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Explicit Expressions of Impedances and Wake Functions

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Abstract

Sections 3.2.4 and 3.2.5 of the Handbook of Accelerator Physics and Engineering on Landau damping are combined and updated. The new addition includes impedances and wakes for multi-layer beam pipe, optical model, diffraction model, and cross-sectional transition.

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3rd edition of Handbook of
Accelerator Physics and Engineering

3.2.4 Explicit Expressions of Impedances and Wake Functions

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See tables in the next pages.

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General Remarks and Notation:

In cylindrically symmetric structures $W'_m(z)$ and $W_m(z)$ denote, respectively, m-th azimuthal multipole longitudinal and transverse wake functions, generated by point charge Q, at distance $-z>0$ behind. $W'_m(z) \equiv dW_m(z)/dz$. $W'_m(z)=0$ and $W_m(z)=0$ when $z>0$ when particle travels at the speed of light. $W'_m(0)=\frac{1}{2}\lim_{z\to 0^-} W'_m(z)$. Longitudinal and transverse momentum kicks on test charge q near pipe axis: $\Delta p_{\|}(z)=-qQW'_0(z)/c$, $\Delta p_{\perp}(z)=-qQ\tilde{r}_{\perp}W_1(z)/c$, where $\tilde{r}_{\perp}$ is (small) offset of the source or exciting charge.

The m-th multipole longitudinal impedance $Z^{\|}_m(k)=\int e^{-ikz/\beta}W^{\|}_m(z)\,dz/(\beta c)$ is related to the m-th multipole transverse impedance, $Z^{\perp}_m(k)=i\int e^{-ikz/\beta}W^{\perp}_m(z)\,dz/(\beta^2 c)$, by $Z^{\|}_m=kZ^{\perp}_m$ ($m\neq 0$), where $k=\omega/c$. Note that $Z^{\|}_m(-k)=Z^{\|*}_m(k)$, $Z^{\perp}_m(-k)=-Z^{\perp *}_m(k)$.

For periodic or translationally invariant structures: steady-state results are given per length L. Unless otherwise stated, structures are cylindrically symmetric with perfectly conducting metallic walls, and with beam pipes of radius b. In many cases, $\beta=v/c$ has been set to 1. $Z_0=\sqrt{\mu_0/\epsilon_0}\approx 377\,\Omega$ is impedance, ϵ_0 electric permittivity, and μ_0 magnetic permeability of free space. 'Pill-box cavity' signifies a pill-box with beam pipes. Here $[\alpha\pm i|\beta|]^{1/n}$ (with α, β real, $n=2$ or 3) is in the 1st/4th quadrant. $H(x)=0,1$ for $x\lessgtr 0$.

For 3D structures with mirror symmetry in x and y, near axis momentum kick in y, $\Delta p_y=-qQ(\tilde{y}W^y_d+yW^y_q)$, with $\tilde{y}$ (y) offset of exciting (test) charge, and W^y_d (W^y_q) dipole (quad) wake terms. Total y wake $W_y=W^y_d+W^y_q$; total y impedance $Z_y=Z^y_d+Z^y_q$.

<table>
<tr><th>Description</th><th>Impedances</th><th>wake</th></tr>
<tr><td>Space-charge: [1] beam radius a in a perfectly conducting beam pipe of radius b, transverse distribution uniform.</td>
<td>$\frac{Z_0^{\|}}{L}=i\frac{Z_0 k g_0}{4\pi\beta^2\gamma^2}\qquad g_0=1+2\ln\frac{b}{a}$
$\frac{Z^{\perp}_{m\neq 0}}{L}=i\frac{Z_0}{2\pi\beta^2\gamma^2 m}\left[\frac{1}{a^{2m}}-\frac{1}{b^{2m}}\right]$</td>
<td>$\frac{W'_0}{L}=\frac{Z_0 c}{4\pi\gamma^2}\left[1+2\ln\frac{b}{a}\right]\delta'(z)$
$\frac{W_{m\neq 0}}{L}=\frac{Z_0 c}{2\pi\gamma^2 m}\left[\frac{1}{a^{2m}}-\frac{1}{b^{2m}}\right]\delta(z)$</td></tr>
<tr><td>Nonuniform distributions: [2] $a^{\|}_{\rm eff}$ is equivalent-uniform-beam radius, $g_0=1+2\ln(b/a^{\|}_{\rm eff})$, while $a^{\perp}_{\rm eff}=[\pi\lambda(0)]^{-1/2}$ is the same when self-force part written as $1/a^{\perp\,2}_{\rm eff}$, $\gamma_e\approx 0.57721$ is Euler's constant.</td>
<td colspan="2">
<table>
<tr><th>Distribution $\lambda(r)$</th><th>g_0 ($m=0$)</th><th>$a^{\|}_{\rm eff}$ ($m=0$)</th><th>$a^{\perp}_{\rm eff}$ ($m=1$)</th></tr>
<tr><td>$\frac{3}{2\pi\hat{r}}\left(1-\frac{r^2}{\hat{r}^2}\right)^{1/2}H(\hat{r}-r)$</td><td>$\frac{8}{3}+2\ln\frac{b}{2\hat{r}}$</td><td>$0.8692\hat{r}$</td><td>$\sqrt{\frac{2}{3}}\hat{r}$</td></tr>
<tr><td>$\frac{1}{2\pi\hat{r}^2}\left(1-\frac{r^2}{\hat{r}^2}\right)H(\hat{r}-r)$</td><td>$\frac{3}{2}+2\ln\frac{b}{\hat{r}}$</td><td>$0.7788\hat{r}$</td><td>$\frac{1}{\sqrt{2}}\hat{r}$</td></tr>
<tr><td>$\frac{2\pi}{\pi^2-4}\cos^2\frac{\pi r}{2\hat{r}}H(\hat{r}-r)$</td><td>$1.921+2\ln\frac{b}{\hat{r}}$</td><td>$0.6309\hat{r}$</td><td>$\frac{\sqrt{\pi^2-4}}{\sqrt{2}\pi}\hat{r}$</td></tr>
<tr><td>$\frac{1}{2\pi\sigma_r^2}e^{-r^2/(2\sigma_r^2)}$</td><td>$\gamma_e+2\ln\frac{b}{\sqrt{2}\sigma_r}$</td><td>$1.7647\sigma_r$</td><td>$\sqrt{2}\sigma_r$</td></tr>
</table>
Image part of $Z^{\perp}_1$ can be written in terms of Laslett's electric image coefficients as $1/b^2\to 2(\xi_{1x,y}-\epsilon_{1x,y})/h^2$ with h denoting half height of vacuum chamber. See Sec.??.</td></tr>
<tr><td>Resistive wall: [1, 3] wall thickness t, dc and ac conductivities σ_c, $\tilde{\sigma}_c$, relaxation time τ; assume $|k|b\gg(s_0/b)^3$, thick walls: $t\gg\delta_c=\sqrt{2/(|k|Z_0\mu_r\sigma_c)}$, the skin depth.</td>
<td colspan="2">$\frac{Z^{\|}_m}{L}=\frac{Z_0/(\pi b^{2m+1})}{(1+\delta_{m0})\sqrt{\frac{iZ_0\tilde{\sigma}_c}{k\mu_r}}-\frac{ibk}{m+1}},\qquad \mu_r=\begin{cases}\text{relative magnetic}\\ \text{permeability}\end{cases}$
$\tilde{\sigma}_c=\sigma_c/(1-ikc\tau)$
Typically, $\tau=27/40/8$ fs for Cu/Ag/Al. Valid for $c\tau/s_0\ll 1$, characteristic distance $s_0=[2b^2\mu_r/(Z_0\sigma_c)]^{1/3}$, $\alpha=[(m+1)(1+\delta_{m0})/2]^{2/3}$.
$\frac{W'_m}{L}=\frac{4Z_0c(m+1)}{\pi b^{2m+2}}\left[\frac{e^{\alpha z/s_0}}{3}\cos\left(\frac{\sqrt{3}\alpha z}{s_0}\right)-\frac{\sqrt{2}}{\pi}\int_0^{\infty}dx\,\frac{x^2e^{\alpha zx^2/s_0}}{x^6+8}\right]$</td></tr>
</table>

