

Die Another Day

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Abstract. The hydra was a many-headed monster from Greek mythology that could grow one or two new heads when one of its heads got cut off. It was the second task of Hercules to kill this monster. In an abstract sense a hydra can be modeled by a tree where the leaves are the heads, and when a head is cut off some subtrees get duplicated. Different hydra species differ by the location of subtrees to be duplicated and by the number of new subtrees grown in each step. Using some deep mathematics, it had been shown that two classes of rather restricted hydra species must always die, independent of the order in which heads are cut off. In this paper we provide an elementary proof which actually gives a complete classification of all hydra species as immortal or doomed. Now, if Hercules had known this...

1 Introduction

According to Greek mythology, the *hydra* was a many-headed monster living in a marsh near Lerna [2]. If one head was cut off, one or two new heads grew from the hydra's body. Nevertheless, in his second task (of twelve, ordered by his cousin Eurystheus) the Greek hero Hercules (a.k.a. Herakles), a son of Zeus, defeated the hydra, although he did not fully play by the rules: his nephew Iolaus burnt the hydra to prevent new heads from growing [4].

Kirby and Paris studied this fight from a graph theoretic point of view and showed that Hercules might even have won without employing unfair tactics (although then he would probably still be fighting today) [1]. They

* The work described in this paper was partially supported by a grant from the National Natural Science Fund China (grant no. 60573025).

proposed to model the hydra as a rooted tree where the heads are the leaves (see Fig. 1). The classical hydra would grow one or two new heads replacing the head that was cut off. Clearly, this kind of hydra cannot die. Instead, Kirby and Paris suggested to duplicate some of the remaining subtrees after a head was cut off. This would add many new heads in one step, so one might suspect that such a hydra would also be immortal.

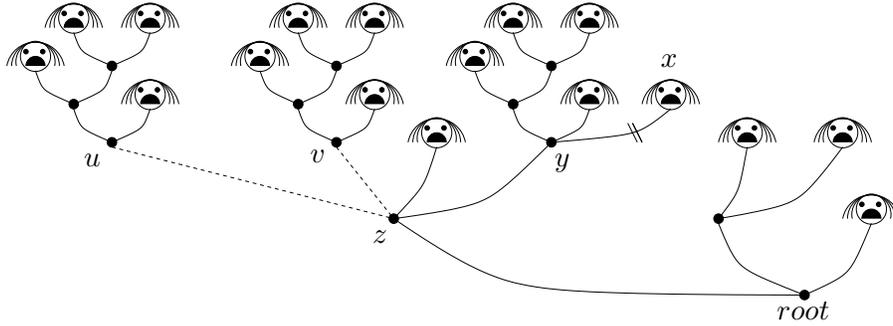


Fig. 1. Cutting off x and growing two new subtrees identical to T_y (without x) from z .

To be more precise, they suggested to duplicate subtrees as follows (we call this hydra species the *i-head hydra*). For a node v let T_v denote the subtree rooted at v . For a leaf x the predecessor y is the *neck*, and the predecessor's predecessor z is the *trunk*. The path from the root to y is the *spine* $S(x)$ of x . When x is cut off in the i -th blow, i new subtrees identical to T_y (without x) will grow from the trunk z . Fig. 1 shows the second blow where cutting off head x will result in two copies of T_y growing out of z , rooted at the new nodes u and v . If x has no siblings, its neck y becomes a leaf, i.e., a new head, and i new heads (the copies of T_y) grow from the trunk z . If y is the root, no new subtrees grow. If x is the root, i.e., the root is the only node (and head) of the hydra, cutting off x will kill the hydra.

A hydra is a *survivor* if it will not die in a finite number of steps for at least one sequence of head cuts. A hydra is *immortal* if it will not die in a finite number of steps for any possible sequence of head cuts. A hydra is *doomed* if it will die in a finite number of steps, for any possible sequence of head cuts.

In a complicated proof using transfinite induction Kirby and Paris showed that the i -head hydra is doomed. Later, Luccio and Pagli gave an elementary combinatorial proof based on a potential function defined on the nodes for the special case of the *2-head hydra* which can only grow two (or any fixed constant number of) new subtrees in each step and posed as an open problem to find an elementary combinatorial proof for the i -head hydra [3].

