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Title: Introduction to Control Part 1.

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Introduction to Control Theory

Part 1. Introduction to Control

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What is control theory?

A study of differential equations, which have user-defined inputs.

Example: Mass - Spring - Damper

$$m\ddot{x}(t) = -kx(t) - b\dot{x}(t) + f(t),$$

where $f(t)$ is a force that we apply.

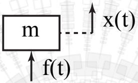
We want to choose inputs in a way that transform a given differential equation into a different equation, one that has certain properties that we want.

A systematic approach to designing state-based inputs (feedback) $f(x, \dot{x}, t)$, which guarantee stable behavior of systems, robust to uncertainties and external disturbances.

Simple Problem: Given a mass (no friction), m, we want to move it to a position 1 meter away, in 1 second, by applying a force:

$$m\ddot{x}(t) = f(t).$$

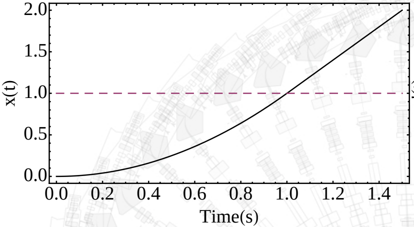
Choose $f(t)$ such that the system's trajectory, $x(t)$, satisfies $x(0) = 0$, $x(1) = 1$, and $\dot{x}(0) = 0$.

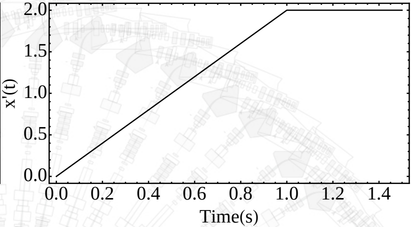


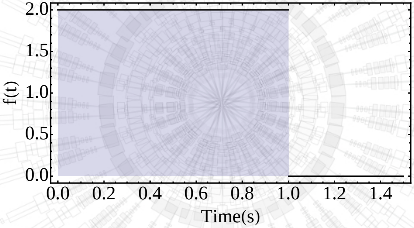
Solution: If $x(t) = t^2$, then $x(0) = 0$, $x(1) = 1$.

- $x(t) = t^2$
- $\dot{x} = 2t$
- $\ddot{x}(t) = 2$
- $m\ddot{x}(t) = 2m$

So we can try $f(t) = 2m$


2.0
1.5
1.0
0.5
0.0
x(t)
0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4
Time(s)


2.0
1.5
1.0
0.5
0.0
x'(t)
0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4
Time(s)

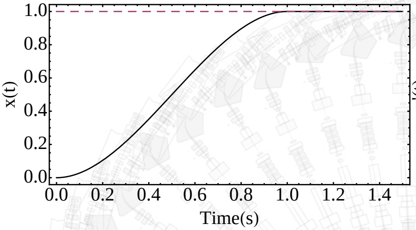

2.0
1.5
1.0
0.5
0.0
f(t)
0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4
Time(s)

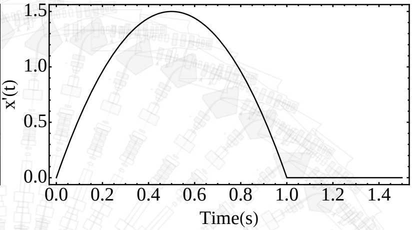
Choose $f(t)$ such that the system's trajectory, $x(t)$, satisfies $x(0) = 0$, $x(1) = 1$, $\dot{x}(0) = 0$, and $\dot{x}(1) = 0$.

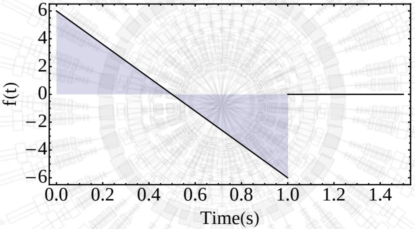
Solution: Let's try, $f(t) = a_1 - a_2 t$.

- $m\ddot{x}(t) = a_1 - a_2 t$
- $\dot{x}(t) = \frac{a_1}{m}t - \frac{a_2 t^2}{2m}$
- $x(t) = \frac{a_1 t^2}{2m} - \frac{a_2 t^3}{6m}$
- $\dot{x}(1) = 0 = \frac{a_1}{m} - \frac{a_2}{2m} \implies a_2 = 2a_1$
- $x(1) = 1 = \frac{3a_1}{2m} - \frac{a_2}{6m} \implies 1 = \frac{3a_1}{2m} - \frac{2a_1}{6m} \implies a_1 = 6m,\ a_2 = 12m$

So we can try $f(t) = 6m - 12mt$


x(t)
Time(s)
0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4
0.0 0.2 0.4 0.6 0.8 1.0


x'(t)
Time(s)
0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4
0.0 0.5 1.0 1.5

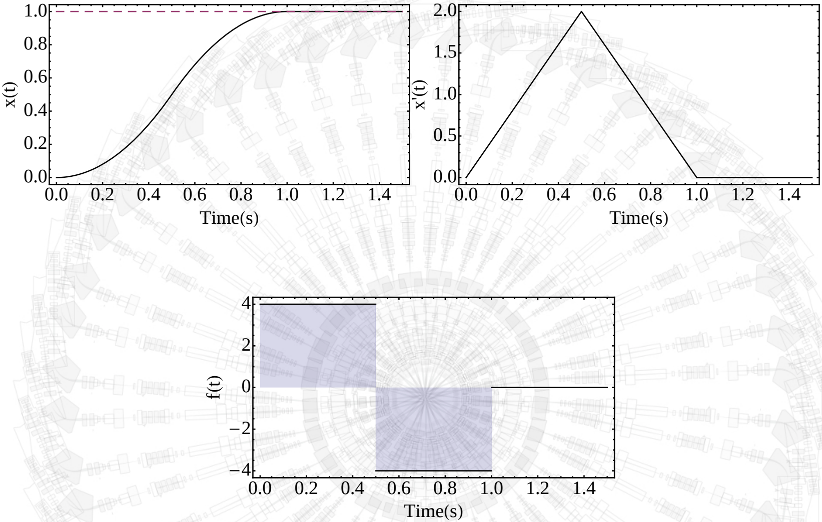

f(t)
Time(s)
0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4
−6 −4 −2 0 2 4 6