Description	Impedances	Wakes
Low frequency: [1] $k \ll 1/s_0$, long range $\lvert z\rvert \gg s_0$.	$\dfrac{Z_m^{\parallel}}{L} = \dfrac{1-\mathrm{sgn}(k)i}{(1+\delta_{0m})\pi\sigma_c\delta_c b^{2m+1}}$ Note : $\quad Z_1^{\perp} = \dfrac{2}{b^2 k} Z_0^{\parallel}$,	$\dfrac{W_m'}{L} = \dfrac{-c}{2\pi b^{2m+1}(1+\delta_{m0})}\sqrt{\dfrac{Z_0\mu_r}{\pi\sigma_c}}\dfrac{1}{(-z)^{3/2}}$ $W_1 = \dfrac{2}{b^2}\displaystyle\int W_0'\,dz$
Low frequency, thin wall: [1] $t \ll \delta_c$ and $\lvert k\rvert \ll 1/\sqrt{bt}$.	$\dfrac{Z_0^{\parallel}}{L} = -\dfrac{Z_0 kt}{2\pi b}, \quad \dfrac{Z_1^{\perp}}{L} = -i\dfrac{Z_0 t}{\pi b^3}$	$\dfrac{W_0'}{L} = -\dfrac{Z_0 tc}{2\pi b}\delta'(z), \ \dfrac{W_1}{L} = -\dfrac{Z_0 ct}{\pi b^3}\delta(z)$
High frequency: [3] $k \gtrsim 1/s_0$, short range $\lvert z\rvert \lesssim s_0$, with $c\tau \gtrsim s_0$. $k_p = \sqrt{Z_0\sigma_c/c\tau}$ is plasma frequency/c.	$\dfrac{Z_m^{\parallel}}{L} = \dfrac{4Z_0 c\tau(m+1)}{\pi b^{2m+1}}$ $\times\dfrac{1-4ikc\tau}{b(1-4ikc\tau)^2 + 32k_p(\alpha c\tau)^2}$	$\dfrac{W_m'}{L} = \dfrac{Z_0 c(m+1)}{\pi b^{2m+2}} e^{z/4c\tau}$ $\times \cos\left[\sqrt{\dfrac{2k_p}{b}}\alpha z\right]$, for α see above
	These formulae depend only on the plasma frequency of the metal. Effects of relative magnetic permeability have not been considered.	
Finite length, lossy insert: [4] of length L, in lossless pipe	If $b^2/s_0^3 k^2 \ll L \ll kb^2, \quad Z_0^{\parallel} = \dfrac{2^{1/2} Z_0}{\pi^{3/2} b}\sqrt{\dfrac{iL}{k}}$, else $Z_0^{\parallel}$ as given above	
Displaced beam: [5] at $\vec{a} = (a_x, a_y)$, rms bunch length σ_ℓ, average current I_b, and $(b/k^2, b, b-a) \gg \delta_c$ and $\gamma \gg 1$.	Wall impedances in last section multiplied by f_z for $Z_0^{\parallel}$ and $f_{x,y}$ for $Z_1^{x,y}$ with $f_z = \dfrac{b^2+a^2}{b^2-a^2}, \quad f_x = \dfrac{b(b^2-a^2+4a_x^2)}{(b^2-a^2)^3}, \quad f_y = \dfrac{b(b^2-a^2+4a_y^2)}{(b^2-a^2)^3}$ Power loss per length traversed is $\dfrac{P}{L} = \dfrac{\Gamma(\frac{3}{4}) I_b^2}{4\pi^2 b\sigma_\ell^{3/2}\sqrt{2\mu_r\sigma_c/Z_0}} f_z$	
Displaced beam between two infinite plates: [5] at $y = \pm h/2$. $\gamma \gg 1$, $[h/k^2, h-2y_0] \gg \delta_c$. Thin dielectric coating of thickness Δh.	$Z_0^{\parallel} = \dfrac{1-\mathrm{sgn}(\omega)i}{\pi h}\sqrt{\dfrac{\lvert\omega\rvert\mu_r Z_0}{2c\sigma_c}} f_z, \quad Z_1^{\perp} = \dfrac{\pi(\mathrm{sgn}(\omega)1-i)}{\sqrt{2\lvert\omega\rvert\sigma_c/(c\mu_r Z_0)}} f_{\perp}$ $f_z = 1 + \dfrac{\pi y_0}{h}\tan\dfrac{\pi y_0}{h}, \quad f_{\perp} = \dfrac{f_z}{h^3\cos^2(\pi y_0/h)}$, beam at $y = y_0$ $Z_0^{\parallel} = -\dfrac{i\omega Z_0(\epsilon_r\mu_r - 1)\Delta h}{\pi c\epsilon_r h} f_z, \quad Z_1^{\perp} = -\dfrac{i\pi Z_0(\epsilon_r\mu_r-1)\Delta h}{\epsilon_r} f_{\perp}$	
Metallic coating on ceramic pipe: [6] compared with all metal pipe $Z_0^{\parallel}(\mathrm{met})$. $t_{m,c} =$ metal/ceramic thickness $\ll b$. $\gamma \gg 1$, $[(\epsilon_r-1)t_c^2, (1-\epsilon_r^{-1})bt_c] \ll \sigma_\ell^2$. Loss P/L is max. at $V = 0.82$.	$Z_0^{\parallel} = Z_0^{\parallel}(\mathrm{met})\dfrac{A+\tanh(\nu t_m)}{1+A\tanh(\nu t_m)}, \quad A = \left(1 - \dfrac{1}{\epsilon_r}\right)\nu t_c, \quad \nu = \dfrac{1-\mathrm{sgn}(\omega)i}{\delta_c}$ $\dfrac{P}{L} = \dfrac{Z_0 I_b^2 t_c(\epsilon_r - 1)}{4\sqrt{\pi} b\sigma_\ell^2\epsilon_r}\left[V - \sqrt{\pi}V^2 e^{V^2}\mathrm{erfc}(V)\right], \quad V = \dfrac{\epsilon_r\sigma_\ell}{(\epsilon_r-1)Z_0\sigma_c t_m t_c}$ Field penetration through pipe, $\dfrac{E_{z,\mathrm{out}}}{E_{z,\mathrm{in}}} = \dfrac{1}{\sqrt{1+4(1-1/\epsilon_r)t_m t_c/\delta_c^2}}$, becomes significant when $t_m \lesssim t_{\mathrm{crit}} = \delta_c^2/t_c$. P/L is at maximum at t_{crit}.	
Elliptical beam pipe:	Low frequency, see [7, 8, 5], high frequency, see [9, 10].	
Rectangular beam pipe:	Low frequency, see [8], high frequency, see [9, 11].	

