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An Overview of Component Qualification using Bayesian Statistics and Energy Methods

Jeffrey L. Dohner

Prepared by
Sandia National Laboratories
Albuquerque, New Mexico 87185 and Livermore, California 94550

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Jeffrey L. Dohner
W88-0, W76-0 systems
Sandia National Laboratories
P.O. box 5800
Albuquerque, New Mexico 87185-1033

ABSTRACT

The below overview is designed to give the reader a limited understanding of Bayesian and Maximum Likelihood (MLE) estimation; a basic understanding of some of the mathematical tools to evaluate the quality of an estimation; an introduction to energy methods and a limited discussion of damage potential. This discussion then goes on to presented a limited presentation as to how energy methods and Bayesian estimation are used together to qualify components[i]. Example problems with solutions have been supplied as a learning aid. Bold letters are used to represent random variables. Un-bolded letter represent deterministic values. A concluding section presents a discussion of attributes and concerns.

Contents:

Figures:

Example Problems:

Tables:

Nominclature:

θ	- vector of parameters
$D = \{x_1, x_2, \dots x_M\}$	- vector of data
$\boldsymbol{x}_i$	- data sample
$f_{\boldsymbol{x}}(x\|\theta)$	- density of $\boldsymbol{x}_i$
$f(\theta\|D)$	- density of θ given D
$f_{\boldsymbol{x}}(D\|\theta)$	- likelihood function
$N(\mu, \sigma^2)$	- normal density
$W(\beta, \eta)$	- Weibull density
$F_{\boldsymbol{x}}(x)$	- cumulative distribution function
$U(0,1)$	- uniform density function
χ^2	- Chi squared statistics
$I(\mathrm{x_i} < x)$	- Indicator function
E_K	- kinetic energy
E_D	- dissipative energy
E_A	- absorbed energy
E_I	- input energy
$a_i(t)$	- acceleration time history
SRS	- shock response spectra
PSD	- power spectral density
M	- margin
S	- severity

Bayesian estimation is a mature field that has been proposed for use with energy methods to qualify system components [ref 1]. The below work is an explanation of the Bayesian approach and limited aspects of estimation theory (sections 1 through 3) followed by a discussion of damage potential (section 4). The use of total energy as a statement of damage is addressed. The final portion of the paper integrates the concept of total energy as an indicator of damage potential and Bayesian estimation as a potential path to component certification.

1) Bayesian Estimation

In sections 1 through 3, common tools used in estimation theory are presented. In section 1, Bayesian and Likelihood estimation is presented and in section 2, a common curve fitting tool is presented. In section 4, the concept of credible bounds is presented as it applies to Bayesian estimation.

In Bayesian estimation, it is desired to predict the joint density of parameters, $f(\theta|D)$, defining a density function $f_{\boldsymbol{x}}(x|\theta)$, given experimental data, $D = \{x_1, x_2, \dots x_M\}$ where $\boldsymbol{x}_i$.are experimental samples assumed to come from $f_{\boldsymbol{x}}(x)$. This prediction is performed using Bayes rule

$$f(\theta|D) = \frac{f_{\boldsymbol{x}}(D|\theta) f_{\boldsymbol{\theta}}(\theta)}{f(D)} \qquad (1)$$

where θ is the vector of parameters in question. For example, if $f_{\boldsymbol{x}}(x) \sim N(\mu, \sigma^2)$, then $\theta = [\mu, \sigma^2]^T$. In Bayesian estimation, these parameters are assumed to be random variables $\boldsymbol{\theta}$. The function $f_{\boldsymbol{\theta}}(\theta)$ is called the prior and the density $f(\theta|D)$ is called the posterior estimate. The function $f_{\boldsymbol{x}}(D|\theta)$ is called the likelihood function. The likelihood function is constructed from data, D and using the prior and the likelihood function, the posterior density, $f(\theta|D)$, can be estimated.

Consider the situation where

$$f_{\boldsymbol{x}}(x) = f_{\boldsymbol{x}}(x|\theta) = \beta \eta^{-\beta} x^{(\beta-1)} e^{-\left(\frac{x}{\eta}\right)^{\beta}}, \qquad (2)$$

a two parameter Weibull density function where $\theta = [\beta, \eta]^T$, $\boldsymbol{x} \sim W(\beta, \eta)$. This density function is attractive in that it is a member of the class of densities where the exponent of the exponential is x to a power where the power is a parameter used to define the density. Therefore, the tails of the density can be estimated by fitting data near to the mean. This is particularly important in reliability and safety considering that such predictions are dominated by the response of the tails.

If a set of components are subjected to an environmental test which drives them to failure, each will fail at different times (or levels), $\boldsymbol{x}_i$. This set of failures represents a collection of data $D = \{x_1, x_2, \dots x_M\}$. Considering that each test does not affect the results of any other test, we can consider each test as being independent of the other allowing us to calculate the likelihood function as

$$f_{\boldsymbol{x}}(D|\theta) = \prod_i^M f_{\boldsymbol{x}}(x_i|\theta) = \prod_i^M \beta\eta^{-\beta} {x_i}^{(\beta-1)} e^{-\left(\frac{x_i}{\eta}\right)^{\beta}}. \qquad (3)$$

where $\boldsymbol{x} \sim W(\beta, \eta)$ is here on assumed.

In many cases, the prior is not a very accurate representation of the density of θ and is often open to interpretation. However, as the amount of data is increased, the significance of inaccuracy in the prior is diminished (see example problem below). For simplicity, it is often assumed that the prior can be represented by independent density functions for each parameter. That is

$$f_{\boldsymbol{\theta}}(\theta) = f_{\boldsymbol{\beta}}(\beta) f_{\boldsymbol{\eta}}(\eta). \qquad (4)$$

Substituting equation 4 and 3 into equation 1, gives

$$f(\theta|D) = \mathrm{K} \prod_i^M \beta\eta^{-\beta} {x_i}^{(\beta-1)} e^{-\left(\frac{x_i}{\eta}\right)^{\beta}} f_\beta(\beta) f_\eta(\eta). \qquad (5)$$

where

$$\frac{1}{K} = f(D) = \iint \prod_i^M \beta\eta^{-\beta} {x_i}^{(\beta-1)} e^{-\left(\frac{x_i}{\eta}\right)^{\beta}} f_\beta(\beta) f_\eta(\eta) d\beta d\eta$$

is used to normalize the density function $f(\theta|D)$.

Following a Bayesian approach, to obtain a updated version of $f_{\boldsymbol{x}}(x|D)$, we simply integrate out θ. That is, using equation 2 and equation 5

$$f_{\boldsymbol{x}}(x|D) = \iint f_{\boldsymbol{x}}(x|\theta) f(\theta|D)\, d\beta d\eta. \qquad (6)$$

Generally, this integration cannot be performed in closed form. Therefore, numerical methods are used. Equation 6 is the estimate of $f_{\boldsymbol{x}}(x)$ given D – our Bayesian result.

A deviation from the Bayesian approach is the maximum likelihood approach. In this approach, we simply use the values in the likelihood function (equation 5), that maximize the function as the estimate of the parameters.

EXAMPLE PROBLEM 1

To illustrate the above discussion, we first developed data which is drawn from the equation 2 density. This is performed by using the cumulative distribution function

$$F_{\boldsymbol{x}}(x) = \int_{-\infty}^{x} f_{\boldsymbol{x}}(x) dx = y = 1 - e^{-\left(\frac{x}{\eta}\right)^{\beta}} \qquad (7)$$

and mapping from a uniform density through its inverse. That is

$$\boldsymbol{x} = \eta \left\{ \ln \left(\frac{1}{1-\boldsymbol{y}} \right) \right\}^{\frac{1}{\beta}} \qquad (8)$$

where $\boldsymbol{y} \sim U(0,1)$. Here, we assume that this represents data that would otherwise come from experimental analysis. A 1000 sample histogram of this data for $\eta^* = 1.0$ and $\beta^* = 2.0$ is given in Figure 1 (i.e. superscript * are exact values).

