LA-UR-16-23837

Title: Uncertainties in the Standard Moments, the Derivation of the Y1 to Y8 parameters, and the Derivation of w1 to w8 for Feynman Histograms.

Author(s): Smith-Nelson, Mark A.
Cutler, Theresa Elizabeth
Hutchinson, Jesson D.

Intended for: Internal Documentation

Issued: 2016-06-02

Uncertainties in the Standard Moments, the Derivation of the Y_1 to Y_8 parameters, and the Derivation of ω_1 to ω_8 for Feynman Histograms.

Mark A. Smith-Nelson, Jesson D. Hutchinson, Theresa E. Cutler.

Introduction

Momentum is a neutron multiplicity analysis software package that calculates a variety of parameters associated with Feynman histograms. While most of these parameters are documented in Cifarelli [1] and Smith-Nelson [2] there are some parameters which are not. Most prominent of these are the uncertainties in the standard moments and this paper will explicitly document these parameters. This paper will also document the higher order Y_n parameters and their associated ω_n functions because they may be useful for future applications.

The generation of what is referred to as Feynman histograms will not be explained here. For a discussion on the generation of Feynman histograms please see Cutler [3].

Uncertainties in the Standard Moments

The j^{th} standard moments for a Feynman histogram are defined as

$$\overline{C_j} = \frac{\sum_{k=0}^{\infty} k^j c_k}{\sum_{k=0}^{\infty} c_k}, \tag{1}$$

where c_k is the number of counts in the bin k of a histogram [4]. The variance for the standard moments is determined by

$$\sigma_{\overline{C_j}}^{\ 2} = \frac{1}{N-1} \frac{\sum_{k=0}^{\infty} \left(k^j - \overline{C_j}\right)^2 c_k}{\sum_{k=0}^{\infty} c_k}, \tag{2}$$

where the parameter N is the number of gates in a histogram [5]. N is calculated by

$$N = \sum_{k=0}^{\infty} c_k . \tag{3}$$

Once the equation for the variance is expanded and simplified the result is

$$\sigma_{\overline{C_j}}^{\ 2} = \frac{1}{N-1}\left(\overline{C_{2j}} - \overline{C_j}^{\,2}\right). \tag{4}$$

For completeness the variances for the first four standard moments are

$$\sigma_{\overline{C_1}}^{\;2} = \frac{1}{N-1}\left(\overline{C_2} - \overline{C_1}^{\,2}\right), \tag{5}$$

$$\sigma_{\overline{C_2}}^{\;2} = \frac{1}{N-1}\left(\overline{C_4} - \overline{C_2}^{\,2}\right), \tag{6}$$

$$\sigma_{\overline{C_3}}^{\;2} = \frac{1}{N-1}\left(\overline{C_6} - \overline{C_3}^{\,2}\right), \text{ and} \tag{7}$$

$$\sigma_{\overline{C_4}}^{\;2} = \frac{1}{N-1}\left(\overline{C_8} - \overline{C_4}^{\,2}\right). \tag{8}$$

Example Histogram

An example histogram is presented in Table 1 and Figure 1. The standard moments and their associated uncertainties for this example are listed in Table 2.

Table 1. The values of the histogram used for illustration purposes.

k	c_k
0	8
1	26
2	64
3	120
4	176
5	199
6	176
7	120
8	64
9	26
10	8
11	2
12	0
Number of gates (N)	989

