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Mohammed-Najib Benbourhim, Patrick Chenin, Abdelhak Hassouni & Jean-Baptiste Hiriart-Urruty, Editors

CALCULATING THE H_∞-NORM OF LARGE SPARSE SYSTEMS VIA CHANDRASEKHAR ITERATIONS AND EXTRAPOLATION *

Younes Chahlaoui[1], Kyle A. Gallivan[1] and Paul Van Dooren[2]

Abstract. We describe an algorithm for estimating the $\mathcal{H}_\infty$-norm of a large linear time invariant dynamical system described by a discrete time state-space model. The algorithm uses Chandrasekhar iterations to obtain an estimate of the $\mathcal{H}_\infty$-norm and then uses extrapolation to improve these estimates.

Résumé. Nous décrivons un algorithme pour estimer la norme $\mathcal{H}_\infty$ d'un système dynamique linéaire à temps invariant de grande dimension décrit par un modèle d'espace d'état discret. L'algorithme emploie des récurrences de Chandrasekhar pour obtenir une estimation de la norme $\mathcal{H}_\infty$ puis emploie l'extrapolation pour améliorer ces estimations.

1. Introduction

We consider the computation of the $\mathcal{H}_\infty$-norm of a $p \times m$ real rational transfer function

$$G(z) := C(zI_n - A)^{-1}B + D \tag{1}$$

of a discrete-time system, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$, with $n \gg m, p$. We start with ideas developed in [2] to approximate the $\mathcal{H}_\infty$-norm using the convergence properties of Chandrasekhar iterations and propose an efficient extrapolation technique to improve those estimates. The performance of the extrapolation is then illustrated using a number of experiments.

The $\mathcal{H}_\infty$-norm of a rational transfer function $\gamma^* := \|G(z)\|_\infty := \sup_{\|u(z)\|_{\ell_2}=1} \|G(z)u(z)\|_{\ell_2}$ is the supremal ℓ_2 norm of the system response $y(z) = G(z)u(z)$ for an arbitrary unit energy input $u(z)$. It is bounded if and only if $G(z)$ is stable [6] and it can then be evaluated from the frequency response using

$$\|G(z)\|_\infty = \max_{\omega\in[0,2\pi]} \|G(e^{j\omega})\|_2.$$

We therefore assume that the given quadruple $\{A, B, C, D\}$ is a real and minimal realization of a stable transfer function $G(z)$. The stability of $G(z)$ implies that all of the eigenvalues of A are strictly inside the unit circle, and hence that $\rho(A) < 1$, where $\rho(A)$ is the spectral radius of A. An important result upon which we rely is the *bounded real lemma*, which states that $\gamma > \|G(z)\|_\infty$ if and only if there exists a positive definite solution P (denoted as $P \succ 0$) to the linear matrix inequality (LMI) [1] :

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[1] SCS, Florida State University, Tallahassee FL 32306-4120, USA;
e-mail: younes.chahlaoui@laposte.net & gallivan@csit.fsu.edu

[2] CESAME, Université catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium; e-mail: vdooren@csam.ucl.ac.be

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$$H(P) := \begin{bmatrix} P - A^T P A - C^T C & -A^T P B - C^T D \\ -B^T P A - D^T C & \gamma^2 I_m - B^T P B - D^T D \end{bmatrix} \succ 0. \tag{2}$$

This suggests that γ^* could be calculated by starting with an initial large value $\gamma > \gamma^*$ for which the LMI has a solution and iterate by decreasing γ until the LMI does not have a solution anymore. The condition $H(P) \succ 0$ implies that the matrix

$$R_\gamma := B^T P B + D^T D - \gamma^2 I_m$$

is negative definite and hence invertible. One shows [6] that the condition $H(P) \succ 0$ is then equivalent to the existence of a solution P for the discrete-time algebraic Riccati equation (DARE) :

$$P = A^T P A + C^T C - (A^T P B + C^T D) R_\gamma^{-1} (B^T P A + D^T C), \tag{3}$$

together with the condition that $-R_\gamma \succ 0$. The solution P of (3) can be obtained from the calculation of an appropriate eigenvalue problem [8]. However, in order to exploit the sparsity of the matrix A to yield an efficient algorithm for large systems, it was recommended in [2] to solve (3) using an iterative scheme known as the Chandrasekhar iteration.