Solution: Let's try,

$$f(t) = \begin{cases} k & 0 < t < 0.5 \\ -k & 0.5 < t < 1 \\ 0 & 1 < t \end{cases} \tag{1}$$

Turns out we need to choose $k = 4m$, can use the force:

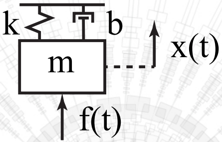
$$f(t) = \begin{cases} 4m & 0 < t < 0.5 \\ -4m & 0.5 < t < 1 \\ 0 & 1 < t \end{cases} \tag{2}$$


1.0
0.8
0.6
0.4
0.2
0.0
x(t)
0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4
Time(s)
2.0
1.5
1.0
0.5
0.0
x'(t)
0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4
Time(s)
4
2
0
−2
−4
f(t)
0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4
Time(s)

Problem: Given

$$m\ddot{x}(t) = -kx(t) - b\dot{x}(t) + f(t),$$

choose $f(t)$ such that the system's trajectory, $x(t)$, arrives at $x(1) = 1$.



Solution: Again, consider $f(t) = a_1 - a_2 t$, so the differential equation is now

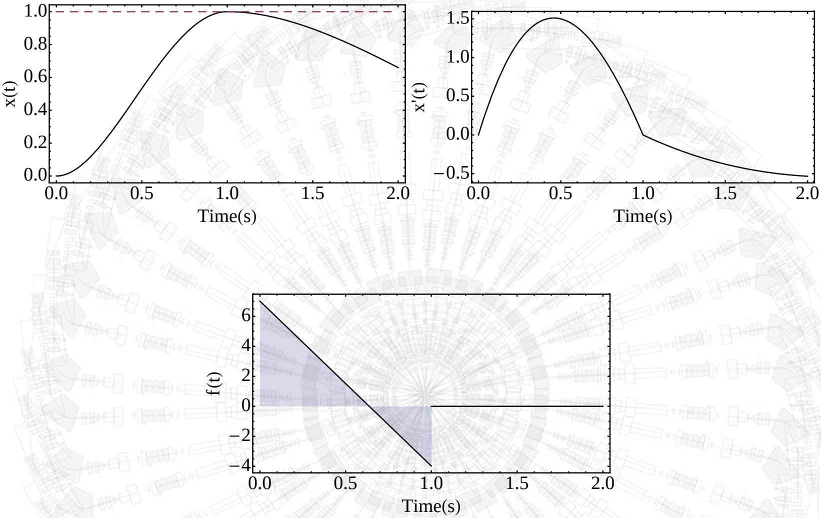
$$m\ddot{x}(t) = -kx(t) - b\dot{x}(t) + a_1 - a_2 t,$$

and we have the constraints $x(0) = 0$, $x(1) = 1$, $\dot{x}(0) = 0$, $\dot{x}(1) = 0$.
After some horrible algebra, we get:

These require exact knowledge of k, b, and m.

$$a_1 = -\frac{e^{-\frac{\sqrt{b^2-4km}}{2m}} k \left(b \left(-1 + e^{\frac{\sqrt{b^2-4km}}{m}} \right) + \left(1 + e^{\frac{\sqrt{b^2-4km}}{m}} - 2e^{\frac{b+\sqrt{b^2-4km}}{2m}} \right) \sqrt{b^2-4km} \right)}{4 \left(\sqrt{b^2-4km} \left(\mathsf{Cosh}\left[\frac{b}{2m}\right] - \mathsf{Cosh}\left[\frac{\sqrt{b^2-4km}}{2m}\right] \right) - k \mathsf{Sinh}\left[\frac{\sqrt{b^2-4km}}{2m}\right] \right)}$$

$$a_2 = -\frac{k^2}{k + \sqrt{b^2-4km} \left(-\mathsf{Cosh}\left[\frac{b}{2m}\right] + \mathsf{Cosh}\left[\frac{\sqrt{b^2-4km}}{2m}\right] \right) \mathsf{Csch}\left[\frac{\sqrt{b^2-4km}}{2m}\right]}$$


1.0
0.8
0.6
0.4
0.2
0.0
x(t)
0.0
0.5
1.0
1.5
2.0
Time(s)
1.5
1.0
0.5
0.0
−0.5
x'(t)
Time(s)
6
4
2
0
−2
−4
f(t)
Time(s)

Oops, we forgot the spring. When $x(1) = 1$ and $\dot{x}(1) = 0$, for

$$m\ddot{x}(t) = -kx(t) - b\dot{x}(t) + f(t),$$

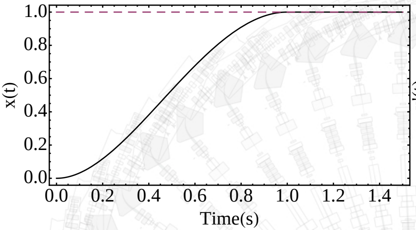
we still have:

$$m\ddot{x}(t) = -k + f(t).$$

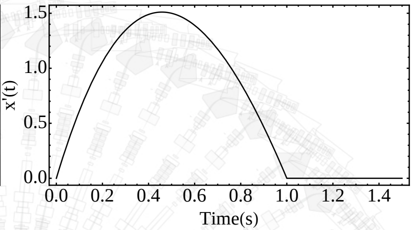
Need to satisfy $\ddot{x}(t \geq 1) = 0 \implies f(t \geq 1) = k$

