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# Stopping Power for Degenerate Electrons

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## Abstract

This is a first attempt at calculating the BPS stopping power with electron degeneracy corrections. Section I establishes some notation and basic facts. Section II outlines the basics of the calculation, and in Section III contains some brief notes on how to proceed with the details of the calculation. The remaining work for the calculation starts with Section III.

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Robert Singleton

Research Notes
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# I. STATISTICS

## A. The Fugacity as an Expansion Parameter

The plasma electrons are assumed to be degenerate and described by a Fermi-Dirac distribution,

$$f_e^{\rm FD} = \frac{1}{e^{\beta_e(E_e-\mu_e)}+1} ~, \tag{1.1}$$

where the electron kinetic energy is $E_e = p_e^2/2m_e$, the electron chemical potential is $\mu_e$, and $\beta_e = 1/T_e$ is the inverse temperature of the electron gas. The ion distribution is assumed to be Maxwell-Boltzmann, so that

$$f_i^{\rm MB} = e^{-\beta_i(E_i-\mu_i)} = z_i\, e^{-\beta_i E_i} ~, \tag{1.2}$$

where the ion kinetic energy is $E_i = p_i^2/2m_i$, the ion chemical potential is $\mu_i$, where $\beta_i = 1/T_i$ is the inverse temperature of the ion gas, and $z_i = e^{\beta_i\mu_i}$ is the ion fugacity parameter. From here on, we drop the superscripts on $f_e$ and $f_i$, and we denote the electron and ion species by the general index $a$, so that $E_a = p_a^2/2m_a$, with the fugacity parameter

$$z_a = e^{\beta_a\mu_a} ~. \tag{1.3}$$

This quantity measures the degeneracy of species $a$. The electron fugacity $z_e$ will serve as an expansion expansion parameter (we will work to all orders in $z_e$), as well as the plasma coupling $g_a = e_a^2\kappa_a/T_a$.

## B. Projectiles and the Stopping Power

In computing the stopping power, we will take the projectile to have a $\delta$-function distribution along a straight line trajectory

$$f_p = f_p(\mathbf{x},\mathbf{p},t) = \mathbb{N}_p\, \delta^{(\nu)}(\mathbf{x}-\mathbf{v}_p t) ~, \tag{1.4}$$

where $\mathbf{v}_p = \mathbf{p}/m_p$, and $\mathbb{N}_p$ is a normalization factor. For simplicity, we assume that the projectile is an ion; for an electron projectile, the associated Fermi blocking terms must be included. We only consider ionic projectiles for now.

### C. The de Broglie Wavelength and Normalization

The number density of a general species $a$ takes the form

$$n_a(\mathbf{x}) = \mathfrak{g}_a \int \frac{d^\nu p}{(2\pi\hbar)^\nu}\, f_a(\mathbf{x},\mathbf{p}) \ , \tag{1.5}$$

where $\mathfrak{g}_a$ is the degeneracy factor for the species, which can be a plasma electron ($e$), a plasma ion ($i$), or a projectile ($p$). We take $\mathfrak{g}_e = 2$ because there are two electron spin-states, while we set $\mathfrak{g}_i = 1$ since the spin states of the classical ions should be counted as distinct states in Maxwell-Boltzmann statistics. For quantum degenerate electrons, however, this is not permitted, and we must include the spin degeneracy factor. Similarly, for the projectile we take $\mathfrak{g}_p = 1$. Since the ion distribution is a Gaussian, the integrals can easily be performed, and we find

$$n_i \ = \ \mathfrak{g}_i\, \frac{z_i}{\lambda_i^\nu} \tag{1.6}$$

$$\lambda_i \ = \ \hbar \left(\frac{2\pi\beta_i}{m_i}\right)^{1/2} \ . \tag{1.7}$$

In three dimensions, the ion chemical potential is therefore $\mu_i = T_i\, \ln\{\lambda_i^3\, n_i/\mathfrak{g}_i\}$.

We can also use (1.5) to calculate the normalization factor in (1.4), since there is only one projectile in the volume,

$$1 = \int d^\nu x\, n_p(\mathbf{x}) = \int d^\nu x \int \frac{d^\nu p}{(2\pi\hbar)^\nu}\, \mathbb{N}_p\, \delta^{(\nu)}(\mathbf{x} - \mathbf{v}_p t) = \mathbb{V}_\nu \cdot \frac{1}{(2\pi\hbar)^\nu}\, \mathbb{N}_p \ , \tag{1.8}$$

or

$$f_p \ = \ \mathbb{N}_p\, \delta^{(\nu)}\,(\mathbf{x} - \mathbf{v}_p t) \tag{1.9}$$

$$\mathbb{N}_p \ = \ \frac{(2\pi\hbar)^\nu}{\mathbb{V}_\nu} \ , \tag{1.10}$$

where $\mathbb{V}_\nu$ is the $\nu$-dimensional volume of a large but finite box containing the system.

Now, let us consider the degenerate electron number density

$$n_e \ = \ \mathfrak{g}_e \int \frac{d^\nu p}{(2\pi\hbar)^\nu}\, f_e(\mathbf{p}) = \mathfrak{g}_e \int \frac{d^\nu p}{(2\pi\hbar)^\nu}\, \frac{1}{e^{\beta_e(E_e(p) - \mu_e)} + 1} \tag{1.11}$$

$$= \ \frac{\mathfrak{g}_e\, \Omega_{\nu-1}}{(2\pi\hbar)^\nu} \int_0^\infty p^{\nu-1} dp\, \frac{1}{e^{\beta_e(E_e(p) - \mu_e)} + 1} \ , \tag{1.12}$$

where the area of a unit hypersphere in $\nu$-dimensions is

$$\Omega_{\nu-1} = \frac{2\pi^{\nu/2}}{\Gamma(\nu/2)} \ , \tag{1.13}$$

and $\Gamma(z)$ is the Gamma function, with $\Gamma(z+1) = z\,\Gamma(z)$ and $\Gamma(1/2) = \sqrt{\pi}$. When $z$ is a positive integer, then $\Gamma(z+1) = z!$, and otherwise one defines [1]

$$\Gamma(z) \;=\; \int_0^\infty \frac{du}{u}\, u^z\, e^{-z} \;. \tag{1.14}$$

Upon making the change of variables to $x = \beta_e p^2/2m_e$ in (1.12), we can express the electron number density in terms of dimensionless function $F$,

$$n_e \;=\; \frac{\mathfrak{g}_e}{\lambda_e^\nu}\, F_\nu(z_e) \tag{1.15}$$

$$F_\nu(z_e) \;\equiv\; \frac{1}{\Gamma(\nu/2)} \int_0^\infty \frac{dx}{x}\, \frac{x^{\nu/2}}{e^x + z_e} \;, \tag{1.16}$$

where the de Broglie wave length of the electron is

$$\lambda_e \;=\; \hbar \left( \frac{2\pi\beta_e}{m_e} \right)^{1/2} \;. \tag{1.17}$$

We therefore need to solve the following equation for $z_e$,

$$F_\nu(z_e) \;=\; n_e \lambda_e^\nu / \mathfrak{g}_e \;. \tag{1.18}$$

Question: Should I put the factor of $\mathfrak{g}_e$ in the definition of $F_\nu$, or show it explicitly? That is to say, given $n_e$ and $\beta_e$, which determines the RHS of Eq. (1.18), we solve numerically for $z_e$. From this we can find the chemical potential $\mu_e = T_e \ln z_e$.

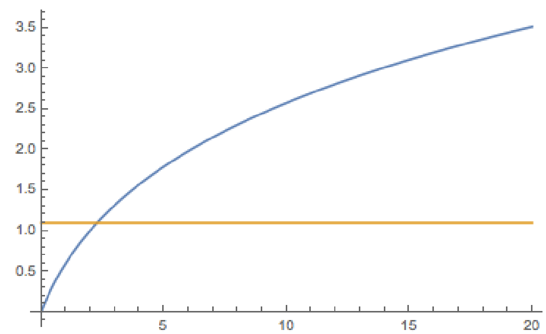


FIG. 1: The Fermi function $F(z)$ vs $z$. The horizontal line is a hypothetical value for $n_e\lambda_e^3/2$. **todo**: plot some real examples, say $T \sim 1\,\text{keV}$ or less, and $n_e \sim 10^{25}\,\text{cm}^{-3}$ or more.

[1] The following asymptotic expansion for small $z$ will eventually be useful, so I record it here:

$$\Gamma(z) \;=\; \frac{1}{z} - \gamma + \frac{1}{2}\left(\gamma^2 + \frac{\pi^2}{6}\right) z + \mathcal{O}(z^2) \;,$$

where the Euler constant is $\gamma = 0.577215664901532860606 5\cdots$.

### D. The Debye Wave-number

The Debye wave number can be written

$$\kappa_a^2 = \beta_a e_a^2 \, \frac{\partial n_a}{\partial(\beta_a \mu_a)} \ . \tag{1.19}$$

For Maxwell-Boltmann ions,

$$\kappa_i^2 = \beta_i e_i^2 \cdot \mathfrak{g}_i \int \frac{d^\nu p}{(2\pi\hbar)^\nu} \, \frac{\partial f_i}{\partial(\beta_i \mu_i)} = \beta_e e^2 \cdot \mathfrak{g}_e \int \frac{d^\nu p}{(2\pi\hbar)^\nu} \, f_e = \beta_i \, e_i^2 n_i \ , \tag{1.20}$$

and for degenerate Fermi-Dirac electrons

$$\kappa_e^2 = \beta_e e^2 \cdot \mathfrak{g}_e \int \frac{d^\nu p}{(2\pi\hbar)^\nu} \, \frac{\partial f_e}{\partial(\beta_e \mu_e)} = \beta_e e^2 \cdot \mathfrak{g}_e \int \frac{d^\nu p}{(2\pi\hbar)^\nu} \, f_e \Big[1 - f_e\Big] \ , \tag{1.21}$$

where I have used

$$\frac{\partial f_i}{\partial(\beta_i \mu_i)} \ = \ f_i \tag{1.22}$$

$$\frac{\partial f_e}{\partial(\beta_e \mu_e)} \ = \ f_e \Big[1 - f_e\Big] \ . \tag{1.23}$$

In a similar manner to (1.16), we have

$$\kappa_e^2 = \beta_e e^2 \, \frac{\mathfrak{g}_e z_e}{\Gamma(\nu/2)} \int_0^\infty \frac{dx}{x} \, \frac{x^{\nu/2} e^x}{(e^x + z)^2} \ . \tag{1.24}$$

## II. SOLUTION ROAD-MAP

The aim of these notes is to calculate the stopping power for a fully ionized plasma with Fermi degenerate electrons and Maxwell-Boltmann ions, exact to all orders in the electron fugacity $z_e = e^{\mu_e/T_e}$. I will first do the calculation in the extreme quantum limit, also known as the first Born approximation. Reference [2] performed a similar calculation for the case of the electron-ion temperature equilibration, and I will base these notes on that work. It might also pay to perform the classical scattering calculation before attempting the scattering to all orders in $\eta$.