In this paper we give such a proof for the more general class of *finite hydra*. As we will see later, the number of subtrees grown in each duplication step is not relevant for the fate of a hydra as long as it is finite, only the locations of the subtrees to be copied do matter. Therefore, we assume that a finite hydra in each step can sprout an arbitrary but finite number of new subtrees. The number of subtrees may either be predetermined (e.g., the 2-head hydra) or given as a function of the structure of the tree or the length of the fight (e.g., the i -head hydra), or the hydra may adapt to the cutting sequence, deciding on the number of tree copies each time a head is cut off.

Pruning a hydra T means to repeatedly cut off heads without duplicating subtrees and to contract paths whose intermediate nodes have degree 2 into single edges (see Fig. 2 for an example). The resulting hydra is then a *subhydra* of T .

Now we can specify which subtrees a finite hydra may copy when losing a head x at neck y on trunk z (see Fig. 2). Let $S(x) = (\text{root} = v_1, v_2, \dots, v_k = y)$ be the spine of x . For $i = k - 1$ (note that we do not allow to grow new trees out of the neck) down to 1, we grow at v_i a finite number of copies of $T_{v_{i+1}}$. Then we prune the hydra arbitrarily.

Theorem 1 (Hydra Theorem). *Any finite hydra is doomed.*

We will prove the Hydra Theorem in Section 2. In Section 3 we will see that a hydra that can copy other subtrees than those allowed for a finite hydra is immortal.

2 Proof of the Hydra Theorem

In this section, all hydras are finite. We start with a few easy observations. First we see that the property of being a subhydra is preserved after cutting off the same heads in both hydras.

If T is a hydra and ρ is a cutting sequence (i.e., a sequence of leaves to be cut), let $T(\rho)$ denote the sequence of tree duplications chosen by T (it is not

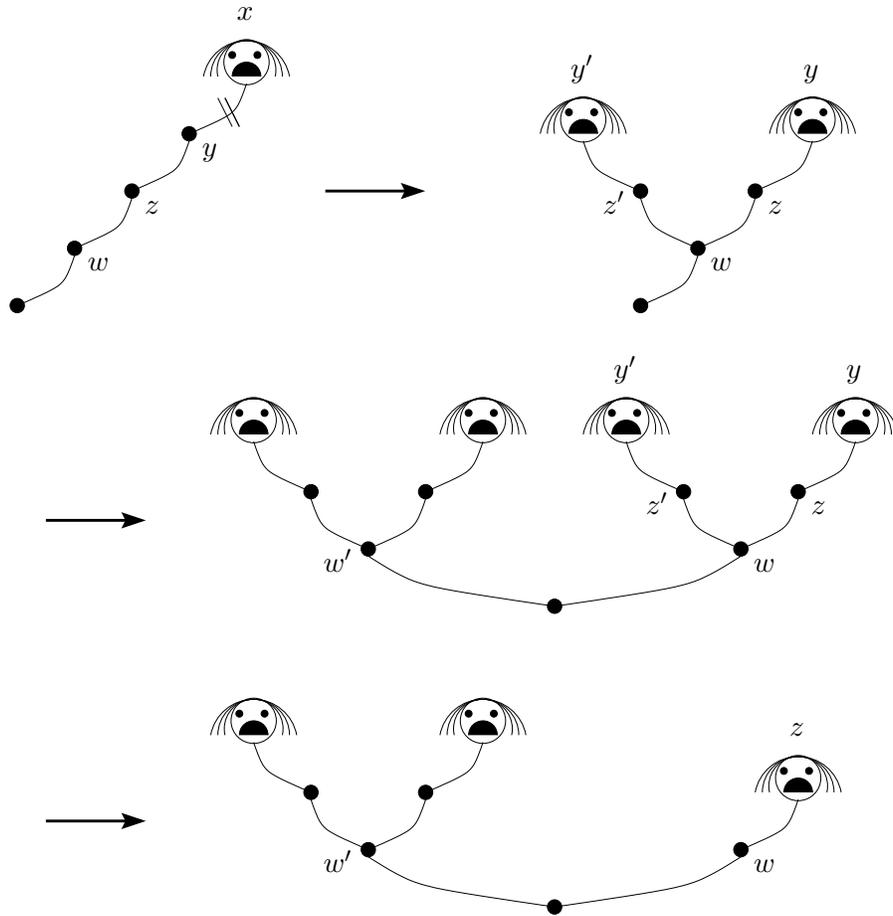


Fig. 2. A finite hydra. If we cut off x , we can first sprout at w a copy of T_z , and then sprout at the root a copy of T_w . Then we prune the right subtree by first contracting the path from w to y' into a single edge and then deleting the leaves y and y' .

