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The Transport Equation in Optically Thick Media: Discussion of IMC and its Diffusion Limit

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The transport equation in optically thick media: discussion of IMC and its diffusion limit ⋆

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Abstract

We discuss the limits of validity of the Implicit Monte Carlo (IMC) method for the transport of thermally emitted radiation. The weakened coupling between the radiation and material energy of the IMC method causes defects in handling problems with strong transients. We introduce an approach to asymptotic analysis for the transport equation that emphasizes the fact that the radiation and material temperatures are always different in time dependent problems, and we use it to show that IMC does not produce the correct diffusion limit. As this is a defect of IMC in the continuous equations, no improvement to its discretization can remedy it.

Key words: Implicit Monte Carlo, radiation transport, diffusion limit

1 Introduction

There are four practical algorithms for the transport of thermally emitted radiation in absorptive media, broadly described in [1], [2], [3] and [4]. The oldest method is flux-limited diffusion [4]. It works well in optically thick media but it loses accuracy in optically thin media. In practical problems, where space is divided into zones and time is divided into time steps, it needs the solution of a number of equations equal to the number of zones in the problem at each time step. This is an additional limitation of the method, but the local coupling between the zones can be exploited to reduce the cost

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of the solution. The second method is Implicit Monte Carlo, IMC [5]. It was introduced at a time when solving a large number of equations (inverting a large matrix) was prohibitively time consuming. It can be solved explicitly and it is "almost" stable. It is still very expensive in optically thick media due to the computer time consumed by effective scattering. We show in this paper that it is inaccurate in time-dependent problems if the Fleck factor, f, is less than unity, even without accounting for the errors introduced by discretization in space and time. The third method is based on tracking light rays along $2n+1$ discrete ordinates, called S_n. It requires the solution of a much larger number of equations than diffusion: specifically they are the number of zones times the number of frequency groups times $(2n+1)$. The number of equations can be reduced to be the same as in diffusion by using Chang's photon-free method [6].

The fourth method is Symbolic Implicit Monte Carlo [7]. The unknowns in it are the material temperatures in each zone at the end of the time step, so their number is equal to that for diffusion, although more densely populated matrices appear when a problem has optically thin regions. As it is fully implicit, [1] it is numerically stable. It is accurate in the continuum limit. When the difference formulation is used, the Monte Carlo noise gets small in thick media [8]. It has been shown both in theory and in practical examples that accurate results are obtained in optically thick media by using a piecewise linear discretization [9], [10]. Theory (and practice) teach us that in multi-dimensional problems, Monte Carlo methods may converge better than deterministic methods, so S_n is not necessarily advantageous in such media.

This paper discusses the limits of validity of the Implicit Monte Carlo method (IMC). We start in Section 2 by outlining our notation and the equations for radiation transport in stationary media in local thermodynamic equilibrium. In Section 3 we outline IMC in some detail. In that section we restrict ourselves to a medium with grey (frequency-independent) absorption and no physical scattering. This makes the equations of IMC simpler and more transparent. (Up to this point the paper presents nothing new.) We stress that the only way to measure the validity of any discretization in space and time is how close its results are to the solution of the continuous equations. So, if we can find limitations and flaws in the continuous equations, no discretization can correct them. We next discuss some "exercises": thermal equilibrium, uniform flux - in space and time - in thick media, and the rate of equilibration between the radiation temperature and the material temperature in a uniform medium.

In Section 4 we start the more formal part of the paper. In that section we treat

[1] It is fully implicit in the treatment of the unknown material temperature as it affects the strength of thermal emission during a time step, but does not treat the frequency spectrum of the emitted radiation implicitly.

a medium with grey absorption and isotropic scattering, in slab geometry. First we investigate the limit of vanishing Fleck factor: where absorption has been completely converted to an isotropic "effective scattering". We derive in this limit the "Eddington" diffusion equation. We then do an asymptotic expansion of the equations for grey absorption and isotropic scattering in optically thick media. We show that the diffusion limit of IMC is *not* the diffusion equation that appears in the literature. We close by solving the radiation transport equations by successive approximations and derive the correct IMC equations that are "not quite" the diffusion equation.

2 Radiation transport equations in stationary media in local thermodynamic equilibrium

In order to introduce our notation we write down the radiation transport equations in stationary media in local thermodynamic equilibrium (LTE) [1], [2], [3], [8]. They describe the propagation of the radiation field in terms of the specific intensity, $I(\mathbf{x}, t; \nu, \mathbf{\Omega})$, where $\mathbf{x}, t$ are the space and time variables, ν is the radiation frequency and $\mathbf{\Omega}$ is a unit vector in the direction of propagation.

$$\frac{1}{c}\frac{\partial I(\mathbf{x}, t; \nu, \mathbf{\Omega})}{\partial t} + \mathbf{\Omega}\cdot\nabla I(\mathbf{x}, t; \nu, \mathbf{\Omega}) = \\ \sigma'_a(\nu, T(\mathbf{x}, t))[B(\nu, T(\mathbf{x}, t)) - I(\mathbf{x}, t; \nu, \mathbf{\Omega})] + Q(I) \quad (1)$$

$B(\nu, T)$ is the thermal (Planck) distribution at the material temperature, $T(\mathbf{x}, t)$, and c is the speed of light. The absorption coefficient, σ'_a, and the scattering term, $Q(I)$, will be defined below. The specific intensity is related to the photon number distribution function $f(\mathbf{x}, t; \nu, \mathbf{\Omega})$ by

$$I(\mathbf{x}, t; \nu, \mathbf{\Omega}) = ch\nu f(\mathbf{x}, t; \nu, \mathbf{\Omega}) \quad , \quad (2)$$

where $h\nu$ is the photon energy.

In Eq. (1), all the variables, I, σ'_a, B are functions of the independent variables, $\mathbf{x}, t; \nu, \mathbf{\Omega}$ and/or $T(\mathbf{x}, t)$. In the following, some of the independent variables will be suppressed.

The black body function and the absorption cross section, corrected for stimulated emission, are

$$B(\nu, T) = \frac{2h\nu^3}{c^2}\left(e^{h\nu/kT} - 1\right)^{-1} \quad (3)$$

and

$$\sigma_a'(\nu,T) = \sigma_a(\nu,T)\left(1 - e^{-h\nu/kT}\right) \quad , \tag{4}$$

with σ_a being the "ordinary" absorption coefficient, per unit distance.

The scattering terms are denoted by

$$\begin{aligned} Q(I) = & \int_0^\infty d\nu' \int_{4\pi} d\mathbf{\Omega}' \frac{\nu}{\nu'} \sigma_s(\nu' \to \nu, \mathbf{\Omega}\cdot\mathbf{\Omega}') I(\nu', \mathbf{\Omega}') \left[1 + \frac{c^2 I(\nu, \mathbf{\Omega})}{2h\nu^3}\right] \\ & - \int_0^\infty d\nu' \int_{4\pi} d\mathbf{\Omega}' \sigma_s(\nu \to \nu', \mathbf{\Omega}\cdot\mathbf{\Omega}') I(\nu, \mathbf{\Omega}) \left[1 + \frac{c^2 I(\nu', \mathbf{\Omega}')}{2h\nu'^3}\right] \quad , \end{aligned} \tag{5}$$

where the $\mathbf{x}, t; T$ dependence of σ_s has been suppressed. There are reciprocity relations among the partial scattering cross sections in Eq. (5). They follow from time reversal invariance of quantum electrodynamics.

The zeroth moment of the intensity gives the radiation energy density

$$U_{rad} = \frac{1}{c} \int_0^\infty d\nu \int_{4\pi} d\mathbf{\Omega}\, I \quad , \tag{6}$$

and its first moment is the radiation flux vector

$$\mathbf{F}_{rad} = \int_0^\infty d\nu \int_{4\pi} d\mathbf{\Omega}\ \mathbf{\Omega}\, I \quad . \tag{7}$$

Interaction of radiation with matter is expressed by the energy conservation law

$$\frac{\partial U_{mat}}{\partial t} = \int_0^\infty d\nu \int_{4\pi} d\mathbf{\Omega}\, \sigma_a'[I - B(\nu,T)] - \int_0^\infty d\nu \int_{4\pi} d\mathbf{\Omega}\, Q(I) + G \quad , \tag{8}$$

where U_{mat} is the energy per unit volume of the material and G is a volume source of energy that heats the material.