### A. Boltzmann Equation: Short Distance

#### 1. Temperature Equilibration

The Boltzmann Equation (BE) gives the rate of change of the electron distribution from scattering, and it is finite in $\nu > 3$ dimensions. The BE with Pauli Blocking is given by (7.1) of Ref. [2]:

$$\boxed{\begin{aligned}\frac{\partial f_e^>}{\partial t} &= \sum\nolimits_i \int \frac{d^\nu p_e'}{(2\pi\hbar)^\nu}\,\frac{d^\nu p_i}{(2\pi\hbar)^\nu}\,\frac{d^\nu p_i'}{(2\pi\hbar)^\nu}\;\big|T_{ei}\big|^2\,(2\pi\hbar)^\nu\,(2\pi\hbar)\,\delta^{(\nu)}\Big(\mathbf{p}_e' + \mathbf{p}_i' - \mathbf{p}_e - \mathbf{p}_i\Big) \qquad (2.1)\\ &\delta\Big(\frac{p_e'^{\,2}}{2m_e} + \frac{p_i'^{\,2}}{2m_i} - \frac{p_e^2}{2m_e} - \frac{p_i^2}{2m_i}\Big)\bigg[f_i(\mathbf{p}_i')f_e(\mathbf{p}_e')\Big[1 - f_e(\mathbf{p}_e)\Big] - f_i(\mathbf{p}_i)f_e(\mathbf{p}_e)\Big[1 - f_e(\mathbf{p}_e')\Big]\bigg]\,.\end{aligned}}$$

The electron energy density in the plasma is given by

$$\mathcal{E}_e^>(\mathbf{x}, t) \;=\; \mathfrak{g}_e \int \frac{d^\nu p}{(2\pi\hbar)^\nu}\,\frac{p^2}{2m_e}\,f_e^>(\mathbf{x}, \mathbf{p}, t)\;, \tag{2.2}$$

where $\mathfrak{g}_e = 2$ is the spin degeneracy of the electron. Using Eq. (2.1), the rate of energy density exchange between electrons and ions is

$$\frac{d\mathcal{E}_{e\mathrm{I}}^>}{dt} \;=\; 2 \int \frac{d^\nu p_e}{(2\pi\hbar)^\nu}\,\frac{p_e^2}{2m_e}\,\frac{\partial f_e^>}{\partial t}\;, \tag{2.3}$$

where I have added the subscript "eI" to emphasize the the electron and ion systems are exchanging energy. The rate (2.3) can now be written in the form

$$
\frac{d\mathcal{E}_{\rm eI}^{>}}{dt} = 2\sum\nolimits_i \int \frac{d^\nu p_e}{(2\pi\hbar)^\nu}\,\frac{d^\nu p_e'}{(2\pi\hbar)^\nu}\,\frac{d^\nu p_i}{(2\pi\hbar)^\nu}\,\frac{d^\nu p_i'}{(2\pi\hbar)^\nu}\,\big|T_{ei}\big|^2\,(2\pi\hbar)^\nu\,\delta^{(\nu)}\Big(\mathbf{p}_e' + \mathbf{p}_i' - \mathbf{p}_e - \mathbf{p}_i\Big)
$$
$$
(2\pi\hbar)\delta\Big(E_e' + E_i' - E_e - E_i\Big)\frac{p_e^2}{2m_e}\left[f_i(\mathbf{p}_i')f_e(\mathbf{p}_e')\Big[1 - f_e(\mathbf{p}_e)\Big] - f_i(\mathbf{p}_i)f_e(\mathbf{p}_e)\Big[1 - f_e(\mathbf{p}_e')\Big]\right] , \tag{2.4}
$$

where $E_a' = p_a'^{\,2}/2m_a$ and $E_a = p_a^2/2m_a$.

▶ We shall exploit the crossing symmetry of the scattering-matrix under $\mathbf{p}_a \leftrightarrow \mathbf{p}_a'$. Formally, crossing symmetry means $T(\mathbf{p}_e', \mathbf{p}_i'; \mathbf{p}_e, \mathbf{p}_i) = T(\mathbf{p}_e, \mathbf{p}_i; \mathbf{p}_e', \mathbf{p}_i')$, which is just the interchange of incoming and outgoing particles in the scattering process. We can combine the two scattering terms in square brackets, replacing the electron kinetic energy in Eq. (2.4) by

$$
\frac{p_e^2}{2m_e} \to \frac{p_e'^{\,2}}{2m_e} - \frac{p_e^2}{2m_e} , \tag{2.5}
$$

thereby giving

$$
\frac{d\mathcal{E}_{\rm eI}^{>}}{dt} = 2\sum\nolimits_i \int \frac{d^\nu p_e}{(2\pi\hbar)^\nu}\,\frac{d^\nu p_e'}{(2\pi\hbar)^\nu}\,\frac{d^\nu p_i}{(2\pi\hbar)^\nu}\,\frac{d^\nu p_i'}{(2\pi\hbar)^\nu}\,\big|T_{ei}\big|^2\,(2\pi\hbar)^\nu\,\delta^{(\nu)}\Big(\mathbf{p}_e' + \mathbf{p}_i' - \mathbf{p}_e - \mathbf{p}_i\Big)
$$
$$
(2\pi\hbar)\,\delta\Big(E_e' + E_i' - E_e - E_i\Big)\left(\frac{p_e'^{\,2} - p_e^2}{2m_e}\right) f_i(\mathbf{p}_i)\,f_e(\mathbf{p}_e)\Big[1 - f_e(\mathbf{p}_e')\Big] . \tag{2.6}
$$

This is Eq. (7.3) of Ref. [2].

▶ Next, we perform the $p_i'$-integral in Eq. (2.6), employing the momentum conserving $\delta$-function to replace $\mathbf{p}_i' = \mathbf{p}_i + \mathbf{p}_e - \mathbf{p}_e'$:

$$
\frac{d\mathcal{E}_{\rm eI}^{>}}{dt} = 2\sum\nolimits_i \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu}\,\frac{d^\nu p_e}{(2\pi\hbar)^\nu}\,\frac{d^\nu p_e'}{(2\pi\hbar)^\nu}\,\big|T_{ei}\big|^2\,\left(\frac{p_e'^{\,2} - p_e^2}{2m_e}\right) \tag{2.7}
$$
$$
(2\pi\hbar)\,\delta\Big(E_e' + E_i' - E_e - E_i\Big)\,f_i(\mathbf{p}_i)\,f_e(\mathbf{p}_e)\Big[1 - f_e(\mathbf{p}_e')\Big]\bigg|_{\mathbf{p}_i' = \mathbf{p}_i + \mathbf{p}_e - \mathbf{p}_e'} .
$$

We perform a coordinate transformation $\mathbf{p}_e, \mathbf{p}_e' \to \bar{\mathbf{p}}, \mathbf{q}$ defined by

$$
\mathbf{q} = \mathbf{p}_e' - \mathbf{p}_e = \mathbf{p}_i - \mathbf{p}_i' \tag{2.8}
$$
$$
\bar{\mathbf{p}} = \frac{1}{2}\Big[\mathbf{p}_e' + \mathbf{p}_e\Big] , \tag{2.9}
$$

and since the Jacobina is uniity, $d^\nu p_e\, d^\nu p_e' = d^\nu \bar{p}\, d^\nu q$, we have

$$\boxed{\begin{aligned} \frac{d\mathcal{E}^{>}_{e\mathrm{I}}}{dt} = 2\sum\nolimits_i \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu}\,\frac{d^\nu \bar{p}}{(2\pi\hbar)^\nu}\,\frac{d^\nu q}{(2\pi\hbar)^\nu}\,\Big|T_{ei}\Big|^2\,(2\pi\hbar)\,\delta\Big(\frac{\mathbf{p}_i\cdot\mathbf{q}}{m_i} - \frac{\bar{\mathbf{p}}\cdot\mathbf{q}}{m_e} - \frac{q^2}{2m_i}\Big) \qquad (2.10) \\ \frac{\bar{\mathbf{p}}\cdot\mathbf{q}}{m_e}\, f_i(\mathbf{p}_i)\, f_e(\mathbf{p}_e)\Big[1 - f_e(\mathbf{p}_e')\Big] \ . \end{aligned}}$$

This is (7.7) in Ref. [2]. When the ions all have the same temperature $T_i = T_{\mathrm{I}}$, the rate can be expressed as

$$\frac{d\mathcal{E}^{>}_{e\mathrm{I}}}{dt} = -\mathcal{C}^{>}_{e\mathrm{I}}\Big(T_e - T_{\mathrm{I}}\Big) \ . \tag{2.11}$$

This defines $\mathcal{C}^{>}_{e\mathrm{I}}$.

### *2. The Born Approximation and the Classical Limit*

For the general scattering event, $p+a \to p'+a'$, the matrix $T$ depends upon the momentum exchange $\mathbf{q}$ the center-of-mass energy $W$, so that $T = T(q^2, W)$. In the extreme quantum limit, or the first Born approximation, the amplitude is independent of $W$, and takes the form

$$T^{\mathrm{Born}}_{pa} = \hbar\,\frac{e_p e_a}{q^2} \ , \tag{2.12}$$

where the momentum transfer is $\mathbf{q} = \mathbf{p}_a - \mathbf{p}_a' = \mathbf{p}_p' - \mathbf{p}_p$.

It will sometime be useful to change variables to the center-of-mass and relative momentum,

$$\mathbf{p} = \frac{m_a\mathbf{p}_p - m_p\mathbf{p}_a}{m_p + m_a} = m_{pa}\mathbf{v}_{pa} \quad \text{with} \quad \mathbf{v}_{pa} = \mathbf{v}_p - \mathbf{v}_a \tag{2.13}$$

$$\mathbf{P} = \frac{m_p\mathbf{p}_p + m_a\mathbf{p}_a}{m_p + m_a} \ . \tag{2.14}$$

The Jacobian is unity, and therefore $d^\nu p_a\, d^\nu p_p = d^\nu P\, d^\nu p$. The product of the momentum and energy conserving $\delta$-functions can be expressed as

$$\delta^{(\nu)}\Big(\mathbf{p}_p' + \mathbf{p}_a' - \mathbf{p}_p - \mathbf{p}_a\Big)\,\delta\Big(\frac{p_p'^{\,2}}{2m_p} + \frac{p_a'^{\,2}}{2m_a} - \frac{p_p^2}{2m_p} - \frac{p_a^2}{2m_a}\Big) \tag{2.15}$$

$$= \delta^{(\nu)}\Big(\mathbf{P}' - \mathbf{P}\Big)\,\delta\left(\frac{p'^{\,2}}{2m_{pa}} - \frac{p^2}{2m_{pa}}\right) \ . \tag{2.16}$$

The cross section $d\sigma$ is related to the scattering matrix $T$ by

$$\int \frac{d^\nu p'}{(2\pi\hbar)^\nu}\,\Big|T_{pa}(q^2, W)\Big|^2\,(2\pi\hbar)\,\delta\left(\frac{p'^{\,2}}{2m_{pa}} - \frac{p^2}{2m_{pa}}\right) = v_{pa}\int d\sigma_{pa} \ , \tag{2.17}$$

where $v_{pa} = |\mathbf{v}_p - \mathbf{v}_a|$ and

$$\mathbf{q} \;=\; \mathbf{p}' - \mathbf{p} \tag{2.18}$$

$$W \;=\; \frac{p^2}{2m_{pa}} \;. \tag{2.19}$$

Note that $\mathbf{q} = \mathbf{p}_p' - \mathbf{p}_p = \mathbf{p}_a - \mathbf{p}_a'$. The classical cross section makes contact with the classical limit through its relation to the impact parameter $b$,

$$d\sigma^{\rm C}_{pa} = \Omega_{\nu-2}\, b^{\nu-1}\, db \;. \tag{2.20}$$

### 3. *Stopping Power*

The BE with Pauli Blocking for a projectile distribution $f_p$ in the plasma with components $a$ (ranging over electrons and ions) is

$$\boxed{\begin{aligned} \frac{\partial f_p^>}{\partial t} \;=\; & \sum\nolimits_a \int \frac{d^\nu p_p'}{(2\pi\hbar)^\nu}\,\frac{d^\nu p_a}{(2\pi\hbar)^\nu}\,\frac{d^\nu p_a'}{(2\pi\hbar)^\nu}\;\big|T_{pa}\big|^2\,(2\pi\hbar)^\nu\,\delta^{(\nu)}\Big(\mathbf{p}_p' + \mathbf{p}_a' - \mathbf{p}_p - \mathbf{p}_a\Big) \qquad (2.21) \\ & (2\pi\hbar)\,\delta\Big(E_p' + E_a' - E_p - E_a\Big)\Bigg[f_p(\mathbf{p}_p')f_a(\mathbf{p}_a') - f_p(\mathbf{p}_p)f_a(\mathbf{p}_a)\Bigg]_{\rm PB} \;, \end{aligned}}$$

where the subscript PB means that the Pauli blocking term is to be included for electrons,

$$f_e^{\rm MB}(p) \to f_e^{\rm FD}(p)\Big[1 - f_e^{\rm FD}(p')\Big] \;. \tag{2.22}$$

