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Prepared for the U.S. Department of Energy under Contract DE-AC02-09CH11466.

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The Hamiltonian Mechanics of Stochastic Acceleration

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(Dated: July 11, 2013)

We show how to find the physical Langevin equation describing the trajectories of particles undergoing collisionless stochastic acceleration. These stochastic differential equations retain not only one-, but two-particle statistics, and inherit the Hamiltonian nature of the underlying microscopic equations. This opens the door to using stochastic variational integrators to perform simulations of stochastic interactions such as Fermi acceleration. We illustrate the theory by applying it to two example problems.

Introduction.— The term “stochastic acceleration” refers to the chaotic motion of particles subjected to a prescribed random force. Such motion occurs in myriad contexts; the turbulent electromagnetic fields present in the interstellar medium and the RF wave fields found in magnetic fusion devices are just two examples. In the astrophysical context, it is thought to be partially responsible for the presence of cosmic rays in our solar system [1]. In the magnetic fusion context, it might explain the presence of certain high-energy tails observed in the National Spherical Torus Experiment when neutral beams are fired into RF-heated plasmas [2].

Robust modeling of stochastic acceleration requires statistical approaches. The dominant approach is to employ the Fokker-Planck equation [3–7] for the one-particle distribution function. However, when studying Richardson dispersion [8, 9], and more generally any phenomenon governed by the two-particle distribution function [10], the one-particle Fokker-Planck equation is insufficient. A Langevin equation for particle trajectories is more appropriate in these cases. Such a Langevin equation would not only govern the evolution of the one- and two-particle distribution functions, but also provide an attractive means to perform Monte Carlo simulations of stochastic acceleration. Currently, there are no satisfactory methods for finding such a Langevin equation [11].

The purpose of this Letter is to describe, for the first time, a systematic procedure for passing from a microscopic description of stochastic acceleration in terms of Hamiltonian equations of motion to the physically-correct Langevin equation for particle trajectories in the long-time limit. We will also show that, aside from reproducing the correct multi-particle statistics, this Langevin equation inherits the Hamiltonian structure of the microscopic dynamics. Specifically, we will show that the Langevin equation is a Hamiltonian stochastic differential equation (SDE) [12]. Thus, this work also provides

the first method for deriving Hamiltonian SDEs from first principles.

We will focus our attention on a general class of stochastic acceleration problems, such as those studied in [3–6]. These consist of a collection of non-interacting particles moving through a prescribed Hamiltonian force field. By assumption, the force will consist of a small-amplitude perturbation superimposed over a time-independent background. The perturbed force felt by a particle will be assumed to have a correlation time much shorter than any bounce time associated with the perturbation, zero mean, and temporally homogeneous statistics. These assumptions preclude treating Coulomb collisions because the polarization field produced by a particle cannot be modeled as a prescribed field; the polarization force depends on the history of a particle’s orbit. They also preclude the treatment of strong turbulence [13].

The main idea.— Mathematically, this type of problem can be described as follows. Each particle moves through a $2n$ -dimensional single-particle phase space M according to a dynamical law given by a time-dependent vector field X_t ; if $z_t \in M$ denotes the trajectory of a particle in M , then

$$\dot{z}_t = X_t(z_t). \quad (1)$$

Because the only forces present are Hamiltonian, X_t must be Hamiltonian in the sense that there is some Poisson bracket $\{\cdot, \cdot\}$ and some time-dependent Hamiltonian, H_t , such that $\dot{z}^i = \{z^i, H_t\}$, where z^i denotes an arbitrary coordinate system on M [14]. By standard mathematical convention, this is written $X_t = X_{H_t}$ [15]. The presumed form of the force then implies $H_t = H_0 + \epsilon h_t$, where $\epsilon \ll 1$, H_0 describes the mean time-independent background, and h_t describes the small-amplitude random perturbation. Moreover, X_{h_t} evaluated on a particle trajectory must have a correlation time τ_{ac} much shorter

than some constant τ , which, in turn, is much shorter than any bounce time associated with the perturbation $\tau_b, \tau_{ac} \ll \tau \ll \tau_b$.

