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LANDAU DAMPING

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Abstract

Section 2.5.8 of the Handbook of Accelerator Physics and Engineering on Landau damping is rewritten. An solvable example is first given to demonstrate the interplay between Landau damping and decoherence. This example is an actual one when the beam oscillatory motion is driven by a wake force. The dispersion relation is derived and its implication on Landau damping is illustrated. The rest of the article touches on the Landau damping of transverse and longitudinal beam oscillations. The stability criteria are given for a bunched beam and the changes of the criteria when the beam is lengthened and becomes unbunched.

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2.5.8 Landau Damping [1, 2, 3, 4]

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Wake force excites a number of collective waves in a beam and displaces it from its equilibrium position. These waves of the beam center exchange energy among themselves, resulting in growth in amplitude for some and damping for some. The spread in oscillation frequency accelerates the damping and decelerates the growth. This process is often called *Landau damping.*

Transverse oscillation of unbunched beam Consider a coasting beam of energy E_0 and betatron frequency ω_β under the influence of a transverse wake W_1. The transverse displacement $y(\theta,t)$ of a particle at azimuthal angle θ around the accelerator ring is given by

$$\left[\left(\frac{\partial}{\partial t}+\omega_0\frac{\partial}{\partial\theta}\right)^2+\omega_\beta^2\right]y(\theta,t)$$
$$=-\frac{eI_0}{\beta cT_0}\int_{-\infty}^{0}dt'W_1(t'-t)\langle y(\theta,t')\rangle, \quad (1)$$

where $T_0=2\pi/\omega_0$ is the revolution period, βc the particle velocity, I_0 is the average beam current, and $\langle\cdots\rangle$ implies averaging over all particles according to the distribution $\rho(\omega_\beta)$. Solution can be obtained in the harmonic-frequency (n-Ω) space via the transformation $\frac{1}{4\pi^2}\int d\theta dt e^{-in\theta+i\Omega t}$. In the upper part of the Ω-plane ($\mathcal{I}m\,\Omega>\omega_g$, with ω_g being the fastest growth rate of $y(\theta,t)$),

$$\tilde y_n(\Omega)=\frac{2i\bar\omega_\beta\kappa\langle\tilde y_n(\Omega)\rangle}{\omega_\beta^2-\hat\omega^2}-\frac{i(\Omega-2n\omega_0)y_{n0}}{2\pi(\omega_\beta^2-\hat\omega^2)}, \quad (2)$$

where the initial beam displacement is $y(\theta,0)=\sum_n y_{n0}e^{in\theta}$, $\dot y(\theta,0)=0$, $\hat\omega=\Omega-n\omega_0$, $\bar\omega_\beta=\langle\omega_\beta\rangle$, $\kappa=ecI_0Z_1^\perp(\Omega)/(2\bar\omega_\beta E_0T_0)$, and $Z_1^\perp$ the transverse impedance. Physically, with $\kappa=\kappa_R+i\kappa_I$, κ_I is the betatron frequency shift due to impedance and κ_R is the growth rate. Analytic continuation into the whole Ω-plane gives

$$\langle\tilde y_n(\Omega)\rangle=-\frac{iy_{n0}(\hat\omega-n\omega_0)\displaystyle\int_C\frac{\rho(\omega_\beta)}{\omega_\beta^2-\hat\omega^2}d\omega_\beta}{H(\Omega)},$$

$$H(\Omega)=1-i2\bar\omega_\beta\kappa\int_C d\omega_\beta\frac{\rho(\omega_\beta)}{\omega_\beta^2-\hat\omega^2}, \quad (3)$$

with integration path C going below the pole at $\hat\omega$ and above the pole at $-\hat\omega$. Back to the θ-t space,

$$\langle y(\theta,t)\rangle=\sum_n\int_W d\Omega\, e^{i(n\theta-\Omega t)}\langle\tilde y_n(\Omega)\rangle, \quad (4)$$

with path W above all poles. Thus the center of the beam consists of many harmonic waves at frequencies determined by the zeroes of $H(\Omega)$.

As an example, consider the Lorentzian distribution $\rho(\omega_\beta)=(S_{\frac12}/\pi)/[(\omega_\beta-\bar\omega_\beta)^2+S_{\frac12}^2]$, where $S_{\frac12}$ is the half-width at half-maximum (HWHM). Since $S_{\frac12}/\bar\omega_\beta\ll1$ and $|\kappa|/\bar\omega_\beta\ll1$, keeping the lowest order, the solution simplifies to

$$\langle y(\theta,t)\rangle=\sum_{n=-\infty}^{\infty}y_{n0}\frac{\bar\omega_\beta-n\omega_0}{2\bar\omega_\beta}\times$$
$$\times\left[e^{in\theta-i(\bar\omega_\beta+n\omega_0+\kappa_I)t}e^{-(S_{\frac12}+\kappa_R)t}+\text{c.c.}\right], \quad (5)$$

which are betatron waves corresponding to betatron sidebands of the revolution harmonics. Here $n>0$ corresponds to fast waves, which are stable because $\kappa_R>0$. For the slow waves with $n<0$, $\kappa_R<0$ and there is stability only when $S_{\frac12}>|\kappa_R|$. We see that the growth initiated by $Z_1^\perp$ is counteracted by the spread in betatron frequency.