I will parallel the treatment given in Ref. [2] as closely as possible. For a charged projectile $p$, the energy of the projectile is given by

$$E_p^> \;=\; 2\int d^\nu x_p \int \frac{d^\nu p_p}{(2\pi\hbar)^\nu}\,\frac{p_p^2}{2m_p}\,f_p^>(\mathbf{x}_p, \mathbf{p}_p, t) = 2\mathbb{V}_\nu \int \frac{d^\nu p_p}{(2\pi\hbar)^\nu}\,\frac{p_p^2}{2m_p}\,f_p^>(\mathbf{p}_p, t) \;, \tag{2.23}$$

where we integrate over all spatial locations of the projectile, with the projectile distribution $f_p(\mathbf{p}, t)$ being independent of space, and $\mathbb{V}_\nu$ is the volume of space. Recall that the projectile distribution is normalized by

$$f_p(\mathbf{x}, \mathbf{p}) = \frac{(2\pi\hbar)^\nu}{\mathbb{V}_\nu}\,\delta^{(\nu)}\!\left(\mathbf{x} - \frac{\mathbf{p}}{m_p}\,t\right) \;. \tag{2.24}$$

It will be convenient to rescale the projectile distribution by the volume factor, $\bar{f}_p = \mathbb{V}_\nu \cdot f_p$, and define

$$\bar{f}_p(\mathbf{x}, \mathbf{p}) = (2\pi\hbar)^\nu\,\delta^{(\nu)}\!\left(\mathbf{x} - \frac{\mathbf{p}}{m_p}\,t\right) \;. \tag{2.25}$$

Using the BE, the stopping power therefore takes the form

$$\frac{dE_p^>}{dt} = 2\sum\nolimits_a \int \frac{d^\nu p_p}{(2\pi\hbar)^\nu} \left(\frac{p_p^2}{2m_p}\right) \int \frac{d^\nu p_p'}{(2\pi\hbar)^\nu} \frac{d^\nu p_a}{(2\pi\hbar)^\nu} \frac{d^\nu p_a'}{(2\pi\hbar)^\nu} \, \big|T_{pa}\big|^2 \tag{2.26}$$
$$(2\pi\hbar)^\nu \, \delta^{(\nu)}\Big(\mathbf{p}_p' + \mathbf{p}_a' - \mathbf{p}_p - \mathbf{p}_a\Big) \, (2\pi\hbar) \, \delta\Big(E_p' + E_a' - E_p - E_a\Big)$$
$$\Big[\bar{f}_p(\mathbf{p}_p') f_a(\mathbf{p}_a') - \bar{f}_p(\mathbf{p}_p) f_a(\mathbf{p}_a)\Big]_{\text{PB}} \, .$$

▶ Use crossing symmetry to make the replacement

$$\left(\frac{p_p^2}{2m_p}\right) \Big[\bar{f}_p(\mathbf{p}_p') f_a(\mathbf{p}_a') - \bar{f}_p(\mathbf{p}_p) f_a(\mathbf{p}_a)\Big]_{\text{PB}} \to \left(\frac{p_p'^2}{2m_p} - \frac{p_p^2}{2m_p}\right) \bar{f}_p(\mathbf{p}_p) \cdot f_a(\mathbf{p}_a)\Big|_{\text{PB}} \, , \tag{2.27}$$

which gives,

$$\frac{dE_p^>}{dt} = 2\sum\nolimits_a \int \frac{d^\nu p_p}{(2\pi\hbar)^\nu} \frac{d^\nu p_p'}{(2\pi\hbar)^\nu} \frac{d^\nu p_a}{(2\pi\hbar)^\nu} \frac{d^\nu p_a'}{(2\pi\hbar)^\nu} \, \big|T_{pb}\big|^2 \tag{2.28}$$
$$(2\pi\hbar)^\nu \, \delta^{(\nu)}\Big(\mathbf{p}_p' + \mathbf{p}_a' - \mathbf{p}_p - \mathbf{p}_a\Big) \, (2\pi\hbar) \, \delta\Big(E_p' + E_a' - E_p - E_a\Big)$$
$$\left(\frac{p_p'^2}{2m_p} - \frac{p_p^2}{2m_p}\right) \bar{f}_p(\mathbf{p}_p) f_a(\mathbf{p}_a)\Big|_{\text{PB}} \, .$$

▶ Integrate $p_p$ over the delta-function $\bar{f}_p$:

$$\boxed{\begin{aligned}\frac{dE_p^>}{dt} &= 2\sum\nolimits_a \int \frac{d^\nu p_p'}{(2\pi\hbar)^\nu} \frac{d^\nu p_a}{(2\pi\hbar)^\nu} \frac{d^\nu p_a'}{(2\pi\hbar)^\nu} \, \big|T_{pb}\big|^2 \left(\frac{p_p'^2}{2m_p} - \frac{p_p^2}{2m_p}\right) f_a(\mathbf{p}_a)\Big|_{\text{PB}} \\ &(2\pi\hbar)^\nu \, \delta^{(\nu)}\Big(\mathbf{p}_p' + \mathbf{p}_a' - \mathbf{p}_p - \mathbf{p}_a\Big) \, (2\pi\hbar) \, \delta\Big(E_p' + E_a' - E_p - E_a\Big) \, .\end{aligned}} \tag{2.29}$$

The stopping power is related to the rate by

$$\frac{dE_p^>}{dx} = \frac{1}{v_p} \frac{dE_p^>}{dt} \, . \tag{2.30}$$

The calculation of the ion contribution from the ionic term goes through just as in Ref. [1], and we express (2.29) as

$$\frac{dE_p^>}{dx} = \frac{dE_p^{e\,>}}{dx} + \frac{dE_p^{\text{I}\,>}}{dx} \, , \tag{2.31}$$

where the ion contribution to the rate is

$$
\begin{aligned}
\frac{dE_p^{\text{I}>}}{dt} \;&=\; 2\sum\nolimits_i \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu}\,\frac{d^\nu p_p'}{(2\pi\hbar)^\nu}\,\frac{d^\nu p_i'}{(2\pi\hbar)^\nu}\;\big|T_{pb}\big|^2\left(\frac{p_p'^{\,2}}{2m_p}-\frac{p_p^2}{2m_p}\right) f_i(\mathbf{p}_i) \\
&\qquad (2\pi\hbar)^\nu\,\delta^{(\nu)}\Big(\mathbf{p}_p'+\mathbf{p}_i'-\mathbf{p}_p-\mathbf{p}_i\Big)\,(2\pi\hbar)\,\delta\Big(E_p'+E_i'-E_p-E_i\Big)\;,
\end{aligned} \tag{2.32}
$$

and the electron contribution is

$$
\begin{aligned}
\frac{dE_p^{e>}}{dt} \;&=\; 2\int \frac{d^\nu p_e}{(2\pi\hbar)^\nu}\,\frac{d^\nu p_p'}{(2\pi\hbar)^\nu}\,\frac{d^\nu p_e'}{(2\pi\hbar)^\nu}\;\big|T_{pb}\big|^2\left(\frac{p_p'^{\,2}}{2m_p}-\frac{p_p^2}{2m_p}\right) f_e(\mathbf{p}_e)\Big[1-f_e(\mathbf{p}_e')\Big] \\
&\qquad (2\pi\hbar)^\nu\,\delta^{(\nu)}\Big(\mathbf{p}_p'+\mathbf{p}_e'-\mathbf{p}_p-\mathbf{p}_e\Big)\,(2\pi\hbar)\,\delta\Big(E_p'+E_e'-E_p-E_e\Big)\;.
\end{aligned} \tag{2.33}
$$

## B. Lenard-Balescu Equation: Long Distance Collective Effects

### *1. Temperature Equilibration*

The Lenard-Balescu equation (LBE) gives the rate of change of the electron distribution from the long distance collective physics, and it is finite in spatial dimensions $n < 3$. The LBE with Paul Blocking is given by (8.1) of Ref. [2],

$$
\begin{aligned}
\frac{\partial f_e^<}{\partial t} \;&=\; -\frac{\partial}{\partial \mathbf{p}_e}\cdot\sum\nolimits_i \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu}\,\frac{d^\nu k}{(2\pi)^\nu}\,\mathbf{k}\left|\frac{e e_i}{k^2\epsilon(k,\mathbf{k}\cdot\mathbf{v}_i)}\right|^2 \pi\delta\Big(\mathbf{k}\cdot\mathbf{v}_i-\mathbf{k}\cdot\mathbf{v}_e\Big) \\
&\qquad \left[\mathbf{k}\cdot\frac{\partial f_i(\mathbf{p}_i)}{\partial \mathbf{p}_i}\,f_e(\mathbf{p}_e)\big[1-f_e(\mathbf{p}_e)\big]-\mathbf{k}\cdot\frac{\partial f_e(\mathbf{p}_e)}{\partial \mathbf{p}_e}\,f_i(\mathbf{p}_i)\right]\;.
\end{aligned} \tag{2.34}
$$

**Question**: Is the momentum on the Pauli Blocking term correct? Here, the dielectric function is

$$
k^2\epsilon(k,\omega) \;=\; \kappa_e^2 + k^2 + F_{\text{I}}(k/\omega) \tag{2.35}
$$

$$
F_{\text{I}}(v) \;=\; -\sum\nolimits_i \beta_i e_i^2 \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu}\,\frac{\hat{\mathbf{k}}\cdot\mathbf{v}_i}{v-\hat{\mathbf{k}}\cdot\mathbf{v}_i+i\eta}\,f_i(\mathbf{p}_i)\;, \tag{2.36}
$$

where $\kappa_e^2$ is the square of the electron Debye number,

$$
\kappa_e^2 = \beta_e e^2\,\frac{\partial n_e}{\partial(\beta_e\mu_e)}\;. \tag{2.37}
$$

See (A9) and (A7) of Ref. [2].

The electron energy density in the plasma is given by

$$\mathcal{E}_e^<(\mathbf{x},t) \;=\; 2\int \frac{d^\nu p}{(2\pi\hbar)^\nu}\,\frac{p^2}{2m_e}\,f_e^<(\mathbf{x},\mathbf{p},t)\;, \tag{2.38}$$

where the factor of 2 arises from the spin degeneracy. Using Eq. (2.34), the rate of energy exchange between electrons and ions becomes

$$\frac{d\mathcal{E}_{e\mathrm{I}}^<}{dt} \;=\; 2\int \frac{d^\nu p_e}{(2\pi\hbar)^\nu}\,\frac{p_e^2}{2m_e}\,\frac{\partial f_e^<}{\partial t}\;. \tag{2.39}$$

From Eq. (2.39) and (2.34)

$$\boxed{\begin{aligned}\frac{d\mathcal{E}_{e\mathrm{I}}^<}{dt} \;=\; -2{\sum}_i\int \frac{d^\nu p_e}{(2\pi\hbar)^\nu}\,\frac{d^\nu p_i}{(2\pi\hbar)^\nu}\,\frac{d^\nu k}{(2\pi)^\nu}\left(\frac{p_e^2}{2m_e}\right)\frac{\partial}{\partial \mathbf{p}_e}\cdot\mathbf{k}\left|\frac{e e_i}{k^2\epsilon(k,\mathbf{k}\cdot\mathbf{v}_i)}\right|^2 \quad (2.40)\\ \pi\delta\Big(\mathbf{k}\cdot\mathbf{v}_i-\mathbf{k}\cdot\mathbf{v}_e\Big)\left[\mathbf{k}\cdot\frac{\partial f_i(\mathbf{p}_i)}{\partial \mathbf{p}_i}\,f_e(\mathbf{p}_e)\big[1-f_e(\mathbf{p}_e)\big]-\mathbf{k}\cdot\frac{\partial f_e(\mathbf{p}_e)}{\partial \mathbf{p}_e}\,f_i(\mathbf{p}_i)\right]\;.\end{aligned}}$$

When the ions all have the same temperature $T_i = T_{\mathrm{I}}$, the rate can be expressed as

$$\frac{d\mathcal{E}_{e\mathrm{I}}^<}{dt} \;=\; -\mathcal{C}_{e\mathrm{I}}^<\Big(T_e - T_{\mathrm{I}}\Big)\;. \tag{2.41}$$

This defines $\mathcal{C}_{e\mathrm{I}}^<$, and the regularized three dimensional rate, to leading and next-to-leading order in the number density, is

$$\mathcal{C}_{e\mathrm{I}}^{\mathrm{BPS}} = \lim_{\nu\to 3}\left[\mathcal{C}_{e\mathrm{I}}^> + \mathcal{C}_{e\mathrm{I}}^<\right]\;. \tag{2.42}$$

