Bounds and algebraic algorithms in differential algebra: the ordinary case*
(extended abstract)

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Abstract. Consider the Rosenfeld-Groebner algorithm for computing a regular decomposition of a radical differential ideal generated by a set of ordinary differential polynomials. This algorithm inputs a system of differential polynomials and a ranking on derivatives and constructs finitely many regular systems equivalent to the original one. The property of regularity allows to check consistency of the systems and membership to the corresponding differential ideals. We propose a bound on the orders of derivatives occurring in all intermediate and final systems computed by the Rosenfeld-Groebner algorithm and outline its proof. We also reduce the problem of conversion of a regular decomposition of a radical differential ideal from one ranking to another to a purely algebraic problem. For the algebraic case, efficient modular and parallel algorithms are currently being developed and implemented.

1 Introduction

Consider the ring of ordinary differential polynomials $\mathbf{k}\{Y\}$, where $\mathbf{k}$ is a differential field of characteristic 0 with differentiation δ, and $Y = \{y_1, \ldots, y_n\}$ is a set whose elements are called differential indeterminates. Let $F \subset \mathbf{k}\{Y\}$ be a set of differential polynomials, then $[F]$ and $\{F\}$ denote the differential and

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radical differential ideals generated by F in $\mathbf{k}\{Y\}$, respectively. A differential ideal may not have a finite generating system, while a radical differential ideal always has one according to the Basis Theorem [14]. One of the central problems in constructive differential algebra is the problem of computing a canonical representation for a radical differential ideal.

The problem, in general, remains open, but an important contribution to it is provided by the Rosenfeld-Gröbner algorithm [2]. This algorithm inputs a set of differential polynomials F and a ranking [10] on the set of derivatives of the indeterminates. Once the ranking is fixed, for each differential polynomial f we can select the derivative of the highest rank present in f, called the *leader* of f; the differential indeterminate present in the leader is called the *leading differential indeterminate* of f and denoted $\operatorname{lv} f$. If f is considered as a univariate polynomial in its leader, the leading coefficient of f is called its *initial*; the initial of any proper derivative of f is called its *separant*.

The selection of leaders in differential polynomials induces differential pseudo-reduction relations w.r.t. these polynomials. By applying differential pseudo-reductions [14, 10] to the elements of F and considering their initials and separants, the Rosenfeld-Gröbner algorithm constructs finitely many systems of the form $F_i = 0,\ H_i \neq 0$, where $F_i, H_i \subset \mathbf{k}\{Y\}$, $i = 1, \ldots, m$. At any intermediate step of the algorithm, these systems are equivalent to F: each solution of $F = 0$ is a solution of $F_i = 0,\ H_i \neq 0$ for some i and vice versa. Speaking algebraically, we have the following representation for the radical differential ideal generated by F:

$$\{F\} = \bigcap_{i=1}^{m} \{F_i\} : H_i^{\infty}.$$

The algorithm terminates when all systems $F_i = 0,\ H_i \neq 0$ become regular [2], which in the ordinary case means that

- F_i is differentially autoreduced,
- H_i is reduced w.r.t. the derivatives of F_i, and
- $H_{F_i} \subseteq H_i$, where H_{F_i} is the set of of initials and separants of the elements of F_i.

In this case, $\{F_i\} : H_i^{\infty} = [F_i] : H_i^{\infty}$, and we obtain a regular decomposition

$$\{F\} = \bigcap_{i=1}^{m} [F_i] : H_i^{\infty}.$$

This decomposition allows to solve the membership problem for $\{F\}$ as follows [2]: $f \in \{F\}$ iff the differential pseudo-remainder of f w.r.t. F_i belongs to the algebraic ideal $(F_i) : H_i^{\infty}$, for all $i \in \{1, \ldots, m\}$.

The complexity of the Rosenfeld-Gröbner algorithm is an open problem. Yet for the corresponding algebraic problem of computing a regular decomposition of a radical algebraic ideal in $\mathbf{k}[Y]$, bounds on complexity are known [16]. Thus, the first natural step towards obtaining complexity bounds in the differential case would be estimating the orders of derivatives occurring in the polynomials computed by the Rosenfeld-Gröbner algorithm. For systems of linear differential polynomials and systems of two differential polynomials in two indeterminates, Ritt [13] has proved that the Jacobi bound on the orders holds (the Rosenfeld-Gröbner algorithm was discovered later, but Ritt's techniques provide the starting point for our analysis of this algorithm).