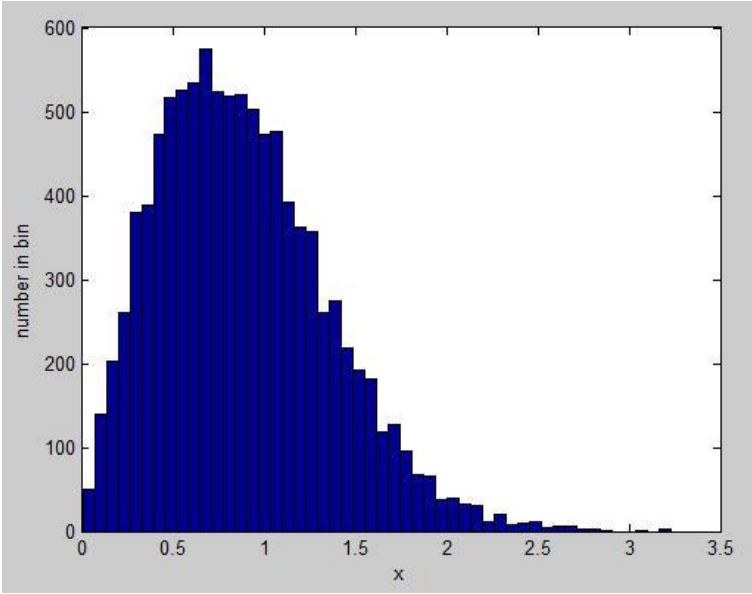

Figure 1. Histogram of data from a Weibull density with $\eta = 1.0$ and $\beta = 2.0$; 1000 points are shown; as the number of points goes to infinity the normalized histogram approaches the true density function

As shown in this figure, a normalized version of the emerging function does looks like a Weibull; nevertheless, the number of samples required to discover this underlying form is very large (100's to 1000's).

One of the benefits of using a Bayesian approach is that if the form of the density is known, the number of samples required to determine the parameters of the density is limited (10's to 100's of samples).

For priors $f_{\boldsymbol{\beta}}(\beta) \sim U(0,5)$ and $f_{\boldsymbol{\eta}}(\eta) \sim U(0,2.5)$ and only 10 samples of data (given in table 1), the function $f(\theta|D)$ can be calculated using equations 1 through 5. A figure of this function is given in Figure 2.

Table 1. Random points from $W(1.0,2.0)$

0.5143	1.1204	1.2134	0.9434	1.4363
0.7779	2.0324	0.5487	1.0832	0.9530

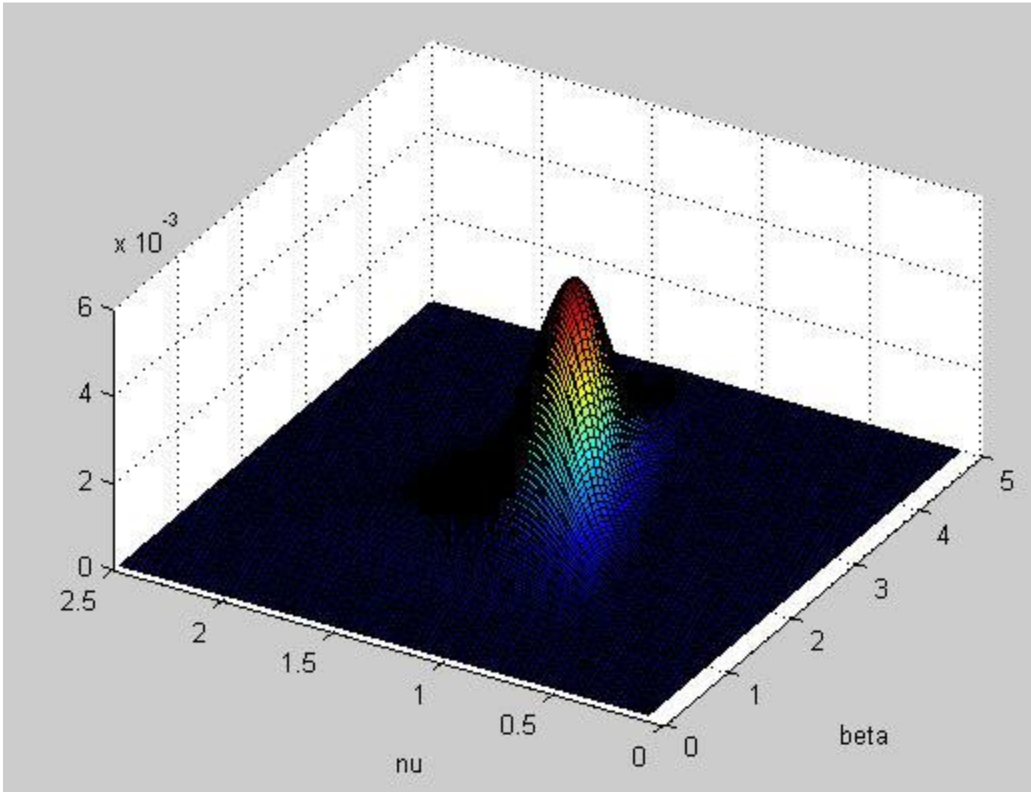

Figure 2. $\boldsymbol{f(\theta|D)\ /K}$ for $\boldsymbol{D = \{x_1, x_2, \ldots x_M\}}$ as given in table 1, *M=10*

This function is similar to the likelihood function $f(D|\theta)$ but is scaled and clipped at the boundaries by the prior. As expected, its maximum value occurs at values very similar to the true values of $\eta^* = 1.0$ and $\beta^* = 2.0$. Using equation 5 and 6, a prediction of the density that this data came from can be made by integrating out the parameters θ. This prediction, $f_{\boldsymbol{x}}(x|D)$, is shown in Figure 3 (green curve). Notice that the prediction is not exact (blue curve) however given that more than an order of magnitude more data was required to produce Figure 1 than to produce Figure 3, it is a relatively good fit.

If all of the data samples are truly from a Weibull density, as the amount of data used to compute the likelihood function increases, the posterior density approaches a Dirac delta function centered at η^* and β^*. Therefore, as the amount of data increases, the prior become less important. This reduces the need to have accurate priors since any prior with non-zero value at η^* and β^* will give the same answer.

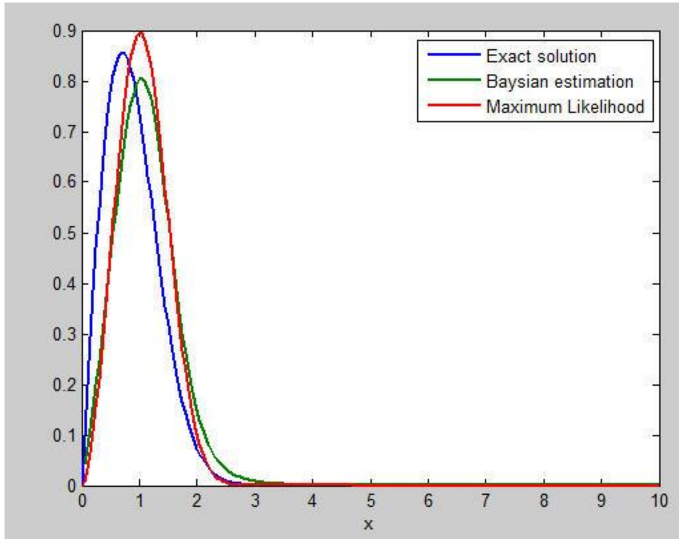

Figure 3. Prediction (green) versus the original (blue) density using 10 data points

The maximum likelihood approach can also be used to estimate the density function for the 10 samples of data given in table 1. This estimate is also given in Figure 3 (red line). The answer is not much different from the Bayesian solution with a uniform prior.

