Extended Abstract: Computational Tasks of Resolution of Singularities

Anne Frühbis-Krüger
FB Mathematik
Universität Kaiserslautern,
67653 Kaiserslautern, Germany

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1 An Illustrated Introduction to the Problem

When studying a variety (or scheme) which is singular, many properties may also be obtained from a non-singular variety which does not differ too much from the original variety. More precisely, given a variety X this approach requires a non-singular variety $\tilde{X}$ and a proper birational morphism $\pi : \tilde{X} \longrightarrow X$ which leaves $X \setminus Sing(X)$ unchanged. In a certain sense we may consider $\tilde{X}$ as a kind of smooth model of our given variety. $\tilde{X}$ (and the process of finding it) is called a resolution of singularities or desingularisation of X. There are many known special cases in which this task can be completed without major difficulties as, for example, the case of toric varieties, where toric blow-ups allow computation by combinatorial methods, or the case of normal surfaces where iterating blowing ups of the singular points and normalization suffices. In the general case, however, this problem has been a central topic in the research of many mathematicians over the last century and is up to now only solved in characteristic zero (on which we focus here), but still open in positive characteristic.
In the case of curves the problem is very accessible to direct methods and has already been solved (over the complex numbers) in the last decade of the 19th century with important contributions e.g. by L. Kronecker, M. Noether and A. Brill. The subsequent step, the case of surfaces, however, already turned out to be more delicate. Here many contributions have been made by the Italian School, among others O.Chisini, G. Albnese and F. Severi. But it was the contribution of H.W. Jung (1908), who studied surfaces (embedded in 3-dimensional space) locally by means of a projection to the plane, which lead to the first rigorous proof of the existence of resolution of singularities of surfaces over $\mathbb{C}$ by R.J.Walker in 1935. These early contributions all followed an analytic approach, whereas a more algebraic point of view entered this field of research with O.Zariski's work (proof of existence of a resolution of singularities over algebraically closed fields of characteristic zero for surfaces in 1939 and for 3-

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dimensional varieties in 1944) enabling a more systematic approach. On these foundations layed by Zariski all newer developments are based to some extent such as e.g. the contributions in low dimensions by S.Abhyankar (1966) and J. de Jong (1996) in positive characteristic and, of course, the breakthrough in characteristic zero, the monumental work of H. Hironaka in 1964 in which he proved resolution of singularities in any dimension.
In the desire to obtain a better understanding of the very complex proof of Hironaka which on one hand introduces Standard Bases and involves highly non-constructive steps on the other hand, new more algorithmic approaches have evolved since the late 1980s with important contributions by the groups of E.Bierstone and P.Milman, of O.Villamayor and S.Encinas and by H.Hauser. Common to all these approaches in characteristic zero is the construction of the proper birational morphism as a sequence of blowing ups whose center is determined by means of an induction on the dimension of the ambient space.

Before we outline the structure of this construction in a little more detail, it seems appropriate to illustrate some technical notions from algebraic geometry (whose proper definition would be beyond the scope of this extended abstract) and have a look at an example in the simplest case, the case of curves where the choice of the center does not pose any problems, since the singular locus is a finite set of points. In figure 1, two examples of resolutions of curves (with simple singularities) are shown: in the top row a desingularization is $V(x^3 - y^4) \subset \mathbb{C}^2$ is achieved by a single blow up of the singular point, in the bottom row we see a resolution of the singularity of $V(x^2 + y^4 - y^5)$ which is improved, but not resolved by the first blow up and requires a second blow up to become non-singular. Comparing the two resolved curves (as they are embedded in the respective ambient spaces), we find another difference: in the lower row the (second) exceptional divisor intersects the curve transversally, in the upper row the exceptional divisor is tangent to the curve. Here the situation in the lower row can not be avoided, since a new exceptional divisor and the transformed curve will always have a common point, but the tangency in the upper row can be improved by further blow ups as shown in figure 2. In the general case, this additional goal is expressed as the fact that the exceptional hypersurfaces arising in the blow ups should be normal crossing (which is roughly speaking the fact that locally at each point the intersection looks like the intersection of coordinate hyperplanes) and should be normal crossing with the transformed variety. In a more rigorous formulation the task of embedded resolution of singularities over a field of characteristic zero can be formulated as

