Impedance scaling for small-angle tapers and collimators

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I. DERIVATION

In this note I will prove that the impedance calculated for a small-angle collimator or taper, of arbitrary 3D profile, has a scaling property that can greatly simplify numerical calculations. This proof is based on the parabolic equation approach to solving Maxwell's equation developed in Refs. [1, 2]

We start from the parabolic equation formulated in [3]. As discussed in [1], in general case this equation is valid for frequencies $\omega \gg c/a$ where a is a characteristic dimension of the obstacle. However, for small-angle tapers and collimators, the region of validity of this equation extends toward smaller frequencies and includes $\omega \sim c/a$.

The parabolic equation is formulated for the *envelope* part of the electromagnetic field

$$\hat{\boldsymbol{E}}(x,y,z,\omega) = \int_{-\infty}^{\infty} dt\, \mathrm{e}^{i\omega t - ikz}\, \boldsymbol{E}(x,y,z,t), \tag{1}$$

where $k = \omega/c$. It is written in terms of the transverse component $\hat{\boldsymbol{E}}_\perp = (\hat{E}_x, \hat{E}_y)$ of the vector $\hat{\boldsymbol{E}}_\perp$,

$$k\frac{\partial}{\partial z}\hat{\boldsymbol{E}}_\perp = \frac{i}{2}\left(\nabla_\perp^2 \hat{\boldsymbol{E}}_\perp - \frac{4\pi}{c}\nabla_\perp \hat{j}_z\right), \tag{2}$$

where z is the coordinate in the direction of motion of the beam, and $\hat{j}_z$ is the Fourier transformed projection of the beam current along z

$$\hat{j}_z(x,y,z,\omega) = \int_{-\infty}^{\infty} dt\, \mathrm{e}^{i\omega t - ikz}\, j_z(x,y,z,t). \tag{3}$$

The longitudinal electric field $\hat{E}_z$ is expressed in terms of $\hat{\boldsymbol{E}}_\perp$

$$\hat{E}_z = \frac{i}{k}\left(\nabla_\perp \cdot \hat{\boldsymbol{E}}_\perp - \frac{4\pi}{c}\hat{j}_z\right). \tag{4}$$

We assume perfect conductivity of the walls. The boundary condition for the electric field requires vanishing tangential component on the wall

$$\boldsymbol{n} \times \hat{\boldsymbol{E}}|_w = 0, \tag{5}$$

where $\boldsymbol{n}$ is the normal vector to the surface of the wall.

The current $\hat{j}_z$ in Eqs. (2) and (4) corresponds to a unit point charge moving with the speed of light along the axis of the system $x = y = 0$, $j_z = qc\delta(x)\delta(y)\delta(z - ct)$. It is given by the following expression

$$\hat{j}_z = q\delta(x)\delta(y). \tag{6}$$

The longitudinal impedance on the z-axis at frequency ω is given by

$$Z(\omega) = -\frac{1}{q}\int_{-\infty}^{\infty} dz \hat{E}_z(0, 0, z, \omega). \tag{7}$$

An equivalent formulation of the impedance problem which avoids singular terms associated with the current $\hat{j}_z$ is the following. We introduce the *vacuum* electric field $\hat{\boldsymbol{E}}_{\text{vac}}$ of the current $\hat{j}_z$ (which satisfies the same Eq. (2), but does not satisfy the boundary condition (5)) and subtract it from $\hat{\boldsymbol{E}}$

$$\hat{\boldsymbol{\mathcal{E}}} = \hat{\boldsymbol{E}} - \hat{\boldsymbol{E}}_{\text{vac}}. \tag{8}$$

The equation for the field $\hat{\boldsymbol{\mathcal{E}}}$, which we call the *radiation* field is

$$\begin{aligned} k\frac{\partial}{\partial z}\hat{\boldsymbol{\mathcal{E}}}_\perp &= \frac{i}{2}\nabla_\perp^2 \hat{\boldsymbol{\mathcal{E}}}_\perp, \\ \hat{\mathcal{E}}_z &= \frac{i}{k}\nabla_\perp \cdot \hat{\boldsymbol{\mathcal{E}}}_\perp, \end{aligned} \tag{9}$$

with the boundary condition

$$\boldsymbol{n} \times \hat{\boldsymbol{\mathcal{E}}}|_w + \boldsymbol{n} \times \hat{\boldsymbol{E}}_{\text{vac}}|_w = 0. \tag{10}$$

The vacuum electric field is perpendicular to the direction of motion (because we consider an ultrarelativistic point charge), and does not contribute to the impedance. Note also that the vacuum field does not depend on z, $\hat{\boldsymbol{E}}_{\text{vac}}(x, y)$.

Let us assume that the geometry of a given surface of the metallic wall is determined by the equation $U(x, y, z) = 0$ with some given function U. Instead of considering one particular shape of the pipe, we consider a family of such pipes, which are defined by various scale lengths L in the longitudinal direction. This means that U is also a function of the parameter L, and it has a special dependence on L:

$$U(x, y, z; L) = V\left(x, y, \frac{z}{L}\right). \tag{11}$$

Varying the parameter L in Eq. (11) we extend or contract the pipe in the z-direction without changing its transverse shape.