In the absence of hydrodynamic work terms or thermal conductivity, the total energy of the radiation field and the material is conserved:

$$\frac{\partial (U_{mat} + U_{rad})}{\partial t} + \nabla\cdot\mathbf{F}_{rad} = G \quad . \tag{9}$$

In the approximation of stationary media, momentum is not conserved. In the general equations of radiation hydrodynamics, it is.

3 Fleck's approximation: Implicit Monte Carlo

The introduction of Fleck's approximation [5], usually called Implicit Monte Carlo (IMC), was an important advance in radiation transport calculations. In this approximation, a strongly emitting and absorbing medium is replaced by a predominantly scattering one. Here we review some details of the method and describe some of the approximations involved.

In their original paper [5], Fleck and Cummings start from Eqs. (1) and (8), without the $Q(I)$ term that represents physical scattering. (In their paper they are Eqs. (3.1a) and (3.1b).) After a series of approximations they arrive at their Eqs. (3.5) and (3.9). Those equations can be written in our notation as

$$\frac{1}{c}\frac{\partial I(\nu)}{\partial t} + \mathbf{\Omega}\cdot\nabla I(\nu) = f\Big[\sigma_a'(\nu)[B(\nu) - I(\nu)]\Big]$$

$$+(1-f)\sigma_a'(\nu)\Big[\Big(\frac{b(\nu)}{\sigma_p}\int_0^\infty d\nu' \int_{4\pi} d\mathbf{\Omega}'\sigma_a'(\nu')I(\nu')\Big) - I(\nu) + G\Big] \tag{10}$$

and

$$\frac{\partial U_{mat}}{\partial t} = f\Big[\int_0^\infty d\nu \int_{4\pi} d\mathbf{\Omega}\,\sigma_a'[I(\nu) - B(\nu)]\Big] + fG \quad . \tag{11}$$

Although these are differential equations, they were derived using time discretization with a time step, Δt. The central idea of IMC is to do a linear extrapolation over a time step Δt; then solve for the new value of the material energy by changing otherwise troublesome time integrals of I and G to Δt times their instantaneous values.

The most important new quantity introduced is the "Fleck factor"

$$f = \frac{1}{1 + \alpha\beta_{mat} c \Delta t \sigma_p} \quad , \tag{12}$$

where α is a number between 0 and 1 that determines the "implicitness" of the approximation, and

$$\beta_{mat} = 4aT_{mat}^3/\rho C_V \quad . \tag{13}$$

β_{mat} is the ratio of the specific heat of a black body radiation field (at the material temperature) to that of the material. The frequency distribution of the radiation, $b(\nu)$, is defined as

$$b(\nu) = \frac{B(\nu)}{4\pi \int_0^\infty d\nu B(\nu)} \quad , \tag{14}$$

and the Planck mean opacity is defined as

$$\sigma_p = \int_0^\infty d\nu \sigma'_a(\nu) b(\nu) \quad . \tag{15}$$

We note that when $f \ll 1$, a large fraction of the volume heating source G is transferred from heating the medium to directly heating the radiation field.

For maximum stability, $\alpha = 1$ is usually chosen. It is clear that for large time steps the Fleck factor becomes small, $f \ll 1$, so the coupling between matter and radiation is weakened.

There are several well-known flaws in the Fleck approximation. First, as discussed above, the coupling between the radiation and the matter is lowered by the factor f. Therefore, transients are stretched out unphysically. Second, boundary layers are thickened unphysically. Instead of being of width $\approx 1/\sigma'_a$ they are wider by a factor $1/\sqrt{f}$. Third, when σ'_a is frequency dependent, f depends on the Planck average opacity, σ_p (instead of the Rosseland mean opacity σ_r). That exaggerates the effective opacity and makes f unrealistically small. Nevertheless, the Fleck approximation strictly conserves energy.

3.1 Grey absorptive medium

Let us start with a grey medium, as Fleck and Cummings do in their original paper [5]. We use the same notation as Densmore and Larsen [11], [13].

In order to make our discussion clearer, we repeat: we treat an absorbing medium, without motion, without scattering, in LTE, and with an opacity that is independent of frequency and temperature. The radiation intensity is denoted by $I(\mathbf{\Omega}) = \int_0^\infty d\nu I(\mathbf{\Omega}, \nu)$, and we denote by B the integral of the black body function at the material temperature, T_{mat}, as

$$B = (c/4\pi)aT_{mat}^4 = \int_0^\infty d\nu B(\nu) \quad .$$

In this notation the original (correct) grey equations of transport, parallel to our Eqs. (1) and (8), without scattering and without external sources, are

$$\frac{1}{c}\frac{\partial I(\mathbf{\Omega})}{\partial t} + \mathbf{\Omega}\cdot\nabla I(\mathbf{\Omega}) + \sigma I(\mathbf{\Omega}) = \sigma B \quad , \tag{16}$$

and

$$\frac{\partial U_{mat}}{\partial t} = \sigma\Big[\int_{4\pi} d\mathbf{\Omega}\,[I(\mathbf{\Omega}) - B]\Big] \quad . \tag{17}$$

In the rest of the paper we write σ instead of σ_a' in order to conform to customary usage.

The parallel (approximate) equations in Fleck's IMC are

$$\frac{1}{c}\frac{\partial I(\mathbf{\Omega})}{\partial t} + \mathbf{\Omega}\cdot\nabla I(\mathbf{\Omega}) + \sigma I(\mathbf{\Omega}) = \\ f\sigma B + (1-f)\sigma\Big[\frac{1}{4\pi}\int_{4\pi} d\mathbf{\Omega} I(\mathbf{\Omega})\Big] \quad , \tag{18}$$

and

$$\frac{\partial U_{mat}}{\partial t} = f\sigma\Big[\int_{4\pi} d\mathbf{\Omega}\,[I(\mathbf{\Omega}) - B]\Big] \quad . \tag{19}$$

Next we use a simple algebraic transformation on Eqs. (18) and (19): we add and subtract $(1-f)\sigma B$ on the right-hand side of the former, while we add and subtact $(1-f)\sigma \int_{4\pi} d\mathbf{\Omega}(I - B)$ on the right hand side of the latter. The result is

$$\frac{1}{c}\frac{\partial I(\mathbf{\Omega})}{\partial t} + \mathbf{\Omega}\cdot\nabla I(\mathbf{\Omega}) = \sigma(B(\mathbf{\Omega}) - \mathbf{I}(\mathbf{\Omega})) \\ + (1-f)\sigma\frac{1}{4\pi}\int_{4\pi} d\mathbf{\Omega}(I(\mathbf{\Omega}) - B) \quad , \tag{20}$$

$$\frac{\partial U_{mat}}{\partial t} = \sigma \int_{4\pi} d\boldsymbol{\Omega} \, (I(\boldsymbol{\Omega}) - B)$$

$$- (1-f)\sigma \int_{4\pi} d\boldsymbol{\Omega}(I(\boldsymbol{\Omega}) - B(\boldsymbol{\Omega}) \quad . \tag{21}$$

There is a clear conclusion from this calculation: The first terms on the right hand side of each equation (20) and (21) are the correct transport equations. The Fleck approximation adds the second term. In fact, the angular integral of the term added to (20) is subtracted from (21), showing that the approximation conserves energy. It also follows that if the additonal terms do not change the results, IMC gives the correct answer. If the added terms do make a difference, IMC gives the wrong answer. This conclusion is independent of the mode of discretization in space and time. By inspection, if $f = 1$ or $I = B$, IMC reduces to the correct radiation tramsport equations.

In optically thin media, it is anybody's guess whether the added terms make a difference. A more important question is what happens in thick media. A simple intuitive answer to this question can be obtained by considering Eq. (19). The heating of the medium is decreased by the Fleck factor. Therefore, whatever happens to the propagation equation, (18), the result cannot be correct.

