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Jian Zheng and Hong Qin

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On the singularity of the Vlasov-Poisson system

Jian Zheng[1] and Hong Qin[1,2]

[1]*CAS Kay Laboratory of Basic Plasma Physics and Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, P. R. China*

[2]*Plamsa Physics Laboratory, Princeton University, Princeton, NJ 08550, USA*

Abstract

The Vlasov-Poisson system can be viewed as the collisionless limit of the corresponding Fokker-Planck-Poisson system. It is reasonable to expect that the result of Landau damping can also be obtained from the Fokker-Planck-Poisson system when the collision frequency ν approaches zero. However, we show that the colllisionless Vlasov-Poisson system is a singular limit of the collisional Fokker-Planck-Poisson system, and Landau's result can be recovered only as the ν approaching zero from the positive side.

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Landau damping is the most fundamental result in plasma physics [1]. In 1946, Landau found that a wave can suffer a damping even in a collisionless plasma when he studied electron oscillations as an initial problem with the linearized Vlasov equation [2]. In his seminal paper, Landau did not give a physical explanation to his result. It seems that Landau damping is just a mathematical result. Since then, Landau damping stimulates many discussions. Now, it is recognized that the Landau damping is due to the wave-particle interactions: waves transfer their energy to the charged particles and thus suffer damping even without any irreversible processes. It is well known that the Vlasov-Poisson system describes the kinetic processes in high-temperature tenuous plasmas in which collisions between charged particles can be neglected. On the other hand, the Vlasov-Poisson system can be viewed as the collisionless limit of the corresponding Fokker-Planck-Poisson system. From this point of view, it is reasonable to expect that Landau's result can also be obtained from the Fokker-Planck-Poisson system when the collision frequency approaches zero.

The linear response of a plasma can be properly described by its permittivity ϵ. In the case of an isotropic plasma, the mean energy Q of a monochromatic electric field dissipated per unit time and volume is given by [3]

$$Q = \frac{\omega \operatorname{Im} \epsilon}{8\pi} |\mathbf{E}|^2. \tag{1}$$

Landau damping means that the permittivity of a collisionless plasma has a positive imaginary part when the frequency is positive,

$$\operatorname{Im} \epsilon_{\text{Landau}}(\omega, k) > 0 \text{ when } \omega > 0. \tag{2}$$

With inclusion of collisions, the plasma permittivity should depend on some collision frequency ν, i.e.,

$$\epsilon = \epsilon(\omega, k; \nu). \tag{3}$$

It is reasonable to expect that Landau's result can be obtained as the collisionless limit of the collisional case, *i.e.*,

$$\epsilon_{\text{Landau}}(\omega, k) = \lim_{\nu \to 0} \epsilon(\omega, k; \nu). \tag{4}$$

From a mathematical point of view, the limit of vanishing ν can be approached from either side of zero. However, we show in this article that Landau's result can be reduced only from $\epsilon(\omega, k; \nu)$ as ν approaching zero from the positive side. When ν approaches zero from the negative side, the permittivity approaches to a different value with a different sign for the imaginary part. In other words, the colllisionless Vlasov-Poisson system is a singular limit of the collisional Fokker-Planck-Poisson system.

We start our discussion from the following one-dimensional Fokker-Planck equation [3],

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial z}-\frac{e}{m}E\frac{\partial f}{\partial v}=C(f)\equiv\nu\frac{\partial}{\partial v}\left(vf+v_T^2\frac{\partial f}{\partial v}\right), \tag{5}$$

where ν is the collision frequency, v_T is the thermal speed, and E is the self-consistent electric field governed by the Poisson equation,

$$\frac{\partial E}{\partial z}=-4\pi n_0 e\left(\int f(t,z,v)dv-1\right). \tag{6}$$