important how we encode this in detail) and let T^ρ denote the hydra after the last cut.

Lemma 2. *Let S be a subhydra of T and let ρ be an arbitrary cutting sequence in S . Then we can apply the same cutting sequence to the corresponding nodes in T , and if T answers with subtree duplications corresponding to those in $S(\rho)$, then S^ρ will be a subhydra of T^ρ .*

Proof. It is straightforward to prove this by induction on the length of ρ . Alternatively, we may cut off heads in T without duplicating subtrees until T is essentially identical to S (we cannot just cut off internal nodes with a single child, these nodes have to be cut off when they become leaves) and then execute ρ in T . \square

Corollary 3. *Let S be a subhydra of T . If T is doomed, then S is doomed. If S is immortal or a survivor, then T is a survivor.*

In light of Corollary 3 we can w.l.o.g. assume that a hydra will always duplicate complete subtrees.

Consider a hydra tree. Whenever we decrease the potential of a leaf w we rename w , let's say to w' , and we say w' was *triggered* by w . If we create a new copy of w when duplicating subtrees, we also say w' was *triggered* by w . We can now build a *hacking tree* T where the nodes correspond to leaves (heads) of the hydra at a certain time and where the children of a node v corresponding to a leaf w of the hydra correspond to the leaves that were triggered by w in one step of the hacking. Note that every node in T will have finite degree (because we only create a finite number of duplicate subtrees in every step). We now need Koenig's Lemma.

Lemma 4 (Koenig's Lemma). *A tree is finite if and only if every node has finite degree and every path is finite.* \square

If we assume the hydra could live forever for some hacking strategy, then the corresponding hacking tree T must have an infinite path P . On P , the first node corresponds to one of the initial heads of the hydra, and every other node was triggered by its preceding node.

3 Immortal Hydras

Also, the new subtrees do not need to grow out of the trunk. When a leaf x is cut off in step i , let y be an arbitrary predecessor of x and z be an arbitrary predecessor of y . Then f_i new subtrees identical to the subtree rooted at y (with x deleted) grow from z .

A normal hydra can only grow subtrees at an ancestor of the cut-off head. A *vicious hydra* can also grow new subtrees at any other node at the same level.

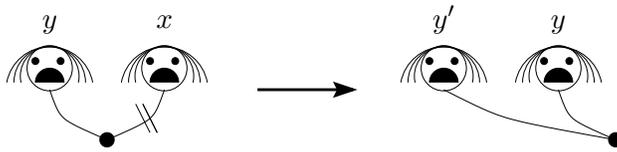


Fig. 3. A hydra that can sprout new subtrees at the neck. If we cut off x , a copy of y will grow and the hydra just looks the same as before, i.e., it is immortal.

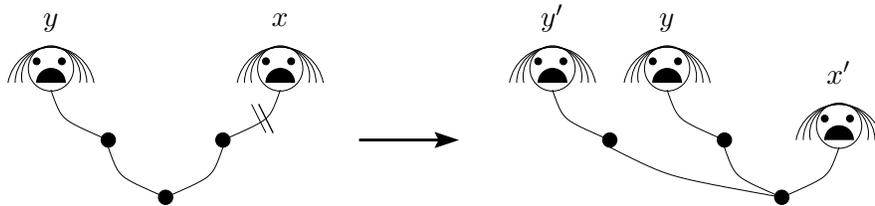


Fig. 4. A hydra that can copy a subtree not containing the head that was cut off. If we cut off x and copy the path from the root to y , the hydra becomes bigger, i.e., it is immortal.