As an example, it has been observed in computer calculations that if there is a boundary between optically thin and optically thick absorbing media, and the thin medium is heated externally, radiation penetrates too deeply into the thick medium.

For later use we write these equations in the difference formulation. We define the difference intensity, $D(\boldsymbol{\Omega}) = I(\boldsymbol{\Omega}) - B$, as in [8].

The transport equations, Eqs. (16) and (17) become

$$\frac{1}{c}\frac{\partial D(\boldsymbol{\Omega})}{\partial t} + \boldsymbol{\Omega}\cdot\nabla D(\boldsymbol{\Omega}) = \sigma D(\boldsymbol{\Omega}) - \frac{1}{c}\frac{\partial B}{\partial t} - \boldsymbol{\Omega}\cdot\nabla B \quad , \tag{22}$$

and

$$\frac{\partial U_{mat}}{\partial t} = \sigma \Big[\int_{4\pi} d\boldsymbol{\Omega} D(\boldsymbol{\Omega}) \Big] \quad . \tag{23}$$

The equations in Fleck's approximation are

$$\frac{1}{c}\frac{\partial D(\mathbf{\Omega})}{\partial t} + \mathbf{\Omega}\cdot\nabla D(\mathbf{\Omega}) = -f\sigma D(\mathbf{\Omega})$$
$$-(1-f)\sigma\Big[D(\mathbf{\Omega}) - \frac{1}{4\pi}\int_{4\pi} d\mathbf{\Omega}' D(\mathbf{\Omega}')\Big] - \frac{1}{c}\frac{\partial B}{\partial t} - \mathbf{\Omega}\cdot\nabla B \quad , \tag{24}$$

and

$$\frac{\partial U_{mat}}{\partial t} = f\sigma\Big[\int_{4\pi} d\mathbf{\Omega} D(\mathbf{\Omega})\Big] \quad . \tag{25}$$

3.2 Analyis of some simple examples

At this point we attempt to clarify the essence of the approximation by comparing the predictions of the original transport equations with IMC using four "exercises".

A/ Thermal equilibrium.

It is seen immediately that both pairs of equations are solved by $I(\mathbf{\Omega}) = B$ when both quantities are independent of space and time. Therefore IMC does have the correct equilibrium limit.

B/ Uniform, time-independent flux.

Suppose that there is a gradient of the radiation density that is constant in space and time. This is a good approximation of the interior of a thick absorbing medium, far from initial and boundary layers.

$$\nabla\cdot B = (c/4\pi)\nabla\cdot(aT^4_{mat}) = const. \tag{26}$$

Eq. (22) reduces to

$$0 = -\sigma D(\mathbf{\Omega}) - \mathbf{\Omega}\cdot\nabla B \quad , \tag{27}$$

or $D(\mathbf{\Omega}) = -(1/\sigma)\mathbf{\Omega}\cdot\nabla B$, as all the other terms are zero. As $\int_{4\pi} d\mathbf{\Omega} D(\mathbf{\Omega}) = 0$, Eq. (23) is also satisfied as $0 = 0$.

A similar treatment of Eq. (24) gives

$$0 = -f\sigma D(\mathbf{\Omega}) - (1-f)\sigma\Big[D(\mathbf{\Omega}) - \frac{1}{4\pi}\int_{4\pi} d\mathbf{\Omega}' D(\mathbf{\Omega}')\Big] - \mathbf{\Omega}\cdot\nabla B \quad . \tag{28}$$

This equation is also solved by $D(\mathbf{\Omega}) = -(1/\sigma)\mathbf{\Omega}\cdot\nabla B$. In order to see this, we note that $f\sigma + (1-f)\sigma = \sigma$ and that $\int_{4\pi} d\mathbf{\Omega}\,\mathbf{\Omega}\cdot\nabla B = 0$. Eq. (25) is again satisfied as $0 = 0$.

The conclusion is that in a grey absorbing medium the *time independent* solution of the transport equations is correct in IMC. More generally, in a grey medium with absorption and isotropic scattering, in steady state, only the total cross section matters for radiation transport. As the solution is independent of the Fleck factor, f, the radiation flux is the same in the correct transport equation and in IMC. Repeating our previous dictum that the correctness of *any* discretization is measured by its closeness to the continuous transport equation, our conclusion is that the correct transport equations and IMC should give the same result, (including the same flux) in time independent problems.

A similar conclusion was reached recently by Castor [12], Densmore [13] and by Larsen, Kumar and Morel [15]. By using the difference formulation here we reach their conclusion more succinctly.

C/ Equilibration between matter and radiation in a uniform grey medium.

We now consider a uniform medium in space and assume that at a given time the material temperature, T_{mat}, and the radiation temperature, T_{rad}, are not equal, but differ by a relatively small amount ΔT. As there are no spatial gradients anywhere, the fluxes are zero. The two temperatures equilibrate in time by exchanging energy between the material and the radiation field, keeping the total energy per unit volume constant.

$$U_{mat} + U_{rad} = const. \tag{29}$$

The derivatives are,

$$\frac{dU_{mat}}{dt} = \frac{dU_{mat}}{dT_{mat}} \cdot \frac{dT_{mat}}{dt} = \rho C_v \frac{dT_{mat}}{dt} \quad , \tag{30}$$

and

$$\frac{dU_{rad}}{dt} = \frac{dU_{rad}}{dT_{rad}} \cdot \frac{dT_{rad}}{dt} = 4aT_{rad}^3 \frac{dT_{rad}}{dt} \quad . \tag{31}$$

From Eq. (29) we conclude that, for our example

$$\frac{dT_{mat}}{dt} = -\frac{4aT_{rad}^3}{\rho C_v}\frac{dT_{rad}}{dt} = -\beta_{rad}\frac{dT_{rad}}{dt} \quad . \tag{32}$$

The notation β_{rad} emphasizes that we used T_{rad} in its definition.

$$dU_{rad}/dU_{mat} \approx 4aT_{rad}^3/\rho C_v = \beta_{rad} \tag{33}$$

From Eq. (19), using the fact that to first order $\int_{4\pi} d\mathbf{\Omega} I(\mathbf{\Omega}) = caT_{rad}^4$ and (always) $\int_{4\pi} d\mathbf{\Omega} B(\mathbf{\Omega}) = caT_{mat}^4$, we get

$$\frac{\partial U_{mat}}{\partial t} = f\sigma ca(T_{rad}^4 - T_{mat}^4) \quad . \tag{34}$$

We now expand Eqs. (30) and (34) to first order in the temperatures. Denoting $\Delta T = T_{rad} - T_{mat}$,

$$\frac{dT_{mat}}{dt} = \frac{f\sigma c}{\rho C_v} 4aT_{rad}^3 \Delta T = f\sigma c\beta_{rad}\Delta T \quad . \tag{35}$$

From Eq. (32) we get

$$\frac{d\Delta T}{dt} = \frac{dT_{rad}}{dt} - \frac{dT_{mat}}{dt} = -\left(1 + \frac{1}{\beta_{rad}}\right)\frac{dT_{mat}}{dt} \quad . \tag{36}$$

Finally

$$\frac{d\Delta T}{dt} = -f\sigma c(1 + \beta_{rad})\Delta T \quad . \tag{37}$$

This equation has the solution

$$\Delta T(t) = \Delta T(0)\, exp[-f\sigma c(1 + \beta_{rad})t] \quad . \tag{38}$$

The conclusion is that if $f < 1$, IMC does not heat the material at the correct rate. This is the root of all the problems with IMC. Of course, if $f = 1$ we are back to the correct equations. As this occurs in continuous radiation transport, no discretization in space and time can correct it.

D/ Generalization to arbitrary time dependent thick grey media

We will discuss our two equations (18) and (19) in detail in Section 4 below. We will see that in thick media the radiation field is close to thermal, but there are problems with the usual derivation of the diffusion equation using "asymptotic expansion". According to some papers the diffusion limit of IMC is independent of the Fleck factor, f. It will be seen that this is not the correct conclusion. Here we present only one decisive step.