### 2. *Stopping Power*

For a projectile $p$, the LB equation is

$$\boxed{\begin{aligned}\frac{\partial f_p^<}{\partial t} \;=\; -\frac{\partial}{\partial \mathbf{p}_p}\cdot{\sum}_a\int \frac{d^\nu p_a}{(2\pi\hbar)^\nu}\,\frac{d^\nu k}{(2\pi)^\nu}\,\mathbf{k}\left|\frac{e_p e_a}{k^2\epsilon(k,\mathbf{k}\cdot\mathbf{v}_a)}\right|^2\pi\delta\Big(\mathbf{k}\cdot\mathbf{v}_a-\mathbf{k}\cdot\mathbf{v}_p\Big) \quad (2.43)\\ \left[\mathbf{k}\cdot\frac{\partial f_a(\mathbf{p}_a)}{\partial \mathbf{p}_a}\,f_p(\mathbf{p}_p)-\mathbf{k}\cdot\frac{\partial f_p(\mathbf{p}_p)}{\partial \mathbf{p}_p}\,f_a(\mathbf{p}_a)\right]_{\mathrm{PB}}\;,\end{aligned}}$$

As in the last section, I will parallel the treatment given in Ref. [2]. Similarly to (2.23), the energy of the projectile is given by

$$E_p^< \;=\; 2\mathbb{V}_\nu\int \frac{d^\nu p_p}{(2\pi\hbar)^\nu}\,\frac{p_p^2}{2m_p}\,f_p^<(\mathbf{p}_p,t)\;, \tag{2.44}$$

where $\mathbb{V}_\nu$ is the volume of space. Equations (2.44) and (2.43)

$$\frac{dE_p^<}{dt} = 2\mathbb{V}_\nu \int \frac{d^\nu p_p}{(2\pi\hbar)^\nu}\,\frac{p_p^2}{2m_p}\,\frac{\partial f_p^<(\mathbf{p}_p,t)}{\partial t} \tag{2.45}$$

$$= -2\sum\nolimits_a \int \frac{d^\nu p_p}{(2\pi\hbar)^\nu}\,\frac{d^\nu p_a}{(2\pi\hbar)^\nu}\,\frac{d^\nu k}{(2\pi)^\nu}\left(\frac{p_p^2}{2m_p}\right)\frac{\partial}{\partial \mathbf{p}_p}\cdot\mathbf{k}\left|\frac{e_p e_a}{k^2\epsilon(k,\mathbf{k}\cdot\mathbf{v}_a)}\right|^2 \tag{2.46}$$

$$\pi\delta\Big(\mathbf{k}\cdot\mathbf{v}_a - \mathbf{k}\cdot\mathbf{v}_p\Big)\left[\mathbf{k}\cdot\frac{\partial f_a(\mathbf{p}_a)}{\partial \mathbf{p}_a}\,\bar{f}_p(\mathbf{p}_p) - \mathbf{k}\cdot\frac{\partial \bar{f}_p(\mathbf{p}_p)}{\partial \mathbf{p}_p}\,f_a(\mathbf{p}_a)\right]_{\text{PB}}\,, \tag{2.47}$$

where $\bar{f}_p = (2\pi\hbar)^\nu\delta^{(\nu)}(\mathbf{x}-\mathbf{v}_p t)$.

▶ Perform the $p_p$-integration. We integrate by parts twice, removing the derivatives from the distribution function $\bar{f}_p$. The first integration gives,

$$\frac{dE_p^<}{dt} = 2\sum\nolimits_a \int \frac{d^\nu p_p}{(2\pi\hbar)^\nu}\,\frac{d^\nu p_a}{(2\pi\hbar)^\nu}\,\frac{d^\nu k}{(2\pi)^\nu}\,\frac{\partial}{\partial p_p^\ell}\left(\frac{p_p^2}{2m_p}\right)\cdot k^\ell\left|\frac{e_p e_a}{k^2\epsilon(k,\mathbf{k}\cdot\mathbf{v}_a)}\right|^2$$

$$\pi\delta\Big(\mathbf{k}\cdot\mathbf{v}_a - \mathbf{k}\cdot\mathbf{v}_p\Big)\left[\mathbf{k}\cdot\frac{\partial f_a(\mathbf{p}_a)}{\partial \mathbf{p}_a}\,\bar{f}_p(\mathbf{p}_p) - \mathbf{k}\cdot\frac{\partial \bar{f}_p(\mathbf{p}_p)}{\partial \mathbf{p}_p}\,f_a(\mathbf{p}_a)\right]_{\text{PB}}\,. \tag{2.48}$$

We next move the differentiation from the second term in square brackets,

$$\frac{dE_p^<}{dt} = 2\sum\nolimits_a \int \frac{d^\nu p_a}{(2\pi\hbar)^\nu}\,\frac{d^\nu k}{(2\pi)^\nu}\,\frac{d^\nu p_p}{(2\pi\hbar)^\nu}\,\bar{f}_p(\mathbf{p}_p)\left[\mathbf{k}\cdot\frac{\partial f_a(\mathbf{p}_a)}{\partial \mathbf{p}_a} + f_a(\mathbf{p}_a)\,\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_p}\right]_{\text{PB}}\frac{\partial}{\partial p_p^\ell}\left(\frac{p_p^2}{2m_p}\right)\cdot$$

$$k^\ell\left|\frac{e_p e_a}{k^2\epsilon(k,\mathbf{k}\cdot\mathbf{v}_a)}\right|^2\pi\delta\Big(\mathbf{k}\cdot\mathbf{v}_a - \mathbf{k}\cdot\mathbf{v}_p\Big) \tag{2.49}$$

or

$$\boxed{\begin{aligned}\frac{dE_p^<}{dt} &= 2\sum\nolimits_a \int \frac{d^\nu p_a}{(2\pi\hbar)^\nu}\,\frac{d^\nu k}{(2\pi)^\nu}\left[\mathbf{k}\cdot\frac{\partial f_a(\mathbf{p}_a)}{\partial \mathbf{p}_a} + f_a(\mathbf{p}_a)\,\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_p}\right]_{\text{PB}}\frac{\partial}{\partial p_p^\ell}\left(\frac{p_p^2}{2m_p}\right)\cdot \\ &\quad k^\ell\left|\frac{e_p e_a}{k^2\epsilon(k,\mathbf{k}\cdot\mathbf{v}_p)}\right|^2\pi\delta\Big(\mathbf{k}\cdot\mathbf{v}_a - \mathbf{k}\cdot\mathbf{v}_p\Big)\end{aligned}} \tag{2.50}$$

Where we have made the substitution $\mathbf{k}\cdot\mathbf{v}_a \to \mathbf{k}\cdot\mathbf{v}_p$ in the dielectric function because of the delta-function. As before, we write

$$\frac{dE_p^<}{dx} = \frac{dE_p^{e\,<}}{dx} + \frac{dE_p^{\text{I}\,<}}{dx}\,, \tag{2.51}$$

where the ion contribution becomes

$$\frac{dE_p^{\mathrm{I}\,<}}{dt} = 2\sum\nolimits_i \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu}\,\frac{d^\nu k}{(2\pi)^\nu}\,\left[\mathbf{k}\cdot\frac{\partial f_i(\mathbf{p}_i)}{\partial \mathbf{p}_i} + f_i(\mathbf{p}_i)\,\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_p}\right]\frac{\partial}{\partial p_p^\ell}\left(\frac{p_p^2}{2m_p}\right)\cdot$$
$$k^\ell\left|\frac{e_p e_i}{k^2\epsilon(k,\mathbf{k}\cdot\mathbf{v}_p)}\right|^2 \pi\delta\Big(\mathbf{k}\cdot\mathbf{v}_i - \mathbf{k}\cdot\mathbf{v}_p\Big)\ , \tag{2.52}$$

and the electron contribution is

$$\frac{dE_p^{e\,<}}{dt} = 2\int \frac{d^\nu p_e}{(2\pi\hbar)^\nu}\,\frac{d^\nu k}{(2\pi)^\nu}\,\left[\mathbf{k}\cdot\frac{\partial f_e(\mathbf{p}_e)}{\partial \mathbf{p}_e} + f_e(\mathbf{p}_e)\Big[1 - f_e(\mathbf{p}_e)\Big]\,\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_p}\right]\frac{\partial}{\partial p_p^\ell}\left(\frac{p_p^2}{2m_p}\right)\cdot$$
$$k^\ell\left|\frac{e_p e_e}{k^2\epsilon(k,\mathbf{k}\cdot\mathbf{v}_p)}\right|^2 \pi\delta\Big(\mathbf{k}\cdot\mathbf{v}_e - \mathbf{k}\cdot\mathbf{v}_p\Big)\ . \tag{2.53}$$

Finally, the stopping power to leading and next-to-leading order (in three dimensions) is then given by

$$\frac{dE_p^{\mathrm{BPS}}}{dx} = \lim_{\nu\to 3}\left[\frac{dE_p^{>}}{dx} + \frac{dE_p^{<}}{dx}\right]\ . \tag{2.54}$$

## III. CALCULATIONAL DETAILS

### A. Ion Temperature Equilibration

This was calculated in Ref. [1].

#### 1. BE

$$\frac{d\mathcal{E}_{\rm eI}^{>}}{dt} = 2\sum\nolimits_i \int \frac{d^\nu p_e}{(2\pi\hbar)^\nu} \frac{d^\nu p_e'}{(2\pi\hbar)^\nu} \frac{d^\nu p_i}{(2\pi\hbar)^\nu} \left|T_{ei}\right|^2 (2\pi\hbar)\, \delta\left(\frac{\mathbf{p}_i\cdot\mathbf{q}}{m_i} - \frac{\bar{\mathbf{p}}\cdot\mathbf{q}}{m_e} - \frac{q^2}{2m_i}\right) \frac{\bar{\mathbf{p}}\cdot\mathbf{q}}{m_e}\, f_e(\mathbf{p}_e) f_i(\mathbf{p}_i)\Big[1 - f_e(\mathbf{p}_e')\Big] \,. \tag{3.1}$$

► Perform the $\mathbf{p}_i$-integration using (E1), in which we integrate perpendicular and parallel to to $\mathbf{q}$. See Ref. [2] in (7.13),

$$\int \frac{d^\nu p_i}{(2\pi\hbar)^\nu}\, f_i(\mathbf{p}_i)\, (2\pi\hbar)\delta\left(\frac{\mathbf{p}_i\cdot\mathbf{q}}{m_i} - \frac{\bar{\mathbf{p}}\cdot\mathbf{q}}{m_e} - \frac{q^2}{2m_i}\right) \tag{3.2}$$

$$= \frac{n_i\lambda_i\, m_i}{q} \exp\left\{-\frac{\beta_i}{2m_i q^2}\, \left(\frac{m_i}{m_e}\,\bar{\mathbf{p}}\cdot\mathbf{q} + \frac{q^2}{2}\right)^2\right\}$$

#### 2. LBE

$$\frac{d\mathcal{E}_{\rm eI}^{<}}{dt} = -2\sum\nolimits_i \int \frac{d^\nu p_e}{(2\pi\hbar)^\nu} \frac{d^\nu p_i}{(2\pi\hbar)^\nu} \frac{d^\nu k}{(2\pi)^\nu} \left(\frac{p_e^2}{2m_e}\right) \frac{\partial}{\partial\mathbf{p}_e}\cdot\mathbf{k} \left|\frac{e e_i}{k^2\epsilon(k,\mathbf{k}\cdot\mathbf{v}_i)}\right|^2 \pi\delta\Big(\mathbf{k}\cdot\mathbf{v}_e - \mathbf{k}\cdot\mathbf{v}_i\Big) \left[\mathbf{k}\cdot\frac{\partial f_i(\mathbf{p}_i)}{\partial\mathbf{p}_i}\, f_e(\mathbf{p}_e)\big[1 - f_e(\mathbf{p}_e)\big] - \mathbf{k}\cdot\frac{\partial f_e(\mathbf{p}_e)}{\partial\mathbf{p}_e}\, f_i(\mathbf{p}_i)\right] \,. \tag{3.3}$$

