L-mode edge plasmas are typically on the order of $\delta n/n_0 \sim 10^{-1}$ inside the last closed flux surface (LCFS),^{17,18} and can be even higher in the scrape off layer (SOL).¹⁹⁻²²

ii) After an H-mode transition, E_r well is formed just inside the LCFS. The pressure gradient scale length and the radial electric field scale length in the edge E_r well is on the order of the ion poloidal gyroradius,²³ $\rho_{\theta i} \equiv v_{Ti} M_i c / (e B_\theta)$.

iii) Not only in H-mode plasmas, but also in some L-mode plasmas, we have $\rho_i/L_p > L_p/R_0$ at the edge. Since ρ_i is a typical unit for the radial scale of microturbulence, and L_p is a length scale of a macroscopic quantity, we have an intriguing situation where one of the primary smallness parameters ρ_i/L_p in the conventional nonlinear gyrokinetic formulation¹ is greater than a ratio between two macroscopic scale lengths.

In retrospect, many theoretically related issues have been addressed in the context of nonlinear gyrokinetic equations for core transport barriers.²⁴ That work, however, only considered the electrostatic fluctuations. The present work provides an extension of the electrostatic nonlinear gyrokinetic equations as presented in Ref. 24 to fully electromagnetic edge turbulence in toroidal geometry, with an ordering applicable to both large fluctuation amplitudes in L-mode and residual fluctuations in the presence of strong $\mathbf{E} \times \mathbf{B}$ in H-mode. It has been shown that the edge turbulence is electromagnetic even for low local values of plasma β .^{14,25} As emphasized in the context of both theory^{26–32} and experiments,^{10–13} $\mathbf{E} \times \mathbf{B}$ flow shear rather than plasma mass flow shear of a particular species plays a more fundamental role in reducing turbulence. It is also natural to deal with the electromagnetic fields, \mathbf{E} and \mathbf{B} , rather than plasma flows, in formulating the nonlinear gyrokinetic equations which are based on the equations of motion for a single charged particle's gyrocenter.

The emphasis in Ref. 24 was on a systematic description of the collective phenomena (i.e., a self-consistent treatment of turbulence) in which the Vlasov equation and the Maxwell's equation are treated on an equal footing with various terms explicitly derived from the general expressions, and with an explicit expression for the total energy invariant which can be used as an indicator of accuracy in numerical simulations. This underlying philosophy followed those of the early modern nonlinear gyrokinetic theories^{3–6} and a recent review.⁷ It's noteworthy that the necessity of extending nonlinear gyrokinetic formulations to edge turbulence in the presence of flow shear has been widely recognized. Recent publications based on the Lie perturbation theory including Refs. 33–35 used the $\mathbf{E} \times \mathbf{B}$ flow, rather than the ion mass flow, following the approach in Ref. 24. On the other hand, the emphasis of Refs. 33–34 was mostly on a systematic manifestation of modern nonlinear gyrokinetic theory in the context of the language of differential geometry. Utilizing noncanonical Hamiltonian theories of guiding center drifts, including that by Littlejohn,^{36–38} more drift terms, including the polarization drift associated with a time-dependent background radial electric field, are kept in the gyrokinetic-Vlasov equation. Ref. 35 has shown that by adopting a modified definition of the $\mathbf{E} \times \mathbf{B}$ flow, higher accuracy in the guiding center drift can be achieved over the formulations in Ref. 33. However, in those recent publications,^{33–35} the explicit form of the gyrokinetic Ampère's law after the integrations over velocity space for

the electrical current has not been derived. Furthermore, the total energy invariant extending that in Ref. 24 has not been derived to date for the electromagnetic case. The recent progress in nonlinear gyrokinetic formulations with flows in tokamak geometry via the Lie perturbation method is summarized in Table I.

In this paper, an energy-conserving and phase-space volume conserving set of the fully electromagnetic nonlinear gyrokinetic Vlasov equation and Maxwell's equation, which is applicable to tokamak edge turbulence, is derived. These conservation properties become more important as long term gyrokinetic simulations well beyond the nonlinear saturation phase are being pursued with recent advances in computational power.³⁹

The principal results of this paper are as follows:

- i.) An energy-conserving set of the fully electromagnetic nonlinear gyrokinetic Vlasov and Maxwell's equations is derived for the first time in the presence of strong $\mathbf{E} \times \mathbf{B}$ flow shear. Symplectic derivation via phase-space Lagrangian Lie-perturbation theory ensures the preservation of the conservation laws.
- ii.) Expressions for the gyrokinetic Maxwell equations alongside corresponding energy invariants are presented for practically useful limiting cases, i.e., the long wavelength limit and for a Maxwellian distribution in μ .
- iii.) In particular, various contributions of both shear-Alfvénic (δA_{\parallel}) and compressional (δB_{\parallel}) electromagnetic fluctuations to the gyrokinetic Poisson equation and the gyrokinetic Ampère's law are explicitly evaluated in the presence of strong $\mathbf{E} \times \mathbf{B}$ flow shear.

The remainder of this paper is organized as follows. In Sec. II, the guiding-center motion in the presence of strong $\mathbf{E} \times \mathbf{B}$ shear is presented. The gyrophase-independent Euler–Lagrange equation for the gyrocenter drift in the presence of the electromagnetic fluctuation is derived in Sec. III. In Sec. IV, an energy conserving set of general gyrokinetic Vlasov-Maxwell equations with strong $\mathbf{E} \times \mathbf{B}$ shear is derived. Their limiting forms for long wavelength fluctuations and for a Maxwellian distribution in μ are also presented respectively. Major emphasis is placed on the rigorous and transparent derivation of the most general result via the phase-space Lagrangian Lie-perturbation theory, and the explicit evaluation of velocity

moments involving the gyrocenter distribution function which appear in the Gyrokinetic Maxwell equations.

II. GUIDING-CENTER DRIFT IN THE PRESENCE OF STRONG $\mathbf{E} \times \mathbf{B}$ FLOW SHEAR

In this section, we present the guiding-center equations of motion in the presence of strong radial electric field shear and steep pressure gradients as observed in most H-mode edges. As emphasized in the previous work on the nonlinear gyrokinetic equations in core transport barriers,²⁴ a formulation in terms of the radial electric field rather than in terms of mass flow is preferred. Since the single particle's guiding center motion is determined by the electromagnetic field rather than the mass flow, this choice is not only natural, but also advantageous in separating the issue of determining the equilibrium ion distribution function (which is also an important issue at the tokamak edge in its own right^{40,41}) from the formulation of the nonlinear gyrokinetic equation for turbulence. Neoclassical equilibrium, i.e., the distribution function in the absence of the turbulence, in the steep pressure gradient edge region, can be calculated numerically as an input for turbulence simulations.⁴²⁻⁴⁵ A massively parallel Monte-Carlo guiding center simulation could tabulate the distribution function in the 4D phase space. We focus only on the issues involving turbulence in this paper without specifying the equilibrium mass flow. This approach is thus conceptually simpler than previous nonlinear gyrokinetic formulations in terms of the relative velocity in the frame moving with the mass flow.⁴⁶⁻⁴⁸

A major emphasis is placed on a rigorous and transparent derivation via the phase-space Lagrangian Lie-perturbation theory. The standard nonlinear gyrokinetic ordering¹ consists of:

$$\omega/\Omega \sim \rho_i k_{\parallel} \sim \delta B/B \sim e\delta\phi/T_i \sim \epsilon$$

and

$$k_{\perp} \rho_i \sim 1,$$

where ω and Ω are the characteristic fluctuation frequency and the ion cyclotron frequency, respectively; k_{\parallel} and k_{\perp} are the components of the wave vector in the parallel and perpendicular direction with respect to the magnetic field; ρ_i is the thermal ion gyroradius; $\delta\phi$ is

the fluctuating electrostatic potential; δB is the fluctuating magnetic field; and $\epsilon \ll 1$ is a small ordering parameter. A tokamak-specific ordering, $B_\theta/B \simeq rq/R \ll 1$, will be used in this work to differentiate the poloidal gyroradius from the gyroradius. Here, r/R is the local inverse aspect ratio, and q is the magnetic safety factor.

For a derivation of the unperturbed phase-space Lagrangian of a charged particle which is relevant for edge plasma conditions including the H-mode state, it is useful to summarize the following key experimental findings. A spontaneous H-mode transition⁹ starts with a rapid increase in the negative radial electric field E_r at the edge just inside the LCFS. Well after the transition, a negative E_r well is formed and the steep ion pressure gradient in the same region becomes large. Quite often, it becomes the dominant contributor to E_r in the radial force balance relation for the main ion species^{23,49,50}, i.e., $E_r \sim \frac{1}{n_i e_i} \frac{\partial P_i}{\partial r}$. For these plasmas, the gradient lengths of E_r and P_i are of the same order as the ion poloidal gyroradius, i.e., $L_E \sim L_p \sim \rho_{\theta i}$. These conditions correspond to $u_E^{(0)} \equiv -\frac{cE_r}{B} \sim u_{*i} \equiv -\frac{c}{n_i e_i B} \frac{\partial P_i}{\partial r} \sim \frac{\rho_i}{L_p} v_{Ti}$, and $\frac{e\Phi^{(0)}}{T_i} \sim 1$. While $\frac{\rho_{\theta i}}{L_p} \sim 1$, we can still identify a small parameter for the unperturbed particle orbit calculations in tokamak plasmas with large E_r shear, except for low aspect ratio experiments such as the National Spherical Torus eXperiment (NSTX),⁵¹

$$\epsilon_E \equiv \frac{\rho_i}{L_E} \sim \frac{\rho_i}{L_p} \sim \frac{B_\theta}{B} \ll 1. \quad (1)$$

We note that ϵ_E is larger than the conventional small parameter δ_B for the unperturbed particle orbit calculations in the absence of large E_r shear,

$$\delta_B \equiv \frac{\rho_i}{L_B} \ll 1, \quad (2)$$

where $L_B^{-1} \equiv \left| \frac{\partial B}{B \partial r} \right|$ originates from the inhomogeneity in the equilibrium \mathbf{B} field.