Let us consider the material heating equation, Eq. (21),

$$\frac{\partial U_{mat}}{\partial t} = f\sigma ca(T_{rad}^4 - T_{mat}^4) \quad .$$

There are two important conclusion that follow by inspection. If the material energy (and its temperature) are time dependent, T_{rad} and T_{mat} cannot be equal. In time dependent problems, if T_{rad} and T_{mat} are given, the heating rate depends on f, therefore it is incorrect if $f < 1$. Conversely, if the heating rate is given, T_{rad} and T_{mat} are incorrect if $f < 1$. If it is assumed that the two temperatures are equal, (as in all the literature known to us) this fact is missed, together with the obvious inconsistency coming from Eq. (21).

4 The "diffusion limit" for grey material with absorption and scattering

We now present an extended analysis of the thick limit for a grey material with absorption and isotropic scattering. We keep our assumption of LTE, of no hydrodynamic motion, no external sources and all cross sections independent of temperature. We start with the grey "Fleck equations", Eqs. (18) and (19). A crucial point is that in time dependent problems there is *no* consistent formulation if we assume that the material (in LTE) has the same temperature as the radiation. Therefore we have to assume that the material and the radiation are at (slightly) different temperatures. In order to make our derivation simpler, we use slab geometry.

The grey IMC equations, (18), (19) become

$$\frac{1}{c}\frac{\partial I(\mu)}{\partial t} + \mu\frac{\partial I(\mu)}{\partial x} + \sigma I(\mu) = \sigma_a B + \sigma_s\Big[\frac{1}{4\pi}2\pi\int_{-1}^{+1} d\mu I(\mu)\Big] \quad , \tag{39}$$

and

$$\frac{\partial U_{mat}}{\partial t} = \sigma_a\Big[2\pi\int_{-1}^{+1} d\mu\,[I(\mu) - B]\Big] \quad . \tag{40}$$

We used the notation, $\mu = cos(\theta)$, $f\sigma = \sigma_a$, $(1-f)\sigma = \sigma_s$ and, of course, $\sigma_a + \sigma_s = \sigma$. (We use a purely absorbing example, but our formulation is equally valid in the presence of isotropic, monochromatic scattering.)

We now introduce some further notation:

$$\Phi_{mat} = \frac{1}{c} 4\pi B \quad , \tag{41}$$

the energy density of a black body radiation field at the material temperature, and

$$\Phi_{rad} = \frac{1}{c} 2\pi \int_{-1}^{+1} d\mu \, I(\mu) \quad . \tag{42}$$

The ratio of the specific heats of the radiation field and the material at the material temperature, β_{mat}, was defined in Eq. (13)

$$\beta_{mat} = \frac{4\pi T_{mat}^3}{\rho C_v} \quad .$$

Then $\partial U_{mat}/\partial t = (1/\beta_{mat}) \partial \Phi_{mat}/\partial t$.

Using this notation, the transport equations are

$$\frac{1}{c}\frac{\partial I(\mu)}{\partial t} + \mu \frac{\partial I(\mu)}{\partial x} + \sigma I(\mu) = \sigma_a \frac{c}{4\pi} \Phi_{mat} + \sigma_s \frac{c}{4\pi} \Phi_{rad} \quad , \tag{43}$$

and

$$\frac{1}{\beta_{mat}} \frac{\partial \Phi_{mat}}{\partial t} = \sigma_a c \left(\Phi_{rad} - \Phi_{mat} \right) \quad . \tag{44}$$

Now we repeat the definition of the radiation flux and the conservation of energy equation.

$$F = 2\pi \int_{-1}^{+1} d\mu \, \mu I \tag{45}$$

$$\frac{\partial}{\partial t} \left(\frac{1}{\beta_{mat}} \Phi_{mat} + \Phi_{rad} \right) = -\frac{\partial}{\partial x} F = -\frac{\partial}{\partial x} 2\pi \int_{-1}^{+1} d\mu \, \mu I(\mu) \tag{46}$$

We decompose I into an isotropic and an anisotropic component.

$$I = 2\pi \int_{-1}^{+1} d\mu \, I + A(\mu) = \frac{c}{4\pi} \Phi_{rad} + A \tag{47}$$

The transport equation (43) becomes, after using $\sigma = \sigma_a + \sigma_s$,

$$\frac{1}{c}\frac{\partial A(\mu)}{\partial t} + \mu\frac{\partial A(\mu)}{\partial x} + \sigma A(\mu) = \\ -\frac{1}{c}\frac{c}{4\pi}\frac{\partial \Phi_{rad}}{\partial t} - \frac{c}{4\pi}\mu\frac{\partial \Phi_{rad}}{\partial x} + \sigma_a\frac{c}{4\pi}(\Phi_{mat} - \Phi_{rad}) \quad . \tag{48}$$

We now split this equation into an angle-dependent and an angle-independent component, by integrating it over angles. The isotropic part is

$$\int_{-1}^{+1} d\mu\, \mu\frac{\partial A(\mu)}{\partial x} = -\frac{c}{4\pi}\frac{\partial \Phi_{rad}}{\partial t} + \frac{c}{4\pi}\sigma_a c(\Phi_{mat} - \Phi_{rad}) \quad . \tag{49}$$

The equation of propagation for asymmetric part is obtained by subtracting (49) from (43)

$$\frac{1}{c}\frac{\partial A(\mu)}{\partial t} + \mu\frac{\partial A(\mu)}{\partial x} + \sigma A(\mu) = -\frac{c}{4\pi}\mu\frac{\partial \Phi_{rad}}{\partial x} + \int_{-1}^{+1} d\mu\, \mu\frac{\partial A(\mu)}{\partial x} \quad . \tag{50}$$

To these two equations we have to add the one for heating the material (44), the definition of the flux (45), and the equation for energy conservation (46).

In order to make things clear, we display the full set of equations (and apologize for the new equation numbers).

$$\sigma_a c(\Phi_{mat} - \Phi_{rad}) = 2\pi\int_{-1}^{+1} d\mu\, \mu\frac{\partial A(\mu)}{\partial x} + \frac{\partial \Phi_{rad}}{\partial t} \tag{51}$$

$$\frac{1}{c}\frac{\partial A(\mu)}{\partial t} + \mu\frac{\partial A(\mu)}{\partial x} + \sigma A(\mu) = -\frac{c}{4\pi}\mu\frac{\partial \Phi_{rad}}{\partial x} + \int_{-1}^{+1} d\mu\, \mu\frac{\partial A(\mu)}{\partial x} \tag{52}$$

$$\frac{1}{\beta_{mat}}\frac{\partial \Phi_{mat}}{\partial t} = \sigma_a c(\Phi_{rad} - \Phi_{mat}) \tag{53}$$

$$F = 2\pi\int_{-1}^{+1} d\mu\mu I = 2\pi\int_{-1}^{+1} d\mu\, \mu A(\mu) \tag{54}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{\beta_{mat}}\Phi_{mat} + \Phi_{rad}\right) = -\frac{\partial F}{\partial x} = -\frac{\partial}{\partial x}2\pi\int_{-1}^{+1} d\mu\,\mu A(\mu) \tag{55}$$

Combining (51) and (53) we see that the energy conservation equation, (55), is automatically satisfied.

4.1 Purely scattering medium

The limit of $f \to 0$ is a monochromatic, isotropically scattering medium. In this limit $\sigma_a = 0$, $\sigma = \sigma_s$. We substitute this into Eqs. (51) - (55). The result is

$$0 = 2\pi\int_{-1}^{+1} d\mu\,\mu\frac{\partial A(\mu)}{\partial x} + \frac{\partial \Phi_{rad}}{\partial t} \quad . \tag{56}$$

Eq. (52) is unchanged:

$$\frac{1}{c}\frac{\partial A(\mu)}{\partial t} + \mu\frac{\partial A(\mu)}{\partial x} + \sigma A(\mu) = -\frac{c}{4\pi}\mu\frac{\partial \Phi_{rad}}{\partial x} + \int_{-1}^{+1} d\mu\mu\frac{\partial A(\mu)}{\partial x} \quad . \tag{57}$$

$$\frac{1}{\beta_{mat}}\frac{\partial \Phi_{mat}}{\partial t} = 0 \tag{58}$$

Eq. (54) is also unchanged:

$$F = 2\pi\int_{-1}^{+1} d\mu\,\mu I = 2\pi\int_{-1}^{+1} d\mu\,\mu A(\mu) \quad . \tag{59}$$

$$\frac{\partial}{\partial t}\Phi_{rad} = -\frac{\partial F}{\partial x} = -\frac{\partial}{\partial x}2\pi\int_{-1}^{+1} d\mu\,\mu A(\mu) \tag{60}$$

The two equations (57) and (60) are identical, so we can use either one of them.

