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On the local non-conforming virtual element spaces

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Abstract

The construction of the *local* non-conforming virtual element spaces is discussed and the isomorphism between their functions and the polynomial moments of such functions is established by a formal argument and two different constructive proofs.

Key words: Virtual element method, shape functions, degrees of freedom.

1. Introduction

The local non-conforming Virtual Element (VE) spaces have a formal definition in terms of a Poisson problem with pure Neumann conditions and their functions are uniquely determined by their moments with respect to some suitable set of polynomials [1]. These moments are usually addressed as the *degrees of freedom*. The connection between the definition of the VE space and the degrees of freedom has been established through the *unisolvence property* but it may be not immediately evident. The goal of this note is to present a construction of the shape functions of the virtual element spaces from their formal definition (Section 2), and discuss why the polynomial moments can be chosen as degrees of freedom (Section 3). Conclusions are offered in Section 4.

2. Shape functions

Let P denote a d-dimensional closed subset of $\mathbb{R}^d$ for $d = 2, 3$ with boundary $\partial\mathsf{P}$. We assume that P is a polygon in 2D and or polyhedron in 3D. The boundary $\partial\mathsf{P}$ is formed by a finite number n_{P} of straight segments (edges) or flat polygons (faces), both denoted by e. The unit normal vector to e is denoted by $\mathbf{n}_{\mathsf{e}}$; the generic unit normal vector to the boundary $\partial\mathsf{P}$ is denoted by $\mathbf{n}_{\mathsf{P}}$. For convenience of exposition, we will normally address e as *an edge* as for $d = 2$, but almost everything in this note with few exceptions that will be explicitly indicated also holds for $d = 3$.

According to [1], the non-conforming VE space of order k is defined as follows:

$$V_h^k(\mathsf{P}) = \left\{ v \in H^1(\mathsf{P}) \,|\, \frac{\partial v}{\partial \mathbf{n}_{\mathsf{e}}} \in \mathbb{P}_{k-1}(\mathsf{e}) \,\forall \mathsf{e} \in \partial\mathsf{P},\, \Delta v \in \mathbb{P}_{k-2}(\mathsf{P}) \right\} \quad \forall k \geq 1, \tag{1}$$

with the usual convention that $\mathbb{P}_{-1}(\mathsf{P}) = \{0\}$ (which occurs for $k = 1$).

For the construction of the shape functions on P we find it convenient to consider the decomposition:

$$V_h^k(\mathsf{P}) = \mathsf{span}\{1\} \oplus \widetilde{V}_h^k(\mathsf{P}), \tag{2}$$

where

$$\widetilde{V}_h^k(\mathsf{P}) = V_h^k(\mathsf{P})/\mathbb{R} = \left\{ v \in H^1(\mathsf{P}) \,|\, \frac{\partial v}{\partial \mathbf{n}_{\mathsf{e}}} \in \mathbb{P}_{k-1}(\mathsf{e})\, \forall \mathsf{e} \in \partial\mathsf{P},\, \Delta v \in \mathbb{P}_{k-2}(\mathsf{P}), \overline{v} = 0 \right\}, \tag{3}$$

where

$$\overline{v} = \frac{1}{|\mathsf{P}|} \int_{\mathsf{P}} v \, dV \tag{4}$$

is the elemental average of v over P.

Each function v of $\widetilde{V}_h^k(\mathsf{P})$ is the solution of the pure Neumann problem:

$$\Delta v = q_v \in \mathbb{P}_{k-2}(\mathsf{P}), \tag{5}$$

$$\frac{\partial v}{\partial \mathbf{n}_{\mathsf{e}}} = r_{v,\mathsf{e}} \in \mathbb{P}_{k-1}(\mathsf{e}) \quad \forall \mathsf{e} \in \partial\mathsf{P}, \tag{6}$$

(with the additional condition that $\overline{v} = 0$ on P) for some polynomial q_v defined on P and some set of polynomials $\{r_{v,\mathsf{e}}\}_{\mathsf{e}\in\partial\mathsf{P}}$, each one of which is defined on a given $\mathsf{e} \in \partial\mathsf{P}$. The polynomials q_v and $\{r_{v,\mathsf{e}}\}_{\mathsf{e}\in\partial\mathsf{P}}$ are required to satisfy the (necessary) compatibility condition

$$\int_{\mathsf{P}} q_v \, dV = \sum_{\mathsf{e}\in\partial\mathsf{P}} \int_{\mathsf{e}} r_{v,\mathsf{e}} \, dS \tag{7}$$

in order for (5)-(6) to be solvable. Indeed, by using (5), the divergence theorem and, then, (6) we obtain:

$$\int_{\mathsf{P}} q_v \, dV = \int_{\mathsf{P}} \Delta v \, dV = \int_{\mathsf{P}} \operatorname{div} \nabla v \, dV = \int_{\partial\mathsf{P}} \frac{\partial v}{\partial \mathbf{n}_{\mathsf{P}}} \, dS = \sum_{\mathsf{e}\in\partial\mathsf{P}} \int_{\mathsf{e}} \frac{\partial v}{\partial \mathbf{n}_{\mathsf{e}}} \, dS = \sum_{\mathsf{e}\in\partial\mathsf{P}} \int_{\mathsf{e}} r_{v,\mathsf{e}} \, dS. \tag{8}$$

Therefore, the solution of problem (5)-(6) exists and is unique up to an additive constant for any choice of the polynomials $q_v \in \mathbb{P}_{k-2}(\mathsf{P})$ and $r_{v,\mathsf{e}} \in \mathbb{P}_{k-1}(\mathsf{e})$ for all $\mathsf{e} \in \mathbb{P}_{k-1}(\mathsf{P})$ satisfying (7).

For an "explicit" construction of the space $\widetilde{V}_h^k(\mathsf{P})$ let us consider the following sets of polynomial functions, which are a (possible) basis of $\mathbb{P}_{k-2}(\mathsf{P})$ and $\mathbb{P}_{k-1}(\mathsf{e})$:

(i) $\mathcal{Q}_{k-2}^{\mathsf{P}} = \{q_\alpha\}_{\alpha\geq 1}$, where, for $k \geq 2$,
- each q_α is a polynomial of degree $\leq k-2$ defined on P;
- $\mathbb{P}_{k-2}(\mathsf{P}) = \mathsf{span}\{q_\alpha\}$ and all q_α are linearly independent;
- $q_1 = 1$ and every q_α for $\alpha \geq 2$ is $L^2(\mathsf{P})$ orthogonal to q_1, i.e., its average on P is zero.