Solution: So, consider

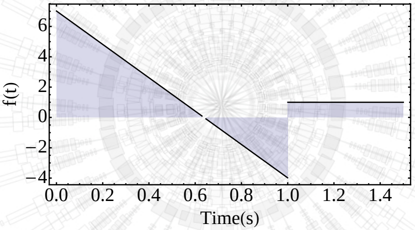
$$f(t) = \begin{cases} a_1 - a_2 t & 0 < t < 1 \\ k & 1 < t \end{cases}$$



x(t)
1.0
0.8
0.6
0.4
0.2
0.0
0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4
Time(s)

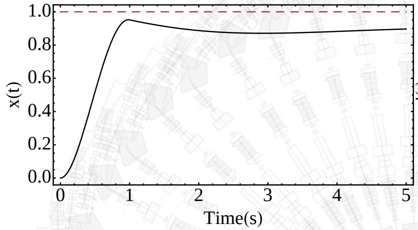


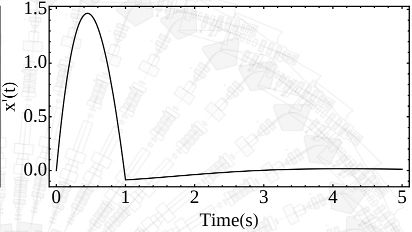
x'(t)
1.5
1.0
0.5
0.0
0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4
Time(s)

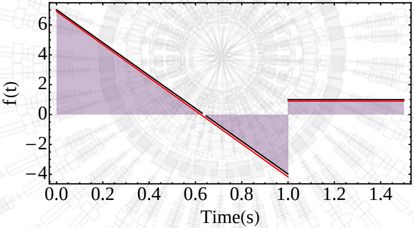


f(t)
6
4
2
0
−2
−4
0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4
Time(s)

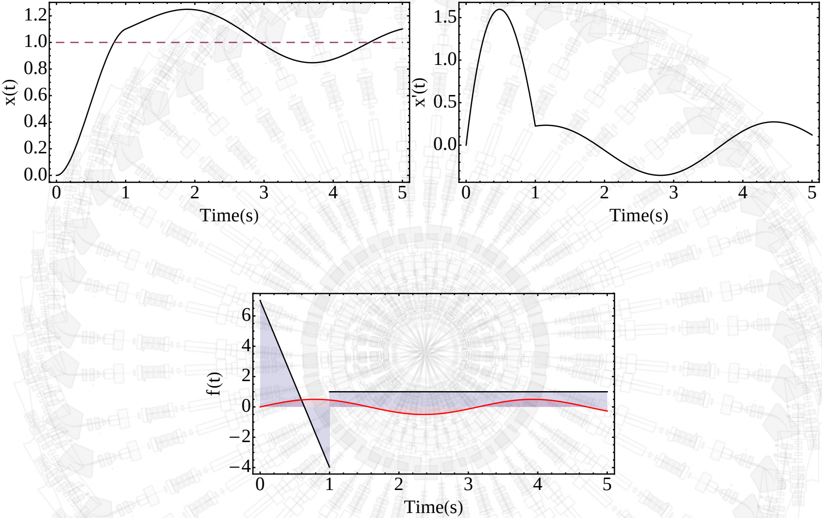
What happens when k, b, and m values in $a_1(k, b, m)$, $a_2(k, b, m)$ are slightly wrong:



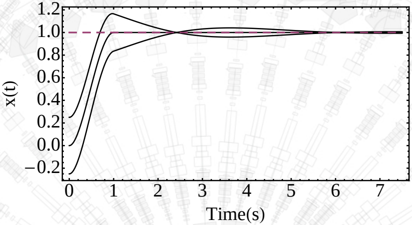


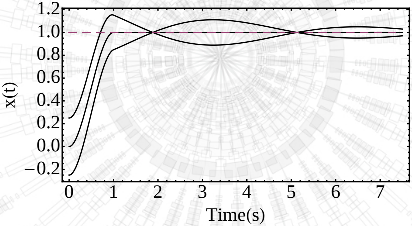


What happens when k, b, and m values are correct, but there is an external disturbance:



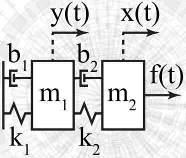
What happens when k, b, and m values are correct, but initial conditions change:

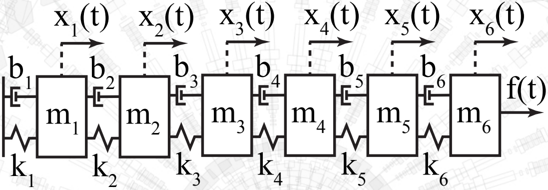




All of the above problems are compounded with larger, more complicated systems

$$\begin{aligned} m_1\ddot{x} &= -k_2(x-y) - b_2\left(\dot{x}-\dot{y}\right) + f(t), \\ m_2\ddot{y} &= -k_1(y) + k_2(x-y) + b_2\left(\dot{x}-\dot{y}\right) - b_1\dot{y}. \end{aligned}$$




x1(t)
x2(t)
x3(t)
x4(t)
x5(t)
x6(t)
b1
b2
b3
b4
b5
b6
m1
m2
m3
m4
m5
m6
f(t)
k1
k2
k3
k4
k5
k6

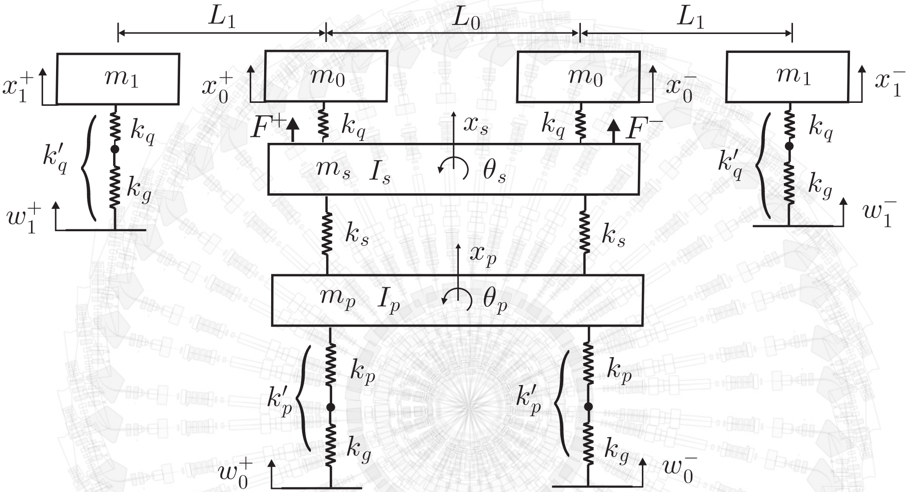


FIG. 2. Lumped mass model of the ILC final focus with the SiD configuration.