The remainder of this paper is organized as follows. The use of the Chandrasekhar recurrences to solve the DARE parameterized by γ is discussed in Section 2, which is followed by Section 3 on convergence and extrapolation. In Section 4 we illustrate our extrapolation method using two numerical examples.

2. Chandrasekhar iteration

Efficient algorithms to solve the DARE have been proposed in the literature [6]. The so-called Chandrasekhar recurrences amount to calculating the solution of the discrete-time Riccati difference equation

$$P_{i+1} = A^T P_i A + C^T C - (A^T P_i B + C^T D) R_i^{-1} (B^T P_i A + D^T C) \tag{4}$$

where

$$R_i := B^T P_i B + D^T D - \gamma^2 I_m,$$

in an efficient manner. Defining also the matrix

$$K_i := B^T P_i A + D^T C, \tag{5}$$

this becomes

$$P_{i+1} = A^T P_i A + C^T C - K_i^T R_i^{-1} K_i. \tag{6}$$

Clearly the difference matrices $\delta P_i := P_{i+1} - P_i$ satisfy

$$\delta P_{i+1} = A^T \delta P_i A - K_{i+1}^T R_{i+1}^{-1} K_{i+1} + K_i^T R_i^{-1} K_i. \tag{7}$$

Using (5-7), one obtains the following identity :

$$\begin{bmatrix} R_{i+1} & K_{i+1} \\ K_{i+1}^T & \delta P_{i+1} + K_{i+1}^T R_{i+1}^{-1} K_{i+1} \end{bmatrix} = \begin{bmatrix} B^T \delta P_i B + R_i & B^T \delta P_i A + K_i \\ A^T \delta P_i B + K_i^T & A^T \delta P_i A + K_i^T R_i^{-1} K_i \end{bmatrix}. \tag{8}$$

Notice that the Schur complement with respect to the (2,2) block is equal to δP_{i+1}. Assume now that for each step i we define the matrices L_i, S_i and G_i according to

$$\delta P_i = L_i^T \Sigma_2 L_i, \quad R_i = S_i^T \Sigma_1 S_i, \quad K_i = S_i^T \Sigma_1 G_i.$$

This, of course, implies that the signature of the matrices R_i and δP_i remains constant for all i. It is shown in [6] that this condition is in fact necessary and sufficient for the Riccati difference equation (4) to converge (see Lemma 3.1 below). An obvious choice is to take $P_0 = 0$, which yields

$$\delta P_0 = P_1 = C^T C - C^T D(D^T D - \gamma^2 I_m)^{-1} D^T C.$$

It also follows from the LMI (2) that we must take $\gamma^2 I_m - D^T D \succ 0$, which implies

$$\delta P_0 = C^T [I + D(\gamma^2 I_m - D^T D)^{-1} D^T] C \succ 0.$$

We thus have that $\Sigma_1 = -I_m$ and $\Sigma_2 = I_\alpha$, where $\alpha \leq p$ is the rank of P_1. The matrix (8) can then be factored as

$$\begin{bmatrix} R_{i+1} & K_{i+1} \\ K_{i+1}^T & \delta P_{i+1} + K_{i+1}^T R_{i+1}^{-1} K_{i+1} \end{bmatrix} = \begin{bmatrix} S_{i+1}^T & 0 \\ G_{i+1}^T & L_{i+1}^T \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} S_{i+1} & G_{i+1} \\ 0 & L_{i+1} \end{bmatrix}. \tag{9}$$

One also easily checks the identity

$$\begin{bmatrix} B^T \delta P_i B + R_i & B^T \delta P_i A + K_i \\ A^T \delta P_i B + K_i^T & A^T \delta P_i A + K_i^T R_i^{-1} K_i \end{bmatrix} = \begin{bmatrix} S_i^T & B^T L_i^T \\ G_i^T & A^T L_i^T \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} S_i & G_i \\ L_i B & L_i A \end{bmatrix}. \tag{10}$$

It follows from the comparison of (9) and (10) that there exists a pseudo-orthogonal transformation Q satisfying

$$\Sigma := \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \quad Q^T \Sigma Q = \Sigma, \tag{11}$$

$$Q \begin{bmatrix} S_i & G_i \\ L_i B & L_i A \end{bmatrix} = \begin{bmatrix} S_{i+1} & G_{i+1} \\ 0 & L_{i+1} \end{bmatrix}.$$

