

Linear Gain for the Microbunching Instability in an RF Compressor

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I. INTRODUCTION

Velocity (or rf) compression has been suggested as a technique for bunch compression [4] complementary to the more established technique involving magnetic chicanes and represents an important research item being investigated at the SPARC test facility [1]. One of the aspects of this technique still not sufficiently understood is its possible impact on the microbunching instability. The purpose of this report is to present the analytical framework for investigating this instability in rf compressors. We use methods similar to those successfully applied to magnetic compressors [2, 3] and derive some integral equations yielding the gain for the instability in linear approximation. The focus here is on the derivation of the relevant equations. Although examples of solutions to these equations are provided we defer a more comprehensive discussion of their implication to a future report. The present study is part of a larger effort for a more comprehensive investigation that eventually will include macroparticle simulations and experiments.

II. SINGLE-PARTICLE DYNAMICS THROUGH AN RF COMPRESSOR

A. Equations of motion

The longitudinal motion of an electron in a travelling wave structure is described by the Hamiltonian $H = \sqrt{m^2c^4 + p_s^2c^2} - e\phi(s, t)$ where $-e$ is the electron charge and $\phi = (E_0/k_{\text{rf}}) \cos(k_{\text{rf}}s - \omega_{\text{rf}}t + \psi_0)$ the electric potential, yielding the longitudinal electric field $E_s = -\partial\phi/\partial s = E_0 \sin(k_{\text{rf}}s - \omega_{\text{rf}}t + \psi_0)$. Our notation uses the coordinate s for the longitudinal direction along the RF structure as we want to reserve z for other uses (see below).

The canonical equations are readily written

$$\frac{ds}{dt} = \frac{p_s c^2}{\sqrt{m^2c^4 + p_s^2c^2}}, \quad (1)$$

$$\frac{dp_s}{dt} = -eE_0 \sin \psi, \quad (2)$$

where we have introduced the phase $\psi = k_{\text{rf}}s - \omega_{\text{rf}}t + \psi_0$.

Next, we are interested in turning s into the independent variable while promoting t to the status of dynamical variable. Also, we select the relativistic factor $\gamma = \sqrt{m^2c^4 + p_s^2c^2}/mc^2$ as the second dynamical variable replacing p_s . The equations for $t(s)$ and $\gamma(s)$ are easily written. The equation for $t(s)$ follows immediately from (1)

$$\frac{dt}{ds} = \frac{\sqrt{m^2c^4 + p_s^2c^2}}{p_s c^2} = \frac{1}{c} \frac{\gamma}{\sqrt{\gamma^2 - 1}}, \quad (3)$$

while

$$\frac{d\gamma}{ds} = \frac{d\gamma}{dp_s} \frac{dp_s}{dt} \frac{dt}{ds} = \frac{dp_s}{dt} \frac{1}{mc^2} = -\frac{eE_0}{mc^2} \sin \psi. \quad (4)$$

We tag a particular solution of (3) and (4) as the orbit of the reference particle, which we denote as (t_r, γ_r) . The orbit of any other particle can be described in terms of the deviation variables $\Delta t = t - t_r$ and $\Delta \gamma = \gamma - \gamma_r$, where (t, γ) are also solutions of (3) and (4). The space separation Δz between an electron and the reference particle is $\Delta z \simeq -c\beta(s)\Delta t$, where $c\beta(s)$ is the beam velocity. The negative sign results from the convention that a particle in the head of the bunch has $\Delta z > 0$. In the following for the only purpose of determining Δz from Δt , we will assume that the beam is sufficiently relativistic that $\beta \simeq 1$ and effectively $\Delta z = -c\Delta t$.

Having introduced the variable $z = -ct$, and the dimensionless parameter $\alpha = eE_0/k_{\text{rf}}mc^2$, we rewrite the equations of motion (3) and (4) as

$$\frac{dz}{ds} = -\frac{\gamma}{\sqrt{\gamma^2 - 1}}, \quad (5)$$

$$\frac{d\gamma}{ds} = -\alpha k_{\text{rf}} \sin(k_{\text{rf}}s + k_{\text{rf}}z + \psi_0). \quad (6)$$

Similarly, we write the equations of motion for the deviation variables ($\Delta z = z - z_r$, $\Delta \gamma = \gamma - \gamma_r$) as

$$\frac{d\Delta z}{ds} = \frac{\gamma_r}{\sqrt{\gamma_r^2 - 1}} - \frac{\gamma_r + \Delta \gamma}{\sqrt{(\gamma_r + \Delta \gamma)^2 - 1}}, \quad (7)$$

$$\frac{d\Delta \gamma}{ds} = \alpha k_{\text{rf}} \sin(k_{\text{rf}}s + k_{\text{rf}}z_r + \psi_0) - \alpha k_{\text{rf}} \sin(k_{\text{rf}}s + k_{\text{rf}}(z_r + \Delta z) + \psi_0). \quad (8)$$

Through first order

$$\frac{d\Delta z}{ds} = \frac{\Delta \gamma}{[\gamma_r^2(s) - 1]^{3/2}}, \quad (9)$$

$$\frac{d\Delta \gamma}{ds} = -\alpha k_{\text{rf}}^2 \cos(k_{\text{rf}}s + k_{\text{rf}}z_r(s) + \psi_0) \Delta z. \quad (10)$$

The solution of the above linear system can be expressed in terms of the transfer matrix $\mathbf{M}$: $\mathbf{x}(s) = \mathbf{M}\mathbf{x}_0$, where $\mathbf{x}(s) = (\Delta z(s), \Delta \gamma(s))$ and $\mathbf{x}_0 = \mathbf{x}(s_0)$. The matrix $\mathbf{M}$ obeys

$$\frac{d\mathbf{M}}{ds} = \mathbf{A}\mathbf{M} \quad (11)$$

with initial condition $\mathbf{M}(s_0) = \mathbf{1}$ and matrix $\mathbf{A}$ defined by

$$\mathbf{A}(s) = \begin{pmatrix} 0 & [\gamma_r^2(s) - 1]^{-3/2} \\ -\alpha k_{\text{rf}}^2 \cos(k_{\text{rf}}s + k_{\text{rf}}z_r(s) + \psi_0) & 0 \end{pmatrix}. \quad (12)$$

Because Eq.'s (7) and (8) are canonical (one can easily verify they can be derived from an effective Hamiltonian) the transformation $\mathbf{M}$ is symplectic.

B. Approximate analytical solution

Exact solutions for the orbit of the reference particle (3) and (4) as well for the the linear equations (9) and (10) in general can be determined only numerically. However, having approximate analytical solutions at our disposal may still be useful.

Write $z = -s + \zeta$, and assume $\gamma \gg 1$ and $\zeta k_{\text{rf}} \ll 1$. We can then Taylor expand Eq.'s (5) and (6) through first order:

$$\frac{d\zeta}{ds} \simeq -\frac{1}{2\gamma^2}, \quad (13)$$

$$\frac{d\gamma}{ds} = -\alpha k_{\text{rf}} \sin(k_{\text{rf}}\zeta + \psi_0) \simeq -\alpha k_{\text{rf}} \sin \psi_0 - \zeta \alpha k_{\text{rf}}^2 \cos \psi_0. \quad (14)$$

Next, introduce the new variable $\hat{\zeta} = \zeta + k_{\text{rf}}^{-1} \tan \psi_0$ in place of ζ . The resulting equations of motion

$$\frac{d\hat{\zeta}}{ds} \simeq -\frac{1}{2\gamma^2}, \quad (15)$$

$$\frac{d\gamma}{ds} = -\hat{\zeta} \alpha k_{\text{rf}}^2 \cos \psi_0, \quad (16)$$

can be derived by an effective Hamiltonian

$$\mathcal{H} = \frac{1}{2\gamma} + \frac{\alpha k_{\text{rf}}^2 \cos \psi_0}{2} \hat{\zeta}^2. \quad (17)$$

Being s -independent, $\mathcal{H}$ is also an invariant. Let us denote $\gamma_0 = \gamma(s=0)$ and $\hat{\zeta}_0 = \hat{\zeta}(s=0)$. Having defined the constant $\bar{\gamma}$ as

$$\frac{1}{2\bar{\gamma}} = \frac{1}{2\gamma_0} + \frac{\alpha k_{\text{rf}}^2 \cos \psi_0}{2} \hat{\zeta}_0^2, \quad (18)$$

and used the invariance of the Hamiltonian we have

$$\hat{\zeta} = \frac{1}{k_{\text{rf}} \sqrt{\alpha \cos \psi_0}} \left(\frac{1}{\bar{\gamma}} - \frac{1}{\gamma} \right)^{1/2}. \quad (19)$$

The above expression can then be inserted in (16) and the equation integrated to determine γ implicitly as a function of s :

$$\left[\sqrt{x(x-1)} + \log(\sqrt{x} + \sqrt{x-1}) \right] \Big|_{x=\gamma_0/\bar{\gamma}}^{x=\gamma/\bar{\gamma}} = -s k_{\text{rf}} \left(\frac{\alpha \cos \psi_0}{\bar{\gamma}^3} \right)^{1/2}. \quad (20)$$

A reasonable approximation of the LHS in the above expression for $x > 1$ up to $x \simeq 3$ is given by

$$\sqrt{x(x-1)} + \log(\sqrt{x} + \sqrt{x-1}) \simeq 2(x-1)^{1/2} + \frac{1}{3}(x-1)^{3/2}. \quad (21)$$

Roughly, this means that with this approximation we can follow the beam up to an energy that about three times larger than the initial energy. This approximation can be used in (20) to find γ as an explicit function of s . The end result is

$$\gamma(s) \simeq \bar{\gamma} \left(4A^{-2/3} + A^{2/3} - 3 \right) \quad (22)$$

with $A = (\sqrt{32 + a^2} - a)/2$ and $a = 3s\hat{k} - [6(x-1)^{1/2} + (x-1)^{3/2}]_{x=\gamma_0/\bar{\gamma}}$, where $\hat{k} = k_{\text{rf}} \sqrt{\alpha \cos \psi_0 / \bar{\gamma}^3}$.