We now define the normal vector to the surface of the pipe, $\boldsymbol{n} = \nabla U$ or

$$\boldsymbol{n} = \boldsymbol{e}_x \frac{\partial V}{\partial x} + \boldsymbol{e}_y \frac{\partial V}{\partial y} + \boldsymbol{e}_z \frac{1}{L} \frac{\partial V}{\partial \zeta}, \tag{12}$$

where we introduced the dimensionless scaled coordinate $\zeta = z/L$, and use notations $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, and $\boldsymbol{e}_z$ for unit vectors in respective directions. We will indicate the dependence of fields versus the parameter L by adding L to the list of arguments and separating it by the semicolon, e.g. $\hat{\boldsymbol{\mathcal{E}}}_\perp(x, y, z, \omega; L)$.

Our goal now is to prove that a solution to the parabolic equation depends on the coordinate z only through the dimensionless variable ζ; more precisely, we will prove that

$$\begin{aligned} \hat{\boldsymbol{\mathcal{E}}}_\perp(x, y, z, \omega; L) &= \boldsymbol{F}_\perp \left(x, y, \frac{z}{L}, \frac{\omega}{L} \right), \\ \hat{\mathcal{E}}_z(x, y, z, \omega; L) &= \frac{1}{L} G \left(x, y, \frac{z}{L}, \frac{\omega}{L} \right), \end{aligned} \tag{13}$$

where $\boldsymbol{F}_\perp$ and G are functions of four arguments. To prove this statement, first we need to show that substituting Eqs. (13) into our equations and the boundary condition results in expressions which involve the coordinate z, the parameter L, and the wavenumber k as combinations z/L and k/L only. Indeed, substituting into Eqs. (9) we find

$$\begin{aligned} \frac{k}{L} \frac{\partial}{\partial \zeta} \boldsymbol{F}_\perp &= \frac{i}{2} \nabla_\perp^2 \boldsymbol{F}_\perp, \\ G &= \frac{iL}{k} \nabla_\perp \cdot \boldsymbol{F}_\perp, \end{aligned} \tag{14}$$

which clearly satisfies our requirement.

We now take a close look at the boundary condition (10). Rewriting it in terms of perpendicular and transverse components of the field we obtain (remember that $\hat{\boldsymbol{E}}_{\text{vac}}$ has only perpendicular components)

$$\begin{aligned} n_z \boldsymbol{e}_z \times \hat{\boldsymbol{\mathcal{E}}}_\perp|_w + \boldsymbol{n}_\perp \times \hat{\boldsymbol{\mathcal{E}}}_\perp|_w + \boldsymbol{n}_\perp \times \boldsymbol{e}_z \hat{\mathcal{E}}_z|_w \\ + n_z \boldsymbol{e}_z \times \hat{\boldsymbol{E}}_{\text{vac}}|_w + \boldsymbol{n}_\perp \times \hat{\boldsymbol{E}}_{\text{vac}}|_w = 0. \end{aligned} \tag{15}$$

The first, third and fourth terms in this equation are perpendicular to $\boldsymbol{e}_z$, and the second and fifth terms are directed in the z-direction. Hence they can be split into two separate equations. The first one is

$$
\begin{aligned}
&n_z \boldsymbol{e}_z \times \hat{\boldsymbol{\mathcal{E}}}_\perp|_w + \boldsymbol{n}_\perp \times \boldsymbol{e}_z \hat{\mathcal{E}}_z|_w + n_z \boldsymbol{e}_z \times \hat{\boldsymbol{E}}_{\text{vac}}|_w = \\
&\frac{1}{L}\frac{\partial V}{\partial \zeta}\boldsymbol{e}_z \times \hat{\boldsymbol{F}}_\perp|_w + \frac{1}{L}\boldsymbol{n}_\perp \times \boldsymbol{e}_z G|_w + \frac{1}{L}\frac{\partial V}{\partial \zeta}\boldsymbol{e}_z \times \hat{\boldsymbol{E}}_{\text{vac}}|_w = 0.
\end{aligned} \tag{16}
$$

The last line, after cancellation of the factor $1/L$, clearly shows that the parameter L does not enter explicitly into it. The second boundary equation is

$$
\begin{aligned}
&\boldsymbol{n}_\perp \times \hat{\boldsymbol{\mathcal{E}}}_\perp|_w + \boldsymbol{n}_\perp \times \hat{\boldsymbol{E}}_{\text{vac}}|_w = \\
&\boldsymbol{n}_\perp \times \hat{\boldsymbol{F}}_\perp|_w + \boldsymbol{n}_\perp \times \hat{\boldsymbol{E}}_{\text{vac}}|_w = 0,
\end{aligned} \tag{17}
$$

and it again does note explicitly contain the parameter L . Our statement is therefore proved.

Substituting the second of Eqs. (13) into (7), we find the scaling property for the longitudinal impedance

$$
Z(\omega; L) = R\left(\frac{\omega}{L}\right), \tag{18}
$$

where R is a function of one variable. Making the Fourier transformation that related the impedance to the wakefield, we also find that the wake has the following scaling property

$$
w(z; L) = Lu\left(zL\right), \tag{19}
$$

where u is a function of one variable.

[1] G. Stupakov, New Journal of Physics **8**, 280 (2006).

[2] G. Stupakov, Preprint SLAC-PUB-13661, SLAC (2009).

[3] G. Stupakov and I. A. Kotelnikov, Phys. Rev. ST Accel. Beams **12**, 000000 (2009).