Tree-Sparse Modeling and Solution of Multistage Stochastic Programs

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Keywords. Tree-sparse programs, multistage stochastic optimization, KKT systems, hierarchical sparsity

1 Extended Abstract

The lecture presents an integrated modeling and solution framework aiming at the robust and efficient solution of very large instances of *tree-sparse programs*. This wide class of nonlinear programs (NLP) is characterized by an underlying tree topology. It includes, in particular, dynamic stochastic programs in scenario tree formulation, *multistage stochastic programs*, where the objective and constraints depend on random future events with known probability distributions.

The basic principle behind the overall concept lies in a suitable nesting of generic and problem-specific algorithmic layers, each handling separate aspects. The large, tree-sparse NLP is tackled by a primal-dual interior method or, as in [1], by an SQP method using a primal-dual interior method as QP solver. These generic methods handle inequalities and, if applicable, nonlinearity and nonconvexity. On the bottom level, the central computational step in every interior iteration consists in calculating a Newton direction from the indefinite *KKT system* representing suitable linearizations of the perturbed Karush–Kuhn–Tucker optimality conditions. Principal features of the tree-sparse approach include the theoretical analysis and algorithmic exploitation of the structure of such KKT systems, based on their natural interpretation as linear-quadratic control problems. This leads to the distinction of generic block sparsity characterizing the entire class of tree-sparse problems, and sub-block sparsity specific to individual instances. The resulting KKT solution algorithm handles the generic block structure according to the general analysis, and the sub-block structure by local sparse matrix techniques [2,3,4]. The tree-sparse framework generalizes earlier work, providing a unified formulation for multistage stochastic programs and various other problems sharing a similar KKT structure, like trajectory optimization problems or the spatial dynamics of multibody systems in descriptor form. Stochastic programming is the primary application field and source of motivation for the generalization. Here the scenario tree reflects the underlying *information structure*, or *nonanticipativity* requirements: every node represents a decision that may depend on the past (the path to the node) but may not anticipate specific realizations of the future. Given the robust overall framework

Dagstuhl Seminar Proceedings 05031
Algorithms for Optimization with Incomplete Information
http://drops.dagstuhl.de/opus/volltexte/2005/71

based on interior methods and possibly SQP methods, the primary goals in algorithm development are *memory efficiency* and *runtime efficiency*, to cope with the enormous size of usual scenario tree problems. The analysis of typical properties of application models leads to the distinction of three major tree-sparse problem types with different regularity properties, each having an associated block-level KKT solution algorithm that can be adapted to the sub-block sparsity of individual instances via specialized data structures and node operations. Thus we arrive at a general modeling and solution framework, providing strong guidelines how to pose a specific problem instance properly and how to construct an associated, highly specialized sparse KKT solver [4,5].

Interior methods are widely known to be well suited for very large-scale linear and nonlinear programming, and have already earlier been considered in stochastic programming. However, their practical performance depends critically on efficient KKT solvers. When our work started, only a single paper [6] addressed the multistage case, using a general-purpose sparse solver with pivoting heuristics adapted to the tree topology. All other approaches aimed at the linear two-stage case only, most of them being based on the very first interior method for stochastic programs [7] and addressing sparsity on a coarse block level. Our approach is the first to systematically develop KKT solution techniques based on a detailed analysis of the rich hierarchical structure in tree-sparse programs, which is then fully exploited. It is also the first that can handle *global constraints* directly and efficiently. Such constraints may couple arbitrary nodes across the tree. In stochastic programs they arise as terminal conditions involving expectations; a prominent application is Markowitz type portfolio management [8], [2], for an application in process engineering see [9]. Other examples of global constraints include periodicity conditions in trajectory optimization or the modeling of kinematic loops in multibody systems.

A coarse comparison to decomposition methods (cf. the survey papers [10,11]) is as follows. Primal (nested Benders) decomposition methods, being based on successive polyhedral approximations of nodal subproblems, are suitable for linearly constrained convex stochastic programs and have proven highly successful in the purely linear case. They are not well suited for the general nonlinear case or for problems with global constraints. Dual decomposition methods have the potential of solving very general stochastic programs, including non-smooth ones. Such methods are very successful in stochastic *integer* programming [12], but only moderately efficient on smooth problems. The proposed tree-sparse approach is efficient on linear as well as nonlinear (smooth) problems, and especially well-suited when global constraints or significant nonlinearities are involved.

Regarding practicability, the suggested specialization of the KKT solver for individual applications would generally be unacceptable if it had to be performed manually. A software tool for that purpose has therefore been developed [13]. This tool generates sparse implementations of the generic block operations, in the form of source code and with emphasis on memory efficiency.

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