In a similar way we can now write an expression for $\hat{\zeta}$. Using again the expression for the Hamiltonian and expanding through second order in $\hat{\zeta}$:

$$\frac{1}{2\gamma^2} = \frac{1}{2} \left(\frac{1}{\bar{\gamma}} - \alpha k_{\text{rf}}^2 \cos \psi_0 \hat{\zeta}^2 \right)^2 \simeq \frac{1}{2} \left(\frac{1}{\bar{\gamma}^2} - \frac{2\alpha k_{\text{rf}}^2 \cos \psi_0 \hat{\zeta}^2}{\bar{\gamma}} \right). \quad (23)$$

We insert in (15) and integrate to obtain

$$\hat{\zeta}(s) = -\frac{1}{\sqrt{2\hat{k}\bar{\gamma}^2}} \tanh \left(\frac{s\hat{k}}{\sqrt{2}} - \text{arctanh}(\sqrt{2\hat{k}\bar{\gamma}^2} \hat{\zeta}_0) \right). \quad (24)$$

Having determined the solution $\hat{\zeta} = \hat{\zeta}(\hat{\zeta}_0, \gamma_0; s)$ and $\gamma = \gamma(\hat{\zeta}_0, \gamma_0; s)$ with initial conditions $(\hat{\zeta}_0, \gamma_0)$, the linear map around a reference orbit can be determined by calculating the Jacobian matrix

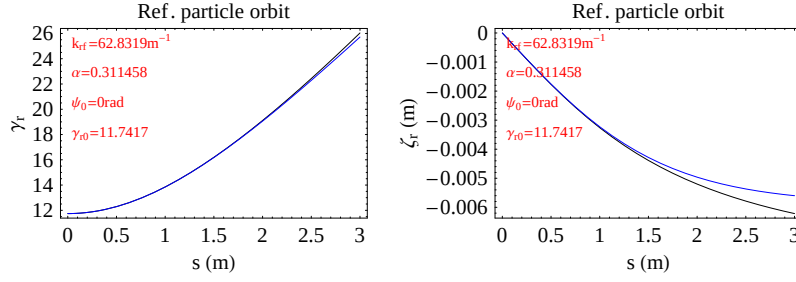


FIG. 1: Reference orbit. Exact numerical solutions (black lines) and approximate analytical (blue lines) solutions.

$$\mathbf{M} = \begin{pmatrix} \frac{\partial \hat{\zeta}}{\partial \hat{\zeta}_0} & \frac{\partial \hat{\zeta}}{\partial \gamma_0} \\ \frac{\partial \gamma}{\partial \hat{\zeta}_0} & \frac{\partial \gamma}{\partial \gamma_0} \end{pmatrix}. \quad (25)$$

with the derivatives evaluated at the initial conditions for the reference orbit $(\hat{\zeta}_0, \gamma_0) = (\hat{\zeta}_{r0}, \gamma_{r0})$. These derivatives can be carried out using a symbolic manipulation program starting from (22) and (24). Manageable expressions are obtained when the initial phase vanishes ($\psi_0 = 0$) and we set $\hat{\zeta}_{r0} = 0$ at $s = 0$. In particular, using $\hat{k} = k_{\text{rf}} \sqrt{\alpha/\gamma^3}$, and the following definition $B = (32 + 9s^2\hat{k}^2)^{1/2} - 3\hat{k}s$ we have

$$M_{11} \simeq \frac{1}{\cosh^2(\hat{k}s/\sqrt{2})}, \quad (26)$$

$$M_{12} \simeq \frac{3s}{4\gamma_{r0}^3 \cosh^2(\hat{k}s/\sqrt{2})} + \frac{\tanh(\hat{k}s/\sqrt{2})}{2\sqrt{2}\gamma_{r0}^3 \hat{k}}, \quad (27)$$

$$M_{21} \simeq \frac{8\hat{k}\gamma_{r0}^3}{(32 + 9s^2\hat{k}^2)^{1/2}} \left(\frac{B^{2/3}}{2^{5/3}} - \frac{2^{5/3}}{B^{2/3}} \right), \quad (28)$$

$$M_{22} \simeq \frac{8\sqrt{2}}{(32 + 9s^2\hat{k}^2)^{1/2}} \left(\frac{B^{1/3}}{2^{5/6}} + \frac{2^{5/6}}{B^{1/3}} \right) - 3. \quad (29)$$

Examples of solutions of Eq.'s (5) and (6) for the reference orbit and (12) for the transfer matrix are reported in Fig.'s 1 and 2. The exact numerical solutions are compared to the approximate analytical expressions worked out in this section. Noticing that in Fig. 2 the two bottom pictures appear to provide a relatively better approximation of the exact solution suggests that perhaps better analytical approximations for M_{11} and M_{12} could be obtained by making use of the relationships $dM_{21}/ds = A_{21}M_{11}$ and $dM_{22}/ds = A_{21}M_{12}$ implied by (11).

III. GAIN CURVE: UNIFORM BEAM

Consider a beam propagating through a drift followed by a RF compressor. The entrance of the drift is positioned at $s = s_0$. We start by studying the model of a beam with longitudinal uniform unperturbed density. For pedagogical reasons in the next two subsections we will consider both the case of a flat-top beam with periodic boundary conditions and a coasting beam. (The two models are essentially the same, but we found it useful to spell out both derivations to straighten out issues of normalization that can be a nuisance when trying to make comparisons with simulations and also to make contact with the case of a beam with gaussian density treated in Sec. IV in the limit of an infinitely long bunch).

A. Flat-top beam with periodic boundary conditions

The dynamical variables in vector notation are $\mathbf{x} = (\Delta z, \Delta\gamma)$. With change of notation from the previous section: from now on we will use z to mean Δz and denote $\Delta\gamma$ with p , i.e. $\mathbf{x} = (z, p)$. In the absence of collective effects the unperturbed dynamics in linear approximation is described by the transfer matrix

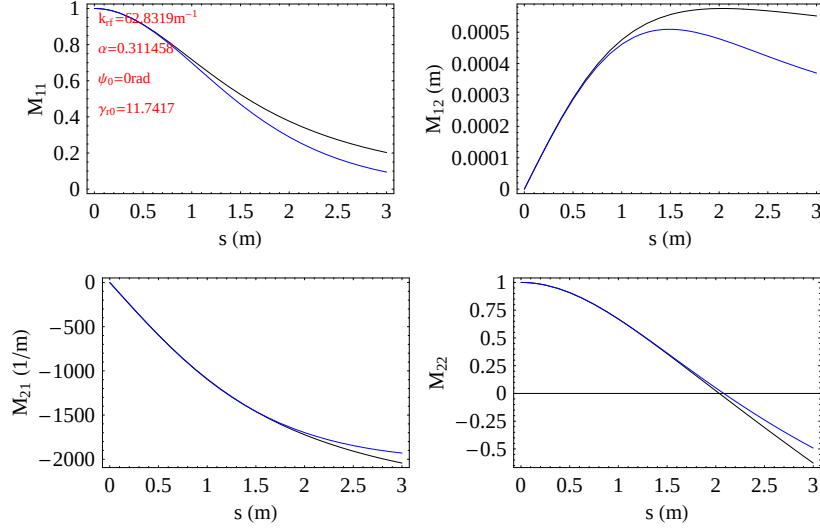


FIG. 2: Entries of the transfer matrix $\mathbf{M}$. Exact numerical solutions (black lines) and approximate analytical solutions (blue lines).

$$\mathbf{M}(s_0 \rightarrow s) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}. \quad (30)$$