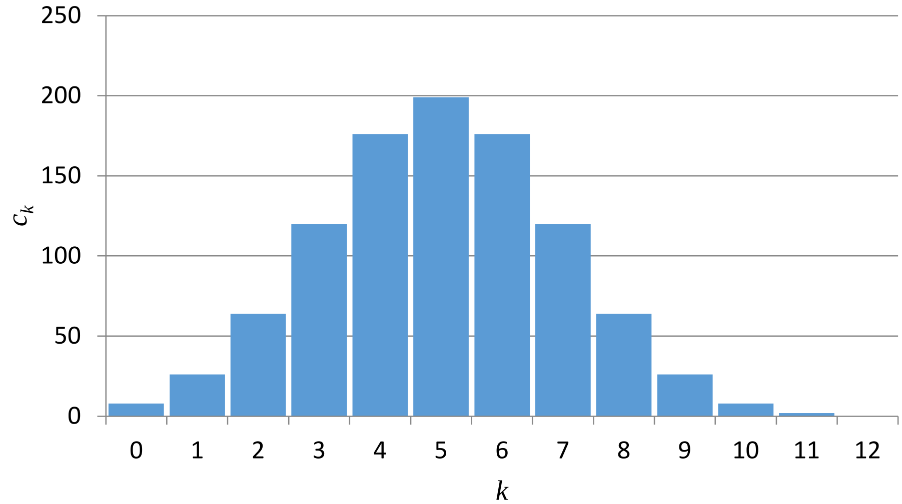

Figure 1. An example histogram used for illustrating the parameters in the uncertainty estimation.

Table 2. The standard moments of the example histogram and the associated uncertainties.

Moment	Value			% Uncertainty
$\overline{C_1}$	5.012	±	0.062	1.2%
$\overline{C_2}$	28.931	±	0.648	2.2%
$\overline{C_3}$	183.495	±	5.992	3.3%
$\overline{C_4}$	1252.203	±	54.931	4.4%

Higher order Y_n Parameters

A set of parameters referred to as the Y_n parameters are used to describe a Feynman histogram. The first three Y_n parameters are explicitly stated in Cifarelli [1] but they do not show the equations for the higher order Y_n parameters. Walston [6] demonstrates the recursive nature of the derivation of these parameters and this method was used to determine the higher order parameters.

The first eight Y_n's were calculated using Maple 18 and are listed below. For comparison the first three Y_n parameters calculated here agree with Cifarelli [1] and so there is confidence that the subsequent Y_n's are correct. The resulting Y_n's are in terms of reduced factorial moments, which are represented as $\overline{m_n}$. The generation of these moments is described in Cutler [3], and Smith-Nelson [2] but for convenience, the first four reduced factorial moments are given below as well.

$$Y_1 = \frac{\overline{m}_1}{\tau}\ , \qquad (9)$$

$$Y_2 = \frac{\overline{m}_2 - \tfrac{1}{2}\overline{m}_1^{2}}{\tau}\,, \qquad (10)$$

$$Y_3 = \frac{\overline{m}_3 - \overline{m}_2\overline{m}_1 + \tfrac{1}{3}\overline{m}_1^{3}}{\tau}\,, \qquad (11)$$

$$Y_4 = \frac{\overline{m}_4 - \overline{m}_3\overline{m}_1 - \tfrac{1}{2}\overline{m}_2^{2} + \overline{m}_2\overline{m}_1^{2} - \tfrac{1}{4}\overline{m}_1^{4}}{\tau}\,, \qquad (12)$$

$$Y_5 = \frac{\overline{m}_5 - \overline{m}_4\overline{m}_1 - \overline{m}_3\overline{m}_2 + \overline{m}_3\overline{m}_1^{2} + \overline{m}_2^{2}\overline{m}_1 - \overline{m}_2\overline{m}_1^{3} + \tfrac{1}{5}\overline{m}_1^{5}}{\tau}\,, \qquad (13)$$

$$Y_6 = \frac{1}{\tau}\left\{\begin{array}{l} \overline{m}_6 - \overline{m}_5\overline{m}_1 - \overline{m}_4\overline{m}_2 + \overline{m}_4\overline{m}_1^{2} - \tfrac{1}{2}\overline{m}_3^{2} + 2\overline{m}_3\overline{m}_2\overline{m}_1 \\ -\overline{m}_3\overline{m}_1^{3} + \overline{m}_2\overline{m}_1^{4} - \tfrac{3}{2}\overline{m}_2^{2}\overline{m}_1^{2} + \tfrac{1}{3}\overline{m}_2^{3} - \tfrac{1}{6}\overline{m}_1^{6} \end{array}\right\}, \qquad (14)$$