Multi-layer pipe wall impedances: [12, 13] Cylindrical beam pipe with N layers, pth layer between $b^{(p-1)}<r<b^{(p)}$ and $b^{(N)}\to\infty$. Layer 1 is vacuum, $a<r<b^{(1)}$, with particle beam of charge Q at $r=a$ and $\theta=0$. $r<a$ is called the 0-th layer. Each layer has its own wavenumber $\nu=k\sqrt{1-\beta^2\varepsilon_1\mu_1}$, $k=\omega/v$ and own properties $\epsilon=\epsilon_0\varepsilon_1=\epsilon_0\epsilon_r(1+i\tan\vartheta_{\rm E})-\frac{\sigma_{\rm dc}}{i\omega(1-i\omega\tau)}$, $\mu=\mu_0\mu_1=\mu_0\mu_r(1+i\tan\vartheta_{\rm M})$; $\vartheta_{\rm E}$, $\vartheta_{\rm M}$ are loss angles, $\sigma_{\rm dc}$ dc conductivity, and τ relaxation time. Actually any frequency dependent ϵ, μ, and conductivity can be assumed. Inside vacuum, $\nu=k/\gamma$; inside conducting metal of skin depth δ_c, $\nu\approx(1-i)/\delta_c$. A user-friendly Mathematica code for computation is available [12]. The derivation is outlined briefly below. In terms of Bessel and Kelvin functions, mth multipole longitudinal fields inside p-th layer:

$$E_s^{(p)}=\cos m\theta\, e^{iks}\left[C_{Ie}^{(p)}I_m\big(\nu^{(p)}r\big)+C_{Ke}^{(p)}K_m\big(\nu^{(p)}r\big)\right],\quad \vec{E}\text{ is electric field}$$

$$G_s^{(p)}=\sin m\theta\, e^{iks}\left[C_{Ig}^{(p)}I_m\big(\nu^{(p)}r\big)+C_{Kg}^{(p)}K_m\big(\nu^{(p)}r\big)\right],\quad \vec{G}=Z_0\vec{H},\ \vec{H}\text{ is magnetic field}$$

Matching E_s, E_θ, G_s, and G_θ at boundary $r=b^{(p)}$ between p-th and $(p+1)$-th layers gives

$$\begin{bmatrix}C_{Ie}^{(p+1)}\\C_{Ke}^{(p+1)}\\C_{Ig}^{(p+1)}\\C_{Kg}^{(p+1)}\end{bmatrix}=M_p^{p+1}\begin{bmatrix}C_{Ie}^{(p)}\\C_{Ke}^{(p)}\\C_{Ig}^{(p)}\\C_{Kg}^{(p)}\end{bmatrix}\ \overset{\text{iteratively}}{\Longrightarrow}\ \begin{bmatrix}C_{Ie}^{(N)}\\C_{Ke}^{(N)}\\C_{Ig}^{(N)}\\C_{Kg}^{(N)}\end{bmatrix}=\mathcal{M}\begin{bmatrix}C_{Ie}^{(1)}\\C_{Ke}^{(1)}\\C_{Ig}^{(1)}\\C_{Kg}^{(1)}\end{bmatrix}\ \text{where}\ \begin{cases}\mathcal{M}\equiv M_{N-1}^{N}M_{N-2}^{N-1}\cdots M_1^2\\ \text{See [12] for explicit}\\ \text{expression of } M_p^{p+1}\end{cases}$$

Since the last layer goes to infinity, $C_{Ie}^{(N)}=C_{Ig}^{(N)}=0$. From the beam region, $C_{Kg}^{(1)}=0$ and $C_{Ke}^{(1)}=-ikQZ_0I_m(ka/\gamma)/[\pi\beta\gamma^2(1+\delta_{m0})]$, one can easily solve for

$$C_{Ie}^{(1)}\equiv-\alpha_1\frac{K_m^{(1)}}{I_m^{(1)}}C_{Ke}^{(1)}=-C_{Ke}^{(1)}\frac{\mathcal{M}_{12}\mathcal{M}_{33}-\mathcal{M}_{32}\mathcal{M}_{13}}{\mathcal{M}_{11}\mathcal{M}_{33}-\mathcal{M}_{13}\mathcal{M}_{31}},\ \text{with}\ I_m^{(1)}=I_m\left(\nu^{(1)}b^{(1)}\right),\ I_m^{(1)}=I_m\left(\nu^{(1)}b^{(1)}\right).$$

With beam at $r=a_1$, $\theta=0$, reduced forces on a unit test charge at $r=a_2>a_1$ and $\theta=\theta_2$ are

$$Z_\parallel=-\int ds\,E_s(a_2,\theta_2,s;\omega)e^{-iks},\quad Z_x=-i\int ds\,[E_x(a_2,\theta_2,s;\omega)-\beta G_y(a_2,\theta_2,s;\omega)]\,e^{-iks}.$$

Space-charge contributions for all multiples ($\alpha_1=1$ or perfectly conducting at $r=b^{(1)}$):

$$Z_\parallel^{\rm SC}=\sum_{m=0}^{\infty}\frac{ikZ_0L\cos m\theta_2}{\pi\beta\gamma^2(1+\delta_{m0})}I_m(x_2)\mathcal{K}_m(x_1),\quad \mathcal{K}_m(x_i)=\left[K_m(x_i)-\frac{K_m^{(1)}}{I_m^{(1)}}I_m(x_i)\right],\quad x_i=\frac{ka_i}{\gamma}$$

$$Z_x^{\rm SC}=\sum_{m=0}^{\infty}\frac{ikZ_0L}{\pi\beta\gamma^3(1+\delta_{m0})}I_m(x_1)\left[\cos\theta_2\cos m\theta_2\mathcal{K}_m(x_2)+\frac{m\gamma}{a_2}\sin\theta_2\sin m\theta_2\mathcal{K}'_m(x_2)\right]$$

The rest are from wall impedances. To any order $a_1^{n_1}a_2^{n_2}$, they are

$$Z_\parallel^{W,n_1,n_2}=-\frac{iL\mu_0\omega}{\pi\beta^2\gamma^2}\left(\frac{ka_1}{2\gamma}\right)^{n_1}\left(\frac{ka_2}{2\gamma}\right)^{n_2}\sum{}'\frac{\cos m\theta_2\bar{\alpha}_1(m)K_m^{(1)}/I_m^{(1)}}{(1+\delta_{m0})(\frac{n_1-m}{2})!(\frac{n_1+m}{2})!(\frac{n_2-m}{2})!(\frac{n_2+m}{2})!}$$

$$Z_x^{W,n_1,n_2}=-\frac{iZ_0L}{\pi\beta\gamma^2a_2}\left(\frac{ka_1}{2\gamma}\right)^{n_1}\left(\frac{ka_2}{2\gamma}\right)^{n_2}\sum{}'\frac{(n_2\cos\theta_2\cos m\theta_2+m\sin\theta_2\sin m\theta_2)\bar{\alpha}_1(m)K_m^{(1)}/I_m^{(1)}}{(1+\delta_{m0})\,(\frac{n_1-m}{2})!(\frac{n_1+m}{2})!(\frac{n_2-m}{2})!(\frac{n_2+m}{2})!}$$

where $\bar{\alpha}_1(m)\equiv1-\alpha_1(m)$, $\sum'$ implies from $m=0$ to $\min(n_1,n_2)$ with n_1-n_2 and n_1-m even. The usual monopole and dipole pipe-wall impedances are

$$Z_0^{\parallel}=Z_\parallel^{W,0,0}=\frac{ikZ_0L}{2\pi\beta\gamma^2}\frac{\bar{\alpha}_1K_0^{(1)}}{I_0^{(1)}}\ \text{and}\ Z_1^{\perp}=\frac{Z_x^{W,1,1}}{a_1}=\frac{iLZ_0k^2}{4\pi\beta\gamma^4}\frac{\bar{\alpha}_1K_1^{(1)}}{I_1^{(1)}}$$

Multi-layer special cases: [13] Pipe wall: $b^{(1)}<r<b^{(2)}=b^{(1)}+t$.

