Fluid Systems I

Seoul National Univ.
School of Mechanical
and Aerospace Engineering

Spring 2011

Fluid Capacitance

$$C = \frac{change\ in\ stored\ mass}{change\ pressure} = \frac{dm}{dp}$$

For a tank,

General fluid capacitance
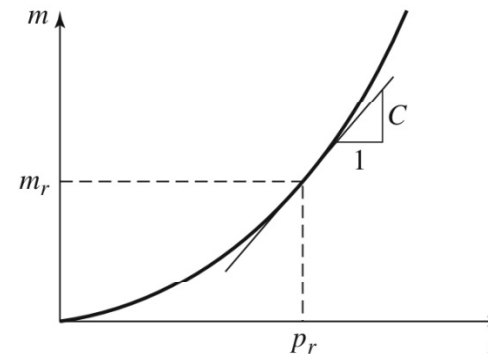
$$C = \rho A \frac{dh}{dp} = \rho A \frac{1}{\rho g} = \frac{A}{g}$$

$$\frac{dm}{dt} = C\frac{dp}{dt} = q_{mi} - q_{mo}$$

$$C\frac{dp}{dt} = \rho A \frac{dh}{dt} = q_{mi} - q_{mo}$$

For liquid level system, C is defined as

$$C = \frac{change\ in\ liquid\ stored}{change\ in\ head}\frac{m^3}{m} = \rho A$$
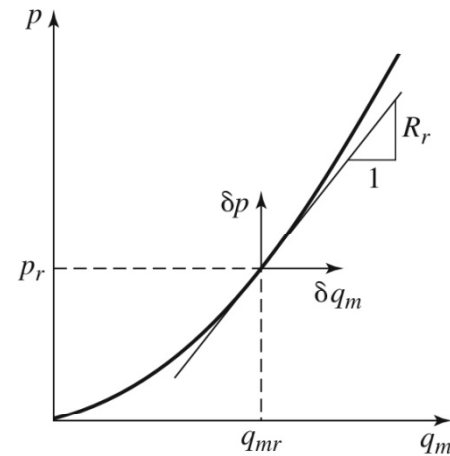
use h(pressure head) , instead of p.

Fluid Resistance

$$R = \frac{change\ in\ pressure}{change\ in\ flow\ rate} = \frac{dp}{dq_m}\bigg|_{q=q_{mr}}$$

$$q_m = \frac{p}{R} \qquad \text{For laminar flow}$$

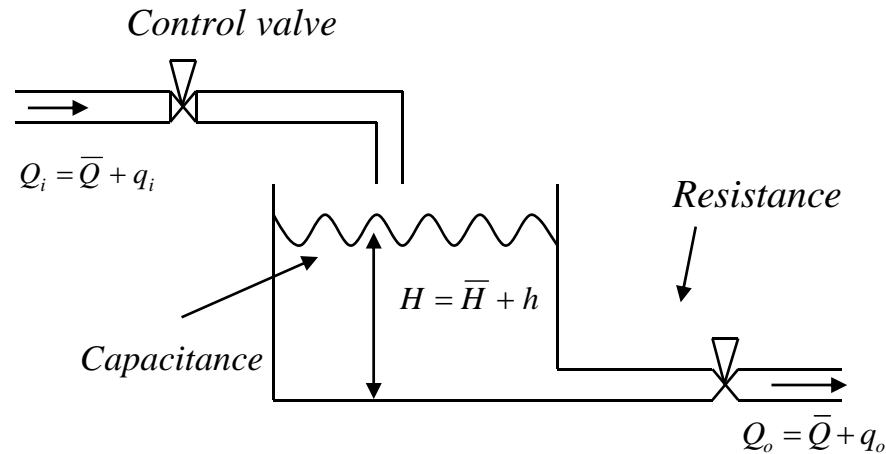$$q_m = \sqrt{\frac{p}{R_l}} \qquad \text{For turbulent flow}$$

For liquid level system, C is defined as

$$R = \frac{change\ in\ level\ difference}{change\ in\ flow\ rate}\ \frac{m}{m^3/s} = \frac{dh}{dq_m}$$