Note that even when $\langle y(\theta,t)\rangle$ is damped to zero, the displacements of individual particles are not. In practice, any small initial displacement of the beam center will be damped immediately if $S_{\frac12}>|\kappa_R|$, ensuring that $Z_1^\perp$ will stop driving the individual displacements. In other words, Landau damping nips any instability growth in the bud.

$H(\Omega)=0$ is called the dispersion relation because it gives frequency Ω as a function of harmonic number n. Since $\pm\hat\omega$ are far apart, the denominator can be linearized to give

$$H(\Omega)=1-i\kappa\int_C\frac{\rho(\omega_\beta)d\omega_\beta}{\omega_\beta-\hat\omega}=0. \quad (6)$$

To obtain the stability contour, $\hat\omega$ is considered to be real, but with a positive infinitesimal imaginary part added. With $u=(\hat\omega-\bar\omega_\beta)/S_{\frac12}$, the dispersion relation is normalized to the HWHM spread,

$$1+\frac{(\Delta\omega)_0}{S_{\frac12}}\Big[f(u)+ig(u)\Big]=0, \quad (7)$$

where $(\Delta\omega)_0=\omega-n\omega_0-\bar\omega_\beta=-i\kappa$. In fact,

$$(\Delta\omega)_0=-\frac{ie\beta I_0}{2E_0T_0}\beta_\perp Z_1^\perp(\bar\omega_\beta+n\omega_0) \quad (8)$$

is the $Z_1^\perp$-driven complex frequency shift in the absence of Landau damping (subscript 0) and $\beta_\perp$ is the betatron function at the impedance. $f(u)+ig(u)$ is called the *transfer function* and can be measured (sec.7.5.3). It is defined as

$$f(u)=\wp\int\frac{\hat\rho(v)dv}{v-u},\quad g(u)=\pi\hat\rho(u), \quad (9)$$

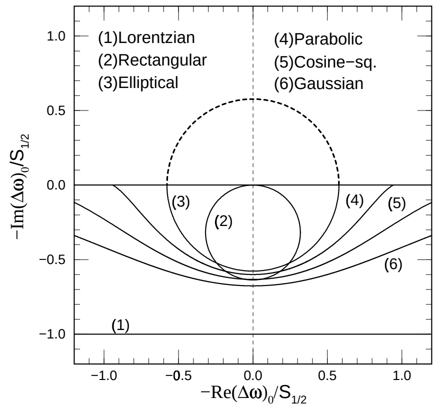

Figure 1: Plot of stability contours for various frequency distributions in the $-(\Delta\omega)_0/S_{\frac{1}{2}}$-plane.

where $\wp$ denotes principal value, $\hat\rho(v)=S_{\frac{1}{2}}\rho(\omega_\beta)$, $v=(\omega_\beta-\bar\omega_\beta)/S_{\frac{1}{2}}$ so that $v=1$ is at the HWHM.

The stability contour is the locus of $-(\Delta\omega)_0/S_{\frac{1}{2}}$ as u is varied from $-\infty$ to ∞. The beam is stable for points lying on the side of the locus containing the origin, or $\mathcal{I}m\,\Delta\omega_0 < S_{\frac{1}{2}}$ for the Lorentzian distribution. Let us consider next the generalized elliptical distribution $\hat\rho(v)=(A_n/a_n)(1-v^2/a_n^2)^n$ when $|v|\le a_n$ and zero otherwise, where $A_n=\Gamma\left(n+\frac{3}{2}\right)/[\sqrt{\pi}\Gamma(n+1)]$ and the parameter $a_n^2=1/(1-2^{-1/n})$ is so chosen that $v=1$ is the HWHM. Note that n need not be an integer. For example, $n=0,\frac{1}{2},1,\frac{3}{2},\infty$ correspond, respectively, to the rectangular, elliptical, parabolic, tri-elliptical, and Gaussian distributions. In addition, $n=2.36$ reproduces closely the cosine-square distribution. Their stability contours are plotted in Fig.1. Notice that all the contours, except the Lorentzian, intersect the $-\mathcal{I}m(\Delta\omega)_0/S_{\frac{1}{2}}$-axis at roughly the same point $-1/\sqrt{3}$. A simplified stability criterion can therefore be represented by a circle centered at the origin with radius equal to the intercept, [6, 7]

$$|(\Delta\omega)_0| < \frac{1}{\sqrt{3}}S_{\frac{1}{2}}F_\perp. \qquad (10)$$

The form factor is $F_\perp=\sqrt{3}$ for the Lorentzian distribution. For the generalized elliptical distribution, $F_\perp=\sqrt{3}a_n/(\pi A_n)$, or 1.103, 1, 1.040, 1.068, 1.097, and 1.174, respectively, for the rectangular, elliptical, parabolic, tri-elliptical, cosine square, and Gaussian distributions.