We will discuss the relation between ϵ_E and δ_B for our problem shortly. Starting from the unperturbed phase-space Lagrangian of a charged particle, one can perform Lie perturbation analysis as described in Refs. 6,7,24,38,48, to obtain the guiding-center phase-space Lagrangian,

$$\gamma_0 \equiv \left(\frac{e}{c} \mathbf{A} + M \mathbf{u}_E + M v_{\parallel} \mathbf{b} \right) \cdot d\mathbf{R} + \frac{\mu B}{\Omega} d\theta - H_0 dt. \quad (3)$$

Here the notations follow mostly those used in Ref. 24. The noncanonical guiding-center coordinates which simplify the phase-space Lagrangian are used. $\mathbf{R} \equiv \mathbf{x} - \boldsymbol{\rho}$, μ is the

guiding-center magnetic moment in the frame moving with $\mathbf{u}_E \equiv c\mathbf{b} \times \nabla\Phi/B$,³⁶ associated with the equilibrium potential Φ . v_{\parallel} is the guiding center parallel velocity which includes the Banos drift,⁵² and θ is the gyro-phase angle. More detailed discussions on the choice of guiding center variables can be found in Ref. 6. On the right-hand side of Eq. (3), the $O(\delta_B)$ term $-\frac{\mu B}{\Omega}[\nabla\mathbf{e}_1 \cdot \mathbf{e}_2 + \frac{1}{2}(\mathbf{b} \cdot \nabla \times \mathbf{b})\mathbf{b}] \cdot d\mathbf{R}$ is ignored for simplicity. The term $\mu\nabla\mathbf{e}_1 \cdot \mathbf{e}_2$ which depends on the choice of perpendicular unit vectors \mathbf{e}_1 and \mathbf{e}_2 , is related to a gyro-gauge invariance.³⁸ In Eq.(3), the guiding-center Hamiltonian up to the order ϵ_E^2 is

$$H_0 = e\Phi + \mu B + (M/2)(v_{\parallel}^2 + u_E^2) + \frac{\mu B}{2\Omega}\mathbf{b} \cdot \nabla \times \mathbf{u}_E, \quad (4)$$

and $\mu\mathbf{b} \cdot \nabla \times \mathbf{u}_E \simeq (\mu/B)\nabla_{\perp}^2\Phi$ describes the finite Larmor-orbit-average reduction of the equilibrium potential.⁴⁸ We note that unlike typical core profiles, the tokamak edge profiles satisfy $\frac{\rho_i}{L_p} > \frac{L_p}{R}$. Therefore, we assume $\epsilon_E^2 > \delta_B$.

In passing, we remark that the trapped ion radial width modification due to E_r shear⁵³⁻⁵⁶ is on the order of unity for our ordering based on typical tokamak H-mode edge plasma parameters. This can be easily shown from the fact that in general toroidal geometry, the banana orbit modification parameter⁵⁷ is given by

$$S \equiv 1 + \frac{mc^2 (RB_{\phi})^2}{e} \frac{\partial}{\partial\psi} \left(\frac{E_r}{RB_{\theta}} \right).$$

On the other hand, the $\mathbf{E} \times \mathbf{B}$ shearing rate in general toroidal geometry^{29,31} is given by

$$\omega_E = \frac{\Delta r_0}{\Delta l_{\perp}} \frac{c(RB_{\theta})^2}{B} \frac{\partial}{\partial\psi} \left(\frac{E_r}{RB_{\theta}} \right).$$

Here, ψ is the poloidal flux representing the radial coordinate via $d\psi = RB_{\theta}dr$, $\Delta r_0 = \Delta\psi_0/RB_{\theta}$ is the radial correlation length, and $\Delta l_{\perp} = RB_{\theta}\Delta\phi/B$ is the correlation length of the ambient turbulence in the direction perpendicular to the field line, but within the flux surface. Therefore, for near isotropic ambient turbulence, $\Delta r_0 \simeq \Delta l_{\perp}$, they are related through⁵⁸

$$S \simeq 1 + \left(\frac{B}{B_{\theta}} \right)^2 \frac{\omega_E}{\Omega_i}.$$

Since $\frac{\omega_E}{\Omega_i} \sim \epsilon_E^2$ from Eq. (1), we have $|S - 1| \sim 1$, and we have an order of unity banana orbit width modification due to $\mathbf{E} \times \mathbf{B}$ shear. It is obvious that the appearance of the particular combination $\frac{E_r}{RB_{\theta}}$ in both the $\mathbf{E} \times \mathbf{B}$ shearing rate and the orbit modification

factor is a consequence of the axisymmetry in tokamak geometry. Based on our ordering, the ion gyro-orbit is near circular in the frame moving with the equilibrium $\mathbf{E} \times \mathbf{B}$ velocity.

In general, the variation of the fundamental one-form, $\gamma \equiv \gamma_\mu dz^\mu = \gamma_i dz^i - h dt$, yields the Euler–Lagrange equation³⁷

$$\left(\frac{\partial \gamma_j}{\partial z^i} - \frac{\partial \gamma_i}{\partial z^j} \right) \frac{dz^j}{dt} = \frac{\partial h}{\partial z^i} + \frac{\partial \gamma_i}{\partial t}. \quad (5)$$

For the unperturbed phase-space Lagrangian given by Eq. (3), the nontrivial components of Eq. (5) are

$$-e\mathbf{B}^* \times \frac{d\mathbf{R}}{dt} - M\mathbf{b} \frac{dv_{\parallel}}{dt} = \nabla[e\Phi + \mu B + (M/2)(v_{\parallel}^2 + u_E^2) + \frac{\mu B}{2\Omega} \mathbf{b} \cdot \nabla \times \mathbf{u}_E].$$

Following the same decomposition procedure described in Ref. 4, one obtains the following gyrocenter equations of motion:

$$\frac{d\mathbf{R}}{dt} = v_{\parallel} \frac{\mathbf{B}^*}{B_{\parallel}^*} + \frac{\mathbf{b}}{eB_{\parallel}^*} \times [e\nabla\Phi + \mu\nabla B + \frac{\mu B}{2\Omega} \nabla(\mathbf{b} \cdot \nabla \times \mathbf{u}_E) + \frac{M}{2} \nabla(u_E^2)], \quad (6)$$

and

$$\frac{dv_{\parallel}}{dt} = -\frac{\mathbf{B}^*}{MB_{\parallel}^*} \cdot [e\nabla\Phi + \mu\nabla B + \frac{\mu B}{2\Omega} \nabla(\mathbf{b} \cdot \nabla \times \mathbf{u}_E) + \frac{M}{2} \nabla(u_E^2)]. \quad (7)$$

Here

$$\mathbf{B}^* \equiv \mathbf{B} + \frac{Mc}{e} \nabla \times (\mathbf{u}_E + v_{\parallel} \mathbf{b}),$$

and

$$B_{\parallel}^* \equiv \mathbf{b} \cdot \mathbf{B}^* = B \left\{ 1 + \frac{\mathbf{b}}{\Omega} \cdot \nabla \times (\mathbf{u}_E + v_{\parallel} \mathbf{b}) \right\}.$$

III. LIE-PERTURBATION ANALYSIS WITH FULLY ELECTROMAGNETIC FLUCTUATIONS

In this section, we introduce the time-dependent electromagnetic fluctuations corresponding to tokamak edge turbulence. It has been almost universally observed^{17,18,59} that the relative density fluctuation amplitude $\frac{\delta n}{n_0}$ increases from the core to the edge monotonically in tokamak plasmas when there are no transport barriers. In the core, the level is often less

than one percent,^{8,11,13} while towards the LCFS, it typically reaches $\sim 10^{-1}$. In the SOL, it can be sometimes as high as 0.25 in the form of long-lived, spatially intermittent blobs.^{22,60} In this paper, we pursue the nonlinear gyrokinetic approach with full ion Larmor radius effects $k_{\perp}\rho_i \sim 1$, and strong turbulence,

$$\epsilon_{\phi} \equiv \frac{\delta f}{f_0} \sim \frac{e\delta\phi}{T_i} \sim \frac{\delta B}{B_0} \ll 1. \quad (8)$$

While the relative magnetic fluctuation level for edge microturbulence, in the absence of low-mode-number MHD activity, is measured to be much smaller than that of the electrostatic fluctuations,²⁰ we order them to be comparable for generality. We perform the perturbation theory derivation up to the second order in the relative fluctuation amplitude, i.e., in ϵ_{ϕ} . By taking the nonlinear mode coupling term comparable to the linear driving term from the equilibrium pressure gradient, we get the mixing length type balance^{61,62}, $\frac{\delta f}{f_0} \sim \frac{1}{k_{\perp}L_p}$ which is equivalent to taking $\epsilon_{\phi} \sim \epsilon_E$ for $k_{\perp}\rho_i \sim 1$. Note that we maintain consistency in ordering by keeping terms up to the second order both in ϵ_{ϕ} and in $\epsilon_E = \rho_i/L_E$.