## B. Ion Stopping Power

### 1. BE

Recall that the BE gives the rate (2.29):

$$\frac{dE_p^>}{dt} = 2\sum\nolimits_a \int \frac{d^\nu p_p'}{(2\pi\hbar)^\nu}\, \frac{d^\nu p_a}{(2\pi\hbar)^\nu}\, \frac{d^\nu p_a'}{(2\pi\hbar)^\nu}\; \big|T_{pb}\big|^2 \left(\frac{p_p'^{\,2}}{2m_p} - \frac{p_p^2}{2m_p}\right) f_a(\mathbf{p}_a)\Big|_{\rm PB} \tag{3.4}$$
$$(2\pi\hbar)^\nu\, \delta^{(\nu)}\Big(\mathbf{p}_p' + \mathbf{p}_a' - \mathbf{p}_p - \mathbf{p}_a\Big)\, (2\pi\hbar)\, \delta\Big(E_p' + E_a' - E_p - E_a\Big)\ ,$$

or the ion and electron contributions are

$$\boxed{\begin{aligned}\frac{dE_p^{\rm I\,>}}{dt} = 2\sum\nolimits_i \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu}\, \frac{d^\nu p_p'}{(2\pi\hbar)^\nu}\, \frac{d^\nu p_i'}{(2\pi\hbar)^\nu}\; \big|T_{pb}\big|^2 \left(\frac{p_p'^{\,2}}{2m_p} - \frac{p_p^2}{2m_p}\right) f_i(\mathbf{p}_i) \\ (2\pi\hbar)^\nu\, \delta^{(\nu)}\Big(\mathbf{p}_p' + \mathbf{p}_i' - \mathbf{p}_p - \mathbf{p}_i\Big)\, (2\pi\hbar)\, \delta\Big(E_p' + E_i' - E_p - E_i\Big)\ ,\end{aligned}} \tag{3.5}$$

and

$$\boxed{\begin{aligned}\frac{dE_p^{e\,>}}{dt} = 2 \int \frac{d^\nu p_e}{(2\pi\hbar)^\nu}\, \frac{d^\nu p_p'}{(2\pi\hbar)^\nu}\, \frac{d^\nu p_e'}{(2\pi\hbar)^\nu}\; \big|T_{pb}\big|^2 \left(\frac{p_p'^{\,2}}{2m_p} - \frac{p_p^2}{2m_p}\right) f_e(\mathbf{p}_e)\Big[1 - f_e(\mathbf{p}_e')\Big] \\ (2\pi\hbar)^\nu\, \delta^{(\nu)}\Big(\mathbf{p}_p' + \mathbf{p}_e' - \mathbf{p}_p - \mathbf{p}_e\Big)\, (2\pi\hbar)\, \delta\Big(E_p' + E_e' - E_p - E_e\Big)\ .\end{aligned}} \tag{3.6}$$

► We perform the $p_p'$-integral, employing the momentum conserving delta-function, to find

$$\frac{dE_p^>}{dt} = 2\sum\nolimits_a \int \frac{d^\nu p_a}{(2\pi\hbar)^\nu}\, \frac{d^\nu p_a'}{(2\pi\hbar)^\nu}\; \big|T_{pb}\big|^2 \tag{3.7}$$
$$(2\pi\hbar)\, \delta\Big(E_p' + E_a' - E_p - E_a\Big) \left(\frac{p_p'^{\,2}}{2m_p} - \frac{p_p^2}{2m_p}\right) f_a(\mathbf{p}_a)\Big|^{\rm PB}_{\mathbf{p}_p' = \mathbf{p}_p + \mathbf{p}_a - \mathbf{p}_a'}\ .$$

**Or**: it might be better to change variables to $\bar{p}$ and $q$.

*2. LBE*

From (2.50):

$$\frac{dE_p^<}{dt} = 2\sum\nolimits_a \int \frac{d^\nu p_a}{(2\pi\hbar)^\nu} \frac{d^\nu k}{(2\pi)^\nu} \left[ \mathbf{k}\cdot \frac{\partial f_a(\mathbf{p}_a)}{\partial \mathbf{p}_a} + f_a(\mathbf{p}_a)\, \mathbf{k}\cdot \frac{\partial}{\partial \mathbf{p}_p}\right]_{\text{PB}} \frac{\partial}{\partial p_p^\ell} \left(\frac{p_p^2}{2m_p}\right) \cdot \\ k^\ell \left| \frac{e_p e_a}{k^2 \epsilon(k, \mathbf{k}\cdot\mathbf{v}_p)}\right|^2 \pi\delta\Big(\mathbf{k}\cdot\mathbf{v}_a - \mathbf{k}\cdot\mathbf{v}_p\Big) \tag{3.8}$$

or in terms of the ion and electron contribution

$$\boxed{\frac{dE_p^{\text{I}\,<}}{dt} = 2\sum\nolimits_i \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu} \frac{d^\nu k}{(2\pi)^\nu} \left[ \mathbf{k}\cdot \frac{\partial f_i(\mathbf{p}_i)}{\partial \mathbf{p}_i} + f_i(\mathbf{p}_i)\, \mathbf{k}\cdot \frac{\partial}{\partial \mathbf{p}_p}\right] \frac{\partial}{\partial p_p^\ell} \left(\frac{p_p^2}{2m_p}\right) \cdot \\ k^\ell \left| \frac{e_p e_i}{k^2 \epsilon(k, \mathbf{k}\cdot\mathbf{v}_p)}\right|^2 \pi\delta\Big(\mathbf{k}\cdot\mathbf{v}_i - \mathbf{k}\cdot\mathbf{v}_p\Big)\ , \qquad (3.9)}$$

and

$$\boxed{\frac{dE_p^{e\,<}}{dt} = 2 \int \frac{d^\nu p_e}{(2\pi\hbar)^\nu} \frac{d^\nu k}{(2\pi)^\nu} \left[ \mathbf{k}\cdot \frac{\partial f_e(\mathbf{p}_e)}{\partial \mathbf{p}_e} + f_e(\mathbf{p}_e)\Big[1 - f_e(\mathbf{p}_e)\Big]\, \mathbf{k}\cdot \frac{\partial}{\partial \mathbf{p}_p}\right] \frac{\partial}{\partial p_p^\ell} \left(\frac{p_p^2}{2m_p}\right) \cdot \\ k^\ell \left| \frac{e_p e_e}{k^2 \epsilon(k, \mathbf{k}\cdot\mathbf{v}_p)}\right|^2 \pi\delta\Big(\mathbf{k}\cdot\mathbf{v}_e - \mathbf{k}\cdot\mathbf{v}_p\Big)\ . \qquad (3.10)}$$

► Do the $d^\nu p_i$ integral and keep the other variables fixed. Use $f_i = z_i \exp\{-\beta_i p_i^2/2m_i\}$, and $\partial f_i/\partial \mathbf{p}_i = \beta_i \mathbf{v}_i\, f_i$, to write

$$\frac{dE_p^{\text{I}\,<}}{dt} = 2\sum\nolimits_i \int \frac{d^\nu k}{(2\pi)^\nu} \frac{d^\nu p_i}{(2\pi\hbar)^\nu} \overbrace{\left[\beta_i\, \mathbf{k}\cdot\mathbf{v}_i\, f_i(\mathbf{p}_i) - f_i(\mathbf{p}_i)\, \mathbf{k}\cdot \frac{\partial}{\partial \mathbf{p}_p}\right]}^{f_i(\mathbf{p}_i)\left[\beta_i\, \mathbf{k}\cdot\mathbf{v}_i - \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_p}\right]} \frac{\partial}{\partial p_p^\ell} \left(\frac{p_p^2}{2m_p}\right) \cdot \\ k^\ell \left| \frac{e_p e_i}{k^2 \epsilon(k, \mathbf{k}\cdot\mathbf{v}_i)}\right|^2 \pi\delta\Big(\mathbf{k}\cdot\mathbf{v}_i - \mathbf{k}\cdot\mathbf{v}_p\Big)\ , \tag{3.11}$$

and using the delta-function to make the replacements $\mathbf{k}\cdot\mathbf{v}_i \to \mathbf{k}\cdot\mathbf{v}_p$ in the integrand, we find

$$\frac{dE_{p\,\text{I}}^<}{dt} = 2\sum\nolimits_i \int \frac{d^\nu k}{(2\pi)^\nu} \frac{d^\nu p_i}{(2\pi\hbar)^\nu}\, f_i(\mathbf{p}_i) \left[\beta_i\, \mathbf{k}\cdot\mathbf{v}_p - \mathbf{k}\cdot \frac{\partial}{\partial \mathbf{p}_p}\right] \frac{\partial}{\partial p_p^\ell} \left(\frac{p_p^2}{2m_p}\right) \cdot \\ k^\ell \left| \frac{e_p e_i}{k^2 \epsilon(k, \mathbf{k}\cdot\mathbf{v}_p)}\right|^2 \pi\delta\Big(\mathbf{k}\cdot\mathbf{v}_i - \mathbf{k}\cdot\mathbf{v}_p\Big)\ . \tag{3.12}$$

The $p_p$-derivatives act on everything to their right.

▶ Perform the $p_i$-integration. We perform the Gaussian integrals $d^{\nu-1}p_\perp$, and integrate $dp_\parallel = m_i\, dv_\parallel$ over the $\delta$-function. The subscript $i$ has been suppressed in the expressions on the right-hand-side. To perform the integral, we decomposed $\mathbf{v}_i$ along the direction defined by $\mathbf{k}$,

$$\mathbf{v}_i = v_\parallel \hat{\mathbf{k}} + \mathbf{v}_\perp \ , \tag{3.13}$$

and we write the $\delta$-function as

$$\delta\Big(\hat{\mathbf{k}} \cdot \mathbf{v}_i - \omega_p/k\Big) = \delta\Big(v_\parallel - v_p \cos\theta\Big) \ . \tag{3.14}$$

The notes algebra_long_distance_1.0.tex does this calculation in detail for a MB plasma, which for the ions becomes

$$\begin{aligned} \frac{dE^<_{p\,\mathrm{I}}}{dt} \;&=\; 2{\sum}_i e_p^2 \int \frac{d^\nu k}{(2\pi)^\nu} \left[\beta_i\, \mathbf{k}\cdot\mathbf{v}_p - \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_p}\right] \frac{\partial}{\partial p_p^\ell}\, \frac{p_p^2}{2m_p} \\ &\qquad \frac{\pi k^\ell}{\left|k^2 \epsilon(k, \mathbf{k}\cdot\mathbf{v}_p)\right|^2}\, \exp\left\{-\frac{1}{2}\,\beta_i m_i v_p^2 \cos^2\theta\right\} \ . \end{aligned} \tag{3.15}$$

and upon dividing by the projectile velocity (or is it $\hat{k}^\ell$?),

$$\boxed{\begin{aligned} \frac{dE^<_{p\,\mathrm{I}}}{dx} \;&=\; 2{\sum}_i \frac{e_p^2}{v_p} \int \frac{d^\nu k}{(2\pi)^\nu} \left[\beta_i\, \mathbf{k}\cdot\mathbf{v}_p - \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_p}\right] \frac{\partial}{\partial p_p^\ell}\, \frac{p_p^2}{2m_p} \\ &\qquad \frac{\pi k^\ell}{\left|k^2 \epsilon(k, \mathbf{k}\cdot\mathbf{v}_p)\right|^2}\, \exp\left\{-\frac{1}{2}\,\beta_i m_i v_p^2 \cos^2\theta\right\} \ . \end{aligned}} \tag{3.16}$$

### Appendix A: Non-degenerate BPS Results

The work of BPS in Ref. [1] assumed Maxwell-Boltmann electrons, and broken the calculation into classical and quantum components, with the quantum being considered a correction to the classical.[2] This section provides some of the salient results.