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2) Weibull Paper

In order to determine if the sample data fits a Weibull, we can plot the data with its estimate and compare the results. For this, Weibull paper is used. Letting $\mathrm{y} = ln\left(ln\left(\frac{1}{1-\mathrm{F_x(x)}}\right)\right)$ and $z = \ln(x)$, equation 6 becomes

$$\mathrm{y} = \beta \mathrm{z} + \beta\, ln(\eta), \qquad (9)$$

a straight line where the slope of this line is β and the intersection of the line with the ordinate is a function η.

Figure 4 contains a picture of Weibull paper. On the ordinate is plotted $ln\left(ln\left(\frac{1}{1-F_x(x)}\right)\right)$ where the values of 100 $F_x(x)$ have been labeled. On the abscissa is the value of $ln(x)$ where the value of x have been labeled. When $x = \eta$, 100 $F_x(x) = 63.2$. Therefore, η is the value of the abscissa with an ordinate value of 63.2 and the slope of line is measured to determine β.

The use of Weibull paper to estimate the parameters of the density assumes that values of $F_x(x)$ are known (or that they can be approximated from measured data). This may require much more data than was used to obtain an estimate of the parameters using the Baysian or likelihood approach.

EXAMPLE PROBLEM 2

Using the data in table 2, we can approximate the function $F_x(x)$ using a frequentist approach. That is

$$F_{\boldsymbol{x}}(x) \sim \sum_{i=1}^{M} I(\mathrm{x_i} < x) \qquad (11)$$

where $I(\mathrm{x_i} < x) = 1.0$ if $\mathrm{x_i} < x$ *and zero otherwise*.

Table 2. Random points from $W(1.0,2.0)$

0.0685	0.1079	0.1084	0.1280	0.1403	0.1660	0.1924	0.2492	0.2607	0.2644
0.3170	0.3423	0.3554	0.3578	0.4058	0.4196	0.4230	0.4337	0.4694	0.4712
0.4724	0.5009	0.5066	0.5276	0.5305	0.5465	0.5577	0.5632	0.5776	0.5982
0.6100	0.6136	0.6207	0.6270	0.6304	0.6310	0.6783	0.7326	0.7591	0.7759
0.7907	0.7928	0.8033	0.8204	0.8258	0.8575	0.8615	0.8686	0.8758	0.8791
0.8797	0.8957	0.9381	0.9524	0.9554	0.9609	0.9663	0.9670	0.9675	0.9728
0.9791	0.9956	1.0335	1.0363	1.0545	1.0637	1.0672	1.0770	1.0843	1.0873
1.0934	1.1049	1.1417	1.1743	1.1838	1.1952	1.2028	1.2443	1.2479	1.2638

These results are given in table 3.

Table 3. An estimate of $\boldsymbol{F_x(x)}$ using table 2 data

x	$F_x(x)$	$100\ F_x(x)$
0.3940	0.1400	14.0
0.7881	0.4000	40.0
1.1821	0.7400	74.0
1.5762	0.9100	91.0

This data can then be plotted on Weibull paper as shown in Figure 4. Again, the slope of this line is β and the value of η is the ordinate value for an abscissa value of 63.2. As shown on this plot, $(\beta,\eta) \sim (\beta^*,\eta^*)$.

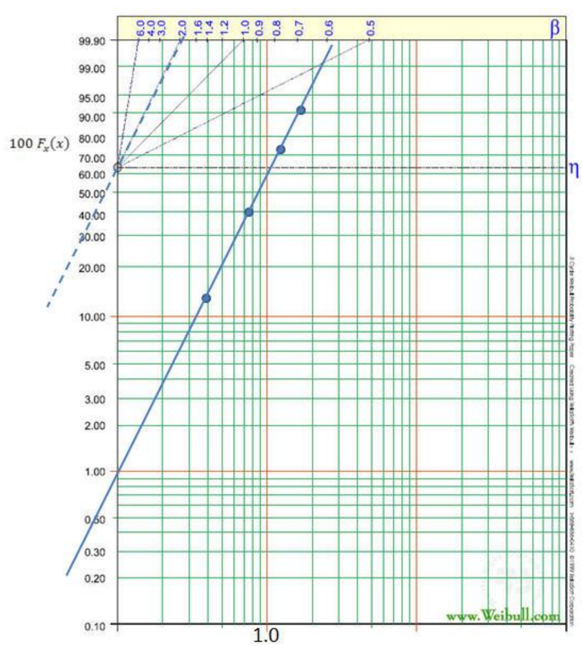

Figure 4. Using Weibull paper to determine $\boldsymbol{\beta}$ and $\boldsymbol{\eta}$

3) Credible bounds

In sections 1 through 3, the parameters of the density function $f_{\boldsymbol{x}}(x)$ were treated in a statistical fashion. Therefore, any result using this density function can also be viewed statistically resulting in *a statistically representation of its statistics*. This is often stated in terms of a confidence or credibility bound. For example, given (β, η), $y = F_{\boldsymbol{x}}(x)$ can be calculated. However since $(\boldsymbol{\beta}, \boldsymbol{\eta})$ are random, $\boldsymbol{y} = \boldsymbol{F}_{\boldsymbol{x}}(x)$ is random with cumulative distribution function $F_{\boldsymbol{y}}(y)$. This implies that there is some finite probability $1 - 2 \propto$ that the true probability, $\boldsymbol{y}$, is within the range $\{y_{min}, y_{max}\}$ where $F_{\boldsymbol{y}}(y_{min}) = \propto$ and $F_{\boldsymbol{y}}(y_{max}) = 1 - \propto$. The range, $\{y_{min}, y_{max}\}$ is referred to as the $100 \propto$ percent confidence bound on the statistic. In Bayesian estimation this bound is not based directly off of the statistics of x but is a function of the parameters of $f_{\boldsymbol{x}}(x)$ which involves a prior. Therefore, this bound is not referred to as a confidence bound but as a credible bound.

The problem of finding a credible bound involves producing random samplings of the random variables and then performing Monte Carlo analysis. A difficulty in this approach is the production of samplings of the random variable when the random variables are not independent. Equation 5 is not Weibull and cannot be represented as the product of marginals (i.e. $\beta$and η are not independent variables).

To overcome this difficulty a non parametric conditional density approach is used. First, the marginal of one of the parameters is determined. Here, assuming a Weibull, we first find the marginal of β as

$$f_{\boldsymbol{\beta}}(\beta) = \int_0^\infty f(\theta|D)\ d\eta. \qquad (12)$$

In return, the cumulative marginal density function of β can be calculated as

$$F_{\boldsymbol{\beta}}(\beta) = \int_0^\beta f_{\boldsymbol{\beta}}(x)\, dx. \qquad (13)$$

To determine a random sampling of β an approach similar to that used in example problem 1 is used where $\boldsymbol{\beta} = F_{\boldsymbol{\beta}}(\boldsymbol{y})^{-1}$ and $\boldsymbol{y} \sim U(0,1)$. Once β is chosen, it is no longer a random variable but is treated as a given. Therefore, since the data is not independent, to find a random sample of η, the given value of β must be considered. This is performed by using the conditional density

$$f_{\boldsymbol{\eta}}(\eta|\beta) = \frac{f(\theta|D,\beta)}{\int f(\theta|D,\beta) d\eta}. \qquad (14)$$

As in equation 13,

$$F_{\boldsymbol{\eta}}(\eta|\beta) = \int_0^\eta f_{\boldsymbol{\eta}}(x|\beta)\, dx \qquad (15)$$

and $\boldsymbol{\eta} = F_{\boldsymbol{\eta}}(\boldsymbol{y}|\beta)^{-1}$ where $\boldsymbol{y} \sim U(0,1)$.