It is precisely this recurrence that is used to compute at low cost the updates of S_i, G_i and L_i. The complexity of the multiplication with $Q \in \mathbb{R}^{m+\alpha \times m+\alpha}$ can be shown to be roughly $2m(m+\alpha)(m+n)$ operations [6] and the matrix products $L_i A$ and $L_i B$ require the product of $\alpha \ll n$ rows with matrices A and B, which is acceptable for large sparse systems. The feedback matrix F_i and closed loop matrix are defined as follows :

$$F_i := R_i^{-1} K_i, \quad A_{F_i} := A - B F_i.$$

Since the matrix R_i is nonsingular, so is S_i and it then follows that $F_i = S_i^{-1} G_i$. This is simpler from a computational point of view since the matrix S_i is already triangular. The closed loop matrix A_{F_i} has a spectral radius $\rho_i := \rho(A_{F_i})$ which determines essentially the convergence of the Riccati difference equation (4). So if γ is overestimated then ρ_i will be smaller than 1, while if γ is underestimated, ρ_i will become larger or equal to 1 and eventually the signature Σ will not be constant, since the DARE does not have a symmetric steady state solution. Since $A - BF_i$ is stable and converges to $A - BF_\gamma$ (where F_γ is the feedback corresponding to the DARE), one can track its spectral radius by the power method applied to $A - BF_i$. One power iteration step amounts to computing

$$x_{k+1} = Ax_k - BS_i^{-1}(G_i x_k)$$

which is quite cheap for sparse matrices A and B. This is precisely used in the extrapolation scheme explained in the next section, where we also discuss the convergence of the Chandrasekhar recurrence.

3. Convergence and extrapolation

A first important result is due to Hassibi and al. [6] and gives necessary and sufficient conditions for the Chandrasekhar iteration to converge for $\gamma > \gamma^*$.

Lemma 3.1. *The Chandrasekhar iteration (11), initialized with $P_0 = 0$ and hence with*

$$\delta P_0 = P_1 = C^T C - C^T D(D^T D - \gamma^2 I_m)^{-1} D^T C$$

converges with constant signature matrix, i.e.

$$\lim_{i\to\infty} L_i = 0 \quad \text{with} \quad \Sigma = \begin{bmatrix} -I_m & 0 \\ 0 & I_\alpha \end{bmatrix},$$

if and only if $\gamma > \gamma^*$. The closed loop matrix $A_{F_\gamma} := A - BF_\gamma$ then has spectral radius $\rho(A_{F_\gamma})$ strictly smaller than one, and $\rho(A_{F_\gamma})$ tends to 1 when γ tends to γ^*.

The Riccati difference equation (4) has many formulations. One useful formulation results from the two-point boundary value problem (see [4] and the references therein). For our case, we have the following result, e.g. proven in [2, 6]:

Lemma 3.2. *When $\bar{R}_\gamma := D^T D - \gamma^2 I_m$ is nonsingular, the Riccati difference equation (4) can be rewritten as*

$$M_1 \begin{bmatrix} I \\ P_{i+1} \end{bmatrix} = M_2 \begin{bmatrix} I \\ P_i \end{bmatrix} (A - BF_i). \tag{12}$$

where

$$M_1 := \begin{bmatrix} A - B\bar{R}_\gamma^{-1} D^T C & 0 \\ -C^T C + C^T D \bar{R}_\gamma^{-1} D^T C & I \end{bmatrix}, \quad M_2 := \begin{bmatrix} I & B\bar{R}_\gamma^{-1} B^T \\ 0 & A^T - C^T D \bar{R}_\gamma^{-1} B^T \end{bmatrix}.$$

Convergence will occur when the pencil $\lambda M_1 - M_2$ (or the matrix $M_1^{-1} M_2$) has no eigenvalues on the unit circle. The feedback matrix F_i then converges to a feedback F_γ such that $A - BF_\gamma$ has eigenvalues in the open unit disc.