> Given a smooth ambient space W and a variety (or scheme) $X \subset W$ with ideal sheaf $\mathcal{I}_X \subset \mathcal{O}_W$, find a sequence
>
> $$W = W_0 \xleftarrow{\pi_1} W_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_r} W_r$$
>
> of blow-ups $\pi_i : W_i \longrightarrow W_{i-1}$ at smooth centers $C_{i-1} \subset W_{i-1}$ such that
>
> (a) The exceptional divisor of the induced morphism $W_i \longrightarrow W$ has only normal crossings and C_i has normal crossings with it.

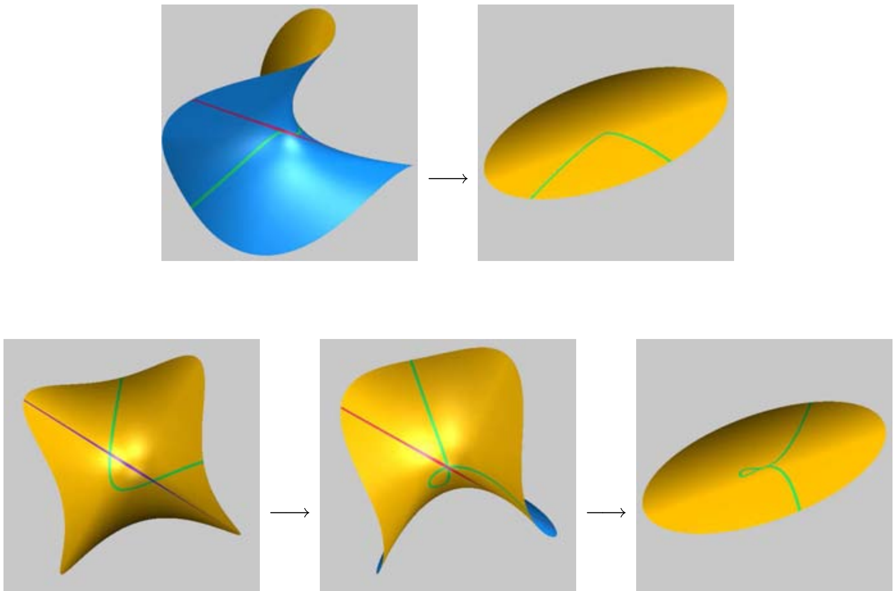

Figure 1: Desingularizations of $V(x^3-y^4)$ (upper row) and $V(x^2+y^4-y^5)$ (lower row). Each of the arrows corresponds to one blow up of the ambient space which corresponds roughly speaking to replacing the center by a projective space $\mathbb{P}^k$ of appropriate dimension k resulting in lines through the center being separated. The preimage of the center under a blow up is referred to as the exceptional divisor or exceptional hypersurface of the blow up and is drawn in a red or purple color in the above images. For each of the above blow ups the center has been chosen to be the only singular point of the respective curve; only the charts which contain a singular point or an intersection of the transformed curve with an exceptional divisor are shown. In the lower row the first exceptional divisor is no longer visible in the interesting chart, but has an intersection with the new exceptional divisor in the other chart which is not included here.

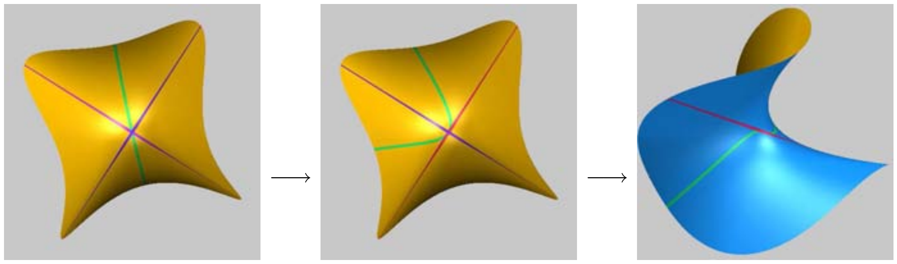

Figure 2: Two further blow ups at the points of tangency allow us to obtain transversal intersections between the exceptional divisors and the transformed curve in the example of the upper row of figure 1. One last blow up (which is not illustrated here) at the point where the 3 lines meet then leads to a normal crossing situation.