Theorem 5. *A vicious hydra can live forever.*

Proof. Consider the hydra in Fig. 6. If we cut off head x , we can grow a subtrees identical to the subtree T_v without x , which is just the path from v to y , at node w which is at the same level as v . If we now cut off the new head y' , u' will become a head and we can grow a new head at z . But now T_y is again identical to its original shape, so we can repeat these steps indefinitely, each time creating a new head in T_w . \square

4 Conclusions

We have completely characterized those hydra that are doomed and those that are immortal.

5 Acknowledgments

The particular style to draw the hydra trees is due to Luccio and Pagli [3]).

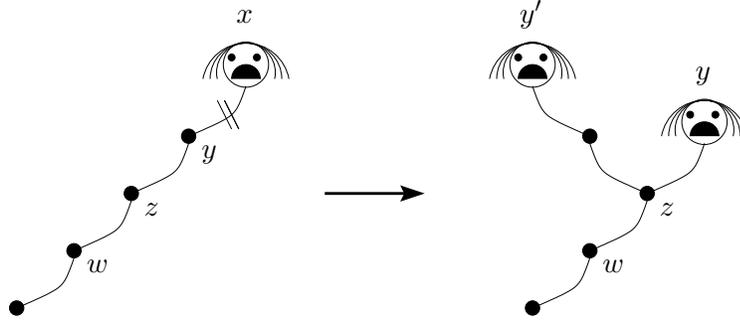


Fig. 5. A hydra that can place a copy of a subtree higher up on the spine. This is the same hydra as in Fig. 2. But this time we sprout the copy of T_z not at w but at z (which is above w) and the hydra becomes bigger as before, i.e., it is immortal.

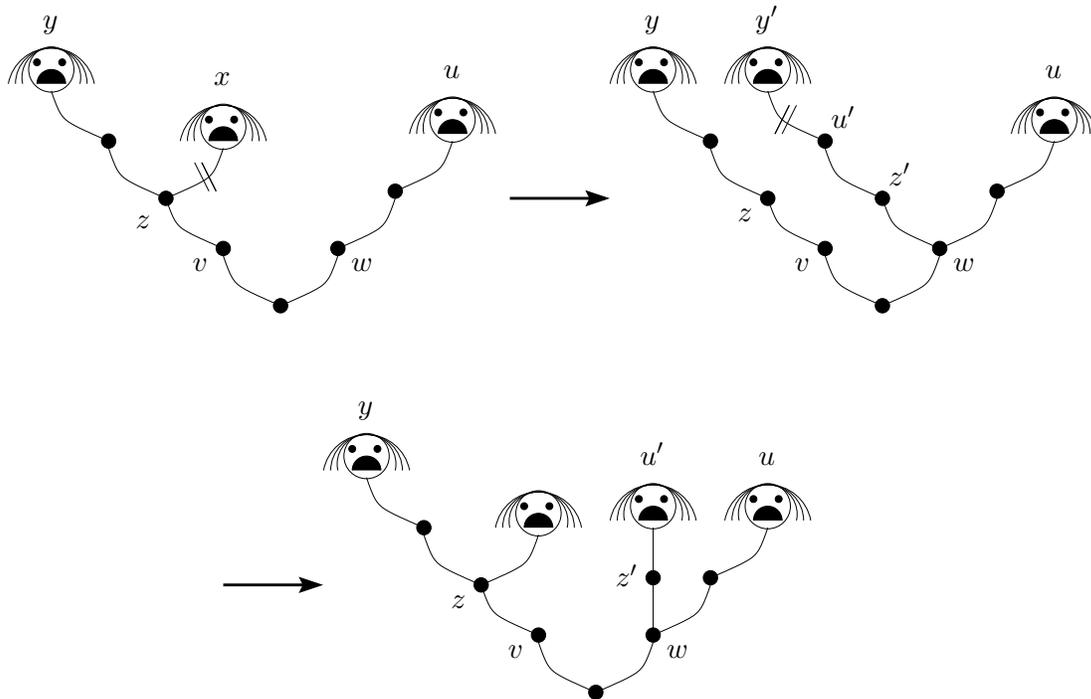


Fig. 6. A hydra that can copy a subtree to somewhere else.

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