EXAMPLE PROBLEM 3

To illustrate how to sample a joint density that does not contain independent variables, we return to example problem 1. Figure 5 is a two dimensional view of the density function $f(\theta|D)$ shown in Figure 2. The parameters are not independent since it is not possible for two marginals to be multiplied together to produce this function.

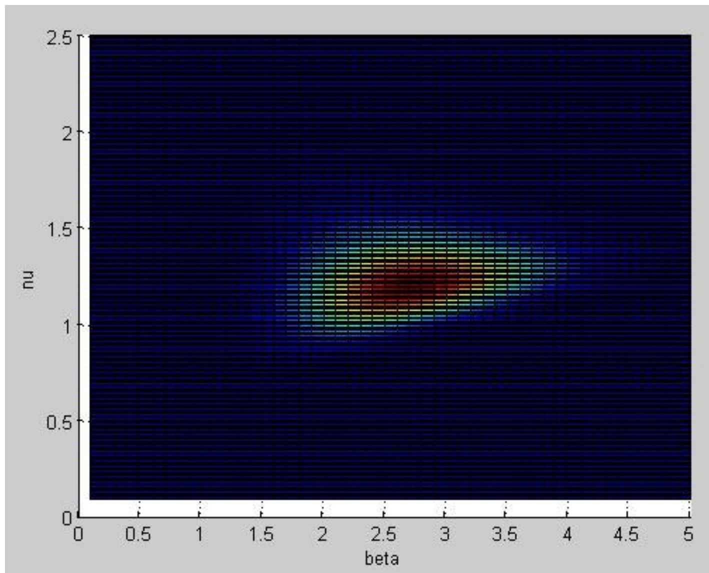

Figure 5. Figure 1 in a 2-D view

Following the above approach, the marginal density for β is shown in Figure 6. Also shown in this figure is a randomly chosen point. The density for η conditioned on this β is also shown in Figure 6. From this density function a random variable for η can be chosen (i.e. by mapping a uniform density through its inverse). The conditional density function for η must be calculated for each randomly sampled value of β.

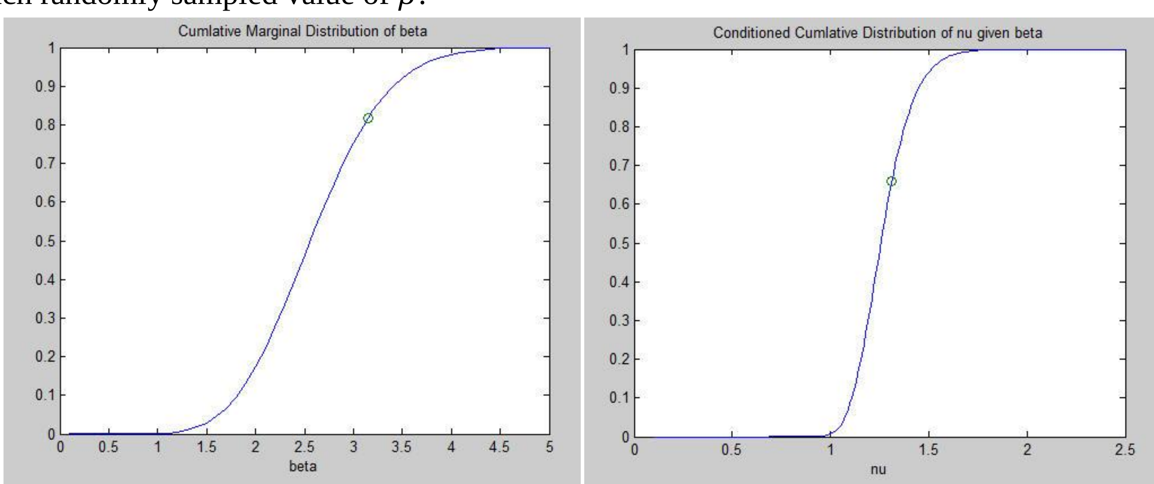

Figure 6. Left: cumulative distribution function for $\boldsymbol{\beta}$; the point represents a random value; Right: the conditional cumulative distribution function for $\boldsymbol{\eta}$ conditioned on the point that was chosen for $\boldsymbol{\beta}$

Figure 7 shows an ensemble of sampled data for the joint density shown given in Figure 5. Using a Parsen's windows[ii], the joint density can be reconstructed (also given in Figure 7). Notice that Figure 5 and the right figure in Figure 7 are very similar as they should be.

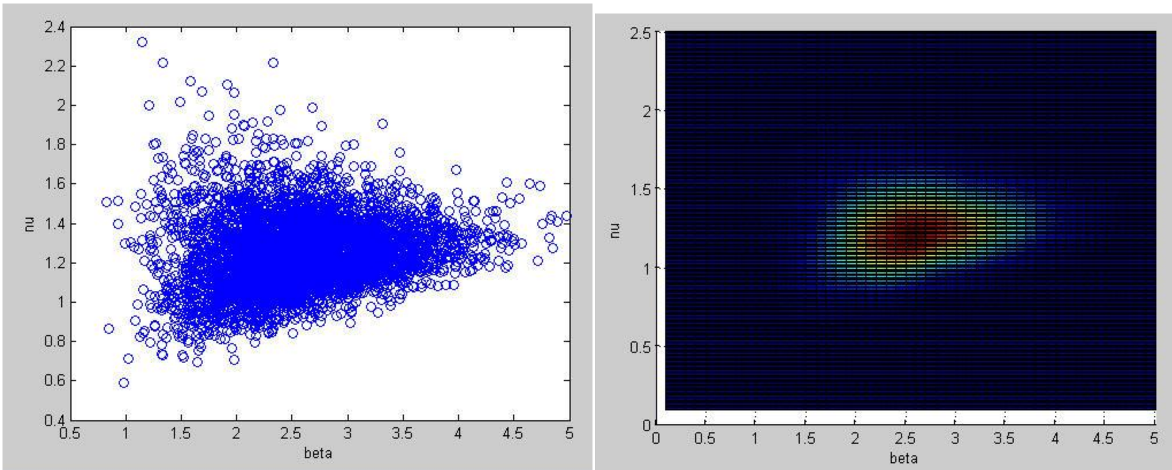

Figure 7. Left: ensemble of sampled data for the joint density in Figure 5; Right: reproduction of joint density function using Parsen's windows

With an ensemble of sampled data, a prediction of credible bounds can be estimated using Monte Carlo. Figure 8 Shows the 5% credible bounds (i.e. $\propto = 0.05$) for $1 - F_{\boldsymbol{x}}(x)$ (reliability) where $F_{\boldsymbol{x}}(x)$ is W(1.0,2.0).