Thin wall: Good for low frequencies. $t\to0$ and E_s does not change across wall. At $r=b^{(3)}$, Case PC: perfectly conducting, Case PM: perfectly magnetic, and Case INF: $b^{(2)}\to\infty$.

$$\bar{\alpha}_1=-\frac{\gamma^2\beta^2(1-\alpha_2)+2ix\gamma\beta/m\zeta}{1+\frac{x^2}{m^2}-\frac{ix\gamma\beta}{m}\left[\frac{2}{\zeta(1-\alpha_2)}+\frac{\zeta(1-\alpha_2)}{2}\right]},\quad \alpha_2^{\rm PC}=-\alpha_2^{\rm PM}=\frac{K_m(y)I_m(x)}{K_m(x)I_m(y)}\underset{m\neq0}{\approx}\left(\frac{b^{(1)}}{b^{(3)}}\right)^2,\quad \alpha_2^{\rm INF}=0$$

$x=kb^{(1)}/\gamma$, $y=kb^{(3)}/\gamma$, $\zeta=Z_0\sigma_ct$, and $m\neq0$.

<table>
<tr><td colspan="3">

Thick wall: Good for high frequencies. At $r = b^{(2)}$, Case PC: perfectly conducting, Case PM: perfectly magnetic, and Case INF: $b^{(2)} \to \infty$. For $m \geq 1$,

$$\bar{\alpha}_1 = \frac{-2\beta^2\gamma^2\left[1 - \frac{(1+i)\Delta Q_\eta}{2m\beta\gamma^2}\right]}{1 - 2ip - \beta\left[\frac{(1-i)Q_\alpha}{m\Delta} - \frac{(1+i)\Delta Q_\eta}{2m}\right] + \frac{Q_\alpha Q_\eta - m^2p^2}{m^2\gamma^2}}, \quad p = \frac{k^2\delta^2}{2}, \quad \Delta = \mu_1\beta k\delta_c,$$

$$Q_\alpha = kb^{(1)}\frac{Q_2 - \alpha_2 P_2}{1-\alpha_2}, \quad Q_\eta = kb^{(1)}\frac{Q_2 - \eta_2 P_2}{1-\eta_2}, \quad Q_2 = \frac{K_m'^{(2)}}{K_m^{(2)}}, \quad P_2 = \frac{I_m'^{(2)}}{I_m^{(2)}}$$

Boundary conditions require $\alpha_2^{\mathrm{PC}} = \eta_2^{\mathrm{PM}} = \dfrac{K_m^{(2,3)} I_m^{(2)}}{I_m^{(2,3)} K_m^{(2)}}, \quad \eta_2^{\mathrm{PC}} = \alpha_2^{\mathrm{PM}} = \dfrac{K_m'^{(2,3)} I_m^{(2)}}{I_m'^{(2,3)} K_m^{(2)}},$

$\alpha_2^{\mathrm{INF}} = \eta_2^{\mathrm{INF}} = 0$, with $I_m^{(p+1,p)} = I_m\left(\nu^{(p+1)} b^{(p)}\right)$, $I_m^{(p)} = I_m\left(\nu^{(p)} b^{(p)}\right)$ and similar definitions for I_m', K_m, and K_m'.

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Electric- and magnetic-dipole approximation: $\bar{\alpha}_1$ can also be derived [14] by approximating beam dipole motion as a superposition of oscillating electric and magnetic dipoles.

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<tr><td>Description</td><td>Impedances</td><td>Wakes</td></tr>
<tr><td colspan="3">

High frequency optical model: [15] High frequency $k \gg 1/h$, short-range $-z \ll h$, transition length $L \ll kh^2$, h is minimum aperture. For tapered transition of angle θ, need $k \gg 1/h\theta$.

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Transitions, shallow cavities, collimators, irises:
(a) Axially symmetric examples: [16]-[18] (i) step-in transition (from d to b), (ii) step-out (from b to d), long collimator, shallow cavity with gap g, (iii) thin iris
(b) 3D, mirror symmetric in x, y: [15] (i) flat step-out transition, aperture $2b$ to $2d$, (ii) *any* step-in transition; iris with small (iii) flat (height $2b$), (iv) elliptical (axes w by b), aperture

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$Z^{\|}$ and $kZ^{\perp}$ are both constants similar for $kZ_d^{\perp}$, $kZ_q^{\perp}$, $W_d^{\perp}$, $W_q^{\perp}$

</td><td>

$$W_{\|} = -Z^{\|} c\delta(z)$$
$$W_{\perp} = -kZ^{\perp} cH(-z)$$

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$$\text{(i)}\quad Z_0^{\|} = Z_1^{\perp} = 0, \quad \text{(ii)}\quad Z_0^{\|} = \frac{Z_0}{\pi}\ln\frac{d}{b}, \quad kZ_1^{\perp} = \frac{Z_0}{\pi}\left(\frac{1}{b^2} - \frac{1}{d^2}\right)$$

$$\text{(iii)}\quad Z_0^{\|} = \frac{Z_0}{\pi}\ln\frac{d}{b}, \quad kZ_1^{\perp} = \frac{Z_0}{2\pi}\left(\frac{1}{b^2} - \frac{b^2}{d^4}\right)$$

where b is small iris or pipe radius, d is large pipe radius. Note: for shallow cavity, waves reflect from outer wall $\Rightarrow g \gtrsim k(d-b)^2$; for collimator, bottom length $\gg kb^2$

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$$\text{(i)}\quad kZ_y = \frac{\pi}{8}Z_0\left(\frac{1}{b^2} - \frac{1}{d^2}\right), \quad Z_q^y = \frac{1}{2}Z_d^y = \frac{1}{3}Z_y$$

$$\text{(ii)}\quad Z_{\|} = Z_{\perp} = 0, \quad \text{(iii)}\quad kZ_y = \frac{Z_0}{2\pi b^2}, \quad Z_q^y = Z_d^y = \frac{1}{2}Z_y$$

$$\text{(iv)}\quad kZ_y = \frac{Z_0}{2\pi b^2}, \quad kZ_d^y = \frac{Z_0}{4\pi b^2}\left(1 + \frac{b^2}{w^2}\right), \quad kZ_q^y = \frac{Z_0}{4\pi b^2}\left(1 - \frac{b^2}{w^2}\right)$$

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High frequency diffraction formulae: $k \gg 1/b$
(a) Deep cavity (Fresnel diffraction) [19, 1], cavity radius d and gap g.