Note that the condition of non-singularity of $\bar{R}_\gamma$ is generically true (i.e., it is singular on a set of measure 0). For $\gamma > \gamma^*$ the matrix $\bar{R}_\gamma$ is guaranteed to be nonsingular since $-R_\gamma \succ 0$ implies $-\bar{R}_\gamma \succ B^T P_i B \succ 0$.

Now let us assume $\gamma > \gamma^*$ and that P is the steady state solution of (4) for that γ. Then as P_i converges to P we can suppose that $\Delta_i := P - P_i$ is small. The following lemma is then proven in [2].

Lemma 3.3. *When the Riccati difference equation (4) converges, each iteration corresponds to an approximate Schur decomposition*

$$\begin{bmatrix} I & 0 \\ -P_i & I \end{bmatrix} M_2^{-1} M_1 \begin{bmatrix} I & 0 \\ P_i & I \end{bmatrix} = \begin{bmatrix} A_{F_i} & X \\ E_{21} & A_{F_i}^{-T} \end{bmatrix}$$

where

$$E_{21} := \Delta_i A_{F_\gamma} - A_{F_\gamma}^{-T} \Delta_i - \Delta_i X \Delta_i, \quad A_{F_i} := A - BF_i,$$

and clearly the spectrum of A_{F_i} converges to that of A_{F_γ} (and hence the stable spectrum of $M_2^{-1} M_1$).

It was shown in [9] that when there exists a positive definite solution P to the DARE (4) then upon convergence we have

$$\Delta_{i+1} \approx A_{F_i}^T \Delta_i A_{F_i},$$

which indicates again that the Chandrasekhar iteration converges linearly. The following lemma [6] shows that such a relation also holds for δP_i.

Lemma 3.4. *Upon convergence, the Chandrasekhar recurrences yield matrices δP_i satisfying*

$$\delta P_{i+1} = A_{F_i}^T \delta P_i A_{F_i} - (F_{i+1} - F_i)^T R_{i+1} (F_{i+1} - F_i) \approx A_{F_i}^T \delta P_i A_{F_i}.$$

Proof. Using the identities

$$B^T\delta P_i B = R_{i+1} - R_i \quad \text{and} \quad B^T\delta P_i A = K_{i+1} - K_i = R_{i+1}F_{i+1} - R_iF_i$$

we easily find that

$$\begin{aligned} A_{F_i}^T\delta P_i A_{F_i} &= A^T\delta P_i A - F_i^T B^T\delta P_i A - A^T\delta P_i BF_i + F_i^T B^T\delta P_i BF_i \\ &= A^T\delta P_i A - F_i^T(R_{i+1}F_{i+1} - R_iF_i) - (F_{i+1}^T R_{i+1} - F_i^T R_i)F_i + F_i^T(R_{i+1} - R_i)F_i \\ &= A^T\delta P_i A + F_i^T R_iF_i - F_{i+1}^T R_{i+1}F_{i+1} + (F_{i+1} - F_i)^T R_{i+1}(F_{i+1} - F_i). \end{aligned}$$

The rest follows from comparing this with (7) and from the fact that $\|F_{i+1} - F_i\| = O(\delta)$ appears quadratically in the equation and hence is negligible. □

It follows from the above lemma that $\|L_{i+1}\|_2 \approx \|L_iA_{F_i}\|_2$, which also shows that the corrections L_i converge linearly to 0. It is also important to note that if $\gamma > \gamma^*$ then A_{F_γ} is stable and since $A - BF_i \approx A_{F_\gamma}$ we can estimate $\rho(A_{F_\gamma})$ using subspace iteration on $A - BF_i$, i.e.,

$$Q_iR_i = (A - BF_i)Q_{i-1}, \quad Q_i^TQ_i = I_k,$$

started with an arbitrary k-dimensional orthogonal basis Q_0. This can be performed at low cost since A is sparse and BF_i is relatively low rank. Moreover, even if $\gamma \leq \gamma^*$ and A_{F_γ} is unstable, the eigenvalues of $Q_{i-1}^T(A - BF_i)Q_{i-1}$ will be close to the dominant spectrum of $A - BF_i$ and according to Lemma 3.3 this will be close to k eigenvalues of the symplectic pencil as long as Δ_i is small.