$$Y_7 = \frac{1}{\tau}\left\{\begin{array}{l} \overline{m}_7 - \overline{m}_6\overline{m}_1 - \overline{m}_5\overline{m}_2 + \overline{m}_5\overline{m}_1^{2} - \overline{m}_4\overline{m}_3 + 2\overline{m}_4\overline{m}_2\overline{m}_1 - \overline{m}_4\overline{m}_1^{3} + \overline{m}_3\overline{m}_2^{2} \\ +\overline{m}_3^{2}\overline{m}_1 + \overline{m}_3\overline{m}_1^{4} - \overline{m}_2\overline{m}_1^{5} + 2\overline{m}_2^{2}\overline{m}_1^{3} - \overline{m}_2^{3}\overline{m}_1 - 3\overline{m}_3\overline{m}_2\overline{m}_1^{2} + \tfrac{1}{7}\overline{m}_1^{7} \end{array}\right\}, \text{ and} \qquad (15)$$

$$Y_8 = \frac{1}{\tau}\left\{\begin{array}{l} \overline{m}_8 - \overline{m}_7\overline{m}_1 - \overline{m}_6\overline{m}_2 + \overline{m}_6\overline{m}_1^{2} - \overline{m}_5\overline{m}_3 + 2\overline{m}_5\overline{m}_2\overline{m}_1 - \overline{m}_5\overline{m}_1^{3} - \tfrac{1}{2}\overline{m}_4^{2} + \overline{m}_4\overline{m}_2^{2} \\ +2\overline{m}_4\overline{m}_3\overline{m}_1 - 3\overline{m}_4\overline{m}_2\overline{m}_1^{2} + \overline{m}_4\overline{m}_1^{4} + \overline{m}_3^{2}\overline{m}_2 - 3\overline{m}_3\overline{m}_2^{2}\overline{m}_1 - \overline{m}_3\overline{m}_1^{5} + 4\overline{m}_3\overline{m}_2\overline{m}_1^{3} \\ -\tfrac{3}{2}\overline{m}_3^{2}\overline{m}_1^{2} + 2\overline{m}_2^{3}\overline{m}_1^{2} - \tfrac{1}{4}\overline{m}_2^{4} + \overline{m}_2\overline{m}_1^{6} - \tfrac{5}{2}\overline{m}_2^{2}\overline{m}_1^{4} - \tfrac{1}{8}\overline{m}_1^{8} \end{array}\right\}. \qquad (16)$$

$$\overline{m}_1 = \frac{\sum_{k=0}^{\infty} k c_k}{\sum_{k=0}^{\infty} c_k}\ , \qquad (17)$$

$$\overline{m}_2 = \frac{\sum_{k=0}^{\infty} k\left(k-1\right) c_k}{2!\sum_{k=0}^{\infty} c_k}\,, \qquad (18)$$

$$\overline{m}_3 = \frac{\sum_{k=0}^{\infty} k\left(k-1\right)\left(k-2\right) c_k}{3!\sum_{k=0}^{\infty} c_k}\,, \text{ and} \qquad (19)$$

$$\overline{m}_4 = \frac{\sum_{k=0}^{\infty} k\left(k-1\right)\left(k-2\right)\left(k-3\right) c_k}{4!\sum_{k=0}^{\infty} c_k}\,. \qquad (20)$$

Shape of the Y_n parameters

The Y_n parameters are fit to equations that are of the form

$$Y_n = R_n \omega_n\left(\lambda, \tau\right). \tag{21}$$

where R_n is the rate of detection of n neutrons from a fission chain, λ is the inverse of the neutron lifetime, and τ is the gatewidth. The derivation of the term ω_n is given in Walston [6] but a simpler formula is referenced by Hutchinson [7], which is

$$\omega_n\left(\lambda, \tau\right) = \sum_{K=0}^{n-1} \binom{n-1}{K} (-1)^K \frac{1-e^{-\lambda\tau K}}{\lambda\tau K}. \tag{22}$$

The first eight derivations of ω_n were calculated using Maple 18 and are given below.