In the following we will use $\mathbf{M}$, $\mathbf{M}(s)$, or $\mathbf{M}(s_0 \rightarrow s)$ interchangeably to denote the transfer map from $s = s_0$ to s and use the notation $\mathbf{M}(s' \rightarrow s)$ to represent the matrix from $s' \geq s_0$ to $s \geq s'$. The transfer map $\mathbf{M}$ will be a combination of the map for a drift and the map for the RF compressor discussed in the previous section. In particular, for large γ the portion of transfer map through a drift reads

$$\mathbf{M}_{\text{drift}}(s_0 \rightarrow s) = \begin{pmatrix} 1 & \gamma_r^{-3} \\ 0 & 1 \end{pmatrix}. \quad (31)$$

We assume that the beam distribution $f(\mathbf{x}_0; s_0) = f_0(\mathbf{x}_0; s_0) + f_1(\mathbf{x}_0; s_0)$ at $s = s_0$ consists of a uniform zero-order smooth density uniform in z and gaussian in p with a chirp h

$$f_0(\mathbf{x}_0; s_0) = \frac{1}{L_{b0}\sqrt{2\pi}\sigma_p} e^{-(p_0 - h z_0)^2 / 2\sigma_p^2} \quad (32)$$

and a first-order perturbation $f_1(\mathbf{x}_0; s_0)$. The bunch has initial length L_{b0} and contains N_b particles; for $f(\mathbf{x}; s)$ we assume periodic boundary conditions in z and normalization to unity.

Consider the evolution of the unperturbed beam density first. The beam density function $f_0(\mathbf{x}_s; s)$ at s is related to the beam density at $f_0(\mathbf{x}_0; s_0)$ at s_0 by $f(\mathbf{x}_s, s) = f(\mathbf{M}^{-1}\mathbf{x}_s; s_0)$. The (normalized) charge density evolves from $\rho(z_0; s_0) = \int_{-\infty}^{\infty} dp_0 f_0(\mathbf{x}_0; s_0) = 1/L_{b0}$ at $s = s_0$ to

$$\begin{aligned}
\rho(z_s; s) &= \int_{-\infty}^{\infty} f(\mathbf{x}_s, s) dp_s \\
&= \frac{1}{L_{b0} \sqrt{2\pi\sigma_p}} \int_{-\infty}^{\infty} e^{-[(M_{21}^{-1}z_s + M_{22}^{-1}p_s) - h(M_{11}^{-1}z_s + M_{12}^{-1}p_s)]^2 / 2\sigma_p^2} dp_s \\
&= \frac{1}{L_{b0} \sqrt{2\pi\sigma_p}} \int_{-\infty}^{\infty} e^{-[z_s(M_{21}^{-1} - hM_{11}^{-1}) + p_s(M_{22}^{-1} - hM_{12}^{-1})]^2 / 2\sigma_p^2} dp_s \\
&= \frac{1}{L_{b0} \sqrt{2\pi\sigma_p}} \int_{-\infty}^{\infty} e^{-[p_s(M_{22}^{-1} - hM_{12}^{-1})]^2 / 2\sigma_p^2} dp_s \\
&= \frac{C}{L_{b0}},
\end{aligned} \tag{33}$$

with the coefficient $C = C(s)$ given by

$$C = \frac{1}{M_{22}^{-1}(s) - hM_{12}^{-1}(s)} = \frac{1}{M_{11}(s) + hM_{12}(s)}, \tag{35}$$

which clearly identifies $C(s)$ as a compression factor. In the above equation the second equality follows from the symplectic properties of the matrix $\mathbf{M}$. The length of the beam at s is $L_b = L_b(s) = L_{b0}/C(s)$.

Define the Fourier integrals of the charge density function at s

$$\tilde{\rho}(k; s) = \frac{1}{L_b} \int_{-L_b/2}^{L_b/2} dz_s e^{-ikz_s} \int_{-\infty}^{\infty} dp f(\mathbf{x}_s; s). \tag{36}$$

The effect of collective forces described by the impedance $Z(k; s)$ (to be specified later) is to change the particle energy according to

$$\frac{dp}{ds} \equiv F(\tilde{\rho}, z_s; s) = -\frac{e^2 N_b}{mc} \sum_{n=-\infty}^{\infty} e^{ikz_s} Z(k; s) \tilde{\rho}(k; s) \tag{37}$$

where the sum $\sum$ is over the discrete set of frequencies $k = k_n = 2\pi n/L_b$, due to the periodic boundary conditions.

The starting point of our analysis is the Vlasov equation expressed in the form of Eq. (10) in the Huang-Kim paper [3]:

$$f(\mathbf{x}_s; s) = f(\mathbf{x}_0; s_0) - \int_{s_0}^s ds' F(\tilde{\rho}; z_{s'}; s') \frac{\partial f(\mathbf{x}_{s'}; s')}{\partial p_{s'}}. \tag{38}$$

Specialized to the linear approximation the above equation reads

$$f_1(\mathbf{x}_s; s) = f_1(\mathbf{x}_0; s_0) - \int_{s_0}^s ds' F(\tilde{\rho}_1; z_{s'}; s') \frac{\partial f_0(\mathbf{x}_{s'}; s')}{\partial p_{s'}}, \tag{39}$$

where the collective force F will depend on the FT $\tilde{\rho}_1$ of the first-order perturbation to the beam density. Next, we want to take the FT of both sides of the above equation. To this end it is convenient to think of both sides as functions of the dynamical variables at current time s :

$$f_1(\mathbf{x}_s; s) = f_1(\mathbf{x}_0(\mathbf{x}_s); s_0) - \int_0^s ds' F(\tilde{\rho}_1; z_{s'}; s') \Big|_{z_{s'}=z_{s'}(\mathbf{x}_s)} \frac{\partial f_0(\mathbf{x}_{s'}; s')}{\partial p_{s'}} \Big|_{\mathbf{x}_{s'}=\mathbf{x}_{s'}(\mathbf{x}_s)}. \tag{40}$$

where $\mathbf{x}_0(\mathbf{x}_s)$ denotes the linear transformation $\mathbf{x}_0 = [\mathbf{M}(s_0 \rightarrow s)]^{-1} \mathbf{x}_s$ and similarly $\mathbf{x}_{s'}(\mathbf{x}_s) = [\mathbf{M}(s' \rightarrow s)]^{-1} \mathbf{x}_s$.

Before proceeding further we make a more definite assumption about the form of the initial perturbation. We assume an expression of the form

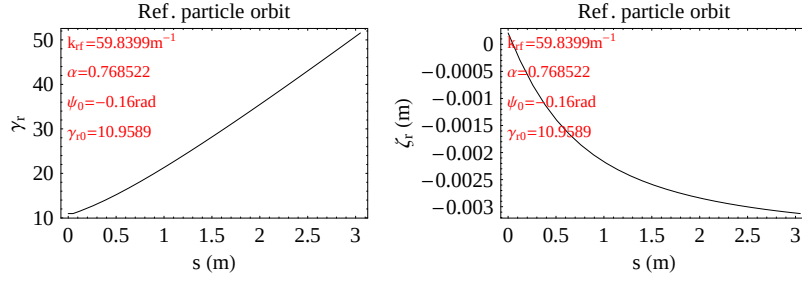


FIG. 3: Reference orbit for a model of beam line consisting of a 1 m drift followed by 3 m long RF compressor (exact numerical solutions). The initial condition for $\hat{\zeta}$ is set by the requirement that the phase is $\psi = \psi_0$ when the reference particle enters the RF compressor.

$$f_1(\mathbf{x}_0; s_0) = \tilde{\rho}_1(k_0; s_0) e^{ik_0 z_0} \frac{e^{-(p_0 - h z_0)^2 / 2 \sigma_p^2}}{\sqrt{2\pi\sigma_p}} + c.c., \quad (41)$$

i.e. consisting of a sinusoidal perturbation to the charge density (while the p density is the same as in the unperturbed distribution).