Figure 1 describes the convergence properties of the Riccati difference equation and the H_∞ approximation algorithm in terms of the spectral radius $\rho(A_{F_\gamma})$ as a function of γ. One can define a region of acceptance for the approximation of γ^* and the width of this region will depend mainly of a tolerance value associated with the convergence/divergence decision. As long as the δP_i remain reasonably small, so will E_{21} and the spectrum of A_{F_i} then gives a reasonable approximation of half of the spectrum of the symplectic pencil. This can be used to detect the symplectic eigenvalues closest to or on the unit circle.

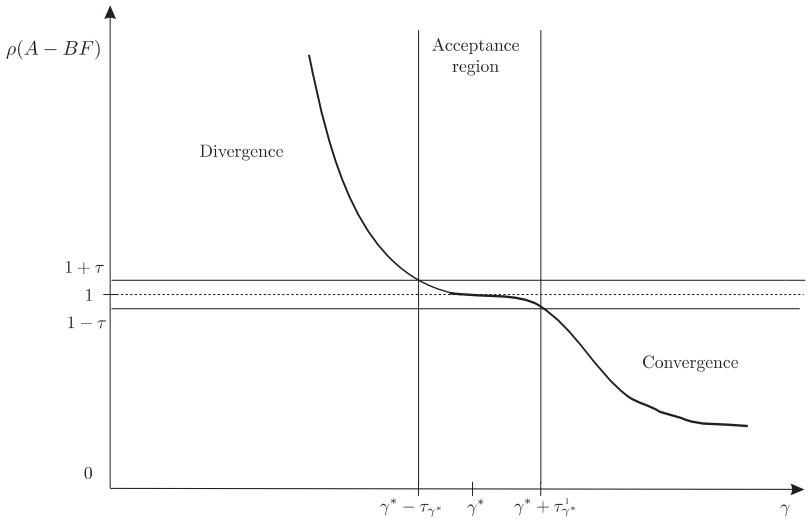

FIGURE 1. Estimate of $\phi(\gamma) := \rho(A - BF_\gamma)$ as a function of γ

The above figure is in a sense misleading since for $\gamma < \gamma^*$, there is no convergence and hence F_γ does in fact not exist. But the matrices $A - BF_i$ diverge and the estimate of their norm evaluated by the power iteration, becomes larger than 1. What is shown in Figure 1 is thus in fact only an estimate. But we can prove the following property of the *true* function $\phi(\gamma)$.

Theorem 3.5. *The function $\phi(\gamma)$ is a continuous (but not necessarily differentiable) function of γ for $\gamma > \gamma^*$.*

Proof. It follows from Lemma 3.3 that $\phi(\gamma) = \rho(A - BF_\gamma)$ where F_γ is the steady state feedback matrix obtained from the DARE. This is therefore the largest generalized eigenvalue of the symplectic pencil $\lambda M_1 - M_2$ inside the unit circle. But the elements of M_1 and M_2 are rational functions of γ and its eigenvalues therefore vary continuously with γ [7]. Also $\phi(\gamma)$ is in general differentiable for almost all $\gamma > \gamma^*$, but it may not be differentiable in a few isolated points. □

Clearly, the convergence/divergence decision plays a crucial role in the choice of the direction of the adaptation of γ. Recall that for a given initial condition P_0, the solution of the discrete-time Riccati equation is given at each instant i by

$$P_i = P_0 + \sum_{k=1}^{i-1} \delta P_k.$$

For a given tolerance τ, we will say that the discrete-time Riccati equation diverges, if one of the following is true:

- the spectral radius $\rho_i = \rho(A_{F_i})$ (estimated by subspace iteration) is larger than $1 + \tau$
- the ratio $\|\delta P_{i+1}\|_2 / \|\delta P_i\|_2 = \|L_{i+1}\|_2^2 / \|L_i\|_2^2$ is larger than $(1 + \tau)^2$. Notice that this is similar to the previous as one has the relation $\|L_{i+1}\|_2 / \|L_i\|_2 \approx \rho(A - BF_i)$,
- the inequality $\gamma^2 I_m - B^T P B - D^T D \succ 0$ does not hold.

By monitoring the convergence using one of these criteria, one can decide if at the steady state (or at least for a large number of steps) we have relative convergence or not and adapt γ in the appropriate direction.