The main point of control theory, is in assuming that our model is slightly wrong, that our measurements of system parameters, such as m, k, and b are slightly wrong, that our actuator dynamics are modeled slightly incorrectly, that we are unaware of external disturbances, and all of our measurements are noisy.

We assume that we are wrong and so we actively monitor our error and provide feedback based on the error, to make our system as robust as possible.

Consider the simple problem

$$\dot{x}(t) = f(t).$$

We will use feedback, so we replace $f(t)$ with $u(x,t)$ and, for notational simplicity, sometimes drop the arguments, writing

$$\dot{x} = u. \tag{3}$$

Also, given a desired set point, x_s, for $x(t)$, we rewrite the dynamics, (3), in terms of the error

$$e = x - x_s, \quad \dot{e} = \dot{x} - \dot{x}_s = \dot{x},$$

we get:

$$\dot{e} = u.$$

Given

$$\dot{e} = u,$$

to force $e(t)$ towards zero, one of the simplest things that we can do is use proportional feedback

$$u = -ke,$$

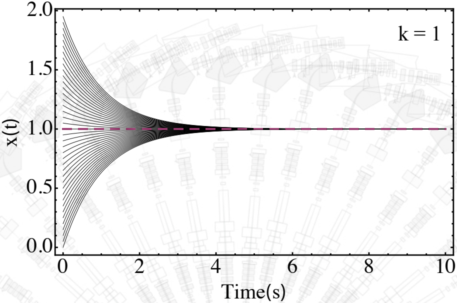
so that the "closed loop" system system is

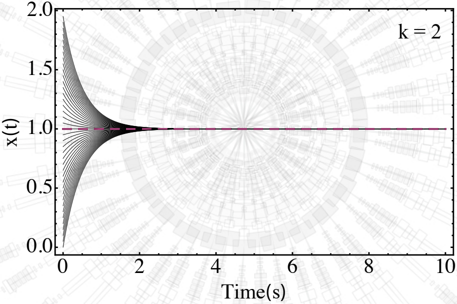
$$\dot{e} = -ke,$$

which has solution

$$e(t) = e(0)e^{-kt}, \qquad (x(t) - x_s) = (x(0) - x_s)\, e^{-kt}.$$

Notice this is independent of initial condition.


k = 1
x(t)
Time(s)
2.0
1.5
1.0
0.5
0.0
0
2
4
6
8
10


k = 2
x(t)
Time(s)
2.0
1.5
1.0
0.5
0.0
0
2
4
6
8
10

If there are un-modeled disturbances,

$$\dot{e} = -ke + n$$

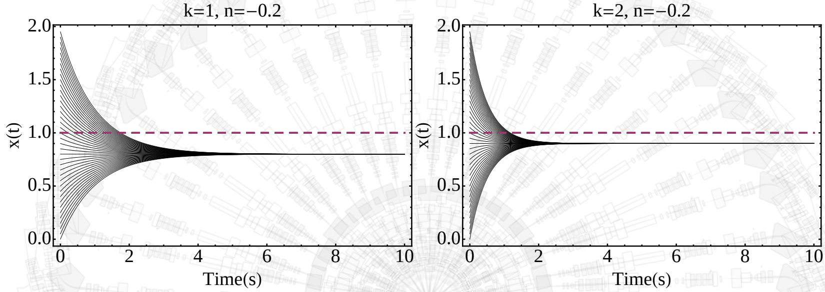
the system has equilibrium at

$$\dot{e} = 0 = -ke + n \qquad e_{\text{final}} = \frac{n}{k},$$

which you can also see from the solution

$$e(t) = \frac{n}{k}\left(1 - e^{-kt}\right) + e(0)e^{-kt}.$$

In this case there will always be a steady state offset.


k=1, n=−0.2
k=2, n=−0.2
x(t)
2.0
1.5
1.0
0.5
0.0
0
2
4
6
8
10
Time(s)

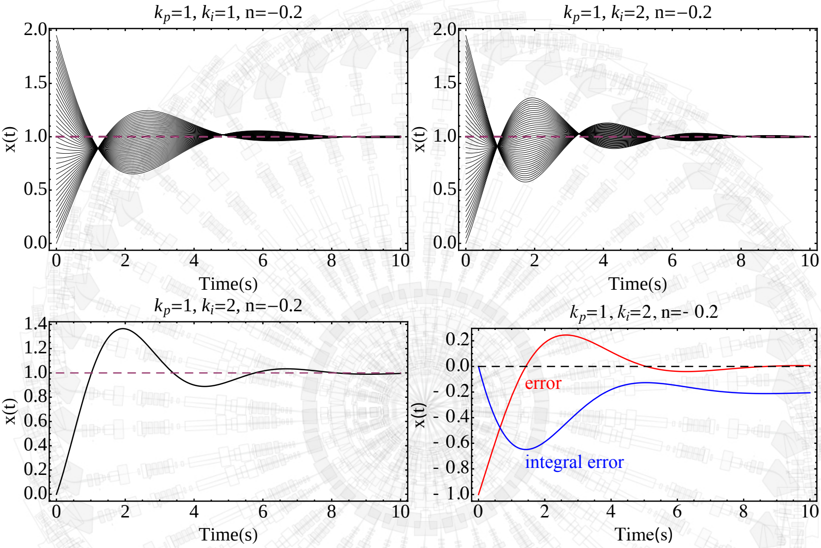
To deal with un-modeled disturbances, we add an additional, integral of the error term