While one could also pursue a drift-kinetic type ordering of $\frac{e\delta\phi}{T_i} \sim 1$ and $k_{\perp}\rho_i \ll 1$, we believe that it's important to treat the relatively short wavelength fluctuations at $k_{\perp}\rho_i \sim 1$, which nonlinearly interact with the longer wavelength fluctuations, accurately even though their amplitudes at saturation are small⁶³. The necessity of resolving edge turbulence down to the scale $k_{\perp}\rho_i \sim 1$ has been demonstrated in Ref. 25. It is also encouraging to note that there seems to be a growing recognition⁶⁴ that the final form of the nonlinear gyrokinetic equation is robust in the drift kinetic regime. A related explicit illustration exists⁶⁵ for the electromagnetic nonlinear gyrokinetic equations in a simple geometry⁵ in detail.

Fluctuations are introduced in the first order phase space Lagrangian in terms of the four-potential $(\delta\phi, \delta\mathbf{A})$.

$$\begin{aligned} \gamma_1 &= \frac{e}{c} \delta\mathbf{A}(\mathbf{R} + \boldsymbol{\rho}, t) \cdot (d\mathbf{R} + d\boldsymbol{\rho}) - e\delta\phi(\mathbf{R} + \boldsymbol{\rho}, t) dt \\ &= \frac{e}{c} (\delta\mathbf{A} \cdot d\mathbf{R} + \delta\mathbf{A} \cdot \frac{\partial\boldsymbol{\rho}}{\partial\mu} d\mu + \delta\mathbf{A} \cdot \frac{\partial\boldsymbol{\rho}}{\partial\theta} d\theta) - e\delta\phi dt. \end{aligned} \quad (9)$$

Then, the Lie-perturbation analysis consists of finding near-identity transformations, order by order, which eliminate the gyro-phase dependence in Eq. (9) introduced by the fact that the fluctuating electromagnetic potentials are functions of the particle position

$\mathbf{x} \equiv \mathbf{R} + \boldsymbol{\rho}$, rather than functions of the guiding center position \mathbf{R} .

$$\Gamma_0 = \gamma_0 + dS_0, \quad (10)$$

$$\Gamma_1 = \gamma_1 - L_1\gamma_0 + dS_1. \quad (11)$$

Here, γ_0 is given by Eq. (3).

$$(L_1\gamma)_\mu = g_1^\nu \left(\frac{\partial \gamma_\mu}{\partial z^\nu} - \frac{\partial \gamma_\nu}{\partial z^\mu} \right), \quad (12)$$

where g_1^ν is the generator of the Lie transformation. With $dS_0 = 0$ and $g_1^t = 0$, Equation (11) yields

$$\begin{aligned} \Gamma_1 = & dS_1 + \frac{e}{c}(\delta\mathbf{A} \cdot d\mathbf{R} + \delta\mathbf{A} \cdot \frac{\partial \boldsymbol{\rho}}{\partial \mu} d\mu + \delta\mathbf{A} \cdot \frac{\partial \boldsymbol{\rho}}{\partial \theta} d\theta) - e\delta\phi dt \\ & + \frac{Mc}{e}g_1^\theta d\mu - \frac{Mc}{e}g_1^\mu d\theta + M\mathbf{b} \cdot \mathbf{g}_1^{\mathbf{R}} dv_{\parallel} - Mg_1^{v_{\parallel}} \mathbf{b} \cdot d\mathbf{R} + \frac{e}{c} \mathbf{g}_1^{\mathbf{R}} \times \mathbf{B}^* \cdot d\mathbf{R} \\ & + \{Bg_1^\mu + Mv_{\parallel}g_1^{v_{\parallel}} + \mathbf{g}_1^{\mathbf{R}} \cdot [e\nabla\Phi + \mu\nabla B + \frac{M}{2}\nabla(u_E^2) + \frac{\mu B}{2\Omega}\nabla(\mathbf{b} \cdot \nabla \times \mathbf{u}_E)]\} dt, \end{aligned} \quad (13)$$

where

$$\mathbf{g}_1^{\mathbf{R}} \equiv (g_1^{\mathbf{R}_1}, g_1^{\mathbf{R}_2}, g_1^{\mathbf{R}_3}).$$

In Eq.(13), one can choose dS_1 and g_1^ν such that all of the $\Gamma_{1\nu}$ vanish except for Γ_{1t} . This choice corresponds to the Hamiltonian approach (rather than the symplectic approach) adopted in Ref. 5. Also, by requiring that there be no θ -dependent term in Γ_{1t} , one obtains

$$\Gamma_{1t} \equiv -e\langle\delta\psi_1\rangle = -e[\langle\delta\phi\rangle - \frac{1}{c}(\bar{v}_{\parallel}\mathbf{b} + \mathbf{u}_E) \cdot \langle\delta\mathbf{A}\rangle - \frac{1}{c}\langle\mathbf{c}_{\perp} \cdot \delta\mathbf{A}\rangle] \quad (14)$$

Here, the contribution from \mathbf{v}_d is neglected, because $\delta_B \ll \epsilon_E$. It's important to note that $\bar{v}_{\parallel} \simeq v_{\parallel} + \frac{e}{mc}\delta A_{\parallel}$, which is close to the canonical momentum including the δA_{\parallel} contribution. The bracket denotes the gyrophase average, i.e., $\langle\delta\psi_1\rangle \equiv (2\pi)^{-1} \oint d\theta \delta\psi_1(\mathbf{R} + \boldsymbol{\rho}, t)$, for instance. $dS_1 = e\Omega^{-1}[\delta\psi_1 - \langle\delta\psi_1\rangle]d\bar{\theta}$ has been chosen according to the gyrokinetic ordering.⁴

The second-order perturbation analysis deals with the equation

$$\Gamma_2 = \gamma_2 - L_1\gamma_1 + \left(\frac{1}{2}L_1^2 - L_2\right)\gamma_0 + dS_2.$$

The resulting equations have not been utilized in most practical applications.⁶⁶⁻⁷¹ However, the resulting quadratic low-frequency ponderomotive like terms are required for energy conservation up to $O(\epsilon_\phi^2)$, in the formulation in terms of the total distribution function.^{3,4} The

importance of keeping those terms for theoretical completeness has been discussed in detail in a recent review article.⁷ The main result of the second order perturbation analysis is the nonlinear modification of the effective potential, which is described below. The derivation is quite similar to that of Ref. 4 and is not repeated here.

Finally, the total phase-space Lagrangian is given by

$$\Gamma = \left(\frac{e}{c} \mathbf{A} + M \mathbf{u}_E + M \bar{v}_{\parallel} \mathbf{b} \right) \cdot d\bar{\mathbf{R}} + \frac{\bar{\mu} B}{\Omega} d\bar{\theta} - (\bar{H}_0 + e \langle \delta\psi \rangle) dt, \quad (15)$$

where the effective potential is

$$\langle \delta\psi \rangle \equiv \langle \delta\psi_1 \rangle + \frac{e}{2Mc^2} \langle |\delta\mathbf{A}|^2 \rangle - \frac{e^2}{2Mc\Omega} \left(\frac{\partial}{\partial \mu} \langle \delta\tilde{\psi}_1^2 \rangle + \frac{c}{e\Omega} \langle \nabla \delta\tilde{\Psi}_1 \times \mathbf{b} \cdot \nabla \delta\tilde{\psi}_1 \rangle \right) \quad (16)$$

with $\delta\tilde{\psi}_1 \equiv \delta\psi_1 - \langle \delta\psi_1 \rangle$, $\delta\tilde{\Psi}_1 \equiv \int d\bar{\theta} \delta\tilde{\psi}_1$, and the overbar is used for the gyro-center variables. Here, the second term from the last reduces to $-\frac{T_i}{2e} \delta u_E^2 / v_{Ti}^2$ in the long wavelength electrostatic limit.^{64,72} The corresponding Euler-Lagrange equation can be obtained by using Eq. (3):

$$-\frac{e}{c} \mathbf{B}^* \times \frac{d\bar{\mathbf{R}}}{dt} - M \mathbf{b} \frac{d\bar{v}_{\parallel}}{dt} = \bar{\nabla} (\bar{H}_0 + e \langle \delta\psi \rangle). \quad (17)$$

Eq. (17) can be decomposed into the following gyrocenter equations of motion:

$$\frac{d\bar{\mathbf{R}}}{dt} = \bar{v}_{\parallel} \frac{\mathbf{B}^*}{B_{\parallel}^*} + \frac{c\mathbf{b}}{eB_{\parallel}^*} \times [e\bar{\nabla}(\Phi + \langle \delta\psi \rangle) + \bar{\mu}\bar{\nabla}B + \frac{\bar{\mu}B}{2\Omega} \bar{\nabla}(\mathbf{b} \cdot \bar{\nabla} \times \mathbf{u}_E) + \frac{M}{2} \bar{\nabla}(u_E^2)], \quad (18)$$

and

$$\frac{d\bar{v}_{\parallel}}{dt} = -\frac{\mathbf{B}^*}{MB_{\parallel}^*} \cdot [e\bar{\nabla}(\Phi + \langle \delta\psi \rangle) + \bar{\mu}\bar{\nabla}B + \frac{\bar{\mu}B}{2\Omega} \bar{\nabla}(\mathbf{b} \cdot \bar{\nabla} \times \mathbf{u}_E) + \frac{M}{2} \bar{\nabla}(u_E^2)]. \quad (19)$$

The $\frac{c\mathbf{b}}{B_{\parallel}^*} \times \bar{\nabla} \langle \delta\psi \rangle$ term on the R.H.S. of Eq. (18) contains terms which are responsible for turbulence-driven radial transport. These include:

- i.) electrostatic $\mathbf{E} \times \mathbf{B}$ transport contained in $\frac{c\mathbf{b}}{B} \times \nabla \langle \delta\phi \rangle$,
- ii.) magnetic flutter transport which is proportional to $-\frac{\mathbf{b}}{B_0} v_{\parallel} \times \nabla \langle \delta A_{\parallel} \rangle \simeq v_{\parallel} \frac{\delta \mathbf{B}_{\perp}}{B}$,

and finally,

- iii.) a compressional magnetic fluctuation driven piece $-\frac{\mathbf{b}}{B} \times \nabla \langle \mathbf{c}_{\perp} \cdot \delta \mathbf{A}_{\perp} \rangle \simeq \frac{c\boldsymbol{\mu}}{e} \mathbf{b} \times \nabla \langle \delta B_{\parallel} \rangle$,
for $k_{\perp} \rho_i \ll 1$.