In order for the algorithm to approximate γ^* to work efficiently there must be an effective method to adapt the value of γ given the observed behavior of the Riccati difference equation. The simplest method is the combination of the Chandrasekhar equations with a bisection method to estimate γ^*. A lower bound for $\gamma^* = \|G(z)\|_\infty$ is easily obtained from $\gamma_{lo} := \|G(e^{j\omega_{lo}})\|_2$ for any frequency ω_{lo}, or from any input/output pair collected over a finite simulation horizon. The idea for estimating γ^* is to run the Chandrasekhar equations for a given $\gamma > \gamma_{lo}$ and check whether or not it converges. If it converges, then $\gamma_{up} := \gamma$ is an overestimate for γ^* and we repeat the process for a new value of γ (say $(\gamma_{lo} + \gamma_{up})/2$) in the interval $[\gamma_{lo}, \gamma_{up}]$ which is known to contain γ^*. If divergence is observed then $\gamma_{lo} := \gamma$ before choosing the next value of γ.

There are several possible avenues to consider to get an effective strategy to update γ to approach γ^* from above. All share the need to model, based on data observed while executing the algorithm, how $\phi(\gamma) := \rho(A - BF_\gamma)$ evolves with γ. Essentially, this is an attempt to determine the function in Figure 1 for $\gamma > \gamma^*$. We have found that for each value of γ, $\rho(A - BF_\gamma)$ can be estimated reasonably well with a few subspace iteration steps. We can therefore estimate the value of the function $\phi(\gamma)$ for several values of γ which can be used to produce a $\gamma > \gamma^*$ closer to γ^* via interpolation.

For any of these strategies, once $\gamma_{old} \to \gamma_{new}$ we must make a choice as to how to proceed with the Chandrasekhar iteration. The simplest approach is to simply restart the iteration with $P_0 = 0$ and monitor convergence/divergence in preparation for the next update of γ.

4. Algorithm and numerical examples

Our algorithm starts by computing a lower bound γ_{lo} obtained from a random simulation for a given input sequence $u(1:k)$. The ratio $\gamma_0 := \|y(1:k)\|_2 / \|u(1:k)\|_2$, where $y(1:k)$ is the corresponding output sequence, is clearly a valid lower bound γ_{lo}. We double this value recursively until we find a value γ_{up} for which the Chandrasekhar iteration converges. Meanwhile γ_{lo} is updated to the last value for which convergence did *not*

occur. Then we choose a value of γ in the interval $[\gamma_{lo}, \gamma_{up}]$ and we test convergence of the corresponding Chandrasekhar iteration. A simple choice is to take the midpoint or a random point in that interval but more educated guesses could be made. If convergence occurs, γ becomes the new γ_{up}; if divergence occurs, γ becomes the new γ_{lo}. When it is too hard to decide convergence or divergence based on a reasonable number of iterations, we choose another value of γ in the current interval.

During this iteration one narrows the interval $[\gamma_{lo}, \gamma_{up}]$ for the true norm γ^*. But meanwhile we also store the spectral radius $\rho(A - BF_\gamma)$ for all values of γ where convergence or divergence was easy to decide. Those values are then used to approximate the function $\phi(\gamma)$ and estimate the value at which it crosses 1. That is our final estimate of the $\mathcal{H}_\infty$ norm of the transfer function.

To illustrate this, we present here two numerical experiments based on stable minimal discrete time systems. Both systems have two inputs and two outputs. The first example was a randomly generated discrete-time system $\{A, B, C, D\}$ of state dimension $n = 100$ and spectral radius $\rho(A) = .8$. The curved line in that plot is the largest singular value of $G(e^{j\omega})$ as a function of the frequency ω. Figure 3 indicates the estimated values of $\phi(\gamma) = \rho(A - BF_\gamma)$ at the values of γ where we ran the Chandrasekhar iteration. The full line interpolating these values is our estimate of $\phi(\gamma)$ and the value at which it crosses 1 is our estimate of γ^*. The true value for γ^* is marked by a full square and is 43.42 in this example. We can see from this example that our match is quite accurate.