</td><td>

$$Z_m^{\|} = \frac{\sqrt{2}Z_0}{(1+\delta_{m0})\pi^{3/2}b^{2m+1}}\sqrt{\frac{ig}{k}}$$
$$Z_1^{\perp} = \frac{2}{b^2 k}Z_0^{\|}$$

Note: no reflections from outer wall $\Rightarrow g \lesssim k(d-b)^2$.

</td><td>

$$W_m' = \frac{\sqrt{2}Z_0 c}{(1+\delta_{m0})\pi^2 b^{2m+1}}\sqrt{\frac{g}{-z}}$$
$$W_1 = \frac{2}{b^2}\int W_0'\, dz$$

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<table>
<tr><th>Description</th><th>Impedances</th><th>Wakes</th></tr>
<tr><td>(b) Periodic array of deep cavities (model for linear accelerator structures): [20]-[24] period L, gap g, outer cavity radius d, with $g \lesssim k(d-b)^2$.</td><td>$$\frac{Z_0^{\parallel}}{L}=\frac{iZ_0}{\pi k b^2}\left[1+\frac{\alpha(g/L)L}{b}\sqrt{\frac{2\pi i}{kg}}\right]^{-1}$$ $$\alpha(\zeta)\approx 1-0.465\sqrt{\zeta}-0.070\zeta$$ $$Z_1^{\perp}=\frac{2}{b^2k}Z_0^{\parallel}$$</td><td>$$\frac{W_0'}{L}=\frac{Z_0c}{\pi b^2}e^{\eta(z)^2}\operatorname{erfc}[\eta(z)]$$ $$\eta(z)=\frac{\alpha L}{b}\sqrt{\frac{-2\pi z}{g}}$$ $$W_1=\frac{2}{b^2}\int W_0'\,dz$$</td></tr>
<tr><td>Numerical fit: [24, 25] valid over larger z range: $-z/L \le 0.15$, $0.34 \le b/L \le 0.69$, $0.54 \le g/L \le 0.89$.</td><td colspan="2">$$W_0'=\frac{Z_0c}{\pi b^2}\exp\left(-\sqrt{\frac{z}{z_0}}\right),\ \ W_1=\frac{4Z_0cz_1}{\pi b^4}\left[1-\left(1+\sqrt{\frac{z}{z_1}}\right)\exp\left(-\sqrt{\frac{z}{z_1}}\right)\right]$$ $$z_0=-0.41\frac{b^{1.8}g^{1.6}}{L^{2.4}},\quad z_1=-0.17\frac{b^{1.79}g^{0.38}}{L^{1.17}}$$</td></tr>
<tr><td>Bethe's dipole moments of a hole of radius a on beam pipe wall [26].</td><td colspan="2">Electric and magnetic dipole moments when wavelength $\gg a$: $$\vec{d}=-\frac{2\epsilon_0}{3}a^3\vec{E}\,,\ \ \vec{m}=-\frac{4}{3\mu_0}a^3\vec{B}$$ $\vec{E}$ and $\vec{B}$ are electric and magnetic flux density at hole when hole is absent. This is a diffraction solution for a thin-wall pipe.</td></tr>
<tr><td rowspan="2">Small 3D obstacle on beam pipe: [27, 28] size $\ll b$, low freq. $k \ll 1/(\text{size})$; ϕ azimuthal angular position of object.</td><td>$$Z_0^{\parallel}=-ikc\mathcal{L},$$ $$Z_1^{\perp}=\frac{4}{b^2k}Z_0^{\parallel}\cos\phi$$</td><td>$$W_0'=-c^2\mathcal{L}\,\delta'(z)$$ $$W_1=\frac{4}{b^2}\int W_0'\cos\phi\ dz$$</td></tr>
<tr><td colspan="2">Inductance $\mathcal{L}=\dfrac{Z_0(\alpha_e+\alpha_m)}{4\pi^2b^2c}$
α_e is electric polarizability, α_m magnetic susceptibility</td></tr>
<tr><td>Elliptical hole: major and minor radii are a and d. $K(m)$ and $E(m)$ are complete elliptical functions of the first and second kind, with $m=1-m_1$ and $m_1=(d/a)^2$. For long ellipse perpendicular to beam, major axis $a \ll b$, beam pipe radius, because the curvature of the beam pipe has been neglected here [29].</td><td colspan="2">$$\alpha_e+\alpha_m=\begin{cases}\dfrac{\pi a^3 m_1^2[K(m)-E(m))]}{3E(m)[E(m)-m_1K(m)]} \\ \dfrac{\pi a^3[E(m)-m_1K(m)]}{3[K(m)-E(m)]}\end{cases}\overset{m\to 1}{\underset{\text{long ellipse}}{\Longrightarrow}}\begin{cases}\dfrac{\pi d^4[\ln(4a/d)-1]}{3a} & \parallel \text{beam},\ d\ll b \\ \dfrac{\pi a^3}{3\,[\ln(4a/d)-1]} & \perp\text{beam},\ a\ll b\end{cases}$$ $$\alpha_e+\alpha_m\ \underset{m\to 0}{\xrightarrow{\text{circular}}}\ \frac{2a^3}{3}\quad \text{circular hole } a=d\ll b$$ Above are for $t \ll a$. When $t \ge a$, $\times 0.56$ when hole is circular and $\times 0.59$ when hole is long-elliptic.
For higher frequency correction, add to $\alpha_e+\alpha_m$ the extra term, $$+\frac{2\pi a^3}{3}\left[\frac{11k^2a^2}{30}\right]\ \text{circular},\ \begin{cases}-\dfrac{\pi a d^2}{3}\left[\dfrac{k^2a^2}{5}\right] & \parallel\text{beam long ellipse} \\ +\dfrac{2\pi a^3}{3}\left[\dfrac{2k^2a^2}{5[\ln(4a/d)-1]}\right] & \perp\text{beam long ellipse}\end{cases}$$</td></tr>
<tr><td>Rectangular slot: length L, width w.</td><td colspan="2">$\alpha_e+\alpha_m=w^3(0.1814-0.0344w/L)\quad t\ll a,\quad \times 0.59$ when $t\ge a$</td></tr>
<tr><td>Rounded-end slot: length L, width w.</td><td colspan="2">$\alpha_e+\alpha_m=w^3(0.1334-0.0500w/L)\quad t\ll a,\quad \times 0.59$ when $t\ge a$</td></tr>
</table>