$$k_i \int_0^t e(\tau) d\tau,$$

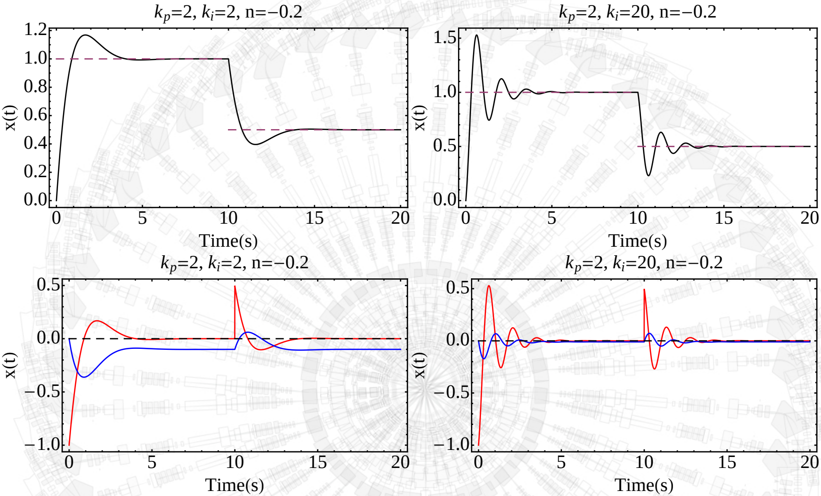
so our overall system now looks like

$$\dot{e}(t) = -k_p e(t) - k_i \int_0^t e(\tau) d\tau + n.$$

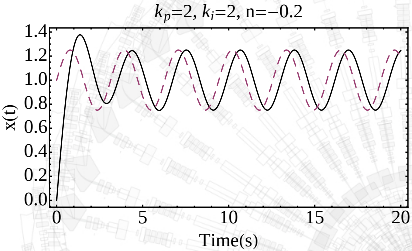
In case of a un-modeled disturbance, the integral term will continue to grow, forcing $e(t)$ towards zero.

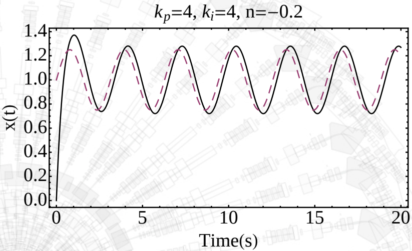


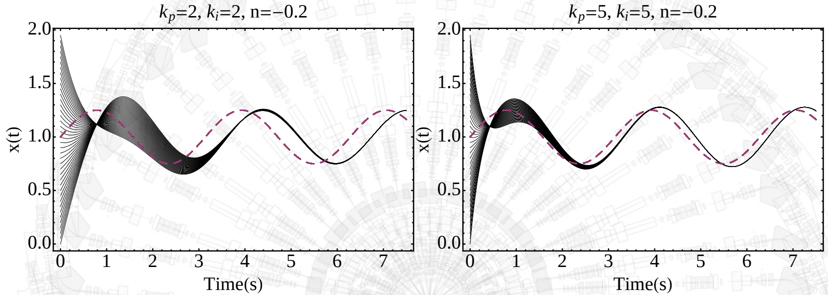
k_p=1, k_i=1, n=−0.2
x(t)
Time(s)
k_p=1, k_i=2, n=−0.2
x(t)
Time(s)
k_p=1, k_i=2, n=−0.2
x(t)
Time(s)
k_p=1, k_i=2, n=- 0.2
error
integral error
x(t)
Time(s)



k_p=2, k_i=2, n=−0.2
x(t)
Time(s)
k_p=2, k_i=20, n=−0.2
x(t)
Time(s)
k_p=2, k_i=2, n=−0.2
x(t)
Time(s)
k_p=2, k_i=20, n=−0.2
x(t)
Time(s)


k_p=2, k_i=2, n=−0.2
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
x(t)
0
5
10
15
20
Time(s)


k_p=4, k_i=4, n=−0.2
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
x(t)
0
5
10
15
20
Time(s)



k_p=2, k_i=2, n=−0.2
k_p=5, k_i=5, n=−0.2
x(t)
Time(s)

Back to the mass-spring-damper system:

$$m\ddot{x}(t) = -kx(t) - b\dot{x}(t) + u. \tag{4}$$

We rewrite the dynamics, (4), in terms of the error

$$e(t) = x(t) - x_s(t), \qquad \dot{e}(t) = \dot{x}(t) - \dot{x}_s(t), \qquad \ddot{e}(t) = \ddot{x}(t) - \ddot{x}_s(t).$$

We get:

$$m\ddot{e}(t) + m\ddot{x}_s(t) = -ke(t) - kx_s(t) - b\dot{e}(t) - b\dot{x}_s(t) + u,$$

which we regroup as

$$\ddot{e} = -\frac{k}{m}e - \frac{b}{m}\dot{e} + \frac{u}{m} - \frac{k}{m}x_s - \frac{b}{m}\dot{x}_s - \ddot{x}_s,$$