The definition of a thick medium is $(1/\sigma)(\partial/\partial x) \ll 1$. Therefore, in the first approximation Eq. (57) reduces to

$$\sigma A(\mu) = -\frac{c}{4\pi}\mu\frac{\partial\Phi_{rad}}{\partial x} \quad . \tag{61}$$

Its solution can be substituted into Eq. (59), giving

$$F = 2\pi\int_{-1}^{+1} d\mu\, \mu A(\mu) = -\frac{1}{\sigma}\frac{c}{4\pi}\frac{\partial\Phi_{rad}}{\partial x}2\pi\int_{-1}^{+1} d\mu\, \mu^2 = -\frac{c}{3\sigma}\frac{\partial\Phi_{rad}}{\partial x} \quad . \tag{62}$$

Finally, from (60) we get the "Eddington" diffusion equation

$$\frac{\partial}{\partial t}\Phi_{rad} = -\frac{c}{3\sigma}\frac{\partial^2\Phi_{rad}}{\partial x^2} \quad . \tag{63}$$

We presented this long derivation in order to emphasize that the diffusion equation *is valid* for a purely scattering medium, that is also the limit of $f \to 0$. In this limit the material is totally decoupled from the radiation. In mathematical terms, the two coupled differential equations for the energy of the material and the energy of the radiation reduce to a single equation for the radiation.

Two more remarks are in order:

First, as the limit is approached and $f \ll 1$ it is clear that equilibration of the material and the radiation gets more and more sluggish. In fact in the limit of $f = 0$ the radiation merrily diffuses through the material without loss of energy.

Second, the asymptotic expansion, presented below, was originally presented for neutron diffusion. See e.g. [17], [18]. There the material density is almost always largely unaffected by the dynamics of the neutron density. Therefore the diffusion equation derived by asymptotic expansion is valid.

4.2 Asymptotic expansion for a thick grey material with absorption and scattering

Next we show the shortcomings of the asymptotic expansion as it was derived in all the literature we are aware of. It should be emphasized that our own paper is not an exception [8].

In optically thick media, in LTE, the photon mean free path $1/\sigma$ is small with respect to the inverse characteristic length $|(1/\Phi)(\partial\Phi/\partial x)| \equiv |(\partial/\partial x))|$ of the change in material properties. We define the small parameter, $\epsilon = (1/\sigma)(\partial/\partial x)$.

We now present our derivation. It is not simple, but we beseech the reader to be patient. We start with the grey "Fleck equations", Eqs. (24), (25). We expand them in power series of ϵ and solve them order by order. A crucial point is that there is *no* consistent expansion to second order, if we assume that the material (in LTE) has the same temperature as the radiation. Therefore we have to assume that the material and the radiation are at (slightly) different temperatures. In order to make our derivation more accessible, we will use the standard transport equations (not the difference formulation). Also, in order to simplify notation we assume slab geometry.

The grey IMC equations, (24), (25) become

$$\frac{1}{c}\frac{\partial I(\mu)}{\partial t} + \mu\frac{\partial I(\mu)}{\partial x} + \sigma I(\mu) = \sigma_a B + \sigma_s\Big[\frac{1}{4\pi}2\pi\int_{-1}^{+1} d\mu I(\mu)\Big] \quad , \tag{64}$$

and

$$\frac{\partial U_{mat}}{\partial t} = \sigma_a\Big[2\pi\int_{-1}^{+1} d\mu\,[I(\mu) - B]\Big] \quad . \tag{65}$$

We used the notation, $\mu = cos\theta$, $f\sigma = \sigma_a$, $(1-f)\sigma = \sigma_s$ and, of course, $\sigma_a + \sigma_s = \sigma$. Other than that, we use our notation from Eqs. (41), (42).

The transport equations are

$$\frac{1}{c}\frac{\partial I(\mu)}{\partial t} + \mu\frac{\partial I(\mu)}{\partial x} + \sigma I(\mu) = \sigma_a\frac{c}{4\pi}\Phi_{mat} + \sigma_s\frac{c}{4\pi}\Phi_{rad} \quad , \tag{66}$$

and

$$\frac{1}{\beta_{mat}}\frac{\partial\Phi_{mat}}{\partial t} = \sigma_a c(\Phi_{rad} - \Phi_{mat}) \quad . \tag{67}$$

For completeness, we display the definition of the radiation flux, and the conservation of energy equation.

$$F = 2\pi \int_{-1}^{+1} d\mu\, \mu I \tag{68}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{\beta_{mat}}\Phi_{mat} + \Phi_{rad}\right) = -\frac{\partial}{\partial x}F = -\frac{\partial}{\partial x}2\pi \int_{-1}^{+1} d\mu\, \mu I(\mu) \tag{69}$$

Next, we expand the variables in power series in ϵ

$$\Phi_{rad} = \Phi_{rad}^{(0)} + \epsilon\Phi_{rad}^{(1)} + \epsilon^2\Phi_{rad}^{(2)} + \dots \tag{70}$$

and use a similar expansion for the other variables: I, Φ_{mat}, F.

The equations themselves are also multiplied by the proper powers of ϵ.

$$\epsilon^2\frac{\partial}{\partial t}\left(\frac{1}{\beta_{mat}}\Phi_{mat} + \Phi_{rad}\right) = -\epsilon\frac{\partial}{\partial x}2\pi \int_{-1}^{+1} d\mu\, \mu I(\mu) \quad [A] \tag{71}$$

$$\epsilon^2\frac{1}{c}\frac{\partial I(\mu)}{\partial t} + \epsilon\mu\frac{\partial I(\mu)}{\partial x} + \sigma I(\mu) = \sigma_a\frac{c}{4\pi}\Phi_{mat} + \sigma_s\frac{c}{4\pi}\Phi_{rad} \quad [B] \tag{72}$$

$$\epsilon^2\frac{1}{\beta_{mat}}\frac{\partial\Phi_{mat}}{\partial t} = \sigma_a c(\Phi_{rad} - \Phi_{mat}) \quad [C] \tag{73}$$

$$F = 2\pi \int_{-1}^{+1} d\mu\, \mu I \quad [D] \tag{74}$$

The definitions of Φ_{rad} and the other variables are now substituted into the equations and we demand that the equations be satisfied order by order in ϵ. We now list the results: for easier reference we label them [A] to [D] and by the order of ϵ, [A0], [A1], etc.