Liquid   Level   Systems

$$R = \frac{change\ in\ level\ difference}{change\ in\ flow\ rate}\ \frac{m}{m^3 / s} \qquad C = \frac{change\ in\ liquid\ stored}{change\ in\ head}\ \frac{m^3}{m}$$
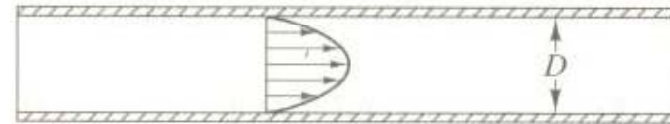
$$Steady\ state\ :\ \ Q_i = \bar{Q} = Q_0 = \frac{\bar{H}}{R}$$

$$consider,\ \ Q_o = \frac{H}{R},\ \ \ \ Q_i - Q_o = C\frac{dH}{dt}$$

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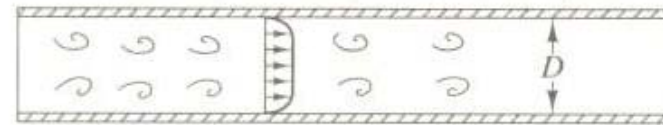and Aerospace Engineering

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Basic Concepts

Laminar flow : When the viscous forces are dominant (slow flow, low Re) they are sufficient enough to keep all the fluid particles in line, then the flow is laminar.

Turbulent flow : It can be interpreted that when the inertial forces dominate over the viscous forces (when the fluid is flowing faster and Re is larger) then the flow is turbulent.

Reynolds number (Re) : The ratio of inertial forces to viscous forces. It is used to determine whether a flow will be laminar or turbulent.

$$\text{Re} = \frac{\rho v D}{\mu}$$

μ : the dynamic viscousity of the fluid

ρ : density of the fluid

D : diameter

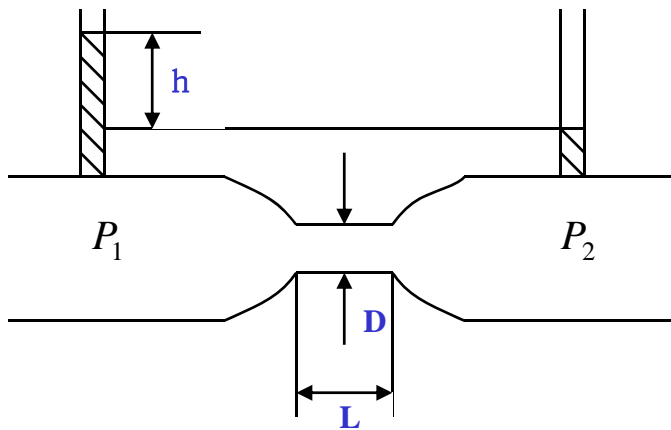$\text{Re} < 2000$: always laminar

$\text{Re} > 4000$: always turbulent

Laminar Flow

Cylindrical pipe

$$P_1 - P_2 = \rho g h, \quad Q \frac{128\nu L}{g\pi D^4} = h$$

$\nu : viscosity,$

$L : length\ of\ pipe$

$D : diameter\ of\ pipe$

$$Q = \frac{h}{R} = K_l h, \qquad R = \frac{128\nu L}{g\pi D^4} \quad [s/m^2]$$

$Q\ : steady - state\ flow\ rate$

$K_l : constant$

$h\ : steady - state\ head$

Laminar Flow

Liquid level dynamics :

$$C \frac{dH}{dt} = Q_i - Q_o$$

$$= \bar{Q} + q_i - (\bar{Q} + q_o) = q_i - q_o$$

$$\frac{dH}{dt} = \frac{d}{dt}(\bar{H} + h) = \frac{dh}{dt}$$

$$Q_o = \bar{Q} + q_o = \frac{\bar{H} + h}{R}, \qquad q_o = \frac{h}{R}$$

$$\rightarrow \quad \frac{dh}{dt} = -\frac{1}{CR}h + \frac{1}{C}q_i$$

Turbulent Flow

$Q_i = \overline{Q} + q_i$

$H = \overline{H} + h$

R

C

$Q_0 = \overline{Q} + q_0$

$$Q = C_d \cdot a \cdot \sqrt{\frac{2}{\rho}(P_1 - P_2)}$$

$\rho : density, \qquad a : area, \qquad C_d : discharge\ coefficient$

$steady\ state : Q_i = \overline{Q} = Q_0 = K_t \sqrt{\overline{H}}$

Turbulent Flow

Liquid level dynamics

$$C\frac{dH}{dt} = Q_i - Q_0 = \overline{Q} + q_i - K\sqrt{H}$$

$$\frac{dH}{dt} = \frac{1}{C}Q_i - \frac{1}{C}K\sqrt{H} = f(Q_i, H)$$

$$f(Q_i, H) = f(\overline{Q}_i, \overline{H}) + \left.\frac{\partial f}{\partial Q_i}\right|_{\overline{Q},\overline{H}}(Q_i - \overline{Q}) + \frac{1}{2!}\frac{\partial^2 f}{\partial Q_i^2}(Q_i - \overline{Q})^2 + \cdots$$