For the construction of $\mathcal{Q}_{k-2}^{\mathsf{P}}$ with $k > 2$ we can choose *any* basis set of $\mathbb{P}_{k-2}(\mathsf{P})/\{1\}$. We include the case for $k = 1$ in the definition above by conventionally taking $\mathcal{Q}_{-1}^{\mathsf{P}} = \emptyset$.

(ii) $\mathcal{R}_{k-1}^{\mathsf{e}} = \{r_{\mathsf{e},\alpha}\}_{\mathsf{e}\in\partial\mathsf{P},\alpha\geq 1}$, where, for $k \geq 1$,
- each $r_{\mathsf{e},\alpha}$ is a polynomial of degree $\leq k-1$ defined on e;
- $\mathbb{P}_{k-1}(\mathsf{e}) = \mathsf{span}\{r_{\mathsf{e},\alpha}\}$ and all $r_{\mathsf{e},\alpha}$ are linearly independent;
- $r_{\mathsf{e},1} = 1$ and $r_{\mathsf{e},\alpha}$ for $\alpha \geq 2$ are $L^2(\mathsf{P})$ orthogonal to $r_{\mathsf{e},1}$, i.e., their average on e is zero.

For the construction of $\mathcal{R}_{k-1}^{\mathsf{e}}$ with $k > 1$ we can choose *any* basis set of $\mathbb{P}_{k-1}(\mathsf{e})/\{1\}$.

Example 1

In 2D, let (x, y) be the usual cartesian coordinates for P and $\xi \in [-|\mathsf{e}|/2, |\mathsf{e}|/2]$ a generic local coordinate defined on e. Then,

(i) for $k = 1$, we have $\mathcal{R}_0^{\mathsf{e}} = \{1\}$ and $\mathcal{Q}_{-1}^{\mathsf{P}} = \emptyset$;

(ii) for $k = 2$, we have $\mathcal{R}_0^{\mathsf{e}} = \{1, \xi/2\}$ and $\mathcal{Q}_0^{\mathsf{P}} = \{1\}$;

(iii) for $k = 3$, we have $\mathcal{R}_1^{\mathsf{e}} = \{1, \xi, \xi^2 - |\mathsf{e}|^2/12\}$, and $\mathcal{Q}_1^{\mathsf{P}} = \{1, x - x_{\mathsf{P}}, y - y_{\mathsf{P}}\}$, where $(x_{\mathsf{P}}, y_{\mathsf{P}})$ are the coordinates of the barycenter of P;

(iv) etc...

In 3D, let x, y, and z denote the usual cartesian coordinates for P and ξ and η two cartesian coordinates with respect to an orthogonal reference system that is locally defined on each face e. Then,

(i) *for $k=1$, we have $\mathcal{R}^{\mathsf{e}}_0=\{1\}$;*
(ii) *for $k=2$, we have $\mathcal{R}^{\mathsf{e}}_0=\{1,\xi-\xi_{\mathsf{e}},\eta-\eta_{\mathsf{e}}\}$, where $(\xi_{\mathsf{e}},\eta_{\mathsf{e}})$ are the coordinates of the barycenter of e, and $\mathcal{Q}^{\mathsf{P}}_0=\{1\}$;*
(iii) *for $k=3$, we have $\mathcal{R}^{\mathsf{e}}_1=\{1,\xi-\xi_{\mathsf{e}},\eta-\eta_{\mathsf{e}},\xi^2-\left\langle\xi^2\right\rangle_{\mathsf{e}},\xi\eta-\langle\xi\eta\rangle_{\mathsf{e}},\eta^2-\left\langle\eta^2\right\rangle_{\mathsf{e}}\}$, where $\langle f\rangle_{\mathsf{e}}$ denotes the average of the function f on e, and $\mathcal{Q}^{\mathsf{P}}_1=\{1,x-x_{\mathsf{P}},y-y_{\mathsf{P}},z-z_{\mathsf{P}}\}$, where $(x_{\mathsf{P}},y_{\mathsf{P}},z_{\mathsf{P}})$ are the coordinates of the barycenter of P;*
(iv) *etc...*

Since $\mathcal{Q}^{\mathsf{P}}_{k-2}$ and $\mathcal{R}^{\mathsf{e}}_{k-1}$ are basis in $\mathbb{P}_{k-2}(\mathsf{P})$ and $\mathbb{P}_{k-1}(\mathsf{e})$ respectively, the cardinality of $\mathcal{Q}^{\mathsf{P}}_{k-2}$ and $\mathcal{R}^{\mathsf{e}}_{k-1}$ coincides with the dimensions of the polynomial spaces $\mathbb{P}_{k-2}(\mathsf{P})$ and $\mathbb{P}_{k-1}(\mathsf{e})$.

Lemma 2.1 *The cardinality of $\mathcal{Q}^{\mathsf{P}}_{k-2}$ for $k\geq 2$ is given by*

$$\mathsf{card}(\mathcal{Q}^{\mathsf{P}}_{k-2})=\begin{cases}\frac{k-1}{1}\,\frac{k}{2} & \text{in 2D},\\[4pt] \frac{k-1}{1}\,\frac{k}{2}\,\frac{(k+1)}{3} & \text{in 3D}.\end{cases}\tag{9}$$

Proof 2.1 *The cardinality of $\mathcal{Q}^{\mathsf{P}}_{k-2}$ for $k\geq 2$ coincides with the dimension of $\mathbb{P}_{k-2}(\mathsf{P})$, the space of bi-variate polynomials in 2D and tri-variate polynomials in 3D. Let $m\geq 2$ be an integer number. The dimension of $\mathbb{P}_m(\mathsf{P})$ is given by*

$$\mathsf{dim}\big(\mathbb{P}_m(\mathsf{P})\big)=\begin{cases}\frac{(m+1)}{1}\,\frac{(m+2)}{2} & \text{in 2D},\\[4pt] \frac{(m+1)}{1}\,\frac{(m+2)}{2}\,\frac{(m+3)}{3} & \text{in 3D}.\end{cases}\tag{10}$$

The assertion of the lemma follows by taking $m=k-2$ in the previous formulas.