<u>Order ϵ^0</u>.

$$\sigma I(\mu)^{(0)} = \sigma_a\frac{c}{4\pi}\Phi_{mat}^{(0)} + \sigma_s\frac{c}{4\pi}\Phi_{rad}^{(0)} \quad [B0] \tag{75}$$

$$0 = \sigma_a c(\Phi_{rad}^{(0)} - \Phi_{mat}^{(0)}) \quad [C0] \tag{76}$$

$$F^{(0)} = 2\pi \int_{-1}^{+1} d\mu \mu I^{(0)} \quad [D0] \tag{77}$$

The solution is

$$\Phi^{(0)}_{rad} = \Phi^{(0)}_{mat} \quad , \tag{78}$$

$$I(\mu)^{(0)} = \frac{c}{4\pi}\Phi^{(0)}_{mat} = \frac{c}{4\pi}\Phi^{(0)}_{rad} \quad , \tag{79}$$

$$F^{(0)} = 0 \quad . \tag{80}$$

<u>Order ϵ</u>.

$$0 = \frac{\partial}{\partial x} 2\pi \int_{-1}^{+1} d\mu \mu I(\mu)^{(0)} = \frac{\partial}{\partial x} F^{(0)} \quad [A1] \tag{81}$$

$$\mu \frac{\partial I(\mu)^{(0)}}{\partial x} + \sigma I(\mu)^{(1)} = \sigma_a \frac{c}{4\pi}\Phi^{(1)}_{mat} + \sigma_s \frac{c}{4\pi}\Phi^{(1)}_{rad} \quad [B1] \tag{82}$$

$$0 = \sigma_a c(\Phi^{(1)}_{rad} - \Phi^{(1)}_{mat}) \quad [C1] \tag{83}$$

$$F^{(1)} = 2\pi \int_{-1}^{+1} d\mu \mu I^{(1)} \quad [D1] \tag{84}$$

[A1] is identically satisfied. From [C1] it follows that

$$\Phi^{(1)}_{rad} = \Phi^{(1)}_{mat} \quad . \tag{85}$$

Then [B1] can be solved. The result is

$$I(\mu)^{(1)} = -\frac{1}{\sigma}\mu \frac{\partial I(\mu)^{(0)}}{\partial x} + \frac{c}{4\pi}\Phi^{(1)}_{rad} = -\frac{1}{\sigma}\mu \frac{\partial}{\partial x}\frac{c}{4\pi}\Phi^{(0)}_{rad} + \frac{c}{4\pi}\Phi^{(1)}_{rad} \quad . \tag{86}$$

From [D1], as $\Phi^{(1)}_{rad}$ gives no flux,

$$F^{(1)} = 2\pi \int_{-1}^{+1} d\mu\, \mu \Big(-\frac{1}{\sigma}\mu \frac{\partial I(\mu)^{(0)}}{\partial x} + \frac{c}{4\pi}\Phi^{(1)}_{rad} \Big) = -\frac{c}{3\sigma}\frac{\partial}{\partial x}\Phi^{(0)}_{rad} \quad . \tag{87}$$

To this order the material does not heat or cool. In fact it is implicitly assumed that $(\partial^2/\partial x^2)\Phi_{rad}^{(0)} = 0$

<u>Order ϵ^2</u>.

$$\frac{\partial}{\partial t}\Big(\frac{1}{\beta_{mat}}\Phi_{mat}^{(0)} + \Phi_{rad}^{(0)}\Big) = -\frac{\partial}{\partial x}2\pi\int_{-1}^{+1} d\mu\,\mu I(\mu)^{(1)} \quad [A2] \tag{88}$$

$$\frac{1}{c}\frac{\partial I(\mu)^{(0)}}{\partial t} + \mu\frac{\partial I(\mu)^{(1)}}{\partial x} + \sigma I(\mu)^{(2)} = \sigma_a\frac{c}{4\pi}\Phi_{mat}^{(2)} + \sigma_s\frac{c}{4\pi}\Phi_{rad}^{(2)} \quad [B2] \tag{89}$$

$$\frac{1}{\beta_{mat}}\frac{\partial\Phi_{mat}^{(0)}}{\partial t} = \sigma_a c(\Phi_{rad}^{(2)} - \Phi_{mat}^{(2)}) \quad [C2] \tag{90}$$

$$F^{(2)} = 2\pi\int_{-1}^{+1} d\mu\,\mu I^{(2)} \quad [D2] \tag{91}$$

This is the lowest order order in ϵ that the material can heat or cool; and if it does, the material and radiation temperatures are necessarily different. We will see that this difference can be eliminated from the solution of [B-2]. This is done in all the published literature we are aware of.

We now proceed to partially solve these equations. From [B2], using (86) we get

$$\begin{aligned}\sigma I(\mu)^{(2)} = -\frac{1}{c}\frac{\partial I(\mu)^{(0)}}{\partial t} - \mu\frac{\partial}{\partial x}\Big(-\frac{1}{\sigma}\mu\frac{\partial I(\mu)^{(0)}}{\partial x} + \frac{c}{4\pi}\Phi_{rad}^{(1)}\Big)& \\ +\sigma_a\frac{c}{4\pi}\Phi_{mat}^{(2)} + \sigma_s\frac{c}{4\pi}\Phi_{rad}^{(2)} =& \\ -\frac{1}{c}\frac{\partial}{\partial t}\frac{c}{4\pi}\Phi_{rad}^{(0)} - \mu\frac{\partial}{\partial x}\Big(-\frac{1}{\sigma}\mu\frac{\partial}{\partial x}\frac{c}{4\pi}\Phi_{rad}^{(0)} + \frac{c}{4\pi}\Phi_{rad}^{(1)}\Big)& \\ +\sigma_a\frac{c}{4\pi}\Phi_{mat}^{(2)} + \sigma_s\frac{c}{4\pi}\Phi_{rad}^{(2)} \quad .&\end{aligned} \tag{92}$$

We now integrate this equation over the solid angle and use the identity that, for any order in ϵ

$$2\pi\int_{-1}^{+1} d\mu\, I(\mu) = c\,\Phi_{rad} \quad . \tag{93}$$

The result, using $2\pi\int_{-1}^{+1} d\mu\,\mu^{2n} = 4\pi/(2n+1)$, and $2\pi\int_{-1}^{+1} d\mu\,\mu^{(2n+1)} = 0$, is

$$\sigma c \Phi_{rad}^{(2)} = -\frac{\partial \Phi_{rad}^{(0)}}{\partial t} + \frac{c}{3\sigma}\frac{\partial^2 \Phi_{rad}^{(0)}}{\partial x^2} + \sigma_a c \Phi_{mat}^{(2)} + \sigma_s c \Phi_{rad}^{(2)} \quad . \tag{94}$$

Using $\sigma = \sigma_a + \sigma_s$, then [C2]

$$\frac{\partial \Phi_{rad}^{(0)}}{\partial t} - \frac{c}{3\sigma}\frac{\partial^2 \Phi_{rad}^{(0)}}{\partial x^2} = \sigma_a c(\Phi_{mat}^{(2)} - \Phi_{rad}^{(2)}) = -\frac{1}{\beta_{mat}}\frac{\partial \Phi_{mat}^{(0)}}{\partial t} \quad . \tag{95}$$

Rearrangement gets the diffusion equation,

$$\frac{\partial}{\partial t}\left(\frac{1}{\beta_{mat}}\Phi_{mat}^{(0)} + \Phi_{rad}^{(0)}\right) = \frac{c}{3\sigma}\frac{\partial^2 \Phi_{rad}^{(0)}}{\partial x^2} \quad . \tag{96}$$

Note again that the diffusion equation depends only on the sum of the absorption and isotropic scattering.

From [D2], using the explicit solution of (92) we get (after leaving out some intermediate steps)

$$F^{(2)} = -\frac{c}{3\sigma}\frac{\partial}{\partial x}\Phi_{rad}^{(1)} \quad . \tag{97}$$

<u>Order ϵ^3</u>.