$$\omega_1 = 1 \ , \tag{23}$$

$$\omega_2 = 1 - \frac{1 - e^{-\lambda\tau}}{\lambda\tau} \ , \tag{24}$$

$$\omega_3 = 1 - \frac{\left(3 - 4e^{-\lambda\tau} + e^{-2\lambda\tau}\right)}{2\lambda\tau} \ , \tag{25}$$

$$\omega_4 = 1 - \frac{\left(11 - 18e^{-\lambda\tau} + 9e^{-2\lambda\tau} - 2e^{-3\lambda\tau}\right)}{6\lambda\tau} \ , \tag{26}$$

$$\omega_5 = 1 - \frac{\left(25 - 48e^{-\lambda\tau} + 36e^{-2\lambda\tau} - 16e^{-3\lambda\tau} + 3e^{-4\lambda\tau}\right)}{12\lambda\tau} \ , \tag{27}$$

$$\omega_6 = 1 - \frac{\left(137 - 300e^{-\lambda\tau} + 300e^{-2\lambda\tau} - 200e^{-3\lambda\tau} + 75e^{-4\lambda\tau} - 12e^{-5\lambda\tau}\right)}{60\lambda\tau} \ , \tag{28}$$

$$\omega_7 = 1 - \frac{\left(147 - 360e^{-\lambda\tau} + 450e^{-2\lambda\tau} - 400e^{-3\lambda\tau} + 225e^{-4\lambda\tau} - 72e^{-5\lambda\tau} + 10e^{-6\lambda\tau}\right)}{60\lambda\tau} \text{, and} \tag{29}$$

$$\omega_8 = 1 - \frac{\begin{pmatrix} 1089 - 2940e^{-\lambda\tau} + 4410e^{-2\lambda\tau} - 4900e^{-3\lambda\tau} \\ +3675e^{-4\lambda\tau} - 1764e^{-5\lambda\tau} + 490e^{-6\lambda\tau} - 60e^{-7\lambda\tau} \end{pmatrix}}{420\lambda\tau} \ . \tag{30}$$

Correlated neutron data was generated to simulate a 4.5 kg sphere of plutonium containing 6% ^{240}Pu [8]. From this data the first four Y_n parameters were calculated and fitted with the appropriate functions. The plots for these are presented below in Figures 2 through 5. It can be see that the shape of the ω_n functions fit the data well.

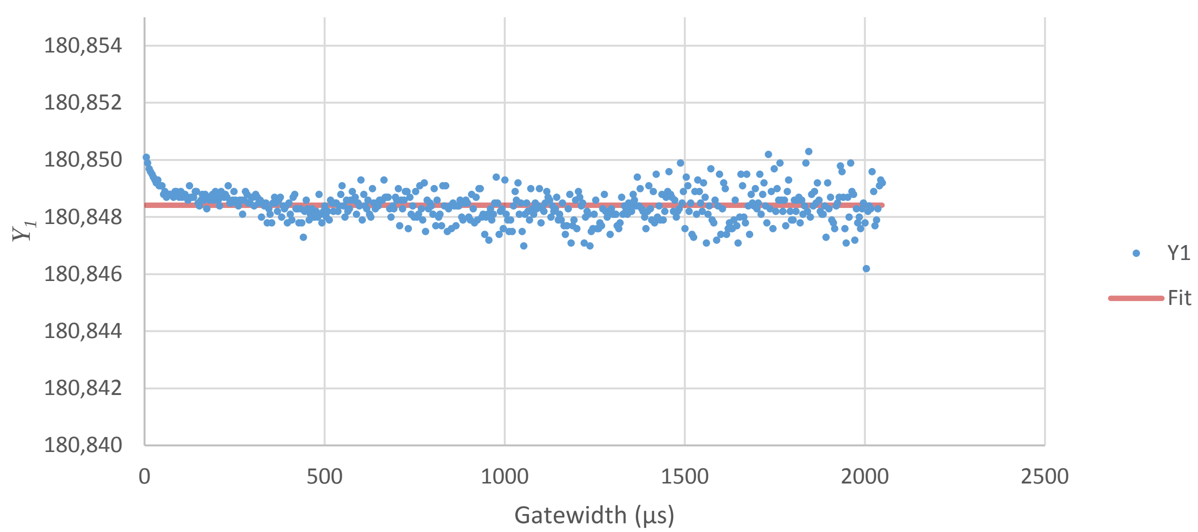

Figure 2. A plot of the Y_1 parameter as a function of gatewidth.