It's important to note that the electromagnetic character of the turbulence does not necessarily imply that the magnetic transport mechanisms described by items ii) and iii) are significant for self-consistent transport carried by collective fluctuations.⁷⁵⁻⁷⁷ Many details and subtleties involved in the electrostatic $\mathbf{E} \times \mathbf{B}$ drift were exhaustively discussed in Ref. 78. Although Eqs. (18) and (19) are mathematically concise, those can be written in the following form which is closer to the result of previous work in terms of the mass flow.^{47,48}

$$\begin{aligned} \frac{d\bar{\mathbf{R}}}{dt} = & \mathbf{u}_E + \bar{v}_{\parallel} \mathbf{b} + \frac{c\mathbf{b}}{eB_{\parallel}^*} \\ & \times [e\bar{\nabla}\langle\delta\psi\rangle + \bar{\mu}\bar{\nabla}B + \frac{\bar{\mu}B}{2\Omega}\bar{\nabla}(\mathbf{b} \cdot \bar{\nabla} \times \mathbf{u}_E) + M(\mathbf{u}_E + \bar{v}_{\parallel}\mathbf{b}) \cdot \bar{\nabla}(\mathbf{u}_E + \bar{v}_{\parallel}\mathbf{b})], \end{aligned} \quad (20)$$

and

$$\begin{aligned} \frac{d\bar{v}_{\parallel}}{dt} = & -\frac{\mathbf{B}^{*(0)}}{MB_{\parallel}^{*(0)}} \cdot [e\bar{\nabla}(\Phi_1 + \langle\delta\psi\rangle) + \bar{\mu}\bar{\nabla}B + \frac{\bar{\mu}B}{2\Omega}\bar{\nabla}(\mathbf{b} \cdot \bar{\nabla} \times \mathbf{u}_E^{(0)}) + M(\mathbf{u}_E^{(0)} + \bar{v}_{\parallel}\mathbf{b}) \cdot \bar{\nabla}(\mathbf{u}_E^{(0)} + \bar{v}_{\parallel}(\mathbf{b}))] \end{aligned}$$

Here, $\mathbf{u}_E^{(0)} \equiv c\mathbf{b} \times \nabla\Phi^{(0)}/B$, $\mathbf{B}^{*(0)} \equiv \mathbf{B} + \frac{Mc}{e}\nabla \times (\mathbf{u}_E^{(0)} + v_{\parallel}\mathbf{b})$, and $B_{\parallel}^{*(0)} \equiv \mathbf{b} \cdot \mathbf{B}^{*(0)}$

Although Eq. (20) is valid for an arbitrary form of Φ , Eq.(21) can be only obtained from Eq. (19) via a perturbative analysis.⁴⁸ The equilibrium electrostatic potential, in general, consists of two parts $\Phi \equiv \Phi_0 + \Phi_1$. In most cases, Φ can be approximated by a flux function $\Phi_0(\psi)$ satisfying $\mathbf{b} \cdot \nabla\Phi_0 = 0$. The poloidal-angle-dependent $\Phi_1(\psi, \Theta)$ can be produced, for instance, by the centrifugal-force-driven charge separation in strongly rotating plasmas.⁷³ While the guiding center drift calculation in Sec. II is valid for $u_E \sim v_{Ti}$, $\Phi_0 = O(\epsilon^{-1})$ and $\Phi_1 = O(1)$, as shown in Ref. 24, we adopted an ordering which is optimal for describing collective fluctuations in tokamak H-mode plasmas. This includes $\frac{e\Phi_0}{T_i} = O(1)$, $\frac{e\Phi_1}{T_i} \sim r/R = O(\epsilon_E)$, and $u_E/v_{Ti} = O(\epsilon_E)$, which allows a strong toroidal rotation with a corresponding Mach number on the order of unity, $u_{\phi} \sim v_{Ti}$, since $B_{\theta}/B = O(\epsilon_E)$. The theory of $\mathbf{E} \times \mathbf{B}$ flow shear suppression of turbulence has been recently extended to include the poloidal-angle-dependent $\Phi_1(\psi, \Theta)$.⁷⁴

IV. NONLINEAR GYROKINETIC VLASOV–MAXWELL SYSTEM

With Eqs. (18) and (19), one can write the gyrokinetic Vlasov equation for the gyrocenter distribution function $F(\bar{\mathbf{R}}, \bar{\mu}, \bar{v}_{\parallel}, t)$,

$$\frac{\partial F}{\partial t} + \frac{d\bar{\mathbf{R}}}{dt} \cdot \bar{\nabla} F + \frac{d\bar{v}_{\parallel}}{dt} \frac{\partial F}{\partial \bar{v}_{\parallel}} = 0. \quad (22)$$

Here, $d\bar{\mu}/dt \equiv 0$ and $\partial F/\partial \bar{\theta} \equiv 0$ have been used. One can also write the gyrokinetic Vlasov equation in the following continuity equation form:

$$\frac{\partial(B_{\parallel}^* F)}{\partial t} + \bar{\nabla} \cdot \left(B_{\parallel}^* \frac{d\bar{\mathbf{R}}}{dt} F \right) + \frac{\partial}{\partial \bar{v}_{\parallel}} \left(B_{\parallel}^* \frac{d\bar{v}_{\parallel}}{dt} F \right) = 0. \quad (23)$$

This is possible because Eqs. (18) and (19) satisfy the following phase-space conservation law:

$$\bar{\nabla} \cdot \left(B_{\parallel}^* \frac{d\bar{\mathbf{R}}}{dt} \right) + \frac{\partial}{\partial \bar{v}_{\parallel}} \left(B_{\parallel}^* \frac{d\bar{v}_{\parallel}}{dt} \right) = 0.$$

The continuity equation form is sometimes more useful in applications which involve taking velocity moments.^{79–81}

We note that various extensions of the gyrokinetic Vlasov equation are conceptually straightforward, once one decides on the generalized ordering. This is because a systematic phase-space Lagrangian derivation of guiding center drift is available, for instance, from Refs. 36–38. On the other hand, expressing the particle charge density and current in terms of the gyrocenter distribution function in the gyrokinetic Maxwell’s equations involves a rather cumbersome pull-back transformation from the gyrocenter coordinate to the particle coordinate. Indeed, in many cases, the gyrokinetic Maxwell’s equations were only presented in their most general form representing the pull-back transformation, without an explicit evaluation of the integrals. However, for many relevant applications including gyrokinetic simulations, explicit evaluations of the particle charge density and current from the gyrocenter distribution function are necessary. This important aspect of explicit representation in the gyrokinetic Poisson equation was recognized and carried out² before the more rigorous Hamiltonian method and the phase-space Lagrangian method were introduced in the nonlinear gyrokinetic formalism.^{3,4} Consideration of energy conservation between particles and fields, in particular, identifying the energy invariant in the gyrokinetic Vlasov-Maxwell system also requires the same level of explicitness in the gyrokinetic Maxwell’s equations

and the gyrokinetic Vlasov equation. Regarding this, the systematic derivation was first achieved in Ref. 3 for electrostatic fluctuations in a straight magnetic field, and then extended to toroidal geometry⁴ and to electromagnetic fluctuations,^{5,6} and finally to the presence of strongly sheared $\mathbf{E} \times \mathbf{B}$ flows.²⁴ It's important to treat the gyrokinetic Vlasov equation, the gyrokinetic Maxwell's equations, and the energy invariant self-consistently on an equal footing. These form three pillars⁷ of nonlinear gyrokinetic theory. We note that a field theoretical variational derivation was also introduced.^{72,82}