and use a proportional - integral controller

$$u = -k_p e - k_i \int_0^t e(\tau)d\tau,$$

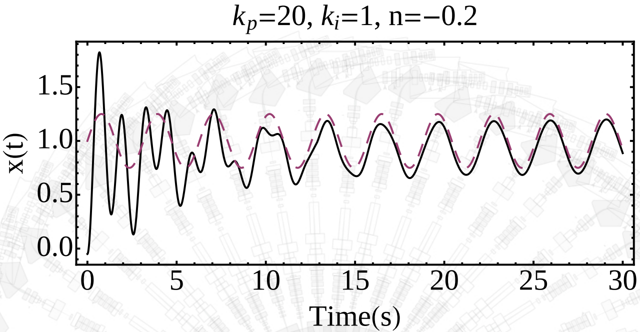
to get the closed-loop system

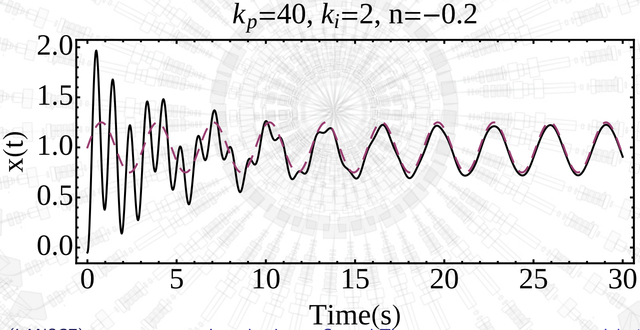
$$\ddot{e} = -\frac{k}{m}e - \frac{b}{m}\dot{e} - \frac{k_p}{m}e - \frac{k_i}{m}\int_0^t e(\tau)d\tau - \frac{k}{m}x_s - \frac{b}{m}\dot{x}_s - \ddot{x}_s,$$

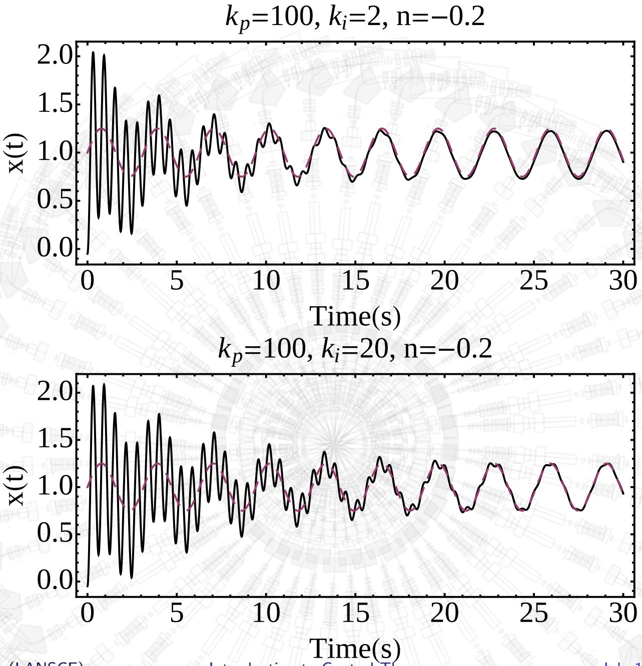
Grouping some terms we get

$$\ddot{e} = -\left(\frac{k + k_p}{m}\right) e - \frac{b}{m}\dot{e} - \frac{k_i}{m}\int_0^t e(\tau)d\tau - \frac{k}{m}x_s - \frac{b}{m}\dot{x}_s - \ddot{x}_s.$$

Increasing the proportional gain, k_p, is equivalent to increasing the spring constant.


k_p=20, k_i=1, n=−0.2
x(t)
0.0
0.5
1.0
1.5
0
5
10
15
20
25
30
Time(s)


k_p=40, k_i=2, n=−0.2
x(t)
0.0
0.5
1.0
1.5
2.0
0
5
10
15
20
25
30
Time(s)


k_p=100, k_i=2, n=−0.2
2.0
1.5
1.0
0.5
0.0
x(t)
0
5
10
15
20
25
30
Time(s)
k_p=100, k_i=20, n=−0.2
2.0
1.5
1.0
0.5
0.0
x(t)
0
5
10
15
20
25
30
Time(s)

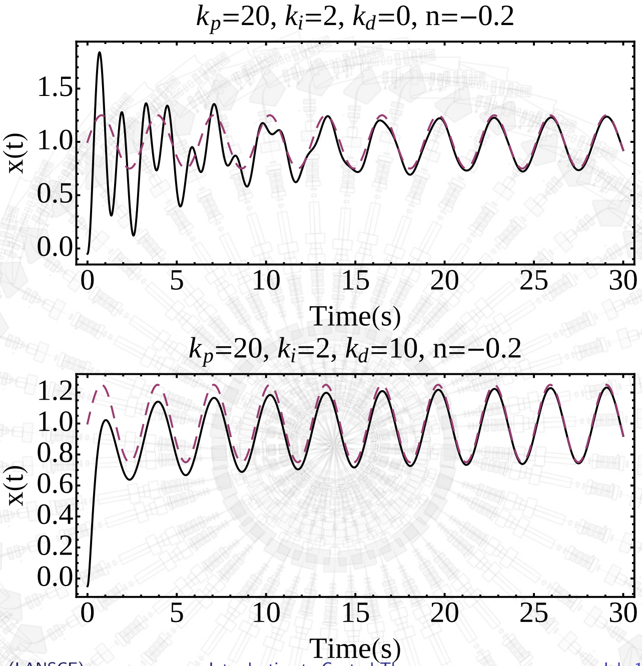
The proportional and integral terms are "too slow." Increasing integral control slows the response too much and we see many repeated overshoots. Increasing proportional causes large initial overshoots. We need to add a third, fast type of control, derivative control:

$$u = -k_d \left(\dot{x}(t) - \dot{x}_s(t) \right) = -k_d \dot{e}(t).$$

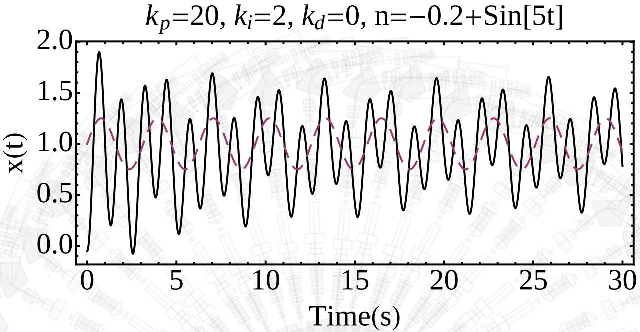
Adding the derivative control term, we get