Lemma 2.2 *The cardinality of $\mathcal{R}^{\mathsf{e}}_{k-1}(\mathsf{e})$ for $k\geq 1$ is given by*

$$\mathsf{card}(\mathcal{R}^{\mathsf{e}}_{k-1})=\begin{cases}\frac{k}{1} & \text{in 2D},\\[4pt] \frac{k}{1}\,\frac{k+1}{2} & \text{in 3D}.\end{cases}\tag{11}$$

Proof 2.2 *The cardinality of $\mathcal{R}^{\mathsf{e}}_{k-1}$ coincides with the dimension of $\mathbb{P}_{k-1}(\mathsf{e})$, the space of uni-variate polynomials in 2D and bi-variate polynomials in 3D (e is $(d-1)$-dimensional if P is d-dimensional). Let $m\geq 1$ be an integer number. The dimension of $\mathbb{P}_m(\mathsf{e})$ is given by*

$$\mathsf{dim}\big(\mathbb{P}_m(\mathsf{e})\big)=\begin{cases}\frac{(m+1)}{1} & \text{in 2D}\\[4pt] \frac{(m+1)}{1}\,\frac{(m+2)}{2} & \text{in 3D}\end{cases}\tag{12}$$

The assertion of the lemma follows by taking $m=k-1$ in the previous formulas.

We define the set of shape functions generating $V^k_h(\mathsf{P})$ by solving directly problem (5)-(6). We have two different kinds of shape functions: those associated with the polynomials in $\mathcal{Q}^{\mathsf{P}}_{k-2}$ when $k\geq 2$ and those associated with the polynomials in $\mathcal{R}^{\mathsf{e}}_{k-1}$ when $k\geq 1$. In both cases, the lowest value of k deserves a special care to satisfy the compatibility condition. Let us start with the latter case. A possible construction of the shape functions associated with $\mathcal{R}^{\mathsf{e}}_{k-1}$ is as follows.

- For $k=1$ and every edge $\mathsf{e}\in\partial\mathsf{P}$ but the last one, the shape functions $\psi_{\mathsf{e},1}$ associated with the polynomials $1=r_{\mathsf{e},1}\in\mathcal{R}^{\mathsf{e}}_0$ are the solutions of the harmonic problem

$$\Delta\psi_{\mathsf{e},1} = 0 \quad \text{in } \mathsf{P} \tag{13}$$

$$\forall \mathsf{e}' \in \partial\mathsf{P} \,:\, \frac{\partial\psi_{\mathsf{e},1}}{\mathbf{n}_{\mathsf{e}'}} = \begin{cases} +\dfrac{1}{|\mathsf{e}|} & \text{if } \mathsf{e}' = \mathsf{e} \\ -\dfrac{1}{|\mathsf{e}^+|} & \text{if } \mathsf{e}' = \mathsf{e}^+ \\ 0 & \text{if } \mathsf{e}' \neq \mathsf{e}, \mathsf{e}^+ \end{cases} \tag{14}$$

where e^+ is the edge consecutive to e (in $3D$ we may take two faces that are consecutive in a local enumeration). The compatibility condition is satisfied because

$$\begin{aligned} 0 &= \int_{\mathsf{P}} \Delta\psi_{\mathsf{e},1}\, dV = \int_{\partial\mathsf{P}} \frac{\partial\psi_{\mathsf{e},1}}{\partial\mathbf{n}_{\mathsf{P}}}\, dS = \sum_{\mathsf{e}'\in\partial\mathsf{P}} \int_{\mathsf{e}'} \frac{\partial\psi_{\mathsf{e},1}}{\partial\mathbf{n}_{\mathsf{e}'}}\, dS = \int_{\mathsf{e}} \frac{\partial\psi_{\mathsf{e},1}}{\partial\mathbf{n}_{\mathsf{e}}}\, dS + \int_{\mathsf{e}^+} \frac{\partial\psi_{\mathsf{e},1}}{\partial\mathbf{n}_{\mathsf{e}^+}}\, dS \\ &= \frac{1}{|\mathsf{e}|}\, |\mathsf{e}| + \Big(-\frac{1}{|\mathsf{e}^+|}\Big)|\mathsf{e}^+| = 0. \end{aligned} \tag{15}$$

Remark 2.1 *We skip the last edge in the previous construction because the function that solves* (13)-(14) *associated with this edge is a linear combination of all the functions associated with the other edges. Indeed, if* $\psi = \sum_{\mathsf{e}\in\partial\mathsf{P}} \psi_{\mathsf{e},1}$*, it is easy to see that* $\Delta\psi = 0$ *and* $\partial\psi/\partial\mathbf{n}_{\mathsf{e}} = 0$*, from which it follows that* $\psi = 0$ *in* $V_h^k(\mathsf{P})/\mathbb{R}$ *(see final Appendix A). For this reason, the dimension of* $V_h^k/\mathbb{R}$ *is reduced by one.*

- For $k \geq 2$, the shape functions $\psi_{\mathsf{e},1}$ are the same as for $k = 1$, while the shape functions $\psi_{\mathsf{e},\alpha}$ for $\alpha = 2, \ldots, \mathsf{card}(\mathcal{R}^{\mathsf{e}}_{k-1})$ are the solutions of the following *harmonic problem*

$$\Delta\psi_{\mathsf{e},\alpha} = 0 \quad \text{in } \mathsf{P}$$

$$\forall \mathsf{e}' \in \partial\mathsf{P} \,:\, \frac{\partial\psi_{\mathsf{e},\alpha}}{\partial\mathbf{n}_{\mathsf{e}'}} = \begin{cases} r_{\mathsf{e},\alpha} & \text{if } \mathsf{e}' = \mathsf{e}, \\ 0 & \text{if } \mathsf{e}' \neq \mathsf{e}. \end{cases}$$

The average of $r_{\mathsf{e},\alpha}$ with $\alpha \geq 2$ on e is zero by construction (see the definition of $\mathcal{R}^{\mathsf{e}}_{k-1}$) and the compatibility condition is satisfied as follows:

$$0 = \int_{\mathsf{P}} \Delta\psi_{\mathsf{e},\alpha} = \int_{\partial\mathsf{P}} \frac{\partial\psi_{\mathsf{e},\alpha}}{\partial\mathbf{n}_{\mathsf{P}}}\, dS = \sum_{\mathsf{e}'\in\partial\mathsf{P}} \int_{\mathsf{e}'} \frac{\partial\psi_{\mathsf{e},\alpha}}{\partial\mathbf{n}_{\mathsf{e}'}}\, dS = \int_{\mathsf{e}} \frac{\partial\psi_{\mathsf{e},\alpha}}{\partial\mathbf{n}_{\mathsf{e}}}\, dS = \int_{\mathsf{e}} r_{\mathsf{e},\alpha}\, dS = 0. \tag{16}$$

A possible construction of the shape functions associated with $\mathcal{Q}^{\mathsf{P}}_{k-2}$ is as follows.