Now, we present the gyrokinetic Maxwell equations in which the ion particle charge density and current are expressed in terms of the gyrocenter distribution function $F(\bar{\mathbf{R}}, \bar{\mu}, \bar{v}_{\parallel}, t)$:

$$\nabla^2(\Phi + \delta\phi) = -4\pi e \left[\int d^6\bar{\mathbf{Z}} \left(F + \bar{g}_1^{\mu} \frac{\partial F}{\partial \bar{\mu}} + \bar{g}_1^{\mathbf{R}} \cdot \bar{\nabla} F \right) \delta^3(\bar{\mathbf{R}} - \mathbf{x} + \bar{\boldsymbol{\rho}}) - n_e(\mathbf{x}, t) \right], \quad (24)$$

$$\begin{aligned} \nabla^2(\mathbf{A} + \delta\mathbf{A}) = & -\frac{4\pi e}{c} \left[\int d^6\bar{\mathbf{Z}} \left((\bar{v}_{\parallel} - \frac{e}{Mc} \delta A_{\parallel}) \mathbf{b} + \mathbf{u}_E + \mathbf{c}_{\perp} \right) \left(F + \bar{g}_1^{\mu} \frac{\partial F}{\partial \bar{\mu}} + \bar{g}_1^{\mathbf{R}} \cdot \bar{\nabla} F \right) \right. \\ & \left. \delta^3(\bar{\mathbf{R}} - \mathbf{x} + \bar{\boldsymbol{\rho}}) - \int d^3\mathbf{v} \mathbf{v} f_e(\mathbf{x}, \mathbf{v}, t) \right], \end{aligned} \quad (25)$$

where $d^6\bar{\mathbf{Z}} \equiv \frac{B_{\parallel}^*}{M} d^3\bar{\mathbf{R}} d\bar{v}_{\parallel} d\bar{\theta} d\bar{\mu}$, $\bar{g}_1^{\mu} = \frac{e}{Mc} (\frac{e}{c} \delta\mathbf{A} \cdot \frac{\partial \bar{\boldsymbol{\rho}}}{\partial \bar{\theta}} + \frac{\partial S_1}{\partial \bar{\theta}})$, $\bar{g}_1^{\mathbf{R}} = -\frac{1}{B_{\parallel}^*} \mathbf{b} \times (\delta\mathbf{A} + \frac{c}{e} \bar{\nabla} S_1)$, $v_{\parallel} \simeq \bar{v}_{\parallel} - \frac{e}{c} \delta A_{\parallel}$ and $S_1 = e\Omega^{-1} \int d\theta (\delta\psi_1 - \langle \delta\psi_1 \rangle)$. It's important to note that in our Hamiltonian formulation, $M\bar{v}_{\parallel}$ is a canonical momentum including $\mathbf{b}^* \cdot \delta\mathbf{A}$. In this case, our formulation is closer to the p_z formulation in Ref. 5 than to the v_z formulation in the same work. In Eq. (24), the first three terms on the right-hand side are the ion particle density $n_i(\mathbf{x}, t)$, written in terms of the gyrocenter distribution function. The first term is the gyroaveraged gyrocenter density contribution, while the second and third terms are the general expression for the polarization density. The global gyrokinetic Vlasov-Maxwell energy is obtained by the Noether method and integration over space, as described in Eq. (50) of Ref. 82, and Eq. (199) of Ref. 7:

$$\begin{aligned} E = & \frac{1}{8\pi} \int d^3\mathbf{x} [|\nabla(\Phi + \delta\phi)|^2 + |\mathbf{B}_0 + \delta\mathbf{B}|^2] + \int d^6\mathbf{z} f_e(\mathbf{z}) \frac{1}{2} m_e \mathbf{v}^2 \\ & + \int d^6\bar{\mathbf{Z}} F_i \left\{ \bar{\mu} B + \frac{M}{2} (u_E^2 + \bar{v}_{\parallel}^2) + \frac{\bar{\mu} B}{2\Omega} \mathbf{b} \cdot \bar{\nabla} \times \mathbf{u}_E - \frac{1}{c} (\langle \delta\mathbf{A} \rangle \cdot \mathbf{u}^* + \langle \delta\mathbf{A} \cdot \mathbf{c}_{\perp} \rangle) \right. \\ & + \frac{e^2}{2Mc^2} \langle |\delta\mathbf{A}|^2 \rangle + \frac{e^3}{2Mc\Omega} \frac{\partial}{\partial \bar{\mu}} [\langle \delta\tilde{\phi}^2 \rangle - \frac{1}{c^2} \langle (\delta\tilde{\mathbf{A}} \cdot \mathbf{u}^* + \delta\tilde{\mathbf{A}} \cdot \mathbf{c}_{\perp})^2 \rangle] \\ & \left. + \frac{e^2}{2M\Omega^2} \langle \bar{\nabla} \delta\tilde{\Phi} \cdot \mathbf{b} \times \bar{\nabla} \delta\tilde{\phi} - \frac{1}{c^2} \bar{\nabla} (\tilde{\boldsymbol{\alpha}} \cdot \mathbf{u}^* + \tilde{\boldsymbol{\beta}}) \cdot \mathbf{b} \times \bar{\nabla} (\delta\tilde{\mathbf{A}} \cdot \mathbf{u}^* + \delta\tilde{\mathbf{A}} \cdot \mathbf{c}_{\perp}) \rangle \right\}, \end{aligned} \quad (26)$$

where $\mathbf{u}^* = \bar{v}_{\parallel} \mathbf{b} + \mathbf{u}_E$, $\tilde{\boldsymbol{\alpha}} = \int d\theta \delta \tilde{\mathbf{A}}$ and $\tilde{\boldsymbol{\beta}} = \int d\theta \delta \tilde{\mathbf{A}} \cdot \mathbf{c}_{\perp}$.

Here, we don't specify the electron dynamics, so as to write the electron kinetic energy in a primitive form. However, depending on the problem considered, either a drift kinetic equation⁸³ or a bounce-averaged kinetic equation^{84,85} can be used when it's appropriate. Sometimes, nonlinearity associated with the magnetically trapped electrons is important.⁸⁶⁻⁸⁹ In Eq. (26), the last term represents the sloshing energy.³ In the total F formulation, the second-order nonlinear correction to the effective potential (which leads to the ponderomotive force) should be kept alongside the sloshing energy in order to ensure energy conservation. Here, $\mathbf{E} \equiv -\nabla(\Phi + \delta\phi)$ is the total electric field, and $\mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B}$ is the total magnetic field. Eq. (26) is a generalization of the result of Ref. 24 to the fully electromagnetic case. Now, we consider limiting cases.

A. Long Wavelength Expression for Arbitrary F:

In the long wavelength limit $k_{\perp} \rho_i \ll 1$, Equations (24) and (25) become

$$\begin{aligned} \nabla^2(\Phi + \delta\phi) = & -4\pi e \{ \bar{N}_i(\mathbf{x}, t) - n_e(\mathbf{x}, t) + \frac{\delta B_{\parallel}}{B} N_i \\ & + \frac{e}{M} \nabla_{\perp} \cdot [\frac{N_i}{\Omega_i^2} \nabla_{\perp} (\delta\phi - \frac{1}{c} \mathbf{u}_E \cdot \delta\mathbf{A})] - \frac{1}{Mc} \nabla_{\perp} \cdot (\frac{J_{i\parallel}}{\Omega_i^2} \nabla_{\perp} \delta A_{\parallel}) \}, \end{aligned} \quad (27)$$

$$\begin{aligned} \nabla^2(A_{\parallel} + \delta A_{\parallel}) = & -\frac{4\pi}{c} \{ \bar{J}_{i\parallel}(\mathbf{x}, t) - j_{e\parallel}(\mathbf{x}, t) + \frac{\delta B_{\parallel}}{B} J_{i\parallel} \\ & + \frac{e}{M} \nabla_{\perp} \cdot [\frac{J_{i\parallel}}{\Omega_i^2} \nabla_{\perp} (\delta\phi - \frac{1}{c} \mathbf{u}_E \cdot \delta\mathbf{A})] - \frac{e^2}{Mc} \nabla_{\perp} \cdot (\frac{\Pi_{i\parallel}}{\Omega_i^2} \nabla_{\perp} \delta A_{\parallel}) \}, \end{aligned} \quad (28)$$

$$\begin{aligned} \nabla^2(\mathbf{A}_{\perp} + \delta\mathbf{A}_{\perp}) = & -\frac{4\pi}{c} \{ \bar{\mathbf{J}}_{i\perp}(\mathbf{x}, t) + \bar{\mathbf{J}}_{iE}(\mathbf{x}, t) - \mathbf{j}_{e\perp}(\mathbf{x}, t) - \mathbf{j}_{eE}(\mathbf{x}, t) + \frac{\delta B_{\parallel}}{B} \mathbf{J}_{iE} \\ & + \frac{e^2}{M} \nabla_{\perp} \cdot [\frac{N_i}{\Omega_i^2} (\nabla_{\perp} (\delta\phi - \frac{1}{c} \mathbf{u}_E \cdot \delta\mathbf{A})) \mathbf{u}_E] - \frac{e}{Mc} \nabla_{\perp} \cdot (\frac{J_{i\parallel}}{\Omega_i^2} (\nabla_{\perp} \delta A_{\parallel}) \mathbf{u}_E) \\ & + \frac{e N_i c}{B} \mathbf{b} \times \nabla_{\perp} (\delta\phi - \frac{1}{c} \mathbf{u}_E \cdot \delta\mathbf{A}) - \frac{J_{i\parallel}}{B} \mathbf{b} \times \nabla_{\perp} \delta A_{\parallel} \\ & + 2c \mathbf{b} \times \nabla_{\perp} (\frac{P_{\perp} \delta B_{\parallel}}{B^2}) \}. \end{aligned} \quad (29)$$