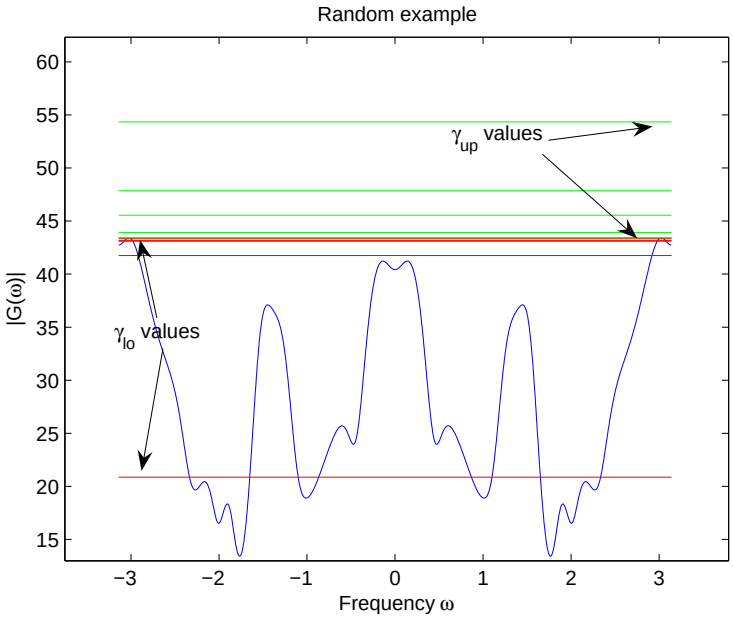

FIGURE 2. Random example : Frequency plot with levels for γ_{lo} and γ_{up} and undecided levels (dashed)

The second experiment is for the CD-player model [3], which we discretized using the zero order hold method of the MATLAB Control Toolbox function C2D. This discretized system $\{A, B, C, D\}$ has state dimension $n = 120$ and spectral radius $\rho(A) = .988$ (its poles are thus very close to the unit circle). One can see in Figure 4 that there are several levels of γ for which it was unclear to decide if the iteration converged or not, within a finite number of steps (we chose 20 steps). Those values correspond to the dashed horizontal lines in Figure 4. Nevertheless, the $\phi(\gamma)$ plot based on the decidable Chandrasekhar runs yield a good plot from which we can estimate γ^*. The true value for γ^* is marked by a full square again and is 2.50×10^5 in this example.

The Chandrasekhar iteration with bisection exploits sparse matrix kernels and low order dense matrices to achieve efficiency for large problems. For the experiments presented here, MATLAB implementations of the