$$\cdots + \left.\frac{\partial f}{\partial H}\right|_{\overline{Q},\overline{H}}(H - \overline{H}) + \frac{1}{2!}\frac{\partial^2 f}{\partial H^2}(H - \overline{H})^2 + \cdots$$

$$= \frac{1}{C}\overline{Q} - \frac{1}{C}K\sqrt{\overline{H}} + \left(\frac{1}{C}q_i - \frac{K}{C \cdot 2\sqrt{\overline{H}}}h\right) + high\ order\ term$$

Turbulent Flow

$$\frac{dH}{dt} = \frac{d\bar{H}}{dt} + \frac{dh}{dt}$$
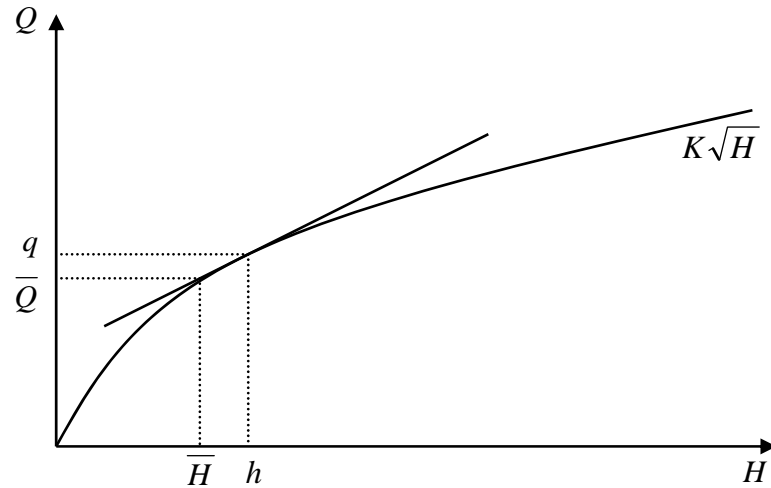
$$\Rightarrow \quad \frac{dh}{dt} = \frac{1}{C}q_i - \frac{K}{C \cdot 2\sqrt{\bar{H}}}h$$

$$q_o = \frac{h}{R}, \qquad R = \frac{2\sqrt{\bar{H}}}{K} = \frac{2\bar{H}}{\bar{Q}}, \quad \left(\bar{Q} = K\sqrt{\bar{H}}\right)$$

$$\therefore \quad \begin{cases} \dfrac{dh}{dt} = -\dfrac{1}{CR}h + \dfrac{1}{C}q_i \\[3mm] q_o = \dfrac{h}{R} \end{cases} \qquad for \ \ small \ \ q_i$$

Linearization

$$Q = \overline{Q} + q = K\sqrt{\overline{H}} + (\quad) \cdot h$$

$$y = f(x), \qquad \overline{y} = f(\overline{x}), \qquad x = \overline{x} + \Delta x$$

$$y = f(\overline{x}) + \left. \frac{\partial f}{\partial x} \right|_{\overline{x}} \cdot \Delta x + \frac{1}{2!} \left. \frac{\partial^2 f}{\partial x^2} \right|_{\overline{x}} \cdot \Delta x^2 + \cdots$$

$$\approx \overline{y} + \left. \frac{\partial f}{\partial x} \right|_{\overline{x}} \cdot \Delta x$$

$$y - \overline{y} = \left. \frac{\partial f}{\partial x} \right|_{\overline{x}} (x - \overline{x}) = K(x - \overline{x})$$

$$y = f(x_1, x_2) \approx f(\overline{x_1}, \overline{x_2}) + \left. \frac{\partial f}{\partial x_1} \right|_{\overline{x_1}, \overline{x_2}} (x_1 - \overline{x_2}) + \left. \frac{\partial f}{\partial x_2} \right|_{\overline{x_1}, \overline{x_2}} (x_2 - \overline{x_2})$$

$$= \overline{y} + K_1(x_1 - \overline{x_1}) + K_2(x_2 - \overline{x_2})$$

$$\therefore \ y - \overline{y} = K_1(x_1 - \overline{x_1}) + K_2(x_2 - \overline{x_2})$$