- For $k = 2$, the shape function ψ_1 associated with the polynomial $q_1 \in \mathcal{Q}^{\mathsf{P}}_0$ is the solution of the problem:

$$\Delta\psi_1 = \frac{n_{\mathsf{P}}}{|\mathsf{P}|} q_1 \quad \text{in } \mathsf{P},$$

$$\frac{\partial\psi_1}{\partial\mathbf{n}_{\mathsf{e}}} = \frac{1}{|\mathsf{e}|} r_{\mathsf{e},1} \quad \text{on every } \mathsf{e} \in \partial\mathsf{P}$$

(recall that n_{P} is the number of edges of $\partial\mathsf{P}$). The compatibility condition is satisfied because:

$$n_{\mathsf{P}} = \int_{\mathsf{P}} \frac{n_{\mathsf{P}}}{|\mathsf{P}|} q_1\, dV = \int_{\mathsf{P}} \Delta\psi_1\, dV = \int_{\partial\mathsf{P}} \frac{\partial\psi_1}{\partial\mathbf{n}_{\mathsf{P}}}\, dS = \sum_{\mathsf{e}\in\partial\mathsf{P}} \int_{\mathsf{e}} \frac{\partial\psi_1}{\partial\mathbf{n}_{\mathsf{e}}}\, dS = \sum_{\mathsf{e}\in\partial\mathsf{P}} \int_{\mathsf{e}} \frac{1}{|\mathsf{e}|} r_{\mathsf{e}1}\, dS = \sum_{\mathsf{e}\in\partial\mathsf{P}} 1 = n_{\mathsf{P}} \tag{17}$$

(recall that $q_1 = 1$ on P and $r_{\mathsf{e},1} = 1$ on e).

- For $k \geq 3$, the shape function ψ_1 is the same as for $k = 2$, while the shape functions ψ_α for $\alpha = 2, \ldots, \mathsf{card}(\mathcal{Q}^{\mathsf{P}}_{k-2})$ are the solution of the following problem

$$\Delta\psi_\alpha = q_\alpha \qquad \text{in } \mathsf{P}$$

$$\frac{\partial\psi_\alpha}{\partial\mathbf{n}_{\mathsf{e}}} = 0 \qquad \text{on every } \mathsf{e}\in\partial\mathsf{P}.$$

The average of $q_\alpha \in \mathcal{Q}^{\mathsf{P}}_{k-2}$ with $\alpha > 2$ on P is zero by construction (see the definition of $\mathcal{Q}^{\mathsf{P}}_{k-2}$) and the compatibility condition is satisfied because

$$0 = \int_{\mathsf{P}} q_\alpha\, dV = \int_{\mathsf{P}} \Delta\psi_\alpha\, dV = \int_{\partial\mathsf{P}} \frac{\partial\psi_\alpha}{\partial\mathbf{n}_{\mathsf{P}}}\, dS = \sum_{\mathsf{e}\in\partial\mathsf{P}} \int_{\mathsf{e}} \frac{\partial\psi_\alpha}{\partial\mathbf{n}_{\mathsf{e}}}\, dS = 0. \tag{18}$$

For $k = 1$, the virtual space $V^1_h(\mathsf{P})$ is generated by the set of shape functions $\big\{1, \{\psi_{\mathsf{e},1}\}_{\mathsf{e}\in\widetilde{\partial\mathsf{P}}}\big\}$ where $\widetilde{\partial\mathsf{P}}$ is the boundary of P without the last edge. When $k \geq 2$, the virtual space $V^k_h(\mathsf{P})$ is generated by the linear combinations of the basis functions 1, ψ_α and $\psi_{\mathsf{e},\alpha}$. More precisely,

$$V^k_h(\mathsf{P}) = \mathsf{span}\Big\{ 1, \{\psi_\alpha\}_{\alpha=1,\ldots\mathsf{card}(\mathcal{Q}^{\mathsf{P}}_{k-2})}, \{\psi_{\mathsf{e},1}\}_{\mathsf{e}\in\widetilde{\partial\mathsf{P}}}, \{\psi_{\mathsf{e},\alpha}\}_{\mathsf{e}\in\partial\mathsf{P},\alpha=2,\ldots\mathsf{card}(\mathcal{R}^{\mathsf{e}}_{k-1})} \Big\}.$$

Assuming conventionally that $\mathrm{card}(\mathcal{Q}^{\mathsf{P}}_{-1}) = 0$ when $k = 1$, for $k \geq 1$ we find that

$$\dim\big(V^k_h(\mathsf{P})\big) = 1 + \mathrm{card}(\mathcal{Q}^{\mathsf{P}}_{k-2}) + \sum_{\mathsf{e}\in\partial\mathsf{P}} \mathrm{card}(\mathcal{R}^{\mathsf{e}}_{k-1}) - 1 = \mathrm{card}(\mathcal{Q}^{\mathsf{P}}_{k-2}) + n_{\mathsf{P}}\,\mathrm{card}(\mathcal{R}^{\mathsf{e}}_{k-1}). \tag{19}$$

From Lemmas 2.1 and 2.2, the dimension of the local virtual element space $V^k_h(\mathsf{P})$ in terms of k and n_{P} is given by the formulas:

$$\dim(V^k_h(\mathsf{P})) = \begin{cases} \frac{k-1}{1}\frac{k}{2} + n_{\mathsf{P}}\frac{k}{1} & \text{in 2D} \\ \frac{k-1}{1}\frac{k}{2}\frac{(k+1)}{3} + n_{\mathsf{P}}\frac{k}{1}\frac{k+1}{2} & \text{in 3D} \end{cases} \tag{20}$$

To ease the notation, in the next section we will use the symbols $N_{k-2} = \mathsf{card}(\mathcal{Q}^{\mathsf{P}}_{k-2})$ with the convention that $N_{-1} = 0$, $M_{k-1} = \mathsf{card}(\mathcal{R}^{\mathsf{e}}_{k-1})$, $P_k = N_{k-2} + n_{\mathsf{P}}\, M_{k-1}$.