The collision term in Eq. (5) conserves the particle number and drives the distribution function to a Maxwellian when $\nu>0$,

$$f_0\propto e^{-v^2/2v_T^2}.$$

Of course, for a high-temperature plasma, a realistic collision term should be the Landau collision integral [4] or the Fokker-Planck equation with the Rosenbluth potentials [5]. However, the model Fokker-Planck equation of (5) is much simpler for the discussion on collisional effects. Lenard and Bernstein first studied the influence of collisions on small-amplitude electron oscillations with Eq. (5) [6]. In recent years, Ng, Bhattachargee and Skiff [7, 8] and Short and Simon [9] also addressed the collisional effect on the eigenmodes of plasma oscillation using Eq. (5). In particular, they investigated the limit of ν approaching zero, and concluded that Landau's result for the Vlasov-Poisson system can be recovered. However, the limit of $\nu\to 0$ in these studies are implicitly assumed to be from the positive side, *i.e.*, $\nu\to +0$. As we will show now, if taking the limit from the negative side, *i.e.*, $\nu\to -0$, we will obtain a very different result. We will first present the mathematical derivation, and then discuss the physical implication of the result.

The linearized Fokker-Planck equation (5) is given by

$$\frac{\partial \delta f}{\partial t}+v\frac{\partial \delta f}{\partial z}-\frac{e}{m}E\frac{\partial f_0}{\partial v}=\nu\frac{\partial}{\partial v}\left(v\delta f+v_T^2\frac{\partial \delta f}{\partial v}\right). \tag{7}$$

where δf is the small perturbation from an equilibrium Maxwellian f_0. The electric field E is determined with the Poisson equation,

$$\frac{\partial E}{\partial z}=-4\pi e n_0\int \delta f dv. \tag{8}$$

Equations (7) and (8) can be solved with the Laplace-Fourier transformation [6], and the plasma permittivity function can be obtained as

$$\epsilon(\omega, k; \nu) = 1 + \frac{1}{\nu^2} \int_0^1 x^{s-1}(1-x)e^{(k/\nu)^2(1-x)}dx. \tag{9}$$

where

$$s = \frac{k^2}{\nu^2} - i\frac{\omega}{\nu}.$$

It would be satisfactory if the limit of $\nu \to 0$ were well-defined and Landau's result could be recovered at this limit. The integral presentation of the permittivity is valid when $\mathrm{Re}\, s > 0$. For any real wave number k, the condition $\mathrm{Re}\, s > 0$ is

$$\mathrm{Im}\, \omega > -\frac{k^2}{\nu} \text{ if } \nu > 0, \tag{10}$$

or

$$\mathrm{Im}\, \omega < -\frac{k^2}{\nu} \text{ if } \nu < 0. \tag{11}$$

In the collsionless limit of $\nu \to 0$, the plasma permittivity of Eq. (9) is well defined in the whole plane of the complex ω. However, we now show that the permittivity $\epsilon(\omega, k; \nu)$ is discontinuous at $\nu = 0$.

First for $\nu > 0$, we change the integral variable as [6]

$$x = e^{-\nu t},$$

then have

$$\epsilon(\omega, k; \nu > 0) = 1 + \frac{1}{\nu} \int_0^\infty e^{-s\nu t}(1 - e^{-\nu t}) \exp\left[(k/\nu)^2(1 - e^{-\nu t})\right] dt. \tag{12}$$

With aid of the series expansion,

$$1 - e^{-\nu t} = -\sum_{n=1}^\infty \frac{(-\nu t)^n}{n!}$$

the integrand on the right hand side of Eq. (12) can be written as

$$\frac{1}{\nu}e^{-s\nu t}(1 - e^{-\nu t}) \exp\left[(k/\nu)^2(1 - e^{-\nu t})\right] = \left[t + \sum_{n=1}^\infty \nu^n Q_n(t)\right] e^{i\omega t - k^2 t^2/2},$$