Our goal in this Letter is to find the correct coarse-grained version of the microscopic equations of motion, X_{H_t} . Specifically, we seek a Langevin equation in the form

$$\delta z_t = X_0(z_t)dt + \sum_{k \geq 1} X_k(z_t) \delta W_t^k \quad (2)$$

whose solutions correctly reproduce the late-time statistical behavior of solutions to the microscopic equations of motion. Here X_k are vector fields on M that must be determined, W^k are independent ordinary Wiener processes, and δ denotes the Stratonovich differential [16] (sometimes also written $\circ d$). We will identify the X_k by demanding that Eq. (2) possess two properties: it must generate the Fokker-Planck equations for the one- and two-particle distribution functions, $f_t(z)$ and $g_t(z_1, z_2)$. Baxendale [17] has proven that a Langevin equation is uniquely determined by its one- and two-particle Fokker-Planck equations. Therefore, these conditions uniquely specify the Langevin equation we seek.

Identifying the Langevin equation. — The one-particle Fokker-Planck equation associated to Eq. (2) is given by [16, 17]

$$\begin{aligned} \frac{\partial f_t}{\partial t} &= -\text{div}(f_t X_0) + \frac{1}{2} \sum_{k \geq 1} \text{div}(\text{div}(f_t X_k) X_k) \\ &= A_1 f_t, \end{aligned} \quad (3)$$

while the two-particle Fokker-Planck equation [17, 18] is given by

$$\begin{aligned} \frac{\partial g_t}{\partial t} &= A_1^{(1)} g_t + A_1^{(2)} g_t \\ &+ \sum_{k \geq 1} \text{div}^{(1)} \text{div}^{(2)} : g_t X_k(z_1) \otimes X_k(z_2). \end{aligned} \quad (4)$$

The divergence operators in these expressions are defined relative to the Liouville volume form and the colon indicates the full contraction of second-rank tensors, $a : b \equiv a^{ij} b_{ij}$. Because these equations follow from Eq. (2) *via* rigorous mathematics, we will refer to them as the mathematical Fokker-Planck equations.

On the other hand, standard coarse-graining procedures [19] lead to the late-time evolution laws for the one- and two-particle distribution functions associated to the microscopic equations of motion, Eq. (1). The physical one-particle Fokker-Planck equation is given by

$$\begin{aligned} \frac{\partial f_t}{\partial t} &= - \left\{ f_t, H_0 + \frac{\epsilon^2}{\tau} \mathbb{E}[s_2] \right\} + \frac{\epsilon^2}{2\tau} \mathbb{E}[\{ \{ f_t, s_1 \}, s_1 \}] \\ &= A_1 f_t, \end{aligned} \quad (5)$$

while the physical two-particle Fokker-Planck equation is given by

$$\frac{\partial g_t}{\partial t} = A_1^{(1)} g_t + A_1^{(2)} g_t + \frac{\epsilon^2}{\tau} \mathbb{E}[\alpha : \mathbf{d}^{(1)} \mathbf{d}^{(2)} g_t]. \quad (6)$$

The notation introduced in these two equations is defined as follows: \mathbb{E} denotes an expectation value; the functions s_1, s_2 are defined by

$$s_1 = \int_0^\tau \exp(\lambda X_{H_0})_* h_{\tau-\lambda} d\lambda \quad (7a)$$

$$s_2 = \frac{1}{2} \int_0^\tau \int_0^a \{ \exp(b X_{H_0})_* h_{\tau-b}, \exp(a X_{H_0})_* h_{\tau-a} \} db da; \quad (7b)$$

$\exp(Y) : M \rightarrow M$ denotes the time-one advance map of the dynamical system defined by the vector field Y ; $(\exp(Y)_* h)(z) \equiv h(\exp(-Y)(z))$; the superscripts indicate which argument of g_t that A_1 and the exterior derivative \mathbf{d} should be applied to; and $\alpha(z_1, z_2) \equiv \mathbb{E}[X_{s_1}(z_1) \otimes X_{s_1}(z_2)]$ is the two-point covariance tensor.

The X_k must be chosen so that the mathematical Fokker-Planck equations, Eqs. (3) and (4), are equivalent to the physical Fokker-Planck equations, Eqs. (5) and (6). However, a direct comparison of these two pairs of equations is difficult with Eqs. (5) and (6) in their current form. To eliminate this issue, we will obtain a special decomposition of the two-point covariance tensor $\alpha(z_1, z_2)$.