Our first result provides a bound for the orders of derivatives occurring in the systems $F_i = 0,\ H_i \neq 0$ (for an arbitrary ranking). Let $m_i(F)$ be the maximal order of a derivative of the i-th indeterminate occurring in F, and let

$$M(F) = \sum_{i=1}^{n} m_i(F).$$

We propose a modification of the Rosenfeld-Gröbner algorithm, in which for every intermediate system $F_i = 0,\ H_i \neq 0$, we have

$$M(F_i \cup H_i) \leq (n-1)!M(F).$$

Our second result is a reduction of the problem of conversion of regular decompositions of radical differential ideals from one ranking to another to a purely algebraic problem. For the algebraic case, efficient modular algorithms are currently being developed [4] and implemented using the `RegularChains` library in Maple [11]; a parallel implementation on a shared memory machine is also in progress [12].

We note that each regular component $[F_i] : H_i^\infty$ can be decomposed further into an intersection of characterizable differential ideals [8] of the form $I_j = [\mathbb{C}_j] : H_{\mathbb{C}_j}^\infty$, where $\mathbb{C}_j$ is an autoreduced subset of I_j of the least rank (called a characteristic set [10] of I_j). Then we obtain a characteristic decomposition $\{F\} = \bigcap_{j=1}^{t} I_j$ of the radical differential ideal. This decomposition makes the membership problem even easier: $f \in \{F\}$ iff the full pseudo-remainder of f w.r.t. $\mathbb{C}_j$ is equal to 0 for all $j \in \{1, \ldots, t\}$.

A prime differential ideal I is characterizable w.r.t. any ranking, and for any characteristic set $\mathbb{C}$ of I, we have $I = [\mathbb{C}] : H_{\mathbb{C}}^\infty$. The minimal differential prime components (called the essential prime components) of a characterizable ideal $I = [\mathbb{C}] : H_{\mathbb{C}}^\infty$ correspond to the minimal prime components of the algebraic ideal $(\mathbb{C}) : H_C^\infty$ [8]: a set $\mathbb{A}$ is a characteristic set of a minimal prime of $(\mathbb{C}) : H_{\mathbb{C}}^\infty$ if and only if $\mathbb{A}$ is a characteristic set of an essential prime component of I; the corresponding algebraic and differential prime components are equal to $(\mathbb{A}) : H_{\mathbb{A}}^\infty$ and $[\mathbb{A}] : H_{\mathbb{A}}^\infty$, respectively. Moreover, the leaders of $\mathbb{A}$ coincide with those of $\mathbb{C}$.

In some sense, a decomposition of a radical differential ideal into characterizable components is the closest we can get to a prime decomposition, when we are restricted to the operations of pseudo-reduction and taking initials and separants. In general, a characteristic decomposition depends on the ranking (an ideal, which is characterizable w.r.t. one ranking, may not be characterizable w.r.t. another). Moreover, the complexity of the computation of a characteristic decomposition also significantly depends on the ranking: characteristic decompositions w.r.t. elimination rankings are, in general, harder to compute than those w.r.t. orderly rankings. Thus, instead of computing a decomposition w.r.t. a given arbitrary ranking directly, by applying the Rosenfeld-Gröbner algorithm, one can first compute a decomposition w.r.t. an orderly ranking, and then convert it by decomposing each orderly characterizable component into an intersection of ideals characterizable w.r.t. the target ranking. We show that the latter conversion step can be performed by a purely algebraic algorithm.

2 Bound on the orders of derivatives

We will modify the Rosenfeld-Gröbner algorithm, in order to ensure the above bound on the orders of derivatives occurring in the intermediate systems of differential polynomials. The modification consists in the following steps:

1. Given a set F of differential polynomials and a ranking, the conventional Rosenfeld-Gröbner algorithm at first computes a characteristic set $\mathbb{C}$ of F, i.e., an autoreduced subset of F of the least rank. We replace this computation by a computation of a weak d-triangular subset of F of the least rank, which we call a *weak differential characteristic set* of F.[5] A set $\mathbb{C} \subset \mathbf{k}\{Y\} \setminus \mathbf{k}$ is called a weak d-triangular set [9, Definition 3.7], if the set of its leaders $\operatorname{ld}\mathbb{C}$ is autoreduced. In the ordinary case, $\mathbb{C}$ is a weak d-triangular set if and only if the leading differential indeterminates $\operatorname{lv} f$, $f \in \mathbb{C}$, are all distinct. The differential pseudo-remainder of a polynomial f w.r.t. a weak d-triangular set $\mathbb{C}$ is defined via [9, Algorithm 3.13].