(b) Let $X_i \subset W_i$ be the strict transform of X. All centers C_i are disjoint from $Reg(X) \subset X_i$, the set of points where X is smooth.[1]

(c) X_r is smooth and has normal crossings with the exceptional divisor of the morphism $W_r \longrightarrow W$.

(d) The morphism $(W_r, X_r) \longrightarrow (W, X)$ is equivariant under group actions.

2 The General Structure of Resolution Algorithms

From the discussion in the previous section the reader could get the (wrong) impression that the key issue is the calculation of the blow up, which is in fact not a big problem since it can be implemented by a single Gröbner Basis calculation[2]. After passing to a covering of the newly introduced projective space by affine charts (to keep the number of variables low), the transforms of the variety and the 'old' exceptional divisors can then be determined by ideal quotients, which are, of course, themselves again Gröbner Basis calculations.

The hard part of the resolution process, however, is the suitable choice of the center. Intuitively, we would like to proceed by always blowing up the worst points. But what are the worst points? A first idea would be to pass to the singular locus and then to its singular locus and so on. Unfortunately, this does not always improve the situation. For example the singular locus of $V(z^2-x^2y^2)$ consists of two lines meeting transversally at the origin (see figure 3, right) but

[1]This is not a typographical error, it is really $Reg(X)$, not $Reg(X_i)$. This condition simply ensures that the sequence of blow-ups is an isomorphism on $Reg(X)$.

[2]This calculation involves $n+k+1$ variables where n is the dimension of the ambient space and k is the number of generators of the ideal of the center.

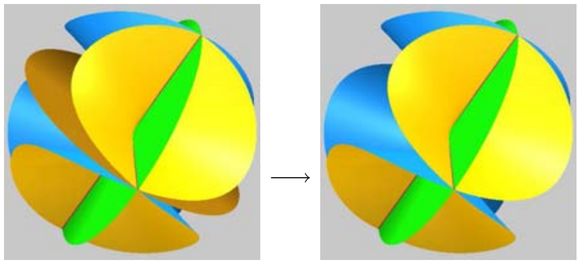

Figure 3: A blow up of the surface $V(z^2 - x^2y^2) \subset \mathbb{C}^3$ at the origin. The green hypersurface in the images is not part of the surface itself, but the hypersurface introduced for the descent in dimension of the ambient space in step (2) below.

blowing up at the origin leads to the same singularity at two different point (each appearing in precisely one chart) with an exceptional hypersurface added (see figure 3, left). To improve this singularity it is necessary to blow up with one of the lines as a center; but as the original situation is symmetric in x and y, there is no obvious choice between the two lines.
Showing that suitable choices of successive centers for resolving the singularities of any given (embedded) variety exist is the key point of Hironaka's work. The constructive approaches to the choice of the centers, which have been found in the 1990s, all follow his central idea which can be formulated in several ways. Here we choose the most algorithmic approach of defining an invariant whose maximal locus is again a variety and subsequently passing to an appropriate new ambient space of lower dimension to construct an auxilliary variety based on the previous maximal locus. Therefore the choice of centers can be divided into 3 main tasks:

(1) determining the maximal locus of an appropriate invariant for a given ambient space W and a given variety X,

(2) finding the new (lower-dimensional) ambient space for a given $X \subset W$ and the previously computed maximal locus,

(3) constructing the new scheme in this ambient space.

The most subtle of these tasks is (2) which is also a key reason that the construction does not hold in positive characteristic. Due to the constraints in the length of this extended abstract, we only have room to sketch the required properties of the hypersurface to be created in step (2) and the idea of the construction of the new variety in (3), before we state the definition of the main ingredient to the invariant of step (1).
As the hypersurface in (2) serves as the new ambient space in the induction on the dimension, it needs to be smooth and it needs to contain the maximal locus found in step (1). Additionally this second condition has to hold after any finite number of blowing ups at points inside the maximal locus as long as the value of the invariant of this locus (computed in the respective transform