We are now ready to integrate over the phase space both sides of Eq. (40) after multiplying by e^{-ikz_s}/L_b . By definition, see (36), the FT of the LHS of Eq. (40) yields $\tilde{\rho}_1(k; s)$. The FT of the first term on the RHS requires more work. First, we carry out the transformation of variables $\mathbf{x}_s \rightarrow \mathbf{x}_0$ and exploit symplecticity $d^2\mathbf{x}_s = d^2\mathbf{x}_0$ to write

$$I_1 \equiv \frac{1}{L_b} \int_{-L_b/2}^{L_b/2} \int_{-\infty}^{\infty} d^2\mathbf{x}_s f_1(\mathbf{x}_0(\mathbf{x}_s); s_0) e^{-ikz_s} = \frac{1}{L_b} \int_{-L_b/2}^{L_b/2} dz_0 \int dp_0 f_1(\mathbf{x}_0; s_0) e^{-ikz_s(\mathbf{x}_0)} \quad (42)$$

where $z_s(\mathbf{x}_0) = [\mathbf{M}\mathbf{x}_0]_1 = M_{11}z_0 + M_{12}p_0$. Let's write $I_1 = I_1^{(a)} + I_1^{(b)}$ where $I_1^{(a)}$ and $I_1^{(b)}$ correspond to the first term on the RHS of (41) and its $c.c.$ respectively. Then insert expression (41) into (42)

$$I_1^{(a)} = \frac{1}{\sqrt{2\pi\sigma_p}} \tilde{\rho}_1(k_0; s_0) \int_{-\infty}^{\infty} dp_0 \frac{1}{L_b} \int_{-L_b/2}^{L_b/2} dz_0 e^{-(p_0 - h z_0)^2 / 2 \sigma_p^2} e^{ik_0 z_0} e^{-ik(M_{11}z_0 + M_{12}p_0)}. \quad (43)$$

Before carrying out the integration over p_0 introduce the change of variable $t = p_0 - h z_0$, yielding

$$\begin{aligned} I_1^{(a)} &= \frac{1}{\sqrt{2\pi\sigma_p}} \tilde{\rho}_1(k_0; s_0) \frac{L_{b0}}{L_b} \frac{1}{L_{b0}} \int_{-L_{b0}/2}^{L_{b0}/2} dz_0 e^{iz_0(k_0 - k/C)} \int_{-\infty}^{\infty} dt e^{-t^2 / 2 \sigma_p^2} e^{ikM_{12}t} \\ &= C(s) \tilde{\rho}_1(k_0; s_0) \delta_{k_0, k/C(s)} e^{-(kM_{12}\sigma_p)^2 / 2}, \end{aligned} \quad (44)$$

with the δ function in the above equation coming from the integral

$$\frac{1}{L_{b0}} \int_{-L_{b0}/2}^{L_{b0}/2} dz_0 e^{iz_0(k_0 - k/C)} = \delta_{k_0, k/C}. \quad (45)$$

Similarly

$$I_1^{(b)} = C(s) \tilde{\rho}_1^*(k_0; s_0) \delta_{-k_0, k/C(s)} e^{-(kM_{12}\sigma_p)^2 / 2}. \quad (46)$$

We can now tackle the evaluation of the FT of the second term in the RHS of Eq. (40):

$$\begin{aligned} I_2 &= -\frac{1}{L_b} \int d^2\mathbf{x}_s e^{-ikz_s} \int_{s_0}^s ds' F(\tilde{\rho}_1; z_{s'}; s') \Big|_{z_{s'}=z_{s'}(\mathbf{x}_s)} \frac{\partial f_0(\mathbf{x}_{s'}; s')}{\partial p_{s'}} \Big|_{\mathbf{x}_{s'}=\mathbf{x}_{s'}(\mathbf{x}_s)} \\ &= -\frac{1}{L_b} \int d^2\mathbf{x}_{s'} e^{-ikz_s(\mathbf{x}_{s'})} \int_{s_0}^s ds' F(\tilde{\rho}_1; z_{s'}; s') \frac{\partial f_0(\mathbf{x}_{s'}; s')}{\partial p_{s'}} \\ &= -\frac{ik}{L_b} \int_{s_0}^s ds' M_{12}(s' \rightarrow s) \int d^2\mathbf{x}_{s'} F(\tilde{\rho}_1; z_{s'}; s') f_0(\mathbf{x}_{s'}; s') e^{-ikz_s(\mathbf{x}_{s'})} \end{aligned} \quad (47)$$

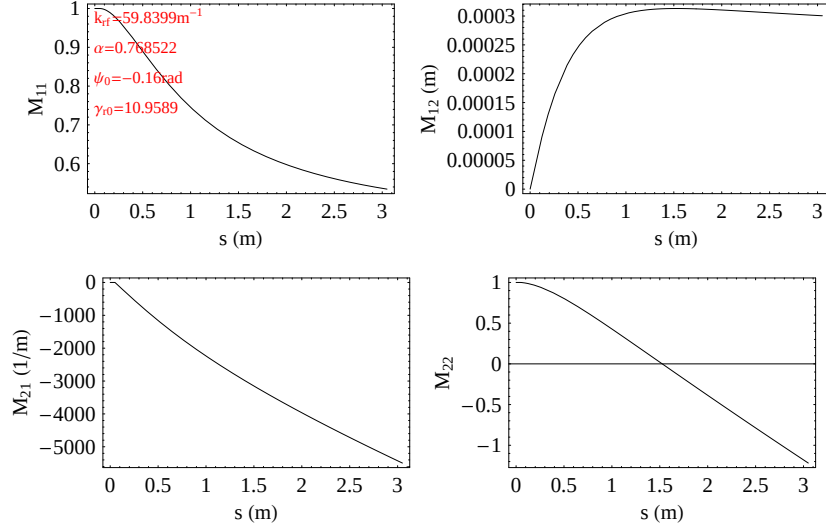


FIG. 4: Entries of the transfer matrix $\mathbf{M}$ (exact numerical solutions).

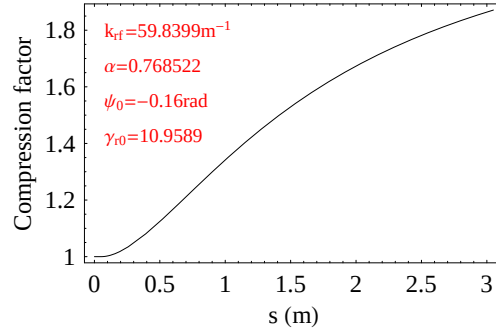


FIG. 5: Compression factor along the beam line. The exit value is 1.87.

The second equality above follows from a change of variables, the third from an integration by parts and $z_s(\mathbf{x}_{s'}) = \mathbf{M}_{11}(s' \rightarrow s)z_{s'} + \mathbf{M}_{12}(s' \rightarrow s)p_{s'}$.

A further change of variables to $\mathbf{x}_0$ yields:

$$I_2 = -\frac{ik}{L_b} \int_0^s ds' M_{12}(s' \rightarrow s) \int d^2\mathbf{x}_0 f_0(\mathbf{x}_0; s_0) F(\tilde{\rho}_1; z_{s'}(\mathbf{x}_0); s') e^{-ikz_s(\mathbf{x}_{s'}(\mathbf{x}_0))} \quad (48)$$

where we have made use of $f_0(\mathbf{x}_{s'}; s') = f_0(\mathbf{x}_0; s_0)$. Observe that in the argument of the exp function in the expression above $z_s(\mathbf{x}_{s'}(\mathbf{x}_0)) = [M(s_0 \rightarrow s)\mathbf{x}_0]_1 = M_{11}z_0 + M_{12}p_0$.

Using the expression (37) for the collective force we find

$$I_2 = \frac{e^2 N_b}{mc} \frac{ik}{L_b} \sum_{k'} \int_0^s ds' M_{12}(s' \rightarrow s) Z(k', s') \tilde{\rho}_1(k', s') \int_{-L_{b0}/2}^{L_{b0}/2} dz_0 \int_{-\infty}^{\infty} dp_0 f_0(\mathbf{x}_0; s_0) e^{ik'z_{s'}(\mathbf{x}_0) - ikz_s(\mathbf{x}_0)} \quad (49)$$

As before we carry out the integration over p upon the change of variables $p_0 \rightarrow p_0 = t + hz_0$. First observe

$$\begin{aligned} k'z_{s'}(\mathbf{x}_0) - kz_s(\mathbf{x}_0) &= k'[M_{11}(s')z_0 + M_{12}(s')p_0] - k[M_{11}(s)z_0 + M_{12}(s)p_0] \\ &= z_0 A + tB, \end{aligned} \quad (50)$$

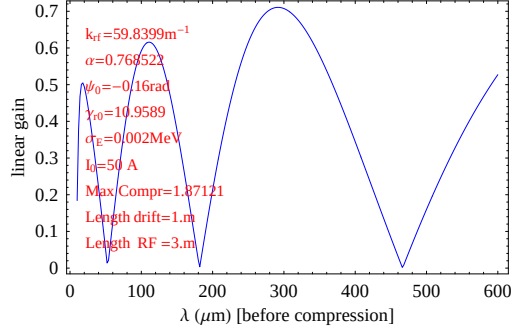


FIG. 6: Linear gain at exit of RF compressor

where the coefficients A and B are defined as

$$\begin{aligned} A &= k' M_{11}(s') - k M_{11}(s) + h[k' M_{12}(s') - k M_{12}(s)] \\ &= k'[M_{11}(s') + h M_{12}(s')] - k[M_{11}(s) + h M_{12}(s)] = k'/C(s') - k/C(s) \end{aligned} \quad (51)$$

$$B = k' M_{12}(s') - k M_{12}(s) \quad (52)$$

and we have used the short-hand notation $M_{ij}(s) = M_{ij}(s_0 \rightarrow s)$ for the entries of the transfer matrix $\mathbf{M}$.