#### 1. Classical

In § 3.1 of Ref. [1], the classical stopping power is broken into short distance and long distance contributions,

$$\frac{dE_a^{\rm C}}{dx} \;=\; \frac{dE_{b,{\rm S}}^{>}}{dx} + \frac{dE_{b,{\rm R}}^{<}}{dx} \;, \tag{A1}$$

where

$$\begin{aligned}\frac{dE_{b,{\rm S}}^{>}}{dx} \;&=\; \frac{e_p^2}{4\pi}\,\frac{\kappa_a^2}{m_p v_p}\left(\frac{m_a}{2\pi\beta_a}\right)^{1/2}\int_0^1 du\, u^{1/2}\exp\left\{-\frac{1}{2}\,\beta_a m_a v_p^2\right\} \\ &\left\{\left[-\ln\left(\beta_a\frac{e_p e_a K}{4\pi}\,\frac{m_a}{m_{pa}}\,\frac{u}{1-u}\right)+2-2\gamma\right]\left[\beta_a M_{pa} v_p^2-\frac{1}{u}\right]+\frac{2}{u}\right\}\end{aligned} \tag{A2}$$

$$\begin{aligned}\frac{dE_{b,{\rm R}}^{<}}{dx} \;&=\; \frac{e_p^2}{4\pi}\,\frac{i}{2\pi}\int_{-1}^{1} d\cos\theta\,\frac{\rho_a(v_p\cos\theta)}{\rho_{\rm tot}(v_p\cos\theta)}\,F(v_p\cos\theta)\ln\left(\frac{F(v_p\cos\theta)}{K^2}\right) \\ &-\frac{e_p^2}{4\pi}\,,\frac{i}{2\pi}\,\frac{1}{\beta_a m_a v_p^2}\,\frac{\rho_a(v_p\cos\theta)}{\rho_{\rm tot}(v_p\cos\theta)}\left[F(v_p)\ln\left(\frac{F(v_p)}{K^2}\right)-F^*(v_p)\ln\left(\frac{F^*(v_p)}{K^2}\right)\right]\;.\end{aligned} \tag{A3}$$

Reference [1] uses the notation $dE_{b,{\rm S}}^{\rm C}/dx$ instead of $dE_{b,{\rm S}}^{<}/dx$, and $e_p$ and $e_a$ are positive absolute values of the charges. We also have

$$\kappa_a^2 \;=\; \beta_a e_a^2 n_a \tag{A4}$$

$$F(u) \;=\; \int_{-\infty}^{\infty} du\,\frac{\rho_{\rm B}(v)}{v-u-i\eta} \;, \tag{A5}$$

where

$$\rho_a(v) \;=\; \kappa_a^2\, v\left(\frac{\beta_a m_a}{2\pi}\right)^{1/2}\exp\left\{-\frac{1}{2}\,\beta_a m_a v^2\right\} \tag{A6}$$

$$\rho_{\rm B}(v) \;=\; \sum\nolimits_a \rho_a(v)\;. \tag{A7}$$

[2] I think it's better to think in terms of the quantum regime, and we should then express the stopping power in terms of a classical "correction" to the quantum stopping power. This is the opposite of how most people write the stopping power, which is classical plus quantum correction.

Also, we define the mass combinations

$$M_{pa} = m_p + m_a \tag{A8}$$

$$\frac{1}{m_{pa}} = \frac{1}{m_p} + \frac{1}{m_a} ~. \tag{A9}$$

### 2. Quantum Corrections

In § 3.2 of Ref. [1], the stopping power is written as the classical piece plus a quantum correction,

$$\frac{dE_a}{dx} = \frac{dE_a^{\rm C}}{dx} + \frac{dE_a^{\rm Q}}{dx} ~, \tag{A10}$$

where

$$\begin{aligned} \frac{dE_a^{\rm Q}}{dx} = &\frac{e_p^2}{4\pi}\,\frac{\kappa_a^2}{2\beta_a m_p v_p^2}\left(\frac{\beta_a m_a}{2\pi}\right)^{1/2}\int_0^\infty dv_{pa}\left[2{\rm Re}\psi(1+i\eta_{pa}) - \ln\eta_{pa}^2\right] \\ &\left\{\left[1 + \frac{M_{pa}}{m_a}\,\frac{v_p}{v_{pa}}\left(\frac{1}{\beta_a m_a v_p v_{pa}} - 1\right)\right]\exp\left\{-\frac{1}{2}\,\beta_a m_a\left(v_p - v_{pa}\right)^2\right\} ~- \right. \\ &\left.\left[1 - \frac{M_{pa}}{m_a}\,\frac{v_p}{v_{pa}}\left(\frac{1}{\beta_a m_a v_p v_{pa}} + 1\right)\right]\exp\left\{-\frac{1}{2}\,\beta_a m_a\left(v_p + v_{pa}\right)^2\right\}\right\}, \end{aligned} \tag{A11}$$

with

$$\eta_{pa} = \frac{e_p e_a}{4\pi\hbar v_{pa}} \tag{A12}$$

$$v_{pa} = |\mathbf{v}_p - \mathbf{v}_a| \tag{A13}$$

$${\rm Re}\psi(1+i\eta) = \sum_{k=1}^\infty \frac{1}{k}\,\frac{\eta^2}{k^2+\eta^2} - \gamma ~. \tag{A14}$$

The quantum correction involves only short-distance physics, while the classical contribution captures both short- and long-distance physics.

### 3. Extreme Quantum Limit

The extreme quantum limit, or the Born approximation, is given by $\eta \ll 1$, which from § 3.4 of BPS is given by

$$\frac{dE_a^{\rm Born}}{dx} = \frac{dE_{b,{\rm B}}^{>}}{dx} + \frac{dE_{b,{\rm R}}^{<}}{dx} ~, \tag{A15}$$

where $dE^<_{b,\mathrm{R}}/dx$ is unchanged from the last section, and the Born approximation gives

$$\frac{dE^>_{b,\mathrm{B}}}{dx} = \frac{e_p^2}{4\pi}\,\frac{\kappa_a^2}{m_p v_p}\left(\frac{m_a}{2\pi\beta_a}\right)^{1/2}\int_0^\infty du\,\exp\left\{-\frac{1}{2}\,\beta_a m_a v_p^2\,u\right\} \tag{A16}$$
$$\left\{\left[-\frac{1}{2}\ln\left(\beta_a\hbar^2 K^2\,\frac{m_a}{m_{pa}^2}\,\frac{u}{1-u}\right)+1-\frac{\gamma}{2}\right]\left[\beta_a M_{pa} v_p^2\,u^{1/2}-u^{-1/2}\right]+u^{-1/2}\right\}$$

**Appendix B: Carlson's Theorem**

One of the most fundamental mathematical underpinnings of the BPS calculation is Carlson's Theorem, which roughly states that if two functions agree on the natural numbers, and their difference is not exponentially large, then the functions are the same. This is the means by which dimensional continuation is defined, and the $\nu \to 3$ limit is to be understood. The following is from Wikipedia. Give generously.

**Theorem 1** (Carlson's Theorem) *Suppose $f(z)$ is an analytic function on $\mathbb{C}$ satisfying the following three conditions:*

*(1) $f$ is an* entire *function of* exponential *type, meaning*

$$\Big|f(z)\Big| \le C\,e^{\tau|z|} \quad \text{for} \quad \mathrm{z} \in \mathbb{C}$$

*for some $C, \tau \in \mathbb{R}^+$.*

*(2) There exists $c < \pi$ such that*

$$\Big|f(iy)\Big| \le C\,e^{c|y|} \quad \text{for} \quad \mathrm{y} \in \mathbb{R}$$

*(3) $f(n) = 0$ for all $n \in \mathbb{N}$*

*Then the function vanishes,* i.e. $f = 0$.

Note: The function $f(z) = \sin\pi z$ satisfies (3) and (1), but note (2); therefore, it does not violate Carlson's Theorem. This is why condition (2) is necessary. The theorem follows from the Phragmen-Lindelof theorem, which itself follows from the maximum-modulus theorem. See Wikipedia for details.

We can use Carlson's Theorem to prove that the gamma function $\Gamma(z)$ defined by (1.14) is the unique analytic continuation to the complex plane of the factorial function $n!$ on the natural numbers $\mathbb{N}$. Suppose $\tilde\Gamma(z)$ is another such function on $\mathbb{C}$ that takes the values $n!$ on $\mathbb{N}$. Then $\Gamma(z) - \tilde\Gamma(z)$ vanishes on the natural numbers (and it can be shown that the function difference doesn't diverge more than exponentially); therefore, $\Gamma(z) - \tilde\Gamma(z) = 0$, *i.e.* $\tilde\Gamma(z) = \Gamma(z)$, *i.e.* $\Gamma(z)$ as defined by (1.14) is the unique analytic continuation of the factorial function. Note: $\Gamma(n+1) = n!$. Note: It seems that (2) is a restriction on the Riemann sheet.

## Appendix C: Kinetic Equations

### 1. The Fokker-Planck Equation from the Boltzmann Equation

Taken from Chapter 11, § 21 p 89 of Ref. [3]. Let $w(\mathbf{p},\mathbf{q})d^3q$ denote the probability of scattering from $\mathbf{p}\to\mathbf{p}-\mathbf{q}$, so that the distribution function $f$ satisfies

$$\frac{\partial f(\mathbf{p},t)}{\partial t} = \int d^3q\Big[w(\mathbf{p}+\mathbf{q},\mathbf{q})f(\mathbf{p}+\mathbf{q},t) - w(\mathbf{p},\mathbf{q})f(\mathbf{p},t)\Big] \ . \tag{C1}$$

We assume that small angle collisions are dominant, *i.e.* collisions change the momentum only on a much smaller scale than the value of the momentum itself. This means that $w(\mathbf{p},\mathbf{q})$ is a sharply peaked function of small $\mathbf{p}$, and we can expand

$$\begin{aligned} w(\mathbf{p}+\mathbf{q},\mathbf{q})f(\mathbf{p}+\mathbf{q},t) = w(\mathbf{p},\mathbf{q})f(\mathbf{p},t) \ + \ \mathbf{q}\cdot\frac{\partial}{\partial\mathbf{p}}\,w(\mathbf{p},\mathbf{q})f(\mathbf{p},t) \ + \\ \frac{1}{2}\,q^\ell q^m\,\frac{\partial^2}{\partial p^\ell\partial p^m}\,w(\mathbf{p},\mathbf{q})f(\mathbf{p},t) + \mathcal{O}(q^3) \ . \end{aligned} \tag{C2}$$

The transport equation then becomes

$$\frac{\partial f(\mathbf{p},t)}{\partial t} = \frac{\partial}{\partial p^\ell}\left[B^\ell\, f + \frac{\partial}{\partial p^m}\Big(C^{\ell m} f\Big)\right] \ , \tag{C3}$$

where

$$B^\ell(\mathbf{p}) \ \equiv \ \int d^3q\, q^\ell w(\mathbf{p},\mathbf{q}) \tag{C4}$$

$$C^{\ell m}(\mathbf{p}) \ \equiv \ \frac{1}{2}\int d^3q\, q^\ell q^m w(\mathbf{p},\mathbf{q}) \ . \tag{C5}$$

By writing

$$A^\ell \equiv \ B^\ell + \frac{\partial C^{\ell m}}{\partial p^m} \ , \tag{C6}$$

we arrive at the form

$$\frac{\partial f(\mathbf{p},t)}{\partial t} = \frac{\partial}{\partial p^\ell}\left[A^\ell\, f + C^{\ell m}\,\frac{\partial f}{\partial p^m}\right] \ . \tag{C7}$$

The time derivative vanishes in equilibrium, when $f = f_0$:

$$f_0(\mathbf{p}) = z\, e^{-\beta p^2/2m} \ , \tag{C8}$$

where $\beta$ is the inverse temperature, $m$ the mass of the particle described by the distribution $f$, and $z = n\lambda^3$ is the fugacity, with $n$ being the number density and $\lambda$ the de Broglie wave length. Then

$$\frac{\partial f_0}{\partial p^m} = -\frac{\beta p^m}{m}\, f_0 = -\beta v^m\, f_0 \ , \tag{C9}$$

and therefore

$$0 = A^\ell\, f_0 + C^{\ell m}\, \frac{\partial f_0}{\partial p^m} = \Big[A^\ell - \beta v^m\, C^{\ell m}\Big] f_0 \quad \Rightarrow \quad A^\ell = \beta v^m\, C^{\ell m} \ . \tag{C10}$$

Thus, Eq. (C7) becomes

$$\frac{\partial f(\mathbf{p},t)}{\partial t} = \frac{\partial}{\partial p^\ell}\, C^{\ell m} \left[\frac{\beta p^m}{m} + \frac{\partial}{\partial p^m}\right] f \ , \tag{C11}$$

which is the Fokker-Planck equation (FPE).