of W) has not dropped. In order to make sure that the normal crossing conditions on the exceptional divisors are not violated, the new hypersurface also needs to be normal crossing with a specific subset of the exceptional divisors and the intersections of the exceptional divisors with the hypersurface need to be normal crossing (implying the normal crossing property of these divisors after descending to the new ambient space). These conditions, however, can in general not be fulfilled by a single hypersurface for all points of the maximal locus simultaneously; such a hypersurface only exists locally. Hence the center has to be constructed locally and the best that can be done is glueing it afterwards, before the subsequent blow up.
After construction of the new ambient space in step (2), a new scheme has to be defined which will play the role of our subvariety. This construction is rather technical and can be done in several ways; common to all is the fact that choosing the hypersurface in step (2) can basically be seen as fixing a main variable. Locally, after choosing appropriate coordinates, the definition of the new scheme is then achieved by constructing a new ideal in the following way: the elements of a suitably chosen set of generators of the ideal of the maximal locus[3] are considered as polynomials in the main variable with coefficents involving the remaining variables, the new ideal is generated by appropriate powers and products of these coefficients.
Up to now we have only considered the induction step on the dimension of the ambient space and used the invariant of step (1) as a black box. We thus obtained a general structure of the invariant defining the center which looks like

$$(inv_n; inv_{n-1}; \dots),$$

where inv_i denotes an invariant being computed w.r.t. an ambient space of dimension i. What is left to be explained is this inv_i for a fixed i. Here again there are several possiblities of which the easiest to explain is of the structure $(ord(I_X), n_E)$ where $ord(I_X)$ denotes the order of the ideal[4] defining the variety/scheme X and n_E counts certain exceptional divisors containing the point at which we want to evaluate. This definition of the ingredients to the invariant is again local, also due to the fact that for subsequent glueing of the local pieces, all data has to be intrinsic.

3 Computational Tasks and Solutions

The algorithmic approach described in the previous section provides algorithmic proofs of resolution of singularities, but unfortunately there are several issues which cause problems in a direct implementation (and also made the very first implementation of a resolution algorithm, which is due to G.Bodnar and J.Schicho rather inefficient):

[3]This chosen set of generators of the maximal locus is by no means minimal. Instead it is chosen in a special way such that it behaves 'well' under blowing up.

[4]This can be viewed as a generalization of the order of a power series. It is the maximal k such that $I_X, x \subset \Updownarrow_x^k$ at the point x where we are evaluating our invariant.

(a) Dramatic increase of the number of charts even after only a few blowing ups, because each blowing up introduces $n - r$ new charts where n is the dimension of the ambient space and r is the dimension of the center.

(b) The hypersurfaces of step (2) only exist locally.

(c) The ingredients of the invariant are of a local nature.

(d) Many of the involved calculations in each step, in particular the blowing ups, involve Gröbner basis calculations.

(e) Glueing of the final result.

The increase of the number of charts in (a) cannot be avoided, since not passing to charts would amount to allowing the number of variables to grow dramatically. And the latter has even worse consequences for the time and memory consumption of the computations than the duplicate work that has to be performed for points appearing in more than one chart. On the other hand, there are always charts which do not contribute any new information to our resolution process, e.g. if the transforms of the variety and the exceptional divisors do not meet any point which is not already appearing in another chart; these charts can safely be suppressed.
Issues (b) and (c) can be tackled by passing to a suitable open covering which of course increases the number of charts even more. Therefore it is essential to glue after computing the maximal locus of the invariant in each dimension and to glue the centers before blowing up instead of calculating a blow up for each open set.
Issue (d) already influenced our approach to issue (a), since we wanted to avoid an increase in the number of variables there. Here there are further important changes to be introduced, e.g. blowing up at a center consisting of several components turns out to be significantly slower than blowing up at each of the components one after the other. This, of course, leads to a (small) rise in the number of charts, but the calculations themselves and even the structures of the calculated objects are simpler - even facilitating subsequent calculations a bit. Additionally, there are several smaller changes to the calculation of the maximal locus of the invariant and more structural considerations, like e.g. which exceptional divisors and which components of the maximal locus to take into account at what step, that also contribute to keeping things as efficient as possible.
The crucial issue for applications of this algorithm is of course the glueing after termination of the algorithm, but it is again beyond the scope of this extended abstract to discuss the particular issues involved in this point.

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