As a first step, notice that if we fix a one-form $\xi \in T_{z_1}^* M$, then we can define a vector field Y_ξ on M by contracting ξ with α on the left according to

$$\begin{aligned} Y_\xi(z_2) &= \alpha(z_1, z_2)(\xi, \cdot) \\ &= \mathbb{E}[\xi(X_{s_1}(z_1)) X_{s_1}(z_2)]. \end{aligned} \quad (8)$$

By forming all possible linear combinations of vector fields of this form, we can construct a (potentially infinite dimensional) linear space of vector fields [20, 21], which we will denote \mathcal{H} ,

$$\mathcal{H} = \{ \text{linear combinations of } Y_\xi, \xi \in T^* M \}. \quad (9)$$

Because each Y_ξ is of the form $Y_\xi(z) = X_{\bar{H}}(z)$ with $\bar{H}(z) = \mathbb{E}[\xi(X_{s_1}(z_0)) s_1(z)]$, and the sum of Hamiltonian vector fields is again Hamiltonian, \mathcal{H} consists entirely of Hamiltonian vector fields. Moreover, following Baxendale [17, 21], we see that \mathcal{H} is a real Hilbert space whose inner product is defined by the formula

$$\begin{aligned} \langle Y_\xi, Y_\eta \rangle_{\mathcal{H}} &= \alpha(z_1, z_2)(\xi, \eta) \\ &= \mathbb{E}[\xi(X_{s_1}(z_1)) \eta(X_{s_1}(z_2))], \end{aligned} \quad (10)$$

where $\xi \in T_{z_1}^* M$ and $\eta \in T_{z_2}^* M$. Therefore we may choose an orthonormal basis $\{e_k\}_{k \geq 1}$ for \mathcal{H} , where each e_k must be of the form $e_k = X_{H_k}$. A simple calculation then leads to the desired decomposition of α :

$$\alpha(z_1, z_2) = \sum_{k \geq 1} X_{H_k}(z_1) \otimes X_{H_k}(z_2). \quad (11)$$

Using this decomposition of the two-point covariance tensor, it is straightforward to manipulate Eqs. (5) and (6) into the same form as Eqs. (3) and (4). After doing so, it is trivial to identify the correct X_k . Indeed, we have found that the physical Langevin equation is given by

$$\delta z_t = X_{\tilde{H}_0}(z_t)dt + \sum_{k \geq 1} X_{\tilde{H}_k}(z_t)\delta W_t^k, \quad (12)$$

where

$$\tilde{H}_0 = H_0 + \frac{\epsilon^2}{\tau}\mathbb{E}[s_2], \quad \tilde{H}_k = \frac{\epsilon}{\sqrt{\tau}}H_k \quad (13)$$

Recall that the X_{H_k} are defined to be an orthonormal basis of the Hilbert space \mathcal{H} defined in Eq. (9).

Because the coefficients in the Langevin equation for stochastic acceleration, Eq. (12), are all Hamiltonian vector fields, this equation is an example of a *stochastic Hamiltonian system*, the foundations of which are developed in [12]. It is in this sense that the Langevin equation for stochastic acceleration inherits the Hamiltonian structure of the microscopic equations.

Example 1. — We will find the physical Langevin equation for two example stochastic acceleration problems. First, consider a single-species, unmagnetized plasma subjected to a random weak electrostatic pulse at τ -second intervals. Assume that the pulses are uniform in space and constant in magnitude, but uniformly and independently distributed in direction. Thus, the k 'th pulse is generated by a potential of the form $\phi_k(\mathbf{x}, t) = (\mathbf{z}_k \cdot \mathbf{x})\phi_o u(t - k\tau)$, where \mathbf{z}_k is a random vector uniformly distributed over the unit sphere and $u(t)$ is a temporal windowing function localized at $t = \tau/2$.

In order to find the Langevin equation governing the plasma dynamics at times much longer than τ , we must (a) calculate s_1 and s_2 using Eqs. (7a) and (7b), (b) find an orthonormal basis $\{X_{H_k}\}_{k \geq 1}$ for the space \mathcal{H} defined in Eq. (9), and (c) write down Eq. (12) with \tilde{H}_0 and \tilde{H}_k calculated using Eq. (13). The results of these three steps are as follows.

(a) A quick calculation shows that

$$s_1 = m_o \mathbf{z} \cdot \mathbf{x} - m_1 \mathbf{z} \cdot \mathbf{v} \quad (14a)$$

$$s_2 = \text{const} \quad (14b)$$

where $m_o = (q/m)\phi_o \int_0^\tau u(s)ds$, $m_1 = (q/m)\phi_o \int_0^\tau (\tau - s)u(s)ds$, and q/m is the charge-to-mass ratio.