Description	Impedances	wake
Annular-ring-shaped cut: inner and outer radii a and $d=a+w$ with $w \ll d$.	$\alpha_e+\alpha_m=\dfrac{\pi^2 d^2 a}{2\ln(32d/w)-4}-\dfrac{\pi^2 w^2(a+d)}{16} \quad t \ll d$ $\alpha_e+\alpha_m=\pi d^2 w-\frac{1}{2}w^2(a+d) \quad t \geq d$	
Half ellipsoidal protrusion with semi axes h radially, a longitudinally, and d azimuthally. ${}_2F_1$ is the hypergeometric function.	$\alpha_e+\alpha_m=2\pi ahd\left[\dfrac{1}{I_b}+\dfrac{1}{I_c-3}\right]$ $I_b={}_2F_1\Big(1,1;\frac{5}{2};1-\frac{h^2}{a^2}\Big)\ , \quad I_c={}_2F_1\Big(1,\frac{1}{2};\frac{5}{2};1-\frac{a^2}{h^2}\Big)\ , \quad \text{if } a=d$ $\alpha_e+\alpha_m=\pi a^3 \quad \text{if } a=d=h\ , \quad \dfrac{2\pi h^3}{3[\ln(2h/a)-1]} \quad \text{if } a=d\ll h$ $\alpha_e+\alpha_m=\dfrac{8h^3}{3}\left[1+\left(\dfrac{4}{\pi}-\dfrac{\pi}{4}\right)\dfrac{a}{h}\right] \quad \text{if } a\ll h=d$ $\alpha_e+\alpha_m=\dfrac{8\pi h^4}{3a}\left[\ln\dfrac{2a}{h}-1\right] \quad \text{if } a\gg h=d$	
Small inductive objects-2D: [27, 30] small cavities, shallow irises, and transitions at low freq. ($h\ll b$, $k\ll 1/h$); h is height of object, g is gap of cavity or length of iris; $\mathcal{L}$ is inductance. For tapered transition pair: θ is taper angle.	$Z_0^{\parallel}=-ikc\mathcal{L}, \quad Z_1^{\perp}=\dfrac{2}{b^2k}Z_0^{\parallel}$	$W_0'=-c^2\mathcal{L}\delta'(z),\ W_1=\dfrac{2}{b^2}\displaystyle\int W_0'\,dz$
	Pill box $g\lesssim h\ll b:\ \mathcal{L}=\dfrac{Z_0}{2\pi cb}\left[gh-\dfrac{g^2}{2\pi}\right]$ Shallow iris $g\lesssim h\ll b:\ \mathcal{L}=\dfrac{Z_0h^2}{4cb}$ Transition pair $g\gg h,\ h\ll b:\ \mathcal{L}=\dfrac{Z_0h^2}{2\pi^2cb}\left(\ln\dfrac{2\pi b}{h}+\dfrac{1}{2}\right)$ Tapered $:\mathcal{L}=\dfrac{Z_0h^2}{\pi^2cb}\left[\ln\left(\dfrac{b\theta}{h}-2\theta\cot\theta\right)+\dfrac{3}{2}-\gamma_e-\psi\left(\dfrac{\theta}{\pi}\right)-\dfrac{\pi}{2}\cot\theta-\dfrac{\pi}{2\theta}\right]$ $\gamma_e\approx 0.57721$ is Euler's constant, $\psi(x)$ is psi function.	
Wall roughness inductive model: [35] 1-D axisymmetric bump on beam pipe, $h(z)$ or 2-D bump $h(z,\theta)$. Valid for low frequency $k\ll$ (bump length or width)$^{-1}$, $h\ll b$, and $\lvert\nabla h\rvert\ll 1$. See also [36]	1-D: $Z_0^{\parallel}=-\dfrac{2ikZ_0}{b}\displaystyle\int_0^{\infty}\kappa\lvert\tilde{h}(\kappa)\rvert^2d\kappa$ with spectrum $\tilde{h}(k)=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}h(z)e^{-ikz}dz$ 2-D: $Z_0^{\parallel}=-\dfrac{4ikZ_0}{b}\displaystyle\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\dfrac{\kappa^2}{\sqrt{\kappa^2+m^2/b^2}}\lvert\tilde{h}_m(\kappa)\rvert^2d\kappa$ with spectrum $\tilde{h}_m(k)=\dfrac{1}{(2\pi)^2}\displaystyle\int_0^{2\pi}d\theta\int_{-\infty}^{\infty}dz\,h(z,\theta)e^{-ikz-im\theta}$ Note: small periodic corrugations model is also used for wall roughness impedance estimation.	
Small periodic corrugations: (a) [31, 32] $L\lesssim h\ll b$, $k\ll 1/h$; L period, h depth, g gap, $\wp$ principal value; $\beta_g c$ group velocity.	$\dfrac{Z_0^{\parallel}}{L}=\dfrac{Z_0}{\pi b^2}\left[\pi k_r\delta(k^2-k_r^2)+i\,\wp\left(\dfrac{k}{k^2-k_r^2}\right)\right], \qquad \dfrac{W_0'}{L}=\dfrac{Z_0c}{\pi b^2}\cos k_rz$ $Z_1^{\perp}=\dfrac{2}{b^2k}Z_0^{\parallel}, \quad k_r=\sqrt{\dfrac{2L}{bgh}}\ ;\ (1-\beta_g)=\dfrac{4hg}{bL}, \quad W_1=\dfrac{2}{b^2}\displaystyle\int W_0'\,dz$	
(b) [33] $L\gg h$, $L\ll b$, $k\ll 1/h$; $k_L=2\pi/L$.	$\dfrac{Z_0^{\parallel}}{L}=\dfrac{Z_0h^2k_L^{3/2}}{8\pi b}(-ik)^{1/2}$	$\dfrac{W_0'}{L}=-\dfrac{Z_0ch^2k_L^3}{16\pi^{3/2}b}\dfrac{1}{(-k_Lz)^{3/2}}$