The integral on z_0 in (49) yields a delta function

$$\frac{1}{L_{b0}} \int_{-L_{b0}/2}^{L_{b0}/2} dz_0 e^{iz_0[k'/C(s') - k/C(s)]} = \delta_{k', kC(s')/C(s)}. \quad (53)$$

This reduces the sum $\sum_{k'}$ to one term with $k' = kC(s')/C(s)$.

$$\begin{aligned} I_2 &= \frac{e^2 N_b}{mc L_b} ik \int_{s_0}^s ds' M_{12}(s' \rightarrow s) Z(k', s') \tilde{\rho}_1(k'; s') \frac{1}{\sqrt{2\pi\sigma_p}} \int_{-\infty}^{\infty} dt e^{-t^2/2\sigma_p^2} e^{itB} \Big|_{k'=kC(s')/C(s)} \\ &= \frac{e^2 N_b}{mc L_b} ik \int_{s_0}^s ds' M_{12}(s' \rightarrow s) Z(k', s') \tilde{\rho}_1(k'; s') e^{-[k' M_{12}(s') - k M_{12}(s)]^2 \sigma_p^2 / 2} \Big|_{k'=kC(s')/C(s)}. \end{aligned} \quad (54)$$

Finally, combining the various pieces we arrive at the following integral equation for the Fourier components of the first-order density perturbation:

$$\begin{aligned} \tilde{\rho}_1(k; s) &= C(s) e^{-[C(s)k_0 M_{12} \sigma_p]^2 / 2} [\tilde{\rho}_1(k_0; s_0) \delta_{k_0, k/C(s)} + \tilde{\rho}_1^*(k_0; s_0) \delta_{-k_0, k/C(s)}] \\ &\quad + \int_{s_0}^s ds' K(s', s) \tilde{\rho}_1(k'; s') \Big|_{k'=kC(s')/C(s)} \end{aligned} \quad (55)$$

with kernel

$$K(s', s) = 4\pi i \frac{I(s)}{I_A} k M_{12}(s' \rightarrow s) \frac{Z(k', s')}{Z_0} e^{-[k' M_{12}(s') - k M_{12}(s)]^2 \sigma_p^2 / 2} \Big|_{k'=kC(s')/C(s)}, \quad (56)$$

where $I(s) = ecN_b/L_b(s) = ecN_b C(s)/L_{b0}$ is the bunch peak current at s , $I_A = ce/r_e \simeq 17\text{kA}$ is the Alfvén current, and Z_0 the vacuum impedance. We have used

$$\frac{e^2 N_b Z_0 c}{mc^2 L_b} = \frac{e^2 N_b}{\varepsilon_0 mc^2 L_b} = 4\pi \frac{e^2 N_b}{4\pi \varepsilon_0 mc^2 L_b} = 4\pi \frac{r_e N_b}{L_b} = 4\pi \frac{r_e ec N_b}{ec L_b} = 4\pi \frac{I(s)}{I_A}. \quad (57)$$

Because of the presence of the delta function the inhomogeneous term in Eq. (55) vanishes unless $k = \pm k_0 C(s)$. Therefore for $k \neq k_0 C(s)$ the solution of the integral equation is $\tilde{\rho}_1(k; s) = 0$. Assuming positive[7] k , $k = k_0 C(s) > 0$ we can rewrite the integral equation as

$$\tilde{\rho}_1(C(s)k_0; s) = C(s)\tilde{\rho}_1(k_0; s_0)e^{-[C(s)k_0 M_{12}\sigma_p]^2/2} + \int_{s_0}^s ds' K(s', s)\tilde{\rho}_1(C(s')k_0; s') \quad (58)$$

with kernel

$$K(s', s) = 4\pi i \frac{I(s)}{I_A} C(s)k_0 M_{12}(s' \rightarrow s) \frac{Z(C(s')k_0, s')}{Z_0} e^{-k_0^2 [C(s')M_{12}(s') - C(s)M_{12}(s)]^2 \sigma_p^2/2}. \quad (59)$$

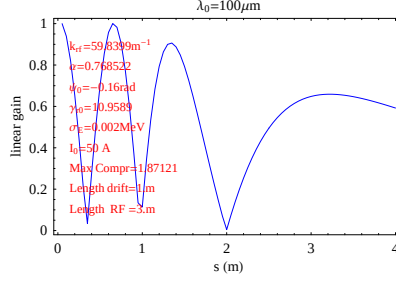


FIG. 7: Linear gain for a single mode of wavelength $100\mu\text{m}$ along the beam line.

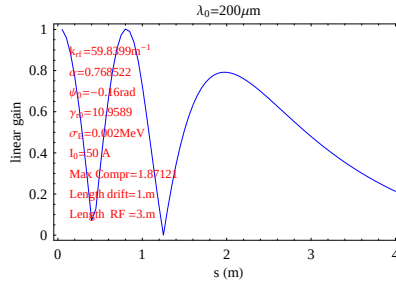


FIG. 8: Linear gain for a single mode of wavelength $200\mu\text{m}$ along the beam line.

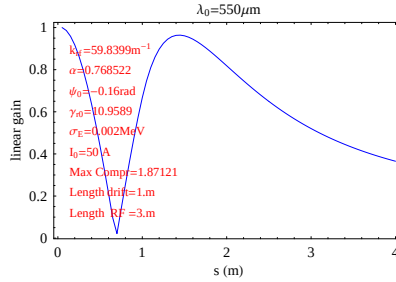


FIG. 9: Linear gain for a single mode of wavelength $550\mu\text{m}$ along the beam line.

The integral equation (58) and the kernel (59) is shown to be formally the same as in the Huang-Kim paper (except for the more general form of the unperturbed dynamics allowed here which results into a more general dependence of the compression factor C on the transfer matrix $\mathbf{M}$) by dividing both sides of Eq. (58) by $C(s)$, writing

$$\frac{\tilde{\rho}_1(C(s)k_0; s)}{C(s)} = \tilde{\rho}_1(k_0; s_0)e^{-[C(s)k_0 M_{12}\sigma_p]^2/2} + \int_{s_0}^s ds' K(s', s) \frac{C(s')}{C(s)} \frac{\tilde{\rho}_1(C(s')k_0; s')}{C(s')}, \quad (60)$$

and setting $b(C(s)k_0; s) = \tilde{\rho}_1(C(s)k_0; s)/C(s)$ to obtain

$$b(C(s)k_0; s) = b(k_0; s_0)e^{-[C(s)k_0 M_{12}\sigma_p]^2/2} + \int_{s_0}^s ds' \hat{K}(s', s)b(C(s')k_0; s') \quad (61)$$

with kernel $\hat{K}(s', s) = K(s', s)[C(s')/C(s)]$:

$$\hat{K}(s', s) = 4\pi i \frac{I(s')}{I_A} C(s)k_0 M_{12}(s' \rightarrow s) \frac{Z(C(s')k_0, s')}{Z_0} e^{-k_0^2[C(s')M_{12}(s') - C(s)M_{12}(s)]^2\sigma_p^2/2}. \quad (62)$$

The only difference between (59) and (62) is in the argument of the peak current: $I(s)$ and $I(s')$ respectively. Notice that in terms of $\tilde{\rho}_1(C(s)k_0; s)$ the conventional definition of the gain curve reads

$$g(k_0; s) = \frac{1}{C(s)} \left| \frac{\tilde{\rho}_1(C(s)k_0; s)}{\tilde{\rho}_1(k_0; s_0)} \right| \quad (63)$$

whereas in terms of b we have

$$g(k_0; s) = \left| \frac{b(C(s)k_0; s)}{b(k_0; s_0)} \right| \quad (64)$$

(i.e. no C factor in the denominator).

For the space-charge impedance we will consider the following two models. The first yields the on-axis longitudinal component of the electric field for a beam with transverse uniform density and circular cross-section:

$$Z(k, r=0) = \frac{iZ_0}{\pi\gamma r_b} \frac{1 - \xi K_1(\xi)}{\xi} \Big|_{\xi=kr_b/\gamma}, \quad (65)$$

where $Z_0 \simeq 120\pi$ is the vacuum impedance, K_1 the modified Bessel function, γ the relativistic factor, and $k = 2\pi/\lambda$ the perturbation wavenumber. A second model yields the longitudinal component of the electric field for the same beam model, $Z_{\text{avg}}(k) = \int_0^{r_b} Z(k, r)rdr/(r_b^2/2)$ averaged over the transverse beam model,

$$Z_{\text{avg}}(k) = \frac{iZ_0}{\pi\gamma r_b} \frac{1 - 2I_1(\xi)K_1(\xi)}{\xi} \Big|_{\xi=kr_b/\gamma}. \quad (66)$$

In Fig.'s 3 through 9 we show a numerical example for a model of beam line consisting of a 1 m drift followed by a 3 m RF compressor. The beam start at 5.6 MeV energy with a $I_0 = 50$ A peak current. We assume a beam transversely uniform with circular cross section with radius $r_b = 0.5$ mm remaining constant through the beam line. Max. compression is 1.87. The gain curve as a function of the wavelength of the initial perturbation is reported in Fig. 6, and shows a max value of about 2.5 in the $\lambda_0 \simeq 200\mu\text{m}$ region. The following three pictures show the evolution along the beam line of the gain for three selected initial modes.