   Unlike characteristic sets, weak characteristic sets satisfy the following property, which is essential for the proof of our bound:

   Lemma 1. *Let F be a set of differential polynomials, and let $\mathbb{C}$ be a weak characteristic set of F. Then* $\operatorname{lv}\mathbb{C} = \operatorname{lv} F$.

2. At the second step, the Rosenfeld-Gröbner algorithm computes the differential pseudo-remainders of $F \setminus \mathbb{C}$ w.r.t. $\mathbb{C}$. The orders of derivatives of non-leading indeterminates (i.e., those not in $\operatorname{lv}\mathbb{C}$) occurring in these pseudo-remainders may be higher than those in F (unless the chosen ranking is orderly). In order to control this growth of orders, we construct a *differential prolongation* of the weak characteristic set $\mathbb{C}$, i.e., an algebraically triangular set $\mathbb{B}$ such that the differential pseudo-reduction of $F \setminus \mathbb{C}$ w.r.t. $\mathbb{C}$ can be replaced by the algebraic pseudo-reduction w.r.t. $\mathbb{B}$. We give the specification of the algorithm computing the differential prolongation, leaving out the details of the computation:

[5] In what follows, we will omit the adjective "differential" from the term "weak differential characteristic set". This should not lead to confusion, since we are not going to use the corresponding algebraic notion.

Algorithm $\mathsf{Differentiate\&Autoreduce}(\mathbb{C}, \{m_i\})$

Input: a weak d-triangular set $\mathbb{C} = C_1, \ldots, C_k$ with $\operatorname{ld} \mathbb{C} = y_1^{(d_1)}, \ldots, y_k^{(d_k)}$, and a set of non-negative integers $\{m_i\}_{i=1}^n$, $m_i \geq m_i(\mathbb{C})$

Output: set $\mathbb{B} = \{B_i^j \mid 1 \leq i \leq k,\ 0 \leq j \leq m_i - d_i\}$ satisfying

$\operatorname{rk} B_i^j = \operatorname{rk} C_i^{(j)}$

$\mathbb{B} \subset [\mathbb{B}^0] \subset [\mathbb{C}] \subset [\mathbb{B}] : H_{\mathbb{B}}^\infty$, where $\mathbb{B}^0 = \{B_i^0 \mid 1 \leq i \leq k\}$

$H_{\mathbb{B}} \subset H_{\mathbb{C}}^\infty + [\mathbb{C}]$, $\quad H_{\mathbb{B}}^\infty H_{\mathbb{C}} \subset H_{\mathbb{B}}^\infty + [\mathbb{B}]$

B_i^j are partially reduced w.r.t. $\mathbb{C} \setminus \{C_i\}$

$m_i(\mathbb{B}) \leq m_i + \sum_{j=1}^k (m_j - d_j)$, $i = k+1, \ldots, n$

or $\{1\}$, if it is detected that $[\mathbb{C}] : H_{\mathbb{C}}^\infty = (1)$

Note that, if the saturated ideal $[\mathbb{C}] : H_{\mathbb{C}}^\infty$ contains 1, the above algorithm may detect this, in which case it returns $\varnothing$ (if this is the case, it is not necessary and does not make sense to compute pseudo-remainders w.r.t. $\mathbb{C}$).

We obtain the following modification of the Rosenfeld-Gröbner algorithm:

Algorithm $\mathsf{RGBound}(F_0, H_0)$

Input: a finite set of differential polynomials F_0, H_0 and a differential ranking

Output: a finite set T of regular systems such that

$$\{F_0\} : H_0^\infty = \bigcap_{(\mathbb{A}, H) \in T} [\mathbb{A}] : H^\infty \quad \text{and}$$

$$M(\mathbb{A} \cup H) \leq (n-1)! M(F_0 \cup H_0) \text{ for } (\mathbb{A}, H) \in T.$$

$T := \varnothing, \quad U := \{(F_0, \varnothing, H_0)\}$
while $U \neq \varnothing$ **do**
  Take and remove any $(F, \mathbb{C}, H) \in U$
  $f :=$ an element of F of the least rank
  $D := \{C \in \mathbb{C} \mid \operatorname{lv} C = \operatorname{lv} f\}$
  $G := F \cup D \setminus \{f\}$
  $\bar{\mathbb{C}} := \mathbb{C} \setminus D \cup \{f\}$
  $\mathbb{B} := \mathsf{Differentiate\&Autoreduce}(\bar{\mathbb{C}}, \{m_i(G \cup \bar{\mathbb{C}} \cup H) \mid 1 \leq i \leq n\})$
  if $\mathbb{B} \neq \{1\}$ **then**
    $\bar{F} := \mathsf{algrem}(G, \mathbb{B}) \setminus \{0\}$
    $\bar{H} := \mathsf{algrem}(H, \mathbb{B}) \cup H_{\mathbb{B}}$
    if $\bar{F} \cap \mathbf{k} = \varnothing$ **and** $0 \notin \bar{H}$ **then**
      if $\bar{F} = \varnothing$ **then** $T := T \cup \{(\mathbb{B}^0, \bar{H})\}$
        else $U := U \cup \{(\bar{F}, \bar{\mathbb{C}}, \bar{H})\}$
      end if
    end if
  end if
  if $\mathbf{s}_f \notin \mathbf{k}$ **then**
    $U := U \cup \{(F \cup \{\mathbf{s}_f\}, \mathbb{C}, H)\}$
    if $\mathbf{i}_f \notin \mathbf{k}$ **then** $U := U \cup \{(F \cup \{\mathbf{i}_f\}, \mathbb{C}, H)\}$ **end if**
  end if
end while
return T