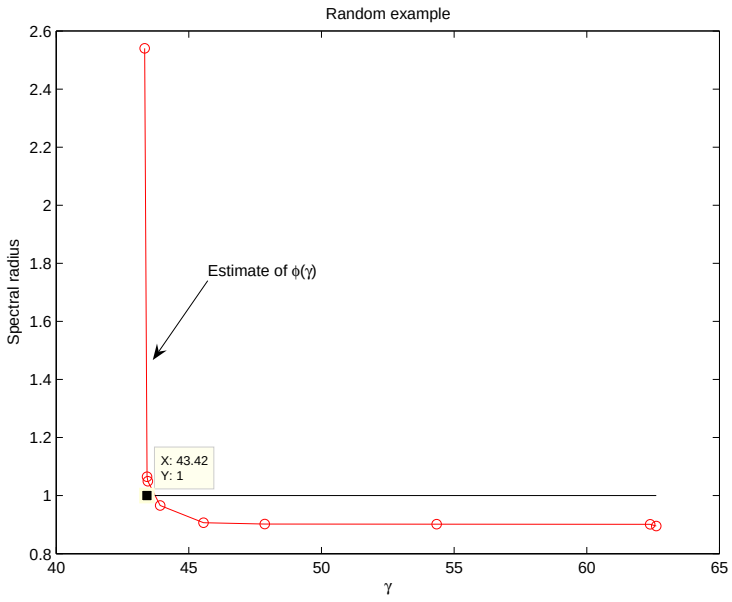

FIGURE 3. Random example : Evolution of $\phi(\gamma)$ as a function of γ

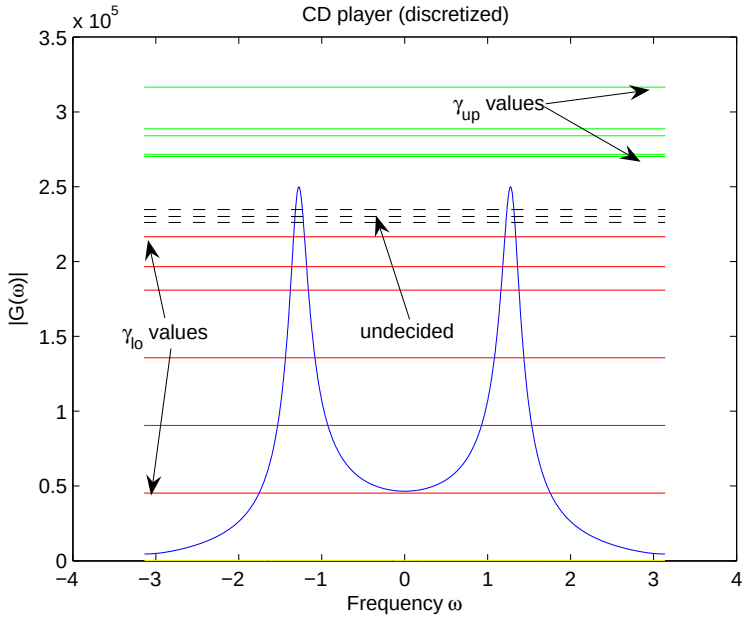

FIGURE 4. CD player : Frequency plot with levels for γ_{lo} and γ_{up} and undecided levels (dashed)

Chandrasekhar iteration without substantial performance tuning was several time faster than other competing methods like the level set method described in [5], or the MATLAB Control Toolbox function NORM(SYS,inf).

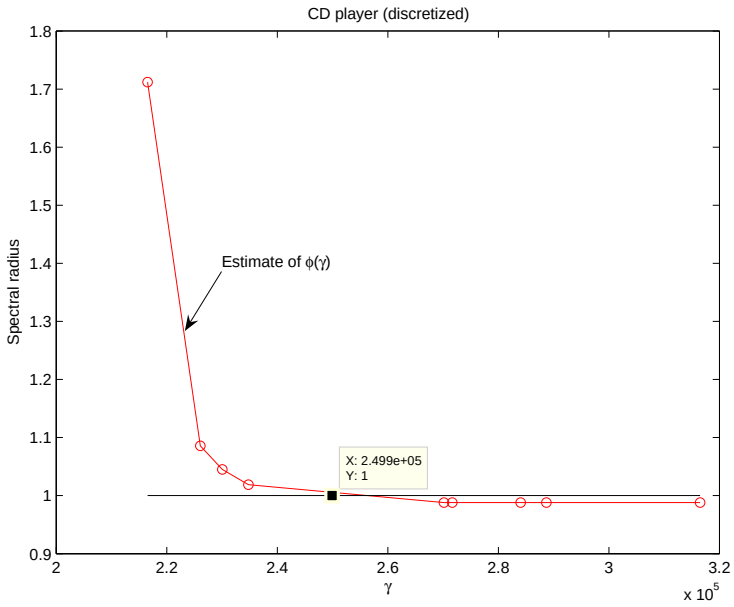

FIGURE 5. CD player : Evolution of $\phi(\gamma)$ as a function of γ

5. Conclusion

We have presented an algorithm based on the Chandrasekhar iteration and initial empirical evidence that it can be used to estimate efficiently $\|G(z)\|_\infty$ for large discrete time linear time invariant dynamical systems. There is still work to do in order to develop a reliable and efficient method. We are currently investigating the influence of the structure of the spectrum of A on the behavior of the algorithm particularly relative to the convergence/divergence decision. We are also investigating an approach that uses the estimate of $\phi(\gamma)$ and estimates of left and right eigenvectors associated with the dominant eigenspaces of $A - BF_\gamma$ to estimate the derivative of $\phi(\gamma)$ to produce a new value of γ. Finally, the relationship between the eigenvalues of $A - BF_i$ and eigenvalues on the unit circle of the associated symplectic pencil can be used to estimate a subset of the unit circle eigenvalues when $\gamma < \gamma^*$. These estimates could be used to develop an update strategy similar to that used in the level set method [5]. The algorithm is easily adapted to estimate the $\mathcal{H}_\infty$ norm of a continuous time system.

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