Noting an apparent symmetry, Eqs. (27) and (28) can be written using a two vector notation,

$$\nabla^2(A^{\alpha} + \delta A^{\alpha}) = -\frac{4\pi}{c} \{ \bar{J}_i^{\alpha}(\mathbf{x}, t) - j_e^{\alpha}(\mathbf{x}, t) + \frac{\delta B_{\parallel}}{B} J_i^{\alpha} + \frac{e}{M} \nabla_{\perp} \cdot [\frac{\Pi_i^{\prime\alpha\beta}}{\Omega_i^2} \nabla_{\perp} \delta A'_{\beta}] \}. \quad (30)$$

Eq. (29) can also be written as:

$$\begin{aligned} \nabla^2(\mathbf{A}_\perp + \delta\mathbf{A}_\perp) = & -\frac{4\pi}{c}\{\bar{\mathbf{J}}_{i\perp}(\mathbf{x}, t) + \bar{\mathbf{J}}_{iE}(\mathbf{x}, t) - \mathbf{j}_{e\perp}(\mathbf{x}, t) - \mathbf{j}_{eE}(\mathbf{x}, t) + \frac{\delta B_\parallel}{B}\mathbf{J}_{iE} \\ & + \frac{e}{Mc}\nabla_\perp \cdot [\frac{J_i^\alpha}{\Omega_i^2}(\nabla_\perp \delta A'_\alpha)\mathbf{u}_E] + \frac{J_i^\alpha}{B}\mathbf{b} \times \nabla_\perp \delta A'_\alpha + 2c\mathbf{b} \times \nabla_\perp(\frac{P_\perp \delta B_\parallel}{B^2})\} \end{aligned} \quad (31)$$

where $A^\alpha = (\Phi, A_\parallel)$, $J^\alpha = (ceN, J_\parallel)$, $\delta A'^\alpha = (\delta\phi - \frac{1}{c}\mathbf{u}_E \cdot \delta\mathbf{A}, \delta A_\parallel)$, and $\Pi_i^{\alpha\beta} = \begin{pmatrix} ceN_i & J_{i\parallel} \\ J_{i\parallel} & e\Pi_{i\parallel}/c \end{pmatrix}$, the metric tensor $g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

In the same limit,

$$\begin{aligned} \langle \delta\psi \rangle \simeq & \langle \delta\psi_1 \rangle + \frac{e}{2Mc^2}\delta A_\parallel^2 - \frac{e\rho_i^2}{2T_i}[\nabla_\perp(\delta\phi - \frac{1}{c}\mathbf{u}_E \cdot \delta\mathbf{A}) - \frac{1}{c}\bar{v}_\parallel \nabla_\perp \delta A_\parallel]^2 \\ & - \frac{1}{B}\mathbf{b} \times [\nabla_\perp(\delta\phi - \frac{1}{c}\mathbf{u}_E \cdot \delta\mathbf{A}) - \frac{1}{c}\bar{v}_\parallel \nabla_\perp \delta A_\parallel] \cdot \delta\mathbf{A}_\perp. \\ \simeq & \delta\phi - \frac{1}{c}(\bar{v}_\parallel \mathbf{b} + \mathbf{u}_E) \cdot \delta\mathbf{A} + \frac{1}{e}\bar{\mu}\delta B_\parallel + \frac{e}{2Mc^2}\delta A_\parallel^2 - \frac{M}{2e}|\delta\mathbf{u}_E + \bar{v}_\parallel \frac{\delta\mathbf{B}_\perp}{B}|^2 \\ & - \frac{1}{c}(\delta\mathbf{u}_E + \bar{v}_\parallel \frac{\delta\mathbf{B}_\perp}{B}) \cdot \delta\mathbf{A}_\perp - \frac{e\rho_i^2}{2T_i} \frac{\mu B}{Mc^2} [2\delta A_\perp \nabla_\perp^2 \delta A_\perp - \nabla_\perp \cdot (\delta A_\parallel \nabla_\perp \delta A_\parallel)], \end{aligned} \quad (32)$$

where $\delta\mathbf{u}_E = \frac{c}{B}\mathbf{b} \times \nabla_\perp(\delta\phi - \frac{1}{c}\mathbf{u}_E \cdot \delta\mathbf{A})$. This simpler expression should be used for Eqs. (18)–(19), which appear in Gyrokinetic Vlasov equation, in this limit.

Equations (27) and (32) reduce to those of Ref. 24 in the electrostatic limit. They also reduce to those of Ref. 7 in the absence of \mathbf{u}_E . Here, we note that a coefficient $n/\Omega^2 \propto n/B^2$ appears inside the divergence operator (∇_\perp) in the last two terms of Eqs. (27)–(28), and the 6th and 7th terms on the R.H.S. of Eq. (29). These expressions are more general than those from Refs. 7 and 24 in toroidal geometry where the $1/B^2$ factor appears outside the divergence operator (∇_\perp). Since we assume that $L_B \gg L_p, L_E$ here (and in other references) the resulting modification is minor, quantitatively. However this general expression is physically appealing, since $v_A^2 \propto B^2/n$ and a close relation exists between the polarization density and the vorticity which appears in reduced fluid equations.⁶⁴ This could serve as a useful guideline when one wants to extend the formulation to a more compact confinement device such as NSTX.⁵¹ The 3rd term on the R.H.S. of Eq. (27), $\frac{\delta B_{\parallel k}}{B}N_i$, can be shown to originate from the $\mathbf{E} \times \mathbf{B}$ drift caused by an induction electric field perpendicular to \mathbf{B}_0 , i.e., $\delta\mathbf{E}_{\perp ind} = -\frac{1}{c}\frac{\partial}{\partial t}\delta\mathbf{A}_\perp$.

Now, we discuss the shielding properties of gyrokinetic plasma in the presence of the equilibrium electric field. As widely recognized, representing the polarization drift as a

shielding term in the gyrokinetic Poisson equation² has provided one of the key computational advantages of the gyrokinetic approach. However, there is an important qualitative difference between the Debye shielding and the polarization shielding, in addition to their magnitudes. In Eq. (27), the Debye shielding term on the left-hand side contains both the equilibrium potential and the perturbed potential, while the polarization density involves $\delta\phi$ only. This is because the polarization density is related to the polarization drift via the continuity equation, and the polarization drift occurs only for the time-varying electric field. Here, it is shown that if one uses a simplified definition of a gyrocenter density, a polarization-density-like term associated with Φ appears²⁴. The gyro-averaged gyrocenter density \bar{N}_i is defined by

$$\bar{N}_i \equiv \int \frac{B_{\parallel}^*}{M} d^3\bar{\mathbf{R}} d\bar{v}_{\parallel} d\bar{\theta} d\bar{\mu} F_i \delta^3(\bar{\mathbf{R}} - \mathbf{x} + \bar{\boldsymbol{\rho}}), \quad (33)$$

and this differs slightly from the gyrocenter density N_i , which is defined by

$$N_i \equiv \int \frac{B_{\parallel}^*}{M} d^3\bar{\mathbf{R}} d\bar{v}_{\parallel} d\bar{\theta} d\bar{\mu} F_i \delta^3(\bar{\mathbf{R}} - \mathbf{x}). \quad (34)$$

In computation, it has been the usual practice to use a simplified gyrocenter density \bar{N}_i^0 with an approximate Jacobian of a phase-space volume element B instead,

$$\bar{N}_i^0 \equiv \int \frac{B}{M} d^3\bar{\mathbf{R}} d\bar{v}_{\parallel} d\bar{\theta} d\bar{\mu} F_i \delta^3(\bar{\mathbf{R}} - \mathbf{x} + \bar{\boldsymbol{\rho}}).$$

Then, using the approximation, $B_{\parallel}^* \equiv B[1 + \frac{\mathbf{b}}{\Omega} \cdot \nabla \times (\mathbf{u}_E + v_{\parallel}\mathbf{b})] \simeq B + \frac{1}{\Omega} \nabla_{\perp}^2 \Phi$, one can write Eq. (33) as

$$\bar{N}_i \simeq \bar{N}_i^0 \left(1 + \frac{e\rho_s^2}{T_e} \nabla_{\perp}^2 \Phi \right). \quad (35)$$

Now, the last term looks like the polarization density associated with Φ . This identification might be useful when one tries to perform a long-time simulation with a slowly time varying Φ .^{90–94}

In passing, we remark that it's possible to formulate gyrokinetics in such a way that the polarization drift appears in the gyrokinetic Vlasov equation⁹⁵ rather than as a polarization density in the gyrokinetic Poisson equation. In Ref. 33–35, the polarization drift associated with a time-varying background appears in the gyrokinetic Vlasov equation, while the polarization density associated with shorter wavelength fluctuations appears in the gyrokinetic Poisson equation as usual. If one tries to describe a transport barrier formation such as an H-mode transition, a distinction between the time-varying background and large-scale

fluctuations becomes rather subtle. In addition, a division between large scale fluctuations and short scale fluctuations seems arbitrary, as noted in previous work.⁶⁵ It's our opinion that treating polarization physics as a shielding term² (i.e., polarization density) has been found to be more efficient for a wide range of numerical applications.³⁹ Other terms which appear in Eqs. (27)-(29) are defined as:

$$\bar{\mathbf{J}}_i \equiv e \int \frac{B_{\parallel}^*}{M} d^3 \bar{\mathbf{R}} d\bar{v}_{\parallel} d\bar{\theta} d\bar{\mu} F_i (\bar{v}_{\parallel} \mathbf{b} + \mathbf{u}_E + \mathbf{c}_{\perp}) \delta^3(\bar{\mathbf{R}} - \mathbf{x} + \bar{\boldsymbol{\rho}}), \quad (36a)$$

$$\mathbf{j}_e \equiv e \int d^3 \mathbf{v} f_e \mathbf{v}(\mathbf{x}, \mathbf{v}, t), \quad (36b)$$

$$J_{i\parallel} \equiv e \int \frac{B_{\parallel}^*}{M} d^3 \bar{\mathbf{R}} d\bar{v}_{\parallel} d\bar{\theta} d\bar{\mu} F_i \bar{v}_{\parallel} \delta^3(\bar{\mathbf{R}} - \mathbf{x}), \quad (36c)$$

$$\Pi_{i\parallel} \equiv \int \frac{B_{\parallel}^*}{M} d^3 \bar{\mathbf{R}} d\bar{v}_{\parallel} d\bar{\theta} d\bar{\mu} F_i \bar{v}_{\parallel}^2 \delta^3(\bar{\mathbf{R}} - \mathbf{x}), \quad (36d)$$

$$P_{\perp} \equiv \int \frac{B_{\parallel}^*}{M} d^3 \bar{\mathbf{R}} d\bar{v}_{\parallel} d\bar{\theta} d\bar{\mu} F_i \bar{\mu} B \delta^3(\bar{\mathbf{R}} - \mathbf{x}). \quad (36e)$$