3. Polynomial moments

The main result of this section is the existence and uniqueness of a function v with an assigned set of moments with respect to the polynomials in $\mathcal{Q}^{\mathsf{P}}_{k-2}$ and $\mathcal{R}^{\mathsf{e}}_{k-1}$. The uniqueness is stated in Proposition 3.1 and Corollary 3.1. The proof of the uniqueness is based on the same argument that is used to prove the unisolvency property in [1]. The existence is stated in Proposition 3.2.

3.1. *Uniqueness*

Let v' and v'' be two functions of $V^k_h(\mathsf{P})$ that have the same moments against the polynomials in $\mathbb{P}_{k-2}(\mathsf{P})$ and $\mathbb{P}_{k-1}(\mathsf{e})$. From the construction of the previous section, we know that there exists a polynomial q' and a set of polynomials $\{r'_{\mathsf{e}}\}_{\mathsf{e}\in\partial\mathsf{P}}$ satisfying the compatibility condition and such that

$$\Delta v' = q' \quad \text{in } \mathsf{P},$$

$$\frac{\partial v'}{\partial\mathbf{n}_{\mathsf{e}}} = r'_{\mathsf{e}} \quad \text{on } \mathsf{e}\in\partial\mathsf{P}.$$

The same characterization is true for v'' by using the polynomials q'' and $\{r''_{\mathsf{e}}\}_{\mathsf{e}\in\partial\mathsf{P}}$. Let $v = v' - v''$. Using the integration by parts we obtain:

$$\int_{\mathsf{P}} |\nabla v'|^2\, dV = \int_{\mathsf{P}} \nabla v\cdot\nabla v\, dV = -\int_{\mathsf{P}} v\Delta v\, dV + \sum_{\mathsf{e}\in\partial\mathsf{P}} \int_{\mathsf{e}} v\frac{\partial v}{\partial\mathbf{n}_{\mathsf{e}}}\, dS.$$

Now, $\Delta v \in \mathbb{P}_{k-2}(\mathsf{P})$, $\partial v/\partial v \in \mathbb{P}_{k-1}(\mathsf{e})$ and the integrals in the last right-hand side are zero as v' and v'' have the same moments on P and on each edge e, and, thus, all the moments of $v = v' - v''$ are zero.

Therefore, $\nabla v = 0$ which implies that $v =$ constant and this constant is zero because it must coincide which any of its moments. From $v' - v'' = 0$ it (obviously) follows that

$$\Delta(v' - v'') = 0 \Rightarrow q' = q''$$

$$\forall \mathsf{e} \in \partial\mathsf{P}: \quad \frac{\partial(v' - v'')}{\partial \mathbf{n}_{\mathsf{e}}} = 0 \Rightarrow r'_{\mathsf{e}} = r''_{\mathsf{e}}.$$

We have, thus, proved the following proposition.

Proposition 3.1 (Uniqueness) *If v' and v'' have the same polynomial moments against $\mathbb{P}_{k-2}(\mathsf{P})$ and $\mathbb{P}_{k-1}(\mathsf{e})$ for each edge e, then $q' = q''$, $r'_{\mathsf{e}} = r''_{\mathsf{e}}$ for each edge e, and, ultimately, $v' = v''$.*

As all the functions of space $V_h^k(\mathsf{P})$ are uniquely characterized by the sets of polynomials $\mathcal{Q}_{k-2}^{\mathsf{P}}$ and $\mathcal{R}_{k-1}^{\mathsf{e}}$, we have the following corollary (whose proof repeats the same argument above and is omitted).

Corollary 3.1 (Uniqueness) *If v' and v'' have the same moments against the polynomials in $\mathcal{Q}_{k-2}^{\mathsf{P}}$ and $\mathcal{R}_{k-1}^{\mathsf{e}}$, then $q' = q''$, $r'_{\mathsf{e}} = r''_{\mathsf{e}}$ for each edge e, and, ultimately, $v' = v''$.*

3.2. *Existence*

Proposition 3.2 (Existence) *Let us consider the real numbers*

- $\{\mu_\alpha\}_{\alpha=1,\mathit{card}(\mathcal{Q}_{k-2}^{\mathsf{P}})}$ *for $k \geq 2$;*
- $\{\mu_{\mathsf{e},\alpha}\}_{\mathsf{e}\in\partial\mathsf{P},\alpha=1,\mathit{card}(\mathcal{R}_{k-1}^{\mathsf{e}})}$ *for $k \geq 1$.*

There exists a function $v \in V_h^k(\mathsf{P})$ such that these numbers are the polynomial moments with respect to the polynomials in $\mathcal{Q}_{k-2}^{\mathsf{P}}$ and $\mathcal{R}_{k-1}^{\mathsf{e}}$:

$$k \geq 2: \quad \frac{1}{|\mathsf{P}|}\int_{\mathsf{P}} v q_\alpha = \mu_\alpha \qquad \textit{for } \alpha = 1, \ldots, \mathit{card}(\mathcal{Q}_{k-2}^{\mathsf{P}}) \tag{21}$$

$$k \geq 1: \quad \frac{1}{|\mathsf{e}|}\int_{\mathsf{e}} v r_{\mathsf{e},\alpha} = \mu_{\mathsf{e},\alpha} \qquad \textit{for every } \mathsf{e} \in \partial\mathsf{P}, \textit{ for } \alpha = 1, \ldots, \mathit{card}(\mathcal{R}_{k-1}^{\mathsf{e}}) \tag{22}$$

A formal argument. An easy counting shows that the total number of moments in (21) and (22) is equal to the dimension of $V_h^k(\mathsf{P})$ in (20).[1] We can thus establish a linear mapping between $V_h^k(\mathsf{P})$ and $\mathbb{R}^{P_k}$ (recall that P_k is the dimension of $V_h^k(\mathsf{P})$). This mapping is *one-to-one* because if all the moments of a function v of $V_h^k(\mathsf{P})$ are zero, then $v = 0$. Indeed, by repeating the uniqueness argument we find $v = 0$. Therefore, this mapping is an isomorphism and each function in $V_h^k(\mathsf{P})$ is uniquely identified by its moments with respect to the polynomial sets $\mathcal{Q}_{k-2}^{\mathsf{P}}$ and $\mathcal{R}_{k-1}^{\mathsf{e}}$. This formal argument is used to prove the unisolvency in [1].