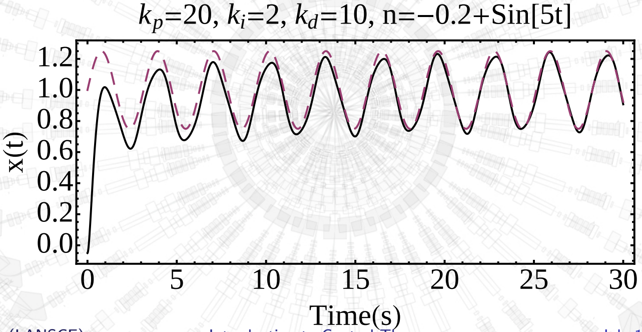
$$\ddot{e} = -\left(\frac{k + k_p}{m} \right) e - \left(\frac{b + k_d}{m} \right) \dot{e} - \frac{k_i}{m} \int_0^t e(\tau) d\tau - \frac{k}{m} x_s - \frac{b}{m} \dot{x}_s - \ddot{x}_s.$$

Notice that, while increasing the proportional gain, k_p, is equivalent to increasing the spring constant, adding derivative gain, k_d is equivalent to increasing the damping factor, which we would expect to slow down the high frequency oscillations.

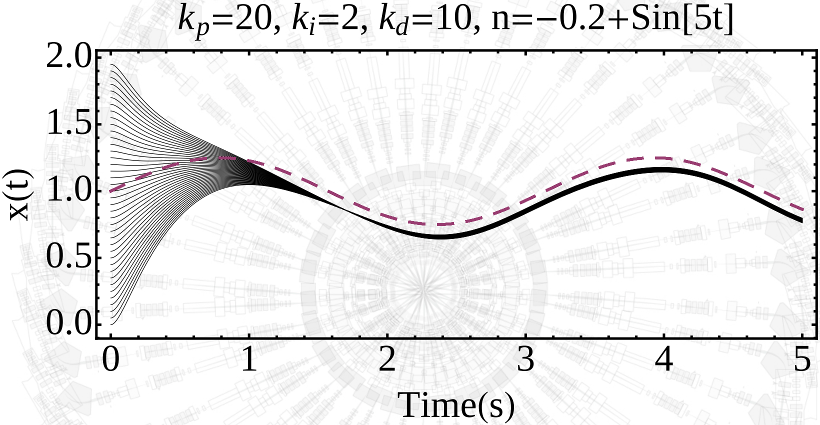


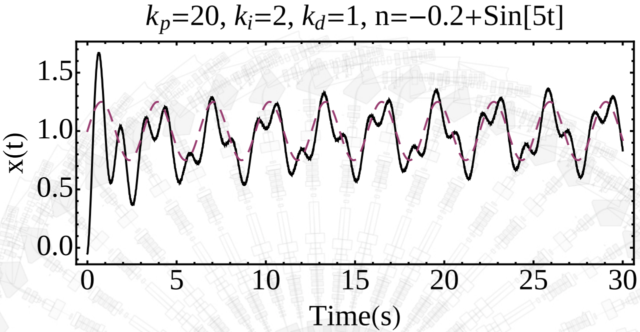
k_p=20, k_i=2, k_d=0, n=−0.2
1.5
1.0
0.5
0.0
x(t)
0
5
10
15
20
25
30
Time(s)
k_p=20, k_i=2, k_d=10, n=−0.2
1.2
1.0
0.8
0.6
0.4
0.2
0.0
x(t)
0
5
10
15
20
25
30
Time(s)

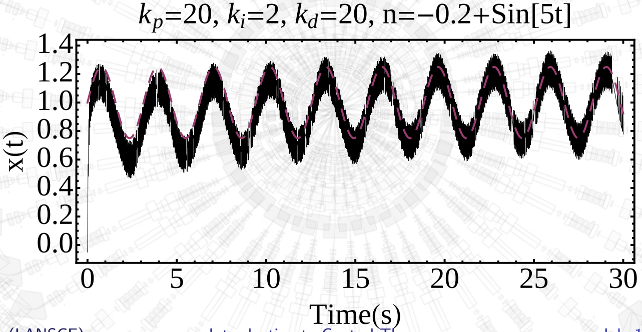

k_p=20, k_i=2, k_d=0, n=−0.2+Sin[5t]
x(t)
2.0
1.5
1.0
0.5
0.0
0
5
10
15
20
25
30
Time(s)


k_p=20, k_i=2, k_d=10, n=−0.2+Sin[5t]
x(t)
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
5
10
15
20
25
30
Time(s)

Finally, utilizing proportional-integral-derivative (PID) control, we can handle a wide range of disturbances and initial conditions, without needing a very detailed knowledge of the system parameters.

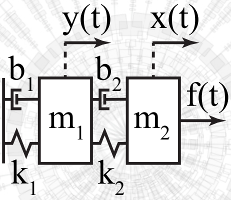



k_p=20, k_i=2, k_d=1, n=−0.2+Sin[5t]
x(t)
1.5
1.0
0.5
0.0
0
5
10
15
20
25
30
Time(s)


k_p=20, k_i=2, k_d=20, n=−0.2+Sin[5t]
x(t)
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
5
10
15
20
25
30
Time(s)

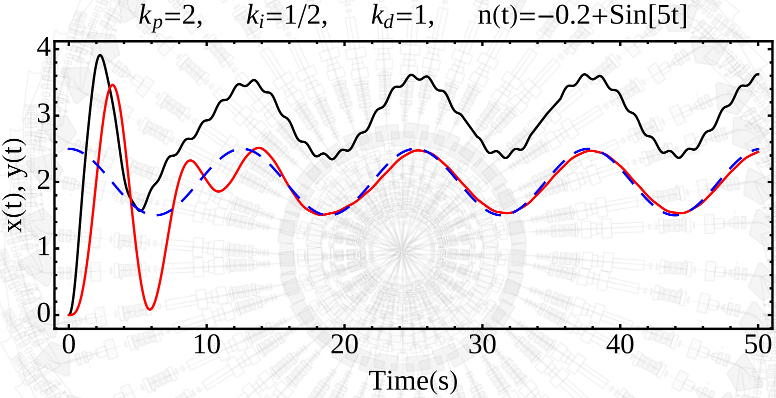
Coupled Mass-Spring-Dampers:

$$
\begin{aligned}
m_2\ddot{y} &= -k_1 y + k_2(x-y) + b_2\left(\dot{x}-\dot{y}\right) - b_1\dot{y}, \\
m_1\ddot{x} &= -k_2(x-y) - b_2\left(\dot{x}-\dot{y}\right) + n(t) + u.
\end{aligned}
$$