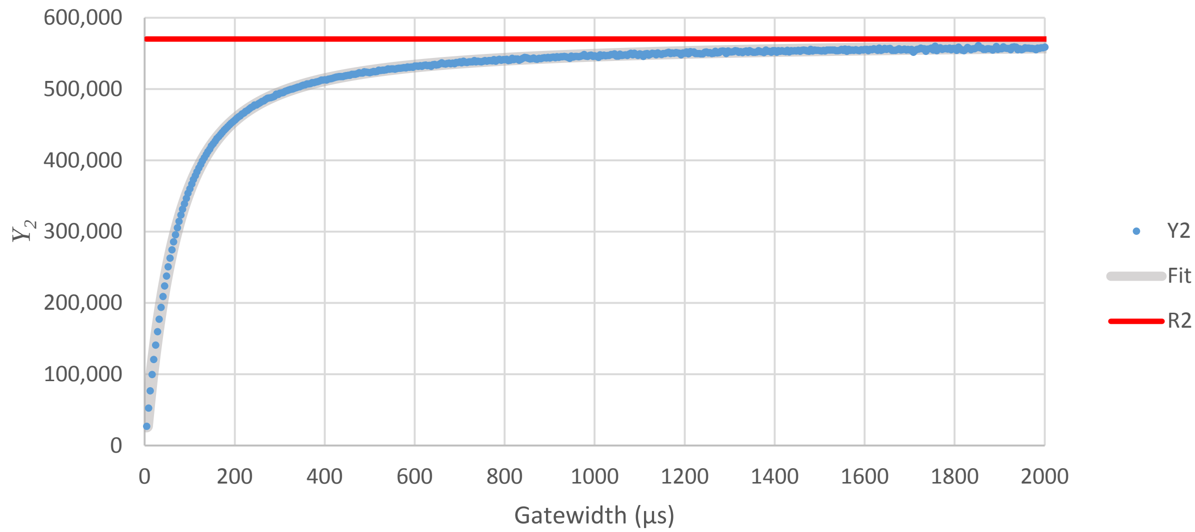

Figure 3. A plot of the Y_2 parameter as a function of gatewidth.

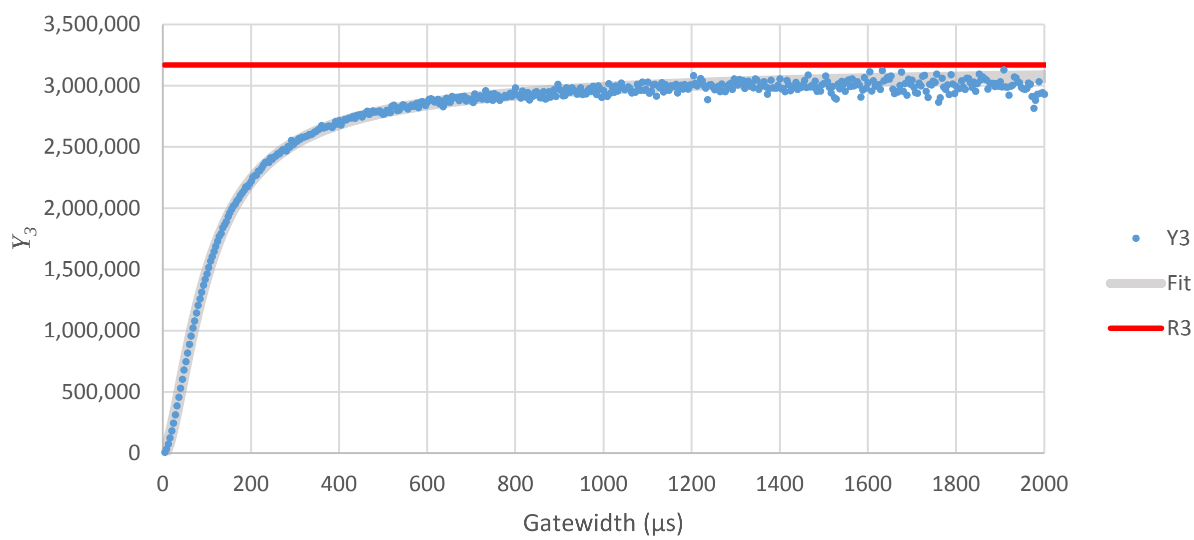

Figure 4. A plot of the Y_3 parameter as a function of gatewidth.