A quasi-constructive proof. We consider a function v of $V_h^k(\mathsf{P})$ and the polynomials q_v and $\{r_{v,\mathsf{e}}\}_{\mathsf{e}\in\partial\mathsf{e}}$ that are related to v by (5)-(6). We decompose the polynomial $q_v \in \mathbb{P}_{k-2}(\mathsf{P})$ into the basis set $\mathcal{Q}_{k-2}^{\mathsf{P}}$:

$$q_v = \sum_{\alpha=1}^{N_{k-2}} a_\alpha(v) q_\alpha, \tag{23}$$

and each polynomial r_{e}^v for $\mathsf{e} \in \partial\mathsf{P}$ into the corresponding basis set $\mathcal{R}_{k-1}^{\mathsf{e}}$:

$$r_{v,\mathsf{e}} = \sum_{\alpha=1}^{M_{k-1}} b_{E,\alpha}(v) r_{\mathsf{e},\alpha}. \tag{24}$$

The coefficients $a_\alpha(v)$ and $b_{\mathsf{e},\alpha}(v)$ are bounded linear functionals of v, an they must satisfy

$$a_1 |\mathsf{P}| = \sum_{\mathsf{e}\in\partial\mathsf{P}} b_{\mathsf{e},1} |\mathsf{e}|, \tag{25}$$

[1] Note that the indices "α" and "e,α" are running throughout the same range.

which comes from imposing the compatibility condition (7). Consider the bounded linear functional on $V_h^k(\mathsf{P})$

$$\mathsf{L}(v) := \overline{v}\mu_1 - |\mathsf{P}| \sum_{\alpha=2}^{N_{k-2}} a_\alpha(v)\mu_\alpha + \sum_{\mathsf{e}\in\partial\mathsf{P}} |\mathsf{e}| \sum_{\alpha=1}^{M_{k-1}} b_{\mathsf{e},\alpha}(v)\mu_{\mathsf{e},\alpha}, \tag{26}$$

and equip $V_h^k(\mathsf{P})$ with the inner product:

$$(u,v)_{V_h^k(\mathsf{P})} := \overline{u}\,\overline{v} + \int_{\mathsf{P}} \nabla u \cdot \nabla v \, dV \tag{27}$$

(recall that $\overline{v}$ and $\overline{u}$ are the elemental averages of u and v, respectively; see also (4)). The Ritz Theorem implies the existence and uniqueness of a function $\widetilde{u}$ in $V_h^k(\mathsf{P})$ such that

$$\mathsf{L}(v) = (\widetilde{u}, v)_{V_h^k(\mathsf{P})} \quad \text{for every } v \in V_h^k(\mathsf{P}). \tag{28}$$

We will show that $\widetilde{u}$ satisfies (21)-(22) by using equation (28) and selecting some particular functions v of $V_h^k(\mathsf{P})$ in (26) and (27),

- Using $v = 1$, we immediately have that the average of $\widetilde{u}$ over P is μ_1. Therefore, $\widetilde{u}$ satisfies (21) with $\alpha = 1$.

– Let v be such that $a_\alpha = 1$ for some given $\alpha > 1$ and $a_{\alpha'} = 0$ for $\alpha' \neq \alpha$ in (23) and all the coefficients $b_{\mathsf{e},\alpha'}$ in (24) are zero. Equation (26) gives

$$\mathsf{L}(v) = \mu_1\overline{v} - \mu_\alpha |\mathsf{P}|. \tag{29}$$

After an integration by parts, and using $\overline{\widetilde{u}} = \mu_1$ equation (27) returns

$$(\widetilde{u}, v)_{V_h^k(\mathsf{P})} = \mu_1\overline{v} - \int_{\mathsf{P}} \widetilde{u}\Delta v \, dV + \sum_{\mathsf{e}\in\partial\mathsf{P}} \int_{\mathsf{e}} \widetilde{u}\frac{\partial v}{\partial \mathbf{n}_{\mathsf{e}}} \, dS = \mu_1\overline{v} - \int_{\mathsf{P}} \widetilde{u} q_\alpha \, dV. \tag{30}$$

Comparing (29) and (30) shows that $\widetilde{u}$ satisfies (21) with $\alpha > 1$.

- Let v be such that $b_{\mathsf{e},1} = 1$ for a given edge e and $b_{\mathsf{e}',\alpha} = 0$ for $\mathsf{e}' \neq \mathsf{e}$ and $\alpha > 1$. We take all the coefficients $a_\alpha = 0$ for $\alpha > 1$, while a_1 is given by relation (25) as $a_1 = b_{\mathsf{e},1}|\mathsf{e}|/|\mathsf{P}|$. Equation (26) gives

$$\mathsf{L}(v) = \mu_1\overline{v} - \mu_1|\mathsf{P}| + \mu_{\mathsf{e},1}|\mathsf{e}|. \tag{31}$$

After an integration by parts, and using $\overline{\widetilde{u}} = \mu_1$ equation (27) gives

$$(\widetilde{u}, v)_{V_h^k(\mathsf{P})} = \mu_1\overline{v} - \int_{\mathsf{P}} \widetilde{u}\Delta v \, dV + \sum_{\mathsf{e}\in\partial\mathsf{P}} \int_{\mathsf{e}} \widetilde{u}\frac{\partial v}{\partial \mathbf{n}_{\mathsf{e}}} \, dS = \mu_1\overline{v} - \mu_1|\mathsf{P}| + \mu_{\mathsf{e},1}|\mathsf{e}|. \tag{32}$$

Comparing (31) and (32) shows that $\widetilde{u}$ satisfies (22) with $\alpha = 1$.