### 2. The Lenard-Balescu Equation from the Boltzmann Equation

This is an exposition of Appendix C of BPS [1]. We start with the Boltzmann scattering kernel (B.1):

$$C_{ab}(\mathbf{p}_a) = \int \frac{d^\nu p_b'}{(2\pi\hbar)^\nu}\, \frac{d^\nu p_a'}{(2\pi\hbar)^\nu}\, \frac{d^\nu p_b}{(2\pi\hbar)^\nu}\, |T_{ab}|^2\, (2\pi\hbar)^\nu \delta^{(\nu)}\big(\mathbf{p}_b' + \mathbf{p}_a' - \mathbf{p}_b - \mathbf{p}_a\big) \tag{C12}$$
$$(2\pi\hbar)\delta(E_b' + E_a' - E_b - E_a)\Big[f_b(\mathbf{p}_b')f_a(\mathbf{p}_a') - f_b(\mathbf{p}_b)f_a(\mathbf{p}_a)\Big] \ .$$

We define the average momentum $\bar{\mathbf{p}}_b$ and the momentum exchange $\mathbf{q}$ by

$$\bar{\mathbf{p}}_b \ = \ \frac{1}{2}\Big[\mathbf{p}_b + \mathbf{p}_b'\Big] \tag{C13}$$

$$\mathbf{q} \ = \ \mathbf{p}_b - \mathbf{p}_b' \tag{C14}$$

When we impose momentum conservation, then we can also write $\mathbf{q} = \mathbf{p}_a' - \mathbf{p}_a$, which is the reason we do not put a subscript on the momentum transfer. We will (i) change variables from $\mathbf{p}_b'$-$\mathbf{p}_b$ to $\bar{\mathbf{p}}_b$-$\mathbf{q}$ (the Jacobian is unity),

$$\mathbf{p}_b \ = \ \bar{\mathbf{p}}_b + \frac{1}{2}\,\mathbf{q} \tag{C15}$$

$$\mathbf{p}_b' \ = \ \bar{\mathbf{p}}_b - \frac{1}{2}\,\mathbf{q} \ , \tag{C16}$$

and (ii) integrate the momentum delta-function over $\mathbf{p}_a'$, *i.e.* make the substitution

$$\mathbf{p}_a' \ = \ \mathbf{p}_a + \mathbf{p}_b - \mathbf{p}_b' = \mathbf{p}_a + \mathbf{q} \ . \tag{C17}$$

Expression (C12) then becomes

$$C_{ab}(\mathbf{p}_a) = \int \frac{d^\nu \bar{p}_b}{(2\pi\hbar)^\nu}\, \frac{d^\nu q}{(2\pi\hbar)^\nu}\, |T_{ab}|^2\, (2\pi\hbar)\delta\bigg(\frac{\mathbf{p}_a\cdot\mathbf{q}}{m_a} - \frac{\bar{\mathbf{p}}_b\cdot\mathbf{q}}{m_b} + \frac{q^2}{2m_a}\bigg) \tag{C18}$$
$$\bigg[f_b\left(\bar{\mathbf{p}}_b - \frac{1}{2}\,\mathbf{q}\right) f_a(\mathbf{p}_a + \mathbf{q}) - f_b\left(\bar{\mathbf{p}}_b + \frac{1}{2}\,\mathbf{q}\right) f_a(\mathbf{p}_a)\bigg] \ .$$

We now expand (C18) in powers of the momentum exchange, working to second order in $\mathbf{q}$. The distribution functions in square brackets become

$$f_b\left(\bar{\mathbf{p}}_b - \frac{1}{2}\,\mathbf{q}\right)\cdot f_a(\mathbf{p}_a+\mathbf{q}) - f_b\left(\bar{\mathbf{p}}_b + \frac{1}{2}\,\mathbf{q}\right)\cdot f_a(\mathbf{p}_a) \tag{C19}$$

$$= \left[1+\frac{1}{2}\,\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{p}_a}\right]\left[\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{p}_a} - \mathbf{q}\cdot\frac{\partial}{\partial\bar{\mathbf{p}}_b}\right] f_a(\mathbf{p}_a) f_b(\bar{\mathbf{p}}_b) + \mathcal{O}(q^3)\ , \tag{C20}$$

and the delta-function becomes

$$\delta\!\left(\frac{\mathbf{p}_a\cdot\mathbf{q}}{m_a} - \frac{\bar{\mathbf{p}}_b\cdot\mathbf{q}}{m_b} + \frac{q^2}{2m_a}\right) = \left[1+\frac{1}{2}\,\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{p}_a}\right]\delta\!\left(\frac{\mathbf{p}_a\cdot\mathbf{q}}{m_a} - \frac{\bar{\mathbf{p}}_b\cdot\mathbf{q}}{m_b}\right)\ . \tag{C21}$$

For a Galilean invariant theory, the scattering matrix is a function of the square of the momentum exchange and the center-of-mass energy, $T = T(q^2, W)$, where

$$W \;=\; \frac{1}{2}\,m_{ab}\,(\mathbf{v}_a - \mathbf{v}_b)^2 = \frac{1}{2}\,m_{ab}\,(\mathbf{v}'_a - \mathbf{v}'_b)^2 \tag{C22}$$

$$= \frac{1}{2}\,m_{ab}\left(\mathbf{v}_a - \bar{\mathbf{v}}_b - \frac{\mathbf{q}}{2m_b}\right)^2\ . \tag{C23}$$

The first order term in $\mathbf{q}$ reads

$$-\frac{1}{2m_b}\,\mathbf{q}\cdot\frac{\partial W}{\partial\mathbf{v}_a} = -\frac{m_{ab}}{2m_b}\Big(\mathbf{v}_a\cdot\mathbf{q} - \mathbf{v}_b\cdot\mathbf{q}\Big) + \mathcal{O}(q^2)\ ; \tag{C24}$$

however, this term does not contribute because of the delta-function, and we can replace $W$ by

$$\bar{W} \;=\; \frac{1}{2}\,m_{ab}\,(\mathbf{v}_a - \bar{\mathbf{v}}_b)^2\ . \tag{C25}$$

When considering the Lenard-Balescu limit and plasma screening, the background plasma breaks Galilean invariance. Rotational invariance is still a good symmetry. Recall that $\mathbf{q} = \hbar\mathbf{k}$:

$$T_{ab}(\bar{W}, k^2, (\mathbf{v}_a + \mathbf{q}/2m_a)\cdot\mathbf{k}) \;=\; \left[1+\frac{1}{2}\,\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{p}_a}\right] T(\bar{W}, k^2, \mathbf{v}_a\cdot\mathbf{k}) + \mathcal{O}(q^2)\ . \tag{C26}$$

Summary: upon dropping the bar from $\bar{\mathbf{p}}_b$ we have

$$C_{ab}(\mathbf{p}_a) = \int \frac{d^\nu p_b}{(2\pi\hbar)^\nu}\,\frac{d^\nu q}{(2\pi\hbar)^\nu}\,|T_{ab}|^2\,(2\pi\hbar)\delta\!\left(\frac{\mathbf{p}_a\cdot\mathbf{q}}{m_a} - \frac{\mathbf{p}_b\cdot\mathbf{q}}{m_b} + \frac{q^2}{2m_a}\right) \tag{C27}$$

$$\left[f_b\left(\mathbf{p}_b - \frac{1}{2}\,\mathbf{q}\right) f_a(\mathbf{p}_a+\mathbf{q}) - f_b\left(\mathbf{p}_b + \frac{1}{2}\,\mathbf{q}\right) f_a(\mathbf{p}_a)\right]\ ,$$

with

$$\delta\Big(\cdots\Big) \;=\; \left[1 + \frac{1}{2}\,\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{p}_a}\right]\delta\left(\frac{\mathbf{p}_a\cdot\mathbf{q}}{m_a} - \frac{\mathbf{p}_b\cdot\mathbf{q}}{m_b}\right) \tag{C28}$$

$$\Big[\cdots\Big] \;=\; \left[1 + \frac{1}{2}\,\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{p}_a}\right]\left[\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{p}_a} - \mathbf{q}\cdot\frac{\partial}{\partial\mathbf{p}_b}\right] f_a(\mathbf{p}_a) f_b(\mathbf{p}_b) \tag{C29}$$

$$T \;=\; \left[1 + \frac{1}{2}\,\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{p}_a}\right] T(\bar{W}, k^2, \mathbf{v}_a\cdot\mathbf{k})\ . \tag{C30}$$

This gives

$$C_{ab}(\mathbf{p}_a) = -\frac{\partial}{\partial\mathbf{p}_a}\cdot\int\frac{d^\nu p_b}{(2\pi\hbar)^\nu}\,\frac{d^\nu k}{(2\pi)^\nu}\,\mathbf{k}\,|\hbar T_{ab}(W,k^2,\mathbf{v}_a\cdot\mathbf{k})|^2\,\pi\,\delta\big(\mathbf{v}_a\cdot\mathbf{q} - \mathbf{v}_b\cdot\mathbf{q}\big) \tag{C31}$$
$$\left[\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_a} - \mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_b}\right] f_a(\mathbf{p}_a) f_b(\mathbf{p}_b)\ ,$$

which is the LBE when we make the identification

$$|\hbar T_{ab}|^2 = \frac{e_a e_b}{k^2\epsilon(k^2,\mathbf{v}_a\cdot\mathbf{k})}\ . \tag{C32}$$

### 3. Convergent Kinetic Equations

This is an exposition of Appendix B on p. 323 of Ref. [1]. This appendix concentrates on the kinetic equation of Gould and DeWitt (GD) [6], although Refs. [6–9] are also relevant. As in the previous Appendix, we start with the Boltzmann scattering kernel (B.1):

$$C_{ab}(\mathbf{p}_a) = \int\frac{d^\nu p_b'}{(2\pi\hbar)^\nu}\,\frac{d^\nu p_a'}{(2\pi\hbar)^\nu}\,\frac{d^\nu p_b}{(2\pi\hbar)^\nu}\,|T_{ab}|^2\,(2\pi\hbar)^\nu\delta^{(\nu)}\big(\mathbf{p}_b' + \mathbf{p}_a' - \mathbf{p}_b - \mathbf{p}_a\big) \tag{C33}$$
$$(2\pi\hbar)\delta(E_b' + E_a' - E_b - E_a)\Big[f_b(\mathbf{p}_b')f_a(\mathbf{p}_a') - f_b(\mathbf{p}_b)f_a(\mathbf{p}_a)\Big]\ .$$

Working in $\nu = 3$ spatial dimensions, we break the scattering kernel into hard and soft contributions,

$$C_{ab}^{\text{converge}}(\mathbf{p}_a) = C_{ab}^{\text{hard}}(\mathbf{p}_a) + C_{ab}^{\text{soft}}(\mathbf{p}_a)\ , \tag{C34}$$

where the hard and soft collision terms have the generic form of scattering kernel (C34). The first term $C_{ab}^{\text{hard}}$ accounts for Coulomb scattering to all orders, with the first Born approximation subtracted out to avoid double counting, as the Born term is included in $C_{ab}^{\text{soft}}$. The Born approximation agrees exactly with the classical approximation in $\nu = 3$, and this is the reason for the subtraction.