Invariants:

1. $\{F_0\} : H_0^\infty = \bigcap_{(F,\mathbb{C},H)\in U} \{F \cup \mathbb{C}\} : H^\infty$
2. for all $(F, \mathbb{C}, H) \in U$:
   - $\mathbb{C}$ is weak d-triangular
   - $F \neq \varnothing$
   - F and H are reduced w.r.t. $\mathbb{C}$

Termination: at each iteration, a triple $(F, \mathbb{C}, H) \in U$ is replaced by at most three triples $(\bar{F}, \bar{\mathbb{C}}, \bar{H}), (F \cup \{\mathbf{i}_f\}, \mathbb{C}, H), (F \cup \{\mathbf{s}_f\}, \mathbb{C}, H)$. For the first triple we have $\operatorname{rk} \bar{\mathbb{C}} < \operatorname{rk} \mathbb{C}$. For the last two, the ranks of the smallest elements of $F \cup \{\mathbf{i}_f\}$ and $F \cup \{\mathbf{s}_f\}$ are less than the rank of the smallest element of F, while $\mathbb{C}$ remains the same.

We give a sketch of the proof of the bound. If $k < n$, we define for an arbitrary set G the quantity

$$M_{\mathrm{lv}\,F}(G) := (n-k)\sum_{i=1}^{k} m_i(G) + \sum_{i=k+1}^{n} m_i(G).$$

Then we show that the pairs $(F, H) \in U$ satisfy the following invariant:

$$\begin{array}{ll} M_{\mathrm{lv}\,F}(F \cup H) \leq (n-1)\ldots(n-k)\cdot M(F_0 \cup H_0), & k < n \\ M(F \cup H) \leq (n-1)! \cdot M(F_0 \cup H_0), & k = n. \end{array}$$

This invariant, together with the straightforward inequality

$$M(F \cup H) \leq M_{\mathrm{lv}\,F}(F \cup H),$$

yields the bound stated in the specification of Algorithm RGBound.

3 Algebraic conversion of characteristic sets of prime differential ideals

We first consider a special case, when the given characterizable ideal $I = [\mathbb{C}] : H_{\mathbb{C}}^\infty$ is prime, and it is required to convert its characteristic set $\mathbb{C}$ from one ranking to another (the problem of efficient conversion of characteristic sets of prime differential ideals from one ranking to another has been addressed in [3, 1, 5]).

Given the orders of derivatives occurring in $\mathbb{C}$, we provide a bound on the orders of derivatives occurring in a characteristic set of I w.r.t. the target ranking. We could use the bound $(n-1)!M(\mathbb{C})$ discussed above, but this bound is too crude, since it applies to a more general case of a radical differential ideal specified by a set of generators. In our particular situation of a prime differential ideal specified by a characteristic set, based on [15, Theorem 24] (if the target ranking is an elimination ranking) or [7, Theorem 6] (for an arbitrary target ranking), we can show that a better bound of $n \cdot \max m_i(\mathbb{C})$ holds.

Using this bound, we find a prime algebraic sub-ideal $J \subset I$, which contains a characteristic set $\bar{\mathbb{C}}$ of I w.r.t. the target ranking. We also present an algorithm which extracts $\bar{\mathbb{C}}$ from an algebraic characteristic set of J w.r.t. the target ranking, thus reducing the problem of computation of $\bar{\mathbb{C}}$ to a purely algebraic problem of computing an algebraic characteristic set of a prime algebraic ideal.

We have carried out a preliminary implementation of this algorithm using the `RegularChains` library in Maple.