- Let v be such that $b_{\mathsf{e},\alpha} = 1$ for some edge e and $\alpha > 1$, $b_{\mathsf{e}',\alpha'} = 0$ for $\mathsf{e}' \neq \mathsf{e}$ or $\alpha' \neq \alpha$ in (24), and all the coefficients $a_{\alpha'} = 0$ in (23). Then, Equation (26) returns

$$\mathsf{L}(v) = \mu_1\overline{v} - \mu_{\mathsf{e},\alpha}|\mathsf{e}|. \tag{33}$$

After an integration by parts, and using $\overline{\widetilde{u}} = \mu_1$ equation (27) returns

$$(\widetilde{u}, v)_{V_h^k(\mathsf{P})} = \mu_1\overline{v} - \int_{\mathsf{P}} \widetilde{u}\Delta v \, dV + \sum_{\mathsf{e}\in\partial\mathsf{P}} \int_{\mathsf{e}} \widetilde{u}\frac{\partial v}{\partial \mathbf{n}_{\mathsf{e}}} \, dS = \mu_1\overline{v} - \int_{\mathsf{e}} \widetilde{u} r_{\mathsf{e},\alpha} \, dS. \tag{34}$$

Comparing (33) and (34) shows that $\widetilde{u}$ satisfies (22) with $\alpha > 1$.

A constructive proof. We consider the expansion of the function v in $\widetilde{V}_h^k(\mathsf{P})$ on the shape functions ψ_β and $\psi_{\mathsf{e},\beta}$ defined in the previous section:

$$v = \sum_{\beta=1}^{N_{k-2}} c_\beta\psi_\beta + \sum_{\mathsf{e}\in\partial\mathsf{P}} \sum_{\beta=1}^{M_{k-1}} c_{\mathsf{e},\beta}\psi_{\mathsf{e},\beta}, \tag{35}$$

so that to determine v we have to determine the coefficients c_α and $c_{\mathsf{e},\alpha}$. From this expansion it follows that:

$$\int_{\mathsf{P}} \nabla v \cdot \nabla\psi_\alpha \, dV = \sum_{\beta=1}^{N_{k-2}} c_\beta \int_{\mathsf{P}} \nabla\psi_\beta \cdot \nabla\psi_\alpha \, dV + \sum_{\mathsf{e}\in\partial\mathsf{P}} \sum_{\beta=1}^{M_{k-1}} c_{\mathsf{e},\beta} \int_{\mathsf{P}} \nabla\psi_{\mathsf{e},\beta} \cdot \nabla\psi_\alpha \, dV \qquad \alpha = 1, \ldots, N_{k-2}, \tag{36}$$

$$\int_{\mathsf{P}} \nabla v \cdot \nabla\psi_{\mathsf{e},\alpha} \, dV = \sum_{\beta=1}^{N_{k-2}} c_\beta \int_{\mathsf{P}} \nabla\psi_\beta \cdot \nabla\psi_{\mathsf{e},\alpha} \, dV + \sum_{\mathsf{e}\in\partial\mathsf{P}} \sum_{\beta=1}^{M_{k-1}} c_{\mathsf{e},\beta} \int_{\mathsf{P}} \nabla\psi_{\mathsf{e},\beta} \cdot \nabla\psi_{\mathsf{e},\alpha} \, dV \qquad \forall \mathsf{e} \in \partial\mathsf{P},\ \alpha = 1, \ldots, M_{k-1}. \tag{37}$$

We consider the matrix

$$\mathsf{M} = \left(\int_{\mathsf{P}} \nabla\phi_\beta \cdot \nabla\phi_\alpha \, dV \right) = \begin{pmatrix} \left[\int_{\mathsf{P}} \nabla\psi_\beta \cdot \nabla\psi_\alpha \, dV \right] & \left[\int_{\mathsf{P}} \nabla\psi_{\mathsf{e},\beta} \cdot \nabla\psi_\alpha \, dV \right] \\ \left[\int_{\mathsf{P}} \nabla\psi_\beta \cdot \nabla\psi_{\mathsf{e},\alpha} \, dV \right] & \left[\int_{\mathsf{P}} \nabla\psi_{\mathsf{e},\beta} \cdot \nabla\psi_{\mathsf{e},\alpha} \, dV \right] \end{pmatrix}, \tag{38}$$

where each term $\big[\ldots\big]$ represents a block sub-matrix, and the vector $\mathbf{c}^T = \big((c_\alpha)^T, (c_{\mathsf{e},\alpha})^T\big)$, which follows the same row partitioning. The right-hand side of (36) and (37) are the vector components $(\mathsf{M}\mathbf{c})_\alpha$ and $(\mathsf{M}\mathbf{c})_{\mathsf{e},\alpha}$. We consider the vector

$$\mathbf{b}^T = \left(\left(\int_{\mathsf{P}} \nabla v \cdot \nabla\psi_\alpha \, dV \right)^T, \left(\int_{\mathsf{P}} \nabla v \cdot \nabla\psi_{\mathsf{e},\alpha} \, dV \right)^T \right). \tag{39}$$

The components of this vector are computable by integration by parts and imposing that the moments of v are given by (21) and (22). For $\alpha = 1, \ldots, N_{k-2}$ we have

$$\int_{\mathsf{P}} \nabla v \cdot \nabla\psi_\alpha \, dV = -\int_{\mathsf{P}} v\Delta\psi_\alpha \, dV + \sum_{\mathsf{e}\in\partial\mathsf{P}} \int_{\mathsf{e}} v \frac{\partial\psi_\alpha}{\partial\mathbf{n}_{\mathsf{e}}} \, dS = -\int_{\mathsf{P}} v q_\alpha \, dV = -\left|\mathsf{P}\right| \mu_\alpha. \tag{40}$$

For $\mathsf{e} \in \partial\mathsf{P}$ and $\alpha = 1$

$$\int_{\mathsf{P}} \nabla v \cdot \nabla\psi_{\mathsf{e}1} = -\int_{\mathsf{P}} v\Delta\psi_{\mathsf{e}1} \, dV + \sum_{\mathsf{e}'\in\partial\mathsf{P}} \int_{\mathsf{e}'} v \frac{\partial\psi_{\mathsf{e}1}}{\partial\mathbf{n}_{\mathsf{e}}} \, dS = -\frac{|\mathsf{e}|}{|\mathsf{P}|} \int_{\mathsf{P}} v + \int_{\mathsf{e}} v \, dS = |\mathsf{e}| \left(-\mu_0 + \mu_{\mathsf{e}1}\right). \tag{41}$$

For $\mathsf{e} \in \partial\mathsf{P}$ and $\alpha = 2, \ldots, M_{k-1}$ we have

$$\int_{\mathsf{P}} \nabla v \cdot \nabla\psi_{\mathsf{e},\alpha} \, dV = -\int_{\mathsf{P}} v\Delta\psi_{\mathsf{e},\alpha} \, dV + \sum_{\mathsf{e}'\in\partial\mathsf{P}} \int_{\mathsf{e}'} v \frac{\partial\psi_\alpha}{\partial\mathbf{n}_{\mathsf{e}'}} \, dS = \int_{\mathsf{e}} v r_{\mathsf{e},\alpha} \, dS = |\mathsf{e}| \, \mu_{\mathsf{e},\alpha}. \tag{42}$$

The linear system of equations (36) and (37) can be rewritten as $\mathsf{M}\mathbf{c} = \mathbf{b}$, whose solution gives the coefficients $\mathbf{c}$ of expansion (35).