(b) Each Y_ξ must be of the form $Y_\xi = X_{g_{\beta\gamma}}$, where

$$g_{\beta\gamma}(\mathbf{x}, \mathbf{v}) = \frac{1}{3}(m_1\boldsymbol{\beta} + m_o\boldsymbol{\gamma}) \cdot (m_1\mathbf{v} - m_o\mathbf{x}), \quad (15)$$

and $\boldsymbol{\beta}, \boldsymbol{\gamma}$ are arbitrary constant 3-component vectors. Using this expression, it is simple to find an orthonormal basis for \mathcal{H} . One is given by $\{X_{\tilde{H}_k}\}_{k=1..3}$, with

$$H_i(\mathbf{x}, \mathbf{v}) = \frac{1}{\sqrt{3}}e_i \cdot (m_1\mathbf{v} - m_o\mathbf{x}), \quad (16)$$

where $\{e_i\}_{i=1..3}$ is the standard basis for \mathbb{R}^3 .

(c) Finally, the physical Langevin equation is given by

$$\delta x^i = v^i dt + \frac{1}{\sqrt{3\tau}}m_1\delta W^i \quad (17a)$$

$$\delta v^i = \frac{1}{\sqrt{3\tau}}m_o\delta W^i, \quad (17b)$$

where $i = 1, 2, 3$.

As is readily verified, the one-particle Fokker-Planck equation for this SDE is given by

$$\frac{\partial f_t}{\partial t} + \mathbf{v} \cdot \nabla f_t = \frac{1}{6\tau}(m_1^2\nabla^2 f_t + m_o m_1 \nabla \cdot \nabla_v f_t + m_o m_1 \nabla_v \cdot \nabla f_t + m_o^2 \nabla_v^2 f_t). \quad (18)$$

On the other hand, given an arbitrary function $\phi(\mathbf{x}, \mathbf{v})$, the SDE

$$\delta x^i = v^i dt + \frac{m_1}{\sqrt{3\tau}}(\cos(\phi)\delta W^{1,i} - \sin(\phi)\delta W^{2,i}) \quad (19a)$$

$$\delta v^i = \frac{m_o}{\sqrt{3\tau}}(\cos(\phi)\delta W^{1,i} - \sin(\phi)\delta W^{2,i}), \quad (19b)$$

where the $W^{1,i}, W^{2,j}$ are six independent ordinary Wiener processes, will also generate Eq. (18). However, when ϕ is not constant, the two-particle Fokker-Planck equation generated by Eq. (19) will differ from the two-point Fokker-Planck equation generated by Eq. (17). This can be verified using Eq. (6). The procedure identified here selects $\phi = 0$ as the physical choice.

The inadequacy of Eq. (19) can also be understood intuitively as follows. Chaotic motions of any two particles experiencing the electrostatic pulses are “synchronized” since the pulses are independent of \mathbf{x} and \mathbf{v} . The Langevin equation (19), on the other hand, desynchronizes particle trajectories by involving additional Wiener processes, in spite of giving the correct one-particle Fokker-Planck equation.

Example 2. — Next, consider a minority population of magnetized fast ions moving through a plane lower-hybrid wave that propagates perpendicular to the magnetic field. Assume the wave has a high harmonic number and a wavelength small compared to a typical ion gyroradius. Karney [22] has shown that the dynamics of the perpendicular velocity of these ions are governed by a canonical time-dependent Hamiltonian system with Hamiltonian

$$H_t = I - \epsilon \sin(\sqrt{2I} \sin \theta - \nu t), \quad (20)$$

where I is the normalized magnetic moment, t the time normalized by the gyroperiod, θ the gyrophase, ν the harmonic number, and ϵ the normalized wave amplitude. Moreover, when ϵ exceeds a threshold value, an ion's motion becomes stochastic. This stochastic motion comes as the result of the effective randomization of the wave

phase felt by an ion after a gyroperiod. Thus, above the stochastic threshold, we can model the wave phase as being randomized every gyroperiod by a random variable η . This allows us to apply the formalism developed in this Letter to find the physical Langevin equation describing the stochastic particle trajectories at times much longer than the gyroperiod.