<table>
<tr><th>Description</th><th>Impedances</th><th>Wakes</th></tr>
<tr><td>Thin dielectric or ferrite layer on pipe: [34] thickness $h \ll b$.</td><td colspan="2">Like small periodic corrugations (a), but $k_r = \left[\frac{2\epsilon_r}{(\epsilon_r\mu_r - 1)bh}\right]^{1/2}$, with relative dielectric constant ϵ_r and magnetic permeability μ_r.</td></tr>
<tr><td rowspan="2">Coherent synchrotron radiation (CSR): [37]-[39] Bunch moves in free space on a circle of radius R; $k \ll \gamma^3/R$. See Sec.??.</td><td>$$\frac{Z_0^{\parallel}}{L} = \frac{Z_0}{2\cdot 3^{1/3}\pi}\Gamma\left(\frac{2}{3}\right)\left[\frac{ik}{R^2}\right]^{1/3}$$</td><td>$$\frac{W_0'}{L} = -\frac{Z_0 c}{2\cdot 3^{4/3}\pi R^{2/3}}\frac{1}{z^{4/3}}$$</td></tr>
<tr><td colspan="2">$\Gamma(2/3) \approx 1.3541$. Note: non-zero wake for test particle ahead of driving particle. $W_0'(0^+)/L \approx 0.1 Z_0 c\gamma^4/R^2$. This is also used to approximate effect at high k for beam in beam pipe; shielded (suppressed) for $k \lesssim R^{1/2}b^{-3/2}$.</td></tr>
<tr><td>Round collimator: (a) [40] low frequency $k \ll 1/d$.</td><td>$$Z_1^{\perp} = -0.3i\frac{Z_0}{d}$$ collimator radius $d \ll b$.</td><td>$$W_1 = -0.3\frac{Z_0 c}{d}\delta(z)$$ collimator radius $d \ll b$.</td></tr>
<tr><td>(b) High frequency $k \gg 1/d$; if tapered, angle $\theta \gg 1/(kd)$.</td><td colspan="2">See optical model formulae (a) above</td></tr>
<tr><td>(c) [41] For any frequency, small angle, $d'(s) \ll 1$, $kdd' \ll 1$, with $d(s)$ pipe profile versus longitudinal position s, and d' is derivative of d with respect to s.</td><td>$$Z_0^{\parallel} = \frac{-iZ_0 k}{4\pi}\int ds\,(d')^2$$ $$Z_1^{\perp} = \frac{-iZ_0}{2\pi}\int ds\left(\frac{d'}{d}\right)^2$$ $\Rightarrow$ symm. tapers of angle $\theta \ll 1$: $$Z_1^{\perp} = \frac{-iZ_0}{\pi}\theta\left(\frac{1}{d} - \frac{1}{b}\right)$$</td><td>$$W_0' = \frac{Z_0 c}{4\pi}\int ds\,(d')^2\delta'(z)$$ $$W_1 = -\frac{Z_0 c}{2\pi}\int ds\left(\frac{d'}{d}\right)^2\delta(z)$$ $$W_1 = -\frac{Z_0 c}{\pi}\theta\left(\frac{1}{d} - \frac{1}{b}\right)\delta(z)$$</td></tr>
<tr><td>Flat collimator: [42] low frequency, small angle, $h'(s) \ll 1$, $h \ll w \ll \ell$, with $h(s)$ vertical profile, w width, ℓ length</td><td>$$Z_y = \frac{-iZ_0 w}{4}\int ds\frac{(h')^2}{h^3}$$</td><td>$$W_y = -\frac{Z_0 c w}{4}\int ds\frac{(h')^2}{h^3}\delta(z)$$</td></tr>
<tr><td rowspan="2">Pill-box cavity — low frequency: [43] cavity radius d, gap g; $S = d/b$. When $g \gg 2(d-b)$, replace g by $d-b$. Valid for $k \ll 1/d$.</td><td>$$Z_0^{\parallel} = -ik\frac{Z_0 g}{2\pi}\ln S$$ $$Z_1^{\perp} = -i\frac{Z_0 g}{\pi b^2}\frac{S^2-1}{S^2+1}$$</td><td>$$W_0' = -\frac{Z_0 c g}{2\pi}\ln S\,\delta'(z)$$ $$W_1 = -\frac{Z_0 c g}{\pi b^2}\frac{S^2-1}{S^2+1}\delta(z)$$</td></tr>
<tr><td colspan="2">Effect will be one half for a step in the beam pipe from radius b to radius d, or vice versa, when $g \gg 2(d-b)$.</td></tr>
<tr><td rowspan="2">Resonator model: [1] for m-th azimuthal mode, with shunt impedance $R_s^{(m)}$, quality factor Q, and resonant frequency k_r.</td><td>$$Z_m^{\parallel} = \frac{R_s^{(m)}}{1 + iQ\,(k_r/k - k/k_r)}$$ $$Z_m^{\perp} = \frac{R_s^{(m)}/k}{1 + iQ\,(k_r/k - k/k_r)}$$</td><td>$$W_m = \frac{R_s^{(m)} c k_r}{Q\bar{k}_r}e^{\alpha z}\sin\bar{k}_r z$$ where $\alpha = k_r/(2Q)$ $$\bar{k}_r = \sqrt{|k_r^2 - \alpha^2|}$$</td></tr>
<tr><td colspan="2">Valid only close to k_r. As $k \to \infty$, $Z_0^{\parallel} \to k^{-1/2}$ for non-periodic cavities and $\to k^{-3/2}$ for an infinite array of cavities. [16, 46]</td></tr>
</table>

<table>
<tr><th>Description</th><th>Impedances</th><th>Wakes</th></tr>
<tr><td>Closed pill-box cavity: [44] resonant frequencies k_{mnp} and “circuit” $(R_s/Q)_{mnp}$ [45], where m, n, p, are azimuthal, radial, longitudinal mode numbers. Cavity radius d and length g; x_{mn} is $n^{\rm th}$ zero of Bessel function J_m.</td><td colspan="2">$$k_{mnp}^2 = \frac{x_{mn}^2}{d^2} + \frac{p^2\pi^2}{g^2}$$ $$\left[\frac{R_s}{Q}\right]_{0np} = \frac{Z_0}{x_{0n}^2 J_0'^2(x_{0n})}\frac{8}{\pi g k_{0np}}\begin{cases} \sin^2\dfrac{gk_{0np}}{2\beta} \times \dfrac{1}{1+\delta_{0p}} & p \text{ even} \\ \cos^2\dfrac{gk_{0np}}{2\beta} & p \text{ odd}\end{cases}$$ $$\left[\frac{R_s}{Q}\right]_{1np} = \frac{Z_0}{J_1'^2(x_{1n})}\frac{2}{\pi g d^2 k_{1np}^2}\begin{cases} \sin^2\dfrac{gk_{1np}}{2\beta} & p \neq 0 \text{ and even} \\ \cos^2\dfrac{gk_{1np}}{2\beta} & p \text{ odd}\end{cases}$$</td></tr>
<tr><td>Curvature impedance: [47] Smooth toroidal chamber of rectangular cross section, width $b-a$, height h, inner radius a, outer radius b, and $R=\frac{1}{2}(a+b)$. As Lorentz factor $\gamma\to\infty$, a contribution remains.</td><td colspan="2">Valid from zero frequency up to just below synchronous resonant modes, i.e., $0<\nu<\sqrt{R/h}$ with $\nu = kh$, $$Z_0^{\parallel} = \frac{ikZ_0h^2}{\pi^2R}\left\{\left[1-e^{-2\pi(b-R)/h}-e^{-2\pi(R-a)/h}\right]\left[1-3\left(\frac{\nu}{\pi}\right)^2\right] +0.05179-0.01355\left(\frac{\nu}{\pi}\right)^2\right\}+\rho kR \approx \frac{ikZ_0h^2}{\pi^2R}\left[A-3B\left(\frac{\nu}{\pi}\right)^2\right].$$ where ρ is quadratic in ν. As $(b-a)/h$ increases, ρ vanishes exponentially and $A\approx B\approx 1$. In general, $A/B\approx 1$ implying ${\rm Im}Z_0^{\parallel}$ changes sign (a node) near $\nu=\pi/\sqrt{3}$.</td></tr>
<tr><td rowspan="2">Kicker with window-frame magnet: [49] width a, height b, length L, beam offset x_0 horizontally, and all image current carried by conducting current plates.</td><td>$$Z_0^{\parallel} = \frac{k^2c^2\mu_0^2L^2x_0^2}{4a^2Z_k}$$ $$Z_1^{\perp} = \frac{kc^2\mu_0^2L^2}{4a^2Z_k}$$</td><td>$$W_0' = -\frac{c^3\mu_0^2L^2x_0^2}{4a^2Z_k}\,\delta_0''(z)$$ $$W_1 = -\frac{c^3\mu_0^2L^2}{4a^2Z_k}\,\delta'(z)$$</td></tr>
<tr><td colspan="2">$Z_k=-ikc\mathcal{L}+Z_g$ with $\mathcal{L}\approx\mu_0bL/a$ the inductance of the windings and Z_g the impedance of the generator and the cable. If the kicker is of C-type magnet, x_0 in $Z_0^{\parallel}$ should be replaced by (x_0+b).</td></tr>
<tr><td rowspan="2">Traveling-wave kicker [49] with characteristic impedance Z_c for the cable, and a window magnet of width a, height b, and length L. Valid for frequency below cutoff.</td><td>$$Z_0^{\parallel}=\frac{Z_c}{4}\left[2\sin^2\frac{\theta}{2}-i\sin\theta\right]$$ $$Z_1^{\perp}=\frac{Z_cL}{4ab}\left[\frac{1-\cos\theta}{\theta}-i\frac{\sin\theta}{\theta}\right]$$</td><td>$$W_0'=\frac{Z_cc}{4}\left[\delta(z)-\delta\left(z+\frac{L}{\beta_{\rm ph}}\right)\right]$$ $$W_1=\frac{Z_c\beta c}{4ab}\left[H(z)-H\left(z+\frac{L}{\beta_{\rm ph}}\right)\right]$$</td></tr>
<tr><td colspan="2">$\theta=kL/\beta_{\rm ph}$ denotes the electrical length of the kicker windings and $\beta_{\rm ph}c=Z_cac/(Z_0b)$ is the matched transmission-line phase velocity of the capacitance-loaded windings. Here, $\beta_{\rm ph}\ll\beta\to 1$, the beam velocity.</td></tr>
</table>