$$\frac{\partial}{\partial t}\left(\frac{1}{\beta_{mat}}\Phi_{mat}^{(1)} + \Phi_{rad}^{(1)}\right) = -\frac{\partial}{\partial x}2\pi\int_{-1}^{+1} d\mu\, \mu I(\mu)^{(2)} \quad [A3] \tag{98}$$

$$\frac{1}{c}\frac{\partial I(\mu)^{(1)}}{\partial t} + \mu\frac{\partial I(\mu)^{(2)}}{\partial x} + \sigma I(\mu)^{(3)} = \sigma_a \frac{c}{4\pi}\Phi_{mat}^{(3)} + \sigma_s \frac{c}{4\pi}\Phi_{rad}^{(3)} \quad [B3] \tag{99}$$

$$\frac{1}{\beta_{mat}}\frac{\partial \Phi_{mat}^{(1)}}{\partial t} = \sigma_a c(\Phi_{rad}^{(3)} - \Phi_{mat}^{(3)}) \quad [C3] \tag{100}$$

$$F^{(3)} = 2\pi\int_{-1}^{+1} d\mu \mu I^{(3)} \quad [D3] \tag{101}$$

We proceed to solve [B3], using (86) and (92)

$$\sigma I(\mu)^{(3)} = -\frac{1}{c}\frac{\partial}{\partial t}\Big(-\frac{1}{\sigma}\mu\frac{\partial}{\partial x}\frac{c}{4\pi}\Phi^{(0)}_{rad} + \frac{c}{4\pi}\Phi^{(1)}_{rad}\Big)$$
$$-\mu\frac{\partial}{\partial x}\frac{1}{\sigma}\Big[-\frac{1}{c}\frac{\partial}{\partial t}\frac{c}{4\pi}\Phi^{(0)}_{rad} - \mu\frac{\partial}{\partial x}\Big(-\frac{1}{\sigma}\mu\frac{\partial}{\partial x}\frac{c}{4\pi}\Phi^{(0)}_{rad} + \frac{c}{4\pi}\Phi^{(1)}_{rad}\Big)$$
$$+\sigma_a\frac{c}{4\pi}\Phi^{(2)}_{mat} + \sigma_s\frac{c}{4\pi}\Phi^{(2)}_{rad}\Big] + \sigma_a\frac{c}{4\pi}\Phi^{(3)}_{mat} + \sigma_s\frac{c}{4\pi}\Phi^{(3)}_{rad} \quad . \tag{102}$$

It simplifies to

$$\sigma I(\mu)^{(3)} = -\frac{1}{c}\frac{\partial}{\partial t}\frac{c}{4\pi}\Phi^{(1)}_{rad}$$
$$-\mu\frac{\partial}{\partial x}\frac{1}{\sigma}\Big[- \mu\frac{\partial}{\partial x}\Big(-\frac{1}{\sigma}\mu\frac{\partial}{\partial x}\frac{c}{4\pi}\Phi^{(0)}_{rad} + \frac{c}{4\pi}\Phi^{(1)}_{rad}\Big)$$
$$+\sigma_a\frac{c}{4\pi}\Phi^{(2)}_{mat} + \sigma_s\frac{c}{4\pi}\Phi^{(2)}_{rad}\Big] + \sigma_a\frac{c}{4\pi}\Phi^{(3)}_{mat} + \sigma_s\frac{c}{4\pi}\Phi^{(3)}_{rad} \quad . \tag{103}$$

We now integrate this equation over the solid angle. The result is

$$\sigma c\Phi^{(3)}_{rad} = -\frac{\partial\Phi^{(1)}_{rad}}{\partial t} + \frac{c}{3\sigma}\frac{\partial^2\Phi^{(1)}_{rad}}{\partial x^2} + \sigma_a c\Phi^{(3)}_{mat} + \sigma_s c\Phi^{(3)}_{rad} \quad . \tag{104}$$

Using $\sigma = \sigma_a + \sigma_s$, then [C3]

$$\frac{\partial\Phi^{(1)}_{rad}}{\partial t} - \frac{c}{3\sigma}\frac{\partial^2\Phi^{(1)}_{rad}}{\partial x^2} = \sigma_a c(\Phi^{(3)}_{mat} - \Phi^{(3)}_{rad}) = -\frac{1}{\beta_{mat}}\frac{\partial\Phi^{(1)}_{mat}}{\partial t} \quad . \tag{105}$$

Rearrangement gets again the diffusion equation,

$$\frac{\partial}{\partial t}\Big(\frac{1}{\beta_{mat}}\Phi^{(1)}_{mat} + \Phi^{(1)}_{rad}\Big) = \frac{c}{3\sigma}\frac{\partial^2\Phi^{(1)}_{rad}}{\partial x^2} \quad . \tag{106}$$

We get the result that an absorbing and scattering medium satisfies the diffusion equation not only to lowest order, but to one higher order; as in a purely absorbing medium investigated by Larsen, Mercer and Morel [14], [15], [16]. Unfortunately they stop here and miss the difference between the material and the radiation temperatures that appear in time-dependent problems.

The flux is, from [D3] and (103)

$$F^{(3)} = 2\pi\int_{-1}^{+1} d\mu\,\mu I^{(3)} = -\frac{c}{5\sigma^3}\frac{\partial^3}{\partial x^3}\Phi^{(0)}_{rad} - \frac{c}{3\sigma}\frac{\sigma_a}{\sigma}\frac{\partial\Phi^{(2)}_{mat}}{\partial x} - \frac{c}{3\sigma}\frac{\sigma_s}{\sigma}\frac{\partial\Phi^{(2)}_{rad}}{\partial x} \quad . \tag{107}$$

Note that $\Phi^{(1)}_{rad} = \Phi^{(1)}_{mat}$ is still not defined, but one can do so.

4.3 Iterative solution

We now "solve" the first four equations for optically thick media by iteration, as in Castor's book [3]. He points out that this method was used originally by Schwarzschild [19] and reviewed by Cox and Giuli [20].

Let us remember that $(1/\sigma)(\partial/\partial x) \approx \epsilon \ll 1$. Also, if the temporal variations are not very large, $(1/\sigma_a c)(\partial/\partial t) \ll 1$. In *all* papers, that use asymptotic expansion (including our own), it was assumed that $(1/\sigma_a c)(\partial/\partial t) \approx \epsilon^2$. We will not need this assumption.

From (52) we get

$$A(\mu) = \frac{1}{\sigma}\left[-\frac{1}{c}\frac{\partial A(\mu)}{\partial t} - \mu\frac{\partial A(\mu)}{\partial x} - \frac{c}{4\pi}\mu\frac{\partial \Phi_{rad}}{\partial x} + \int_{-1}^{+1} d\mu\, \mu\frac{\partial A(\mu)}{\partial x}\right] \ . \tag{108}$$

Similarly, from (51) and (53) we get

$$\Phi_{mat} - \Phi_{rad} = \frac{1}{\sigma_a c}\left[2\pi\int_{-1}^{+1} d\mu\, \mu\frac{\partial A(\mu)}{\partial x} + \frac{\partial \Phi_{rad}}{\partial t}\right] \ , \tag{109}$$

$$\Phi_{rad} - \Phi_{mat} = \frac{1}{\sigma_a c}\left[\frac{1}{\beta_{mat}}\frac{\partial \Phi_{mat}}{\partial t}\right] \ . \tag{110}$$

Now two important remarks and a conjecture. First, we have made no approximations yet, but the orders of magnitude of the terms are getting apparent. Second, as it was pointed out in this paper *ad nauseam*, if the material and/or the radiation temperatures are time dependent, there has to be a difference between Φ_{rad} and Φ_{mat}. Also, it is clear that the presence of σ_a in these equations is the source of the difference between IMC and the correct transport equation. The latter corresponds to $f = 1$ or $\sigma_a = \sigma$. Third, if we eliminate the temperature difference, as well as σ_a from the equations, we get

$$2\pi\int_{-1}^{+1} d\mu\, \mu\frac{\partial A(\mu)}{\partial x} + \frac{\partial \Phi_{rad}}{\partial t} = -\frac{1}{\beta_{mat}}\frac{\partial \Phi_{mat}}{\partial t} \ . \tag{111}$$

We strongly suspect that this elimination led all previous authors to overlook this important point and led some of them to the erroneous conclusion that IMC does give the correct answers even in time dependent problems and even in the continuum limit.