4 Algebraic conversion of differential regular decompositions

Now consider the general case, when we are given an arbitrary characterizable differential ideal $I = [\mathbb{C}] : H_{\mathbb{C}}^{\infty}$ and need to compute its characteristic decomposition w.r.t. another ranking. Since the essential prime components of I correspond to the minimal primes of the algebraic ideal $(\mathbb{C}) : H_{\mathbb{C}}^{\infty}$, and thus their characteristic sets can be computed from $\mathbb{C}$ without applying differentiations, we have the bound $M = n \cdot \max m_i(\mathbb{C})$ for the characteristic sets of the essential primes of I w.r.t. the target ranking.

Let $d = \max_{f \in \mathbb{C}}(M - \operatorname{ord} \operatorname{ld} f)$, where $\operatorname{ld} f$ denotes the leading derivative of f w.r.t. the initial ranking and $\operatorname{ord} \operatorname{ld} f$ is its order, and let

$$\mathbb{C}^{(d)} = \{f^{(k)} \mid f \in \mathbb{C},\ 0 \le k \le d\}.$$

Consider the algebraic ideal $J = (\mathbb{C}^{(d)}) : H_{\mathbb{C}}^{\infty}$. This ideal is characterizable w.r.t. the initial ranking and satisfies the following properties:

- $(\mathbb{C}) : H_{\mathbb{C}}^{\infty} \subset J \subset I$;
- the minimal primes of J correspond to the essential primes of I (and to the minimal primes of the algebraic ideal $(\mathbb{C}) : H_{\mathbb{C}}^{\infty}$);
- every minimal prime of J contains a differential characteristic set of the corresponding essential prime of I w.r.t. any ranking.

Applying a purely algebraic (and factorization-free) algorithm, we compute a decomposition $J = J_1' \cap \ldots \cap J_l'$ into algebraic "bi-characterizable" components, i.e., ideals characterizable w.r.t. both initial and target rankings (one could also compute a universal characteristic decomposition [6] of J, but for our purposes characterizability w.r.t. the initial and target rankings is sufficient).

We observe that a component J_i', whose characteristic set w.r.t. the initial ranking has a set of leaders distinct from $\operatorname{ld} \mathbb{C}^{(d)}$, is a redundant component, i.e., $J = \cap_{j \ne i} J_j'$. So, we can assume that the characteristic sets of J_i' have leaders equal to $\operatorname{ld} \mathbb{C}^{(d)}$ for all $i = 1, \ldots, l$. We prove then that every minimal prime component Q of J_i' is also a minimal prime component of J, hence it corresponds to an essential prime component $P \supset Q$ of I.

Now, due to the choice of d, every minimal prime of $J = (\mathbb{C}^{(d)}) : H_{\mathbb{C}}^{\infty}$ contains a differential characteristic set of the corresponding essential prime of I w.r.t. any ranking. We take the algebraic characteristic set of J_i' w.r.t. the target ranking and extract from it a weak characteristic set $\mathbb{T}_i$. We autoreduce $\mathbb{T}_i$ differentially, obtaining $\mathbb{B}_i$, and consider the saturated ideal $I_i' = [\mathbb{B}_i] : H_{\mathbb{B}_i}^\infty$ (here autoreduction, initials, and separants are considered w.r.t. the target ranking). We show that $\mathbb{B}_i$ is a characteristic set of I_i' w.r.t. the target ranking, and the essential primes of I_i' are those essential primes of I that contain the minimal primes of J_i'. Therefore, we obtain a characteristic decomposition w.r.t. the target ranking:

$$I = \bigcap_{i=1}^{l} I_i' = \bigcap_{i=1}^{l} [\mathbb{B}_i] : H_{\mathbb{B}_i}^\infty.$$

5 Conclusions

It is essential to understand, to which extent the computation of a characteristic decomposition of a radical differential ideal can be obtained by purely algebraic means, i.e., extracted from an algebraic characteristic decomposition of a differential prolongation of a given system of differential polynomials. We have shown how to do this for the problem of conversion of a characteristic decomposition from one ranking to another, in the ordinary case. The partial case remains open.

For the more general problem of computing a characteristic decomposition of a radical differential ideal from a system of generators, we conjecture that its reduction to a purely algebraic problem is equivalent to the Ritt problem [14]. However, a precise formulation of this conjecture yet has to be written.

We also suggest that the bounds on the orders of derivatives proposed in this paper provide an initial step towards obtaining a bound on the complexity of the Rosenfeld-Gröbner algorithm.

On the experimental side, our next goal is to implement the algebraic algorithm for converting differential characteristic decompositions from one ranking to another. This will allow us to see whether the benefits of efficient algebraic methods outweigh the costs associated with the increase in the number of variables and polynomials resulting from the differential prolongation.

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