Since $m_e v_{\parallel} \simeq m_e \bar{v}_{\parallel} + \frac{e}{c} \delta A_{\parallel}$, the collisionless skin depth term $(\frac{\omega_{pe}}{c})^2 \delta A_{\parallel}$ will appear explicitly on the R.H.S. of Eq. (28), if the current density $j_{e\parallel}$ is defined by the moment of \bar{v}_{\parallel} . We can see that the electrons contribute to the collisionless skin depth term dominantly, so it is reasonable to neglect that of the ions in Eq. (25). Turbulence at the scale of the collisionless skin depth was simulated^{96,97} and measured in experiments.⁹⁸

The corresponding energy invariant in this long wavelength limit is

$$\begin{aligned} E = & \frac{1}{8\pi} \int d^3 \mathbf{x} [|\nabla(\Phi + \delta\phi)|^2 + |\mathbf{B}_0 + \delta\mathbf{B}|^2] + \int d^6 \mathbf{z} f_e(\mathbf{z}) \frac{1}{2} m_e \mathbf{v}^2 \\ & + \int d^6 \bar{\mathbf{Z}} F_i \{ \bar{\mu} B + \frac{M}{2} (u_E^2 + \bar{v}_{\parallel}^2) + \frac{\bar{\mu} B}{2\Omega} \mathbf{b} \cdot \bar{\nabla} \times \mathbf{u}_E - \frac{1}{c} (\langle \delta \mathbf{A} \rangle \cdot \mathbf{u}^* + \langle \delta \mathbf{A} \cdot \mathbf{c}_{\perp} \rangle) \\ & + \frac{e^2}{2Mc^2} (\delta A_{\parallel})^2 + \frac{e^2 \rho_i^2}{2T_i} [(\nabla_{\perp} \delta\phi)^2 - \frac{1}{c^2} (\bar{v}_{\parallel} \nabla_{\perp} \delta A_{\parallel} + \nabla_{\perp} (\delta \mathbf{A}_{\perp} \cdot \mathbf{u}_E))^2 \\ & - \frac{\mu B}{Mc^2} (2\delta A_{\perp} \nabla_{\perp}^2 \delta A_{\perp} - \nabla_{\perp} \cdot (\delta A_{\parallel} \nabla_{\perp} \delta A_{\parallel})) \\ & + \frac{e}{cB} \mathbf{b} \times (\bar{v}_{\parallel} \nabla_{\perp} \delta A_{\parallel} + \nabla_{\perp} (\delta \mathbf{A}_{\perp} \cdot \mathbf{u}_E)) \cdot \delta \mathbf{A} \}. \end{aligned} \quad (37)$$

B. Arbitrary wavelength Expressions for Maxwellian F in $\bar{\mu}$:

Now, we consider arbitrary values of $k_{\perp} \rho_i$, but assume that F is Maxwellian in $\bar{\mu}$ (i.e., $F \propto \exp(-\bar{\mu} B/T)$). Note that this approximation is a bit more general than a linearization

with a Maxwellian equilibrium distribution function F_0 , but inclusive of the latter. In this limit, after expanding in Fourier components, Equations (24) and (25) become

$$\begin{aligned}
\nabla^2(\Phi + \delta\phi) = & -4\pi e\{\bar{N}_i - n_e - \sum_k \exp(i\mathbf{k} \cdot \mathbf{x}) \\
& [\frac{1 - \Gamma_0}{T_i}(eN_i(\delta\phi_k - \frac{1}{c}\mathbf{u}_{E0} \cdot \delta\mathbf{A}_k) - \frac{1}{c}J_{i\parallel}\delta A_{\parallel k}) + (\Gamma_1 - \Gamma_0)\frac{\delta B_{\parallel k}}{B}N_i \\
& + \frac{e}{T_i}(\Gamma_1 - \Gamma_0)i\rho_i^2\mathbf{k}_\perp \cdot (e(\nabla_\perp N_i)(\delta\phi_k - \frac{1}{c}\mathbf{u}_{E0} \cdot \delta\mathbf{A}_k) - \frac{1}{c}(\nabla_\perp J_{i\parallel})\delta A_{\parallel k}) \\
& + \frac{1}{B}(2\Gamma_1 + k_\perp^2\rho_i^2(\Gamma_0 + \Gamma_2 - 2\Gamma_1))(\mathbf{b} \times \nabla_\perp N_i) \cdot \delta\mathbf{A}_{\perp k}\}], \tag{38}
\end{aligned}$$

$$\begin{aligned}
\nabla^2(A_{\parallel} + \delta A_{\parallel}) = & -\frac{4\pi}{c}\{\bar{J}_{i\parallel} - j_{e\parallel} - \sum_k \exp(i\mathbf{k} \cdot \mathbf{x}) \\
& [\frac{e}{T_i}(1 - \Gamma_0)(J_{i\parallel}(\delta\phi_k - \frac{1}{c}\mathbf{u}_{E0} \cdot \delta\mathbf{A}_k) - \frac{e}{c}\Pi_{i\parallel}\delta A_{\parallel k}) + (\Gamma_1 - \Gamma_0)\frac{\delta B_{\parallel k}}{B}J_{i\parallel} \\
& + \frac{e}{T_i}(\Gamma_1 - \Gamma_0)i\rho_i^2\mathbf{k}_\perp \cdot ((\nabla_\perp J_{i\parallel})(\delta\phi_k - \frac{1}{c}\mathbf{u}_{E0} \cdot \delta\mathbf{A}_k) - \frac{e}{c}(\nabla_\perp \Pi_{i\parallel})\delta A_{\parallel k}) \\
& + \frac{1}{B}(2\Gamma_1 + k_\perp^2\rho_i^2(\Gamma_0 + \Gamma_2 - 2\Gamma_1))(\mathbf{b} \times \nabla_\perp J_{i\parallel}) \cdot \delta\mathbf{A}_{\perp k}\}], \tag{39}
\end{aligned}$$

$$\begin{aligned}
\nabla^2(\mathbf{A}_\perp + \delta\mathbf{A}_\perp) = & -\frac{4\pi}{c}\{\bar{\mathbf{J}}_{i\perp} + \bar{\mathbf{J}}_{iE} - \mathbf{j}_{e\perp} - \mathbf{j}_{eE} - \sum_k \exp(i\mathbf{k} \cdot \mathbf{x}) \\
& [\frac{e}{T_i}(1 - \Gamma_0)(eN_i(\delta\phi_k - \frac{1}{c}\mathbf{u}_{E0} \cdot \delta\mathbf{A}_k) - \frac{1}{c}J_{i\parallel}\delta A_{\parallel k})\mathbf{u}_{E0} + (\Gamma_1 - \Gamma_0)\frac{\delta B_{\parallel k}}{B}\mathbf{j}_{E0} \\
& + \frac{e}{T_i}(\Gamma_1 - \Gamma_0)i\rho_i^2\mathbf{k}_\perp \cdot (e(\nabla_\perp N_i)(\delta\phi_k - \frac{1}{c}\mathbf{u}_{E0} \cdot \delta\mathbf{A}_k) - \frac{1}{c}(\nabla_\perp J_{i\parallel})\delta A_{\parallel k})\mathbf{u}_{E0} \\
& + \frac{c}{B}(\Gamma_1 - \Gamma_0)i\mathbf{b} \times \mathbf{k}_\perp (eN_i(\delta\phi_k - \frac{1}{c}\mathbf{u}_{E0} \cdot \delta\mathbf{A}_k - \frac{1}{c}J_{i\parallel}\delta A_{\parallel})) \\
& + \frac{e^2}{Mc}N_i(2\Gamma_1 + k_\perp^2\rho_i^2(\Gamma_0 + \Gamma_2 - 2\Gamma_1))\delta\mathbf{A}_{\perp k} + \frac{2cT_i}{B^2}(\Gamma_1 - \Gamma_0)\delta B_{\parallel k}\mathbf{b} \times \nabla_\perp N_i \\
& + \frac{c}{B}(1 - \Gamma_0 + \Gamma_1)\mathbf{b} \times (e(\nabla_\perp N_i)(\delta\phi_k - \frac{1}{c}\mathbf{u}_{E0} \cdot \delta\mathbf{A}_k) - \frac{1}{c}(\nabla_\perp J_{i\parallel})\delta A_{\parallel k}) \\
& - \frac{2c}{B}(\Gamma_1 + k_\perp^2\rho_i^2(\Gamma_1 - \Gamma_0))\hat{\mathbf{k}}_\perp \cdot (e(\nabla_\perp N_i)(\delta\phi_k - \frac{1}{c}\mathbf{u}_{E0} \cdot \delta\mathbf{A}_k) - \frac{1}{c}(\nabla_\perp J_{i\parallel})\delta A_{\parallel k})\delta\hat{\mathbf{A}}_\perp \\
& + \frac{e}{B}(2\Gamma_1 + k_\perp^2\rho_i^2(\Gamma_0 + \Gamma_2 - 2\Gamma_1))(\mathbf{b} \times \nabla_\perp N_i) \cdot \delta\mathbf{A}_{\perp k}\mathbf{u}_{E0} \\
& + \frac{cT_i}{B^2}k_\perp^2\rho_i^2(3\Gamma_0 + \Gamma_2 - 4\Gamma_1)i\mathbf{k}_\perp \cdot (\nabla_\perp N_i)\delta\mathbf{A}_{\perp k}\}]. \tag{40}
\end{aligned}$$