The first approximation in ϵ is

$$A(\mu)^{(1)} = -\frac{c}{4\pi}\mu\frac{1}{\sigma}\frac{\partial\Phi_{rad}^{(1)}}{\partial x} \quad , \tag{112}$$

$$\Phi_{rad}^{(1)} - \Phi_{mat}^{(1)} = \frac{1}{\sigma_a c}\left[\frac{1}{\beta_{mat}}\frac{\partial\Phi_{mat}^{(1)}}{\partial t}\right] = -\frac{1}{\sigma_a c}\left[\frac{\partial\Phi_{rad}^{(1)}}{\partial t}\right] \quad , \tag{113}$$

$$F^{(1)} = 2\pi\int_{-1}^{+1} d\mu\,\mu A(\mu)^{(1)} = -2\pi\int_{-1}^{+1} d\mu\,\mu\frac{c}{4\pi}\mu\frac{1}{\sigma}\frac{\partial\Phi_{rad}^{(1)}}{\partial x} = -\frac{c}{3\sigma}\frac{\partial\Phi_{rad}^{(1)}}{\partial x} \quad . \tag{114}$$

Substituting this into (55) we get

$$\frac{\partial}{\partial t}\left(\frac{1}{\beta_{mat}}\Phi_{mat}^{(1)} + \Phi_{rad}^{(1)}\right) = -\frac{\partial F^{(1)}}{\partial x} = \frac{\partial}{\partial x}\frac{c}{3\sigma}\frac{\partial\Phi_{rad}^{(1)}}{\partial x} \quad . \tag{115}$$

This is the diffusion equation. It is a result good to first order in ϵ. Note that only the total cross section, σ appears in this equation. It is easy to conclude from this equation that IMC results in the correct diffusion of radiation. This is where the literature usually stops. In reality this equation has to be solved together with Eq. (113) and there the result of IMC is different from the correct one. In other words, whenever the radiation temperature is time dependent, $\Phi_{mat} \neq \Phi_{rad}$ and the temperature difference depends on the Fleck factor, f.

In order to illuminate this point a little more, we eliminate Φ_{mat} from the diffusion equation using Eq. (113); and emphasize that the result is good only to first order.

$$\frac{1}{\beta_{mat}}\frac{\partial\Phi_{rad}^{(1)}}{\partial t} + \frac{\partial\Phi_{rad}^{(1)}}{\partial t} = \frac{c}{3\sigma}\frac{\partial^2\Phi_{rad}^{(1)}}{\partial x^2} - \frac{1}{\sigma_a c}\frac{1}{\beta_{mat}}\frac{\partial^2\Phi_{rad}^{(1)}}{\partial t^2} \tag{116}$$

We see that there is a correction to the diffusion equation, with σ_a appearing instead of σ in the second term on the right hand side. If we use the usual "asymptotic" expansion in powers of ϵ we miss this difference. As $\sigma_a = f\sigma$, the influence of the incorrect second term becomes large as f becomes small. This is typical of IMC in optically thick media.

Similarly, we can eliminate Φ_{rad} from the diffusion equation using Eq. (113). The result is

$$\frac{1}{\beta_{mat}}\frac{\partial\Phi^{(1)}_{mat}}{\partial t}+\frac{\partial\Phi^{(1)}_{mat}}{\partial t}=\frac{c}{3\sigma}\frac{\partial^2\Phi^{(1)}_{mat}}{\partial x^2}-\frac{1}{\sigma_a c}\frac{1}{\beta_{mat}}\frac{\partial^2\Phi^{(1)}_{mat}}{\partial t^2}+\frac{1}{\sigma_a c}\frac{1}{\beta_{mat}}\frac{c}{3\sigma}\frac{\partial^3\Phi^{(1)}_{mat}}{\partial x^2\partial t} \quad . \tag{117}$$

It is also instructive to calculate the flux in terms of Φ_{mat}

$$F^{(1)}=-\frac{c}{3\sigma}\frac{\partial\Phi^{(1)}_{mat}}{\partial x}-\frac{c}{3\sigma}\frac{1}{\sigma_a c}\frac{1}{\beta_{mat}}\frac{\partial^2\Phi^{(1)}_{mat}}{\partial t\partial x} \quad . \tag{118}$$

We see that when $f<1$ there is an enhanced flux if Φ_{mat} increases in time. That is the reason that radiation penetrates a cold medium too fast in IMC.

The second order approximation is obtained by using the first approximation on the right hand side of Eqs. (108), (109) and (110) The calculation is straightforward, but it involves some algebra. We quote here the results.

$$A(\mu)^{(2)}=-\frac{c}{4\pi}\left[\mu\frac{1}{\sigma}\frac{\partial\Phi_{rad}}{\partial x}-\mu\frac{1}{\sigma^2 c}\frac{\partial^2\Phi^{(2)}_{rad}}{\partial t\partial x}-\left(\mu^2-\frac{2}{3}\right)\frac{1}{\sigma^2}\frac{\partial^2\Phi^{(2)}_{rad}}{\partial x^2}\right] \tag{119}$$

$$\Phi^{(2)}_{rad}-\Phi^{(2)}_{mat}=-\frac{1}{\sigma_a c}\left[\frac{\partial\Phi^{(2)}_{rad}}{\partial t}-\frac{c}{3\sigma}\frac{\partial^2\Phi^{(2)}_{rad}}{\partial x^2}\right] \tag{120}$$

$$F^{(2)}=2\pi\int_{-1}^{+1}d\mu\,\mu A(\mu)^{(2)}=-\frac{c}{3\sigma}\left[\frac{\partial\Phi^{(2)}_{rad}}{\partial x}-\frac{1}{\sigma c}\frac{\partial^2\Phi^{(2)}_{rad}}{\partial t\partial x}\right] \tag{121}$$

We remark that these results stand alone; they do not have to be added to the first approximation as in asymptotic expansion. Energy conservation yields the modified diffusion equation

$$\frac{\partial}{\partial t}\left(\frac{1}{\beta_{mat}}\Phi^{(2)}_{mat}+\Phi^{(2)}_{rad}\right)=-\frac{\partial F^{(2)}}{\partial x}=\frac{\partial}{\partial x}\frac{c}{3\sigma}\left[\frac{\partial\Phi^{(2)}_{rad}}{\partial x}-\frac{1}{\sigma c}\frac{\partial^2\Phi^{(2)}_{rad}}{\partial t\partial x}\right] \quad . \tag{122}$$

We can again use Eq. (120) to eliminate $\Phi^{(2)}_{mat}$. The result is

$$\frac{1}{\beta_{mat}}\frac{\partial\Phi^{(2)}_{rad}}{\partial t}+\frac{\partial\Phi^{(2)}_{rad}}{\partial t}=$$
$$\frac{c}{3\sigma}\frac{\partial^2\Phi^{(2)}_{rad}}{\partial x^2}-\frac{1}{\sigma_a c}\frac{1}{\beta_{mat}}\frac{\partial^2\Phi^{(2)}_{rad}}{\partial t^2}-\frac{c}{3\sigma}\left(\frac{1}{\sigma c}-\frac{1}{\beta_{mat}\sigma_a c}\right)\frac{\partial^3\Phi^{(2)}_{rad}}{\partial t\partial x^2}\quad . \tag{123}$$

This is a very complicated equation. We could write an equation for Φ_{mat}, but it is even more complicated.

5 Summary and conclusions

This paper analyzes the limits of Fleck's IMC radiation transport in simple cases, without the approximations that are introduced by discretization in space and time. In some sense, therefore, the paper is not practical. Nevertheless, the only valid comparison for practical results is their closeness to the solution of the continuous differential equation. Therefore if an approximate method does not get the correct continuum equations in some cases; it is sure that *any* discretization of that method will get the wrong answer.

There are some cases that IMC *does* get the correct limit. The most important ones are: a complete thermal equilibrium and a steady state, that is non-uniform in space.

It is well known that IMC weakens the coupling between material and radiation, by the Fleck factor f. Therefore it is interesting to investigate the validity of the approximation in time dependent problems. It is pointed out that even in optically thick media, far from boundaries, the "effective" temperature of the radiation and the material temperature are always different. That difference is wrong in IMC. Using this "insight", the paper shows that even in grey media IMC does *not* satisfy the correct diffusion equation, except in a purely scattering medium, when $f \to 0$. In that case the "Eddington" diffusion equation is obtained.

Then, using asymptotic expansion, we derive the differential equation that is satisfied by IMC in thick stationary media. It is "not quite" the correct diffusion equation.

The result that IMC does not give the correct diffusion equation is shown in two different ways, because all the literature (that we know of) got it wrong.

Fortunately, there are methods that are shown to give the correct results and that are actually faster than IMC in practical cases [10] .

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