As in the previous example, the first step is to calculate s_1 and s_2 . Set $\tau = 2\pi$ and adopt the rough approximation

$$\sum_{n=-\infty}^{\infty} \frac{J_n}{\nu - n} \exp(in\theta) \approx \frac{J_{n_o}}{\delta} \exp(in_o\theta), \quad (21)$$

where $\nu = n_o + \delta$, $|\delta| < \frac{1}{2}$, and $J_n = J_n(\sqrt{2I})$ denotes the Bessel function of the first kind [23]. This approximation amounts to selecting the most slowly varying term in the sum in Eq. (21). Then, upon directly evaluating the integrals in Eqs. (7a) and (7b), the resulting expressions for s_1 and $\mathbb{E}[s_2]$ are

$$s_1 = 2\pi \text{sinc}(\pi\delta) J_{n_o} \sin(n_o\theta + \eta) \quad (22a)$$

$$\begin{aligned} \mathbb{E}[s_2] &= \frac{\pi}{2} \sum_{m=-\infty}^{\infty} \frac{J_{m+1}^2 - J_{m-1}^2}{m - \nu} \\ &+ \frac{\pi}{2} \text{sinc}(2\pi\delta) \frac{J_{n_o+1}^2 - J_{n_o-1}^2}{\delta}, \end{aligned} \quad (22b)$$

where η is a random variable uniformly distributed over the interval $[0, 2\pi]$ and $\text{sinc}(x) = \sin(x)/x$.

Next, the space \mathcal{H} can be constructed using the above expression for s_1 . In this case, \mathcal{H} is two-dimensional and has a basis $\{X_{H_1}, X_{H_2}\}$, where

$$H_1(I, \theta) = \sqrt{2\pi} \text{sinc}(\pi\delta) J_{n_o}(\sqrt{2I}) \cos(n_o\theta) \quad (23a)$$

$$H_2(I, \theta) = \sqrt{2\pi} \text{sinc}(\pi\delta) J_{n_o}(\sqrt{2I}) \sin(n_o\theta). \quad (23b)$$

Finally, the coefficients for the Langevin equation, Eq. (12), can be derived using Eq. (13). The result is

$$\begin{aligned} \delta I &= \epsilon \sqrt{\pi} \text{sinc}(\pi\delta) n_o J_{n_o}(\sqrt{2I}) \\ &\times (\sin(n_o\theta) \delta W^1 - \cos(n_o\theta) \delta W^2) \end{aligned} \quad (24a)$$

$$\begin{aligned} \delta\theta &= \left(1 + \frac{\epsilon^2}{2\pi} \frac{\partial}{\partial I} \mathbb{E}[s_2] \right) dt \\ &+ \left(\epsilon \sqrt{\frac{\pi}{2I}} \text{sinc}(\pi\delta) J'_{n_o}(\sqrt{2I}) \right. \\ &\left. \times (\cos(n_o\theta) \delta W^1 + \sin(n_o\theta) \delta W^2) \right). \end{aligned} \quad (24b)$$

The diffusion of the magnetic moment I predicted by Eq. (24) has already been studied by Karney [22]. However, Eq. (24) extends and compliments Karney's results by predicting the appropriate diffusion in gyrophase, as well as the correct two-particle statistics.

Concluding remarks. — We have shown how to derive the physical Langevin equation for particle trajectories undergoing stochastic acceleration. This SDE correctly generates the correct one- and two-particle Fokker-Planck equations and inherits the Hamiltonian structure of the microscopic equations of motion. This inheritance is theoretically satisfying because it is a direct consequence of demanding consistency with the physical one- and two-particle Fokker-Planck equations. It also highlights the importance of Hamiltonian Langevin equations.

A Hamiltonian Langevin equation [12] is a Stratonovich SDE of the form given in Eq. (12). If a loop of initial conditions for this SDE evolves under a given realization of the noise, then the action of that loop is constant in time. In addition, these equations arise from a stochastic action principle [12] for which Noether's theorem applies. Thus, by showing the physical Langevin equation is Hamiltonian, we have also identified potentially powerful tools for the analysis of stochastic acceleration. In particular, using the methods of Bou-Rabee [24], the stochastic action principle can be used to develop variational integrators for Eq. (12). Because these integrators are known to possess superior long-term statistical fidelity [25], this approach may prove to be useful in Monte Carlo simulations of stochastic acceleration.

The authors would like to express their appreciation to I. Dodin, J. A. Krommes, J. Parker, and G. W. Hammett. This work was supported by DOE contracts DE-AC02-09CH11466 and DE-FG02-04ER41289.

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