where $Q_n(t)$ is a finite power series of t. It can be verified that both the sum $\sum_n \nu^n Q_n$ and the integral $\int_0^\infty t^n \exp(i\omega t - k^2t^2/2)dt$ converge uniformly for any finite real k . The permittivity in Eq. (12) can be written as

$$\begin{aligned}\epsilon(\omega,k;\nu>0) &= 1+\int_0^\infty t\exp\left(i\omega t-\frac{1}{2}k^2t^2\right)dt+\sum_{n=1}^{\infty}\nu^n\int_0^\infty Q_n(t)\exp\left(i\omega t-\frac{1}{2}k^2t^2\right)dt\\ &= 1+\frac{\partial}{\partial(i\omega)}\int_0^\infty \exp\left(i\omega t-\frac{1}{2}k^2t^2\right)dt+O(\nu). \qquad (13)\end{aligned}$$

It is easy to verify that

$$\int_0^\infty e^{i\omega t-k^2t^2/2}dt=-\frac{i}{\sqrt{2}k}Z(\omega/\sqrt{2}k),$$

where $Z(\xi)$ is the plasma dispersion function [10]. Thus the permittivity as ν approaching zero from the positive sides is

$$\epsilon(\omega,k;\nu\to+0)=1+\frac{1}{k^2}\left[1+\frac{\omega}{\sqrt{2}k}Z\left(\frac{\omega}{\sqrt{2}k}\right)\right]+O(\nu), \qquad (14)$$

where use is made of the following fact

$$Z'(\xi)=-2(1+\xi Z).$$

Equation (14) is the well-known Landau result.

Now let's look at the case of $\nu<0$. We introduce a new integral variable

$$x=e^{\nu t}.$$

The permittivity (9) can now be written as

$$\epsilon(\omega,k;\nu<0)=1+\frac{1}{\nu}\int_0^\infty e^{s\nu t}(e^{\nu t}-1)\exp\left[(k/\nu)^2(1-e^{\nu t})\right]dt. \qquad (15)$$

In the limit of $\nu\to-0$, for any real frequency ω, we obtain

$$\epsilon(\omega,k;\nu\to-0)=1+\int_0^\infty te^{(-i\omega t-k^2t^2/2)}dt=1+\frac{1}{k^2}\left[1+\frac{\omega}{\sqrt{2}k}Z^*\left(\frac{\omega}{\sqrt{2}k}\right)\right]. \qquad (16)$$

In this case, the imaginary part of the permittivity becomes

$$\mathrm{Im}\,\epsilon(\omega,k;\nu\to-0)=-\sqrt{\frac{\pi}{2}}\frac{\omega}{k^3}e^{-\omega^2/2k^2},$$

which implies that the plasma is emissive instead of absorptive in this case. Equations (14) and (16) clear show that $\epsilon(\omega,k;\nu\to+0)\neq\epsilon(\omega,k;\nu\to-0)$. Therefore, we conclude that the collisionless Vlasov-Poisson system is a singular limit of the collisional Fokker-Planck-Poisson system.

The fact that a negative collision frequency corresponds to an instability should not come as a surprise. It is actually very simple to understand. It is easy to verify that the eignfunctions and eigenvalues of the collision term $C(f)$ in Eq. (5) are given by

$$F_n(v) = e^{-v^2/2v_T^2} H_n(v/v_T),$$
$$\lambda_n = -n\nu, \ (n = 0, 1, 2 \cdots)$$

where $H_n(x)$ is the $n-$th order Hermite polynomial. With aid of the eigenfunctions and eigenvalues, the equation

$$\frac{\partial f}{\partial t} = C(f)$$

can be easily solved,

$$f = \sum_{n=0}^{\infty} a_n F_n e^{-n\nu t}.$$

where the factors a_n are determined with the initial condition. When $\nu > 0$, the distribution function relaxes to a Maxwellian in sufficient time, indicating that the unperturbed distribution f_0 itself in Eq. (7) is stable. On the other hand, if $\nu < 0$, the system is unstable: any small departure from a Maxwellian exponentially grows with time, leading to that the plasma becomes emissive. This is natural and expected. What is surprising here is that the limit of $\nu \to 0$ is a singular point of the Fokker-Planck-Poisson system. In particular for a Maxwellian plasma, as $\nu \to +0$ the system is stable with a finite damping rate, and as $\nu \to -0$ the system is unstable with a finite growth rate.