Gould and DeWitt (GD) define the hard scattering amplitude by

$$|T_{ab}^{\text{hard}}|^2 \equiv |T_{\text{D}}|^2 - |T_{\text{D}}^{(1)}|^2 \ , \tag{C35}$$

where $T_{\text{D}}$ is the scattering amplitude for the Debye screened Coulomb potential, and $T_{\text{D}}^{(1)}$ is the corresponding first Born approximation for that potential. The amplitude $T_{\text{D}}$ is asymptotic to the exact scattering amplitude $T$ at large momentum transfer

$$\mathbf{q} = \mathbf{p}_a' - \mathbf{p}_a = \mathbf{p}_b - \mathbf{p}_b' \ , \tag{C36}$$

but $T_{\text{D}}$ misses the correct small-q physics involving screening. Debye screening renders each term in $T^{\text{hard}}$ separately finite at large distances. Furthermore, the subtraction of the two terms in $T^{\text{hard}}$ renders this amplitude finite (at large distances) in the limit $\kappa_{\text{D}} \to 0$. Finally, GD define the soft scattering amplitude to include screening effects,

$$T_{ab}^{\text{soft}} \equiv \hbar \, \frac{e_a e_b}{q^2 \, \epsilon(\mathbf{q}/\hbar, \Delta E/\hbar)} \ , \tag{C37}$$

where

$$\Delta E = \frac{p_a'^2}{2m_a} - \frac{p_a^2}{2m_a} = \frac{p_b^2}{2m_b} - \frac{p_b'^2}{2m_b} \ . \tag{C38}$$

We now have

$$|T_{ab}^{\text{converge}}|^2 = |T_{\text{D}}|^2 - |T_{\text{D}}^{(1)}|^2 + |T_{ab}^{\text{soft}}|^2 \ . \tag{C39}$$

Converting to wave number using $\mathbf{q} = \hbar\mathbf{k}$, and defining $\omega = \Delta E/\hbar$, we can write

$$T_{ab}^{\text{soft}} = \frac{1}{\hbar} \, \frac{e_a e_b}{k^2 \, \epsilon(\mathbf{k}, \omega)} \ . \tag{C40}$$

### 4. The Stopping Power and Other Rates from the Fokker-Planck Equation

The Fokker-Planck equation is given by (4.1) in BPS [p 267]:

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\right] f(\mathbf{x},\mathbf{p},t) = \sum\nolimits_a \frac{\partial}{\partial p^\ell}\, C_a^{\ell m}(\mathbf{x},\mathbf{p},t) \left[\beta_a v^m + \frac{\partial}{\partial p^m}\right] f(\mathbf{x},\mathbf{p},t) \ . \tag{C41}$$

Consider a kinetic quantity $q(\mathbf{x},\mathbf{p})$, and define

$$\mathcal{Q}(\mathbf{x},t) \ = \ \int \frac{d^\nu p}{(2\pi\hbar)^\nu}\, q(\mathbf{x},\mathbf{p}) f(\mathbf{x},\mathbf{p},t) \tag{C42}$$

$$\mathcal{F}^k(\mathbf{x},t) \ = \ \int \frac{d^\nu p}{(2\pi\hbar)^\nu}\, q(\mathbf{x},\mathbf{p})\, v^k f(\mathbf{x},\mathbf{p},t) \ , \tag{C43}$$

where $v^k = p^k/m$. This gives,

$$\frac{\partial \mathcal{Q}}{\partial t} + \boldsymbol{\nabla}\cdot\mathcal{F} = \sum\nolimits_a \int \frac{d^\nu p}{(2\pi\hbar)^\nu}\, q(\mathbf{p}) \frac{\partial}{\partial p^\ell}\, C_a^{\ell m} \left[\beta_a v^m + \frac{\partial}{\partial p^m}\right] f \ , \tag{C44}$$

and upon integrating by parts, we can express

$$\frac{\partial \mathcal{Q}}{\partial t} + \boldsymbol{\nabla}\cdot\mathcal{F} = -\sum\nolimits_a \int \frac{d^\nu p}{(2\pi\hbar)^\nu}\, \frac{dQ_a}{dt}\, f \ , \tag{C45}$$

where

$$\frac{dQ_a}{dt} = \left[\beta_a v^m - \frac{\partial}{\partial p^m}\right] C_a^{\ell m}\, \frac{\partial q}{\partial p^\ell} \ . \tag{C46}$$

When the distribution $f$ is a $\delta$-function over the trajectory, then the integral may be performed and we have

$$\frac{\partial \mathcal{Q}}{\partial t} + \boldsymbol{\nabla}\cdot\mathcal{F} = -\sum\nolimits_a \frac{dQ_a}{dt} \ . \tag{C47}$$

## Appendix D: Numerical Constants and Quantities

### 1. Constants

Euler's constant is

$$\gamma = 0.57721566490153286060651\cdots \ . \tag{D1}$$

Terms involving $\hbar$ and $c$:

$$c \;=\; 2.997 * \times 10^8\,\mathrm{cm/s} \tag{D2}$$

$$\hbar \;=\; 6.582 \times 10^{-16}\,\mathrm{eV\,s} = 6.582 \times 10^{-19}\,\mathrm{keV\,s} \tag{D3}$$

$$\hbar c \;=\; 1974\,\mathrm{eV\,\AA} = 1.974 \times 10^{-8}\,\mathrm{keVcm} \tag{D4}$$

$$a_0 \;=\; 0.5292\,\mathrm{\AA} = 5.292\;\times 10^{-9}\,\mathrm{cm} \tag{D5}$$

$$B_e \;=\; \frac{e^2}{4\pi \cdot 2a_0} = 13.6\,\mathrm{eV} \tag{D6}$$

We shall never require the numerical value of the electric charge $e$ for any calculation, so I won't bother to record its value. Instead, I will express the square of the charge $e^2$ in terms of the binding energy of the hydrogen atom and the Bohr radius: $e^2 = \big(e^2/a_0\big) \cdot a_0$. Everyone remembers that the binding energy of hydrogen is 13.6 eV, and that the radius of a hydrogen atom is about 0.5 Å (0.5292). For example, the ion Debye wave number can be written

$$\kappa_i^2 \;=\; \beta_i e_i^2 n_i = \beta_i \frac{Z_i^2 e^2}{8\pi a_0}\,(8\pi a_0) n_i = \frac{Z_i^2 B_e}{T_i}\,(8\pi n_i a_0)\;. \tag{D7}$$

Note that the units are trivially correct. Some masses:

$$m_e \;=\; 511.00\,\mathrm{keV/c^2} \tag{D8}$$

$$m_p \;=\; 938.28\,\mathrm{MeV/c^2} \tag{D9}$$

$$m_n \;=\; 939.57\,\mathrm{MeV/c^2} \tag{D10}$$

$$m_{\mathrm{AMU}} \;=\; 931.50\,\mathrm{MeV/c^2} \tag{D11}$$

We define $N_{\mathrm{A}} m_{\mathrm{AMU}} = 1\,\mathrm{g}$, where $N_{\mathrm{A}} = 6.02 \times 10^{23}$

### 2. Physical Quantities

For a plasma of inverse temperature $\beta = 1/T$, the de Broglie wave length $\lambda$ of a particle of mass $m$ is defined by

$$\lambda \;=\; \hbar \left(\frac{2\pi\beta}{m}\right)^{1/2} \; . \tag{D12}$$

For the electron and protons at inverse temperature $\beta$,

$$\lambda_e \;=\; 2.1889 \times 10^{-9}\, \beta^{1/2}\, \text{cm} \quad = 0.356954 \cdot a_0 \beta^{1/2} \tag{D13}$$

$$\lambda_e \;=\; 5.10823 \times 10^{-11}\, \beta^{1/2}\, \text{cm} = 9.65275 \times 10^{-3} \cdot a_0 \beta^{1/2} \tag{D14}$$

with $\beta_e$ in inverse keV.

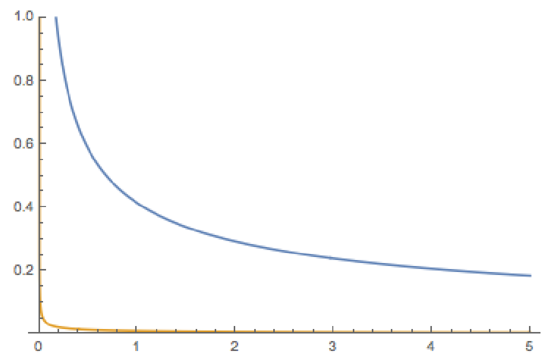


FIG. 2: $\lambda_e/a_0$ (blue) and $\lambda_p/a_0$ vs $T = 1/\beta$ in keV. The ions are always classical, and the electrons are quantum

The Debye wave number is:

$$\kappa^2 \;=\; \beta e^2 n = \frac{B_e}{T}\,(8\pi n a_0) \; . \tag{D15}$$

Note that the units are correct.

## Appendix E: Some Algebra

### 1. The $\mathbf{p}_i$ Integration

We will often require the integration of a Gaussian with a delta-function along a transverse direction, and it is convenient to give the integral in two alternative forms:

$$\int \frac{d^\nu p_i}{(2\pi\hbar)^\nu}\, f_i(\mathbf{p}_i)\,(2\pi\hbar)\,\delta\left(\mathbf{v}_i\cdot\hat{\mathbf{k}} - V\right) = n_i\lambda_i\, m_i\, \exp\left\{-\frac{1}{2}\,\beta_i m_i\, V^2\right\} \tag{E1}$$

$$\int \frac{d^\nu p_i}{(2\pi\hbar)^\nu}\, e^{-\beta_i p_i^2/2m_i}\,(2\pi\hbar)\,\delta\left(\mathbf{v}_i\cdot\hat{\mathbf{k}} - V\right) = \frac{m_i}{\lambda_i^{\nu-1}}\, \exp\left\{-\frac{1}{2}\,\beta_i m_i\, V^2\right\} \tag{E2}$$

where $f_i = z_i\, e^{-p_i^2/2m_i}$ is the Maxwell-Boltzmann distribution. To perform the integral, we decomposed $\mathbf{v}_i$ along the direction defined by $\hat{\mathbf{k}}$,

$$\mathbf{v}_i = v_\parallel \hat{\mathbf{k}} + \mathbf{v}_\perp\ , \tag{E3}$$

and since $\mathbf{p}_i = m_i \mathbf{v}_i$, we shall write $p_i^2 = p_\perp^2 + m_i^2 v_\parallel^2$. We now do the integrals in the normal and parallel directions,

$$z_i \int \frac{d^\nu p_\perp}{(2\pi\hbar)^{\nu-1}}\, e^{-\beta_i p_\perp^2/2m_i} \cdot m_i \int dv_\parallel e^{-\frac{1}{2}\beta_i m_i\, v_\parallel^2}\,\delta\,(v_\parallel - V) \tag{E4}$$

$$= \frac{z_i}{\lambda_i^{\nu-1}} \cdot m_i \exp\left\{-\frac{1}{2}\,\beta_i m_i V^2\right\} = n_i m_i \lambda_i\, \exp\left\{-\frac{1}{2}\,\beta_i m_i V^2\right\}\ , \tag{E5}$$

where we have used

$$n_i = \frac{z_i}{\lambda_i^\nu}\ . \tag{E6}$$

### 2. Fermi Function

In three dimensions, we define the Fermi function

$$F(z) \equiv \frac{2z}{\sqrt{\pi}} \int_0^\infty dx \, \frac{x^{1/2}}{e^x + z} \ , \tag{E7}$$

which satisfies

$$\frac{n_e \lambda_e^3}{2} = F(z_e) \ , \tag{E8}$$

or

$$z_e = F^{-1}(n_e \lambda_e^3/2) \ . \tag{E9}$$

Express $F$ as a series:

$$\frac{1}{e^x + z} = e^{-x} \frac{1}{1 + z e^{-x}} = e^{-x} \sum_{\ell=0}^{\infty} (-1)^\ell \, z^\ell e^{-\ell z} \ , \tag{E10}$$

where $|z| < 1$.

$$F(z) \ = \ \frac{2z}{\sqrt{\pi}} \int_0^\infty dx \, x^{1/2} e^{-x} \sum_{\ell=0}^{\infty} (-1)^\ell \, z^\ell e^{-\ell z} \tag{E11}$$

$$= \ \frac{2}{\sqrt{\pi}} \sum_{\ell=0}^{\infty} (-1)^\ell \, z^{\ell+1} \int_0^\infty dx \, x^{1/2} e^{-(\ell+1) z} \tag{E12}$$

$$= \ -\frac{2}{\sqrt{\pi}} \sum_{\ell=1}^{\infty} (-1)^{\ell+1} \, z^\ell \int_0^\infty dx \, x^{1/2} e^{-\ell z} \tag{E13}$$

For the Debye wave number: In three dimensions, we define

$$D(z) \equiv \frac{2z}{\sqrt{\pi}} \int_0^\infty dx \, \frac{x^{1/2} \, e^x}{(e^x + z)^2} \tag{E14}$$

To do: large and small $z$ limits, and write python module for $F$ and $F^{-1}$.

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