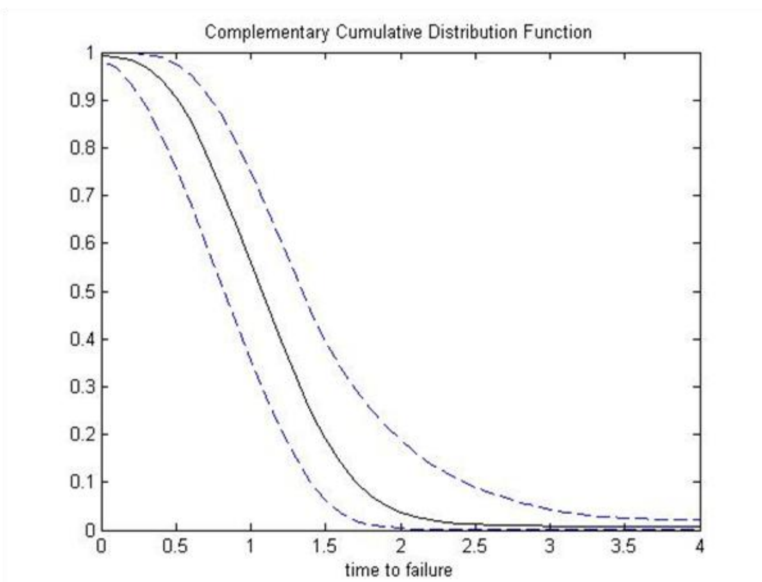

Figure 8. 5% credible bounds for the complementary cumulative distribution function for data with a posterior density $\boldsymbol{f}(\boldsymbol{\theta}|\boldsymbol{D})$ where $\boldsymbol{D}$ is given in table 1

4) Damage Potential

In sections 1 through 3, a number of common estimation tools where presented to fit or to determine the quality of an assumed distribution given experimental data. In this section, the concept of damage potential will be presented as will the concept of total energy as an indicator of this potential. In the next section, the results of this section will be integrated with the above sections to present a proposed method of component certification.

Components must survive environments as specified by the stockpile to target sequence (STS). These environments are divided into two sets of environments - normal environments and abnormal environments. Normal environments are those environments which the weapon system is expected to operate reliably within (these include hostile environments by definition). Abnormal environments are environments for which the weapon system is not required to operate reliably within, but in which, it must be safe (i.e. the required minimum possibility of a blinding white flash). The weapon is not expected to encounter abnormal environments during its life but might encounter such an environment. Abnormal environments would include plane crashes, fuel fires and hydrostatic crush.

Components within the weapon system must be designed to function such that the overall reliability and safety of the weapon system meets probabilistic requirements as defined in the military characteristics (MCs). These statistical requirements are different for the normal and abnormal environments and for different aspects of the life of the weapon system.

System level insults (as defined in the STS) determine component level responses via a transmission path from the system environment to the component. In the past, component environments were deduced by subjecting the weapon system to an experimental system level insult and then measuring the response of a prototype or a mass mockup of the component. This response was then used to define specifications for component design through a number of different methods. These specifications were presented in terms of quantities designed to signify the potential to damage the component. In this regard, these quantities were a statement of damage potential. By following this process, the testing of components could be transferred from the system level to the laboratory resulting in a more cost effective certification process.

Here we focus on two quantities used to specify damage potential – a shock response spectrum and an energy spectrum. As shown in figure 9 (and as stated above) the response of a component (or mockup) is measured while the system is subjected to a system level insult. This acceleration response is then applied to a math model to calculate a form of damage potential. For discussion here, this math model is a set of second order linear oscillators with the same acceleration input at the base of each. Each second order oscillator is tuned to a different natural frequency $\omega_{n,i}$ where $\omega_{n,1} < \omega_{n,2} < \omega_{n,3} \ldots$ and so on. The min-max SRS measure of damage potential is a plot of the maximum acceleration of each mass as a function of natural frequency. The hypothesis behind damage potential using the SRS approach is that a more severe insult to the component can be constructed if its SRS envelops the SRS of the component response seen during a system level insult.

An SRS is an attractive method for representing damage potential considering that several SRS's can be enveloped to determine a component environment $a_b(t)$ that has "greater" damage potential than any enveloped environment. Enveloping is important considering that not every acceleration environment can be produced in the laboratory. However, by enveloping a set of measured SRS's with an SRS that can be produced in a laboratory, we can state that if the component survives the laboratory test, it will survive in the system level environment.

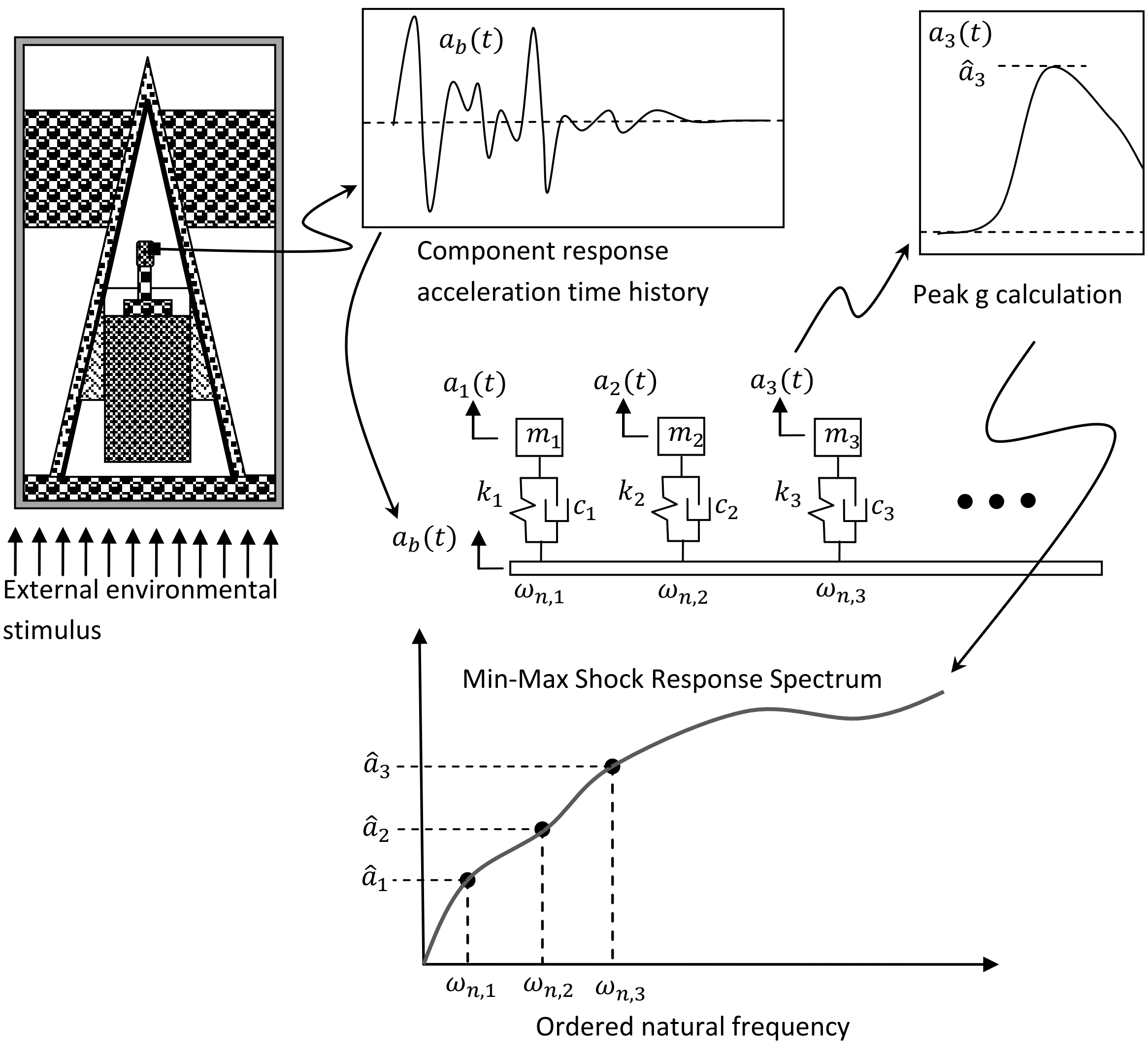

Figure 9. A min-max SRS is determine by applying an enforce acceleration at the base of a set of second order oscillators and determining the peak acceleration response; The SRS is a plot of these peaks as a function of natural frequency

The SRS method of stating damage potential for shock inputs is a legacy method with a large amount of experimental data that justifies its use. Nevertheless, there are many criticisms to its use.