Summary of Liquid Level Systems

Laminar flow

$$\frac{dh}{dt} = -\frac{1}{CR}h + \frac{1}{C}q_i$$

$$\overline{Q} = K \cdot \overline{H} \qquad R = \frac{\overline{H}}{\overline{Q}}$$

$$R = \frac{128\nu L}{g\pi D^4}$$

$$q_o = \frac{h}{R}$$

Turbulent flow

$$\frac{dh}{dt} = -\frac{1}{CR}h + \frac{1}{C}q_i$$
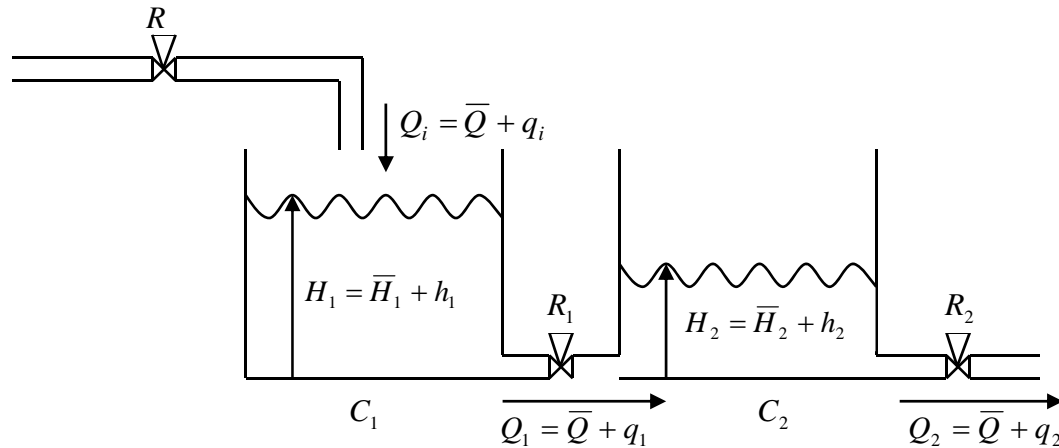
$$q_o = \frac{h}{R}$$

$$R = \frac{2\overline{H}}{\overline{Q}} = \frac{2\sqrt{\overline{H}}}{K} = \frac{2\sqrt{\overline{H}}}{C_d a\sqrt{2g}}$$

$$\left(\because \overline{Q} = K\sqrt{\overline{H}} = C_d a\sqrt{\frac{2}{\rho}\rho g \overline{H}} = C_d a\sqrt{2g}\sqrt{\overline{H}} \right)$$

Liquid Level Systems with Interaction

Steady state :

$$Q_i = \bar{Q} = Q_1 = Q_2$$

$$= \frac{\bar{H}_1 - \bar{H}_2}{R_1} = \frac{\bar{H}_2}{R_2}$$

Liquid level dynamics :

$$C_1 \frac{dH_1}{dt} = \bar{Q} + q_i - \left(\bar{Q} + q_1 \right) \qquad \rightarrow \quad C_1 \frac{dh_1}{dt} = q_i - q_1$$

$$Q_1 = \bar{Q} + q_1 = \frac{1}{R_1} \left(\bar{H}_1 + h_1 - (\bar{H}_2 + h_2) \right) \quad \rightarrow \quad q_1 = \frac{h_1 - h_2}{R_1}$$

$$C_2 \frac{dH_2}{dt} = \bar{Q} + q_1 - \left(\bar{Q} + q_2 \right) \qquad \rightarrow \quad C_2 \frac{dh_2}{dt} = q_1 - q_2$$

$$Q_2 = \bar{Q} + q_2 = \frac{\bar{H}_2 + h_2}{R_2} \qquad\qquad \rightarrow \quad q_2 = \frac{h_2}{R_2}$$

Liquid Level Systems with Interaction

$$\therefore \begin{cases} \dfrac{dh_1}{dt} = -\dfrac{1}{C_1}\left(\dfrac{h_1 - h_2}{R_1}\right) + \dfrac{1}{C_1} q_i \\[4mm] \dfrac{dh_2}{dt} = \dfrac{1}{C_2}\left(\dfrac{h_1 - h_2}{R_1}\right) - \dfrac{1}{C_2}\dfrac{h_2}{R_2} \end{cases}$$

$$Transfer\ Function = \frac{Q_2(S)}{Q_i(S)} = \frac{1}{R_1 C_1 R_2 C_2 S^2 + \left(R_1 C_1 + R_2 C_2 + R_2 C_1\right) S + 1}$$