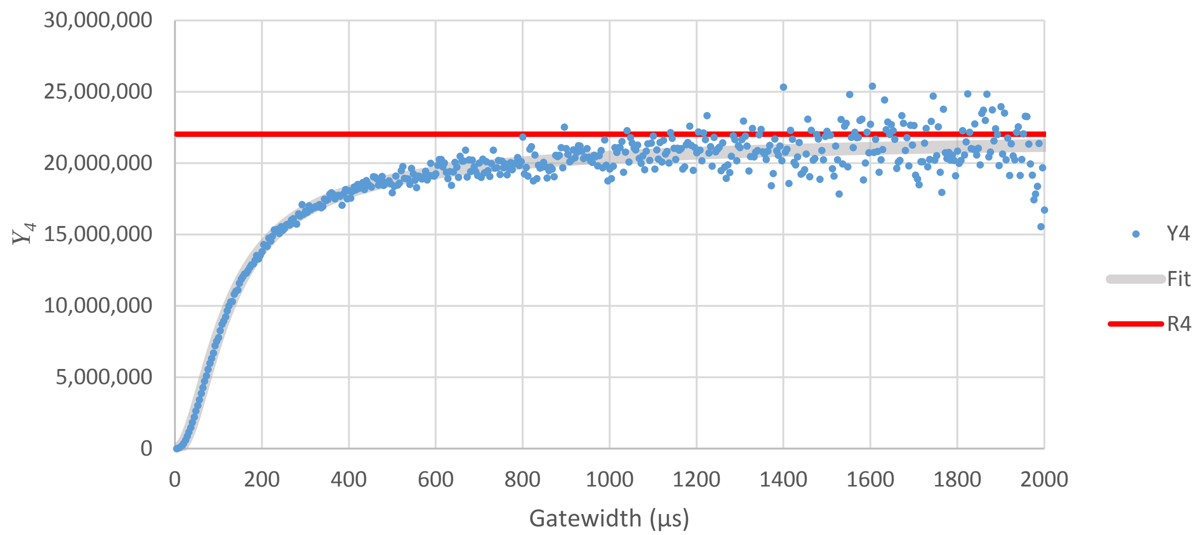

Figure 5. A plot of the Y_4 parameter as a function of gatewidth.

Summary

The uncertainties in the standard moments has been documented. The higher order Y_n parameters of Y_1 to Y_8 as functions of the reduced factorial moments have also been calculated here. The shape of the Y_n parameters as a function of gatewidth have also been generated and explicitly given.

References

1) D. Cifarelli, and W. Hage. "Models for a Three-Parameter Analysis of Neutron Signal Correlation Measurements for Fissile Material Assay." Nucl. Inst. Meth. A251, 550-563 (1986).
2) M. Smith-Nelson, T. Burr, J. Hutchinson, and T. Cutler. "Uncertainties of the Y_n Parameters of the Hage-Cifarelli Formalism." Los Alamos National Laboratory Report: LA-UR-15-21365 (2015).
3) T. Cutler, M. Smith-Nelson, J. Hutchinson. "Deciphering the Binning Method Uncertainty in Neutron Multiplicity Measurements." Los Alamos National Laboratory Report: LA-UR-14-23374 (2014).
4) A. Robba, E.Dowdy, H. Atwater. "Neutron Multiplication Measurements Using Moments of the Neutron Counting Distribution." Nucl. Inst. Meth. 215, 473-479 (1983).
5) E. Dowdy, G. Hansen, A. Robba, J. Pratt. "Proceedings 2nd Annual ESARDA Symposium on Safeguards and Nuclear Material Management." Edinburgh, Scotland (1980).
6) S. Walston. "The Idiot's Guide to the Statistical Theory of Fission Chains." Lawrence Livermore National Laboratory Report: LLNL-TR-414245 (2009).
7) J. Hutchinson, M. Smith-Nelson, T. Grove. "Uncertainty Analysis of Subcritical Benchmark Experiments Using the Hage-Cifarelli Formalism." Los Alamos National Laboratory Report: LA-UR-13-26060 (2013).
8) B. Richard, J. Hutchinson. "Nickel-Reflected Plutonium-Metal-Sphere Subcritical Measurements." Nuclear Criticality Safety Program Report: FUND-NCERC-PU-HE3-MULT-001 (Sept 2014).