If the frequency spread $S_{\frac{1}{2}}$ is due to a momentum spread $\Delta\delta_{\frac{1}{2}}$, the simplified stability criterion becomes

$$|Z_\perp(n\omega_0+\bar\omega_\beta)| < \frac{4\pi E_0\xi_{\rm eff}\Delta\delta_{\frac{1}{2}}}{\sqrt{3}e\beta I_0\beta_\perp}F_\perp, \qquad (11)$$

where $\xi_{\rm eff}=\xi-\eta(n+\nu_\beta)$ with ξ the chromaticity, ν_β the betatron tune, and η the slip factor. For a broad-band impedance rolling off at frequency ω_c, the substitution $n=\omega_c/\omega_0$ can be made.

Transverse oscillation of a single bunch When the bunch is very much shorter than the wavelength of the perturbing $Z_1^\perp$, the bunch can be viewed as a single macro-particle, [3] oscillating transversely with frequency Ω. The dispersion relation is the same as Eq. (6), but with $\hat\omega=\Omega$. The simplified stability criterion is still Eq. (10). The complex mode frequency shift due to wake effect in the absence of Landau damping is now

$$(\Delta\omega)_0=(\Omega-\bar\omega_\beta)_0=\frac{e^2N_B\beta_\perp\mathcal{W}}{2\beta^2E_0T_0}, \qquad (12)$$

$$\mathcal{W}=\sum_{k=1}^{\infty}W_1(-kC)e^{i\bar\omega_\beta kT_0}$$

$$=-\frac{i\beta}{T_0}\sum_{p=-\infty}^{\infty}Z_1^\perp(p\omega_0+\bar\omega_\beta), \qquad (13)$$

where N_B is the number of particles in the bunch.

When the length of the bunch is taken into account, there will be many more modes of transverse oscillation along the bunch. Head-tail instabilities can happen (Sec.2.5.6) and can be Landau damped when the growth rate $\lesssim S_{\frac{1}{2}}$.

Stronger impedance may cause two modes to merge, resulting in transverse mode-coupling instability (Sec.2.5.6). The growth is fast once above threshold and Landau damping usually does not help.

When the bunch is very much longer than the wavelength of $Z_1^\perp$ and the growth rate is much faster than the synchrotron frequency, locally the bunch can be viewed as unbunched. Therefore the stability criterion of Eq. (10) applies. [7] However, we must replace I_0 by the local peak current and $\Delta\delta_{\frac{1}{2}}$ by the local momentum spread.

Longitudinal oscillation of unbunched beam This case is unique in that there is no external focusing frequency. However, the longitudinal impedance $Z_0^\parallel$ does alter the the particle's energy and therefore its revolution frequency. The collective frequency shift, $(\Delta\omega)_0$, is very similar to a synchrotron oscillation and is given, in the absence of Landau damping, by [5]

$$(\Delta\omega)_0^2=(\Omega-n\omega_0)^2=\frac{ien\eta I_0\omega_0^2}{2\pi\beta^2E_0}Z_0^\parallel(n\omega_0). \quad (14)$$

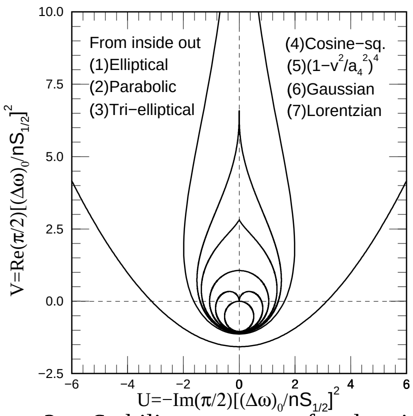

Figure 2: Stability contours for longitudinal unbunched beam in the $\pi(\Delta\omega)_0^2/(2n^2S_{\frac{1}{2}}^2)$-plane.