Remark 3.1 *Note that* M *is non-singular because* $\widetilde{V}_h^k(\mathsf{P})$ *does not contain the constants.*

4. Conclusion

We discussed a possible construction of the shape functions for the local virtual element spaces of any order k and their relation with moments with respect to polynomials up to the same order. These moments are unisolvent in the virtual element space and are considered as degrees of freedom in practical implementations. The shape functions are the solution of a set of Poisson problems with pure Neumann boundary conditions, where the right-hand side and the boundary data are taken in given basis for polynomials of degree (up to) $k-2$ on the element and $k-1$ on each edge of the element's boundary. Special care is deserved to ensure that the compatibility condition (necessary to solve a Poisson problem with only Neumann conditions) be satisfied.

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References

[1] B. Ayuso de Dios, K. Lipnikov, and G. Manzini. The non-conforming virtual element method, 2014. Submitted.

Appendix A. Proof of the linear independence of $\{\psi_{\mathsf{e},1}\}_{\mathsf{e}\in\widetilde{\partial\mathsf{P}}}$

We represent the constant data associated with the n_{P} edges of $\partial\mathsf{P}$ in (14) through the vectors w_i in $\mathbb{R}^{n_{\mathsf{P}}}$. We also conveniently number these edges by introducing the index "i", which runs from 1 to $n_{\mathsf{P}}-1$; e.g., e_i for $i=1,\dots,n_{\mathsf{P}}-1$ is the i-th edge of $\widetilde{\partial\mathsf{P}}$ (which we recall that denotes the boundary of P without the last edge). A one-to-one and onto correspondence is just established between these vectors and the shape functions $\psi_{\mathsf{e}_i,1}$:

$$w_1 = \left(\frac{1}{|\mathsf{e}_1|},\ -\frac{1}{|\mathsf{e}_2|},\ 0,\ 0,\ 0,\ \dots,\ 0\right)^T \longleftrightarrow \psi_{\mathsf{e}_1,1}, \tag{A.1}$$

$$w_2 = \left(0,\ \frac{1}{|\mathsf{e}_2|},\ -\frac{1}{|\mathsf{e}_3|},\ 0,\ 0,\ \dots,\ 0\right)^T \longleftrightarrow \psi_{\mathsf{e}_2,1}, \tag{A.2}$$

until

$$w_{n_{\mathsf{P}}-1} = \left(0,\ 0,\ 0,\ \dots,\ \frac{1}{|\mathsf{e}_{n_{\mathsf{P}}-1}|},\ -\frac{1}{|\mathsf{e}_{n_{\mathsf{P}}}|}\ \right)^T \longleftrightarrow \psi_{\mathsf{e}_{n_{\mathsf{P}}-1},1} \tag{A.3}$$

Since the vectors $w_1, w_2, w_3, \dots, w_{n_{\mathsf{P}}-1}$ are linearly independent, so are the corresponding functions $\psi_{\mathsf{e}_1,1}$, $\psi_{\mathsf{e}_2,1}$, $\psi_{\mathsf{e}_3,1}, \dots, \psi_{\mathsf{e}_{n_{\mathsf{P}}-1},1}$. However, vector

$$w_{n_{\mathsf{P}}} = \left(-\frac{1}{|\mathsf{e}_1|},\ 0,\ 0,\ 0,\ \dots,\ +\frac{1}{|\mathsf{e}_{n_{\mathsf{P}}}|}\right)^T, \tag{A.4}$$

which would correspond to $\psi_{\mathsf{e}_{n_{\mathsf{P}}},1}$, is such that

$$\sum_{i=1}^{n_{\mathsf{P}}} w_i = 0, \tag{A.5}$$

so the last function $\psi_{\mathsf{e}_{n_{\mathsf{P}}},1}$ cannot be linearly independent of the other $\psi_{\mathsf{e}_i,1}$ for $i=1,\dots,n_{\mathsf{P}}-1$. As noted in Remark 2.1, function $\psi=\sum_{i=1}^{n_{\mathsf{P}}}\psi_{\mathsf{e}_i,1}$ corresponding to $\sum_{i=1}^{n_{\mathsf{P}}} w_i$ is such that $\Delta\psi=0$ and $\partial\psi/\partial\mathbf{n}_{\mathsf{e}}=0$ for every $\mathsf{e}\in\partial\mathsf{P}$, from which it follows that $\psi=0$ in $V_h^k(\mathsf{P})/\mathbb{R}$.

Note that all vector w_i are such that the sum of their elements is zero. The completion of the vectors $\{w_1, w_2, \dots, w_{n_{\mathsf{P}}-1}\}$ to a basis of $\mathbb{R}^{n_{\mathsf{P}}}$ requires an n_{P}-th vector the sum of whose elements is not zero, as for example: [2]

$$w_{n_{\mathsf{P}}} = \left(\frac{1}{|\mathsf{e}_1|}, \frac{1}{|\mathsf{e}_2|}, \frac{1}{|\mathsf{e}_3|}, \dots \frac{1}{|\mathsf{e}_{n_{\mathsf{P}}-1}|}, \frac{1}{|\mathsf{e}_{n_{\mathsf{P}}}|},\right) \tag{A.6}$$

but such a vector cannot be representative of the normal derivative of a function of $V_h^k(\mathsf{P})$ on $\partial\mathsf{P}$ because it violates the compatibility condition.

[2] Otherwise all vectors in $\mathbb{R}^{n_{\mathsf{P}}}$ would be such that the sum of their elements is zero and this fact is obviously false.