Description	Impedances	Wakes
Strip-line BPMs (pair): [48] length L, angle each subtending to pipe axis ϕ_0, forming transmission lines of characteristic impedance Z_c with pipe.	$Z_0^{\parallel}=2Z_c\left[\dfrac{\phi_0}{2\pi}\right]^2\left[2\sin^2 kL-i\sin 2kL\right]$ $Z_1^{\perp}=\left[\dfrac{Z_0^{\parallel}}{k}\right]_{\rm pair}\dfrac{1}{b^2}\left[\dfrac{4}{\phi_0}\right]^2\sin^2\dfrac{\phi_0}{2}$	$W_0'=2Z_c\,c\left[\dfrac{\phi_0}{2\pi}\right]^2\left[\delta(z)-\delta(z+2L)\right]$ $W_1=\dfrac{8Z_c\,c}{\pi^2 b^2}\sin^2\dfrac{\phi_0}{2}\left[H(z)-H(z+2L)\right]$
	The strip-lines are assumed to terminate with impedance Z_c at the upstream end.	

Wakes for a Gaussian Bunch:

The *bunch wakes* of a bunch with longitudinal charge distribution λ_z, are given by $\mathcal{W}_m'(z)=\int_{-\infty}^{0}W_m'(x)\lambda_z(z-x)\,dx$, $\mathcal{W}_m(z)=\int_{-\infty}^{0}W_m(x)\lambda_z(z-x)\,dx$. In the following we give bunch wakes of a Gaussian bunch [$\lambda_z=e^{-(z/\sigma_z)^2/2}/(\sqrt{2\pi}\sigma_z)$, with σ_z the rms bunch length] for wakefield forms found in the tables above, and also give their first moments $\langle\mathcal{W}\rangle=\int_{-\infty}^{\infty}\mathcal{W}(z)\lambda_z(z)dz$ and the rms $\mathcal{W}_{\rm rms}=\sqrt{\langle\mathcal{W}^2\rangle-\langle\mathcal{W}\rangle^2}$. Here the z dependence alone is considered and the wake coefficient is scaled out; for a specific problem, the appropriate coefficients, found in the tables above, need to be included at the end.

Note: for power law wakes with $-2<\alpha<-1$, $\mathcal{W}$ is obtained using integration by parts [38]. It is assumed that in the range $|z|\ll\sigma_z$ the wake form changes so that $\int_{-\infty}^{\infty}W(z)\,dz=0$. Consequently, $\mathcal{W}$ can be obtained without knowing the details of W at very short range.

Wake form, W	Bunch wake, $\mathcal{W}$	$\langle\mathcal{W}\rangle$	$\mathcal{W}_{\rm rms}$
Circuit Models:			
Resistive: $\delta(z)$	$\dfrac{1}{\sqrt{2\pi}\sigma_z}\,e^{-(z/\sigma_z)^2/2}$	$\dfrac{1}{2\sqrt{\pi}\sigma_z}$	$\dfrac{0.111}{\sigma_z}$
Inductive: $\delta'(-z)$	$\dfrac{z}{\sqrt{2\pi}\sigma_z^3}\,e^{-(z/\sigma_z)^2/2}$	0	$\dfrac{1}{\sqrt{6\pi}\,3^{1/4}\sigma_z^2}$
Capacitive: $H(-z)$	$\dfrac{1}{2}\left[1+\mathrm{erf}\left(\dfrac{-z}{\sqrt{2}\sigma_z}\right)\right]$	$\dfrac{1}{2}$	$\dfrac{1}{2\sqrt{3}}$
Power Law: $(-z)^\alpha$	$f(-z/\sigma_z)\sigma_z^\alpha$, with $f(x)$ given by (upper/lower sign for $x\gtrless 0$):		
Low freq. resistive wall (W_m) and Fresnel diffraction (W_m'): $\alpha=-\frac{1}{2}$	$\sqrt{\dfrac{\pi\lvert x\rvert}{8}}e^{-x^2/4}\left[I_{-1/4}\pm I_{1/4}\right]\Big\vert_{x^2/4}$	$\dfrac{0.723}{\sqrt{\sigma_z}}$	$\dfrac{0.292}{\sqrt{\sigma_z}}$
Fresnel diffraction (W_m): $\alpha=\frac{1}{2}$	$\sqrt{\dfrac{\pi}{32}}\displaystyle\int_{-\infty}^{x}\lvert y\rvert^{1/2}e^{-y^2/4}\left[I_{-1/4}\pm I_{1/4}\right]\Big\vert_{y^2/4}dy$	$0.489\sqrt{\sigma_z}$	$0.374\sqrt{\sigma_z}$
Low freq. resistive wall (W_m') and small periodic corrugations (W_0'): [50] $\alpha=-\frac{3}{2}$	$\sqrt{\dfrac{\pi\lvert x\rvert^3}{8}}e^{-x^2/4}\left[I_{1/4}-I_{-3/4}\pm I_{-1/4}\mp I_{3/4}\right]\Big\vert_{x^2/4}$	$\dfrac{-0.489}{\sigma_z^{3/2}}$	$\dfrac{0.516}{\sigma_z^{3/2}}$
CSR (W_0'): z^α with $\alpha=-\frac{4}{3}$ (note: $z>0$)	$-\dfrac{3}{\sqrt{2\pi}}\displaystyle\int_0^{\infty}\dfrac{(x+y)e^{-(x+y)^2/2}}{y^{1/3}}dy$	$\dfrac{-0.758}{\sigma_z^{4/3}}$	$\dfrac{0.532}{\sigma_z^{4/3}}$
Resonator Model: $\left\{\begin{matrix}\sin(-k_r z)\\ \cos(k_r z)\end{matrix}\right\}e^{\alpha_r z}$, with k_r, α_r, real	$\mathcal{W}=f(-z/\sigma_z)$, with $f(x)=\dfrac{1}{2}\,e^{-(k_r^2-\alpha_r^2)\sigma_z^2/2-\alpha_r\sigma_z x}$ $\times\left\{\begin{matrix}\mathrm{Im}\\ \mathrm{Re}\end{matrix}\right\}e^{ik_r\sigma_z(x-\alpha_r\sigma_z)}\left\{1+\mathrm{erf}\left[\dfrac{(ik_r-\alpha_r)\sigma_z+x}{\sqrt{2}}\right]\right\}$ $\langle\mathcal{W}\rangle=\dfrac{1}{2}\,e^{-(k_r^2-\alpha_r^2)\sigma_z^2}\left\{\begin{matrix}\mathrm{Im}\\ \mathrm{Re}\end{matrix}\right\}e^{-i2k_r\alpha_r\sigma_z^2}\left\{1+\mathrm{erf}\left[(ik_r-\alpha_r)\sigma_z\right]\right\}$		