- The SRS cannot be tied back to the military characteristics of the weapon system which are stated in terms of probabilities. The SRS is a hard bound where the component either survives the bounding environment specified by the SRS or it does not.
- The SRS does not allow for damage accumulation. The SRS damage potential criterion is the same for a component that is shocked once or multiple times[iii].
- There is no provision for including temporal information. The SRS is not unique. The same SRS could have come from different waveforms[iv].
- The SRS statement of damage potential cannot be compared to other statements of damage potential. In particular, it is difficult to make a connection between an SRS and a power spectral density (PSD).
- The SRS is very heuristic. This is no strong theoretical argument connecting peak acceleration of a set of second order systems to the failure mechanisms of a component. The reason for using an SRS is based primary upon a plethora of experimental evidence.

As a result, alternatives to the SRS method have been sought. One alternative is to use energy as a statement of damage potential. The energy spectrum starts with the same set of oscillators shown in Figure 9. The difference between the energy spectrum and the SRS is that the energy spectrum is stated in terms of energy into an oscillator, not in terms of its peak acceleration response. Figure 10 shows the response of a single second order oscillator. Given the enforced acceleration,$a_b(t)$, the energy distribution per unit mass of this oscillator obeys the relationship[v]

$$E_K + E_D + E_A = E_I \qquad (16)$$

where $\frac{E_k}{m_i} = \frac{v_i^2}{2}$, is the kinetic energy; $\frac{E_D}{m_i} = \frac{1}{m_i}\int c_i v_i^2 dt$, is the dissipated energy; $\frac{E_A}{m_i} = \frac{1}{m_i}\int k_i x_i v_i dt$, is the absorbed energy; $\frac{E_I}{m_i} = -\int a_b v_i dt = \frac{E_K+E_D+E_A}{m_i}$; is the input energy and where $v_i(t) = \int a_i(t)dt$ and $x_i(t) = \int v_i(t)dt$.

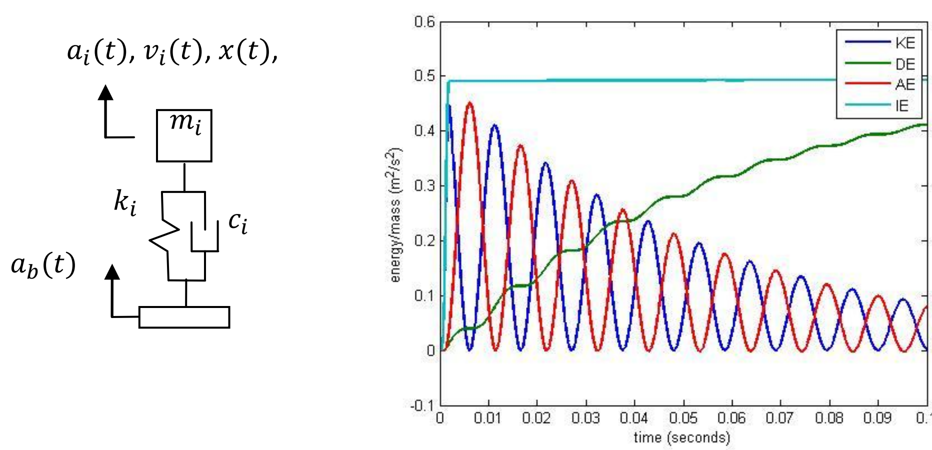

Figure 10. Second order oscillator and energy response; the input is a 0.002 second, 1000g Haversine; the system is tuned to a natural frequency of 300 rad/sec (48Hz) with modal damping at 3%

As expected, the kinetic energy and absorbed energy oscillate and fall in amplitude as the dissipated energy rises. Over time, the total input energy is dissipated. As stated above,

$$\frac{E_I}{m_i} = -\int a_b v_i dt. \qquad (17)$$

Following the work of Ordaz et. al.[vi] ,

$$V_i(j\omega) = A(j\omega)H_v(j\omega) \qquad (18a)$$

where

$$H_v(j\omega) = -\frac{j\omega}{\Omega^2 - \omega^2 + 2j\zeta\omega\Omega} \qquad (18b)$$

$V_i(j\omega) = \mathcal{F}(v_i(t))$, $A_i(j\omega) = \mathcal{F}(a_i(t))$, $\Omega = \sqrt{\frac{k_i}{m_i}}$, $\zeta = \frac{c_i}{2m_i}$ and

$$\mathcal{F}^{-1}\big(V_i(j\omega)\big) = v_i(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V_i(j\omega)e^{j\omega t}\, d\omega \qquad (18c)$$

Substituting equation 18b into equation 18a, and the result into equation 18c and 17, gives

$$\frac{E_I}{m_i} = -\frac{1}{2\pi}\int a_b(t)\int_{-\infty}^{\infty} A(j\omega)H_v(j\omega)e^{j\omega t}\, d\omega\, dt = -\frac{1}{2\pi}\int_{-\infty}^{\infty} A(j\omega)H_v(j\omega)\left[\int a_b(t)e^{j\omega t} dt\right] d\omega = -\frac{1}{2\pi}\int_{-\infty}^{\infty} A(j\omega)A(-j\omega)H_v(j\omega)\, d\omega = -\frac{1}{2\pi}\int_{-\infty}^{\infty} |A(j\omega)|^2 H_v(j\omega)\, d\omega\,. \qquad (19a)$$

From equation 18b, $H_v(j\omega)$, is conjugate symmetric. Therefore,[vii]

$$\frac{E_I}{m_i} = -\frac{1}{\pi}\int_0^{\infty} |A(j\omega)|^2 \Re e[H_v(j\omega)]\, d\omega. \qquad (19b)$$

This can also be written as

$$\frac{E_I}{m_i} = -\frac{1}{2\pi}\int_{-\infty}^{\infty} |H(j\omega)|^2\, d\omega \qquad (19c)$$

where

$$H(j\omega) = A(j\omega)H_v(j\omega). \qquad (19d)$$

Equation 19b states that the energy input to the i^{th} second order oscillator is the negative of the integrated filtered acceleration signal energy. This is the input energy spectrum of the acceleration time history $a_i(t)$.

EXAMPLE PROBLEM 4

To illustrate the above equations, we will calculate the energy spectra for a Haversine where

$$a_b(t) = \frac{1}{2}\left\{1 - \cos\left(2\pi \frac{t}{0.002}\right)\right\}$$

The input time history is shown in Figure 11a. The magnitude of the Fourier integral transform (calculated using DFTs) is shown in Figure 11b, and the input energy spectrum is shown in Figure 11c. Notice that the input energy spectrum shows a response for frequencies beyond where there is content in the Fourier integral transform (similar to an SRS response). Also notice that there is ringing in the response at DC. This is due to the fact that $\lim_{\omega,\omega_n \to 0} \Re e[H_v(j\omega)] \to \infty$.