Landau damping arises from a spread in the revolution frequency. For a distribution $\rho(\omega_0)$ with mean $\bar\omega_0$, the dispersion relation is [6]

$$1-(\Delta\omega)_0^2\int_C\frac{\rho(\omega_0)d\omega_0}{(\Omega-n\omega_0)^2}=0, \qquad (15)$$

where C goes below the pole at Ω/n. At the onset of instability, integration by parts and normalization to the HWHM revolution frequency spread $S_{\frac{1}{2}}$ result in

$$1-\frac{(\Delta\omega)_0^2}{n^2S_{\frac{1}{2}}^2}\int\frac{\hat\rho'(v)dv}{v-u-i\epsilon}=0, \qquad (16)$$

where $\epsilon\to0^+$, $v=(\omega_0-\bar\omega_0)/S_{\frac{1}{2}}$, $nu=(\Omega-n\bar\omega_0)/S_{\frac{1}{2}}$, and $\hat\rho(v)=S_{\frac{1}{2}}\rho(\omega_0)$ so that $v=1$ is the HWHM. If the spread in ω_0 comes from the spread in energy ΔE, then $S_{\frac{1}{2}}=(|\eta|\bar\omega_0/\beta^2)(\Delta E/E_0)_{\frac{1}{2}}$. The dispersion relation becomes, in the U-V notations,

$$1-\frac{2i}{\pi}\mathrm{sgn}(\eta)(U+iV)\int\frac{\hat\rho'(v)dv}{v-u-i\epsilon}=0, \quad (17)$$

$$U+iV=\frac{eI_0\beta^2Z_0^{\|}(n\bar\omega_0)/n}{4|\eta|E_0(\Delta E/E_0)_{\frac{1}{2}}^2}. \qquad (18)$$

The stability contour can be traced by varying u from $-\infty$ to ∞, and is depicted in Fig. 2 for various distributions when $\eta<0$. Except for the Lorentzian distribution, all contours intersect the V-axis at roughly $V_{in}=-1$. A simplified stability criterion will therefore be the approximation of the contours as circles passing through V_{in}, or [6]

$$\left|\frac{Z_0^{\|}}{n}\right|<\frac{4|\eta|E_0}{e\beta^2I_0}\left[\frac{\Delta E}{E_0}\right]_{\frac{1}{2}}^2F_{\|}. \qquad (19)$$

For the Lorentzian distribution, the form factor is $F_{\|}=\pi/2$. For the generalized elliptical distribution $F_{\|}=\pi a_n^2/(4n+2)$, which amounts to 1.047, 1.047, 1.061, 1.080, 1.097, and 1.133, respectively, for the elliptical, parabolic, tri-elliptical, cosine square, $(1-v^2/a_4^2)^4$, and Gaussian distribution. The simplified stability criterion for the Gaussian distribution can also be written in the close form as $|Z_0^{\|}/n|<2\pi|\eta|E_0\sigma_E^2/(e\beta^2I_0)$, where σ_E is the fractional rms energy spread.

Longitudinal oscillation of a single bunch When the bunch is very much shorter than the wavelength of $Z_0^{\|}$, it can be approximated by a macro-particle. [3] In the presence of the synchrotron frequency ω_s, the problem is similar to that of the transverse. As a result, the dispersion relation is still Eq. (6) but with $\hat\omega=\Omega$. $(\Delta\omega)_0$ is the shift from the mean synchrotron frequency $\bar\omega_s$ (potential-well distortion included),

$$(\Delta\omega)_0=(\Omega-\bar\omega_s)_0=\frac{e^2N_B\eta c\mathcal{W}}{2\beta E_0T_0\bar\omega_s}, \qquad (20)$$

$$\mathcal{W}=\sum_{k=1}^{\infty}e^{i\bar\omega_skT_0}W_0''(-kC)$$
$$=\frac{i}{C}\sum_{p=-\infty}^{\infty}(p\omega_0+\bar\omega_s)Z_0^{\|}(p\omega_0+\bar\omega_s), \qquad (21)$$

W_0' being the longitudinal wake function and W_0'' its derivative. Stability contours is given by Fig. 1, and Eq. (10) remains a simplified stability criterion with $S_{\frac{1}{2}}$ being the HWHM spread in ω_s.

For finite bunch length, there will be many longitudinal modes of oscillation. Longitudinal head-tail instability can occur (Sec.2.5.6) but can be Landau damped by the spread in ω_s.

When $Z_0^{\|}$ is strong enough, two modes of oscillation can merge and longitudinal mode-coupling instability occurs (Sec.2.5.6). Once above threshold, the growth is fast and Landau damping usually does not help.

When the bunch is much longer than the wavelength of $Z_0^{\|}$ and the synchrotron oscillation period is much longer than the instability growth time, the beam can be viewed as unbunched locally. Thus dispersion relation (17) and stability condition (19) apply [7]. However, the average current I_0 must be replaced by the local peak current and the energy spread ΔE by the local energy spread.

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