This interesting results can also be derived by investigating the eigenvalue problem of the Fokker-Plank-Poisson system. Let $\delta f = e^{-i\omega t}\delta\hat{f}$, where ω is the eigen-frequency. By expanding the perturbed distribution function with the eigenfunctions of the collision term $C(f)$, i.e., $\delta\hat{f} = \sum_{n=0}^{\infty} a_n F_n$, we can recast Eqs. (7) and (8) as an eigenvalue problem [7, 11],

$$\mathbb{A}\mathbf{a} = \omega\mathbf{a}, \tag{17}$$

where $\mathbf{a}$ is a vector defined as

$$\mathbf{a} = (a_0, a_1, \cdots),$$

and $\mathbb{A}$ is an array defined as

$$\mathbb{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 1+k^{-2} & -i\nu & \sqrt{2} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & -2i\nu & \sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & -3i\nu & \sqrt{4} & \cdots \\ 0 & 0 & 0 & \sqrt{4} & -4i\nu & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{18}$$

Because of $\mathbb{A}(-\nu) = \mathbb{A}^*(\nu)$, it is easy to show that if $\omega_n(\nu)$ is an eigenvalue of $\mathbb{A}(\nu)$, $\omega_n^*(\nu)$ is an eigenvalue of $\mathbb{A}(-\nu)$. Ng et al have demonstrated that when $\nu \to +0$ the damping rate of the least damped eignemode returns to the Landau result [7]. Therefore, when $\nu \to -0$, the mode is unstable, and the growth equals to the corresponding Landau damping rate.

In summary, we have re-studied the permittivity of a collisionless plasma by solving a linearized model Fokker-Planck-Poisson system and taking the collisionless limit of $\nu \to 0$. Using two different approaches, we have shown that the $\nu \to 0$ limit of collisional permittivity is singular. While the $\nu \to +0$ corresponds to the classical Landau damping, the $\nu \to -0$ limit corresponds to a different permittivity with a different sign of the imaginary part. For example, for a Maxwellian plasma, the limit of $\nu \to +0$ corresponds to the classical Landau damping with a non-vanishing damping rate, and the limit of $\nu \to -0$ corresponds to a unstable case with a non-vanishing growth-rate. This conclusion is consistent with previous results on collisional eigenmodes obtained by Ng *et al.* [7, 8]. The existence of such a singularity begs the question of whether the Vlasov-Poisson system, as a collisionless limit of the Fokker-Planck-Poisson system, is a physically well-posed model. Given the Vlasov-Poisson system without knowing whether it is the limit of $\nu \to +0$ or limit of $\nu \to -0$ of the Fokker-Planck-Poisson system, it is difficult to determine if an perturbation will be dumped or unstable. Of course, one may argue that the collision frequency is always positive and it is meaningless to discuss the limit of $\nu \to -0$. However, we emphasize that negative collision frequency can never be ruled out. One can image, at least theoretically, that an externally applied effect can have the tendency to move the system away from the thermal equilibrium. For example, the effect of an appropriately designed RF system can be approximated by a collision operator with a negative collision frequency [12]. With these physical considerations in mind, it seems that we do have a singularity at hand. It is therefore interesting and necessary to investigate how this apparent singularity can be resolved both mathematically and physically.

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Information Services
Princeton Plasma Physics Laboratory
P.O. Box 451
Princeton, NJ 08543

Phone: 609-243-2245
Fax: 609-243-2751
e-mail: pppl_info@pppl.gov
Internet Address: http://www.pppl.gov