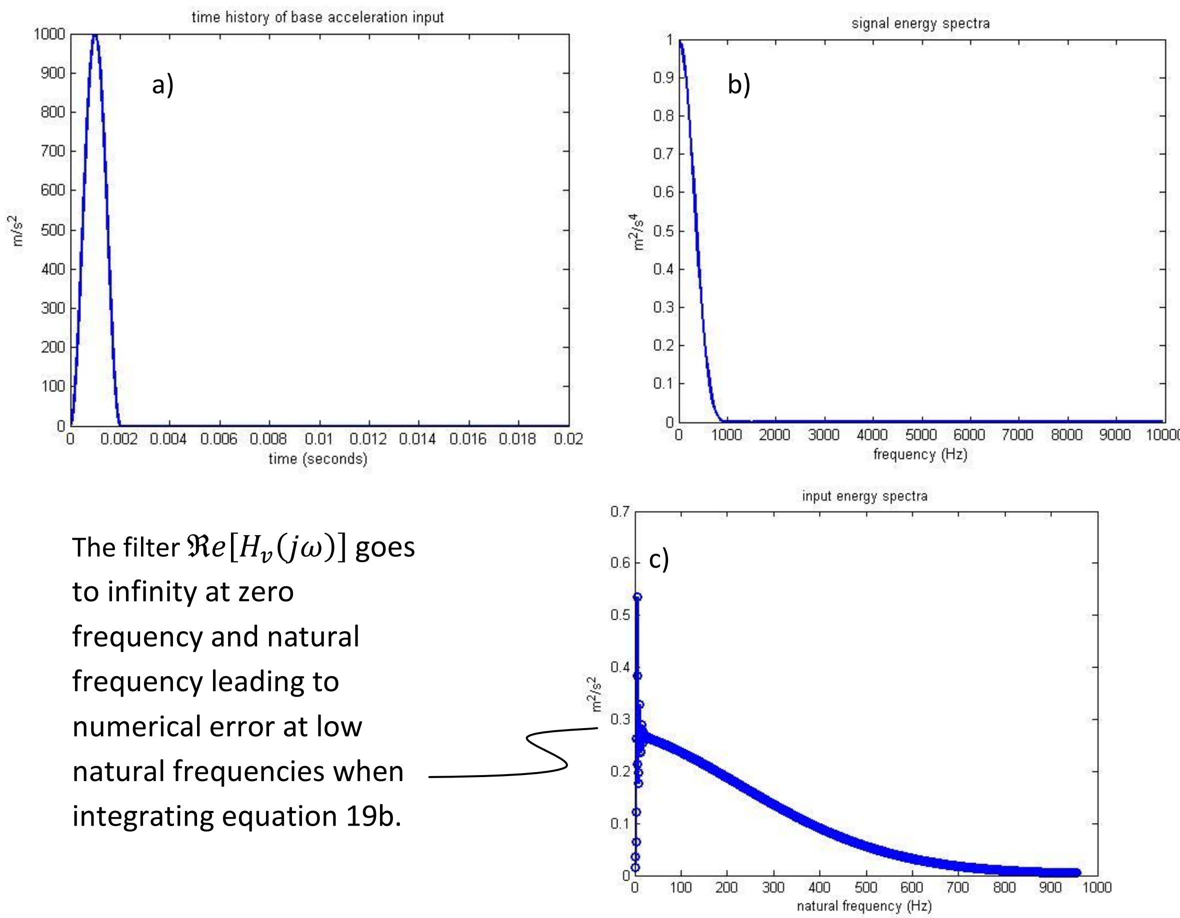

Figure 11. Upper left a): time history of acceleration input; Upper right b): frequency response of acceleration input: Lower c): input energy spectra; the ringing at zero natural frequency is due to numerical error

To further illustrate the energy spectrum, consider the input

$$a_b(t) = \sum_{i=1}^{3} A_i e^{-\alpha_i t}\{\sin(2\pi f_i\, t)\}$$

where the parameters are given in the below table

i	A_i	α_i	f_i
1	1000	100	500
2	1000	150	200
3	900	100	350

Then the time history, frequency spectrum and energy spectrum are as given below.

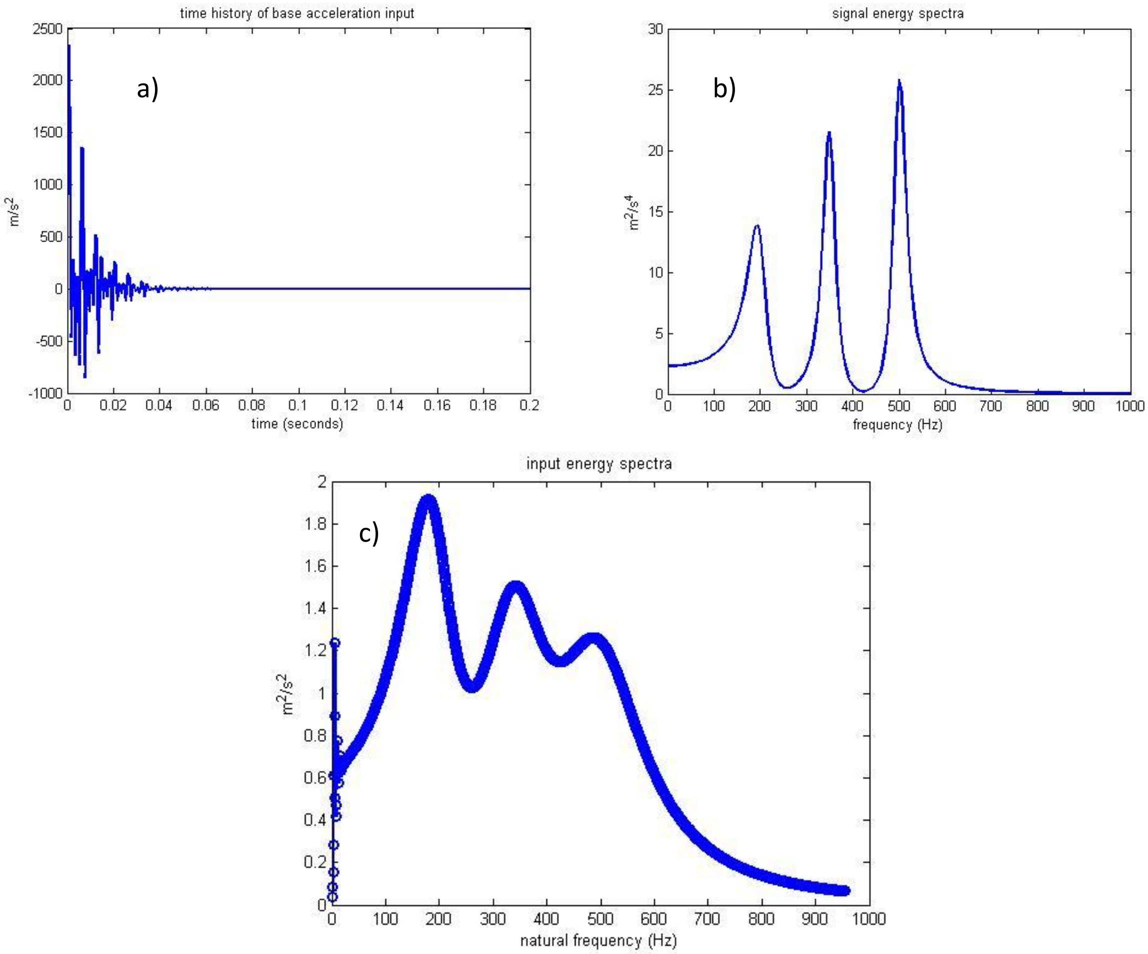

Figure 12. Upper left a): time history of acceleration input; Upper right b): frequency response of acceleration input: Lower c): input energy spectra

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5) Severity

In the initial portions of this paper, it was shown how Bayesian estimation could be used to predict the parameters of a two parameter Weibull density. This was performed with the use of a likihood function (equations 3) which was constructed using experimental data (D). The variable, x, in the Weibull density can stand for a number of different physical quantities. Traditionally, x, represents the time to failure of a component subjected to a continuous stimulus. However, using an energy approach, the variable x represents severity, a function of the energy spectrum. Following reference 1, severity, S, is defined to be the expected value of margin, M, where margin is defined to be

$$\mathrm{M} = \frac{\sum_{i=1}^{N} \Lambda_i^2 E_2^I(f_{n,i})}{\sum_{i=1}^{N} \Lambda_i^2 E_1^I(f_{n,i})}, \qquad \text{(eq. 20)}$$

N is the number of modes considered, Λ_{i} is the modal participation factor for the i^{th} component mode, $E_2^I(f_{n,i})$ is the energy spectrum of the component response at the i^{th} natural frequency of the component, and $E_1^I(f_{n,i})$ is the energy spectrum of a specification at the i^{th} natural frequency. The modal participation factors and the natural frequencies for this calculation must come from an experimental realization or from a numerical representation of the component.