B. Coasting beam

The form of the beam distribution $f(\mathbf{x}_0; s_0) = f_0(\mathbf{x}_0; s_0) + f_1(\mathbf{x}_0; s_0)$ is the same as before with the zero-order smooth density uniform in z given by

$$f_0(\mathbf{x}_0; s_0) = \frac{1}{\sqrt{2\pi}\sigma_p} e^{-(p_0 - hz_0)^2/2\sigma_p^2}. \quad (67)$$

The normalization is such that $n_0 dz \int_{-\infty}^{\infty} dp_0 f(\mathbf{x}; s)$ gives the no. of particles in the interval dz , where n_0 is the particle line density.

Define the Fourier integrals of the charge density function at s

$$\tilde{\rho}(k; s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz_s e^{-ikz_s} \int_{-\infty}^{\infty} dp f(\mathbf{x}_s; s). \quad (68)$$

with the effect of collective forces described by the impedance $Z(k; s)$ described by

$$\frac{dp}{ds} \equiv F(\tilde{\rho}, z_s; s) = -\frac{e^2 n_0}{mc} \int_{-\infty}^{\infty} dk e^{ikz_s} Z(k; s) \tilde{\rho}(k; s). \quad (69)$$

For the initial perturbation we take

$$f_1(\mathbf{x}_0; s_0) = A e^{ik_0 z_0} \frac{e^{-(p_0 - h z_0)^2 / 2 \sigma_p^2}}{\sqrt{2\pi\sigma_p}} + c.c., \quad (70)$$

Following the derivation in the preceding section, we have

$$I_1 = \frac{A}{2\pi\sqrt{2\pi}\sigma_p} \int_{-\infty}^{\infty} dz_0 e^{iz_0(k_0 - k/C)} \int_{-\infty}^{\infty} dt e^{-t^2/2\sigma_p^2} e^{ikM_{12}t} = \left[A\delta\left(k_0 - \frac{k}{C(s)}\right) + A^*\delta\left(k_0 + \frac{k}{C(s)}\right) \right] e^{-(kM_{12}\sigma_p)^2/2},$$

where $\delta(\cdot)$ is the Dirac function and

$$I_2 = \frac{e^2 n_0}{mc} ik \int_{s_0}^s ds' C(s') M_{12}(s' \rightarrow s) Z(k', s') \tilde{\rho}_1(k'; s') \exp\left(-\frac{\sigma_p^2}{2} [k' M_{12}(s') - k M_{12}(s)]^2\right) \Big|_{k'=kC(s')/C(s)} \quad (71)$$

The integral equation obeyed by $\tilde{\rho}_1(k; s)$ is then

$$\tilde{\rho}_1(k; s) = e^{-k^2 M_{12}^2 \sigma_p^2 / 2} \left[A\delta\left(k_0 - \frac{k}{C(s)}\right) + A^*\delta\left(k_0 + \frac{k}{C(s)}\right) \right] + \int_{s_0}^s ds' K(s', s) \tilde{\rho}_1(kC(s')/C(s); s') \quad (72)$$

with kernel

$$K(s', s) = 4\pi i \frac{I(s')}{I_A} k M_{12}(s' \rightarrow s) \frac{1}{Z_0} Z\left(k \frac{C(s')}{C(s)}, s'\right) \exp\left(-\frac{\sigma_p^2}{2} \frac{k^2}{C(s)^2} [C(s') M_{12}(s') - C(s) M_{12}(s)]^2\right). \quad (73)$$

One should look for solutions of (72) in the space of generalized functions (or distributions). Such solutions will have the form

$$\tilde{\rho}_1(k; s) = b(k; s) \delta\left(k_0 \pm \frac{k}{C(s)}\right), \quad (74)$$

with the ordinary functions b obeying the equation

$$b(C(s)k_0; s) = A e^{-[Ck_0 M_{12} \sigma_p]^2 / 2} + \int_{s_0}^s ds' K(s', s) b(C(s')k_0; s') \quad (75)$$

with kernel

$$K(s', s) = 4\pi i \frac{I(s')}{I_A} C(s) k_0 M_{12}(s' \rightarrow s) \frac{1}{Z_0} Z(C(s')k_0, s') \exp\left(-\frac{\sigma_p^2 k_0^2}{2} [C(s') M_{12}(s') - C(s) M_{12}(s)]^2\right). \quad (76)$$

Notice that a proper definition of gain in terms of the quantity $\tilde{\rho}_1(k; s)$ involves an integration over frequencies:

$$g(k_0; s) = \frac{1}{C(s)} \frac{|\int_{-\infty}^{\infty} dk \tilde{\rho}_1(k; s)|}{|A|} \quad (77)$$

Inserting (74) into the above equation we obtain

$$g(k_0; s) = \frac{|b(C(s)k_0; s)|}{|A|}, \quad (78)$$

consistent with (64).

IV. GAIN CURVE: BUNCHED BEAM MODEL

In this section we will explore a bunched model for the beam with bi-gaussian density in phase space. We are in part motivated by results in [6] where a better agreement with numerical simulations for low-energy transport in a drift was claimed using this model.

In comparison to the uniform beam model there are two complications. The first is that the problem involves multifrequencies, as the Fourier modes cease to be orthogonal. A practical consequence is that the kernel in the integral equation for the modes analogous to (58) will contain an additional integration over frequencies, introducing a numerical complication.

A second and perhaps more limiting complication is that the evolution of the smooth part (or zero-order) density of the beam will respond to the presence of the collective effects. This dynamics is difficult to determine self-consistently in the framework of our model and can only be approximated by resorting to some simplifying assumptions. Here is our strategy.

Let us begin with defining the smooth part of the density function at $s = s_0$:

$$f_0(\mathbf{x}_0; s_0) = \frac{1}{2\pi\sigma_p\sigma_z} e^{-z_0^2/\sigma_z^2} e^{-(p_0 - h z_0)^2/2\sigma_p^2}. \quad (79)$$

In first approximation this evolves according to the transfer matrix $\mathbf{M}$ for motion under external forces defined in Sec.II. We then use this transformation to determine the charge density at later times

$$\rho_0(z; s) = \frac{B(s)}{\sqrt{2\pi}\sigma_z} e^{-z^2 B(s)^2/2\sigma_z^2}, \quad (80)$$

where the coefficient B is defined as

$$B(s) = \frac{1}{\left[[M_{11}(s) + h M_{12}(s)]^2 + M_{12}(s)^2 \frac{\sigma_p^2}{\sigma_z^2} \right]^{1/2}}. \quad (81)$$

In the limit where $M_{12}(s)^2 \frac{\sigma_p^2}{\sigma_z^2}$ is negligible small, which should always be the case in applications of interest for x-ray FELs, then $B(s) \simeq C(s)$.

We can then use this expression to calculate the resulting energy changes due to space-charge. We linearize this contribution and add it to the dynamics under the external forces. This model will not account for the beam expansion caused by space charge but will capture, in first approximation, the built-up of an energy-position correlation, which may affect compression in the RF structure. Having defined the FT of the longitudinal charge density for a beam with unlimited support as

$$\tilde{\rho}(k; s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz e^{-ikz} \int_{-\infty}^{\infty} dp f(\mathbf{x}; s), \quad (82)$$

the effect of collective forces described by the impedance $Z(k; s)$ is to modify the energy as in

$$\frac{dp}{ds} \equiv F(\tilde{\rho}, z; s) = -\frac{e^2 c N_b}{mc^2} \int_{-\infty}^{\infty} dk e^{ikz} Z(k; s) \tilde{\rho}(k; s). \quad (83)$$

The FT of the smooth density (80) is

$$\tilde{\rho}_0(k) = \frac{1}{2\pi} e^{-k^2 \sigma_z^2/2B^2}. \quad (84)$$

We are now interested in computing the linear term (in z) of $F(\tilde{\rho}_0, z; s)$, which is straightforwardly obtained by expanding e^{ikz} in (83) through first order:

$$F(\tilde{\rho}_0, z; s) \simeq -\frac{e^2 c N_b}{mc^2} \left(\int_{-\infty}^{\infty} dk Z(k; s) \tilde{\rho}_0(k; s) + iz \int_{-\infty}^{\infty} dk k Z(k; s) \tilde{\rho}_0(k; s) \right). \quad (85)$$

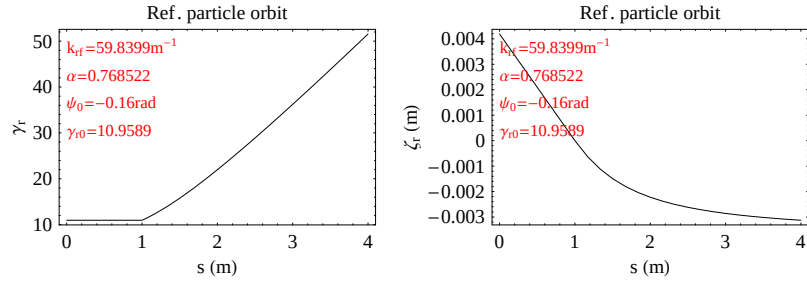


FIG. 10: Bi-gaussian beam. Reference orbit for a model of beam line consisting of a 1 m drift followed by 3 m long RF compressor (exact numerical solutions). The initial condition for ζ is set by the requirement that the phase is $\psi = \psi_0$ when the reference particle enters the RF compressor.