Coupled Mass-Spring-Dampers:

$$u = -k_p e_y - k_i \int_0^t e_y(\tau) d\tau - k_d \dot{e}_y, \qquad e_y = y(t) - y_s(t)$$



State Space Form

In general, a mass m subject to a force, $F(x, \dot{x}, t)$, has dynamics

$$F(x, \dot{x}, t) = ma = m\ddot{x}, \tag{5}$$

if we define

$$x_1 \equiv x, \qquad x_2 \equiv \dot{x}_1 = \dot{x},$$

we can rewrite (5) as

$$\begin{aligned} \dot{x}_1 &= x_2, && (6) \\ \dot{x}_2 &= \frac{1}{m} F(x_1, x_2, t). && (7) \end{aligned}$$

Example: Driven Mass-Spring-Damper: The equation of motion

$$m\ddot{x} = -kx - b\dot{x} + u, \tag{8}$$

can be rewritten as

$$\begin{aligned} \dot{x}_1 &= x_2, && (9) \\ \dot{x}_2 &= -\frac{k}{m} x_1 - \frac{b}{m} x_2 + \frac{1}{m} u. && (10) \end{aligned}$$

Example: Coupled Mass-Spring-Dampers: The equations of motion are:

$$m_1\ddot{y} = -k_1 y - b_1\dot{y} + k_2(x-y) + b_2(\dot{x}-\dot{y}) + u_1, \tag{11}$$

$$m_2\ddot{x} = -k_2(x-y) - b_2(\dot{x}-\dot{y}) + u_2. \tag{12}$$

Defining state variables

$$x_1 \equiv y, \quad x_2 \equiv \dot{y}, \quad x_3 \equiv x, \quad x_4 \equiv \dot{x},$$

we rewrite (11), (12) as

$$\dot{x}_1 = x_2, \tag{13}$$

$$\dot{x}_2 = -\frac{k_1}{m_1}x_1 - \frac{b_1}{m_1}x_2 + \frac{k_2}{m_1}(x_3-x_1) + \frac{b_2}{m_1}(x_4-x_2) + \frac{1}{m_1}u_1, \tag{14}$$

$$\dot{x}_3 = x_4, \tag{15}$$

$$\dot{x}_4 = -\frac{k_2}{m_2}(x_3-x_1) - \frac{b_2}{m_2}(x_4-x_2) + \frac{1}{m_2}u_2. \tag{16}$$

Combining like terms, we rewrite the equations of motion as

$$\dot{x}_1 = x_2, \tag{17}$$

$$\dot{x}_2 = -\left(\frac{k_1}{m_1}+\frac{k_2}{m_1}\right)x_1+\left(\frac{b_1}{m_1}-\frac{b_2}{m_1}\right)x_2+\frac{k_2}{m_1}x_3+\frac{b_2}{m_1}x_4+\frac{1}{m_1}u_1, \tag{18}$$

$$\dot{x}_3 = x_4, \tag{19}$$

$$\dot{x}_4 = \frac{k_2}{m_2}x_1+\frac{b_2}{m_2}x_2-\frac{k_2}{m_2}x_3-\frac{b_2}{m_2}x_4+\frac{1}{m_2}u_2. \tag{20}$$

which we rewrite as

$$\underbrace{\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}}_{\dot{\mathbf{x}}}=\underbrace{\begin{bmatrix}0 & 1 & 0 & 0\\ -\left(\frac{k_1}{m_1}+\frac{k_2}{m_1}\right) & \left(\frac{b_1}{m_1}-\frac{b_2}{m_1}\right) & \frac{k_2}{m_1} & \frac{b_2}{m_1}\\ 0 & 0 & 0 & 1\\ \frac{k_2}{m_2} & \frac{b_2}{m_2} & -\frac{k_2}{m_2} & -\frac{b_2}{m_2}\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}}_{\mathbf{x}}+\underbrace{\begin{bmatrix}0&0&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&1&0&0\end{bmatrix}}_{B}\underbrace{\begin{bmatrix}u_1\\ u_2\\ u_3\\ u_4\end{bmatrix}}_{\mathbf{u}}, \tag{21}$$

finally resulting in the concise linear matrix differential equation

$$\dot{\mathbf{x}} = A\mathbf{x}+B\mathbf{u}. \tag{22}$$

In general, a linear time-invariant system can be written in the form

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}}_{\mathbf{u}} \quad (23)$$

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}. \quad (24)$$

Typically, a linear feedback control of the form

$$\mathbf{u} = -K\mathbf{x}$$

is used, resulting in a closed loop system of the form

$$\dot{\mathbf{x}} = (A - BK)\,\mathbf{x},$$

which has solution

$$\mathbf{x}(t) = e^{(A-BK)t}\mathbf{x}(0).$$

Therefore, stability is guaranteed if $A - BK$ is Hurwitz:

$$\mathrm{Re}\,\{\lambda_i\,(A - BK)\} < 0. \quad (25)$$

Example

$$\dot{x} = x + u, \quad u = -2x \tag{26}$$

$$\dot{x} = -x \implies x(t) = e^{-t}x(0). \tag{27}$$

Example

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{28}$$

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\mathbf{u}} \tag{29}$$

$$\mathbf{u} = -K\mathbf{x} = -\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} \tag{30}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}}_{A-BK} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{31}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} e^{-t}x_1(0) \\ e^{-3t}x_2(0) \end{bmatrix} \tag{32}$$

$$\lambda_i(A - BK) = \{-1, \quad -3\} \tag{33}$$