Once again, Eqs. (38) and (39) can be combined in a covariant fashion with a two-vector notation,

$$\begin{aligned}\nabla^2(A^\alpha + \delta A^\alpha) = & -\frac{4\pi}{c}\{\bar{J}_i^\alpha - j_e^\alpha - \sum_k \exp(i\mathbf{k} \cdot \mathbf{x})[\frac{e}{T_i}(1 - \Gamma_0)\Pi_i^{\prime\alpha\beta}\delta A'_{\beta k} \\ & + (\Gamma_1 - \Gamma_0)\frac{\delta B_{\parallel k}}{B}J_i^\alpha + \frac{e}{T_i}(\Gamma_1 - \Gamma_0)i\rho_i^2\mathbf{k}_\perp \cdot (\nabla_\perp \Pi_i^{\prime\alpha\beta})\delta A'_{\beta k} \\ & + \frac{1}{B}(2\Gamma_1 + k_\perp^2\rho_i^2(\Gamma_0 + \Gamma_2 - 2\Gamma_1))(\mathbf{b} \times \nabla_\perp J_i^\alpha) \cdot \delta \mathbf{A}_{\perp k}]\},\end{aligned}\quad (41)$$

Eq. (40) can also be written as:

$$\begin{aligned}\nabla^2(\mathbf{A}_\perp + \delta \mathbf{A}_\perp) = & -\frac{4\pi}{c}\{\bar{\mathbf{J}}_{i\perp} + \bar{\mathbf{J}}_{iE} - \mathbf{j}_{e\perp} - \mathbf{j}_{eE} - \sum_k \exp(i\mathbf{k} \cdot \mathbf{x})[\frac{e}{cT_i}((1 - \Gamma_0)J_i^\alpha \delta A'_\alpha \\ & + (\Gamma_1 - \Gamma_0)\frac{\delta B_{\parallel k}}{B}\mathbf{j}_{E0} + (\Gamma_1 - \Gamma_0)i\rho_i^2\mathbf{k}_\perp \cdot (\nabla_\perp J_i^\alpha)\delta A'_\alpha)\mathbf{u}_{E0} \\ & + \frac{c}{B}(\Gamma_1 - \Gamma_0)i\mathbf{b} \times \mathbf{k}_\perp J_i^\alpha \delta A'_\alpha \\ & + \frac{e^2}{Mc}N_i(2\Gamma_1 + k_\perp^2\rho_i^2(\Gamma_0 + \Gamma_2 - 2\Gamma_1))\delta \mathbf{A}_{\perp k} + \frac{2cT_i}{B^2}(\Gamma_1 - \Gamma_0)\delta B_{\parallel k}\mathbf{b} \times \nabla_\perp N_i \\ & + \frac{c}{B}(1 - \Gamma_0 + \Gamma_1)\mathbf{b} \times (\nabla_\perp J_i^\alpha)\delta A'_\alpha - \frac{2c}{B}(\Gamma_1 + k_\perp^2\rho_i^2(\Gamma_1 - \Gamma_0))\hat{\mathbf{k}}_\perp \cdot (\nabla_\perp J_i^\alpha)\delta A'_\alpha \delta \hat{\mathbf{A}}_\perp \\ & + \frac{e}{B}(2\Gamma_1 + k_\perp^2\rho_i^2(\Gamma_0 + \Gamma_2 - 2\Gamma_1))(\mathbf{b} \times \nabla_\perp N_i) \cdot \delta \mathbf{A}_{\perp k}\mathbf{u}_{E0} \\ & + \frac{cT_i}{B^2}k_\perp^2\rho_i^2(3\Gamma_0 + \Gamma_2 - 4\Gamma_1)i\mathbf{k}_\perp \cdot (\nabla_\perp N_i)\delta \mathbf{A}_{\perp k}]\}.\end{aligned}\quad (42)$$

where $\hat{\mathbf{k}}_\perp = \mathbf{k}_\perp/|\mathbf{k}_\perp|$, $\delta \hat{\mathbf{A}}_\perp = \delta \mathbf{A}_\perp/|\delta \mathbf{A}_\perp|$, $\Gamma_n(b) = I_n(b)e^{-b}$, and I_n is the modified Bessel function of order n , where $b = k_\perp^2\rho_i^2$. The corresponding energy invariant can be written as

$$\begin{aligned}E = & \frac{1}{8\pi} \int d^3\mathbf{x} [|\nabla(\Phi + \delta\phi)|^2 + |\mathbf{B} + \delta\mathbf{B}|^2] + \int d^6\mathbf{z} f_e(\mathbf{z}) \frac{1}{2} m_e (v_\parallel \mathbf{b} + \mathbf{c}_\perp + \mathbf{u}_E)^2 \\ & + \int d^6\bar{\mathbf{z}} F_i [\bar{\mu}B + \frac{M}{2}(u_E^2 + \bar{v}_\parallel^2) + \frac{\bar{\mu}B}{2\Omega} \mathbf{b} \cdot \bar{\nabla} \times \mathbf{u}_E - \frac{1}{c}(\langle \delta \mathbf{A} \rangle \cdot \mathbf{u}^* + \langle \delta \mathbf{A} \cdot \mathbf{c}_\perp \rangle)] \\ & + \frac{e}{2T_i} \sum_k (1 - \Gamma_0) [en_0 |\delta\phi_k|^2 - \frac{1}{c^2} (e\pi_{i0\parallel} |\delta A_{\parallel k}|^2 + \mathbf{u}_{E0} \cdot \delta \mathbf{A}_{-k} (\mathbf{j}_{E0} \cdot \delta \mathbf{A}_k + 2j_{i0\parallel} \delta A_{\parallel k}))] \\ & + \frac{e^2 n_0}{2Mc^2} \sum_k [|\delta A_{\parallel k}|^2 + (2\Gamma_1 + k_\perp^2\rho_i^2(\Gamma_0 + \Gamma_2 - 2\Gamma_1)) \frac{|\delta B_{\parallel k}|^2}{k_\perp^2}] \\ & - \frac{1}{cB} \sum_k (\Gamma_0 - \Gamma_1) \delta B_{\parallel -k} (\mathbf{j}_{E0} \cdot \delta \mathbf{A}_k + J_{i\parallel} \delta A_{\parallel k}),\end{aligned}\quad (43)$$

with

$$n_0 \equiv \int \frac{B_{\parallel}^*}{M} d\bar{v}_{\parallel} d\bar{\theta} d\bar{\mu} F_{i0}, \quad (44a)$$

$$j_{i0\parallel} \equiv e \int \frac{B_{\parallel}^*}{M} d\bar{v}_{\parallel} d\bar{\theta} d\bar{\mu} F_{i0} \bar{v}_{\parallel}, \quad (44b)$$

$$\pi_{i0\parallel} \equiv \int \frac{B_{\parallel}^*}{M} d\bar{v}_{\parallel} d\bar{\theta} d\bar{\mu} F_{i0} \bar{v}_{\parallel}^2. \quad (44c)$$

Once again, Eqs. (41) and (43) are the fully electromagnetic generalization of the electrostatic results in Ref. 24. The symplectic derivation, which preserves Poincare invariance⁹⁹ automatically ensures the various conservation laws.

TABLE I: Progress in Nonlinear Gyrokinetic Formulations in a Torus with Flows via Lie Perturbation Method

	Brizard '95 ⁴⁸	Hahm '96 ²⁴	Qin, et al., '06 ³³ , '07 ³⁴	Kawamura and Fukuyama '08 ³⁵	This work
Characteristics of Fluctuations appearing in Gyrokinetic Vlasov Equation	electromagnetic	electrostatic	electromagnetic	electromagnetic	electromagnetic
Explicit Evaluation* of Gyrokinetic Poisson's Equation		for electrostatic fluctuations	for electrostatic fluctuations	for electrostatic fluctuations	for electromagnetic fluctuations
Explicit Evaluation* of Gyrokinetic Parallel Ampère's law					for electromagnetic fluctuations
Explicit Evaluation* of Gyrokinetic Perpendicular Ampère's law					for electromagnetic fluctuations
Expression for total energy invariant		yes			yes

Vacant slots indicate that the specific item has not been performed in that particular paper.

* Explicit evaluation here means performing the integration over velocity space following the pull back transformation, for quantities such as polarization density and magnetization terms.

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