The first integral on the RHS of the above equation vanishes ($Z(k; s)$ is purely imaginary and therefore $Z(-k) = Z(k)^* = -Z(k)$ is an odd function in k while $\tilde{\rho}_0(k; s)$ is even in k). As for the second term, specializing the calculation to the impedance model of Eq. (66) we have

$$F(\tilde{\rho}_0, z; s) = z \frac{e^2 c N_b}{m c^2} \frac{Z_0 \gamma}{\pi r_b^3} \frac{1}{2\pi} \int_{-\infty}^{\infty} dx [1 - 2I_1(x)K_1(x)] e^{-\eta^2 x^2/2}, \quad (86)$$

where $\eta = \gamma \sigma_z / (r_b C)$. A good approximation of the above integral can be obtained if $\eta \gg 1$ by using

$$[1 - 2I_1(x)K_1(x)] \simeq -\frac{x^2}{2} \left[\log\left(\frac{x}{2}\right) + \gamma_E - \frac{1}{4} \right]. \quad (87)$$

We have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dx [1 - 2I_1(x)K_1(x)] e^{-\eta^2 x^2/2} \simeq \frac{1}{\eta^3} \frac{\log \eta}{2\sqrt{2\pi}}, \quad (88)$$

and find

$$F(\tilde{\rho}_0, z; s) = 2z \frac{I_0 B}{I_A} \frac{B^2}{\gamma^2 \sigma_z^2} \log(\gamma \sigma_z / r_b B), \quad (89)$$

where I_0 is the initial peak current defined as $I_0 = e c N_b / \sqrt{2\pi} \sigma_z$; $I_0 B(s)$ is then the peak current at s .

This modifies the equations of motion for determining the transfer matrix (9) and (10)

$$\frac{dz}{ds} = \frac{p}{(\gamma_r^2 - 1)^{3/2}}, \quad (90)$$

$$\frac{dp}{ds} = \left[\frac{2I_0 B(s)}{I_A} \frac{B(s)^2}{\gamma_r^2 \sigma_z^2} \log\left(\frac{\gamma_r \sigma_z}{r_b B(s)}\right) - \alpha k_{rf}^2 \cos(k_{rf} s + k_{rf} z_r + \psi_0) \right] z. \quad (91)$$

We will denote the transfer matrix associated with this system of equations as $\hat{\mathbf{M}}$. We can possibly try to improve on the inconsistency of this model in the following way. After solving for the transfer matrix $\hat{\mathbf{M}}$ associated with (90) and (91) we determine the resulting compression coefficient

$$\hat{B} = \frac{1}{\left[[\hat{M}_{11}(s) + h \hat{M}_{12}(s)]^2 + \hat{M}_{12}(s)^2 \frac{\sigma_p^2}{\sigma_z^2} \right]^{1/2}}. \quad (92)$$

Next, we insert (92) into (91) in place of B , recalculate the transfer matrix $\hat{\mathbf{M}}$ and continue with this procedure hoping to achieve convergence. (Of course, it will have to be verified that indeed a convergence can be achieved).

We can now proceed and write an integral equation for first order density perturbations of a bi-gaussian beam. We start by assuming a small perturbation to (79) of the form.

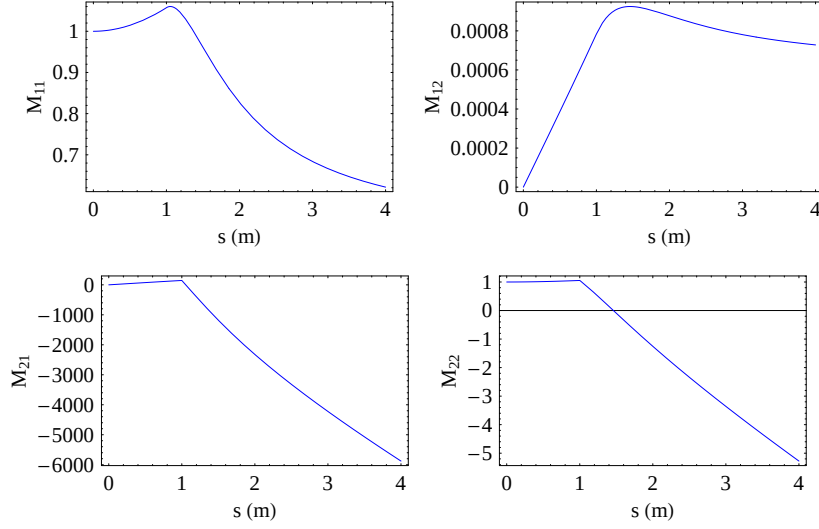


FIG. 11: Bi-gaussian beam. Entries of the transfer matrix $\mathbf{M}$ (exact numerical solutions).

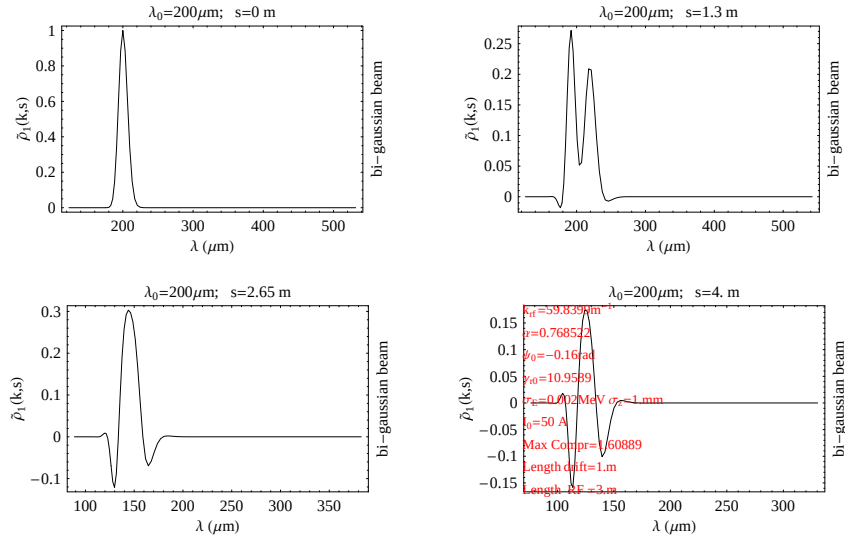


FIG. 12: Bi-gaussian beam. Snap-shots of the solution $\tilde{\rho}_1(k; s)$ of the integral equation as a function of $\lambda = 2\pi/k$ of the integral equation along the beam line. The initial perturbation has wavelength $\lambda_0 = 2\pi/k_0$. The solution remains relatively narrow with the peak evolving as $\lambda = \lambda_0/\hat{C}(s)$.

$$f_1(\mathbf{x}_0; s_0) = 2\pi\tilde{\rho}_1(k_0; s_0)e^{ik_0z_0}\frac{e^{-z_0^2/2\sigma_z^2}}{\sqrt{2\pi}\sigma_z}\frac{e^{-(p_0-hz_0)^2/2\sigma_p^2}}{\sqrt{2\pi}\sigma_p} + c.c. \quad (93)$$

The 2π coefficient is required for consistency with (82).