Returning to equation 7 and assuming a Weibull density, the probability of the failure of a component can now be represented as

$$F_{\boldsymbol{s}}(S) = 1 - e^{-\left(\frac{S}{\eta}\right)^{\beta}} \qquad \text{(eq. 21)}$$

which can be determined experimentally by subjecting a number of components to increasing levels of S until failure occurs, recording these S values at failure and then using Bayesian or likelihood estimate to determine the parameters $(\boldsymbol{\beta}, \boldsymbol{\eta})$. As presented above, this also allows for a prediction of credible bounds.

Equation 21 can be used to determine the probability of a component failing in any environment of known severity. This, in return, can be used in fault tree analysis to determine if the reliability of the weapon system is met or in a safety analysis to determine if Walkse or normal environments MCs are satisfied.

EXAMPLE PROBLEM 5

Consider the situation where the acceleration response in a given component direction in a bounding environment is given by the time response in Figure 12a. This time history represents a specification that the component must not fail to within a given level of probability. Nevertheless, this is not a time history that can be reproduced within a laboratory environment.

The time history in Figure 11a can be produced within a laboratory environment using a drop table and programming material. It is desired to relate these two insults in terms of their severity. To do this, the uncertainty, natural frequencies and participation factors at the point of the acceleration measurement must be known.

A numerical model of the component is constructed and the natural frequencies and participation factors are calculated as given in the below table. Here, it is assumed that the uncertainty in the natural frequencies and participation factors are minimal and therefore the margin is the severity. Also shown in this table are the values of the input energy spectrum of the specification and laboratory insult.

i	$f_{n,i}$	Λ_{i}	$E_2^I(f_{n,i})$ laboratory spectrum	$E_1^I(f_{n,i})$ specification spectrum
1	520	0.3	1.1649	0.0504
2	210	0.1	1.5727	0.1831
3	365	0.9	1.4261	0.1060

Using equation 20, the severity of the laboratory insult is 0.072. Relative to the energy metric, this is an under test.

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6) Discussion

A number of different topics have been cover in this paper. Each of these has a different level of maturity and is acceptable to different levels.

Bayesian and likelihood estimation

Bayesian estimation is a very mature area of analysis that is built on a strong mathematical framework with historical benefits and complications.

- **The priors used in Bayesian estimation are almost never known:** Nevertheless, for the situation where there is a large amount of data, this shouldn't make much of a difference considering that as the amount on information grows, the solution is less and less a function of the prior. However, for limited data, this is not the situation. One way around this problem is to assume that nothing is known about the parameters other than the form of the density function and that the parameters of the density function are bounded. Using maximum entropy, this infers a uniform prior density function across the bounded range. The result is that the estimate is the expected value of the parameters within the range. Another method to determine these parameters is the likelihood method which simply takes the value of the parameters at the peak value of parameter density. The problem is that the resulting likelihood estimate does not have to equal the expected value of the function derived from the Bayesian approach and therefore, there is always a questions as to which is the better estimate – the Bayesian solution or the Likelihood solution.
- **For both the Bayesian and likelihood methods, the form of the density function must be known:** In general, this is a difficult assumption to justify, but without large amount of data and with limited options, such an approach is often taken.

Damage potential

Experimental analysis in the laboratory is less costly and time consuming than full scale experimental analysis. Therefore, there is a strong desire to take any experimental analysis of a component out of the full scale environment and to move it into the laboratory. Nevertheless, moving from the full scale environment to a laboratory environment is froth with potential errors.

- **There are only a limited number of insults that can be produced in a laboratory environment:** More common types of insults are Haversine shocks (produced by drop tables), frequency limited random excitations (produced by shaker tables), and resonant plate tests (produced using a Hopkinson bars and resonant plates). None of these acceleration time histories exactly mimic time histories seen by components in real environments. Moreover, none of the machines that produce these time histories mimic the mechanical impedance seen by components in full scale environments. Therefore, there is a need to connect the potential for components to fail within the laboratory to the potential for components to fail in system environments. This is done using the SRS and/or energy methods (and other methods not discussed here). As a result, the quality of the results is highly dependent on the underlying assumptions used in each of these methods to quantify damage and no method is perfect.
- **The SRS uses the peak acceleration response of the parallel resonator system given in figure 9 to deduce an enveloping laboratory insult: Nevertheless, the form of this parallel system cannot be generalized to the physics of all components and peak acceleration cannot be generalized to all forms of damage.** In this regard, the SRS is not based upon a strong theoretical argument that encompasses many forms of physics and failure. Nevertheless, SRS's are broadly used in experimental analysis to certify components throughout the engineering world. This is not because it has a strong theoretical basis, but because they have a strong historical basis which is back up by extensive experimental evidence.

  The above energy method uses the same parallel resonator system as the SRS to gage the potential for energy flow into the component, but goes further by including the physics of the component to determine if the component is susceptible to this flow. For example, the energy spectrum of the specification may be band limited over a range of natural frequencies for which the component cannot accept energy (i.e. $\Lambda_i^2 E_2^I(f_{n,i}) \sim 0$). For this type of insult the severity is zero and therefore so is the damage potential. The SRS assumes (quite conservatively) there is a mode of vibration at every frequency and therefore there is some potential for damage to the component even for this type of situation.

- **The energy method (as described above) is a proposed improvement over the SRS but is not a panacea:** The key assumption is that insults with equal energy flow into the component will have equal damage potential.

- As a minor note, the modal damping in equation 18b is not a function of component dynamics. Therefore, the margin (equation 20) does not truly represent the ratio of energies into the component in the presence of the specification over the energy into the component in the presence of a laboratory insult.
- On a stronger note, the component natural frequencies used in the laboratory will not be those of the component within the weapon system unless the support of the component in the laboratory has a compliance which is similar to that in the system, a difficult objective to achieve.
- On the strongest note, two inputs may have the same energy into the component but each may excite different modes of vibration – one which is susceptible to failure and one which is not. In this regard, they do not have the same damage potential.
- There is also a limitation of physics. Equation 20 assumes that the system is linear and rational of which no real system is.

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[i] T.S. Edwards, W76/Mk4 and W76-1/Mk4A Dynamic environments Capability Study, OUO, unreleased

[ii] R.O. Duda, P. e. Hart, D.G. Stork, Pattern Classification, Second Edition, John Wiley & Sons, New York, 2001

[iii] D. Smallwood, presentation from the 80th Shock and Vibration Symposium Oct. 2009

[iv] Same as reference iii.

[v] Same as reference iii.

[vi] M. Ordaz, B. Hueta, E. Reinoso, Exact computation of input-energy spectra from Fourier amplitude spectra, Earthquake Engineering and Structural Dynamics, 2003,; 32:587-605

[vii] Reference iii presents this result incorrectly

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