We then multiply both sides of (39) by $e^{-ikz}/2\pi$ and integrate over the phase space. Going through the same derivation as in the preceding section we find the following equation

$$\tilde{\rho}_1(k; s) = e^{-[k\hat{M}_{12}(s)\sigma_p]^2/2}[\tilde{\rho}_1(k_0; s_0)e^{-[k_0-k/\hat{C}(s)]^2\sigma_z^2/2} + \tilde{\rho}_1^*(k_0; s_0)e^{-[k_0+k/\hat{C}(s)]^2\sigma_z^2/2}] + \int_{s_0}^s ds' \int_{-\infty}^{\infty} dk' K(k', k; s', s)\tilde{\rho}_1(k; s') \quad (94)$$

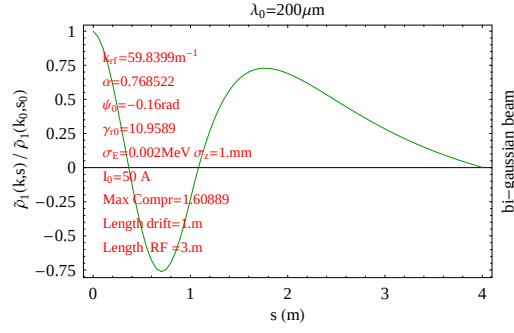


FIG. 13: Bi-gaussian beam. Solution of the integral equation integrated over the frequencies (and normalized to the initial value $s = s_0$ along the beam line i.e. $\int_{-\infty}^{\infty} dk \tilde{\rho}_1(k; s) / \int_{-\infty}^{\infty} dk \tilde{\rho}_1(k; s_0)$). [y-label of figure is corrupted]

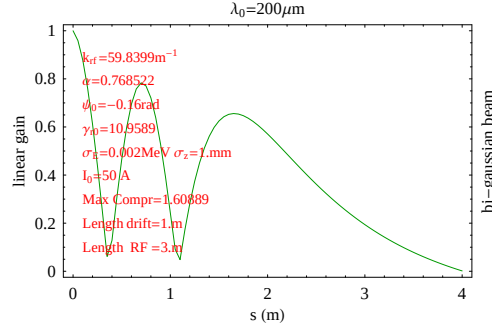


FIG. 14: Gain corresponding to the solution reported in Fig. 13

with kernel

$$K(k', k; s', s) = 2i \frac{I_0}{I_A} \sqrt{2\pi} \sigma_z k \hat{M}_{12}(s' \rightarrow s) \frac{Z(k', s')}{Z_0} \times \exp \left(-\frac{\sigma_z^2}{2} \left[\frac{k'}{\hat{C}(s')} - \frac{k}{\hat{C}(s)} \right]^2 \right) \exp \left(-\frac{\sigma_P^2}{2} [k' \hat{M}_{12}(s') - k \hat{M}_{12}(s)]^2 \right), \quad (95)$$

and compression factor $\hat{C}$ defined as

$$\hat{C} = \frac{1}{\hat{M}_{11}(s) + h \hat{M}_{12}(s)}. \quad (96)$$

To make contact with results from the previous sections we are interested in considering the limit of a long bunch $\sigma_z \rightarrow \infty$. To this end observe

$$\lim_{\sigma_z \rightarrow \infty} \sqrt{2\pi} \sigma_z \exp \left(-\frac{\sigma_z^2}{2} \left[\frac{k'}{\hat{C}(s')} - \frac{k}{\hat{C}(s)} \right]^2 \right) = 2\pi \hat{C}(s') \delta \left(k' - k \frac{\hat{C}(s')}{\hat{C}(s)} \right). \quad (97)$$

Therefore, the reduced kernel reads

$$\begin{aligned} K_{\text{red}}(k; s', s) &= \int_{-\infty}^{\infty} dk' K(k', k; s', s) \\ &= 4\pi i \frac{I_0 \hat{C}(s')}{I_A} k \hat{M}_{12}(s' \rightarrow s) \frac{Z(k', s')}{Z_0} \exp \left(-\frac{\sigma_P^2}{2} [k' \hat{M}_{12}(s') - k \hat{M}_{12}(s)]^2 \right) \bigg|_{k' = k \hat{C}(s') / \hat{C}(s)}, \end{aligned} \quad (98)$$

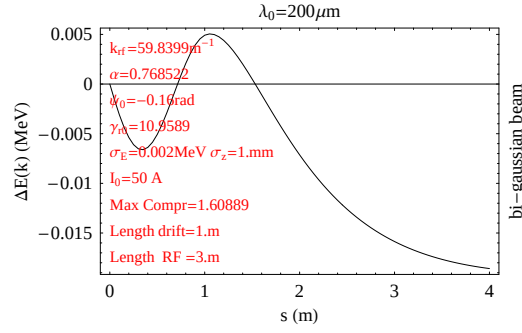


FIG. 15: Amplitude of energy modulation $\Delta E = mc^2 \int_{-\infty}^{\infty} \Delta \tilde{p}(k_s, s) dk_s$ induced by the charge density modulation solving the integral equation (assuming an initial max. 5% density modulation on the density profile).

identical to (73). In this limit the inhomogeneous term of Eq. (94) vanishes unless $k = k_0 \hat{C}(s)$, in which case the exp term equals unity. Therefore, for $k \neq k_0 \hat{C}(s)$ the solution for the integral equation is $\tilde{\rho}_1(k, s) = 0$ while for $k = k_0 \hat{C}(s)$ the equation becomes formally identical to that of a coasting beam.

A possible definition for the gain of mode k_0 is

$$\begin{aligned} g(k_0; s) &= \frac{1}{\hat{C}(s)} \frac{|\int_{-\infty}^{\infty} dk \tilde{\rho}_1(k; s)|}{|\int_{-\infty}^{\infty} dk \tilde{\rho}_1(k; s_0)|} \\ &= \frac{1}{\hat{C}(s)} \frac{|\int_{-\infty}^{\infty} dk \tilde{\rho}_1(k; s)|}{|\tilde{\rho}_1(k_0; s_0)| \int_{-\infty}^{\infty} dk e^{-[k_0 - k/\hat{C}(s_0)]^2 \sigma_z^2/2}} = \frac{\sigma_z}{\hat{C}(s)} \frac{|\int_{-\infty}^{\infty} dk \tilde{\rho}_1(k; s)|}{|\tilde{\rho}_1(k_0; s_0)|}, \end{aligned} \quad (99)$$

having used $\hat{C}(s_0) = 1$ and $\hat{M}_{12}(s_0) = 0$. This definition is consistent with (77) in the limit $\sigma_z \rightarrow \infty$.

A relevant quantity is the first-order energy change induced by the exit of the RF compressor by the first-order density modulation. From (83) we have

$$\Delta p(s, z) = \int_{s_0}^s \frac{dp}{ds'} ds' = -\frac{e^2 N_b}{mc} \int_{s_0}^s ds' \int_{-\infty}^{\infty} dk' e^{ik'z} Z(k'; s') \tilde{\rho}_1(k'; s') \quad (100)$$

with Fourier components

$$\begin{aligned} \Delta \tilde{p}(k_s, s) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik_s z} \Delta p(s, z) dz \\ &= -\frac{e^2 N_b Z_0}{mc} \int_{s_0}^s \frac{Z(k_{s'}; s')}{Z_0} \tilde{\rho}_1(k_{s'}; s') ds' \\ &= -4\pi \frac{I_0}{I_A} \sigma_z \sqrt{2\pi} \int_{s_0}^s \frac{Z(k_{s'}; s')}{Z_0} \tilde{\rho}_1(k_{s'}; s') ds'. \end{aligned} \quad (101)$$

If we assume that the wavelength $2\pi/k$ is much smaller than the bunch length σ_z so that the solution to the integral equation $\tilde{\rho}_1(k; s)$ is narrowly peaked at $k = k_0 C(s)$ then the energy modulation $\Delta p(s, z)$ will be almost sinusoidal with amplitude

$$\Delta p(s, z) = \int_{-\infty}^{\infty} e^{ik_s z} \Delta \tilde{p}(k_s, s) dk_s \simeq e^{iC(s)k_0 z} \int_{-\infty}^{\infty} \Delta \tilde{p}(k_s, s) dk_s. \quad (102)$$

Numerical examples of solutions of the integral equation for a bi-gaussian beam of rms length $\sigma_z = 1\text{mm}$ are presented in Fig.'s 10 through 15.

V. CONCLUSIONS

We have presented a derivation for the integral equations yielding the gain curve for the microbunching instability developing in an rf compressor as the result of longitudinal space charge. We considered both the case of a bunched and coasting beam. The latter is of course less realistic but yields equations that are simpler to solve and may nonetheless be sufficiently accurate for the high-frequency region of the noise spectrum. The preliminary numerical examples to illustrate the form of the solutions to the above integral equations show that for reasonable values of beam parameters the instability gain appear to remain below unity. Work is underway to combine the results from linear theory discussed in this report and macroparticle simulations for a more definite evaluation of the microbunching instability in rf compressors.

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 - [7] Having found the solution of the following integral equation $\tilde{\rho}_1(k; s)$ for positive k we can then reconstruct the entire function from $\tilde{\rho}_1(-k; s) = \tilde{\rho}_1^*(k; s)$.

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